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ISOMETRY GROUPS OF PROPER METRIC SPACES

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ABSTRACT. Given a locally compact Polish space X, a necessary and sufficient condition for a group G of homeomorphisms of X to be the full isometry group of (X, d) for some proper metric d on X is given. It is shown that every locally compact Polish group G acts freely on $G \times X$ as the full isometry group of $G \times X$ with respect to a certain proper metric on $G \times X$, where X is an arbitrary locally compact Polish space having more than one point such that $(\mathrm{card}(G), \mathrm{card}(X)) \neq (1, 2)$. Locally compact Polish groups which act effectively and almost transitively on complete metric spaces as full isometry groups are characterized. Locally compact Polish non-Abelian groups on which every left invariant metric is automatically right invariant are characterized and fully classified. It is demonstrated that for every locally compact Polish space X having more than two points the set of all proper metrics d such that $\mathrm{Iso}(X, d) = \{\mathrm{id}_X\}$ is dense in the space of all proper metrics on X.

1. Introduction

Throughout this paper, a metric on a metrizable space (or a metric space) is *proper* iff all closed balls (with respect to this metric) are compact. A topological group or space is *Polish* if it is completely metrizable and separable. The set of all proper metrics on a locally compact Polish space X which induce the topology of X is denoted by $\mathrm{Metr}_c(X)$ (for such X, $\mathrm{Metr}_c(X)$ is non-empty; see e.g. [19]). The neutral element of a group G is denoted by e_G. The identity map on a set X is denoted by id_X. For every metric space (Y, ϱ), $\mathrm{Iso}(Y, \varrho)$ stands for the group of all (bijective) isometries of (Y, ϱ).

Isometry groups (equipped with the topology of pointwise convergence) of separable complete metric spaces are useful 'models' for studying Polish groups. On the one hand, they are defined and appear in topology quite naturally. On the other hand, thanks to the result of Gao and Kechris [8], every Polish group may be 'represented' as (that is, is isomorphic to) the (full) isometry group $\mathrm{Iso}(X, d)$ of some separable complete metric space (X, d). It may be of great importance to know how to build the space (X, d) (or how 'nice' the topological space X can be) such that $\mathrm{Iso}(X, d)$ is isomorphic to a given Polish group G. Natural questions which arise when dealing with this issue are the following:

(Q1) Does there exist compact X for compact G?
(Q2) Does there exist locally compact X for locally compact G?
(Q3) Is every Lie group isomorphic to the isometry group of a manifold (with respect to some compatible metric)?

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(Q4) (Melleray [15]) Is every compact Lie group the isometry group of some compact Riemannian manifold?
(Q5) Can X be (metrically) homogeneous? (That is, can G act effectively and transitively on any X as the full isometry group?)

Melleray [15] improved the original proof of Gao and Kechris and answered in the affirmative question (Q1). Later Malicki and Solecki [13] solved problem (Q2) by showing that each locally compact Polish group is (isomorphic to) the isometry group of some proper metric space. In all these papers the construction of the crucial metric space X is complicated and based on the techniques of the so-called Katětov maps, and it may turn out that for a 'nice' group G (e.g. a connected Lie group) the space X contains a totally disconnected open (non-empty) subset. In the present paper we deal with locally compact Polish groups and propose a new approach to the above topic. Using new ideas, we solve in the affirmative problem (Q3) (which is closely related to (Q4)) and characterize locally compact Polish groups which are isomorphic to (full) isometry groups acting transitively on proper metric spaces (see Theorem 1.3, especially points (i) and (ii))—this answers question (Q5) for locally compact Polish groups. Our main results in this direction are:

Theorem 1.1. *Let G be a locally compact Polish group and X be a locally compact Polish space with $(\operatorname{card}(G), \operatorname{card}(X)) \neq (1,2)$. Let $G \times X \ni (g,x) \mapsto g.x \in X$ be a (continuous) proper **non-transitive** action of G on X such that for some point $\omega \in X$:*

(F1) G acts freely at ω,
(F2) G acts effectively on $X \setminus G.\omega$.

Then there exists $d \in \operatorname{Metr}_c(X)$ such that $\operatorname{Iso}(X,d)$ consists precisely of all maps of the form $x \mapsto a.x$ $(a \in G)$.

Corollary 1.2. *Let $(G,\cdot)$ be a locally compact Polish group and X be a locally compact Polish space having more than one point for which $(\operatorname{card}(G), \operatorname{card}(X)) \neq (1,2)$. There exists $d \in \operatorname{Metr}_c(G \times X)$ such that $\operatorname{Iso}(G \times X, d)$ consists precisely of all maps of the form $(g,x) \mapsto (ag,x)$ $(a \in G)$. In particular, $\operatorname{Iso}(G \times X, d)$ acts freely on $G \times X$ and is isomorphic to G.*

Theorem 1.3. *For a locally compact Polish group $(G,\cdot)$, the following conditions are equivalent:*

(i) there exists a complete metric space (X,d) and an effective action $G \times X \ni (g,x) \mapsto g.x \in X$ such that $\operatorname{Iso}(X,d)$ consists precisely of all maps of the form $x \mapsto a.x$ $(a \in G)$ and $G.b$ is dense in X for some $b \in X$,
(ii) there exists a left invariant metric $\varrho \in \operatorname{Metr}_c(G)$ such that $\operatorname{Iso}(G,\varrho)$ consists precisely of all natural left translations of G on itself; in particular, $\operatorname{Iso}(G,\varrho)$ is isomorphic to G and acts freely, transitively and properly on G,
(iii) one of the following two conditions is fulfilled:
  (a) G is Boolean; that is, $x^2 = e_G$ for each $x \in G$,
  *(b) G is non-Abelian and there is **no** open normal Abelian subgroup H of G of index 2 such that $x^2 = p$ for any $x \in G \setminus H$, where $p \in H$ is (independent of x and) such that $p^2 = e_G \neq p$.*

Moreover, in each of the following cases condition (ii) *is fulfilled:*

- *G is non-solvable,*
- *G is non-Abelian and its center is either trivial or non-Boolean,*
- *G is non-Abelian and connected.*

Proposition 1.4. *Every locally compact Polish Abelian group $(G, \cdot)$ admits an invariant metric $\varrho \in \mathrm{Metr}_c(G)$ such that $\mathrm{Iso}(G, \varrho)$ consists precisely of all maps of the forms $x \mapsto ax$ and $x \mapsto ax^{-1}$ $(a \in G)$.*

Corollary 1.2 answers in the affirmative question (Q3): every (separable) Lie group G is isomorphic to the isometry group of $G \times \{-1, 1\}$ as well as of $G \times (\mathbb{R}/\mathbb{Z})$ and $G \times \mathbb{R}$ (for certain proper metrics). All these spaces are manifolds (even Lie groups), and the first two of them are compact provided the group G is as well. What is more, if G is a connected non-Abelian Lie group, it is isomorphic to its own isometry group with respect to a certain left invariant proper metric, by Theorem 1.3.

It is worth mentioning that by studying problems discussed above we have managed to find (and classify) *all* locally compact Polish non-Abelian groups on which every left invariant metric is automatically right invariant (see Corollary 3.15). It is fascinating and unexpected that up to isomorphism there are only a countable number of such groups, each of them is of exponent 4 and totally disconnected and among them only three are infinite: one compact, one discrete and one non-compact non-discrete. Explicit descriptions are given in Remark 3.16.

To formulate the main result of the paper (which characterizes isometry groups on proper metric spaces), let us introduce a few necessary notions. Some of them are well known.

Definition 1.5. Let X and Y be topological spaces and let $\mathcal{F}$ be a collection of transformations of X into Y.

- (cf. [7, p. 162]) $\mathcal{F}$ is said to be *evenly continuous* iff for any $x \in X$, $y \in Y$ and a neighbourhood W of y there exist neighbourhoods U and V of x and y (respectively) such that conditions $f \in \mathcal{F}$ and $f(x) \in V$ imply $f(U) \subset W$.
- $\mathcal{F}$ is *pointwise precompact* iff for each $x \in X$ the closure (in Y) of the set $\mathcal{F}.x := \{f(x)\colon\ f \in \mathcal{F}\}$ is compact.

Definition 1.6. Let X and Y be arbitrary sets and let $\mathcal{F}$ be a collection of functions of X into Y. The *symmetrized 2-hull* of $\mathcal{F}$ is the family $\mathsf{H}_2(\mathcal{F})$ of all functions $g\colon X \to Y$ such that for any two points x and y of X there is $f \in \mathcal{F}$ with $\{g(x), g(y)\} = \{f(x), f(y)\}$. Notice that $\mathsf{H}_2(\mathcal{F}) \supset \mathcal{F}$.

Let X be a locally compact Polish space. Since X is also σ-compact, the space $\mathcal{C}(X, X)$ of all continuous functions of X into itself is Polish when equipped with the compact-open topology (that is, the topology of uniform convergence on compact sets). We shall always consider $\mathcal{C}(X, X)$ with this topology. According to the Ascoli-type theorem (see e.g. [7, Theorem 3.4.20]), a set $\mathcal{F} \subset \mathcal{C}(X, X)$ is compact iff $\mathcal{F}$ is closed, evenly continuous and pointwise precompact.

We are now ready to formulate the main result of the paper. (Notice that below it is not assumed that the group G is a topological group.)

Theorem 1.7. *Let X be a locally compact Polish space and G be a group of homeomorphisms of X such that* $(\mathrm{card}(G), \mathrm{card}(X)) \neq (1, 2)$*. The following conditions are equivalent:*

(i) *there exists $d \in \mathrm{Metr}_c(X)$ such that $\mathrm{Iso}(X, d) = G$,*

(ii) *there exists a G-invariant metric in $\mathrm{Metr}_c(X)$, and for any G-invariant metric $d \in \mathrm{Metr}_c(X)$ and each $\varepsilon > 0$ there is $\varrho \in \mathrm{Metr}_c(X)$ such that $d \leqslant \varrho \leqslant (1+\varepsilon)d$ and $\mathrm{Iso}(X, \varrho) = G$,*

(iii) *each of the following three conditions is fulfilled:*
   (Iso1) *G is closed in the space $\mathcal{C}(X, X)$,*
   (Iso2) *for every compact set K in X the family $\mathcal{D}_K = \{h \in G\colon\ h(K) \cap K \neq \varnothing\}$ is evenly continuous and pointwise precompact,*
   (Iso3) $\mathsf{H}_2(G) = G$.

Point (ii) in the above result asserts much more than just the existence of a proper metric ϱ with $\mathrm{Iso}(X, \varrho) = G$. It says that by 'almost' preserving the geometry of the space, we may approximate any proper G-invariant metric by such metrics. The only difficult part of Theorem 1.7 is the implication '(iii) $\implies$ (ii)'. We shall prove it in the next section involving Baire's theorem and a very recent result by Abels, Manoussos and Noskov [1].

Further consequences of Theorem 1.7 are stated below.

Corollary 1.8. *Let X be a locally compact Polish space and let $\mathcal{G} = \{\mathrm{Iso}(X, d)\colon\ d \in \mathrm{Metr}_c(X)\}$. Then for any non-empty family $\mathcal{F} \subset \mathcal{G}$, $\bigcap \mathcal{F} \in \mathcal{G}$.*

Corollary 1.9. *For every locally compact Polish space X having more than two points, the set of all metrics $d \in \mathrm{Metr}_c(X)$ for which $\mathrm{Iso}(X, d) = \{\mathrm{id}_X\}$ is dense (in the topology of uniform convergence on compact subsets of $X \times X$) in $\mathrm{Metr}_c(X)$.*

The paper is organized as follows. The next section is devoted to the proof of Theorem 1.7. It also contains short proofs of Theorem 1.1 and Corollary 1.2. The proofs of Corollaries 1.8 and 1.9 are left to the reader as simple exercises (they are immediate consequences of Theorem 1.7). In the last section we study locally compact Polish groups G which satisfy condition (ii) of Theorem 1.3 and prove this result as well as Proposition 1.4.

In general, isometry groups of locally compact separable complete metric spaces may not be locally compact and may not act properly on the underlying spaces. However, if (X, d) is a connected locally compact metric space, then the isometry group of (X, d) is locally compact and acts properly on X (see [5]). This result remains true when we replace the connectedness of X by the properness of the metric d (see [8]). Other results in this topic may be found in [14].

It is already known that a proper action of a locally compact Polish group G on a locally compact Polish space X admits a G-invariant proper metric on X (see [1]). It was proved much earlier that every locally compact Polish group admits a left invariant proper metric (see [18]). These are the two main tools of our work. For other results on constructing proper or G-invariant metrics the reader is referred to [11] and [19].

Remark 1.10. A careful reader noticed that in some of the results stated above a strange condition that $(\mathrm{card}(G), \mathrm{card}(X)) \neq (1, 2)$ appears. It is an interesting phenomenon that this trivial case — when $(\mathrm{card}(G), \mathrm{card}(X)) = (1, 2)$ — is the only exception for these theorems to hold. (We leave it as a very simple exercise that the above condition is necessary whenever it appears in the statement.)

Notation and terminology. In this paper all considered topological spaces (unless otherwise stated) are Polish. By a *map* we mean a continuous function. All isomorphisms between topological groups as well as actions of topological groups on topological spaces are assumed to be continuous (unless otherwise stated). We use the multiplicative notation for all groups. Let G be a topological group, X be a topological space and let $G \times X \ni (g,x) \mapsto g.x \in X$ be an action. We call G a *Boolean* group iff $g^2 = e_G$ for each $g \in G$. (Every Boolean group is Abelian.) For $g \in G$, $x \in X$, $A \subset G$ and $B \subset X$ we write $A.B := \{a.b\colon\ a \in A,\ b \in B\}$, $g.B := \{g\}.B$ and $A.x := A.\{x\}$. The action is *effective* (on a set $Y \subset X$) iff $g.x = x$ for each x (belonging to Y) implies $g = e_G$. It is *free* (at a point $\omega \in X$) iff $g.x = x$ for some $x \in X$ (for $x = \omega$) implies $g = e_G$. The action is *transitive* (resp. *almost transitive*) iff $G.x = X$ (resp. iff $G.x$ is dense in X) for some $x \in X$. Finally, it is *proper* if the map $\Phi\colon G \times X \ni (g,x) \mapsto (x, g.x) \in X \times X$ is proper, that is, if $\Phi^{-1}(K)$ is compact for any compact $K \subset X \times X$. This is equivalent to (see [1]):

($\star$) for any compact set $K \subset X$, $D_K := \{g \in G\colon\ g.K \cap K \neq \varnothing\}$ is compact.

A metric d on X is *G-invariant* if $d(g.x, g.y) = d(x,y)$ for all $g \in G$ and $x,y \in X$. When G and X are locally compact and Polish, and the action is proper, the set of all G-invariant metrics $d \in \operatorname{Metr}_c(X)$ is non-empty ([1]) and we denote it by $\operatorname{Metr}_c(X|G)$.

Whenever X is a locally compact Polish space and G is a group of homeomorphisms of X, G acts naturally on X by $\varphi.x = \varphi(x)$. What is more, the function $\mathcal{C}(X,X) \times \mathcal{C}(X,X) \ni (f,g) \mapsto g \circ f \in \mathcal{C}(X,X)$ is continuous ([7, Theorem 3.4.2]) and thus $(G, \circ)$ is a topological group iff the inverse is continuous on G. But this is always true for locally compact groups with continuous multiplication, by a theorem of Ellis [6] (for more general results in this fashion consult [20], [4], [16], [2]). However, in the context of isometry groups of proper metric spaces the continuity of the inverse is an elementary excercise.

Let (X,d) be a metric space. The closed d-ball in X with center at $a \in X$ and of radius $r > 0$ is denoted by $\bar{B}_d(a,r)$; and $d \oplus d$ stands for the 'sum' metric on $X \times X$, that is, $(d \oplus d)((x,y),(x',y')) = d(x,x') + d(y,y')$. For a function $f\colon X \to \mathbb{R}$ we put

$$\operatorname{Lip}_d(f) = \sup\left\{\frac{|f(x) - f(y)|}{d(x,y)}\colon\ x,y \in X,\ x \neq y\right\} \in [0,\infty],$$

provided $\operatorname{card}(X) > 1$ and $\operatorname{Lip}_d(f) = 0$ otherwise. The function f is *non-expansive* iff $\operatorname{Lip}_d(f) \leqslant 1$. For every $b \in X$ let $e_b\colon X \to \mathbb{R}$ be the so-called *Kuratowski* map corresponding to b; that is, $e_b(x) = d(b,x)$. For a non-empty set $A \subset X$ we denote by $\operatorname{dist}_d(x,A)$ the d-distance of a point x from A, i.e.

$$\operatorname{dist}_d(x,A) = \inf_{a \in A} d(x,a).$$

It is well known (and easy to prove) that both e_b and $\operatorname{dist}_d(\cdot, A)$ are non-expansive maps. Also, the maximum and the minimum of finitely many non-expansive (real-valued) maps is non-expansive. These facts will be applied later.

For any set X, Δ_X is the diagonal of X; that is, $\Delta_X = \{(x,x)\colon\ x \in X\}$. A subset K of $X \times X$ is said to be *symmetric* if $(y,x) \in K$ for every $(x,y) \in K$.

Beside all the aspects discussed above, all notions and notation which appeared earlier are obligatory.

2. Characterization of isometry groups with respect to proper metrics

Our first aim is to show Theorem 1.7. Its proof will be preceded by several auxiliary results.

From now on, X is a fixed locally compact Polish (non-empty) space and G is a group of homeomorphisms of X such that

$$(\mathrm{card}(G), \mathrm{card}(X)) \neq (1, 2). \tag{2.1}$$

We equip G with the topology inherited from $\mathcal{C}(X, X)$. Further, we put

$$\mathcal{R}_G = \{(x, y; f(x), f(y))\colon\ f \in G,\ x, y \in X\} \\ \cup \{(x, y; f(y), f(x))\colon\ f \in G,\ x, y \in X\}, \tag{2.2}$$

and for $x, y \in X$,

$$G^s(x, y) = \{(f(x), f(y))\colon\ f \in G\} \cup \{(f(y), f(x))\colon\ f \in G\}.$$

We begin with the following already known result whose proof we omit (see the notes in the introductory section; use the main result of [1] to conclude the non-emptiness of the set $\mathrm{Metr}_c(X|G)$).

Proposition 2.1. *If G satisfies conditions* (Iso1)–(Iso2) *of* Theorem 1.7*, then G is a locally compact topological group, the natural action of G on X is proper and the set* $\mathrm{Metr}_c(X|G)$ *is non-empty.*

Lemma 2.2. *If G satisfies conditions* (Iso1)–(Iso2) *of* Theorem 1.7*, then $\mathcal{R}_G$ is a closed equivalence relation on $X \times X$ and for each $(x, y) \in X \times X$ the equivalence class of (x, y) with respect to $\mathcal{R}_G$ coincides with $G^s(x, y)$.*

Proof. We leave this as an exercise that $\mathcal{R}_G$ is an equivalence relation and that $G^s(x, y)$ is the equivalence class of (x, y). Here we shall focus only on the closedness of $\mathcal{R}_G$ in $(X \times X)^2$. By Proposition 2.1, the action of G on X is proper and thus the function $X \times G \ni (x, f) \mapsto (x, f(x)) \in X \times X$ is a closed map (cf. [7, Theorem 3.7.18]). Consequently, the set $W = \{(x, f(x))\colon\ x \in X,\ f \in G\}$ is closed in $X \times X$. So, the note that

$$\mathcal{R}_G = \{(x, y; z, w) \in (X \times X)^2\colon\ (x, z; y, w) \in W \times W\ \vee\ (x, w; y, z) \in W \times W\}$$

completes the proof. □

The next lemma is the only result in the proof of which the strange condition (2.1) is used. This lemma will find an application later.

Lemma 2.3. *Let $u\colon X \to X$ be a function such that for any two distinct points x and y in X there exists $f \in G$ such that $\{u(x), u(y)\} = \{f(x), f(y)\}$. Then $u \in \mathsf{H}_2(G)$.*

Proof. According to Definition 1.6, we only need to check that $u(x) \in G.x$ for every $x \in X$. If G acts transitively on X, the assertion immediately follows since then $G(x) = X$ for any $x \in X$. Thus we assume that G acts non-transitively. This means that $G(x) \neq X$ for any $x \in X$.

First assume that $\mathrm{card}(G) > 1$. Then there is $g \in G$ and $c \in X$ with $g(c) \neq c$. So, by assumption, there is $f \in G$ with $\{u(c), u(g(c))\} = \{f(c), f(g(c))\} \subset G(c)$ and hence $u(c) \in G(c)$. Now let $x \in X \setminus G(c)$. Then necessarily $x \neq c$ and thus there is $f \in G$ with $\{u(x), u(c)\} = \{f(x), f(c)\}$. Since $c \notin G(x)$ and $u(c) \in G(c)$, it follows that $u(c) = f(c)$ and therefore $u(x) = f(x) \in G(x)$. Finally, if $x \in G(c)$, take $a \in X \setminus G(c)$ and a function $f \in G$ such that $\{u(x), u(a)\} = \{f(x), f(a)\}$. Now, as before, since $u(a) \in G(a)$ and $a \notin G(x)$, we obtain that $u(a) = f(a)$ and consequently $u(x) = f(x) \in G(x)$ as well.

Now assume that $G = \{\mathrm{id}_X\}$. By (2.1), $\mathrm{card}(X) > 2$. Let $x \in X$ be arbitrary. Take two distinct points $y, z \in X \setminus \{x\}$. By assumption, $\{u(x), u(y)\} = \{x, y\}$ and $\{u(x), u(z)\} = \{x, z\}$, which yields that $u(x) = x$, and we are done. □

Lemma 2.4. *If G satisfies conditions* (Iso1)–(Iso2) *of* Theorem 1.7, *then $\mathsf{H}_2(G)$ is a group of homeomorphisms, closed in $\mathcal{C}(X,X)$, and every G-invariant metric on X is $\mathsf{H}_2(G)$-invariant as well.*

Proof. The last part of the lemma immediately follows from the definition of $\mathsf{H}_2(G)$. Further, it follows from Proposition 2.1 that G acts properly on X, and thus there is $d \in \mathrm{Metr}_c(X|G)$. Then—by the first argument of the proof—each member of $\mathsf{H}_2(G)$ is isometric with respect to d. This yields that $\mathsf{H}_2(G) \subset \mathcal{C}(X,X)$. Moreover, it follows from Lemma 2.2 that $G^s(x,y)$ is closed in $X \times X$ for any $x, y \in X$. We infer from this that $\mathsf{H}_2(G)$ is closed in $\mathcal{C}(X,X)$ (since $\mathsf{H}_2(G)$ consists of all functions $u\colon X \to X$ such that $(u(x), u(y)) \in G^s(x,y)$ for all $x, y \in X$). Hence it suffices to show that each member u of $\mathsf{H}_2(G)$ is a bijection and $u^{-1} \in \mathsf{H}_2(G)$. To see this, fix $a \in X$ and take $f \in G$ such that $u(a) = f(a)$. Then the map $v := f^{-1} \circ u\colon (X,d) \to (X,d)$ is isometric and $v(a) = a$. This implies that v sends each closed d-ball around a, which is compact, into itself. Since every isometric map of a compact metric space into itself is onto (see e.g. [12]), v, and consequently u, is a bijection. Finally, for arbitrary points x and y of X take $f \in G$ such that $\{u(u^{-1}(x)), u(u^{-1}(y))\} = \{f(u^{-1}(x)), f(u^{-1}(y))\}$. Then $f^{-1} \in G$ and $\{u^{-1}(x), u^{-1}(y)\} = \{f^{-1}(x), f^{-1}(y)\}$. This yields that $u^{-1} \in \mathsf{H}_2(G)$, and we are done. □

The statement of the next lemma is complicated. However, this result is our key tool and will be applied in its full form in a part of the proof of Theorem 1.7.

Lemma 2.5. *Let (Y, ϱ) be a metric space, a and b be two distinct points of Y and let K be a closed symmetric non-empty set in $Y \times Y$ such that $(a,b) \notin K$. Further, let $\varepsilon > 0$ and let D be a dense subset of $[0,\infty)$. Then there are $\delta > 0$, $\alpha \in D$ and a map $u\colon Y \to \mathbb{R}$ such that:*

(L1) $\mathrm{Lip}_\varrho(u) \leqslant 1 + \varepsilon$ *and* $|u(x) - u(y)| \leqslant \alpha$ *for all* $x, y \in Y$,
(L2) $|u(x) - u(y)| = \alpha > \varrho(x,y)$ *for each* $(x,y) \in \bar{B}_\varrho(a,\delta) \times \bar{B}_\varrho(b,\delta)$,
(L3) $\sup_{(x,y)\in K} |u(x) - u(y)| < \alpha$.

Proof. Decreasing ε, if needed, we may and do assume that

$$\varepsilon < \frac{1}{4}\min(1, \varrho(a,b)) \quad \text{and} \quad [\bar{B}_\varrho(a,2\varepsilon) \times \bar{B}_\varrho(b,2\varepsilon)] \cap K = \varnothing \tag{2.3}$$

(here we use the closedness of K). Everywhere below in this proof δ is a positive number less than ε. Let $A_\delta = \bar{B}_\varrho(a,\delta)$, $B_\delta = \bar{B}_\varrho(b,\delta)$, $c_\delta = \varrho(a,b) - 2\delta > 0$ and let $u_\delta\colon Y \to \mathbb{R}$ be given by the formula

$$u_\delta(y) = \min(\mathrm{dist}_\varrho(y, A_\delta), c_\delta) - \delta \min(\mathrm{dist}_\varrho(y, A_\delta), \mathrm{dist}_\varrho(y, B_\delta), c_\delta).$$

Note that:

$$\mathrm{Lip}_\varrho(u_\delta) \leqslant 1 + \delta, \tag{2.4}$$

$$\varrho(x, y) \in [c_\delta, c_\delta + 4\delta] \quad \text{for} \quad (x, y) \in A_\delta \times B_\delta, \tag{2.5}$$

$$\lim_{\delta\to 0} c_\delta = \varrho(a, b). \tag{2.6}$$

It follows from (2.3), (2.5) and the fact that $\delta < \varepsilon$ that

$$u_\delta(Y) \subset [0, c_\delta], \quad u_\delta^{-1}(\{0\}) = A_\delta \quad \text{and} \quad u_\delta^{-1}(\{c_\delta\}) = B_\delta. \tag{2.7}$$

Now let $(x, y) \in K$. We infer from (2.3) and the symmetry of K that

$$\min(\varrho(z, a), \varrho(z, b)) \geqslant 2\varepsilon \quad \text{for} \quad z \in \{x, y\}.$$

So, for such z we have $\mathrm{dist}_\varrho(z, A_\delta) \geqslant \varepsilon$ and $\mathrm{dist}_\varrho(z, B_\delta) \geqslant \varepsilon$. This combined with the inequality $\varepsilon < c_\delta$ (cf. (2.3)) gives $u_\delta(z) \in [0, c_\delta - \delta\varepsilon]$, which shows that

$$\sup_{(x,y)\in K} |u_\delta(x) - u_\delta(y)| \leqslant c_\delta - \delta\varepsilon < c_\delta. \tag{2.8}$$

The final function u may be taken in the form $u = \lambda u_\delta$ for small enough δ and suitably chosen $\lambda > 1$ (so that $\lambda c_\delta \in D$). It follows from (2.4)–(2.8) that it is possible to do this. The details are left to the reader. □

As a consequence of Lemma 2.5 we obtain

Lemma 2.6. *Let (Y, ϱ) be a metric space, K and L be two disjoint symmetric closed non-empty subsets of $Y \times Y$ such that $L \cap \Delta_Y = \varnothing$ and $\varrho\big|_L \equiv \mathrm{const}$. Then for every $\varepsilon > 0$ there exists a metric ϱ_ε on Y such that $\varrho \leqslant \varrho_\varepsilon \leqslant (1 + \varepsilon)\varrho$ and*

$$\sup_{(x,y)\in K} \varrho_\varepsilon(x, y) \neq \sup_{(x,y)\in L} \varrho_\varepsilon(x, y).$$

Proof. We may assume that $\sup_{(x,y)\in K} \varrho(x, y) = \sup_{(x,y)\in L} \varrho(x, y) =: c$ (because otherwise we may put $\varrho_\varepsilon = \varrho$). Take $(a, b) \in L$, note that $\varrho(a, b) = c$ and apply Lemma 2.5 to obtain a map $u\colon Y \to \mathbb{R}$ with properties (L1)–(L3). Now it suffices to define ϱ_ε by $\varrho_\varepsilon(x, y) = \max(\varrho(x, y), |u(x) - u(y)|)$. Then

$$\sup_{(x,y)\in K} \varrho_\varepsilon(x, y) = \max(c, \sup_{(x,y)\in K} |u(x) - u(y)|) < |u(a) - u(b)| \leqslant \sup_{(x,y)\in L} \varrho_\varepsilon(x, y).$$

□

The next lemma is obvious and we omit its proof.

Lemma 2.7. *Let $d \in \mathrm{Metr}_c(X|G)$.*

(a) *For any $x, y \in X$, d is constant on $G^s(x, y)$.*
(b) *Let ϱ be a metric on X such that $d \leqslant \varrho \leqslant Md$ for some $M > 1$. Let $\varrho_G\colon X \times X \to [0, \infty)$ be given by*

$$\varrho_G(x, y) = \sup_{f\in G} \varrho(f(x), f(y)). \tag{2.9}$$

Then $\varrho_G \in \mathrm{Metr}_c(X|G)$, $d \leqslant \varrho_G \leqslant Md$ and $\varrho_G(a, b) = \sup_{(x,y)\in G^s(a,b)} \varrho(x, y)$ for any $a, b \in X$.

By the above result, whenever $d \in \mathrm{Metr}_c(X|G)$ and $K = G^s(x, y)$ for some $x, y \in X$, $d(K)$ consists of a single number. For simplicity, we shall write $d[K]$ to denote this number. In the next two results we shall use the transformation $\varrho \mapsto \varrho_G$ defined by the formula (2.9).

Proposition 2.8. *Assume G satisfies conditions* (Iso1)–(Iso2) *of* Theorem 1.7. *For any $d' \in \mathrm{Metr}_c(X|G)$ and each $\varepsilon > 0$ there exists a metric $\varrho \in \mathrm{Metr}_c(X|G)$ such that $d' \leqslant \varrho \leqslant (1+\varepsilon)d'$, and whenever U is an open subset of $(X \times X) \setminus \Delta_X$ with $\varrho\big|_U \equiv \mathrm{const}$, then there are two distinct points x and y in X for which $U \subset G^s(x,y)$.*

Proof. We may assume that $\mathrm{card}(X) > 2$. Let $\{(\xi_n,\eta_n)\}_{n=1}^{\infty}$ be a dense subset of $(X \times X) \setminus \Delta_X$. We arrange all members of the collection $\{G^s(\xi_n,\eta_n)\colon\ n \geqslant 1\}$ in a one-to-one sequence (finite or not) $(K_n)_{n=0}^N$ (where $N \in \{0,1,2,\dots,\infty\}$). We conclude from Lemma 2.2 that the sets $K_0, K_1, \dots$ are closed, symmetric, and disjoint from Δ_X, as well as pairwise disjoint. We shall now construct sequences $(d_n)_{n=1}^N$ and $(s_n)_{n=1}^N$ such that

(1_n) $d_n \in \mathrm{Metr}_c(X|G)$, $d_{n-1} \leqslant d_n \leqslant (1+s_{n-1})d_{n-1}$ with $d_0 = d'$, and $s_0 = \frac{\varepsilon}{2}$,
(2_n) $c_n := \min\{|d_n[K_j] - d_n[K_l]|\colon\ j,l \in \{0,\dots,n\},\ j \neq l\} > 0$,
(3_n) $0 < 8\max(1, d_n[K_0],\dots,d_n[K_{n+1}])s_n \leqslant \min(s_{n-1}, c_n)$ and $\prod_{j=0}^n (1+s_j) < 1+\varepsilon$.

It follows from Lemma 2.6 that there is a metric ϱ_0 such that $d_0 \leqslant \varrho_0 \leqslant (1+s_0)d_0$ and $\sup_{(x,y)\in K_0} \varrho_0(x,y) \neq \sup_{(x,y)\in K_1} \varrho_0(x,y)$. Put $d_1 = (\varrho_0)_G$. It follows from Lemma 2.7 that conditions (1_1)–(2_1) are fulfilled. Now choose s_1 so that (3_1) is satisfied as well.

Suppose that we have defined d_n and s_n for some positive $n < N$. If $d_n[K_{n+1}] \notin \{d_n[K_0],\dots,d_n[K_n]\}$, we put $d_{n+1} = d_n$. Otherwise there is a unique $s \in \{0,\dots,n\}$ such that $d_n[K_{n+1}] = d_n[K_s]$. Another application of Lemma 2.6 gives a metric ϱ_n on X such that $\sup_{(x,y)\in K_{n+1}} \varrho_n(x,y) \neq \sup_{(x,y)\in K_s} \varrho_n(x,y)$ and $d_n \leqslant \varrho_n \leqslant (1+s_n)d_n$. We put $d_{n+1} = (\varrho_n)_G$. As before, we see that (1_{n+1}) is satisfied and that $d_{n+1}[K_s] \neq d_{n+1}[K_{n+1}]$. Let us check that (2_{n+1}) is satisfied too. Since $|d_{n+1} - d_n| \leqslant s_n d_n$, for $j = 0,\dots,n+1$ we have, by (3_n), $|d_{n+1}[K_j] - d_n[K_j]| \leqslant \frac{1}{4}c_n$. Therefore, for $j \in \{0,\dots,n\} \setminus \{s\}$, we obtain

$$\begin{aligned}|d_{n+1}[K_{n+1}] - d_{n+1}[K_j]| &\geqslant |d_n[K_{n+1}] - d_n[K_j]| - \frac{1}{2}c_n\\ &= |d_n[K_s] - d_n[K_j]| - \frac{1}{2}c_n \geqslant \frac{1}{2}c_n > 0.\end{aligned}$$

Similarly, when $j,l \in \{0,\dots,n\}$ are different, then $|d_{n+1}[K_j] - d_{n+1}[K_l]| \geqslant |d_n[K_j] - d_n[K_l]| - \frac{1}{2}c_n \geqslant \frac{1}{2}c_n > 0$. This shows (2_{n+1}). Now, as before, choose s_{n+1} so that (3_{n+1}) is fulfilled.

Having the sequences $(d_n)_{n=0}^N$ and $(s_n)_{n=0}^N$, use (1_n) and (3_n) to show that

(4_n) $d_0 \leqslant d_n \leqslant (1+\varepsilon)d_0$ and $s_n \leqslant 2^{m-n}s_m$ for $m \in \{0,\dots,n\}$

for each n. We define the final metric ϱ as the pointwise limit of the d_n's. Precisely, when N is finite, put $\varrho = d_N$ and note that, by (2_N), the numbers $\varrho[K_0],\dots,\varrho[K_N]$ are distinct. If $N = \infty$, let $\varrho(x,y) = \lim_{n\to\infty} d_n(x,y)$ (the limit exists by (1_n) and (4_n)). Observe that in both the cases $\varrho \in \mathrm{Metr}_c(X|G)$ and $d' \leqslant \varrho \leqslant (1+\varepsilon)d'$. We claim that also for $N = \infty$ the numbers $\varrho[K_0], \varrho[K_1], \dots$ are distinct. To see this, take two integers p and q such that $0 \leqslant q < p$. It then follows that

$$|d_p[K_p] - d_p[K_q]| \geqslant c_p > 0 \tag{2.10}$$

(see (2_p)). For $j \in \{p,q\}$ we have

$$0 \leqslant d_{p+1}[K_j] - d_p[K_j] \leqslant s_p d_p[K_j] \leqslant \frac{1}{8}c_p, \tag{2.11}$$

and for $n > p$:

$$0 \leqslant d_{n+1}[K_j] - d_n[K_j] \leqslant s_n d_n[K_j] \leqslant \frac{1}{2}s_{n-1} \leqslant \frac{1}{2^{n-p}}s_p \leqslant \frac{1}{2^{n-p}} \cdot \frac{1}{8}c_p. \tag{2.12}$$

So, (2.11) and (2.12) give $0 \leqslant \varrho[K_j] - d_p[K_j] \leqslant \frac{1}{4}c_p$ ($j \in \{p,q\}$). But this, combined with (2.10), implies that $|\varrho[K_p] - \varrho[K_q]| \geqslant \frac{1}{2}c_p > 0$, and we are done.

To complete the proof, assume that $U \subset (X \times X) \setminus \Delta_X$ is open and non-empty and $\varrho\big|_U \equiv \text{const}$. Since $\{(\xi_n, \eta_n)\}_{n=1}^{\infty}$ is dense in $(X \times X) \setminus \Delta_X$, so is the set $\bigcup_{j=0}^{N} K_j$. Hence there is a non-negative integer $j \leqslant N$ such that $U \cap K_j \neq \varnothing$. Then, by the assumption on U, $\varrho\big|_U \equiv \varrho[K_j]$. Since $U \setminus K_j$ is open and $U \cap K_l = \varnothing$ for $l \neq j$ (because $\varrho[K_j] \neq \varrho[K_l]$), $U \setminus K_j$ is empty, and we are done. □

Lemma 2.9. *Assume G satisfies conditions* (Iso1)–(Iso2) *of* Theorem 1.7. *Let $\varrho \in \mathrm{Metr}_c(X|G)$ be such that whenever U is an open subset of $(X \times X) \setminus \Delta_X$ with $\varrho\big|_U \equiv$ const, then there are two distinct points x and y in X for which $U \subset G^s(x,y)$. Let a and b be two distinct points of X and let $\Omega_r = \{(x,y) \in X \times X\colon\ \mathrm{dist}_{\varrho\oplus\varrho}((x,y), G^s(a,b)) < r\}$, where $r > 0$. For each $\varepsilon > 0$ there is a metric $\lambda \in \mathrm{Metr}_c(X|G)$ such that $\varrho \leqslant \lambda \leqslant (1+\varepsilon)\varrho$ and $(g(a), g(b)) \in \Omega_r$ for every $g \in \mathrm{Iso}(X,\lambda)$.*

Proof. We may assume that $\Omega_r \neq X \times X$. Fix $s \in (0,r)$ and let F be the closure of Ω_s in $X \times X$. Observe that

$$\Omega_s \subset F \subset \Omega_r, \quad (f \times f)(\Omega_s) = \Omega_s \text{ for all } f \in G \quad \text{and} \quad \Omega_s \text{ is symmetric.} \tag{2.13}$$

Let $\mathcal{L}$ be the collection of all sets $G^s(x,y)$ whose interior is non-empty. By Lemma 2.2 and the separability of X, the family $\mathcal{L}$ is countable (finite or not) and thus the set $D = (0,\infty) \setminus \{\varrho[L]\colon\ L \in \mathcal{L}\}$ is dense in $[0,\infty)$. Finally, put $K = (X \times X) \setminus \Omega_s$ and notice that K is closed, symmetric (thanks to (2.13)), non-empty and $(a,b) \notin K$. Let $\delta > 0$, $\alpha \in D$ and $u\colon X \to \mathbb{R}$ be as in Lemma 2.5 (applied for $Y = X$). Define a metric λ_0 on X by $\lambda_0(x,y) = \max(\varrho(x,y), |u(x) - u(y)|)$ and put $\lambda = (\lambda_0)_G$. It follows from Lemma 2.7 that $\lambda \in \mathrm{Metr}_c(X|G)$ and $\varrho \leqslant \lambda \leqslant (1+\varepsilon)\varrho$. We claim that λ is the metric we searched for. We argue by a contradiction. Suppose there is $g \in \mathrm{Iso}(X,\lambda)$ such that $(g(a), g(b)) \notin \Omega_r$. Then $(g(a), g(b)) \notin F$. By the continuity of g, there are open neighbourhoods

$$U \subset \bar{B}_\varrho(a,\delta) \quad \text{and} \quad V \subset \bar{B}_\varrho(b,\delta) \tag{2.14}$$

of a and b (respectively) such that

$$[g(U) \times g(V)] \cap F = \varnothing. \tag{2.15}$$

Fix arbitrary $(x,y) \in U \times V$. We infer from (2.15) that $(g(x), g(y)) \notin F$ and hence $(g(x), g(y)) \in K$. Consequently (see the second property in (2.13)), for any $f \in G$, $(f(g(x)), f(g(y))) \in K$. So, by (L3) (see Lemma 2.5):

$$\sup_{f \in G} |u(f(g(x))) - u(f(g(y)))| < \alpha. \tag{2.16}$$

At the same time, by (L1)–(L2) and (2.14), $|u(f(x)) - u(f(y))| \leqslant \alpha = |u(x) - u(y)| > \varrho(x,y)$. We infer from this (and from the G-invariance of ϱ) that $\lambda(x,y) = \alpha$. Since $g \in \mathrm{Iso}(X,\lambda)$, $\lambda(g(x), g(y)) = \alpha$. This combined with (2.16) and the G-invariance of ϱ yields that $\varrho(g(x), g(y)) = \alpha$. Since x and y are arbitrary, $\varrho\big|_P \equiv \alpha$, where $P = g(U) \times g(V)$. P is an open non-empty subset of $(X \times X) \setminus \Delta_X$, since g is a homeomorphism and $\alpha \in D$ (thus $\alpha \neq 0$). So, it follows from the property of

ϱ that there exist distinct points p and q in X such that $P \subset G^s(p,q)$. But then $G^s(p,q) \in \mathcal{L}$ and consequently $\alpha = \varrho[G^s(p,q)] \notin D$, which is a contradiction and finishes the proof. $\square$

For $d \in \mathrm{Metr}_c(X|G)$ let $\Delta(d)$ be the set of all metrics ϱ such that $d \leqslant \varrho \leqslant Md$ for some constant $M > 1$; and let $\Delta_G(d) = \Delta(d) \cap \mathrm{Metr}_c(X|G)$. For two metrics $\varrho, \varrho' \in \Delta(d)$ we define their distance $\Lambda_d(\varrho, \varrho')$ as the least non-negative constant C such that $|\varrho - \varrho'| \leqslant Cd$. The following result is left to the reader as a simple exercise.

Lemma 2.10. *Let $d \in \mathrm{Metr}_c(X|G)$ and $\varrho_1, \varrho_2, \ldots, \varrho \in \Delta(d)$.*

(A) *Λ_d is a complete metric on $\Delta(d)$ and $\Delta_G(d)$ is a closed set in $(\Delta(d), \Lambda_d)$.*
(B) *$\lim_{n\to\infty} \Lambda_d(\varrho_n, \varrho) = 0$ iff there is a sequence $(\varepsilon_n)_{n=1}^\infty$ of positive real numbers such that $(1+\varepsilon_k)^{-1}\varrho \leqslant \varrho_k \leqslant (1+\varepsilon_k)\varrho$ for each k and $\lim_{n\to\infty} \varepsilon_n = 0$. If $\lim_{n\to\infty} \Lambda_d(\varrho_n, \varrho) = 0$, then the metrics $\varrho_1, \varrho_2, \ldots$ converge uniformly to ϱ on each of the sets $\{(x,y) \in X \times X\colon\ d(x,y) \leqslant r\}$ $(r > 0)$.*
(C) *For any $C > 1$ and $\varrho \in \Delta_G(d)$ the set $\{\varrho' \in \Delta_G(d)\colon\ \varrho \leqslant \varrho' \leqslant tC\varrho$ for some $t \in (0,1)\}$ is dense (in the topology of $(\Delta(d), \Lambda_d)$) in the set $\{\varrho' \in \Delta_G(d)\colon\ \varrho \leqslant \varrho' \leqslant C\varrho\}$.*

From now on, $\Delta(d)$ and all its subsets are equipped with the topology induced by the metric Λ_d.

Proposition 2.11. *Suppose G satisfies conditions* (Iso1)–(Iso2) *of* Theorem 1.7. *Let $d \in \mathrm{Metr}_c(X|G)$ and Ω be an open set in $X \times X$. For any two distinct points a and b of X the set $A_d(a,b;\Omega)$ consisting of all metrics $\varrho \in \Delta_G(d)$ such that $(g(a), g(b)) \in \Omega$ for every $g \in \mathrm{Iso}(X,\varrho)$ is $\mathcal{G}_\delta$ in $\Delta_G(d)$.*

Proof. Fix $\omega \in X$. For $n \geqslant 1$ let U_n consist of all metrics $\varrho \in \Delta_G(d)$ such that $(g(a), g(b)) \in \Omega$ whenever $g \in \mathrm{Iso}(X,\varrho)$ is such that $\max(d(g(\omega),\omega), d(g^{-1}(\omega),\omega)) \leqslant n$. Observe that $A_d(a,b;\Omega) = \bigcap_{n=1}^\infty U_n$, and thus it suffices to show that the set $\Delta_G(d) \setminus U_n$ is closed for every n. Fix $m \geqslant 1$ and let $\varrho_1, \varrho_2, \ldots \in \Delta_G(d) \setminus U_m$ be a sequence which converges to some $\varrho \in \Delta_G(d)$. Then there are a sequence $(g_n)_{n=1}^\infty \subset \mathcal{C}(X,X)$ and a number $M > 0$ such that for every $n \geqslant 1$:

$$d \leqslant \varrho_n \leqslant Md, \qquad g_n \in \mathrm{Iso}(X,\varrho_n), \tag{2.17}$$

$$\max(d(g_n(\omega),\omega), d(g_n^{-1}(\omega),\omega)) \leqslant m, \tag{2.18}$$

$$(g_n(a), g_n(b)) \in (X \times X) \setminus \Omega. \tag{2.19}$$

We conclude from (2.17) that $\max(\mathrm{Lip}_d(g_n), \mathrm{Lip}_d(g_n^{-1})) \leqslant M$ for any n. But then, thanks to (2.18), $g_n(x), g_n^{-1}(x) \in \bar{B}_d(\omega, Md(x,\omega) + m)$ for all $x \in X$ and $n \geqslant 1$. Since the metric d is proper, we see that the family $\mathcal{F} := \{g_n\colon\ n \geqslant 1\} \cup \{g_n^{-1}\colon\ n \geqslant 1\}$ is evenly continuous as well as pointwise precompact. So, by the Ascoli-type theorem, the closure of $\mathcal{F}$ in $\mathcal{C}(X,X)$ is compact. Hence, passing to a subsequence, we may assume that the sequences $g_1, g_2, \ldots$ and $g_1^{-1}, g_2^{-1}, \ldots$ converge uniformly on compact sets to some maps g and h, respectively. But then $\max(d(g(\omega),\omega), d(h(\omega),\omega)) \leqslant m$ (by (2.18)), $(g(a), g(b)) \notin \Omega$ (cf. (2.19)) and $g \circ h = h \circ g = \mathrm{id}_X$, which means that g is bijective and $h = g^{-1}$. To end the proof, it suffices to show that $g \in \mathrm{Iso}(X,\varrho)$ (because then $\varrho \notin U_m$). This simply follows from (2.17):

$$|\varrho_n(g_n(x),g_n(y)) - \varrho(g(x),g(y))| \leqslant \Lambda_d(\varrho_n,\varrho)d(g_n(x),g_n(y)) \\ + |\varrho(g_n(x),g_n(y)) - \varrho(g(x),g(y))| \to 0 \quad (n \to \infty).$$

But $\varrho_n(g_n(x),g_n(y)) = \varrho_n(x,y)$ and, similarly as above,

$$\lim_{n\to\infty} |\varrho_n(x,y) - \varrho(x,y)| = 0,$$

which gives $\varrho(g(x),g(y)) = \varrho(x,y)$. □

Theorem 1.7 will easily be concluded from the following

Theorem 2.12. *Suppose G satisfies conditions* (Iso1)–(Iso2) *of* Theorem 1.7. *Then for every $d \in \mathrm{Metr}_c(X|G)$ and each $M > 1$ the set $\{\varrho \in \Delta_G(d)\colon\ \varrho \leqslant Md,\ \mathrm{Iso}(X,\varrho) = \mathsf{H}_2(G)\}$ is dense and a $\mathcal{G}_\delta$-set in $\Delta_G^M(d) := \{\varrho \in \Delta_G(d)\colon\ \varrho \leqslant Md\}$.*

Proof. We may and do assume that $\mathrm{card}(X) > 1$. Below we continue the notation of Proposition 2.11. Let $\{(a_n,b_n)\}_{n=1}^\infty$ be a dense set in $(X \times X) \setminus \Delta_X$. It is clear that $\Delta_G^M(d)$ is closed in $\Delta_G(d)$ and thus it is a complete metric space (with respect to Λ_d; cf. Lemma 2.10). For any n and m put

$$\Omega_{n,m} = \{(x,y) \in X \times X\colon\ \mathrm{dist}_{d\oplus d}((x,y),G^s(a_n,b_n)) < 2^{-m}\}.$$

It is clear that $\Omega_{n,m}$ is open in $X \times X$ and thus it follows from Proposition 2.11 that $A_{n,m} := A_d(a_n,b_n;\Omega_{n,m}) \cap \Delta_G^M(d)$ is a $\mathcal{G}_\delta$ set in $\Delta_G^M(d)$. Hence, thanks to Baire's theorem, it suffices to show that:

(C1) $A_{n,m}$ is dense in $\Delta_G^M(d)$ for all $n,m \geqslant 1$,
(C2) $(\Delta :=)\{\varrho \in \Delta_G(d)\colon\ \varrho \leqslant Md,\ \mathrm{Iso}(X,\varrho) = \mathsf{H}_2(G)\} = \bigcap_{n,m=1}^\infty A_{n,m}$.

We begin with (C1). Fix positive integers n and m. By Lemma 2.10, the set

$$W = \{\varrho \in \Delta_G(d)\colon\ \varrho \leqslant tMd \text{ for some } t \in (0,1)\}$$

is dense in $\Delta_G^M(d)$. So, take a metric $d' \in W$ and $t \in (0,1)$ such that $d' \leqslant tMd$. Fix arbitrary $\varepsilon > 0$ such that $t(1+\varepsilon)^2 \leqslant 1$. Now apply Proposition 2.8 to obtain a suitable metric ϱ (see the statement of Proposition 2.8). Since then $d \leqslant \varrho$, we have $\Omega_r \subset \Omega_{n,m}$, where $r = 2^{-m}$ and

$$\Omega_r = \{(x,y) \in X \times X\colon\ \mathrm{dist}_{\varrho\oplus\varrho}((x,y),G^s(a_n,b_n)) < r\}.$$

Finally, apply Lemma 2.9 to obtain a metric $\lambda \in \mathrm{Metr}_c(X|G)$ such that $\varrho \leqslant \lambda \leqslant (1+\varepsilon)\varrho$ and

$$(g(a_n),g(b_n)) \in \Omega_r(\subset \Omega_{n,m}) \quad \text{for every } g \in \mathrm{Iso}(X,\lambda). \tag{2.20}$$

Then necessarily

$$d \leqslant d' \leqslant \varrho \leqslant \lambda \leqslant (1+\varepsilon)\varrho \leqslant (1+\varepsilon)^2 d' \leqslant (1+\varepsilon)^2 tMd \leqslant Md,$$

which yields that $\lambda \in \Delta_G^M(d)$ and consequently $\lambda \in A_{n,m}$ (by (2.20)). What is more, the above estimations imply that $|d' - \lambda| \leqslant \varepsilon(2+\varepsilon)d' \leqslant \varepsilon(2+\varepsilon)Md$ and hence $\Lambda_d(d',\lambda) \leqslant \varepsilon(2+\varepsilon)M$, which may be arbitrarily small. This shows (C1).

Now we pass to (C2). Since

$$\mathsf{H}_2(G) = \{f\colon X \to X|\quad \forall x,y \in X\colon\ (f(x),f(y)) \in G^s(x,y)\},$$

we clearly have $\Delta \subset \bigcap_{n,m=1}^\infty A_{n,m}$. To prove the converse inclusion, take $\varrho \in \Delta_G^M(d)$ which belongs to each of the $A_{n,m}$'s, fix $g \in \mathrm{Iso}(X,\varrho)$ and notice that then, since

$G^s(a_n,b_n) = \bigcap_{m=1}^{\infty} \Omega_{n,m}$ (by the closedness of $G^s(a_n,b_n)$; see Lemma 2.2), for all $n \geqslant 1$ we have $(g(a_n),g(b_n)) \in G^s(a_n,b_n)$. Equivalently, $(a_n,b_n;g(a_n),g(b_n)) \in \mathcal{R}_G$ (cf. (2.2)) for any $n \geqslant 1$. Therefore (since g is continuous, $\mathcal{R}_G$ is closed in $(X \times X)^2$ (see Lemma 2.2) and the set $\{(a_n,b_n)\colon\ n \geqslant 1\}$ is dense in $(X \times X) \setminus \Delta_X$), $(x,y;g(x),g(y)) \in \mathcal{R}_G$ for any two distinct points x and y of X. This means that we may apply Lemma 2.3, which gives $g \in \mathsf{H}_2(G)$. Since $\mathsf{H}_2(G) \subset \mathrm{Iso}(X,\varrho)$ by Lemma 2.4, the proof is complete. □

Proof of Theorem 1.7. The most difficult part — implication '(iii) $\implies$ (ii)' — immediately follows from Proposition 2.1 and Theorem 2.12, while implication '(ii) $\implies$ (i)' is trivial. Let us briefly explain that (iii) is implied by (i). If $G = \mathrm{Iso}(X,d)$ for a proper metric d on X, then the natural action of G on X is proper, by a theorem of Gao and Kechris [8], which gives (Iso2) (see ($\star$) in the subsection 'Notation and terminology'). Point (Iso3) follows from Lemma 2.4. Finally, (Iso1) is well known and may be shown in the following way. If functions $g_1, g_2, \ldots \in \mathrm{Iso}(X,d)$ converge uniformly on compact sets to a function $g \in \mathcal{C}(X,X)$, then necessarily g is isometric with respect to d and it suffices to check that g is a surjection. Fixing a point $\omega \in X$, we see that there is $M > 0$ such that $d(g_n(\omega),\omega) \leqslant M$ and consequently $d(g_n^{-1}(\omega),\omega) \leqslant M$ for every n. Then $g_n^{-1}(x) \in \bar{B}_d(\omega, d(x,\omega)+M)$ for each $x \in X$, and hence the sequence $g_1^{-1}, g_2^{-1}, \ldots$ contains a subsequence which converges uniformly on compact sets to some $h\colon X \to X$. Then $g \circ h = \mathrm{id}_X$ and we are done. □

Corollary 2.13. *For any connected locally compact Polish space X,*

$$\{\mathrm{Iso}(X,d)\colon\ d \textit{ is a compatible metric}\} = \{\mathrm{Iso}(X,\varrho)\colon\ \varrho \in \mathrm{Metr}_c(X)\}. \tag{2.21}$$

Proof. It follows from the connectedness of X and the theorem of van Dantzig and van der Waerden [5] that for every compatible metric d on X, $G := \mathrm{Iso}(X,d)$ is locally compact and acts properly on X. One simply concludes that therefore conditions (Iso1)–(Iso2) are satisfied. Hence, according to Lemma 2.4, $\mathsf{H}_2(G)$ is a group of homeomorphisms and (consequently) $\mathsf{H}_2(G) \subset \mathrm{Iso}(X,d) = G$. This shows that (Iso3) is fulfilled as well. So, there is $\varrho \in \mathrm{Metr}_c(X)$ for which $\mathrm{Iso}(X,\varrho) = \mathrm{Iso}(X,d)$ (by Theorem 1.7). This proves the inclusion '$\subset$' in (2.21). Since the converse one is obvious, the proof is complete. □

For simplicity, let us call a group G of homeomorphisms of a locally compact Polish space X an *iso-group of transformations* iff G satisfies conditions (Iso1)–(Iso3), or, equivalently (thanks to Theorem 1.7), if there is a metric $d \in \mathrm{Metr}_c(X)$ such that $G = \mathrm{Iso}(X,d)$.

Proof of Theorem 1.1. Denote by $\tilde{G} \subset \mathcal{C}(X,X)$ the group of all maps of the form $x \mapsto a.x$ ($a \in G$). It suffices to show that $\tilde{G}$ satisfies conditions (Iso1)–(Iso3), thanks to Theorem 1.7. Since the action of G is proper, (Iso2) is fulfilled (cf. ($\star$)) and the set $G.\omega$ is closed. Indeed, the properness of the action of G implies that the function $\Phi\colon G \times X \ni (g,x) \mapsto (g.x,x) \in X \times X$ is a closed map and therefore the set $\Phi(G \times \{\omega\})$ is closed in $X \times X$, but $\Phi(G \times \{\omega\}) = (G.\omega) \times \{\omega\}$.

Let us briefly check that $\tilde{G}$ is closed in $\mathcal{C}(X,X)$. If $\lim_{n\to\infty} g_n.x = u(x)$ for any $x \in X$, then there is $a \in G$ such that $u(\omega) = a.\omega$. It follows from (F1) and the properness of the action that the function $G \ni g \mapsto g.\omega \in G.\omega$ is a homeomorphism and hence $\lim_{n\to\infty} g_n = a$, which yields that $u(x) = a.x$ for each $x \in X$.

Now we pass to (Iso3). Suppose $f \in \mathsf{H}_2(\tilde{G})$. Then

$$f(x) \in G.x \tag{2.22}$$

for any $x \in X$. By (F1) and (2.22), for each $\alpha \in G$ there is a **unique** $\beta_\alpha \in G$ such that

$$f(\alpha.\omega) = (\beta_\alpha \alpha).\omega. \tag{2.23}$$

Let $a = \beta_{e_G}$. We shall show that $f(x) = a.x$ for every $x \in X$. First assume that $x \notin G.\omega$. Then $\{f(x), f(\omega)\} = \{g.x, g.\omega\}$ for some $g \in G$. This implies, thanks to (2.22)–(2.23), that $g = a$ and $f(x) = a.x$ (since $x \notin G.\omega$). Now let $x = \alpha.\omega$ for some $\alpha \in G$. Taking into account (2.23), we have to show that $\beta_\alpha = a$. Since the action of G is non-transitive, the set $D = X \setminus G.\omega$ is non-empty. For each $y \in D$ we have $\{f(y), f(x)\} = \{g.y, g.x\}$ for some $g \in G$. We conclude from this that $g = \beta_\alpha$ and $f(y) = \beta_\alpha.y$ (because $f(x) \notin D$ and β_α is unique). At the same time, $f(y) = a.y$, by the previous part of the proof. Hence $(a^{-1}\beta_\alpha).y = y$ for each $y \in D$. Now an application of (F2) yields that $a^{-1}\beta_\alpha = e_G$ and consequently $\beta_\alpha = a$. □

Proof of Corollary 1.2. It suffices to observe that the function $G \times (G \times X) \ni (a, (g, x)) \mapsto (ag, x) \in G \times X$ is a (continuous) free and non-transitive (since $\operatorname{card}(X) > 1$) proper action and to apply Theorem 1.1. □

Other consequences of Theorem 1.1 are stated below.

Corollary 2.14. *Let G be a locally compact Polish group, X be a locally compact Polish space and let $G \times X \ni (g, x) \mapsto g.x \in X$ be a proper action of G on X such that for some point $\omega \in X$, G acts freely at ω and the G-orbit $G.\omega$ of ω is non-open in X. Then the maps $x \mapsto g.x$ ($g \in G$) form an iso-group of transformations.*

Proof. It follows from the assumptions that $G.\omega \neq X$ and $\operatorname{card}(X) \geqslant \aleph_0$. Thus, according to Theorem 1.1, it suffices to show that G acts effectively on $X \setminus G.\omega$. But if $g \in G$ is such that $g.x = x$ for every $x \in X \setminus G.\omega$, then $g.z = z$ for some $z \in G.\omega$ (since $X \setminus G.\omega$ is non-closed) and hence $g = e_G$. □

Corollary 2.15. *Let G be a locally compact Polish group, X be a locally compact Polish space having more than one point and let $G \times X \ni (g, x) \mapsto g.x \in X$ be a proper effective action of G on X. Let $X \sqcup G$ denote the topological disjoint union of X and G and $\tilde{G} \subset \mathcal{C}(X \sqcup G, X \sqcup G)$ be the set of all maps $\psi_a\colon X \sqcup G \to X \sqcup G$ with $a \in G$ of the form: $\psi_a(x) = a.x$ for $x \in X$ and $\psi_a(g) = ag$ for $g \in G$. Then $\tilde{G}$ is an iso-group of transformations.*

The above result is a special case of Theorem 1.1 and its proof is left to the reader.

3. Isometry groups of homogeneous proper metric spaces

Throughout this section, G is a fixed topological group. For basic information on topological groups the reader is referred to any classical book on this topic, e.g. [17] or [9, 10].

By $\kappa_G\colon G \to G$ we denote the map $x \mapsto x^{-1}$. For $a \in G$ let $L_a\colon G \ni x \mapsto ax \in G$ and let $\mathcal{L}(G) = \{L_a\colon\ a \in G\} \subset \mathcal{C}(G, G)$. It is clear that:

- $\mathcal{L}(G)$ is a group of homeomorphisms,

- $\mathcal{L}(G)$ satisfies conditions (Iso1)–(Iso2) (cf. the proof of Theorem 1.1) provided G is locally compact and Polish,
- $\mathcal{L}(G)$ acts freely and transitively on G.

The main goal of this section is to determine $\mathsf{H}_2(\mathcal{L}(G))$. As a consequence, we shall characterize all topological groups G satisfying $\mathsf{H}_2(\mathcal{L}(G)) = \mathcal{L}(G)$ (see Theorem 3.5). For this purpose we introduce the following

Definition 3.1. Let H be a topological Abelian group and $p \in H$ be such that $p^2 = e_H \neq p$. We define a multiplication on $H \times \{-1, 1\}$ as follows:

$$(x, j) \cdot (y, k) = \begin{cases} (xy, jk) & \text{if } j = 1, \\ (xy^{-1}, jk) & \text{if } j = -1,\ k = 1 \\ (xy^{-1}p, jk) & \text{if } j = k = -1. \end{cases} \qquad (x, y \in H,\ j, k \in \{-1, 1\}),$$

Straightforward calculations show that $(H \times \{-1, 1\}, \cdot)$ is a topological group when it is equipped with the product topology. We shall denote it by $H \times_p \{-1, 1\}$. Observe that $(e_H, 1)$ is the neutral element of $H \times_p \{-1, 1\}$, and for any $x \in H$, $(x, -1)^{-1} = (xp, -1)$.

To avoid repetition, every pair (H, p) where H is a topological Abelian group and $p \in H$ is such that $p^2 = e_H \neq p$ will be called a *base pair*. Two base pairs (H, p) and (K, q) are *isomorphic* if there exists an isomorphism of H onto K which sends p to q.

The following is left to the reader as an exercise.

Lemma 3.2. *Let (H, p) be a base pair and let $K = H \times_p \{-1, 1\}$.*

(a) *The set $\tilde{H} := H \times \{1\}$ is an open normal subgroup of K of index 2 and the function $H \ni x \mapsto (x, 1) \in \tilde{H}$ is an isomorphism of topological groups.*
(b) *For each $a \in K \setminus \tilde{H}$, $a^2 = \tilde{p} := (p, 1)$ and $\tilde{p}^2 = e_K$.*
(c) *For any $x, y \in H$, $(x, -1) \cdot (y, 1) \cdot (x, -1)^{-1} = (y^{-1}, 1)$, $(x, 1) \cdot (y, -1) \cdot (x, 1)^{-1} = (x^2 y, -1)$ and $(x, -1) \cdot (y, -1) \cdot (x, -1)^{-1} = (x^2 y^{-1}, -1)$. In particular, if H is non-Boolean, the center $Z(K)$ of K coincides with $\{(x, 1)\colon\ x \in H,\ x^2 = e_H\} = \{z \in K\colon\ z^2 = e_K\}$.*
(d) *K is non-Boolean. K is Abelian iff H is Boolean.*

For further studies, let us introduce one more useful notation: let $\mathsf{H}_2(\mathcal{L}(G), e_G)$ stand for the set of all $f \in \mathsf{H}_2(\mathcal{L}(G))$ which map e_G onto e_G.

In the following result the group K is equipped with no topology.

Lemma 3.3. *Let $K = \mathsf{H}_2(\mathcal{L}(G))$ and $K_0 = \mathsf{H}_2(\mathcal{L}(G), e_G)$.*

(a) *$(K, \circ)$ is a group.*
(b) *K_0 is a Boolean subgroup of K.*
(c) *For every $f \in K$ there is a unique pair $(a, f_0) \in G \times K_0$ such that $f = L_a \circ f_0$.*
(d) *Let $f \in K_0$. Then:*

$$\{f(x), f(x^{-1})\} = \{x, x^{-1}\} \qquad (x \in G), \tag{3.1}$$

and for every (possibly non-closed) subgroup D of G, $f(D) \subset D$ and $f\big|_D \in \mathsf{H}_2(\mathcal{L}(D), e_D)$.

Proof. Point (c) as well as the fact that $(K, \circ)$ is a semigroup are left as simple exercises. Consequently, K_0 is a semigroup as well.

We start with (d). For any $x \in G$ there is $a \in G$ such that $\{f(x), f(e_G)\} = \{ax, a\}$. We conclude from this that $f(x) \in \{x, x^{-1}\}$ and consequently $f(x^{-1}) \in \{x, x^{-1}\}$ as well. Further, there is $b \in G$ such that $\{f(x), f(x^{-1})\} = \{bx, bx^{-1}\}$, which yields that either $f(x) \neq f(x^{-1})$ or $x = x^{-1}$. Both these cases give (3.1) and hence $f(f(x)) = x$. This shows (b). Now (b) and (c) imply (a). Finally, if D is a subgroup of G, (3.1) shows that $f(D) \subset D$. So, if x and y belong to D and $a \in G$ is such that $\{f(x), f(y)\} = \{ax, ay\}$, then $a \in Dx^{-1} = D$ and therefore $f\big|_D \in \mathsf{H}_2(\mathcal{L}(D), e_D)$. □

Lemma 3.4. *For every topological Abelian group H,*

$$\mathsf{H}_2(\mathcal{L}(H)) = \mathcal{L}(H) \cup \{L_a \circ \kappa_H\colon\ a \in H\}.$$

Proof. Thanks to Lemma 3.3, it suffices to show that $\mathsf{H}_2(\mathcal{L}(H), e_H) = \{\mathrm{id}_H, \kappa_H\}$. Observe that $\{x^{-1}, y^{-1}\} = \{(x^{-1}y^{-1}) \cdot x, (x^{-1}y^{-1}) \cdot y\}$ for any $x, y \in H$, and thus $\kappa_H \in \mathsf{H}_2(\mathcal{L}(H))$. So, it remains to check that if $f \in \mathsf{H}_2(\mathcal{L}(H), e_H)$ is different from the identity map, then $f = \kappa_H$. By assumption, there is $b \in H$ such that $f(b) \neq b$. By (3.1), it is enough to check that if $f(x) = x$, then $x = x^{-1}$, or—equivalently—that $x^2 = e_H$.

Assume $f(x) = x$. We infer from (3.1) that $f(b) = b^{-1} \neq b$. Since $f \in \mathsf{H}_2(\mathcal{L}(H))$, there is $a \in H$ for which $\{ax, ab\} = \{f(x), f(b)\}(= \{x, b^{-1}\})$. Notice that $ax \neq x$, because otherwise $a = e_H$ and $b^{-1} = ab = b$, which is false. Hence $ax = b^{-1}$ and $ab = x$ from which we easily deduce that $x^2 = e_H$. □

Proof of Proposition 1.4. We know that $\mathcal{L}(H)$ satisfies (Iso1)–(Iso2) and hence, by Lemma 2.4, so does $\mathsf{H}_2(\mathcal{L}(H))$. What is more, it is obvious that $\mathsf{H}_2(\mathsf{H}_2(\mathcal{F})) = \mathsf{H}_2(\mathcal{F})$ for an arbitrary family $\mathcal{F}$ of functions. Now it suffices to apply Lemma 3.4 and Theorem 1.7 to get the assertion. □

The main result of this section is the following

Theorem 3.5. *Let G be a topological group, $K = \mathcal{L}(G)$ and $L = \mathsf{H}_2(K)$. Then L is a group of homeomorphisms of G and exactly one of the following three conditions holds:*

(a) *G is either Boolean or non-Abelian and isomorphic to **no** group of the form $H \times_p \{-1, 1\}$ where (H, p) is a base pair. In that case $L = K$.*
(b) *G is either Abelian non-Boolean or isomorphic to a group of the form $H \times_p \{-1, 1\}$ where (H, p) is a base pair and $\{x^2\colon\ x \in H\} \not\subset \{e_H, p\}$. In that case K is a normal subgroup of L; $\mathsf{H}_2(K, e_G)$ is isomorphic to $\{-1, 1\}$ and consists of two automorphisms of G; and for any $a \in G$ and $f \in \mathsf{H}_2(K, e_G)$, $f \circ L_a \circ f^{-1} = L_{f(a)}$.*
(c) *G is isomorphic to a group of the form $(H \times_p \{-1, 1\}) \times_{\tilde{p}} \{-1, 1\}$ where (H, p) is a base pair, H is Boolean and $\tilde{p} = (p, 1)$. In that case K is non-normal in L and $\mathsf{H}_2(K, e_G)$ is isomorphic to $\{-1, 1\}^3$ and contains κ_G (which is not an automorphism).*

The above result implies, among other things, that there is a unique topology τ on $L = \mathsf{H}_2(\mathcal{L}(G))$ finer than the topology of pointwise convergence such that (L, τ) is a topological group and the function $G \ni x \mapsto L_x \in L$ is an embedding (namely, $\tau =$ the topology of pointwise convergence). What is more, in this topology $\mathcal{L}(G)$ is open in L.

For simplicity, we shall call every group G satisfying condition (a) of Theorem 3.5 an *iso-group*; and if G satifies condition (c), it will be called *iso-singular.*

The proof of Theorem 3.5 is quite elementary. However, it is not so short. We shall precede it by a few auxiliary results. Under the notation of Theorem 3.5, it follows from point (c) of Lemma 3.3 that connection $\mathsf{H}_2(\mathcal{L}(G), e_G) = \{\mathrm{id}_G\}$ implies that $K = L$. This is true if G is Boolean; on the other hand, for non-Boolean Abelian groups we have that $\mathsf{H}_2(\mathcal{L}(G), e_G) = \{\mathrm{id}_G, \kappa_G\}$, by Lemma 3.4. In the sequel we shall show, among other things, that if G is non-Abelian and $\mathsf{H}_2(\mathcal{L}(G), e_G) \neq \{\mathrm{id}_G\}$, then G has a very special form: it is isomorphic to $H \times_p \{-1, 1\}$ for some base pair (H, p) with non-Boolean H.

Lemma 3.6. *For a topological group G the following conditions are equivalent:*

(i) *there exists a base pair (H, p) such that G is isomorphic to $H \times_p \{-1, 1\}$,*

(ii) *there is an Abelian (possibly non-closed) subgroup K of G of index 2 such that $x^2 = q$ for any $x \in G \setminus K$ and some $q \in K$ such that $q^2 = e_G \neq q$.*

Moreover, if G, K and q are as in (ii)*, then K is open and normal (in G) and G is isomorphic to $K \times_q \{-1, 1\}$.*

Proof. Implication '(i) $\Longrightarrow$ (ii)' follows from Lemma 3.2. Here we shall focus on the converse implication. Under the assumptions of (ii), K is normal, there is $b \in G \setminus K$ and $G \setminus K = Kb$. Let us check that the formulas $(x, 1) \mapsto x$ and $(x, -1) \mapsto xb$ define an isomorphism Φ of $K \times_q \{-1, 1\}$ onto G. It is clear that Φ is a bijection. Observe that for each $z \in Kb$, $z^4 = q^2 = e_G$, and therefore $z^{-1} = z^3 = zq = qz$. This means that for each $x \in K$, $xqb = (xb)q = (xb)^{-1} = b^{-1}x^{-1} = qbx^{-1}$ and consequently $xb = bx^{-1}$. So, for any $x, y \in K$ we have $\Phi(x, 1) \cdot \Phi(y, k) = \Phi((x, 1) \cdot (y, k))$ for each $k \in \{-1, 1\}$ and

$$\Phi(x, -1) \cdot \Phi(y, 1) = xby = xy^{-1}b = \Phi(xy^{-1}, -1) = \Phi((x, -1) \cdot (y, 1)),$$

$$\Phi(x, -1) \cdot \Phi(y, -1) = xbyb = xy^{-1}b^2 = xy^{-1}q = \Phi(xy^{-1}q, 1) = \Phi((x, -1) \cdot (y, -1)),$$

which shows that Φ is a (possibly discontinuous) homomorphism. Notice that Φ is a homeomorphism iff K is open in G, iff K is closed (being of index 2 in G). So, to complete the proof, we only need to verify the closedness of K. If G is non-Abelian, then the closure of K differs from G (since K is Abelian) and thus it has to coincide with K, since K is of index 2. Finally, if G is Abelian, then K is Boolean and $K = \{x \in G\colon\ x^2 = e_G\}$, which yields the closedness of K. $\square$

Lemma 3.7. *Let (H, p) be a base pair and $G = H \times_p \{-1, 1\}$. Let $\Phi_G\colon G \to G$ be given by $\Phi_G(x, 1) = (x, 1)$ and $\Phi_G(x, -1) = (xp, -1)(= (x, -1)^{-1})$ $(x \in H)$. Then $\Phi_G \in \mathsf{H}_2(\mathcal{L}(G), e_G) \setminus \{\mathrm{id}_G\}$; Φ_G is both a homeomorphism and an automorphism of G and*

$$\Phi_G \circ L_a \circ \Phi_G^{-1} = L_{\Phi_G(a)} \tag{3.2}$$

for each $a \in G$.

Proof. Since $p \neq e_H$, $\Phi_G \neq \mathrm{id}_G$. It is clear that $\Phi_G(e_G) = e_G$ and Φ_G is a homeomorphism. For $x, y \in H$ we have $\{\Phi_G(x, 1), \Phi_G(y, 1)\} = \{e_G \cdot (x, 1), e_G \cdot (y, 1)\}$, $\{\Phi_G(x, 1), \Phi_G(y, -1)\} = \{(xyp, -1) \cdot (x, 1), (xyp, -1) \cdot (y, -1)\}$ and

$$\{\Phi_G(x, -1), \Phi_G(y, -1)\} = \{(p, 1) \cdot (x, -1), (p, 1) \cdot (y, -1)\},$$

which gives $\Phi_G \in \mathsf{H}_2(\mathcal{L}(G))$. A verification that Φ_G is an automorphism and satisfies condition (3.2) is left to the reader. $\square$

Lemma 3.8. *Let G be non-Abelian. The following conditions are equivalent:*

(i) $\kappa_G \in \mathsf{H}_2(\mathcal{L}(G))$,
(ii) *for any $x, y \in G$, $x^2 = y^2$ or $xy = yx$,*
(iii) *there is a non-trivial topological Boolean group H and $p \in H \setminus \{e_H\}$ such that G is isomorphic to $(H \times_p \{-1, 1\}) \times_{\tilde{p}} \{-1, 1\}$ where, as usual, $\tilde{p} = (p, 1)$. In particular,* $\operatorname{card}(\{x^2\colon\ x \in G\}) = 2$.

Proof. The equivalence of (i) and (ii) is straightforward. To see that (iii) is followed by (ii), first note that $K = H \times_p \{-1, 1\}$ is Abelian (by Lemma 3.2) and that $\{y^2\colon\ y \in K\} = \{\tilde{p}, e_K\}$. Thus, K is non-Boolean, and therefore the center of $L := K \times_{\tilde{p}} \{-1, 1\}$ coincides with $\{z \in L\colon\ z^2 = e_L\}$. Consequently, the assertion of (ii) now easily follows since $\{z^2\colon\ z \in L\} = \{(\tilde{p}, 1), e_L\}$. The main point is to show that (iii) is implied by (ii).

Assume G satisfies (ii). Since G is non-Abelian, there are points $a, b, c \in G$ such that

$$ab \neq ba \qquad \text{and} \qquad c^2 \neq e_G. \tag{3.3}$$

Everywhere below, x, y and z denote arbitrary elements of G and Z denotes its center. Let $p = c^2 (\neq e_G)$ and $K = Z \cup cZ$. We divide the proof into a few steps.

S1. If $xy \neq yx$, $zx = xz$ and $yz = zy$, then $z^2 = e_G$.

Proof: It follows from (ii) that $x^2 = y^2$. Similarly, since $(zx)y \neq z(yx) = y(zx)$, another application of (ii) gives $y^2 = (zx)^2 = z^2x^2 = z^2y^2$, and therefore $z^2 = e_G$.

S2. $z \in Z(G) \iff z^2 = e_G$.

Proof: Implication '$\implies$' follows from (3.3) and S1. To see the converse, assume $z^2 = e_G$. If $x^2 \neq e_G$, (ii) gives $zx = xz$. Finally, if $x^2 = e_G$, then $c^2 \neq x^2$ and $c^2 \neq z^2$ (cf. (3.3)) and hence, again thanks to (ii), $cx = xc$ and $cz = zc$. Since $c^2 \neq e_G$, S1 yields $zx = xz$.

S3. $\{x^2\colon\ x \in G\} = \{p, e_G\}$, $p^2 = e_G$ and K is an Abelian subgroup of G.

Proof: It follows from S2 that there is $u \in G$ for which $cu \neq uc$. Then, by (ii), $u^2 = p$. Assume $x^2 \neq p$. Other applications of (ii) give: $xc = cx$ (since $x^2 \neq c^2$) and $xu = ux$ (since $x^2 \neq u^2$). Now S1 implies that $x^2 = e_G$. This proves the first assertion of S3, which is followed by the remainder (because $c^2 = p \in Z \subset K$).

S4. $K = \{x \in G\colon\ xc = cx\}$.

Proof: Since K is Abelian, we only need to check that if $xc = cx$, then $x \in K$. Taking into account S2, we may assume that $x^2 \neq e_G$ and hence, by S3, $x^2 = p$. Then $(xc)^2 = x^2c^2 = p^2 = e_G$, and consequently $xc \in Z$ or, equivalently, $x \in Zc^{-1} = Zc^3 = Zpc = Zc \subset K$.

S5. $xc \in \{cx, pcx\} \cup Z$.

Proof: We may assume that $xc \neq cx$ (and thus $x^2 = p$, by (ii)) and $(xc)^2 \neq e_G$ (by S2). Then $(xc)^2 = p$ (cf. S3) and consequently $xcxc = c^2$ from which we deduce that $xcx = c$, $xc = cx^3$ (by S3) and finally $xc = cpx = pcx$.

S6. $x, y \notin K \implies xy \in K$.

Proof: It follows from S4 that $xc \neq cx$ and $yc \neq cy$. Now since $xc, yc \notin K$, S5 gives $cx = pxc$ and $cy = pyc$. So, $c(xy) = pxcy = p^2xyc = (xy)c$ and therefore $xy \in K$ (by S4).

We are now ready to complete the proof. It follows from S3 and S6 that K is an Abelian subgroup of G which has index 2. What is more, S3 and S2 show that $x^2 = p \in K$ for any $x \notin K$ (since then $x \notin Z$). So, Lemma 3.6 yields that G is isomorphic to $K \times_p \{-1,1\}$. But it is easily seen (since $K = Z \cup cZ$ and $c^2 = p \in Z$) that K is isomorphic to $Z \times_p \{-1,1\}$ (again apply Lemma 3.6). Observe that under the identification of K with $Z \times_p \{-1,1\}$ given in the proof of Lemma 3.6, the point $p \in K$ corresponds to $\tilde{p} = (p,1) \in Z \times_p \{-1,1\}$ and thus G is isomorphic to $(Z \times_p \{-1,1\}) \times_{\tilde{p}} \{-1,1\}$. □

Lemma 3.9. *Suppose $\kappa_G \notin \mathsf{H}_2(\mathcal{L}(G))$ and let $f \in \mathsf{H}_2(\mathcal{L}(G), e_G)$ be different from id_G. Then:*

(P1) *the set $K := \{x \in G\colon\ f(x) = x\}$ is an Abelian subgroup of G of index 2,*
(P2) *there is $p \in K \setminus \{e_G\}$ such that $p^2 = e_G$ and for any $x \in K$ and $y \in G \setminus K$, $xyx = y$ and $f(y) = y^{-1}$,*
(P3) *$\{x^2\colon\ x \in K\} \not\subset \{p, e_G\}$,*
(P4) *G is isomorphic to $K \times_p \{-1,1\}$.*

Proof. Let $U = \{x \in G\colon\ f(x) \neq x\}$, $V = \{x \in G\colon\ f(x) \neq x^{-1}\}$ and $W = \{x \in G\colon\ x^2 = e_G\}$. It follows from (3.1) that U, V and W are pairwise disjoint, $G = U \cup V \cup W$ and $x \in X \iff x^{-1} \in X$ whenever $x \in G$ and $X \in \{U, V, W\}$. Further, the assumptions imply that U and V are non-empty. Observe also that $K = G \setminus U$. For further usage, fix $u \in U$ and $v \in V$ and put $p := u^2$. As was done previously, we divide the proof into steps. Everywhere below, x, y and z denote arbitrary elements of G.

S1. Suppose $xy = yx$. If $x \in U$, then $f(y) = y^{-1}$; if $x \in V$, then $f(y) = y$.
Proof: Let C be the group generated by x and y. We infer from Lemma 3.3 that $f\big|_C \in \mathsf{H}_2(\mathcal{L}(C), e_C)$. Since C is Abelian, Lemma 3.4 yields that $f\big|_C = \mathrm{id}_C$ or $f\big|_C = \kappa_C$. Now the assertion easily follows.

S2. If $x \in U$ and $y \in K$, then $yxy = x$.
Proof: Let $a \in G$ be such that $(\{x^{-1}, y\} =)\{f(x), f(y)\} = \{ax, ay\}$. If $y = ay$, then $a = e_G$ and $x^{-1} = x$, which is false. Thus $y = ax$ and $x^{-1} = ay$ from which we may deduce that indeed $yxy = x$.

S3. If $x, y \notin V$, then $xy = yx$ or $x^2 = y^2$.
Proof: Notice that then $f(x) = x^{-1}$ and $f(y) = y^{-1}$, and use the fact that $f \in \mathsf{H}_2(\mathcal{L}(G))$ (cf. points (i) and (ii) in Lemma 3.8).

S4. If $x \in U$, then $x^4 = e_G$.
Proof: Suppose, to the contrary, that $x^4 \neq e_G$. It follows from S1 that $f(x^2) = x^{-2} (\neq x^2)$ and hence $x^2 \in U$. Now S2 gives $vxv = x$ and $vx^2v = x^2$ as well. So, $vx^2v = xvxv$, and consequently $vx = xv$. Another usage of S1 yields that $f(v) = v^{-1}$, which is false.

S5. If $x \in U$, then $xy, yx \in U$ or $x^2 = y^2$.
Proof: If $xy \notin U$, then S2 implies that

$$(xy)x(xy) = x$$

and thus $yx^2y = e_G$, $x^2 = y^{-2}$, $y^2x^2 = e_G$ and finally, by S4, $y^2 = x^2$. Similarly, if $yx \notin U$, then $y^2 = x^2$, which completes the proof of S5.

S6. $x, y \in U$, $z \in K \implies (xy)z = z(xy)$.
Proof: It follows from S2 that $zxz = x$ and $zyz = y$. But then $yzy^{-1} = z^{-1}$ and $z = xz^{-1}x^{-1}$. So, $z = x(yzy^{-1})x^{-1} = (xy)z(xy)^{-1}$ and we are done.

S7. $x, y \in U \implies xy \in K$.

Proof: It suffices to show that if $x, y, xy \in U$ for some x and y, then $f = \kappa_G$. Fix z. It follows from (3.1) that $f(z) \in \{z, z^{-1}\}$. So, we only need to check that if $f(z) = z$, then $f(z) = z^{-1}$ as well. But this is immediate: if $f(z) = z$, then S6 gives $(xy)z = z(xy)$ and thus, since $xy \in U$, $f(z) = z^{-1}$, by S1.

S8. $\{x^2\colon\ x \in U\} = \{p\}$ and $p^2 = e_G \neq p$.

Proof: Let $x \in K$. By S7, $xu \notin U$, and hence, thanks to S5, $x^2 = u^2 (= p)$. Finally, $p^2 = u^4 = e_G$, by S4, and, of course, $p \neq e_G$ since $u \notin W$.

S9. W coincides with the center of G.

Proof: It may be deduced from S1 that each element of U commutes with no element of V and therefore the center of G is contained in W. Conversely, if $x^2 = e_G$ and y is arbitrary, we have the following four possibilities:

(1°) $y \in U$; then $y^2 \neq x^2$ and hence $yx = xy$, by S3,

(2°) $y^2 \neq p$; then $y^2 \neq u^2$ and therefore $yu \in U$ (by S5), and it follows from (1°) that x commutes with both u and yu which easily implies that $xy = yx$,

(3°) $y^2 = p$ and $(xy)^2 \neq p$; then, by (2°), xy commutes with x, and consequently $xy = yx$,

(4°) $y^2 = (xy)^2 = p$; then $xyx = y$, and therefore $xy = yx$, since $x^2 = e_G$.

S10. There is $q \in K$ such that $q^2 \notin \{p, e_G\}$.

Proof: Since $\kappa_G \notin \mathsf{H}_2(\mathcal{L}(G))$, Lemma 3.8 implies that there are $x, y \in G$ such that $xy \neq yx$ and $x^2 \neq y^2$. Then $x, y \notin W$ (thanks to S9) and there is $q \in \{x, y\}$ such that $q^2 \neq p$. Then $q^2 \notin \{p, e_G\}$ and consequently $q \in K$ (by S8).

S11. K is a subgroup of G.

Proof: Let q be as in S10. It suffices to check that K coincides with the centralizer of q, that is, $K = \{x \in G\colon\ xq = qx\}$. Since $q \in V$, the inclusion '$\supset$' follows from S1. To see the converse, first note that $qu \in U$, since $q^2 \neq u^2$ (cf. S5). As mentioned earlier, $u^{-1} \in U$ as well. So, if $x \in K$, an application of S6 gives $(qu \cdot u^{-1})x = x(qu \cdot u^{-1})$, and we are done.

S12. K is Abelian.

Proof: The map $\varphi\colon K \ni x \mapsto uxu^{-1} \in G$ is a homomorphism. However, since $u \in U$, it follows from S2 that $xux = u$ for $x \in K$, and therefore $\varphi(x) = x^{-1}$. So, κ_K is a homomorphism from which we infer the assertion of S12.

Now we are ready to finish the proof. It follows from S11, S12 and S7 that $p \in K$ and point (P1) is fulfilled. Further, S8, S2 and the definition of K give point (P2), while (P3) is covered by S10. So, an application of Lemma 3.6 yields (P4), and we are done. □

Lemma 3.10. *Let* (H, p), G *and* Φ_G *be as in* Lemma 3.7. *If* $\{x^2\colon\ x \in H\} \not\subset \{p, e_H\}$, *then* $\mathsf{H}_2(\mathcal{L}(G), e_G) = \{\Phi_G, \mathrm{id}_G\}$.

Proof. It follows from the assumptions that H is non-Boolean and thus G is non-Abelian. What is more, we conclude from point (iii) of Lemma 3.8 that $\kappa_G \notin \mathsf{H}_2(\mathcal{L}(G))$. Thus, if $f \in \mathsf{H}_2(\mathcal{L}(G), e_G) \setminus \{\mathrm{id}_G\}$, Lemma 3.9 implies that $V = \{x \in G\colon\ f(x) = x\}$ is an Abelian subgroup of G of index 2, $xyx = y$ for any $x \in V$ and $y \in G \setminus V$, and f coincides with κ_G on $G \setminus V$. So, to convince the reader that $f = \Phi_G$, we only need to check that $V = H \times \{1\}$. Since both these groups have index 2, it is enough to verify that $V \subset H \times \{1\}$. But if $(a, -1) \in V$ for some $a \in H$, then V (being Abelian) is contained in the centralizer of $(a, -1)$, which coincides with $\{(ax, -1)\colon\ x \in H,\ x^2 = e_H\} \cup \{(x, 1)\colon\ x \in H,\ x^2 = e_H\}$ (cf. point (c) of Lemma 3.2). Hence, if $b \in H$ is such that $b^2 \notin \{p, e_H\}$, then $(b, 1) \notin V$ and therefore $(a, -1) \cdot (b, 1) \cdot (a, -1) = (b, 1)$. But this is false since $(a, -1) \cdot (b, 1) \cdot (a, -1) = (b^{-1}p, 1)$ and $b^2 \neq p$. The proof is complete. □

Our last result necessary for giving a proof of Theorem 3.5 is the following

Lemma 3.11. *Let H be a non-trivial topological Boolean group, $p \in H \setminus \{e_H\}$ and let $G = (H \times_p \{-1, 1\}) \times_{\tilde{p}} \{-1, 1\}$. Then $\kappa_G \in \mathsf{H}_2(\mathcal{L}(G), e_G)$, $\mathcal{L}(G)$ is non-normal in $\mathsf{H}_2(\mathcal{L}(G))$ and $\mathsf{H}_2(\mathcal{L}(G), e_G)$ consists of homeomorphisms and is isomorphic to $\{-1, 1\}^3$.*

Proof. We have already shown (in Lemma 3.8) that $\kappa_G \in \mathsf{H}_2(\mathcal{L}(G))$. We conclude from this (and the non-commutativity of G) that $\mathcal{L}(G)$ is non-normal (because $(\kappa_G \circ L_a \circ \kappa_G^{-1})(x) = xa^{-1}$ for any $a, x \in G$, and thus $\kappa_G \circ L_a \circ \kappa_G^{-1} \notin \mathcal{L}(G)$ whenever a does not belong to the center of G).

We shall naturally identify G, as a set, with $H \times \{-1, 1\} \times \{-1, 1\}$. Put $Z = H \times \{1\} \times \{1\}$, $a = (e_H, -1, 1)$, $b = (e_H, 1, -1)$ and $c = ab$. Then Z is the center of G (since $H \times_p \{-1, 1\}$ in non-Boolean; see Lemma 3.2), the sets Z, aZ, bZ and cZ are pairwise disjoint and their union is G. What is more, since $x^2 = \tilde{\tilde{p}}(= (p, 1, 1))$ for any $x \notin Z$, the sets H_a, H_b and H_c are subgroups of G where $H_x = xZ \cup Z$ for $x \in \{a, b, c\}$. It is clear that each of these groups is open and Abelian. Thus, if $f \in \mathsf{H}_2(\mathcal{L}(G), e_G)$, then $f\big|_{H_x}$ coincides with id_{H_x} or κ_{H_x} for $x = a, b, c$ (see Lemmas 3.3 and 3.4). This implies that f is a homeomorphism and that $\operatorname{card}(\mathsf{H}_2(\mathcal{L}(G), e_G)) \leqslant 8$. Since the group $\mathsf{H}_2(\mathcal{L}(G), e_G)$ is Boolean (cf. Lemma 3.3), it remains to show that $\operatorname{card}(\mathsf{H}_2(\mathcal{L}(G), e_G)) = 8$. We leave it as an exercise that whenever $f\colon G \to G$ is such that $f\big|_Z = \mathrm{id}_Z$ and $f\big|_{xZ} \in \{\mathrm{id}_G\big|_{xZ}, \kappa_G\big|_{xZ}\}$ for $x \in \{a, b, c\}$, then $f \in \mathsf{H}_2(\mathcal{L}(G), e_G)$, which finishes the proof. (For example, use the fact that for each $x \in \{a, b, c\}$, H_x is of index 2 and $y^2 = \tilde{\tilde{p}}$ for $y \notin H_x$ to show that G is 'naturally' isomorphic to $H_x \times_{\tilde{\tilde{p}}} \{-1, 1\}$; cf. Lemma 3.6. Then apply Lemma 3.7 to conclude that $f_x \in \mathsf{H}_2(\mathcal{L}(G), e_G)$, where $f_x(y) = y$ for $y \in H_x$ and $f_x(y) = y^{-1}$ otherwise. Finally, check that by composing f_a, f_b, f_c and κ_G one may obtain any of functions mentioned above.) □

Proof of Theorem 3.5. Let us briefly sum up all facts already established to conclude the whole assertion. The case when G is Abelian (Boolean or not) directly follows from Lemma 3.4. Therefore, we may assume G is non-Abelian. We have three possibilities:

- $\kappa_G \in \mathsf{H}_2(\mathcal{L}(G), e_G)$; in that case use Lemmas 3.8 and 3.11 to get that G is isomorphic to $(H \times_p \{-1, 1\}) \times_{\tilde{p}} \{-1, 1\}$ for some base pair (H, p) with Boolean H and that $\mathsf{H}_2(\mathcal{L}(G), e_G)$ consists of 8 homeomorphisms;
- $\kappa_G \notin \mathsf{H}_2(\mathcal{L}(G), e_G)$ and $\mathsf{H}_2(\mathcal{L}(G), e_G) \neq \{\mathrm{id}_G\}$; in that case use Lemmas 3.9, 3.10 and 3.7 to conclude that G is isomorphic to $H \times_p \{-1, 1\}$ for some base pair (H, p) with $\{x^2\colon\ x \in H\} \not\subset \{p, e_H\}$ and that $\mathsf{H}_2(\mathcal{L}(G), e_G)$ consists of two homeomorphic automorphisms of G;
- $\mathsf{H}_2(\mathcal{L}(G), e_G) = \{\mathrm{id}_G\}$; in that case use Lemma 3.7 to infer that G is not isomorphic to any group of the form $H \times_p \{-1, 1\}$ where (H, p) is a base pair.

Notice that all assumptions (about the form of $\mathsf{H}_2(\mathcal{L}(G), e_G)$) in the above cases, as well as all their conclusions, are mutually exclusive, hence points (a), (b) and (c) of the theorem are satisfied and there is no other possibility. (In all above situations the fact that $\mathsf{H}_2(\mathcal{L}(G))$ consists of homeomorphisms may simply be deduced from point (c) of Lemma 3.3.) Now the proof is complete. $\square$

Corollary 3.12. *Let (H, p) and (K, q) be two base pairs.*

(A) *The topological groups $H \times_p \{-1, 1\}$ and $K \times_q \{-1, 1\}$ are isomorphic iff the base pairs (H, p) and (K, q) are isomorphic (cf.* Definition 3.1*).*
(B) *If H and K are Boolean, then the topological groups $(H \times_p \{-1, 1\}) \times_{\tilde{p}} \{-1, 1\}$ and $(K \times_q \{-1, 1\}) \times_{\tilde{q}} \{-1, 1\}$ are isomorphic iff the base pairs (H, p) and (K, q) are isomorphic.*

Proof. The sufficiency of the latter condition in both points (A) and (B) is immediate. Let us show the necessity in (A). For simplicity, put $G = H \times_p \{-1, 1\}$ and $L = K \times_q \{-1, 1\}$. Below $\tilde{H} = H \times \{1\} \subset G$ and $\tilde{K} = K \times \{1\} \subset L$. Let $\Psi \colon G \to L$ be an isomorphism. Since (H, p) and $(\tilde{H}, \tilde{p})$ (respectively (K, q) and $(\tilde{K}, \tilde{q})$) are isomorphic, it suffices to show that $(\tilde{H}, \tilde{p})$ and $(\tilde{K}, \tilde{q})$ are isomorphic. We have three possibilities:

(1°) G is Abelian. Then so is L, and thus H and K are Boolean. Therefore $\Psi(\tilde{H}) = \tilde{K}$ since both $\tilde{H}$ and $\tilde{K}$ consist of all elements of order 2. Consequently, $\Psi(\tilde{p}) = \tilde{q}$, since $x^2 = \tilde{p}$ for $x \in G \setminus \tilde{H}$ and $y^2 = \tilde{q}$ for $y \in L \setminus \tilde{K}$, and we are done.
(2°) $\operatorname{card}(\{x^2\colon\ x \in G\}) > 2$. Then $\operatorname{card}(\{y^2\colon\ y \in L\}) > 2$ as well, and consequently $\{h^2\colon\ h \in H\} \not\subset \{p, e_H\}$ and $\{k^2\colon\ k \in K\} \not\subset \{q, e_K\}$. Then $f := \Psi \circ \Phi_G \circ \Psi^{-1} \in \mathsf{H}_2(\mathcal{L}(L), e_L) \setminus \{\mathrm{id}_L\}$ and therefore $f = \Phi_L$, by Lemma 3.10. But then $\Psi(\tilde{H}) = \tilde{K}$, because $\tilde{H} = \{x \in G\colon\ \Phi_G(x) = x\}$, and similarly for $\tilde{K}$. Consequently, $\Psi(\tilde{p}) = \tilde{q}$ (cf. point (1°)) and we are done.
(3°) G is non-Abelian and $\{x^2\colon\ x \in G\} = \{\tilde{p}, e_G\}$. Then L is non-Abelian and $\{y^2\colon\ y \in L\} = \{\tilde{q}, e_L\}$ as well, and therefore $\Psi(\tilde{p}) = \tilde{q}$, $\{h^2\colon\ h \in H\} = \{p, e_H\}$ and $\{k^2\colon\ k \in K\} = \{q, e_K\}$ (notice that both H and K are non-Boolean). Put $Z = \{h \in H\colon\ h^2 = e_H\}$, $W = \{k \in K\colon\ k^2 = e_K\}$, $\tilde{Z} = Z \times \{1\} \subset \tilde{H}$ and $\tilde{W} = W \times \{1\} \subset \tilde{K}$. Then $\Psi(\tilde{Z}) = \tilde{W}$ since $\tilde{Z}$ and $\tilde{W}$ coincide with the centra of G and L, respectively (cf. Lemma 3.3). So, (Z, p) and (W, q) are isomorphic, and therefore $(Z \times_p \{-1, 1\}, \tilde{p})$ and $(W \times_q \{-1, 1\}, \tilde{q})$ are isomorphic as well. Finally, Lemma 3.6 implies that (H, p) is isomorphic to $(Z \times_p \{-1, 1\}, \tilde{p})$ and (K, q) to $(Z \times_q \{-1, 1\}, \tilde{q})$.

So, the proof of (A) is complete, while point (B) simply follows from (A). $\square$

Corollary 3.13. *Up to isomorphism, there are only countably many locally compact Polish iso-singular groups. Among them only three are infinite: one compact, one discrete and one non-compact non-discrete. Each such group G is of exponent 4 (that is, $x^4 = e_G$ for any $x \in G$) and totally disconnected. Also, if G is finite, it is uniquely determined (up to isomorphism) by its cardinality.*

Proof. By a theorem of Braconnier [3] (see also [9, (25.29)]), every locally compact Boolean group is isomorphic to the group of the form

$$\{-1, 1\}^{\alpha} \times \{-1, 1\}^{\oplus \beta}, \tag{3.4}$$

where α and β are arbitrary cardinals (when one of them is equal to 0, omit the suitable factor), $\{-1,1\}^\alpha$ is the (full) Cartesian product of α copies of $\{-1,1\}$ equipped with the product topology and $\{-1,1\}^{\oplus\beta}$ is the *weak direct product* of β copies of $\{-1,1\}$ (cf. [9, (2.3)]) equipped with the discrete topology. That is, if $\operatorname{card}(Y) = \beta$, $\{-1,1\}^{\oplus\beta}$ may be represented as the subgroup of $\{-1,1\}^Y$ (taken without inheriting the topology and) consisting of all functions $f\colon Y \to \{-1,1\}$ for which the set $\{y \in Y\colon\ f(y) = -1\}$ is finite. When the group (3.4) is Polish, both cardinals α and β do not exceed $\aleph_0$. So, up to isomorphism, there are only a countable number of non-trivial locally compact Polish Boolean groups: $B(n) := \{-1,1\}^n$ $(n = 1,2,3,\ldots)$, $B(\omega) := \{-1,1\}^{\aleph_0}$, $B(\sigma) := \{-1,1\}^{\oplus\aleph_0}$ and $B(\infty) := B(\sigma) \times B(\omega)$. Each of these groups may naturally be represented as the group of $\{-1,1\}$-valued sequences (finite or not): $B(n)$, $B(\omega)$, $B(\sigma)$ and $B(\infty)$ consist of sequences numbered by, respectively, $J(n) = \{1,\ldots,n\}$, $J(\omega) = J(\sigma) = \{1,2,\ldots\}$ and $J(\infty) = \mathbb{Z}$, where

$$B(\infty) = \{(x_k)_{k\in\mathbb{Z}} \subset \{-1,1\}|\quad \exists k_0 \in \mathbb{Z} \quad \forall k < k_0\colon\ x_k = 1\}$$

(note also that the topology of $B(\infty)$ is finer than the one of pointwise convergence). It follows from the definition of an iso-singular group and Corollary 3.12 that it is enough to show that whenever $k \in \{1,2,\ldots,\omega,\sigma,\infty\}$ and $p,q \in B(k)$ are different from the neutral element, then there exists an automorphism $\tau\colon B(k) \to B(k)$ which sends p to q. This may be provided as follows. There are $n,m \in J(k)$ such that $p_m = q_n = -1$. Put $H_p = \{x \in B(k)\colon\ x_m = 1\}$ and similarly $H_q = \{x \in B(k)\colon\ x_n = -1\}$. It is clear that H_p and H_q are isomorphic and open in G. So, if $\eta\colon H_p \to H_q$ is an isomorphism, then the formulas $\tau(\xi) = \eta(\xi)$ for $\xi \in H_p$ and $\tau(\xi) = \eta(\xi p)q$ otherwise well defines the automorphism we searched for. Further details are left to the reader (see also Remark 3.16 below). □

Other consequences of Theorem 3.5 are stated below.

Corollary 3.14. *Let G be a locally compact Polish group.*

- *G is an iso-group iff there is a left invariant metric $d \in \operatorname{Metr}_c(G)$ such that $\operatorname{Iso}(G,d) = \mathcal{L}(G)$.*
- *If G is iso-singular, there is a left invariant metric $d \in \operatorname{Metr}_c(G)$ such that the set $\operatorname{Iso}(G,d;e_G) := \{f \in \operatorname{Iso}(G,d)\colon\ f(e_G) = e_G\}$ contains precisely 8 functions.*
- *If G is non-iso-singular and not an iso-group, there is a left invariat metric $d \in \operatorname{Metr}_c(G)$ such that $\operatorname{card}(\operatorname{Iso}(G,d;e_G)) = 2$.*

Proof. Just apply Theorems 3.5 and 2.12 (notice that G is an iso-group if and only if $\mathsf{H}_2(\mathcal{L}(G)) = \mathcal{L}(G)$). □

Corollary 3.15. *For a locally compact Polish group G the following conditions are equivalent:*

(i) *every left invariant pseudometric on G (possibly having nothing in common with the topology of G) is right invariant,*
(ii) *every left invariant metric in $\operatorname{Metr}_c(G)$ is right invariant,*
(iii) *G is either Abelian or iso-singular.*

In particular, up to isomorphism, there are only countably many (locally compact Polish) non-Abelian groups G which satisfy (i).

Proof. It is well known (and quite an easy exercise) that a left invariant pseudometric on a group G is right invariant as well iff κ_G is isometric with respect to this pseudometric. Therefore the conclusion simply follows from Theorems 3.5 and 2.12 and Corollary 3.13. $\square$

Remark 3.16. As we announced in the introductory part, we are now able to give explicit descriptions of all locally compact Polish non-Abelian groups on which every left invariant metric is automatically right invariant. In what follows, we preserve the notation introduced in the proof of Corollary 3.13.

Let $p(1) = -1 \in B(1)$, $p(2) = (-1, 1) \in B(2)$, and for $n > 2$ let $p(n) = (-1, 1, 1, \dots, 1) \in B(n)$; further, let $p(\omega) = p(\sigma) = (-1, 1, 1, 1, \dots) \in B(\sigma) \subset B(\omega)$ (the inclusion is purely set-theoretic, i.e. it does not imply that the topology is inherited), and finally let $p(\infty) = (p_m)_{m\in\mathbb{Z}}$ be such that $p_0 = -1$ and $p_m = 1$ otherwise. Now for $k \in \{1, 2, \dots, \omega, \sigma, \infty\}$ let

$$IS(k) = (B(K) \times_{p(k)} \{-1, 1\}) \times_{\widetilde{p(k)}} \{-1, 1\}$$

(where, as usual, $\widetilde{p(k)} = (p(k), 1)$; cf. Definition 3.1). Corollary 3.15 combined with the argument presented in the proof of Corollary 3.13 shows that the $IS(k)$'s are the only groups under the question.

Corollary 3.15 may simply be generalized as follows.

Proposition 3.17. *For a metrizable topological group G the following conditions are equivalent:*

(i) *every left invariant pseudometric on G (possibly having nothing in common with the topology of G) is right invariant,*
(ii) *every left invariant compatible metric on G is right invariant,*
(iii) *G is either Abelian or iso-singular.*

Proof. It follows from Theorem 3.5 that (iii) is equivalent to the fact that $\kappa_G \in \mathsf{H}_2(\mathcal{L}(G))$. We have also already mentioned in the proof of Corollary 3.15 that a left invariant pseudometric is right invariant iff it is preserved by κ_G. We infer from these remarks that (i) follows from (iii). Since (ii) is obviously implied by (i), to end the proof it remains to show that if $\kappa_G \notin \mathsf{H}_2(\mathcal{L}(G))$, then there is a compatible metric on G which is not preserved by κ_G. To this end, it suffices to apply Lemma 3.18 stated below for $f = \kappa_G$. $\square$

Lemma 3.18. *Let G be a metrizable topological group and d be a left invariant compatible metric on G. For any $f \notin \mathsf{H}_2(\mathcal{L}(G))$ and $\varepsilon > 0$ there is a left invariant metric ϱ on G such that $d \leqslant \varrho \leqslant (1+\varepsilon)d$ and $f \notin \operatorname{Iso}(G, \varrho)$.*

Proof. We may assume $f \in \operatorname{Iso}(X, d)$ (because otherwise it suffices to put $\varrho = d$). Since $f \notin \mathsf{H}_2(\mathcal{L}(G))$, there are two points a and b in G such that $(f(a), f(b)) \notin \{(ga, gb)\colon\ g \in G\} \cup \{(gb, ga)\colon\ g \in G\}$. Equivalently, $(f(a), f(b)) \notin K$, where $K := \{(x, y) \in G \times G\colon\ x^{-1}y = a^{-1}b\ \vee\ y^{-1}x = b^{-1}a\}$. We conclude that $a \neq b$. Observing that K is a closed symmetric subset of $G \times G$, we may apply Lemma 2.5 to obtain a function $u\colon G \to \mathbb{R}$ such that $\operatorname{Lip}_d(u) \leqslant 1+\varepsilon$, $|u(f(a)) - u(f(b))| > d(a, b)$

and $|u(f(a)) - u(f(b))| > \sup_{(x,y)\in K} |u(x) - u(y)|$. It is now easily seen that the function $\varrho\colon G \times G \to \mathbb{R}$ given by

$$\varrho(x,y) = \max\Bigl(d(x,y), \sup_{g\in G} |u(gx) - u(gy)|\Bigr) \qquad (x, y \in G)$$

is a well-defined metric having all postulated properties (since $\varrho(f(a), f(b)) > \varrho(a,b)$). $\square$

Remark 3.19. Proposition 3.17 favours groups of the form $G := (H \times_p \{-1,1\}) \times_{\tilde{p}} \{-1,1\}$ with Boolean H among the topological non-Abelian groups. However, the fact that the above G is independent (up to isomorphism) of the choice of $p \in H \setminus \{e_H\}$ (as it is in the case of the locally compact Polish G; see the proof of Corollary 3.13) is no longer true in general, even in the class of Polish groups. To be convinced of that, take a non-trivial connected Polish Boolean group B and put $H = B \times \{-1,1\}$. Further, let $p = (b,1)$ and $q = (b,-1)$, where $b \in B \setminus \{e_B\}$. Then the topological groups $(H \times_p \{-1,1\}) \times_{\tilde{p}} \{-1,1\}$ and $(H \times_q \{-1,1\}) \times_{\tilde{q}} \{-1,1\}$ are non-isomorphic. Indeed, Corollary 3.12 asserts that these groups are isomorphic iff so are the base pairs (H,p) and (H,q). But p belongs to the component of H containing e_H, while q does not, and therefore these base pairs are non-isomorphic.

Before we pass to the proof of Theorem 1.3, let us show the following

Proposition 3.20. *Let G be a topological group, (X,d) be a metric space and $G \times X \ni (g,x) \mapsto g.x \in G \times X$ be a (possibly discontinuous) effective action of G on X such that*

$$\mathrm{Iso}(X,d) = \{M_g\colon\ g \in G\}, \tag{3.5}$$

where for $g \in G$, $M_g\colon X \ni x \mapsto g.x \in X$, and for some $b \in X$ one of the following two conditions is fulfilled:

(T1) *$G.b$ is dense in X and (X,d) is complete or*

(T2) *$G.b = X$.*

Then G is an iso-group.

Proof. We argue by a contradiction. Suppose G is not an iso-group. Let $K = \{g \in G\colon\ g.b = b\}$. It is clear that K is a subgroup of G. What is more, it follows from (T1) (since M_g is continuous), (T2) and the effectivity of the action that for each $a \in G$,

$$(\forall x \in G\colon\ xax^{-1} \in K) \implies a = e_G. \tag{3.6}$$

We claim that $K = \{e_G\}$. When G is Abelian, this immediately follows from (3.6). On the other hand, if G is non-Abelian, there is a base pair (H,p) such that H is non-Boolean and G is isomorphic to $H \times_p \{-1,1\}$. Let us identify these two groups. Let Z be the center of G. Property (3.6) implies that $K \cap Z = \{e_G\}$. So, $K \subset \tilde{H}(= H \times \{1\})$ because for $x \in G \setminus \tilde{H}$, $x^2 = \tilde{p} \in Z$. Since for $a \in \tilde{H}$, $xax^{-1} = a$ for $x \in \tilde{H}$ and $xax^{-1} = a^{-1}$ otherwise (cf. Lemma 3.2), we see that indeed $K = \{e_G\}$. Consequently, the action is free at b and the function $\Phi\colon G \ni g \mapsto g.b \in G.b$ is a bijection. For simplicity, let X_0 and d_0 stand for, respectively, $G.b$ and the restriction of d to $X_0 \times X_0$.

Since G is not an iso-group, there is $f \in \mathsf{H}_2(\mathcal{L}(G))$ which does not belong to $\mathcal{L}(G)$. Put $u_0 := \Phi \circ f \circ \Phi^{-1}\colon X_0 \to X_0$ and observe that $u_0 \in \mathsf{H}_2(\mathrm{Iso}(X_0,d_0))$, by (3.5). This means that u_0 is isometric (with respect to d_0) and consequently

$u_0 \in \mathrm{Iso}(X_0, d_0)$, being a bijection. Now when (T1) is fulfilled, we see that there is $u \in \mathrm{Iso}(X, d)$ which extends u_0; otherwise (i.e. if (T2) holds) we put $u = u_0 \in \mathrm{Iso}(X, d)$. We conclude from (3.5) that there is $a \in G$ such that $u(x) = a.x$ for any $x \in X$. But then for every $g \in G$,

$$f(g) = \Phi^{-1}(u_0(\Phi(g))) = \Phi^{-1}(u(g.b)) = \Phi^{-1}(ag.b) = ag,$$

and hence $f = L_a \in \mathcal{L}(G)$, a contradiction. $\square$

We are now ready to give a short

Proof of Theorem 1.3. Implication '(ii) $\Longrightarrow$ (i)' is immediate; Proposition 3.20 and Lemma 3.6 show that (iii) is implied by (i); and applications of Theorem 3.5, Lemma 3.6 and Corollary 3.14 show that (ii) follows from (iii).

Finally, the remainder of the theorem is a consequence of Lemma 3.6 and the equivalence between points (ii) and (iii) (of the theorem): every group G of the form $H \times_p \{-1, 1\}$ (where H is non-Boolean) is solvable (since $\tilde{H}$ is normal Abelian and $G/\tilde{H}$ is Abelian as well), disconnected ($\tilde{H}$ is open) and has non-trivial Boolean center (by Lemma 3.2). $\square$

Taking into account Theorems 1.3, 3.5 and Lemma 3.18, the following question arises:

Problem 3.21. Does every metrizable iso-group G admit a compatible left invariant metric d such that $\mathrm{Iso}(X, d)$ consists precisely of all left translations of G?

References

[1] H. Abels, A. Manoussos, and G. Noskov, *Proper actions and proper invariant metrics*, J. Lond. Math. Soc. (2) **83** (2011), no. 3, 619–636, DOI 10.1112/jlms/jdq091. MR2802502 (2012e:54050)

[2] Ahmed Bouziad, *Every Čech-analytic Baire semitopological group is a topological group*, Proc. Amer. Math. Soc. **124** (1996), no. 3, 953–959, DOI 10.1090/S0002-9939-96-03384-9. MR1328341 (96f:22002)

[3] Jean Braconnier, *Sur les groupes topologiques localement compacts*, J. Math. Pures Appl. (9) **27** (1948), 1–85 (French). MR0025473 (10,11c)

[4] Nikolaus Brand, *Another note on the continuity of the inverse*, Arch. Math. (Basel) **39** (1982), no. 3, 241–245, DOI 10.1007/BF01899530. MR682451 (84b:22001)

[5] D. van Dantzig and B. L. van der Waerden, *Über metrisch homogene Räume*, Abh. Math. Sem. Hamburg **6** (1928), 367–376.

[6] Robert Ellis, *A note on the continuity of the inverse*, Proc. Amer. Math. Soc. **8** (1957), 372–373. MR0083681 (18,745d)

[7] Ryszard Engelking, *General topology*, 2nd ed., Sigma Series in Pure Mathematics, vol. 6, Heldermann Verlag, Berlin, 1989. Translated from the Polish by the author. MR1039321 (91c:54001)

[8] Su Gao and Alexander S. Kechris, *On the classification of Polish metric spaces up to isometry*, Mem. Amer. Math. Soc. **161** (2003), no. 766, viii+78. MR1950332 (2004b:03067)

[9] Edwin Hewitt and Kenneth A. Ross, *Abstract harmonic analysis. Vol. I*, 2nd ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 115, Springer-Verlag, Berlin, 1979. Structure of topological groups, integration theory, group representations. MR551496 (81k:43001)

[10] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis II* (A Series of Comprehensive Studies in Mathematics, Vol. 152), Springer-Verlag, Berlin, 1997.

[11] J.-L. Koszul, *Lectures on groups of transformations*, Notes by R. R. Simha and R. Sridharan. Tata Institute of Fundamental Research Lectures on Mathematics, No. 32, Tata Institute of Fundamental Research, Bombay, 1965. MR0218485 (36 #1571)

[12] A. Lindenbaum, *Contributions à l'étude de l'espace métrique I*, Fund. Math. **8** (1926), 209–222.

[13] Maciej Malicki and Sławomir Solecki, *Isometry groups of separable metric spaces*, Math. Proc. Cambridge Philos. Soc. **146** (2009), no. 1, 67–81, DOI 10.1017/S0305004108001631. MR2461868 (2009j:54057)

[14] Antonios Manoussos and Polychronis Strantzalos, *On the group of isometries on a locally compact metric space*, J. Lie Theory **13** (2003), no. 1, 7–12. MR1958572 (2004a:54045)

[15] Julien Melleray, *Compact metrizable groups are isometry groups of compact metric spaces*, Proc. Amer. Math. Soc. **136** (2008), no. 4, 1451–1455, DOI 10.1090/S0002-9939-07-08727-8. MR2367119 (2008m:54048)

[16] Helmut Pfister, *Continuity of the inverse*, Proc. Amer. Math. Soc. **95** (1985), no. 2, 312–314, DOI 10.2307/2044534. MR801345 (87a:22004)

[17] L. Pontrjagin, *Topological Groups*, Princeton University Press, Princeton, 1946.

[18] Raimond A. Struble, *Metrics in locally compact groups*, Compositio Math. **28** (1974), 217–222. MR0348037 (50 #535)

[19] Robert Williamson and Ludvik Janos, *Constructing metrics with the Heine-Borel property*, Proc. Amer. Math. Soc. **100** (1987), no. 3, 567–573, DOI 10.2307/2046449. MR891165 (88m:54039)

[20] W. Żelazko, *A theorem on B_0 division algebras*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **8** (1960), 373–375 (English, with Russian summary). MR0125901 (23 #A3198)

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