

PACKING DIMENSION OF IMAGES OF ADDITIVE LÉVY PROCESSES

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ABSTRACT. In this paper we generalize the packing dimension profiles defined by Khoshnevisan, Schilling, and Xiao (2012) to higher dimensions and use it to compute the packing dimension of an arbitrary image of an additive Lévy process.

1. INTRODUCTION

An N -parameter random field $X = \{X(t) : t \in \mathbb{R}_+^N\}$ on \mathbb{R}^d is an *additive Lévy process* if each sample path of X satisfies:

$$(1.1) \quad X(t) = X_1(t_1) + \cdots + X_N(t_N) \quad \forall t = (t_1, \dots, t_N) \in \mathbb{R}_+^N,$$

where X_1, \dots, X_N are independent one-parameter Lévy processes on \mathbb{R}^d . Let Ψ_1, \dots, Ψ_N denote the Lévy exponents of X_1, \dots, X_N , respectively. Then the characteristic function of X is $\mathbb{E}[e^{i\xi \cdot X(t)}] = \exp(-\sum_{j=1}^N t_j \Psi_j(\xi))$, for all $\xi \in \mathbb{R}^d$.

Additive Lévy processes have been used to study Lévy sheets as well as the intersection and self-intersection of Lévy processes. For example, see Dalang and Walsh [1, 2], Fitzsimmons and Salisbury [5], Kendall [7], and Khoshnevisan and Xiao [10], to cite some of the references.

In this paper we compute the packing dimension of the images of an arbitrary additive Lévy process. For a nonrandom Borel set $F \subseteq \mathbb{R}_+^N$, the random image $X(F)$ usually exhibits fractal structure, and it is natural to calculate the Hausdorff dimension and packing dimension of $X(F)$. (See Falconer [3] for definitions of Hausdorff and packing dimension.) When $N = 1$, there is a rich literature for computing the Hausdorff dimension $\dim_{\text{H}} X(F)$ of $X(F)$; see Khoshnevisan and Xiao [11] and its extensive reference. On the other hand, there have been few results about the packing dimension $\dim_{\text{p}} X(F)$ of $X(F)$. In a recent paper, Khoshnevisan, Schilling, and Xiao [9] defined a new family of packing dimension profiles Dim_{κ} and showed that

$$(1.2) \quad \dim_{\text{p}} X(F) = \text{Dim}_{\kappa} F \quad \text{a.s.}$$

When $N > 1$, a formula for the Hausdorff dimension $\dim_{\text{H}}(F)$ was given by Yang [16]. However, there is little known about the packing dimension $\dim_{\text{p}} X(F)$.

The packing dimension profiles defined in [9] are probabilistic but have analytic significance as well. They can be regarded as an extension of the packing dimension profiles defined by Falconer and Howroyd [4] and Howroyd [6]. However this extension is restricted to subsets of \mathbb{R} while the dimension profiles of [4] and [6] are defined for subsets of \mathbb{R}^N for all $N \geq 1$. It is natural to ask whether we can

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generalize the dimension profiles of [9] to subsets of \mathbb{R}^N for $N > 1$ so that the new dimension profiles extend that of [4] and [6].

In this paper, we generalize the packing dimension profiles defined in [9] to higher dimensions and derive a multiparameter version of (1.2). We also show that our definition gives probabilistic interpretation to the packing dimension profiles of Falconer and Howroyd [4] and Howroyd [6]. The extension from $N = 1$ to $N \geq 1$ is not straightforward since the structure of \mathbb{R}^N is richer than \mathbb{R} if $N > 1$. In particular, the main theorem of [9] relies on a stopping time argument, while we cannot define stopping times for general multiparameter processes. In order to overcome this difficulty, we employ the potential theory for additive Lévy processes that was developed in Khoshnevisan, Xiao, and Zhong [14].

The rest of this paper is organized as follows. In Section 2, we define a family of box dimension profiles and packing dimension profiles, and present the main result. In Section 3, we prove our main result. Finally, we give some applications in Section 4.

2. DIMENSION PROFILES

2.1. Box dimension profiles. Recall that $X = \{X(t) : t \in \mathbb{R}_+^N\}$ is an additive Lévy process on \mathbb{R}^d . Define the *associated process* $\tilde{X} = \{\tilde{X}(t) : t \in \mathbb{R}^N\}$ by

$$(2.1) \quad \tilde{X}(t) := \sum_{j=1}^N \operatorname{sgn}(t_j) X_j(|t_j|) \quad \forall t \in \mathbb{R}^N.$$

It follows that

$$(2.2) \quad \forall s, t \in \mathbb{R}_+^N : \tilde{X}(t - s) \text{ has the same distribution as } X(t) - X(s),$$

and the Fourier transform of $\tilde{X}(t)$ is given by

$$(2.3) \quad \mathbb{E}[e^{i\xi \cdot \tilde{X}(t)}] = \exp\left(-\sum_{j=1}^N |t_j| \Psi_j(\operatorname{sgn}(t_j)\xi)\right) \quad \forall \xi \in \mathbb{R}^d.$$

For every $x \in \mathbb{R}^d$, let $|x| := \max_{1 \leq k \leq d} |x_k|$ denote the l^∞ norm of x . Then $B(x, \varepsilon) := \{y \in \mathbb{R}^d : |x - y| < \varepsilon\}$ is the l^∞ open ball centered at x with radius $\varepsilon > 0$. For every $\varepsilon > 0$, define

$$(2.4) \quad \kappa_\varepsilon(t) := \mathbb{P}\{\tilde{X}(t) \in B(0, \varepsilon)\}.$$

It follows that $\kappa_\varepsilon(t)$ is nondecreasing in ε for every fixed t . Moreover, the standard theory of Lévy processes ensures that $\{X_j(t_j)\}_{t_j \geq 0}$ is continuous in probability for all $1 \leq j \leq N$. Thus $\kappa_\varepsilon(t)$ is continuous in t for every fixed ε . We consider the family of functions $\kappa := \{\kappa_\varepsilon\}_{\varepsilon > 0}$.

Definition 2.1. For every bounded Borel set $F \subseteq \mathbb{R}^N$, the *box dimension profile* $\overline{\dim}_\kappa F$ of F with respect to the family κ is defined as

$$(2.5) \quad \overline{\dim}_\kappa F := \sup \left\{ \eta > 0 : \liminf_{\varepsilon \downarrow 0} \varepsilon^{-\eta} \inf_{\mu \in \mathcal{P}(F)} \iint \kappa_\varepsilon(t - s) \mu(ds) \mu(dt) = 0 \right\},$$

where $\mathcal{P}(F)$ denotes the collection of all Borel probability measures supported on F .

Remark 2.2. When $N = 1$, we have

$$(2.6) \quad \kappa_\varepsilon(t) = \mathbb{P}\{X(|t|) \in B(0, \varepsilon)\} = \mathbb{P}\{X(t) \in B(0, \varepsilon)\},$$

for all $\varepsilon > 0$ and $t \geq 0$. This shows that our definition of box dimension profiles generalize that of Khoshnevisan, Schilling, and Xiao [9]. For this reason, throughout this paper, we will use the same notation for dimension profiles as those in [9].

A routine check shows that $\overline{\dim}_\kappa$ has the following properties:

- (i) $\overline{\dim}_\kappa$ is *monotone*, that is, $\overline{\dim}_\kappa F \leq \overline{\dim}_\kappa G$ if $F \subseteq G$;
- (ii) $\overline{\dim}_\kappa$ is *finitely stable*, that is, $\overline{\dim}_\kappa \bigcup_{l=1}^n G_l = \max_{1 \leq l \leq n} \overline{\dim}_\kappa G_l$;
- (iii) $\overline{\dim}_\kappa$ is *translation invariant*, that is, $\overline{\dim}_\kappa F = \overline{\dim}_\kappa (F + t)$ for all $t \in \mathbb{R}^N$, where $= \{t + s : s \in F\}$.

We may also express $\overline{\dim}_\kappa F$ in potential-theoretic terms:

$$(2.7) \quad \overline{\dim}_\kappa F = \overline{\lim}_{\varepsilon \downarrow 0} \frac{\log Z_\kappa(\varepsilon; F)}{\log \varepsilon},$$

where $Z_\kappa(\varepsilon; F)$ is the minimum κ_ε -energy over F :

$$(2.8) \quad Z_\kappa(\varepsilon; F) := \inf_{\mu \in \mathcal{P}(F)} \iint \kappa_\varepsilon(t - s) \mu(ds) \mu(dt).$$

The following technical lemma simplifies the calculation of $Z_\kappa(\varepsilon; F)$. It is similar to Theorem 4.1 of Khoshnevisan and Xiao [13].

Lemma 2.3. *For every bounded Borel set $F \subseteq \mathbb{R}^N$, let \overline{F} denote the closure of F . Then*

$$(2.9) \quad \inf_{\mu \in \mathcal{P}(\overline{F})} \iint \kappa_\varepsilon(t - s) \mu(ds) \mu(dt) = \inf_{\mu \in \mathcal{P}_f(F)} \iint \kappa_\varepsilon(t - s) \mu(ds) \mu(dt),$$

where $\mathcal{P}_f(F)$ denotes the collection of all finitely-supported [discrete] probability measures on F .

Proof. It suffices to prove

$$(2.10) \quad \inf_{\nu \in \mathcal{P}_f(F)} \iint \kappa_\varepsilon(t - s) \nu(ds) \nu(dt) \leq \inf_{\mu \in \mathcal{P}(\overline{F})} \iint \kappa_\varepsilon(t - s) \mu(ds) \mu(dt),$$

as the converse inequality is trivial. Choose and fix any $\mu \in \mathcal{P}(\overline{F})$. For every $n \geq 1$, define

$$(2.11) \quad I_n(\bar{k}) := \left[\frac{k^1}{2^n}, \frac{k^1 + 1}{2^n} \right) \times \cdots \times \left[\frac{k^N}{2^n}, \frac{k^N + 1}{2^n} \right),$$

where $\bar{k} = (k^1, \dots, k^N) \in \mathbb{Z}^N$. Let

$$(2.12) \quad K_n(F) := \{\bar{k} \in \mathbb{Z}^N : I_n(\bar{k}) \cap \overline{F} \neq \emptyset\}.$$

Note that $K_n(F)$ is finite since F is bounded. For each $\bar{k} \in K_n(F)$, choose some $t_{n, \bar{k}} \in I_n(\bar{k}) \cap \overline{F}$. Define a new probability measure μ_n supported on $\{t_{n, \bar{k}} : \bar{k} \in K_n(F)\}$ by

$$(2.13) \quad \mu_n(t_{n, \bar{k}}) = \mu(I_n(\bar{k}) \cap \overline{F}).$$

In other words, we concentrate the mass $\mu(I_n(\bar{k}) \cap \bar{F})$ to the singleton $t_{n,\bar{k}}$. Since $\kappa_\varepsilon(t)$ is continuous in t for fixed $\varepsilon > 0$ and \bar{F} is compact, $\kappa_\varepsilon(t)$ is uniformly continuous on \bar{F} . Therefore

$$(2.14) \quad \lim_{n \rightarrow \infty} \left| \iint \kappa_\varepsilon(t-s)\mu(ds)\mu(dt) - \iint \kappa_\varepsilon(t-s)\mu_n(ds)\mu_n(dt) \right| = 0.$$

Next, since F is dense in \bar{F} , for each $t_{n,\bar{k}}$ we can choose some $s_{n,\bar{k}} \in F$ such that $|t_{n,\bar{k}} - s_{n,\bar{k}}| < 2^{-(n+1)}$. To avoid complication, we can require the $s_{n,\bar{k}}$'s to be distinct. Define a new probability measure ν_n on $\{s_{n,\bar{k}} : \bar{k} \in K_n(F)\}$ by

$$(2.15) \quad \nu_n(s_{n,\bar{k}}) = \mu_n(t_{n,\bar{k}}).$$

It follows immediately that $\nu_n \in \mathcal{P}_f(F)$. By the uniform continuity of $\kappa_\varepsilon(t)$ and (2.14), we have

$$(2.16) \quad \lim_{n \rightarrow \infty} \left| \iint \kappa_\varepsilon(t-s)\nu_n(ds)\nu_n(dt) - \iint \kappa_\varepsilon(t-s)\mu(ds)\mu(dt) \right| = 0.$$

This further implies that

$$(2.17) \quad \inf_{\nu \in \mathcal{P}_f(F)} \iint \kappa_\varepsilon(t-s)\nu(ds)\nu(dt) \leq \iint \kappa_\varepsilon(t-s)\mu(ds)\mu(dt).$$

Since $\mu \in \mathcal{P}(\bar{F})$ is arbitrary, we can take \inf over $\mathcal{P}(\bar{F})$ on the right hand side of the above inequality to complete the proof. \square

Lemma 2.4. *For every bounded Borel set $F \subseteq \mathbb{R}^N$, let \bar{F} denote the closure of F . Then*

$$(2.18) \quad \overline{\dim}_\kappa F = \overline{\dim}_\kappa \bar{F}.$$

Proof. Since $\mathcal{P}_f(F) \subseteq \mathcal{P}(F) \subseteq \mathcal{P}(\bar{F})$, Lemma 2.3 implies that

$$(2.19) \quad \inf_{\mu \in \mathcal{P}(\bar{F})} \iint \kappa_\varepsilon(t-s)\mu(ds)\mu(dt) = \inf_{\mu \in \mathcal{P}(F)} \iint \kappa_\varepsilon(t-s)\mu(ds)\mu(dt).$$

Then (2.18) follows from Definition 2.1. \square

Remark 2.5. The above lemma implies that $\overline{\dim}_\kappa$ is not σ -stable in general. For example, consider $F = \mathbb{Q}_+^N \cap [0, 1]^N$. On one hand, for every $t \in \mathbb{R}^N$ the only Borel probability measure supported on $\{t\}$ is the Dirac measure δ_t . Then we can compute $\overline{\dim}_\kappa \{t\}$ by definition and obtain $\overline{\dim}_\kappa \{t\} = 0$. On the other hand, $\overline{\dim}_\kappa F = \overline{\dim}_\kappa \bar{F} = \overline{\dim}_\kappa [0, 1]^N$ according to Lemma 2.4. We will show in Corollary 4.1 that if κ is determined by a d -dimensional isotropic additive α -stable process X with $0 < \alpha \leq 2$, then $\overline{\dim}_\kappa [0, 1]^N = \alpha \text{Dim}_{d/\alpha}^{\text{FH}} [0, 1]^N$, where Dim_s^{FH} denotes the s -dimensional packing dimension profile of Falconer and Howroyd [4]. Since $\text{Dim}_{d/\alpha}^{\text{FH}} [0, 1]^N = \min(d/\alpha, N) > 0$, we see that $\overline{\dim}_\kappa F > \sup_{t \in F} \overline{\dim}_\kappa \{t\}$.

2.2. An equivalent definition of box dimension profiles. In general it is hard to use Definition 2.1 to compute the box dimension profile of any given set since the hitting probability $\mathbb{P}\{\tilde{X}(t) \in B(0, \varepsilon)\}$ is only explicitly computable when X has nice properties. However, we may express a box dimension profile in terms of the characteristic exponent Ψ of the underlying additive Lévy process X .

For every Borel probability measure μ on \mathbb{R}^N , and for all $\xi \in \mathbb{R}^d$, define the “energy form”:

$$(2.20) \quad \mathcal{E}_\mu(\xi) := \iint_{\mathbb{R}_+^N \times \mathbb{R}_+^N} \exp\left(-\sum_{j=1}^N |t_j - s_j| \Psi_j(\operatorname{sgn}(t_j - s_j)\xi)\right) \mu(ds)\mu(dt).$$

An application of Fubini’s theorem shows that

$$(2.21) \quad \begin{aligned} \mathcal{E}_\mu(\xi) &= \iint_{\mathbb{R}_+^N \times \mathbb{R}_+^N} \mathbb{E}[e^{i\xi \cdot (X(t) - X(s))}] \mu(ds)\mu(dt) \\ &= \mathbb{E}\left[\left|\int e^{i\xi \cdot X(t)} \mu(dt)\right|^2\right]. \end{aligned}$$

Thus it follows that $0 \leq \mathcal{E}_\mu(\xi) \leq 1$.

The following proposition uses the characteristic exponent of X to compute a box dimension profile. It is a generalization of Theorem 2.6 of [9] to the multiparameter setting. The proof of Theorem 2.6 in [9] works here and needs minor changes. See also the proof of Theorem 1.1 in Khoshnevisan and Xiao [12].

Proposition 2.6. *For every compact set $F \subseteq \mathbb{R}^N$,*

$$(2.22) \quad \overline{\dim}_\kappa F = \sup \left\{ \eta > 0 : \lim_{\varepsilon \downarrow 0} \varepsilon^{-\eta} \inf_{\mu \in \mathcal{P}(F)} \int_{\mathbb{R}^d} \frac{\mathcal{E}_\mu(\xi/\varepsilon)}{\prod_{k=1}^d (1 + \xi_k^2)} d\xi = 0 \right\}.$$

2.3. Packing dimension profiles. We can regularize the box dimension profiles $\overline{\dim}_\kappa$ defined in (2.5) to produce a family of packing type dimension profiles.

Definition 2.7. For every Borel set $F \subseteq \mathbb{R}^N$, the *packing dimension profile* $\operatorname{Dim}_\kappa F$ of F with respect to the family κ is defined as

$$(2.23) \quad \operatorname{Dim}_\kappa F := \inf \left\{ \sup_{k \geq 1} \overline{\dim}_\kappa F_k : F \subseteq \bigcup_{k \geq 1} F_k, F_k \text{ is a bounded Borel set} \right\}.$$

One can verify that $\operatorname{Dim}_\kappa$ has the following properties:

- (i) $\operatorname{Dim}_\kappa$ is *monotone*, that is, $\operatorname{Dim}_\kappa F \leq \operatorname{Dim}_\kappa G$ if $F \subseteq G$;
- (ii) $\operatorname{Dim}_\kappa$ is σ -*stable*, that is, $\operatorname{Dim}_\kappa \bigcup_{n=1}^\infty G_n = \sup_{n \geq 1} \operatorname{Dim}_\kappa G_n$.

We can also associate $\operatorname{Dim}_\kappa$ to a “packing measure with respect to the family κ ”. In order to do that, we borrow some ideas from Howroyd [6].

Definition 2.8. For every Borel set $F \subseteq \mathbb{R}^N$ and real number $\delta > 0$, we say that a sequence of triples $(\omega_i, t_i, \varepsilon_i)_{i \geq 1}$ is a (κ, δ) -*packing* of F if for all $i \geq 1$: (i) $\omega_i \geq 0$; (ii) $t_i \in F$; (iii) $0 < \varepsilon_i \leq \delta$; and (iv) $\sum_{j \geq 1} \omega_j \kappa_{\varepsilon_j}(t_i - t_j) \leq 1$.

Definition 2.9. For every constant $s > 0$, the *s-dimensional packing measure* $P^{s,\kappa}(F)$ of $F \subseteq \mathbb{R}^N$ with respect to the family κ is defined as

$$(2.24) \quad P^{s,\kappa}(F) := \inf \left\{ \sum_{k \geq 1} P_0^{s,\kappa}(F_k) : F \subseteq \bigcup_{k \geq 1} F_k, F_k \text{ is a bounded Borel set} \right\},$$

where $P_0^{s,\kappa}$ is a *premeasure* defined by

$$(2.25) \quad P_0^{s,\kappa}(F) := \lim_{\delta \downarrow 0} P_\delta^{s,\kappa}(F)$$

and

$$(2.26) \quad P_\delta^{s,\kappa}(F) := \sup \left\{ \sum_{i \geq 1} \omega_i \varepsilon_i^s : (\omega_i, t_i, \varepsilon_i)_{i \geq 1} \text{ is a } (\kappa, \delta)\text{-packing of } F \right\},$$

with the convention that $\inf \emptyset := \infty$ and $\sup \emptyset := 0$.

Definition 2.10. For every Borel set $F \subseteq \mathbb{R}^N$, the *packing dimension profile* $\text{P-dim}_\kappa F$ of F with respect to the family κ is defined as

$$(2.27) \quad \text{P-dim}_\kappa F := \inf \{s > 0 : P^{s,\kappa}(F) = 0\}.$$

The two packing dimension profiles defined in (2.23) and (2.27) coincide. This can be shown by adapting the proof of Theorem 26 of Howroyd [6]. For completeness we record it here.

First we introduce an equivalent formulation of $Z_\kappa(\varepsilon; F)$ (see (2.8)) for fixed $\varepsilon > 0$ and bounded Borel set $F \subseteq \mathbb{R}^N$. A sequence of pairs $(\omega_i, t_i)_{i=1}^k$ is a *size ε weighted κ -packing* of F if for all $i = 1, \dots, k$: (i) $t_i \in F$; (ii) $\omega_i \geq 0$; and (iii) $\sum_{j=1}^k \omega_j \kappa_\varepsilon(t_j - t_i) \leq 1$. Define the quantity $N_\kappa(\varepsilon; F)$ by

$$(2.28) \quad N_\kappa(\varepsilon; F) := \sup \left\{ \sum_{i=1}^k \omega_i : (\omega_i, t_i)_{i=1}^k \text{ is a size } \varepsilon \text{ weighted } \kappa\text{-packing of } F \right\},$$

with the convention that $\sup \emptyset := 0$. Then the following technical lemma holds.

Lemma 2.11. For every bounded Borel set $F \subseteq \mathbb{R}^N$ and real number $\varepsilon > 0$,

$$(2.29) \quad N_\kappa(\varepsilon; F) = 1/Z_\kappa(\varepsilon; F).$$

Proof. On one hand, for all $\eta < N_\kappa(\varepsilon; F)$, (2.28) implies the existence of a size ε weighted κ -packing $(\omega_i, t_i)_{i=1}^k$ of F such that $\sum_{i=1}^k \omega_i > \eta$. Let $W = \sum_{i=1}^k \omega_i$ and define a discrete probability measure μ on $\{t_i\}_{i=1}^k$ by $\mu(t_i) = \omega_i/W$. It follows immediately that $\mu \in \mathcal{P}(F)$. Then elementary calculation shows that

$$(2.30) \quad Z_\kappa(\varepsilon; F) \leq \sum_{i=1}^k \frac{\omega_i}{W} \sum_{j=1}^k \frac{\omega_j}{W} \kappa_\varepsilon(t_i - t_j) \leq \sup_{1 \leq i \leq k} \sum_{j=1}^k \frac{\omega_j}{W} \kappa_\varepsilon(t_i - t_j) \leq \frac{1}{W}.$$

Therefore $N_\kappa(\varepsilon; F) \leq 1/Z_\kappa(\varepsilon; F)$.

On the other hand, for all $\eta > Z_\kappa(\varepsilon; F)$, Lemma 2.3 implies the existence of a sequence $\{t_i\}_{i=1}^k \subseteq F$ together with a discrete probability measure μ on $\{t_i\}_{i=1}^k$ such that $\sum_{i,j=1}^k \kappa_\varepsilon(t_i - t_j) \omega_i \omega_j < \eta$, where $\omega_i = \mu(t_i)$ for $1 \leq i \leq k$. Let $a_{i,j} = \kappa_\varepsilon(t_i - t_j)$. Then the $k \times k$ matrix $A := (a_{i,j})_{1 \leq i,j \leq k}$ is symmetric. Let $\mathbf{u} := (u_1, \dots, u_k)' \in \mathbb{R}^k$ be a column vector and define

$$(2.31) \quad J := \min \left\{ \mathbf{u}' A \mathbf{u} : \mathbf{u} = (u_1, \dots, u_k)' \text{ such that } \sum_{i=1}^k u_i = 1 \text{ and } u_i \geq 0 \right\}.$$

A fact of linear algebra (see Proposition 3 of [6] for instance) shows that if $\mathbf{u}^* \in \mathbb{R}_+^k$, $\sum_{i=1}^k u_i^* = 1$ and $\mathbf{u}^{*'} A \mathbf{u}^* = J$, then $(A \mathbf{u}^*)_i = J$, provided that $u_i^* > 0$. If we select the sequence of pairs $(v_j, s_j)_{j=1}^l$ by removing from the sequence $(u_i^*, t_i)_{i=1}^k$ all those

pairs (u_i^*, t_i) with $u_i^* = 0$, then we see that $(v_j/J, s_j)_{j=1}^l$ is a size ε weighted κ -packing of F . It follows that

$$(2.32) \quad N_\kappa(\varepsilon; F) \geq \sum_{j=1}^l \frac{v_j}{J} = \frac{1}{J}.$$

Since $J \leq \sum_{i,j=1}^k \kappa_\varepsilon(t_i - t_j)\omega_i\omega_j < \eta$, we have $N_\kappa(\varepsilon; F) \geq 1/Z_\kappa(\varepsilon; F)$. This completes the proof. \square

Proposition 2.12. *For every Borel set $F \subseteq \mathbb{R}^N$, we have*

$$(2.33) \quad \text{Dim}_\kappa F = \text{P-dim}_\kappa F.$$

Proof. Consider an arbitrary bounded Borel set $E \subseteq \mathbb{R}^N$. For each $\varepsilon > 0$, let $(\omega_i, t_i)_{i=1}^k$ be a size ε weighted κ -packing of E . Then $(\omega_i, t_i, \varepsilon)_{i=1}^k$ is trivially a (κ, ε) -packing of E . It follows from this fact, (2.26), and (2.28) that for all $s > 0$,

$$(2.34) \quad P_\varepsilon^{s,\kappa}(E) \geq \varepsilon^s N_\kappa(\varepsilon; E) = \varepsilon^s (Z_\kappa(\varepsilon; E))^{-1},$$

where we used Lemma 2.11 to derive the last equality. Now if $0 < \eta < \text{Dim}_\kappa F$, then for every sequence of bounded Borel sets $\{F_k\}_{k \geq 1}$ with $F \subseteq \bigcup_{k \geq 1} F_k$, we can find some F_n such that $\overline{\text{dim}}_\kappa F_n > \eta$. Combining with (2.34) and (2.7), we get $P_0^{\eta,\kappa}(F_n) > 0$. Therefore $\text{P-dim}_\kappa F \geq \eta$. Let $\eta \uparrow \text{Dim}_\kappa F$ to obtain $\text{Dim}_\kappa F \leq \text{P-dim}_\kappa F$.

On the other hand, if $0 < \eta < \text{P-dim}_\kappa F$, then for every sequence of bounded Borel sets $\{F_k\}_{k \geq 1}$ with $F \subseteq \bigcup_{k \geq 1} F_k$, we can find some F_n such that $P_0^{\eta,\kappa}(F_n) > 0$. For every fixed θ satisfying $0 < \theta < \eta$, we can find some a such that $0 < a < P_0^{\eta,\kappa}(F_n)$. Since $P_\varepsilon^{\eta,\kappa}(F_n)$ decreases to $P_0^{\eta,\kappa}(F_n)$ as $\varepsilon \downarrow 0$, we have $P_\varepsilon^{\eta,\kappa}(F_n) > a$ for each $\varepsilon > 0$. Then (2.26) implies the existence of a (κ, ε) -packing $(\omega_i, t_i, r_i)_{i=1}^\infty$ of F_n such that $\sum_{i=1}^\infty \omega_i r_i^\eta > a$. For each $m \geq 1$, define

$$(2.35) \quad K_m = \sum_{\{i: 2^{-m} < r_i \leq 2^{1-m}\}} \omega_i.$$

Then there must be a positive integer $M(\varepsilon)$ such that

$$(2.36) \quad K_{M(\varepsilon)} > 2^{(M(\varepsilon)-1)\theta} (1 - 2^{\theta-\eta})a,$$

since otherwise $\sum_{m=1}^\infty K_m 2^{(1-m)\eta} \leq \sum_{m=1}^\infty 2^{(m-1)\theta} (1 - 2^{\theta-\eta})a 2^{(1-m)\eta} = a$. Let $\{i(j)\}_{j=1}^J$ be a subsequence such that $2^{-M(\varepsilon)} < r_{i(j)} \leq 2^{1-M(\varepsilon)}$ for all $1 \leq j \leq J$ and $\sum_{j=1}^J \omega_{i(j)} \geq 2^{(M(\varepsilon)-1)\theta} (1 - 2^{\theta-\eta})a$. The fact that $(\omega_i, t_i, r_i)_{i=1}^\infty$ is a (κ, ε) -packing of F_n implies that $\sum_{j \geq 1} \omega_j \kappa_{r_j}(t_i - t_j) \leq 1$ for all $i \geq 1$. Since $\kappa_\varepsilon(t)$ is nondecreasing in ε for fixed t and $r_{i(j)} \leq 2^{1-M(\varepsilon)}$ for all $1 \leq j \leq J$, we see that $(\omega_{i(j)}, t_{i(j)})_{j=1}^J$ is a size $2^{-M(\varepsilon)}$ weighted κ -packing of F_n . Therefore

$$(2.37) \quad N_\kappa(2^{-M(\varepsilon)}; F_n) \geq 2^{(M(\varepsilon)-1)\theta} (1 - 2^{\theta-\eta})a.$$

Since $M(\varepsilon) \uparrow \infty$ as $\varepsilon \downarrow 0$, by taking logarithms and then limsup as $\varepsilon \downarrow 0$, we obtain

$$(2.38) \quad \overline{\text{dim}}_\kappa F_n \geq \liminf_{\varepsilon \downarrow 0} \frac{\log Z_\kappa(2^{-M(\varepsilon)}; F_n)}{\log 2^{-M(\varepsilon)}} = \lim_{M \rightarrow \infty} \frac{\log N_\kappa(2^{-M}; F_n)}{-\log 2^{-M}} \geq \theta.$$

Let $\theta \uparrow \eta$ to see that $\text{Dim}_\kappa F \geq \text{P-dim}_\kappa F$. This completes the proof. \square

The connection between packing dimension profiles Dim_κ and the family of measures $\{P^{s,\kappa}\}_{s \geq 0}$ can be used to prove the following technical lemma, which will be used later.

Lemma 2.13. *If a Borel set $F \subseteq \mathbb{R}^N$ satisfies $\text{Dim}_\kappa F > s$ for some $s \geq 0$, then there exists a compact set $K \subseteq F$ such that $\text{Dim}_\kappa(K \cap G) \geq s$ for all open sets $G \subseteq \mathbb{R}^N$ with $K \cap G \neq \emptyset$.*

Proof. Since $\text{Dim}_\kappa F > s$, for every collection of bounded Borel sets $\{F_n\}_{n \geq 1}$ with $F \subseteq \bigcup_{n \geq 1} F_n$, we can find some F_n such that $\text{Dim}_\kappa F_n > s$. This implies $P^{s,\kappa}(F_n) = \infty$. Then we can use the proof of Theorem 22 of Howroyd [6] to complete the argument. \square

2.4. Main result. With the box dimension profiles and packing dimension profiles defined so far, we are able to present our main result regarding the packing dimension of images of additive Lévy processes.

Theorem 2.14. *Let $X := \{X(t) : t \in \mathbb{R}_+^N\}$ be an additive Lévy process in \mathbb{R}^d and let $\overline{\text{dim}}_M$ denote the upper box-counting dimension. Then for all nonrandom bounded Borel sets $F \subseteq \mathbb{R}_+^N$:*

$$(2.39) \quad \overline{\text{dim}}_M X(F) = \overline{\text{dim}}_\kappa F \text{ a.s. and}$$

$$(2.40) \quad \text{dim}_p X(F) = \text{Dim}_\kappa F \text{ a.s.}$$

We will prove this theorem in the next section.

3. PROOF OF THE MAIN THEOREM

In this section we strive to prove Theorem 2.14. The proof relies on finding upper bounds for the probabilities of the events where the underlying additive Lévy process hits small balls. In order to estimate these probabilities, we adapt the methods of section 4 of Khoshnevisan, Xiao, and Zhong [14]. We first borrow certain results from [14] without proofs.

3.1. Preliminaries. Recall that $X = \{X_t : t \in \mathbb{R}_+^N\}$ is an additive Lévy process on \mathbb{R}^d . We assume that the underlying probability space is $(\Omega, \mathcal{F}, \mathbb{P})$. Define \mathbb{P}_x to be the law of $x + X$ for any $x \in \mathbb{R}^d$, and let \mathbb{E}_x be the corresponding expectation operator. Furthermore, we define a sigma-finite measure \mathbb{P}_{λ_d} and a corresponding expectation operator \mathbb{E}_{λ_d} by

$$(3.1) \quad \mathbb{P}_{\lambda_d}\{A\} = \int_{\mathbb{R}^d} \mathbb{P}_x\{A\} dx \quad \forall A \subseteq \mathbb{R}^d \text{ Borel,}$$

$$(3.2) \quad \mathbb{E}_{\lambda_d}[Z] = \int_{\mathbb{R}^d} \mathbb{E}_x[Z] dx \quad \forall Z : \Omega \rightarrow \mathbb{R}_+ \text{ bounded measurable.}$$

We also extend the idea of filtrations. Let $\Pi = \{1, \dots, N\}$. For all $A \subseteq \Pi$ define the partial order $\preccurlyeq_{(A)}$ on \mathbb{R}^N by

$$(3.3) \quad s \preccurlyeq_{(A)} t \iff \begin{cases} s_i \leq t_i, & \text{for all } i \in A, \\ s_i > t_i, & \text{for all } i \in A^C. \end{cases}$$

We may also write $t \succcurlyeq_{(A)} s$ for $s \preccurlyeq_{(A)} t$. It follows that the partial orders $\{\preccurlyeq_{(A)} : A \subseteq \Pi\}$ totally order \mathbb{R}^N in the sense that, for all $s, t \in \mathbb{R}^N$, we can find $A \subseteq \Pi$ such that $s \preccurlyeq_{(A)} t$. For each $A \subseteq \Pi$, we define $\mathcal{F}^A(t)$ to be the σ -field generated

by $\{X(r) : r \preccurlyeq_{(A)} t\}$. We can, and will assume, that $\mathcal{F}^A(t)$ is \mathbb{P}_x -complete for all $x \in \mathbb{R}^d$, and each $\mathcal{F}^A(t)$ is $\preccurlyeq_{(A)}$ -right continuous.

For any Borel probability measure $\mu \in \mathcal{P}(\mathbb{R}_+^N)$, define the *occupation measure* \mathbb{O}_μ on \mathbb{R}^d via the prescription

$$(3.4) \quad \mathbb{O}_\mu(A) := \int_{\mathbb{R}_+^N} \mathbb{1}_A(X(s))\mu(ds) \quad \forall \text{ Borel set } A \subseteq \mathbb{R}^d.$$

Then by standard integration theory, we have

$$(3.5) \quad \mathbb{O}_\mu(f) = \int_{\mathbb{R}_+^N} f(X(s))\mu(ds),$$

for all nonnegative measurable or bounded measurable function $f : \mathbb{R}^d \rightarrow \mathbb{R}$. Now for fixed $\mu \in \mathcal{P}(\mathbb{R}_+^N)$, bounded or nonnegative function $f : \mathbb{R}^d \rightarrow \mathbb{R}$, and $A \subseteq \Pi$, define the process $\mathcal{M}_\mu^A f$ by

$$(3.6) \quad \mathcal{M}_\mu^A f(t) := \mathbb{E}_{\lambda_d}[\mathbb{O}_\mu(f) | \mathcal{F}^A(t)] \quad \forall t \in \mathbb{R}_+^N.$$

The following lemma is from Lemma 4.2 of [14].

Lemma 3.1. *For all measurable functions $f : \mathbb{R}^d \rightarrow \mathbb{R}_+$ in $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ with its Fourier transform $\hat{f} \in L^1(\mathbb{R}^d)$, and for all $\mu \in \mathcal{P}(\mathbb{R}_+^N)$, the following holds:*

$$(3.7) \quad \mathbb{E}_{\lambda_d} \left[\sup_{s \in \mathbb{Q}_+^N} |\mathcal{M}_\mu^A f(s)|^2 \right] \leq 4^N (2\pi)^{-d} \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 \mathcal{E}_\mu(\xi) d\xi,$$

where \mathcal{E}_μ is defined in (2.20).

Finally, recall the associated process $\tilde{X} = \{\tilde{X}(t) : t \in \mathbb{R}_+^N\}$ of X defined in (2.1). We define a family of operators $P = \{P_t : t \in \mathbb{R}^N\}$ on $L^\infty(\mathbb{R}^d)$ by

$$(3.8) \quad P_t f(x) := \mathbb{E}[f(\tilde{X}(t) + x)] \quad \forall f \in L^\infty(\mathbb{R}^d) \text{ and } x \in \mathbb{R}^d.$$

Proposition 3.2 of [14] states the Markov property of X in terms of P_t and we record it here for future reference.

Lemma 3.2. *Suppose that $A \subseteq \Pi$ and that $s \preccurlyeq_{(A)} t$, where s and t are both in \mathbb{R}_+^N . Then for every measurable function $f : \mathbb{R}^d \rightarrow \mathbb{R}_+$,*

$$(3.9) \quad \mathbb{E}_{\lambda_d}[f(X(t)) | \mathcal{F}^A(s)] = P_{t-s} f(X(s)),$$

\mathbb{P}_{λ_d} -a.s., where P_t is defined in (3.8).

3.2. Hitting probabilities. For every $\delta > 0$ and Borel set $F \subseteq \mathbb{R}_+^N$, let F^δ denote the closed δ -enlargement of F , that is, $F^\delta := \{y \in \mathbb{R}_+^N : \text{dist}(y, F) \leq \delta\}$, where $\text{dist}(y, F) := \inf_{x \in F} |x - y|$. Let λ_d denote the Lebesgue measure on \mathbb{R}^d . For fixed $\varepsilon > 0$ and Borel set $F \subseteq \mathbb{R}_+^N$, we want to estimate $\mathbb{E}[\lambda_d((X(F))^\varepsilon)]$. It turns out that this quantity equals the probability of the event that the underlying additive Lévy process hits a small ball.

Proposition 3.3. *For every bounded Borel set $F \subseteq \mathbb{R}_+^N$ and small number $\varepsilon > 0$, we have*

$$(3.10) \quad \mathbb{E}[\lambda_d((X(F))^\varepsilon)] \leq \frac{C\varepsilon^d}{Z_\kappa(\varepsilon; F)},$$

where $Z_\kappa(\varepsilon; F)$ is defined in (2.8) and the constant C depends only on N and d .

Proof. We use the idea of the proof for Theorem 2.1 of [14]. Choose and fix some $\Delta \notin \mathbb{R}_+^N$. For each $\delta > 0$, let T^δ denote some measurable and $(\mathbb{Q}_+^N \cap F^\delta) \cup \{\Delta\}$ -valued function on Ω , such that $T^\delta \neq \Delta$ if and only if $|X(T^\delta)| < \varepsilon/2$. Then for each $k > 0$ define a set function

$$(3.11) \quad \mu^{\delta,k}(\cdot) := \frac{\mathbb{P}_{\lambda_d}\{T^\delta \in \cdot, T^\delta \neq \Delta, |X(0)| \leq k\}}{\mathbb{P}_{\lambda_d}\{T^\delta \neq \Delta, |X(0)| \leq k\}}.$$

Since $B(0, \varepsilon)$ is open and X_j is right-continuous for all $1 \leq j \leq N$, an application of Fubini's theorem implies that

$$(3.12) \quad \begin{aligned} \mathbb{P}_{\lambda_d}\{T^\delta \neq \Delta, |X(0)| \leq k\} &= \mathbb{P}_{\lambda_d}\{X(F^\delta) \cap B(0, \varepsilon/2) \neq \emptyset, |X(0)| \leq k\} \\ &= \int_{[-k,k]^d} \mathbb{P}\{X(F^\delta) \cap B(-x, \varepsilon/2) \neq \emptyset\} dx \\ &= \mathbb{E}[\lambda_d((X(F^\delta) \oplus B(0, \varepsilon/2)) \cap [-k, k]^d)]. \end{aligned}$$

Since the last term is positive and finite, we see that $\mu^{\delta,k}$ is a probability measure on F^δ .

Now for every $A \subseteq \Pi$ and nonnegative measurable or bounded measurable function $f : \mathbb{R}^d \rightarrow \mathbb{R}$, we have

$$(3.13) \quad \begin{aligned} \mathcal{M}_{\mu^{\delta,k}}^A f(s) &\geq \mathbb{E}_{\lambda_d} \left[\int_{t \succ_{(A)} s} f(X(t)) \mu^{\delta,k}(dt) \middle| \mathcal{F}^A(s) \right] \\ &= \int_{t \succ_{(A)} s} \mathbb{E}_{\lambda_d}[f(X(t)) | \mathcal{F}^A(s)] \mu^{\delta,k}(dt) \\ &= \int_{t \succ_{(A)} s} P_{t-s} f(X(s)) \mu^{\delta,k}(dt), \end{aligned}$$

\mathbb{P}_{λ_d} -almost surely, where the last equality follows from Lemma 3.2. We obtain the following by taking sup over all positive rational numbers and noticing that $T^\delta \in \mathbb{Q}_+^N \cup \{\Delta\}$ and $|X(T^\delta)| < \varepsilon/2$ on $\{T^\delta \neq \Delta, |X(0)| \leq k\}$:

$$(3.14) \quad \begin{aligned} &\sup_{s \in \mathbb{Q}_+^N} \mathcal{M}_{\mu^{\delta,k}}^A f(s) \\ &\geq \sup_{s \in \mathbb{Q}_+^N} \int_{t \succ_{(A)} s} P_{t-s} f(X(s)) \mu^{\delta,k}(dt) \\ &\geq \int_{t \succ_{(A)} T^\delta} P_{t-T^\delta} f(X(T^\delta)) \mu^{\delta,k}(dt) \cdot \mathbb{1}\{T^\delta \neq \Delta, |X(0)| \leq k\} \\ &\geq \int_{t \succ_{(A)} T^\delta} \inf_{x \in \mathbb{R}^d, |x| \leq \varepsilon/2} P_{t-T^\delta} f(x) \mu^{\delta,k}(dt) \cdot \mathbb{1}\{T^\delta \neq \Delta, |X(0)| \leq k\}, \end{aligned}$$

\mathbb{P}_{λ_d} -almost surely. Summing over all $A \subseteq \Pi$ gives

$$(3.15) \quad \begin{aligned} &\sum_{A \subseteq \Pi} \sup_{s \in \mathbb{Q}_+^N} \mathcal{M}_{\mu^{\delta,k}}^A f(s) \\ &\geq \int_{\mathbb{R}_+^N} \inf_{x \in \mathbb{R}^d, |x| \leq \varepsilon/2} P_{t-T^\delta} f(x) \mu^{\delta,k}(dt) \cdot \mathbb{1}\{T^\delta \neq \Delta, |X(0)| \leq k\}, \end{aligned}$$

\mathbb{P}_{λ_d} -almost surely. Note that the null set is independent of the choice of $\varepsilon > 0$. By squaring and taking \mathbb{P}_{λ_d} -expectations on both sides of the above inequality, and

noticing the special choice of $\mu^{\delta,k}$, we obtain the following:

$$\begin{aligned}
 (3.16) \quad & \mathbb{E}_{\lambda_d} \left[\left| \sum_{A \subseteq \Pi} \sup_{s \in \mathbb{Q}_+^N} \mathcal{M}_{\mu^{\delta,k}}^A f(s) \right|^2 \right] \\
 & \geq \int_{\mathbb{R}_+^N} \left[\int_{\mathbb{R}_+^N} \inf_{|x| \leq \varepsilon/2} P_{t-s} f(x) \mu^{\delta,k}(dt) \right]^2 \mu^{\delta,k}(ds) \cdot \mathbb{P}_{\lambda_d} \{T^\delta \neq \Delta, |X(0)| \leq k\} \\
 & \geq \left[\int_{\mathbb{R}_+^N} \int_{\mathbb{R}_+^N} \inf_{|x| \leq \varepsilon/2} P_{t-s} f(x) \mu^{\delta,k}(dt) \mu^{\delta,k}(ds) \right]^2 \cdot \mathbb{P}_{\lambda_d} \{T^\delta \neq \Delta, |X(0)| \leq k\},
 \end{aligned}$$

where the last inequality follows from the Cauchy-Schwarz inequality. On one hand, the elementary inequality $(\sum_{k=1}^n a_k)^2 \leq n \sum_{k=1}^n a_k^2$ shows that

$$\begin{aligned}
 (3.17) \quad & \mathbb{E}_{\lambda_d} \left[\left| \sum_{A \subseteq \Pi} \sup_{s \in \mathbb{Q}_+^N} \mathcal{M}_{\mu^{\delta,k}}^A f(s) \right|^2 \right] \leq 2^N \sum_{A \subseteq \Pi} \mathbb{E}_{\lambda_d} \left[\left| \sup_{s \in \mathbb{Q}_+^N} \mathcal{M}_{\mu^{\delta,k}}^A f(s) \right|^2 \right] \\
 & \leq 16^N (2\pi)^{-d} \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 \mathcal{E}_{\mu^{\delta,k}}(\xi) d\xi,
 \end{aligned}$$

where we used Lemma 3.1 to derive the last inequality. On the other hand, we choose our f to be the Fourier transform of the ε -scaled Pólya distribution, that is,

$$(3.18) \quad f_\varepsilon(x) = \prod_{k=1}^d \left(1 - \frac{|x_k|}{2\varepsilon} \right)^+ \quad \forall x \in \mathbb{R}^d,$$

where $a^+ := \max(a, 0)$ for all real numbers a . Then the Fourier transform of f_ε is given by

$$(3.19) \quad \hat{f}_\varepsilon(\xi) = \prod_{k=1}^d \left(\frac{1 - \cos(2\varepsilon\xi_k)}{\varepsilon\xi_k^2} \right) \quad \forall \xi \in \mathbb{R}^d.$$

It follows that both f_ε and \hat{f}_ε are continuous, nonnegative, and in $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. In particular, combining with (2.20), we get

$$(3.20) \quad \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 \mathcal{E}_\mu(\xi) d\xi \leq \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 d\xi,$$

for all $\mu \in \mathcal{P}(\mathbb{R}_+^N)$, and

$$\begin{aligned}
 (3.21) \quad & \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 d\xi = \int_{\mathbb{R}^d} \prod_{k=1}^d \left(\frac{1 - \cos(2\varepsilon\xi_k)}{\varepsilon\xi_k^2} \right)^2 d\xi = \prod_{k=1}^d \left(\int_{\mathbb{R}} \frac{(1 - \cos(2\varepsilon\xi_k))^2}{\varepsilon^2 \xi_k^4} d\xi_k \right) \\
 & = \left(8\varepsilon \int_{\mathbb{R}} \frac{(1 - \cos y)^2}{y^4} dy \right)^d = (8C_0)^d \varepsilon^d,
 \end{aligned}$$

where $C_0 := \int_{\mathbb{R}} y^{-4} (1 - \cos y)^2 dy < \infty$.

From (3.8) and (3.18), we see that $P_{t-s} f_\varepsilon(x)$ is continuous in (t, s, x) . Since $\overline{B(0, \varepsilon/2) \times F^{\delta_0} \times F^{\delta_0}}$ is compact (say $\delta_0 = 1$), $P_{t-s} f_\varepsilon(x)$ is uniformly continuous on $\overline{B(0, \varepsilon/2) \times F^{\delta_0} \times F^{\delta_0}}$. In particular, $g(t, s) := \inf_{x \in \mathbb{R}^d, |x| \leq \varepsilon/2} P_{t-s} f_{\varepsilon/2}(x)$ is

continuous on $F^{\delta_0} \times F^{\delta_0}$. Since the family of probability measures $\{\mu^{1/n,m}\}_{n,m \geq 1}$ are tight, and $F^\delta \downarrow \bar{F}$ as $\delta \downarrow 0$, an application of Prohorov's theorem, together with the continuity of $\inf_x P_{t-s} f_\varepsilon(x)$, imply that there exists some $\mu \in \mathcal{P}(\bar{F})$, such that, along certain subsequence $\delta' \downarrow 0$ and $k' \uparrow \infty$,

$$(3.22) \quad \begin{aligned} & \iint_{|x| \leq \varepsilon/2} \inf_{|x| \leq \varepsilon/2} P_{t-s} f_\varepsilon(x) \mu^{\delta',k'}(dt) \mu^{\delta',k'}(ds) \\ & \rightarrow \iint_{|x| \leq \varepsilon/2} \inf_{|x| \leq \varepsilon/2} P_{t-s} f_\varepsilon(x) \mu(dt) \mu(ds). \end{aligned}$$

Furthermore, (3.12) shows that

$$(3.23) \quad \lim_{\delta \downarrow 0} \lim_{k \uparrow \infty} \mathbb{P}_{\lambda_d} \{T^\delta \neq \Delta, |X(0)| \leq k\} = \mathbb{E}[\lambda_d((X(\bar{F}))^{\varepsilon/2})].$$

Here \bar{F} denotes the closure of F . Now we combine (3.16), (3.17), (3.21), (3.22) and (3.23) to get

$$(3.24) \quad \mathbb{E}[\lambda_d((X(\bar{F}))^{\varepsilon/2})] \cdot \iint_{|x| \leq \varepsilon/2} \inf_{|x| \leq \varepsilon/2} P_{t-s} f_\varepsilon(x) \mu(dt) \mu(ds) \leq 16^N (2\pi)^{-d} (8C_0)^d \varepsilon^d,$$

where $C_0 = \int_{\mathbb{R}} y^{-4} (1 - \cos y)^2 dy < \infty$.

Finally, we notice that $1 - (2\varepsilon)^{-1}|z| \geq \frac{1}{2}$ whenever $z \in B(0, \varepsilon)$. Consequently, $2^{-d} \mathbb{1}_{B(0, \varepsilon)}(z) \leq f_\varepsilon(z)$ for all $z \in \mathbb{R}^d$. Let $z := \tilde{X}(t-s) + x$, use (3.8), and take expectation to find

$$(3.25) \quad \begin{aligned} P_{t-s} f_\varepsilon(x) &= \mathbb{E}[f_\varepsilon(\tilde{X}(t-s) + x)] \\ &\geq 2^{-d} \mathbb{E}[\mathbb{1}_{B(0, \varepsilon)}(\tilde{X}(t-s) + x)] = 2^{-d} \mathbb{P}\{|\tilde{X}(t-s) + x| \leq \varepsilon\} \\ &\geq 2^{-d} \mathbb{P}\{|\tilde{X}(t-s)| < \varepsilon/2\} \cdot \mathbb{1}\{|x| \leq \varepsilon/2\}, \end{aligned}$$

where the last line follows from the triangle inequality. Taking inf over $|x| \leq \varepsilon/2$ gives

$$(3.26) \quad \inf_{x \in \mathbb{R}^d, |x| \leq \varepsilon/2} P_{t-s} f_\varepsilon(x) \geq 2^{-d} \mathbb{P}\{|\tilde{X}(t-s)| < \varepsilon/2\} = 2^{-d} \kappa_{\varepsilon/2}(t-s).$$

We combine the above inequality with (3.24) and change $\varepsilon/2$ to ε to obtain

$$(3.27) \quad \mathbb{E}[\lambda_d((X(\bar{F}))^\varepsilon)] \leq \frac{16^N (2\pi)^{-d} (8C_0)^d 2^d (2\varepsilon)^d}{\inf_{\nu \in \mathcal{P}(\bar{F})} \iint \kappa_\varepsilon(t-s) \nu(dt) \nu(ds)}.$$

The proposition is proved, thanks to Lemma 2.3. □

3.3. Proof of Theorem 2.14. With the help of Proposition 3.3, we can adapt the proof of Theorem 2.7 of Khoshnevisan, Schilling, and Xiao [9] for one-parameter Lévy processes to the current multiparameter setting. Equation (2.39) can be proved with minor changes and we only prove equation (2.40). For that purpose, we need the following technical lemma.

Lemma 3.4. *For all compact sets $F \subseteq \mathbb{R}_+^N$, $\dim_p(\overline{X(F)}) = \dim_p(X(F))$ almost surely, where \bar{E} denotes the closure of a set E .*

Proof. It is straightforward to see that $\dim_p(\overline{X(F)}) \geq \dim_p(X(F))$ almost surely. Then we recall that $X(t) = \sum_{j=1}^N X_j(t_j)$ for $t \in \mathbb{R}_+^N$ and each X_j is a one-parameter

Lévy process. Notice that with probability one

$$(3.28) \quad \{t \in F : X \text{ is discontinuous at } t\} \subseteq \bigcup_{j=1}^N \{t \in F : X_j(t_j) \neq X_j(t_j-)\}.$$

Furthermore, for each $1 \leq j \leq N$, let $Y^j(t) := \sum_{1 \leq l \leq N, l \neq j} X_l(t_l) + X_j(t_j-)$. Since there are at most countably many jumps for each X_j , and \dim_p is σ -stable, it follows that for all $1 \leq j \leq N$,

$$(3.29) \quad \begin{aligned} \dim_p(\{Y^j(t) : t \in F, X_j(t_j) \neq X_j(t_j-)\}) \\ = \dim_p(\{X(t) : t \in F, X_j(t_j) \neq X_j(t_j-)\}) \leq \dim_p X(F) \quad \text{a.s.} \end{aligned}$$

Since $\overline{X(F)} \subseteq X(F) \cup (\bigcup_{j=1}^N \{Y^j(t) : t \in F, X_j(t_j) \neq X_j(t_j-)\})$ almost surely, the σ -stability of \dim_p implies that $\dim_p \overline{X(F)} \leq \dim_p(X(F))$ almost surely. This completes the proof. \square

Proof of Theorem 2.14: (2.40). Definition 2.7 implies that for every $\eta > \text{Dim}_\kappa F$ there exists a sequence of bounded Borel sets $\{F_n\}_{n \geq 1}$ such that

$$(3.30) \quad F \subseteq \bigcup_{n=1}^\infty F_n \quad \text{and} \quad \sup_{n \geq 1} \overline{\dim}_\kappa F_n < \eta.$$

Since $X(F) \subseteq \bigcup_{n \geq 1} X(F_n)$, (2.39) implies that

$$(3.31) \quad \dim_p X(F) \leq \sup_{n \geq 1} \overline{\dim}_M X(F_n) = \sup_{n \geq 1} \overline{\dim}_\kappa F_n < \eta \quad \text{a.s.}$$

Let $\eta \downarrow \text{Dim}_\kappa F$ to obtain $\dim_p X(F) \leq \text{Dim}_\kappa F$ almost surely.

Next we prove the converse inequality, that is, $\dim_p X(F) \geq \text{Dim}_\kappa F$ almost surely. Without loss of generality, we may assume that $\text{Dim}_\kappa F > 0$; otherwise, there is nothing to prove. For every $0 < \eta < \text{Dim}_\kappa F$, Lemma 2.13 implies that there exists a compact subset $K \subseteq F$, depending on η , such that

$$(3.32) \quad \overline{\dim}_\kappa(K \cap [s, t]) \geq \eta,$$

for all $s, t \in \mathbb{Q}_+^N$ and $s \prec_{(\Pi)} t$ (see (3.3)) such that $K \cap [s, t] \neq \emptyset$, where $[s, t] := \{r \in \mathbb{R}_+^N : s \prec_{(\Pi)} r \prec_{(\Pi)} t\}$. For each $1 \leq j \leq N$, let T_j denote the set

$$(3.33) \quad \begin{aligned} \{t \in K : X_j(t_j-) \neq X_j(t_j) \text{ and there is no sequence } \{t^n\}_{n \geq 1} \subseteq K \\ \text{such that } t_j^n \downarrow t_j \text{ as } n \uparrow \infty\}. \end{aligned}$$

Since each X_j has at most countably many jumps almost surely, each T_j is the union of at most countably many hyperplanes in \mathbb{R}^N almost surely. By a similar argument as in Lemma 3.4, we may remove the set $\bigcup_{1 \leq j \leq N} \{X(t) : t \in T_j\}$ from $\overline{X(K)}$ such that the remaining set $\overline{X(K)} \setminus (\bigcup_{1 \leq j \leq N} \{X(t) : t \in T_j\})$ has the same packing dimension as $\overline{X(K)}$. Moreover $\overline{X(K)} \setminus (\bigcup_{1 \leq j \leq N} \{X(t) : t \in T_j\})$ is closed almost surely. Therefore, without loss of generality, we may assume that $\overline{X(K)}$ is almost surely a closed set and for every $x \in X(K)$ there is a subset $A \subseteq \Pi$ and a sequence $\{t^n\}_{n \geq 1} \subseteq K$ such that $t^n \prec_{(A)} t^{n+1}$, $t_j^n \neq t_j^{n+1}$ for all $1 \leq j \leq N$ and $n \geq 1$, and $x = \lim_{n \rightarrow \infty} X(t^n)$. Then for every fixed $a \in \mathbb{Q}^d$ and $r \in \mathbb{Q}_+$, if $\overline{X(K)} \cap B(a, r) \neq \emptyset$, then we can find (random) $s, t \in \mathbb{Q}_+^N$ such that $s \prec_{(\Pi)} t$, $K \cap [s, t] \neq \emptyset$, and $X(K \cap [s, t]) \subseteq B(a, r)$. Since

$$(3.34) \quad \mathbb{P}\{\overline{\dim}_M X(K \cap [s, t]) = \overline{\dim}_\kappa(K \cap [s, t]) \text{ for all } s \prec_{(\Pi)} t \in \mathbb{Q}_+^N\} = 1,$$

from our choice of K (see (3.32)), we see that

$$(3.35) \quad \overline{\dim}_M(\overline{X(K)} \cap B(a, r)) \geq \eta \quad \text{a.s. on the event } \{\overline{X(K)} \cap B(a, r) \neq \emptyset\}.$$

This further implies that the event

$$(3.36) \quad \left\{ \overline{\dim}_M(\overline{X(K)} \cap B(a, r)) \geq \eta \text{ whenever } \overline{X(K)} \cap B(a, r) \neq \emptyset \right. \\ \left. \text{for all } a \in \mathbb{Q}^d \text{ and } r \in \mathbb{Q}_+ \right\}$$

happens with probability one. Now Baire’s category theorem (Proposition 3.6 of Falconer [3]) shows that $\dim_p \overline{X(K)} \geq \eta$ almost surely. Finally, thanks to Lemma 3.4, we have $\dim_p X(K) \geq \eta$ almost surely. Let $\eta \uparrow \text{Dim}_\kappa F$ to obtain $\dim_p X(F) \geq \text{Dim}_\kappa F$ almost surely. \square

4. APPLICATIONS

In this section we apply Theorem 2.14 to certain additive Lévy processes in order to derive some interesting corollaries. First we present the promised connection between the packing dimension profiles defined in (2.23) and that of Falconer and Howroyd [4] and Howroyd [6].

Corollary 4.1. *Let X be a d -dimensional additive Lévy process so that each X_j is isotropic α -stable for some $0 < \alpha \leq 2$. Then for every nonrandom bounded Borel set $F \subseteq \mathbb{R}_+^N$, we have*

$$(4.1) \quad \text{Dim}_\kappa F = \alpha \text{Dim}_{d/\alpha}^{\text{FH}} F,$$

where Dim_s^{FH} denotes the s -dimension packing dimension profiles of Falconer and Howroyd [4]. In particular,

$$(4.2) \quad \dim_p X(F) = \alpha \text{Dim}_{d/\alpha}^{\text{FH}} F \quad \text{a.s.}$$

Proof. Since $\mathbb{E}[e^{i\xi \cdot X(t)}] = \exp(-(t_1 + \dots + t_N) \|\xi\|^\alpha)$ for all $\xi \in \mathbb{R}^d$, we see that $X(t) \stackrel{d}{=} X_1(t_1 + \dots + t_N)$ for all $t \in \mathbb{R}_+^N$, where “ $\stackrel{d}{=}$ ” means equal in distribution. This and the symmetry of each X_j together imply that

$$(4.3) \quad \kappa_\varepsilon(t) = \mathbb{P}\{|\tilde{X}(t)| < \varepsilon\} = \mathbb{P}\{|X_1(|t_1| + \dots + |t_N|)| < \varepsilon\} \quad \forall t \in \mathbb{R}^N.$$

Since X_1 is α -stable, it is well known that there exist constants $0 < C'_1 < C'_2 < \infty$ such that uniformly for all $r > 0$ and $\varepsilon \in (0, 1)$,

$$(4.4) \quad C'_1 \left(\frac{\varepsilon}{r^{1/\alpha}} \wedge 1 \right)^d \leq \mathbb{P}\{|X_1(r)| < \varepsilon\} \leq C'_2 \left(\frac{\varepsilon}{r^{1/\alpha}} \wedge 1 \right)^d.$$

Then (4.3) and (4.4) imply the existence of nonrandom constants $0 < C_1 < C_2 < \infty$ such that

$$(4.5) \quad C_1 \left(\frac{\varepsilon}{N^{1/\alpha} |t|^{1/\alpha}} \wedge 1 \right)^d \leq \kappa_\varepsilon(t) \leq C_2 \left(\frac{\varepsilon}{|t|^{1/\alpha}} \wedge 1 \right)^d,$$

uniformly for all $t \in \mathbb{R}^N \setminus \{0\}$ and $\varepsilon \in (0, 1)$. It follows from this and (2.5) that

$$(4.6) \quad \overline{\dim}_\kappa F \\ = \sup \left\{ \eta > 0 : \liminf_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^{\eta/\alpha}} \inf_{\mu \in \mathcal{P}(F)} \iint \left[\left(\frac{\varepsilon}{|t-s|} \right)^{d/\alpha} \wedge 1 \right] \mu(dt) \mu(ds) = 0 \right\}.$$

By Theorem 4.1 of Khoshnevisan and Xiao [13], we obtain

$$(4.7) \quad \overline{\dim}_\kappa F = \alpha \text{B-}\overline{\text{Dim}}_{d/\alpha} F,$$

where $\text{B-}\overline{\text{Dim}}_s F$ denotes the s -dimensional box dimension profiles of Howroyd [6]. By applying a similar argument as the proof of (2.40), we deduce that $\text{Dim}_\kappa F = \alpha \text{P-}\overline{\text{Dim}}_{d/\alpha} F$, where $\text{P-}\overline{\text{Dim}}_s F$ denotes the s -dimension packing dimension profiles of Howroyd [6]. Thanks to Theorem 1.1 of [13], $\text{P-}\overline{\text{Dim}}_s F = \text{Dim}_s^{\text{FH}} F$ for all analytic sets $F \subseteq \mathbb{R}^N$ and $s > 0$. Therefore $\text{Dim}_\kappa F = \alpha \text{Dim}_{d/\alpha}^{\text{FH}} F$. Finally, (4.2) follows from Theorem 2.14. \square

Next we use Corollary 4.1 to compute the Falconer-Howroyd packing dimension profiles of a family of random sets. In order to do so, we recall that a subordinator $\{S(t)\}_{t \geq 0}$ is a one-parameter Lévy process such that for almost all sample paths S is nondecreasing and $S(t) \geq 0$ for all $t \geq 0$. The Laplace exponent Φ of a subordinator S is defined by $\mathbb{E}[e^{-\lambda S(t)}] = e^{-t\Phi(\lambda)}$. Our next corollary is a direct extension of Corollary 1.3 of [9]. The proof is the same and we omit it.

Corollary 4.2. *Let $S(t) = (S_1(t), \dots, S_N(t))$, where S_1, \dots, S_N are independent subordinators with Laplace exponents Φ_1, \dots, Φ_N , respectively. Then for all $s \geq \frac{1}{2}$,*

$$(4.8) \quad \text{Dim}_s^{\text{FH}} S([0, 1]) = s(1 - \theta) \quad a.s.,$$

where

$$(4.9) \quad \theta := \lim_{\lambda \uparrow \infty} \frac{1}{\log \lambda} \log \left(\int_1^\lambda \frac{dx}{\sum_{j=1}^N \Phi_j(x^{1/s})} \right).$$

In particular, if S_1, \dots, S_N have the same distribution, then

$$(4.10) \quad \text{Dim}_s^{\text{FH}} S([0, 1]) = \text{Dim}_s^{\text{FH}} S_1([0, 1]) \quad a.s.$$

Equation (4.10) is not surprising since all N subordinators are driven by the same time t . Next we will let the subordinators driven by their own times and compute the Falconer-Howroyd packing dimension profiles again.

Corollary 4.3. *Let $S(t) = (S_1(t_1), \dots, S_N(t_N))$ for $t \in \mathbb{R}_+^N$, where S_1, \dots, S_N are independent subordinators with Laplace exponents Φ_1, \dots, Φ_N , respectively. Then for all $s \geq \frac{1}{2}$,*

$$(4.11) \quad \text{Dim}_s^{\text{FH}} S([0, 1]^N) = s(1 - \theta) \quad a.s.,$$

where

$$(4.12) \quad \theta := \lim_{\lambda \uparrow \infty} \frac{1}{\log \lambda} \log \left(\int_1^\lambda \frac{dx}{\prod_{j=1}^N \Phi_j(x^{1/s})} \right).$$

In particular, if S_1, \dots, S_N have the same distribution, then

$$(4.13) \quad \text{Dim}_s^{\text{FH}} S([0, 1]^N) \leq N \cdot \text{Dim}_s^{\text{FH}} S_1([0, 1]).$$

We can prove Corollary 4.3 by the same method as in Corollary 4.2, provided that there is a multiparameter version of Theorem 1.1 of Khoshnevisan and Xiao [12]. One way to derive the said multiparameter analog is to first find a formula for additive Lévy processes similar to that of Theorem 6 of Taylor [15] and then employ the method of [12]. However, we can also achieve this directly from Proposition 2.6, as the following shows.

Lemma 4.4. $\text{Dim}_\kappa [0, 1]^N = \overline{\text{dim}}_\kappa [0, 1]^N$.

Proof. For every l^∞ open ball $B(t, \varepsilon)$, define $F(t, \varepsilon) := B(t, \varepsilon) \cap [0, 1]^N$. If $F(t, \varepsilon)$ is nonempty, then we can find finitely many points $t^1, \dots, t^n \in \mathbb{R}^N$ such that $[0, 1]^N \subseteq \bigcup_{i=1}^n G_i(t, \varepsilon)$, where $G_i(t, \varepsilon) := \{s + t^i : s \in F(t, \varepsilon)\}$. The monotonicity, finite stability, and translation invariance of $\overline{\text{dim}}_\kappa$ together imply that

$$(4.14) \quad \overline{\text{dim}}_\kappa F(t, \varepsilon) \leq \overline{\text{dim}}_\kappa [0, 1]^N \leq \max_{1 \leq i \leq n} \overline{\text{dim}}_\kappa G_i(t, \varepsilon) = \overline{\text{dim}}_\kappa F(t, \varepsilon).$$

Since $B(t, \varepsilon)$ is arbitrary, we see that $\overline{\text{dim}}_\kappa [0, 1]^N \cap V = \overline{\text{dim}}_\kappa [0, 1]^N$ for every open set $V \subseteq \mathbb{R}^N$, provided $V \cap [0, 1]^N \neq \emptyset$. Since $[0, 1]^N$ is compact, an application of Baire’s category theorem shows that $\text{Dim}_\kappa [0, 1]^N = \overline{\text{dim}}_\kappa [0, 1]^N$. \square

Lemma 4.5. *Let X be an N -parameter additive Lévy process in \mathbb{R}^d with characteristic function $\mathbb{E}[e^{i\xi \cdot X(t)}] = \exp(-\sum_{j=1}^N t_j \Psi_j(\xi))$. Furthermore, we assume that $\Psi_j(\xi) > 0$ for all $\xi \in \mathbb{R}^d$ and $1 \leq j \leq N$. For each $\varepsilon > 0$, define*

$$(4.15) \quad W(\varepsilon) = \int_{\mathbb{R}^d} \frac{\rho(\xi/\varepsilon)}{\prod_{k=1}^d (1 + \xi_k^2)} d\xi \quad \text{with} \quad \rho(z) = \prod_{j=1}^N \text{Re} \left(\frac{1}{1 + \Psi_j(z)} \right).$$

Then

$$(4.16) \quad \text{dim}_p X([0, 1]^N) = \sup\{\eta > 0 : \liminf_{\varepsilon \downarrow 0} \varepsilon^{-\eta} W(\varepsilon) = 0\} = \liminf_{\varepsilon \downarrow 0} \frac{\log W(\varepsilon)}{\log \varepsilon} \quad a.s.$$

Proof. For each $\varepsilon > 0$, define

$$(4.17) \quad V(\varepsilon) = \inf_{\mu \in \mathcal{P}([0, 1]^N)} \int_{\mathbb{R}^d} \frac{\mathcal{E}_\mu(\xi/\varepsilon)}{\prod_{k=1}^d (1 + \xi_k^2)} d\xi,$$

with $\mathcal{E}_\mu(\xi)$ defined in (2.20). Thanks to Theorem 2.14, Lemma 4.4 and Proposition 2.6, it suffices to show that there exist some nonrandom constants $0 < c < C < \infty$, not depending on ε , such that

$$(4.18) \quad cW(\varepsilon) \leq V(\varepsilon) \leq CW(\varepsilon) \quad \forall \varepsilon > 0.$$

In order to show this, define for each $t \in \mathbb{R}^N$ and $t_0 \in \mathbb{R}$

$$(4.19) \quad f_\varepsilon(t, t_0) := e^{-|t_0|} \int_{\mathbb{R}^d} \exp \left(-\sum_{j=1}^N |t_j| \Psi_j(\text{sgn}(t_j)\xi/\varepsilon) \right) \frac{d\xi}{\prod_{k=1}^d (1 + \xi_k^2)}.$$

From Fubini’s theorem, we see that the Fourier transform of f_ε satisfies

$$(4.20) \quad \hat{f}_\varepsilon(z) = 2^{N+1} \int_{\mathbb{R}^d} \text{Re} \left(\frac{1}{1 - iz_0} \right) \prod_{j=1}^N \text{Re} \left(\frac{1}{\Psi_j(\xi/\varepsilon) - iz_j} \right) \frac{d\xi}{\prod_{k=1}^d (1 + \xi_k^2)},$$

for all $z \in \mathbb{R}^{1+N}$. Moreover, since $\Psi_j(\xi) > 0$ for all $\xi \in \mathbb{R}^d$ and $1 \leq j \leq N$, $\hat{f}_\varepsilon(z)$ is positive for all $z \in \mathbb{R}^{N+1}$. Thus we can apply Theorem 3.1 of Khoshnevisan [8] to derive that

$$(4.21) \quad \begin{aligned} & \inf_{\mu \in \mathcal{P}([0, 1]^N \times \{0\})} \iint f_\varepsilon(t - s, t_0 - s_0) \mu(dt \times dt_0) \mu(ds \times ds_0) \\ &= \int_{[0, 1]^N} f_\varepsilon(t, 0) dt = \int_{\mathbb{R}^d} \prod_{j=1}^N \frac{1 - \exp(-\Psi_j(\xi/\varepsilon))}{\Psi_j(\xi/\varepsilon)} \frac{d\xi}{\prod_{k=1}^d (1 + \xi_k^2)}. \end{aligned}$$

On one hand, the elementary facts

$$(4.22) \quad \lim_{x \rightarrow \infty} \frac{1 - e^{-x}}{x} / \frac{1}{1+x} = 1 \quad \text{and} \quad \lim_{x \downarrow 0} \frac{1 - e^{-x}}{x} / \frac{1}{1+x} = 1$$

imply that there exist nonrandom constants $0 < c < C < \infty$ such that $c/(1+x) \leq (1 - e^{-x})/x \leq C/(1+x)$ for all $x \geq 0$. On the other hand, we have

$$(4.23) \quad \inf_{\mu \in \mathcal{P}([0,1]^N \times \{0\})} \iint f_\varepsilon(t - s, t_0 - s_0) \mu(dt \times dt_0) \mu(ds \times ds_0) = V(\varepsilon).$$

Now (4.18) follows from these two facts and (4.21). □

Proof of Corollary 4.3. Let X be an additive Lévy process in \mathbb{R} , independent from S . Furthermore, let X_j be isotropic stable with index $\alpha = 1/s$ for $1 \leq j \leq N$. Define $Y(t) := \sum_{j=1}^N X_j(S_j(t_j))$ for $t \in \mathbb{R}_+^N$. Then Y is an additive Lévy process in \mathbb{R} with characteristic function $\mathbb{E}[e^{izX(t)}] = \exp(-\sum_{j=1}^N t_j \Phi_j(|z|^\alpha))$, where Φ_j is the Laplace exponent of S_j . On one hand, we can apply Corollary 4.1, conditionally on the process S , to derive

$$(4.24) \quad \dim_p X(S([0, 1])) = \alpha \text{Dim}_{1/\alpha}^{\text{FH}} S([0, 1]) \quad \text{a.s.}$$

On the other hand, we can apply Lemma 4.5 directly to the additive Lévy process Y to derive

$$(4.25) \quad \begin{aligned} \dim_p X(S([0, 1])) &= \dim_p Y([0, 1]) \\ &= \overline{\lim}_{\varepsilon \downarrow 0} \frac{1}{\log \varepsilon} \log \left(\int_0^\infty \frac{dx}{(1+x^2) \prod_{j=1}^N (1 + \Phi_j((x/\varepsilon)^\alpha))} \right) \quad \text{a.s.} \end{aligned}$$

Now by applying to the last term a similar analysis of Corollary 1.3 of Khoshnevisan, Schilling, and Xiao [9], we have

$$(4.26) \quad \begin{aligned} &\overline{\lim}_{\varepsilon \downarrow 0} \frac{1}{\log \varepsilon} \log \left(\int_0^\infty \frac{dx}{(1+x^2) \prod_{j=1}^N (1 + \Phi_j((x/\varepsilon)^\alpha))} \right) \\ &= \overline{\lim}_{\varepsilon \downarrow 0} \frac{1}{\log \varepsilon} \log \left(\varepsilon \int_1^{\varepsilon^{1/\alpha}} \frac{dx}{\prod_{j=1}^N \Phi_j((x/\varepsilon)^\alpha)} \right). \end{aligned}$$

It follows that

$$(4.27) \quad \dim_p X(S([0, 1])) = 1 - \theta \quad \text{a.s.}$$

Therefore (4.11) follows from this and (4.24).

When S_1, \dots, S_N have the same distribution, Jensen's inequality implies that

$$(4.28) \quad \begin{aligned} \theta &= \underline{\lim}_{\lambda \uparrow \infty} \frac{1}{\log \lambda} \log \left((\lambda - 1) \int_1^\lambda \frac{(\lambda - 1)^{-1} dx}{\Phi(x^{1/s})^N} \right) \\ &\geq \underline{\lim}_{\lambda \uparrow \infty} \frac{1}{\log \lambda} \log \left((\lambda - 1) \left(\int_1^\lambda \frac{(\lambda - 1)^{-1} dx}{\Phi(x^{1/s})} \right)^N \right) \\ &= 1 - N + N \underline{\lim}_{\lambda \uparrow \infty} \frac{1}{\log \lambda} \log \left(\int_1^\lambda \frac{dx}{\Phi(x^{1/s})} \right). \end{aligned}$$

Thus $\text{Dim}_s^{\text{FH}} S([0, 1]^N) \leq N \cdot \text{Dim}_s^{\text{FH}} S_1([0, 1])$. □

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