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SPACES OF GEODESICS OF PSEUDO-RIEMANNIAN SPACE FORMS AND NORMAL CONGRUENCES OF HYPERSURFACES

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Abstract. We describe natural Kähler or para-Kähler structures of the spaces of geodesics of pseudo-Riemannian space forms and relate the local geometry of hypersurfaces of space forms to that of their normal congruences, or Gauss maps, which are Lagrangian submanifolds.

The space of geodesics $L^{\pm}(\mathbb{S}^{n+1}_{p,1})$ of a pseudo-Riemannian space form $\mathbb{S}^{n+1}_{p,1}$ of non-vanishing curvature enjoys a Kähler or para-Kähler structure $(\mathbb{J}, \mathbb{G})$ which is in addition Einstein. Moreover, in the three-dimensional case, $L^{\pm}(\mathbb{S}^{n+1}_{p,1})$ enjoys another Kähler or para-Kähler structure $(\mathbb{J}', \mathbb{G}')$ which is scalar flat. The normal congruence of a hypersurface $\mathcal{S}$ of $\mathbb{S}^{n+1}_{p,1}$ is a Lagrangian submanifold $\bar{\mathcal{S}}$ of $L^{\pm}(\mathbb{S}^{n+1}_{p,1})$, and we relate the local geometries of $\mathcal{S}$ and $\bar{\mathcal{S}}$. In particular $\bar{\mathcal{S}}$ is totally geodesic if and only if $\mathcal{S}$ has parallel second fundamental form. In the three-dimensional case, we prove that $\bar{\mathcal{S}}$ is minimal with respect to the Einstein metric $\mathbb{G}$ (resp. with respect to the scalar flat metric $\mathbb{G}'$) if and only if it is the normal congruence of a minimal surface $\mathcal{S}$ (resp. of a surface $\mathcal{S}$ with parallel second fundamental form); moreover $\bar{\mathcal{S}}$ is flat if and only if $\mathcal{S}$ is Weingarten.

Introduction

After the seminal paper of N. Hitchin ([14]) describing the natural complex structure of the space of oriented straight lines of Euclidean 3-space, several invariant structures on the space of geodesics of certain Riemannian manifolds and their submanifolds have recently been explored by different authors (see [4], [8], [11], [9], [10], [15], [16], [17], [23], [24]). In [1], a unified viewpoint has been given to this question, classifying all invariant Riemannian, symplectic, complex and para-complex structures that may exist on the space of geodesics of a number of spaces: the Euclidean and pseudo-Euclidean spaces, the Riemannian and pseudo-Riemannian space forms and the complex and quaternionic space forms. One of the interesting issues about the spaces of geodesics is that the normal congruence (or Gauss map) of a one-parameter family of parallel hypersurfaces in some space is a Lagrangian submanifold of the corresponding space of geodesics.

The purpose of this paper is twofold: first, to give a more precise picture of the structure of the space of geodesics of pseudo-Riemannian space forms, and second to study in detail the relationships between the pseudo-Riemannian geometry of a one-parameter family of parallel hypersurfaces and that of its normal congruence.

In particular, we describe the natural Kähler or para-Kähler structure of the space of geodesics of pseudo-Riemannian space forms of non-vanishing curvature

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and prove that the corresponding metric $\mathbb{G}$ is Einstein (Theorem 2.1). The space of geodesics of pseudo-Riemannian three-dimensional space forms, which is four-dimensional, is specific since *(i)* it is the only dimension for which the space of geodesics of flat pseudo-Euclidean spaces enjoys an invariant metric (see [23], [1]), and *(ii)* in the non-flat case it enjoys another natural complex or para-complex structure, which in turn defines a neutral metric $\mathbb{G}'$. We prove that $\mathbb{G}'$ is scalar flat and locally conformally flat (Theorem 2.4).

Next we turn our attention to the relation between one-parameter families of parallel hypersurfaces in pseudo-Riemannian space forms and their normal congruences. We first check that an n-dimensional geodesic congruence $\bar{\mathcal{S}}$ is Lagrangian if and only it orthogonally crosses a hypersurface $\mathcal{S}$ (Theorem 2.10), and therefore all the hypersurfaces $\mathcal{S}_t$ parallel to $\mathcal{S}$ and to its polar. Given a one-parameter family of parallel hypersurfaces $(\mathcal{S}_t)$ and its normal congruence $\bar{\mathcal{S}}$, we relate the first and second fundamental forms of $(\mathcal{S}_t)$ to those of $\bar{\mathcal{S}}$ (Theorems 2.11 and 2.16). These formulas imply several interesting corollaries: $\bar{\mathcal{S}}$ is totally geodesic (either with respect to $\mathbb{G}$ or $\mathbb{G}'$) if and only if the hypersurfaces $\mathcal{S}_t$ have parallel second fundamental form; in the three-dimensional case, $\bar{\mathcal{S}}$ is minimal with respect to $\mathbb{G}$ if and only if one of the parallel surfaces $\mathcal{S}_t$ is minimal (Corollary 2.14); $\bar{\mathcal{S}}$ is minimal with respect to $\mathbb{G}'$ if and only if the parallel surfaces $\mathcal{S}_t$ are totally geodesic (Corollary 2.17); the induced metric on $\bar{\mathcal{S}}$ is flat if and only if the surfaces $\mathcal{S}_t$ are Weingarten (Corollary 2.18). We also exhibit three families of Lagrangian surfaces which are marginally trapped with respect to $\mathbb{G}$ or $\mathbb{G}'$ (Corollary 2.20).

The paper is organised as follows: Section 1 provides some useful preliminaries and Section 2 gives the precise statements of results; Section 3 deals with the geometry of the spaces of geodesics, while Section 4 is devoted to normal congruences of hypersurfaces.

1. Preliminaries

1.1. **Hypersurfaces in pseudo-Riemannian space forms.** Consider the real space $\mathbb{R}^{n+2}$ and endowed with the canonical pseudo-Riemannian metric of signature $(p, n+2-p)$, where $0 \leq p \leq n+1$:

$$\langle ., . \rangle_p := -\sum_{i=1}^{p} dx_i^2 + \sum_{i=p+1}^{n+2} dx_i^2,$$

and the $(n+1)$-dimensional quadric

$$\mathbb{S}^{n+1}_{p,\epsilon} = \left\{ x \in \mathbb{R}^{n+2} \middle| \, \langle x, x \rangle_p^2 = \epsilon \right\},$$

where $\epsilon = \pm 1$. The metric induced on $\mathbb{S}^{n+1}_{p,\epsilon}$ by the canonical inclusion $\mathbb{S}^{n+1}_{p,\epsilon} \hookrightarrow (\mathbb{R}^{n+2}, \langle ., . \rangle_p)$ has signature $(p, n+1-p)$ if $\epsilon = 1$ and $(p-1, n+2-p)$ if $\epsilon = -1$, and has constant sectional curvature $K = \epsilon$. Conversely, it is known (see [18]) that any pseudo-Riemannian manifold with constant sectional curvature is, up to a scaling of the metric, locally isometric to one of these quadrics. The transformation

$$\begin{array}{cccc} \mathtt{A}: & \mathbb{R}^{n+2} & \to & \mathbb{R}^{n+2} \\ & (x_1, ..., x_p, x_{p+1}, ..., x_{n+2}) & \mapsto & (x_{p+1}, ..., x_{n+2}, x_1, ..., x_{p+1}) \end{array}$$

defines an anti-isometry of $\mathbb{S}^{n+1}_{p,\epsilon}$ onto $\mathbb{S}^{n+1}_{n+2-p,-\epsilon}$. It is therefore sufficient to study the case $\epsilon = 1$. The two Riemannian space forms are (i) the sphere $\mathbb{S}^{n+1} := \mathbb{S}^{n+1,1}_{0,1}$, which is the only compact quadric, and (ii) the hyperbolic space $\mathbb{H}^{n+1} :=$

$\mathtt{A}(\mathbb{S}^{n+1}_{n+1,1}) \cap \{x \in \mathbb{R}^{n+2} | x_1 > 0\}$) ($\mathbb{S}^{n+1}_{1,-1}$ and $\mathbb{S}^{n+1}_{n+1,1}$ are the only non-connected quadrics). Analogously, the two Lorentzian space forms are the de Sitter space $d\mathbb{S}^{n+1} := \mathbb{S}^{n+1}_{1,1}$ and the anti-de Sitter space $Ad\mathbb{S}^{n+1} := \mathbb{S}^{n+1}_{2,-1} = \mathtt{A}(\mathbb{S}^{n+1}_{n,1})$.

Let $\varphi : \mathcal{M}^n \to \mathbb{S}^{n+1}_{p,1}$ be a smooth map from an orientable n-dimensional manifold $\mathcal{M}^n$. We set $g := \varphi^* \langle .,. \rangle_p$ for the induced metric on $\mathcal{M}^n$. We shall always assume that φ is a pseudo-Riemannian immersion, i.e. g is non-degenerate. This is equivalent to the existence of a unit normal vector field along the immersed hypersurface $\mathcal{S} := \varphi(\mathcal{M}^n)$ that we will denote by N. The curvature of $\mathcal{S}$ may be equivalently described by two tensors: the second fundamental form h with respect to N, i.e. $h(X,Y) = g(\nabla_X Y, N)$, where ∇ denotes the Levi-Civita connection of $\langle .,. \rangle_p$; the shape operator defined by $AX = -dN(X)$. They are related by the formula $g(AX,Y) = h(X,Y)$. The shape operator A is not necessarily real diagonalizable since it is symmetric with respect to the possibly indefinite metric g. More precisely, A may be of three types: real diagonalizable, complex diagonalizable, or not diagonalizable at all. In the two-dimensional case, we shall use the existence of a canonical form for A, i.e. the existence of a frame (e_1, e_2) such that the matrices of g and A take a simple form (see [20]):

- real diagonalizable case:
$$g = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix},$$
with $\epsilon_1, \epsilon_2 = \pm 1$;
- complex diagonalizable case
$$g = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} H & \lambda \\ -\lambda & H \end{pmatrix},$$
with non-vanishing λ;
- non-diagonalizable case:
$$g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} H & 1 \\ 0 & H \end{pmatrix}.$$

1.2. **Parallel hypersurfaces.** It will be convenient to introduce some notation: we set $(\mathtt{cos}\epsilon, \mathtt{sin}\epsilon) := (\cos, \sin)$ if $\epsilon = 1$ and $(\mathtt{cos}\epsilon, \mathtt{sin}\epsilon) := (\cosh, \sinh)$ if $\epsilon = -1$. Given $t \in \mathbb{R}$, the image of
$$\varphi_t := \mathtt{cos}\epsilon(t)\varphi + \mathtt{sin}\epsilon(t) N,$$
when an immersion, is parallel to $\mathcal{S}$. When A is invertible, the map $N : \mathcal{M}^n \to \mathbb{S}^{n+1}_{p,\epsilon}$, where $\epsilon := |N|^2_p$, is an immersion and its image $\mathcal{S}' := N(\mathcal{M}^n)$ is called the *polar* of $\mathcal{S}$. If $\epsilon = 1$, we have $\varphi_{\pi/2} = N$; hence the polar of $\mathcal{S}$ is parallel to $\mathcal{S}$. If $\epsilon = -1$, $\mathcal{S}' \in \mathbb{S}^{n+1}_{p,-1} = \mathtt{A}(\mathbb{S}^{n+1}_{n+2-p,1})$. In all cases, a unit normal vector field along $\mathcal{S}_t = \varphi_t(\mathcal{M}^n)$ is
$$N_t := \mathtt{cos}\epsilon(t) N - \epsilon \mathtt{sin}\epsilon(t)\varphi,$$
which, when an immersion, is parallel to $\mathcal{S}'$.

Lemma 1.1. *Let $\varphi : \mathcal{M}^2 \to \mathbb{S}^3_{p,1}$ be an immersion with mean curvature H and Gaussian curvature K, which satisfies the following linear Weingarten equation:*
$$\left| \frac{2H}{K - \epsilon} \right| = C,$$

where $C \in [0,\infty)\cup\{\infty\}$ and $(\epsilon, C) \neq (-1,1)$. Then there exists a minimal immersed hypersurface which is parallel to $\mathcal{S} := \varphi(\mathcal{M}^2)$ or to its polar $\mathcal{S}'$.

Proof. We first compute

$$d\varphi_t = \mathtt{cos}\epsilon(t) d\varphi + \mathtt{sin}\epsilon(t) dN = (\mathtt{cos}\epsilon(t) Id - \mathtt{sin}\epsilon(t) A) \circ d\varphi.$$

Observe that φ_t is an immersion if and only if $\mathtt{cos}\epsilon(t) Id - \mathtt{sin}\epsilon(t) A$ is invertible. When this is the case, we have

$$A_t = -dN_t = (\mathtt{cos}\epsilon(t) A + \epsilon \mathtt{sin}\epsilon(t) Id) \circ (\mathtt{cos}\epsilon(t) Id - \mathtt{sin}\epsilon(t) A)^{-1}.$$

A straightforward calculation shows that the mean curvature $H_t = trA_t$ of φ_t vanishes if and only if $\mathtt{cos}\epsilon(2t) 2H + \mathtt{sin}\epsilon(2t)(\epsilon - K)$ vanishes as well. If $\epsilon = 1$ we get the vanishing of H_{t_0} setting $t_0 := \frac{1}{2}\tan^{-1}\left(\frac{2H}{K-1}\right)$. If $\epsilon = -1$ and $\left|\frac{2H}{K+1}\right| < 1$, the same occurs with $t_0 := \frac{1}{2}\tanh^{-1}\left(\frac{2H}{K+1}\right)$. Finally, if $\epsilon = -1$ and $\left|\frac{2H}{K+1}\right| > 1$, we easily check that $N_{t_0} := \cosh(t_0)N - \epsilon\sinh(t_0)\varphi$, where $t_0 := \frac{1}{2}\coth^{-1}\left(\frac{2H}{K+1}\right)$, is minimal. □

We shall denote by $\mathtt{arctan}\epsilon$ the integral of the map $\frac{1}{1+\epsilon t^2}$, i.e.

$$\mathtt{arctan}\epsilon(t) = \begin{cases} \tan^{-1}(t) \text{ if } \epsilon = 1, \\ \tanh^{-1}(t) \text{ if } \epsilon = -1, |t| < 1, \\ \coth^{-1}(t) \text{ if } \epsilon = -1, |t| > 1. \end{cases}$$

1.3. **Lagrangian submanifolds.** We first recall the definition of a Lagrangian submanifold:

Definition 1.2. Let $(\mathcal{N}, \omega)$ be a $2n$-dimensional symplectic manifold. An immersion $\varphi : \mathcal{M}^n \to \mathcal{N}$ is said to be *Lagrangian* if $\varphi^*\omega = 0$.

We refer the reader to [2] or [7] for material about para-complex geometry (sometimes referred to as *split-complex* or *bi-Lagrangian* geometry). By a *pseudo-Kähler* or a *para-Kähler* manifold, we mean a manifold equipped with a complex or para-complex structure $\mathbb{J}$ and a compatible pseudo-Riemannian metric $\mathbb{G}$, i.e. such that $\mathbb{G}(\mathbb{J}., \mathbb{J}.) = \epsilon\mathbb{G}(.,.)$. Here, $\epsilon = 1$ in the complex case and $\epsilon = -1$ in the para-complex case. In other words $\mathbb{J}$ is an isometry in the complex case and an anti-isometry in the para-complex case. It is furthermore required that the *symplectic form* $\omega := \epsilon\mathbb{G}(\mathbb{J}.,.)$ be closed.[1] Observe that the metric $\mathbb{G}$ is determined by the pair $(\mathbb{J}, \omega)$ via the equation $\mathbb{G} := \omega(., \mathbb{J}.)$.

It is well known that the extrinsic curvature of a Lagrangian submanifold in a Kähler manifold $(\mathcal{N}, \mathbb{J}, \mathbb{G})$ is described by the tri-symmetric tensor $h(X,Y,Z) := \mathbb{G}(D_X Y, \mathbb{J}Z)$, where D denotes the Levi-Civita connection of $\mathbb{G}$ (see [3]). It turns out that the same fact holds in the para-Kähler case. Since the proof is similar, it is omitted.

Lemma 1.3. *Let $\mathcal{L}$ be a non-degenerate, Lagrangian submanifold of a pseudo-Kähler or para-Kähler manifold $(\mathcal{N}, \mathbb{J}, \mathbb{G}, \omega)$. Denote by D the Levi-Civita connection of $\mathbb{G}$. Then the map $h(X,Y,Z) := \mathbb{G}(D_X Y, \mathbb{J}Z)$ is tensorial and tri-symmetric,*

[1] Of course the factor ϵ is unessential here and is put in order to simplify further exposition. In particular, this convention allows us to recover, in the case of $\mathbb{R}^2$, the "natural" objects $\mathbb{G} := dx^2 + \epsilon dy^2$, $\mathbb{J}(\partial_x, \partial_y) := (\partial_y, -\epsilon\partial_x)$ and $\omega := dx \wedge dy$.

i.e.

$$h(X,Y,Z) = h(Y,X,Z) = h(X,Z,Y).$$

2. Statement of results

2.1. Structures of the space of geodesics of pseudo-Riemannian space-forms.

Let x be a point of $\mathbb{S}^{n+1}_{p,1}$ and $v \in T_x\mathbb{S}^{n+1}_{p,1} = x^\perp$ a unit vector tangent to x. Setting $\epsilon := |v|^2_p$, the unique geodesic γ of $\mathbb{S}^{n+1}_{p,1}$ passing through x with velocity v is the periodic curve parametrized by $\gamma(t) = \mathtt{cos}\epsilon(t)x + \mathtt{sin}\epsilon(t)v$.

The set $L^+(\mathbb{S}^{n+1}_{p,1})$ of positive oriented geodesics of $\mathbb{S}^{n+1}_{p,1}$ identifies with the Grassmannian $Gr^+(n+2,2)$ of oriented two-planes of $\mathbb{R}^{n+2}$ with positive induced metric, while the set $L^-(\mathbb{S}^{n+1}_{p,1})$ of negative oriented geodesics of $\mathbb{S}^{n+1}_{p,1}$ identifies with the Grassmannian $Gr^-(n+2,2)$ of oriented two-planes of $\mathbb{R}^{n+2}$ with indefinite induced metric:

$$L^+(\mathbb{S}^{n+1}_{p,1}) \simeq Gr^+_p(n+2,2) \simeq \left\{x\wedge y \in \Lambda^2(\mathbb{R}^{n+2}) \big| (x,y)\in T\mathbb{S}^{n+1}_{p,1}, \langle y,y\rangle_p = 1\right\},$$

$$L^-(\mathbb{S}^{n+1}_{p,1}) \simeq Gr^-_p(n+2,2) \simeq \left\{x\wedge y \in \Lambda^2(\mathbb{R}^{n+2}) \big| (x,y)\in T\mathbb{S}^{n+1}_{p,1}, \langle y,y\rangle_p = -1\right\}.$$

Observe that the anti-isometry $\mathtt{A}$ induces a canonical one-to-one correspondence between $Gr^-_p(n+2,2)$ and $Gr^-_{n+2-p}(n+2,2)$, hence between $L^-(\mathbb{S}^{n+1}_{p,1})$ and

$$L^-(\mathbb{S}^{n+1}_{n+2-p,1}) = L^-(\mathtt{A}(\mathbb{S}^{n+1}_{p,-1})).$$

We may regard $L^+(\mathbb{S}^{n+1}_{p,1})$ and $L^-(\mathbb{S}^{n+1}_{p,1})$ as two submanifolds of the pseudo-Euclidean space

$$\Lambda^2(\mathbb{R}^{n+2}) := \mathrm{Span}\left\{e_i\wedge e_j, 1\le i<j\le n+2\right\} \simeq \mathbb{R}^{\frac{(n+2)(n+1)}{2}},$$

where $(e_1,...,e_{n+2})$ denotes the canonical basis of $\mathbb{R}^{n+2}$. This viewpoint allows us to define in a natural way several structures on $L^\pm(\mathbb{S}^{n+1}_{p,1})$: first, we use the fact that $\Lambda^2(\mathbb{R}^{n+2})$ is equipped with the flat pseudo-Riemannian metric

$$\langle\langle x\wedge y, x'\wedge y'\rangle\rangle := \langle x,x'\rangle_p\langle y,y'\rangle_p - \langle x,y'\rangle_p\langle y,x'\rangle_p;$$

we shall denote by $\mathbb{G}$ the induced metric on $L^\pm(\mathbb{S}^{n+1}_{p,1})$, i.e. $\mathbb{G} = \iota^*\langle\langle .,.\rangle\rangle$, where $\iota : L^\pm(\mathbb{S}^{n+1}_{p,1}) \to \Lambda^2(\mathbb{R}^{n+2})$ is the canonical inclusion. Second, observe that a positive (resp. indefinite) oriented plane is equipped with a canonical complex (resp. para-complex) structure J. Explicitly, given $\bar{x} = x\wedge y \in Gr^\pm_p(n+2,2)$, with $|x|^2_p = 1$ and $|y|^2_p = \epsilon$, we set $\mathrm{J}x = y$ and $\mathrm{J}y = -\epsilon x$. In particular, $\mathrm{J}^2 = \epsilon Id$. On the other hand, a tangent vector to $\iota(L^\pm(\mathbb{S}^{n+1}_{p,1}))$ at the point $\bar{x}$ takes the form $x\wedge X + y\wedge Y$, where $X, Y \in \bar{x}^\perp$. We then define:

$$\mathbb{J}(x\wedge X + y\wedge Y) := (\mathrm{J}x)\wedge X + (\mathrm{J}y)\wedge Y = y\wedge X - \epsilon x\wedge Y.$$

It is straightforward that $\mathbb{J}^2 = \epsilon Id\big|_{\bar{x}}$, i.e. $\mathbb{J}$ is an almost complex or para-complex structure.

Theorem 2.1. *$(L^+(\mathbb{S}^{n+1}_{p,1}), \mathbb{J}, \mathbb{G})$ is a $2n$-dimensional pseudo-Kähler manifold with signature $(2p, 2(n-p))$ and $(L^-(\mathbb{S}^{n+1}_{p,1}), \mathbb{J}, \mathbb{G})$ is a $2n$-dimensional para-Kähler manifold, hence with neutral signature (n,n). In both cases, the metric $\mathbb{G}$ is Einstein, with scalar curvature $\bar{S} = \epsilon 2n^2$, and is never conformally flat.*

Remark 2.2. It is not difficult to check that $\mathbb{G}$ and $\mathbb{J}$ are invariant under the natural action of the group $SO(n+2-p,p)$ of isometries of $\mathbb{S}^{n+1}_{p,1}$. Such invariant structures have been studied with the Lie algebra formalism in [1], where in particular it is proved that such an invariant pseudo-Riemannian metric and complex or para-complex structure are unique on $L^{\pm}(\mathbb{S}^{n+1}_{p,1})$, for $n \geq 3$. The fact that $\mathbb{G}$ is Einstein has been proved in [19] in the spherical case.

Remark 2.3. The complex structure of $L^{+}(\mathbb{S}^{n+1}_{p,1})$ may be alternatively described by identifying $L^{+}(\mathbb{S}^{n+1}_{p,1})$ with the hyperquadric

$$\Big\{[z_1 : ... : z_{n+2}] \Big| \; -\sum_{i=1}^{p} z_i^2 + \sum_{i=p+1}^{n+2} z_i^2 = 0\Big\}$$

of the pseudo-complex projective space $\mathbb{CP}^{n+1}_p$ (see [21]).

In the three-dimensional case, $L^{\pm}(\mathbb{S}^3_{p,1})$ enjoys other natural structures, which may de defined as follows: since the orthogonal two-plane $\bar{x}^{\perp}$ admits a canonical orientation (that orientation compatible with the orientations of $\bar{x}$ and $\mathbb{R}^4$), it enjoys a canonical complex or para-complex structure J' (depending on whether the induced metric on $\bar{x}^{\perp}$ is positive or indefinite). Hence we set

$$\mathbb{J}'(x \wedge X + y \wedge Y) := x \wedge (\mathrm{J}'X) + y \wedge (\mathrm{J}'Y).$$

We therefore get another almost complex or para-complex structure on $L^{\pm}(\mathbb{S}^3_{p,1})$. Finally, we introduce one more tensor: we want to define a pseudo-Riemannian structure $\mathbb{G}'$ on $L^{\pm}(\mathbb{S}^3_{p,1})$ with the requirement that the pair $(\mathbb{J}', \mathbb{G}')$ induces the same symplectic structure, up to sign, than that of $(\mathbb{J}, \mathbb{G})$. In other words, we require that $\omega(.,.) := \epsilon'\mathbb{G}'(\mathbb{J}'.,.)$ be the same as $\omega(.,.) := \epsilon\mathbb{G}(\mathbb{J}.,.)$. Hence, we must have:

$$\mathbb{G}' = \omega(.,\mathbb{J}'.) = \epsilon\mathbb{G}(\mathbb{J}.,\mathbb{J}'.) = -\epsilon\mathbb{G}(.,\mathbb{J}\circ\mathbb{J}'.).$$

It turns out that this defines another Kähler or para-Kähler structure:

Theorem 2.4. *The two-form $\mathbb{G}' := -\epsilon\mathbb{G}(.,\mathbb{J}'\circ\mathbb{J}.)$ is symmetric and therefore defines a pseudo-Riemannian metric on $L^{\pm}(\mathbb{S}^3_{p,1})$. The Levi-Civita connection of $\mathbb{G}'$ is the same as that of $\mathbb{G}$, and the structures $(\mathbb{J}, \mathbb{G})$ and $(\mathbb{J}', \mathbb{G}')$ share the same symplectic form ω. Moreover, $(L^{+}(\mathbb{S}^3), \mathbb{J}', \mathbb{G}')$, $(L^{+}(\mathbb{H}^3), \mathbb{J}', \mathbb{G}')$, $(L^{-}(d\mathbb{S}^3), \mathbb{J}', \mathbb{G}')$ and $(L^{+}(Ad\mathbb{S}^3), \mathbb{J}', \mathbb{G}')$ are pseudo-Kähler manifolds while $(L^{+}(d\mathbb{S}^3), \mathbb{J}', \mathbb{G}')$ and $(L^{-}(Ad\mathbb{S}^3), \mathbb{J}', \mathbb{G}')$ are para-Kähler manifolds. In all cases, the metric $\mathbb{G}'$ has neutral signature $(2,2)$, is scalar flat and locally conformally flat.*

Remark 2.5. The properties of $\mathbb{G}'$ have been derived in [9] in the case of hyperbolic space.

The fact that $(\mathbb{J}, \mathbb{G})$ and $(\mathbb{J}', \mathbb{G}')$ share both the same Levi-Civita connection and the symplectic form implies that they also share some distinguished classes of submanifolds:

Corollary 2.6. *Lagrangian surfaces, flat and totally geodesic submanifolds in $L^{\pm}(\mathbb{S}^3_{p,1})$, are the same for $(\mathbb{J}, \mathbb{G})$ and $(\mathbb{J}', \mathbb{G}')$.*

Remark 2.7. In some cases, these invariant structures may be defined in a more intuitive way. For example, using the direct sum of self-dual and anti-self-dual bivectors in $(\Lambda^2(\mathbb{R}^4, \langle ., .\rangle_0), \langle\langle ., .\rangle\rangle) \simeq (\mathbb{R}^6, \langle ., .\rangle_0)$, one can prove that $L^+(\mathbb{S}^3) \simeq \mathbb{S}^2 \times \mathbb{S}^2$ and that

$$\mathbb{G} = \langle ., .\rangle_0 \oplus \langle ., .\rangle_0, \qquad \mathbb{G}' = \langle ., .\rangle_0 \oplus -\langle ., .\rangle_0,$$

$$\mathbb{J} = \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}, \qquad \mathbb{J}' = \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix},$$

where $(j, \langle ., .\rangle_0)$ is the canonical Kähler structure of $\mathbb{S}^2$ (see [5]). Analogously, in $(\Lambda^2(\mathbb{R}^4, \langle ., .\rangle_2), \langle\langle ., .\rangle\rangle) \simeq (\mathbb{R}^6, \langle ., .\rangle_3)$ the Hodge operator is para-complex and we still have a direct sum of self-dual and anti-self-dual bivectors. A computation then shows that $L^+(Ad\mathbb{S}^3) \simeq \mathbb{H}^2 \times \mathbb{H}^2$ and $L^-(Ad\mathbb{S}^3) \simeq d\mathbb{S}^2 \times d\mathbb{S}^2$, and again we could describe $(\mathbb{J}, \mathbb{G})$ and $(\mathbb{J}', \mathbb{G}')$ as product structures built from the canonical Kähler and para-Kähler structures of $\mathbb{H}^2$ and $d\mathbb{S}^2$ respectively. On the other hand, the Hodge operator being complex in $(\Lambda^2(\mathbb{R}^4, \langle ., .\rangle_1), \langle\langle ., .\rangle\rangle) \simeq (\mathbb{R}^6, \langle ., .\rangle_2)$, there is no natural direct sum of it into eigenspaces, and it does not seem possible a priori to describe $L^\pm(d\mathbb{S}^3)$ and $L^+(\mathbb{H}^3)$ as a Cartesian product of two surfaces.

Remark 2.8. Since the two complex or para-complex structures $\mathbb{J}$ and $\mathbb{J}'$ commute, their composition $\mathbb{J}'' := \mathbb{J} \circ \mathbb{J}'$ defines one more invariant structure: if $\mathbb{J}$ and $\mathbb{J}'$ are both complex or both para-complex, then $\mathbb{J}''$ is complex, and if $\mathbb{J}$ and $\mathbb{J}'$ are of different types, $\mathbb{J}''$ is para-complex. The two-form $\mathbb{G}'' := \omega(., \mathbb{J}'')$ is not symmetric, so there is no pseudo- or para-Kähler structure associated to $\mathbb{J}''$.

Observe also that the triple $(\mathbb{J}, \mathbb{J}', \mathbb{J}'')$ is *not* a para-quaternionic structure, since $\mathbb{J}$ and $\mathbb{J}'$ commute rather than anti-commute. The case $L^-(Ad\mathbb{S}^3)$ excepted, this triple is what is called an *almost product bi-complex* structure in [6].

TABLE 1. Structures on $L^\pm(\mathbb{S}^3_{p,1})$

Space form	Space of geodesics	(ϵ, ϵ')	Signature of $\mathbb{G}$	$\mathbb{J}$	$\mathbb{J}'$	$\mathbb{J}''$
$\mathbb{S}^3_{0,1} = \mathbb{S}^3$	$L(\mathbb{S}^3)$	$(1, 1)$	$(+, +, +, +)$	complex	complex	para
$\mathbb{S}^3_{1,1} = d\mathbb{S}^3$	$L^+(d\mathbb{S}^3)$	$(1, -1)$	$(+, -, +, -)$	complex	para	complex
	$L^-(d\mathbb{S}^3) \simeq L^-(\mathbb{H}^3)$	$(-1, -1)$	$(+, +, -, -)$	para	complex	complex
$\mathbb{S}^3_{2,1} \simeq Ad\mathbb{S}^3$	$L^+(Ad\mathbb{S}^3)$	$(1, 1)$	$(-, -, -, -)$	complex	complex	para
	$L^-(Ad\mathbb{S}^3)$	$(-1, -1)$	$(+, -, -, +)$	para	para	para
$\mathbb{S}^3_{3,1} \simeq \mathbb{H}^3$	$L^-(\mathbb{H}^3) \simeq L^-(d\mathbb{S}^3)$	$(-1, 1)$	$(-, -, +, +)$	para	complex	complex

2.2. **Normal congruences of immersed hypersurfaces as Lagrangian submanifolds.**

Definition 2.9. Let $\mathcal{S}$ be an immersed surface of pseudo-Riemannian space form $\mathbb{S}^{n+1}_{p,1}$ with unit normal vector N. The *normal congruence* (or *Gauss map*) $\bar{\mathcal{S}}$ of $\mathcal{S}$ is the set of geodesics crossing $\mathcal{S}$ orthogonally in the direction N.

Theorem 2.10. *Let φ be a pseudo-Riemannian immersion of an orientable manifold $\mathcal{M}^n$ in pseudo-Riemannian space form $\mathbb{S}^{n+1}_{p,1}$ with unit normal vector N. Then the normal congruence of $\mathcal{S} := \varphi(\mathcal{M}^n)$, i.e. the image of the immersion $\bar{\varphi} : \mathcal{M}^n \to L^{\pm}(\mathbb{S}^{n+1}_{p,1})$ defined by $\bar{\varphi} = \varphi \wedge N$, is Lagrangian with respect to ω. In this case, $\bar{\mathcal{S}}$ is also the normal congruence of the hypersurfaces parallel to $\mathcal{S}$ and to its polar $\mathcal{S}'$. Conversely, let $\bar{\varphi} : \mathcal{M}^n \to L^{\pm}(\mathbb{S}^{n+1}_{p,1})$ be an immersion of a simply connected n-manifold. Then $\bar{\mathcal{S}}$ is the normal congruence of an immersed hypersurface of $\mathbb{S}^{n+1}_{p,1}$ if and only if $\bar{\varphi}$ is Lagrangian.*

In view of this result, it is natural to relate the geometry of a Lagrangian submanifold to that of the corresponding hypersurface of $\mathbb{S}^{n+1}_{p,1}$.

Theorem 2.11. *Let φ be a pseudo-Riemannian immersion of an orientable manifold $\mathcal{M}^n$ in pseudo-Riemannian space form $\mathbb{S}^{n+1}_{p,1}$ with unit normal vector N. Set $|N|^2_p := \epsilon$, and denote by A the shape operator of φ with respect to N and by ∇^g the Levi-Civita connection of g. Then the induced metric $\bar{g} := \bar{\varphi}^*\mathbb{G}$, with $\bar{\varphi} = \varphi \wedge N$, is given by the following formula:*

$$\bar{g} = \epsilon g + g(A., A.).$$

In particular, $\bar{g}$ is non-degenerate if and only if $\epsilon Id + A^2$ is invertible.

Moreover, the extrinsic curvatures h of $\mathcal{S} := \varphi(\mathcal{M}^n)$ and of $\bar{h}$ of $\bar{\mathcal{S}} := \bar{\varphi}(\mathcal{M}^n)$ are related by the formula

$$\bar{h} = \epsilon \nabla^g h.$$

In particular, the normal congruence $\bar{\mathcal{S}}$ is totally geodesic if and only if $\mathcal{S}$ has parallel second fundamental form.

Remark 2.12. The fact that the tensor $\bar{h}$ of $\bar{\mathcal{S}}$ is tri-symmetric is equivalent to the Codazzi equation for the hypersurface $\mathcal{S}$.

Corollary 2.13. *If the shape operator A of $\mathcal{S}$ is real diagonalizable (this is always the case if $\epsilon' = 1$), the mean curvature vector of $\bar{\mathcal{S}}$ with respect to $\mathbb{G}$ is*

$$\vec{H} = -\frac{\epsilon}{n}\mathbb{J}\bar{\nabla}\left(\sum_{i=1}^{n} \mathrm{arctan}\epsilon(\kappa_i)\right),$$

where $\kappa_1, ..., \kappa_n$ are the principal curvatures of $\mathcal{S}$ and $\bar{\nabla}$ is the gradient with respect to the induced metric $\bar{g}$. In particular, if $\mathcal{S}$ is isoparametric (i.e. its principal curvatures are constant) or austere (i.e. the set of its principal curvatures is symmetric with respect to 0), then its normal congruence $\bar{\mathcal{S}}$ is $\mathbb{G}$-minimal.

Corollary 2.14. *If $n = 2$, the mean curvature vector of $\bar{\mathcal{S}}$ with respect to $\mathbb{G}$ is*

$$\vec{H} = -\frac{\epsilon}{2}\mathbb{J}\bar{\nabla}\,\mathrm{arctan}\epsilon\left(\frac{2H}{1-\epsilon K}\right),$$

where H and K denote the mean curvature and the Gaussian curvature of $\mathcal{S}$ respectively. In particular, $\bar{\mathcal{S}}$ is $\mathbb{G}$-minimal if and only if it is the normal congruence of a minimal surface.

Remark 2.15. Corollaries 2.13 and 2.14 have been proved in [22] in the spherical case. The fact that the mean curvature vector takes the form $\vec{H} = \frac{\epsilon}{n}\mathbb{J}\bar{\nabla}\beta$, where β is an $\mathbb{S}^1$-valued map, is due to the fact that the metric $\mathbb{G}$ is Einstein (cf [13]). The map β is called the *Lagrangian angle* of the submanifold $\bar{\mathcal{S}}$.

In the three-dimensional case, it is natural to study the pseudo-Riemannian geometry of Lagrangian surfaces of $L^{\pm}(\mathbb{S}^3_{p,1})$ with respect to the metric $\mathbb{G}'$ described in Theorem 2.4.

Theorem 2.16. *Let φ be a pseudo-Riemannian immersion of an orientable surface $\mathcal{M}^2$ in pseudo-Riemannian space form $\mathbb{S}^3_{p,1}$ with shape operator A and unit normal vector N. Then the induced metric $\bar{g}' := \bar{\varphi}^*\mathbb{G}'$, with $\bar{\varphi} = \varphi \wedge N$, is given by the following formula:*

$$\bar{g}' = g(., (A\mathrm{J}' - \mathrm{J}'A).).$$

Moreover,

- *If A is real diagonalizable, the metric $\bar{g}'$ is degenerate at umbilic points of $\mathcal{S} := \varphi(\mathcal{M}^2)$ and indefinite elsewhere. the null directions of $\bar{g}'$ are the principal directions of $\mathcal{S}$.*
- *If A is complex diagonalizable, the metric $\bar{g}'$ is everywhere definite.*
- *If A is not diagonalizable, the metric $\bar{g}'$ is everywhere degenerate.*

When $\bar{g}'$ is not degenerate, the extrinsic curvatures h and $\bar{h}$ of $\mathcal{S}$ and $\bar{\mathcal{S}} := \bar{\varphi}(\mathcal{M}^2)$ are related by the formula

$$\bar{h} = \epsilon \nabla^g h.$$

In particular, the normal congruence $\bar{\mathcal{S}}$ of $\mathcal{S}$ is totally geodesic if and only if $\mathcal{S}$ has parallel second fundamental form.

Corollary 2.17. *$\bar{\mathcal{S}}$ is $\mathbb{G}'$-minimal if and only if it is totally geodesic, i.e. $\mathcal{S}$ has parallel second fundamental form. In addition A is real diagonalizable, $\mathcal{S}$ is the set of equidistant points to a geodesic of $\mathbb{S}^3_{p,1}$.*

Corollary 2.18. *The induced metric $\bar{g}'$ is flat (and the metric $\bar{g}$ as well by Corollary* 2.6*) if and only if the surface $\mathcal{S}$ is Weingarten, i.e. there exists a functional relation $f(H, K) = 0$ satisfied by the mean curvature and the Gaussian curvature of $\mathcal{S}$.*

Remark 2.19. Corollaries 2.17 and 2.18 have been proved in the case of hyperbolic space in [8] and [10] respectively. Corollary 2.18 has been proved in the case of Euclidean space in [12].

Corollary 2.20. *If the shape operator A of $\mathcal{S}$ is not diagonalizable, then its normal congruence $\bar{\mathcal{S}}$ is a $\mathbb{G}$-marginally trapped surface, i.e. the mean curvature vector of $\bar{\mathcal{S}}$ with respect to $\mathbb{G}$ is null. If $\mathcal{S}$ is a tube (i.e. the set of equidistant points to an arbitrary curve of $\mathbb{S}^3_{p,1}$) or a surface of revolution, then its normal congruence $\bar{\mathcal{S}}$ is a $\mathbb{G}'$-marginally trapped surface.*

3. The geometry of the space of geodesics

3.1. The Einstein metric $\mathbb{G}$ (Proof of Theorem 2.1).

3.1.1. *The second fundamental form of h^ι and the complex structure $\mathbb{J}$.* Let $\bar{x} := x \wedge y \in L^{\pm}(\mathbb{S}^{n+1}_{p,1})$ with $|x|^2_p = 1$ and $|y|^2_p = \epsilon$ and let $(e_1, ..., e_n)$ be an orthonormal basis of the orthogonal complement of $x \wedge y$. We set $\epsilon_i := |e_i|^2_p$ and $\epsilon_{n+i} := \epsilon\epsilon_i$. Then an orthonormal basis $(E_a)_{1\leq a\leq 2n}$ of $T_{\bar{x}}L^{\pm}(\mathbb{S}^{n+1}_{p,1})$, with $\mathbb{G}(E_a, E_a) = \epsilon_a$, is given by

$$E_i := x \wedge e_i \quad \text{and} \quad E_{n+i} := y \wedge e_i.$$

Fix the index i and introduce the curve

$$\gamma_i(t) := x \wedge y_i(t) := x \wedge (\texttt{cos}\epsilon_{n+i}(t)\, y + \texttt{sin}\epsilon_{n+i}(t)\, e_i).$$

In particular, $\gamma_i(0) = \bar{x}$ and $\gamma_i'(0) = E_i$. Introduce furthermore the following orthonormal frame $\bar{V} = (\bar{v}_1, ..., \bar{v}_{2n})$ along γ_i:

$$\begin{array}{lll} \bar{v}_j(t) := x \wedge e_j, & \bar{v}_{n+j}(t) := y_i(t) \wedge e_j, & \text{if } j \neq i, \\ \bar{v}_i(t) := x \wedge y_i'(t), & \bar{v}_{n+i}(t) := y_i(t) \wedge y_i'(t). & \end{array}$$

Observe that $\bar{v}_a(0) = E_a$, $\forall a, 1 \leq a \leq 2n$, and that the frame $\bar{V}$ is parallel along γ_i. On the other hand, it is easily checked that $D_{E_i}\mathbb{J} = \mathbb{J}D_{E_i}$, so $\mathbb{J}$ is parallel, and therefore integrable.

We now proceed to compute the second fundamental form of the immersion ι.

Proposition 3.1. *The second fundamental form of the embedding* $\iota : L^{\pm}(\mathbb{S}^{n+1}_{p,1}) \to \Lambda^2(\mathbb{R}^{n+2})$ *is given by the formula*

$$h^{\iota}(v \wedge V, w \wedge W) = -\langle v, w\rangle_p \langle V, W\rangle_p \bar{x} + \varpi(v, w) V \wedge W,$$

where ϖ *is the symplectic form of the plane* $\bar{x}$ *defined by* $\varpi(.,.) = \epsilon\langle \mathrm{J}.,.\rangle_p$.

Proof. We have

$$\begin{array}{rcl} h^{\iota}(E_i, E_j) & = & \left(D_{\bar{\gamma}_i'}\bar{v}_j\right)^{\perp} = \bar{v}_j'(0) = -\epsilon_{n+i}\delta_{ij}\bar{x}, \\ h^{\iota}(E_i, E_{n+j}) & = & \left(D_{\bar{\gamma}_i'}\bar{v}_{n+j}\right)^{\perp} = \bar{v}_{n+j}'(0) = e_i \wedge e_j. \end{array}$$

An analogous computation, using the curve $\gamma_{n+i}(t) = (\texttt{cos}\epsilon_i(t)\, x - \texttt{sin}\epsilon_i(t)\, e_i) \wedge y$, implies that

$$h^{\iota}(E_{n+i}, E_{n+j}) = -\epsilon_i \delta_{ij} \bar{x}.$$

The claimed formula follows from the bi-linearity of h^{ι}. □

3.1.2. *The curvature of* $\mathbb{G}$. We use the Gauss equation and Proposition 3.1 in order to compute the curvature tensor $\bar{R}$ of $\mathbb{G}$: for $1 \leq a, b, c, d \leq 2n$, we have

$$\mathbb{G}(\bar{R}(E_a, E_b)E_c, E_d) = \langle\langle h^{\iota}(E_a, E_c), h^{\iota}(E_b, E_d)\rangle\rangle - \langle\langle h^{\iota}(E_a, E_d), h^{\iota}(E_b, E_c)\rangle\rangle.$$

For example, we calculate

$$\mathbb{G}(\bar{R}(E_i, E_j)E_k, E_l) = \epsilon_{n+i}\epsilon_{n+j}\langle\langle \bar{x}, \bar{x}\rangle\rangle(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{kl}) = \epsilon\epsilon_i\epsilon_j(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{kl}).$$

This expression vanishes unless $\{k, l\} = \{i, j\}$ and $i \neq j$, in which case it becomes

$$\mathbb{G}(\bar{R}(E_i, E_j)E_i, E_j) = -\mathbb{G}(R(E_i, E_j)E_j, E_i) = \epsilon\epsilon_i\epsilon_j.$$

It is now easy to calculate the Ricci curvature of $\bar{R}$:

$$\begin{aligned} \overline{Ric}(E_i, E_j) &= \sum_{a=1}^{2n} \mathbb{G}^{aa}\mathbb{G}(\bar{R}(E_i, E_a)E_j, E_a) \\ &= \sum_{k=1}^{n} \left(\mathbb{G}^{kk}\mathbb{G}(\bar{R}(E_i, E_k)E_j, E_k) + \mathbb{G}^{n+k,n+k}\mathbb{G}(\bar{R}(E_i, E_{n+k})E_j, E_{n+k})\right) \\ &= \sum_{k=1, k\neq i}^{n} \epsilon_k(\delta_{ij}\epsilon\epsilon_k\epsilon_i) + \sum_{k=1}^{n} \epsilon_{n+k}(\delta_{ik}\delta_{jk}\epsilon_i\epsilon_k) \\ &= \epsilon n\, \mathbb{G}_{ij}. \end{aligned}$$

Analogous calculations show that $\overline{Ric}(E_{n+i}, E_{n+j}) = \epsilon n\, \mathbb{G}_{n+i,n+j}$ and that $\overline{Ric}(E_i, E_{n+j}) = 0$. Hence the metric $\mathbb{G}$ is Einstein, with constant scalar curvature $\bar{S} = \epsilon 2n^2$.

Finally, since $\mathbb{G}$ is Einstein, the Weyl tensor is given by the formula

$$W^{\mathbb{G}} = \mathbb{G}(\bar{R}.,.) - \frac{\bar{S}}{4n(2n-1)}\mathbb{G} \circ \mathbb{G} = \mathbb{G}(\bar{R}.,.) - \frac{\epsilon n}{2(2n-1)}\mathbb{G} \circ \mathbb{G}.$$

It is easily seen, for example, that $\mathbb{G} \circ \mathbb{G}(E_i, E_j, E_{n+i}, E_{n+j})$ vanishes. On the other hand, if $i \neq j$, $\mathbb{G}(\bar{R}(E_i, E_j)E_{n+i}, E_{n+j}) = \epsilon_i \epsilon_j$, so $W^{\mathbb{G}}$ does not vanish and therefore $\mathbb{G}$ is never conformally flat.

3.2. **The scalar flat metric $\mathbb{G}'$ in dimension $n = 2$ (Proof of Theorem 2.4).** We are going to express all the relevant tensors in the orthonormal basis (E_1, E_2, E_3, E_4) of $T_{\bar{x}} L^{\pm}(\mathbb{S}^{n+1}_{p,1})$. Observe first that the matrix of $\mathbb{G}$ in this basis is $diag(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) = diag(\epsilon_1, \epsilon_2, \epsilon\epsilon_1, \epsilon\epsilon_2)$ and that

$$\mathbb{J} = \begin{pmatrix} 0 & 0 & -\epsilon & 0 \\ 0 & 0 & 0 & -\epsilon \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbb{J}' = \begin{pmatrix} 0 & -\epsilon' & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\epsilon' \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

It follows that

$$\epsilon \mathbb{J}\mathbb{J}' = \epsilon \mathbb{J}'\mathbb{J} = \begin{pmatrix} 0 & 0 & 0 & \epsilon' \\ 0 & 0 & -1 & 0 \\ 0 & -\epsilon\epsilon' & 0 & 0 \\ \epsilon & 0 & 0 & 0 \end{pmatrix}.$$

Hence, taking into account that $\epsilon' = \epsilon_1 \epsilon_2$, the matrix of the bilinear form $\mathbb{G}' := -\epsilon \mathbb{G}(., \mathbb{J} \circ \mathbb{J}'.)$ in the basis $(E_a)_{1 \leq a \leq 4}$ is

$$\mathbb{G}' = \begin{pmatrix} 0 & 0 & 0 & \epsilon_2 \\ 0 & 0 & -\epsilon_2 & 0 \\ 0 & -\epsilon_2 & 0 & 0 \\ \epsilon_2 & 0 & 0 & 0 \end{pmatrix}.$$

The fact that $\mathbb{G}$ and $\mathbb{G}'$ have the same Levi-Civita connection follows from the next lemma:

Lemma 3.2. *Let $(\mathcal{N}, \mathbb{G})$ be a pseudo-Riemannian manifold with Levi-Civita connection D and T a symmetric, D-parallel $(1,1)$ tensor. Then the Levi-Civita connection of the pseudo-Riemannian metric $\mathbb{G}'(.,.) := \mathbb{G}(., T.)$ is D.*

Proof. Elementary using local coordinates and the explicit formula for the Christoffel symbols. ☐

Since $\mathbb{G}$ and $\mathbb{G}'$ have the same Levi-Civita connection, they have the same curvature tensor $\bar{R}$. Therefore,

$$\mathbb{G}'(\bar{R}(.,.).,.) := -\epsilon \mathbb{G}(\bar{R}(.,.).,\mathbb{J} \circ \mathbb{J}'.).$$

Then by an elementary calculation we obtain

$$-\overline{Ric}'(X, Y) = \overline{Ric}(X, Y) = \epsilon 2\mathbb{G}(X, Y).$$

It follows that the scalar curvature of $\mathbb{G}'$ vanishes:

$$\bar{S}' = \sum_{a,b=1}^{4} (\mathbb{G}')^{ab} \overline{Ric}'(E_a, E_b) = -2\epsilon \sum_{a,b=1}^{4} (\mathbb{G}')^{ab} \mathbb{G}_{ab} = 0.$$

It may be interesting to point out that the Ricci curvature of $\mathbb{G}'$ is non-negative in the case of $L(\mathbb{S}^3)$, non-positive in the case of $L^+(Ad\mathbb{S}^3)$, and indefinite in the other cases.

Finally, since $\mathbb{G}'$ is scalar flat, its Weyl tensor is given by the formula

$$W^{\mathbb{G}'} = \mathbb{G}'(\bar{R}.,.) - \frac{1}{2}\overline{Ric}' \circ \mathbb{G}' = \mathbb{G}(\bar{R}., \epsilon \mathbb{J} \circ \mathbb{J}'.) - \epsilon \mathbb{G} \circ \mathbb{G}'.$$

We may calculate, for example, that

$$\begin{array}{rcl} W^{\mathbb{G}'}(E_1, E_2, E_2, E_4) & = & \mathbb{G}(\bar{R}(E_1, E_2)E_1, \epsilon \mathbb{J} \circ \mathbb{J}' E_4) - \epsilon\, \mathbb{G} \circ \mathbb{G}'(E_1, E_2, E_2, E_4) \\ & = & -\epsilon' \epsilon \epsilon_1 \epsilon_2 + \epsilon\, \mathbb{G}(E_2, E_2) \mathbb{G}'(E_1, E_4) \\ & = & -\epsilon + \epsilon \epsilon_2 \epsilon_2 \\ & = & 0. \end{array}$$

It is easily checked in the same manner that the other components of the Weyl tensor vanish. The metric $\mathbb{G}'$ is therefore locally conformally flat.

4. Normal congruences of hypersurfaces and Lagrangian submanifolds

4.1. **Lagrangian submanifolds are normal congruences (Proof of Theorem 2.10).** Let $\varphi : \mathcal{M}^n \to \mathbb{S}^{n+1}_{p,1}$ be an immersed, orientable hypersurface with non-degenerate metric and unit normal vector N and introduce the map

$$\begin{array}{rccc} \bar{\varphi} : & \mathcal{M}^n & \to & L^{\pm}(\mathbb{S}^{n+1}_{p,1}), \\ & x & \mapsto & \varphi(x) \wedge N(x). \end{array}$$

In the following, we shall often allow the abuse of notation in identifying a tangent vector X to $\mathcal{M}^n$ with its image $d\varphi(X)$, a vector tangent to $\mathbb{S}^{n+1}_{p,1}$, and therefore an element of $\mathbb{R}^{n+2}$. We furthermore set $\bar{X} := d\bar{\varphi}(X)$, so that

$$\bar{X} := d\bar{\varphi}(X) = d(\varphi \wedge N)(X) = d\varphi(X) \wedge N + \varphi \wedge dN(X) = X \wedge N + AX \wedge \varphi.$$

It follows that $\bar{\varphi}$ is Lagrangian since

$$\begin{array}{rcl} \omega(\bar{X}, \bar{Y}) & = & \epsilon \mathbb{G}(X \wedge (\mathrm{J}N) + AX \wedge (\mathrm{J}\varphi), Y \wedge N + AY \wedge \varphi) \\ & = & \epsilon \left(\langle X, AY \rangle_p \langle \mathrm{J}N, \varphi \rangle_p + \langle AX, Y \rangle_p \langle \mathrm{J}\varphi, N \rangle_p \right) \\ & = & -\langle X, AY \rangle_p + \langle AX, Y \rangle_p = 0. \end{array}$$

Conversely, let $\bar{\mathcal{S}}$ be an n-dimensional geodesic congruence, i.e. the image of an immersion $\bar{\varphi} : \mathcal{M}^n \to L^{\pm}(\mathbb{S}^{n+1}_{p,1})$. We shall investigate under which condition there exists a hypersurface $\mathcal{S}$ of $\mathbb{S}^{n+1}_{p,1}$ which intersects orthogonally the geodesics $\bar{\varphi}(x)$, $\forall x \in \mathcal{M}^n$. For this purpose set $\bar{\varphi}(x) := e_1(x) \wedge e_2(x)$ with $|e_1|^2_p = 1$ and $|e_2|^2_p = \epsilon$. Let $\varphi : \mathcal{M}^n \to \mathbb{S}^{n+1}_{p,1}$ such that $\varphi(x) \in \bar{\varphi}(x)$, $\forall x \in \mathcal{M}^n$. Therefore there exists $t : \mathcal{M}^n \to \mathbb{S}^1$, such that $\varphi(x) = e_1(x)\mathtt{cos}\epsilon(t(x)) + e_2(x)\mathtt{sin}\epsilon(t(x))$. Remember that J denotes the complex or para-complex structure on $\bar{\varphi}(x)$, in particular $\mathrm{J}\varphi = e_2 \mathtt{cos}\epsilon\,(t) - \epsilon e_1 \mathtt{sin}\epsilon\,(t)$. It is easily seen that $\mathcal{S}$ intersects the geodesic $\bar{\varphi}(x) =$

$e_1(x) \wedge e_2(x) = \varphi(x) \wedge \mathrm{J}\varphi(x)$ orthogonally at the point $\varphi(x)$ if and only if the following vanishes:

$$\langle d\varphi, \mathrm{J}\varphi\rangle_p = |\mathrm{J}\varphi|_p^2 dt + \langle de_1, e_2\rangle_p \mathtt{cos}\epsilon^2(t) - \epsilon\langle de_2, e_1\rangle_p \mathtt{sin}\epsilon^2(t) = \epsilon dt + \langle de_1, e_2\rangle_p.$$

Hence, $\bar{\mathcal{S}}$ is the normal congruence of $\mathcal{S}$ if and only if there exists $t : \mathcal{M}^n \to \mathbb{S}^1$ such that $\langle de_1, e_2\rangle_p = -\epsilon dt$. Since $\mathcal{M}^n$ is simply connected, it is sufficient to have $d\langle de_1, e_2\rangle_p = 0$. On the other hand,

$$\mathbb{J}d\bar{\varphi} = \mathbb{J}(de_1 \wedge e_2 + e_1 \wedge de_2) = -de_1 \wedge e_1 + e_2 \wedge de_2,$$

so that

$$\omega(d\bar{\varphi}(X), d\bar{\varphi}(Y)) = \langle\langle de_1(X) \wedge e_2 + e_1 \wedge de_2(X), -de_1(Y) \wedge e_1 + e_2 \wedge de_2(Y)\rangle\rangle$$

$$= -\langle de_1(X), de_2(Y)\rangle_p + \langle de_1(Y), de_2(X)\rangle_p = -d\langle de_1, e_2\rangle_p(X, Y).$$

We conclude that t, and thus φ as well, exists if and only if $\bar{\varphi}$ is Lagrangian. Of course, the choice of different constants of integration when solving t corresponds to different, parallel hypersurfaces.

4.2. **Geometry of Lagrangian submanifolds with respect to the Einstein metric $\mathbb{G}$.**

4.2.1. *The induced metric $\bar{g} = \bar{\varphi}^*\mathbb{G}$ and the second fundamental form $\bar{h}$ (Proof of Theorem* 2.11*).* Using the description of the metric $\mathbb{G}$ given in Section 3.1, we have:

$$\begin{aligned}
\bar{g}(X,Y) &= \mathbb{G}(X \wedge N + AX \wedge \varphi, Y \wedge N + AY \wedge \varphi)\\
&= \langle X, Y\rangle_p\langle N, N\rangle_p - \langle X, N\rangle_p\langle Y, N\rangle_p + \langle \varphi, Y\rangle_p\langle AX, N\rangle_p - \langle \varphi, N\rangle_p\langle AX, Y\rangle_p\\
&\quad + \langle X, \varphi\rangle_p\langle N, AY\rangle_p - \langle X, AY\rangle\langle N, \varphi\rangle_p + \langle AX, AY\rangle_p\langle \varphi, \varphi\rangle_p - \langle AY, \varphi\rangle_p\langle AX, \varphi\rangle_p\\
&= \epsilon g(X, Y) + g(AX, AY).
\end{aligned}$$

We now discuss the degeneracy of $\bar{g}$: suppose there exist X such that

$$\bar{g}(X, Y) = \epsilon g(X, Y) + g(AX, AY) = g(\epsilon X + A^2X, Y)$$

vanishes $\forall\, Y \in T\mathcal{M}$. Since the metric g is non-degenerate, it follows that $\epsilon X + A^2X$ vanishes. Hence $\epsilon Id + A^2$ is not invertible. If A is diagonalizable, the eigenvalues of A^2 are non-negative, so we must have $\epsilon = -1$.

Next, denoting by ∇ (resp. D) the flat connection of $\mathbb{R}^{n+2}$ (resp. $\Lambda^2(\mathbb{R}^{n+2})$), we have

$$D_{\bar{X}}\bar{Y} = (\nabla_X Y) \wedge N + (\nabla_X AY) \wedge \varphi,$$

so

$$\begin{aligned}
\bar{h}(X, Y, Z) &= \mathbb{G}(\nabla_X Y \wedge N + \nabla_X AY \wedge \varphi, Z \wedge (\mathrm{J}N) + AZ \wedge (\mathrm{J}\varphi))\\
&= \langle \nabla_X Y, AZ\rangle_p\langle N, N\rangle_p - \epsilon\langle \nabla_X AY, Z\rangle_p\langle \varphi, \varphi\rangle_p\\
&= \epsilon\Big(h(\nabla_X Y, Z) - \big(X(\langle AY, Z\rangle_p) - \langle AY, \nabla_X Z\rangle_p\big)\Big)\\
&= \epsilon(\nabla_X h)(Y, Z).
\end{aligned}$$

4.2.2. *The mean curvature vector in the diagonalizable case (Proof of Corollary 2.13).* Assume that A is real diagonalizable and let $(e_1, ..., e_n)$ be an orthonormal frame $(e_1, ..., e_n)$ on $(T\mathcal{M}, g)$, with $\epsilon_i := g(e_i, e_i)$ and such that $Ae_i = \kappa_i e_i$, where $\kappa_1, ..., \kappa_n$ are the principal curvatures of $\mathcal{S}$.

We introduce the notation $\omega^i_{jk} := g(\nabla_{e_i} e_j, e_k)$. In particular, ω^i_{jk} is anti-symmetric in its lower indices. It follows that

$$\bar{g}(e_i, e_j) = 0 \text{ if } i \neq j, \quad \text{and} \quad \bar{g}(e_i, e_i) = \epsilon\epsilon_i + \epsilon_i \kappa_i^2 = \epsilon_i(\epsilon + \kappa_i^2).$$

Moreover, if $j \neq k$,

$$\bar{h}(e_i, e_j, e_k) = \epsilon\Big(h(\nabla_{e_i} e_j, e_k) + h(e_j, \nabla_{e_i} e_k) - e_i(h(e_j, e_k))\Big) = \epsilon(\kappa_k - \kappa_j)\omega^i_{jk}$$

and

$$\bar{h}(e_i, e_j, e_j) = \epsilon\Big(2h(\nabla_{e_i} e_j, e_j) - e_i(h(e_j, e_j))\Big) = -\epsilon\epsilon_j e_i(\kappa_j).$$

For further use, observe that the tri-symmetry of $\bar{h}$, or equivalently the Codazzi equation of the immersion φ, implies

$$(\kappa_j - \kappa_i)\omega^i_{ij} = \epsilon_j e_i(\kappa_j).$$

Since the basis $(e_1, ..., e_n)$ is orthogonal with respect to the metric $\bar{g}$, we have

$$\mathbb{G}(n\vec{H}, \mathbb{J}d\bar{\varphi}(e_i)) = \sum_{j=1}^n \frac{\bar{h}(e_j, e_j, e_i)}{\bar{g}(e_j, e_j)} = -\sum_{j=1}^n \frac{e_i(\kappa_j)}{1 + \epsilon\kappa_j^2} = e_i(\beta),$$

where $\beta := -\sum_{j=1}^n \mathtt{arctan}\epsilon(\kappa_j)$. Hence $\vec{H} = \frac{\epsilon}{n}\mathbb{J}\bar{\nabla}\beta$.

Clearly the immersion $\bar{\varphi}$ is $\mathbb{G}$-minimal if and only if the map β is constant. This happens of course if the principal curvatures of $\mathcal{S}$ are constant, i.e. it is isoparametric. Moreover, if $\mathcal{S}$ is austere, i.e. the set of the principal curvatures is symmetric with respect to 0, the Lagrangian angle β vanishes because the function $\mathtt{arctan}\epsilon$ is odd. This completes the proof of Corollary 2.13.

4.2.3. *The mean curvature vector in the two-dimensional case (Proof of Corollary 2.14).* Here and in the next section, we shall make use of canonical form of A (see Section 1.1 and [20]).

The real diagonalizable case

By the computation of the previous section:

$$\beta = -\left(\mathtt{arctan}\epsilon(\kappa_1) + \mathtt{arctan}\epsilon(\kappa_2)\right) = \mathtt{arctan}\epsilon\left(\frac{2H}{1 - \epsilon K}\right)$$

which is the required expression of the Lagrangian angle β. We now prove that if β is constant, the assumptions of Lemma 1.1 are satisfied. Assume by contradiction that $(\epsilon, \frac{2H}{K-\epsilon}) = (-1, \pm 1)$. It follows that $\frac{\kappa_1+\kappa_2}{\kappa_1\kappa_2+1} = \pm 1$, which in turn implies that $|\kappa_1|$ or $|\kappa_2| = 1$. Therefore, $-Id + A^2$ is not invertible, and the metric $\bar{g}$ is degenerate by Theorem 2.11. Since this situation is excluded a priori, we may use Lemma 1.1 and conclude that there exists a minimal hypersurface parallel to $\mathcal{S}$ or its polar $\mathcal{S}'$, and therefore whose normal congruence is $\bar{\mathcal{S}}$.

The complex diagonalizable case

Using the normal form of A (Section 1.1), a quick computation shows that

$$h = \begin{pmatrix} -H & -\lambda \\ -\lambda & H \end{pmatrix} \quad \text{and} \quad \bar{g} = \begin{pmatrix} -\epsilon - H^2 + \lambda^2 & -2H\lambda \\ -2H\lambda & \epsilon + H^2 - \lambda^2 \end{pmatrix}.$$

Hence, using the fact that

$$\nabla_{e_1} e_1 = \omega^1_{12} e_2, \qquad \nabla_{e_1} e_2 = \omega^1_{12} e_1,$$
$$\nabla_{e_2} e_1 = \omega^2_{12} e_2, \qquad \nabla_{e_2} e_2 = \omega^2_{12} e_1,$$

we calculate[2]

$$\bar{h}_{111} = \epsilon(-2\lambda\omega^1_{12} + e_1(H)), \qquad \bar{h}_{112} = \epsilon(-2\lambda\omega^2_{12} + e_2(H)) = \epsilon e_1(\lambda),$$
$$\bar{h}_{122} = -\epsilon(2\lambda\omega^1_{12} + e_1(H)) = \epsilon e_2(\lambda), \qquad \bar{h}_{222} = -\epsilon(2\lambda\omega^2_{12} + e_2(H)).$$

Hence

$$\begin{aligned}\mathbb{G}(2\vec{H}, \mathbb{J}d\bar{\varphi}(e_1)) &= \frac{(\epsilon + H^2 - \lambda^2)(\bar{h}_{111} - \bar{h}_{122}) + 4H\lambda\bar{h}_{112}}{-(\epsilon + H^2 - \lambda^2)^2 - 4H^2\lambda^2} \\ &= -\frac{2\epsilon(\epsilon + H^2 - \lambda^2)e_1(H) + \epsilon 4H\lambda e_1(\lambda)}{1 + H^4 + \lambda^4 + 2\epsilon H^2 - 2\epsilon\lambda^2 - 2H^2\lambda^2 + 4H^2\lambda^2} \\ &= -\frac{2(1 + \epsilon H^2 - \epsilon\lambda^2)e_1(H) + \epsilon 4H\lambda e_1(\lambda)}{1 + H^4 + \lambda^4 + \epsilon 2(H^2 - \lambda^2) + 2H^2\lambda^2}.\end{aligned}$$

In the same way, we get

$$\mathbb{G}(2\vec{H}, \mathbb{J}d\bar{\varphi}(e_2)) = -\frac{2(1 + \epsilon H^2 - \epsilon\lambda^2)e_2(H) + \epsilon 4H\lambda e_2(\lambda)}{1 + H^4 + \lambda^4 + \epsilon 2(H^2 - \lambda^2) + 2H^2\lambda^2}.$$

On the other hand, using the fact that $K = H^2 + \lambda^2$,

$$d\beta = d\mathtt{arctan}\epsilon\left(\frac{2H}{1 - \epsilon H^2 - \epsilon\lambda^2}\right).$$

It follows that $\mathbb{G}(2\vec{H}, \mathbb{J}.) = d\beta$, which is equivalent to $2\vec{H} = \epsilon\mathbb{J}\bar{\nabla}\beta$, the required formula. If $\epsilon = -1$ we have, using the fact that $\lambda \neq 0$,

$$\left|\frac{2H}{K+1}\right| = \frac{2|H|}{H^2 + \lambda^2 + 1} < \frac{2|H|}{H^2 + 1} \leq 1.$$

Therefore, if $\bar{\mathcal{S}}$ is $\mathbb{G}$-minimal, i.e. β is constant, we may again use Lemma 1.1 to conclude that there exists a minimal surface parallel to φ or N. Hence we have proved Corollary 2.14 in this complex diagonalizable case.

The non-diagonalizable case

By a calculation similar to that of the previous cases we have:

$$\mathbb{G}(2\vec{H}, \mathbb{J}d\bar{\varphi}(e_1)) = 0 \quad \text{and} \quad \mathbb{G}(2\vec{H}, \mathbb{J}d\bar{\varphi}(e_2)) = 2\frac{e_2(H)}{1 + \epsilon H^2}.$$

On the other hand, we have $d\beta = \frac{2dH}{1+\epsilon H^2}$. Taking into account that $e_1(H)$ vanishes, we deduce that $\mathbb{G}(2\vec{H}, \mathbb{J}.) = d\beta$, which is equivalent to the required formula. If $\epsilon = -1$ we have, using the fact that $|H| \neq 1$,

$$\left|\frac{2H}{K+1}\right| = \frac{2|H|}{1 + H^2} < 1.$$

Therefore, we may use Lemma 1.1 again and complete the proof of Corollary 2.14.

[2]The fact that we obtain two different expressions for $\bar{h}_{112}$ and $\bar{h}_{122}$ accounts for the Codazzi equation.

4.3. **Geometry of Lagrangian surfaces with respect to the scalar flat metric $\mathbb{G}'$.**

4.3.1. *The metric $\bar{g}'$ and the second fundamental form $\bar{h}$ (Proof of Theorem 2.16).* Using the description of the metric $\mathbb{G}'$ given in Section 3.2 and by a calculation analogous to that of Section 4.2.1, we obtain

$$\bar{g}'(X,Y) = -\epsilon\mathbb{G}(X \wedge N + AX \wedge \phi, \mathbb{J}'\mathbb{J}(Y \wedge N + AY \wedge \phi)) = g(X, (-\mathrm{J}'A + A\mathrm{J}')Y),$$

which gives the claimed formula for $\bar{g}'$. We now discuss the degeneracy and the signature of $\bar{g}'$, which depend on the type of the shape operator A:

The real diagonalizable case

Write g and A in canonical form, with (e_1, e_2) an oriented, orthonormal local frame. It follows that $\mathrm{J}'e_1 = e_2, \mathrm{J}'e_2 = -\epsilon' e_1$. We easily get

$$\bar{g}' = \begin{pmatrix} 0 & \epsilon_2(\kappa_2 - \kappa_1) \\ \epsilon_2(\kappa_2 - \kappa_1) & 0 \end{pmatrix}.$$

In particular, $\bar{g}'$ is degenerate at umbilic points and indefinite otherwise.

The complex diagonalizable case

Write g and A in canonical form. It follows that $\mathrm{J}'e_1 = e_2, \mathrm{J}'e_2 = e_1$ (here $\epsilon' = -1$ since the metric g is indefinite). Hence

$$\bar{g}' = \begin{pmatrix} -2\lambda & 0 \\ 0 & -2\lambda \end{pmatrix},$$

which shows that the metric $\bar{g}'$ is everywhere definite.

The non-diagonalizable case

Writing g and A in canonical form and observing that $\mathrm{J}'e_1 = e_1, \mathrm{J}'e_2 = -e_2$, we get

$$\bar{g}' = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix},$$

which shows that the metric $\bar{g}'$ is everywhere degenerate. In particular, we do not need to take into consideration the case of A being non-diagonalizable in the proofs of Corollaries 2.17 and 2.18.

4.3.2. *The mean curvature vector and the Proof of Corollary 2.17.* Again we split the proof into two cases:

The real diagonalizable case

It has been seen in Section 4.2.2 that $\bar{h}_{ijj} := \bar{h}(e_i, e_j, e_j) = -\epsilon\epsilon_j e_i(\kappa_j)$. It follows that

$$\mathbb{G}'(2\vec{H}', \mathbb{J}'d\bar{\varphi}(e_1)) = \frac{\bar{h}_{112}}{\bar{g}'(e_1,e_2)} = \frac{-\epsilon\epsilon_1 e_2(\kappa_1)}{\epsilon_2(\kappa_2 - \kappa_1)} = -\epsilon\epsilon' \frac{e_2(\kappa_1)}{\kappa_2 - \kappa_1}$$

and

$$\mathbb{G}'(2\vec{H}', \mathbb{J}'d\bar{\varphi}(e_2)) = \frac{\bar{h}_{122}}{\bar{g}'(e_1,e_2)} = \frac{-\epsilon\epsilon_2 e_1(\kappa_2)}{\epsilon_2(\kappa_2 - \kappa_1)} = -\epsilon \frac{e_1(\kappa_2)}{\kappa_2 - \kappa_1}.$$

Hence

$$\vec{H}' = \frac{-\epsilon}{2(\kappa_2 - \kappa_1)^2}\Big(\epsilon_1 e_1(\kappa_2)\mathbb{J}'d\bar{\varphi}(e_1) + \epsilon_2 e_2(\kappa_1)\mathbb{J}'d\bar{\varphi}(e_2)\Big).$$

In particular, we see that if $\bar{\mathcal{S}}$ is $\mathbb{G}'$-minimal, both $e_1(\kappa_2)$ and $e_2(\kappa_1)$ vanish. We now use the Codazzi equation derived in Section 4.2.2:

$$\left\{ \begin{array}{rcl} (\kappa_2 - \kappa_1)\omega_{12}^1 & = & \epsilon_2 e_1(\kappa_2), \\ (\kappa_2 - \kappa_1)\omega_{12}^2 & = & \epsilon_1 e_2(\kappa_1). \end{array} \right.$$

Since we assume that the metric $\bar{g}'$ is not degenerate, $\kappa_2 - \kappa_1$ does not vanish. Therefore the $\mathbb{G}'$-minimality condition implies the vanishing of ω_{12}^1 and ω_{12}^2, i.e. the flatness of g. The next step consists of using the Gauss equation with respect to the immersion $\varphi : \mathcal{M}^2 \to \mathbb{S}^3_{p,1}$, giving

$$g(R^g(e_1, e_2)e_1, e_2) = \epsilon h(e_1, e_1)h(e_2, e_2) - \epsilon h(e_1, e_2)h(e_1, e_2) + K_{\mathbb{S}^3_{p,1}} = \epsilon\epsilon'\kappa_1\kappa_2 + 1.$$

Hence $\kappa_1\kappa_2 = -\epsilon\epsilon'$. Taking into account the vanishing of $e_1(\kappa_2)$ and $e_2(\kappa_1)$, it implies that both principal curvatures are constant, non-vanishing and different from ± 1. In particular, $\mathcal{S}$ has parallel second fundamental form and $\bar{\mathcal{S}}$ is totally geodesic.

In the real diagonalizable case we are able to give a more precise characterization of surfaces with parallel second fundamental form: introducing the map $\varphi_t := \mathtt{cos}\epsilon(t)\varphi + \mathtt{sin}\epsilon(t)N$ and differentiating, we get

$$d\varphi_t(e_2) = \mathtt{cos}\epsilon(t)d\varphi(e_2) + \mathtt{sin}\epsilon(t)dNe_2 = (\mathtt{cos}\epsilon(t) - \kappa_2\mathtt{sin}\epsilon(t))d\varphi(e_2).$$

Hence, choosing t_0 such that $\frac{\mathtt{cos}\epsilon(t_0)}{\mathtt{sin}\epsilon(t_0)} = \kappa_2 = -\epsilon(\kappa_1)^{-1}$ yields the vanishing of $d\varphi_{t_0}(e_2)$. Defining local coordinates (s_1, s_2) on $\mathcal{M}^2$ such that $\partial_{s_1} = e_1$ and $\partial_{s_2} = e_2$, we claim that the curve $\gamma(s_1) := \varphi_{t_0}(s_1, s_2)$ is a geodesic of $\mathbb{S}^3_{p,1}$. To see this, we calculate the acceleration of γ in $\mathbb{R}^4$:

$$\gamma'' = \frac{\mathtt{cos}\epsilon(t_0) - \kappa_1\mathtt{sin}\epsilon(t_0)}{\epsilon\mathtt{cos}\epsilon(t_0)}(-\mathtt{sin}\epsilon(t_0)N - \mathtt{cos}\epsilon(t_0)\varphi),$$

which is collinear to γ. Hence γ is a geodesic and $\varphi(\mathcal{M}^2)$ is a tube over γ.

The complex diagonalizable case

Since the basis (e_1, e_2) is orthogonal with respect to $\bar{g}'$, the $\mathbb{G}'$-minimality of $\bar{\mathcal{S}}$ is equivalent to the vanishing of

$$\bar{h}_{111} + \bar{h}_{122} = -4\epsilon\lambda\omega_{12}^1 \quad \text{and} \quad \bar{h}_{112} + \bar{h}_{222} = -4\epsilon\lambda\omega_{12}^2$$

(the coefficients $\bar{h}_{ijk}$ have been determined in Section 4.2.3). Hence ω_{12}^1 and ω_{12}^2 vanish and g is flat. Again we use the Gauss equation with respect to the immersion $\varphi : \mathcal{M}^2 \to \mathbb{S}^3_{p,1}$, obtaining

$$\begin{aligned} g(R^g(e_1, e_2)e_1, e_2) &= \epsilon\big(h(e_1, e_1)h(e_2, e_2) - h(e_1, e_2)h(e_1, e_2)\big) + K_{\mathbb{S}^3_{p,1}} \\ &= -\epsilon(H^2 + \lambda^2) + 1; \end{aligned}$$

hence $H^2 + \lambda^2 = \epsilon$. On the other hand, the Codazzi equation becomes a Cauchy-Riemann system satisfied by the pair (H, λ):

$$\begin{cases} e_1(H) &= -e_2(\lambda), \\ e_2(H) &= e_1(\lambda), \end{cases}$$

so by the Liouville theorem, H and λ are constant, which implies that $\bar{h}$ vanishes.

4.3.3. *Flat Lagrangian surfaces: Proof of Corollary* 2.18. Again we consider two cases:

The real diagonalizable case

In order to characterize the flatness of $\bar{g}' := \bar{\varphi}^*\mathbb{G}'$, we shall use the Gauss equation twice, first with respect to the immersion $\bar{\varphi} : \mathcal{M}^2 \to L^\pm(\mathbb{S}^3_{p,1})$, and then with respect to the embedding $\iota : L^\pm(\mathbb{S}^3_{p,1}) \to \Lambda^2(\mathbb{R}^4)$.

First, using the principal frame (e_1, e_2) introduced in the previous section, we have

$$\begin{aligned} K^{\bar{g}'} &= \bar{g}'(R^{\bar{g}'}(\bar{e}_1, \bar{e}_2)\bar{e}_1, \bar{e}_2) \\ &= \mathbb{G}'(\vec{h}(\bar{e}_1, \bar{e}_2), \vec{h}(\bar{e}_1, \bar{e}_2)) - \mathbb{G}'(\vec{h}(\bar{e}_1, \bar{e}_1), \vec{h}(\bar{e}_2, \bar{e}_2)) + \mathbb{G}'(\bar{R}(\bar{e}_1, \bar{e}_2)\bar{e}_1, \bar{e}_2), \end{aligned}$$

where $\vec{h} : T\bar{\mathcal{S}} \times T\bar{\mathcal{S}} \to N\bar{\mathcal{S}}$ denotes the second fundamental form of the immersion $\bar{\varphi}$ with respect to the metric $\mathbb{G}'$. In other words, $\mathbb{G}'(\vec{h}(X, Y), \mathbb{J}Z) = \bar{h}(X, Y, Z)$. We have

$$\vec{h}(\bar{e}_i, \bar{e}_j) = \frac{\bar{h}_{ij2}N_1 + \bar{h}_{ij1}N_2}{\epsilon_1(\kappa_2 - \kappa_1)},$$

so that

$$\mathbb{G}'(\vec{h}(\bar{e}_1, \bar{e}_2), \vec{h}(\bar{e}_1, \bar{e}_2)) = 2\epsilon_1 \frac{\bar{h}_{112}\bar{h}_{122}}{\kappa_2 - \kappa_1}$$

and

$$\mathbb{G}'(\vec{h}(\bar{e}_1, \bar{e}_1), \vec{h}(\bar{e}_2, \bar{e}_2)) = \epsilon_1 \frac{\bar{h}_{111}\bar{h}_{222} + \bar{h}_{112}\bar{h}_{122}}{\kappa_1 - \kappa_2}.$$

Hence

$$\begin{aligned} \mathbb{G}'(\vec{h}(\bar{e}_1, \bar{e}_2), \vec{h}(\bar{e}_1, \bar{e}_2)) - \mathbb{G}'(\vec{h}(\bar{e}_1, \bar{e}_1), \vec{h}(\bar{e}_2, \bar{e}_2)) &= \epsilon_1 \frac{2\bar{h}_{112}\bar{h}_{122} - \bar{h}_{111}\bar{h}_{222} + \bar{h}_{112}\bar{h}_{122}}{\kappa_1 - \kappa_2} \\ &= \epsilon_2 \frac{e_2(\kappa_1)e_1(\kappa_2) - e_1(\kappa_1)e_2(\kappa_2)}{\kappa_1 - \kappa_2} \\ &= -\epsilon_2 \frac{(d\kappa_1 \wedge d\kappa_2)(e_1, e_2)}{\kappa_1 - \kappa_2}. \end{aligned}$$

We now proceed to calculate $\mathbb{G}'(\bar{R}(\bar{e}_1, \bar{e}_2)\bar{e}_1, \bar{e}_2)$. We have

$$\bar{e}_i = d\varphi(e_i) \wedge N + \varphi \wedge dN(e_i) = -E_{2+i} - \kappa_i E_i.$$

Then we easily get that $h^\iota(\bar{e}_1, \bar{e}_1) = -\epsilon_1(\epsilon + \kappa_1^2)\bar{x}$ and $h^\iota(\bar{e}_1, \bar{e}_2) = (\kappa_1 - \kappa_2)e_1 \wedge e_2$. Analogously we may check that $h^\iota(\bar{e}_1, \epsilon\mathbb{J}' \circ \mathbb{J}\bar{e}_2)$ is collinear to $\bar{x}$, while $h^\iota(\bar{e}_2, \epsilon\mathbb{J}' \circ \mathbb{J}\bar{e}_2)$ is collinear to $e_1 \wedge e_2$.

It follows that, again using the Gauss equation and the fact that the metric $\langle\langle ., . \rangle\rangle$ is flat,

$$\begin{aligned} \mathbb{G}'(\bar{R}(\bar{e}_1, \bar{e}_2)\bar{e}_1, \bar{e}_2) &= -\mathbb{G}'(\bar{R}(\bar{e}_1, \bar{e}_2)\bar{e}_1, \epsilon\mathbb{J}' \circ \mathbb{J}\bar{e}_2) \\ &= -\Big(\langle\langle h^\iota(\bar{e}_1, \mathbb{J}' \circ \mathbb{J}\bar{e}_2, h^\iota(\bar{e}_2, \bar{e}_1))\rangle\rangle - \langle\langle h^\iota(\bar{e}_1, \bar{e}_1), h^\iota(\bar{e}_2, \epsilon\mathbb{J}' \circ \mathbb{J}'\bar{e}_2)\rangle\rangle\Big) \\ &= 0. \end{aligned}$$

We conclude that the metric $\bar{g}'$ (and therefore $\bar{g}$ as well) is flat if and only if $d\kappa_1 \wedge d\kappa_2$ vanishes, i.e. $\mathcal{S}$ is Weingarten.

The complex diagonalizable case

The calculations are analogous to the real diagonalizable case and left to the reader.

4.4. Marginally trapped Lagrangian surfaces: Proof of Corollary 2.20.

4.4.1. *$\mathbb{G}$-marginally trapped Lagrangian surfaces.* We have seen in Section 4.2.3 that if the shape operator A of φ is not diagonalizable, then $\bar{g}(e_1, e_1)$ vanishes. It follows that $d\bar{\varphi}(e_1)$, and therefore $\mathbb{J}d\bar{\varphi}(e_1)$ as well, is a $\mathbb{G}$-null vector. We have also seen that $\mathbb{G}(2\vec{H}, \mathbb{J}d\bar{\varphi}(e_1))$ vanishes, so $\vec{H}$, a vector of the plane $N\bar{\mathcal{S}}$ spanned by $\mathbb{J}d\bar{\varphi}(e_1)$ and $\mathbb{J}d\bar{\varphi}(e_2)$, must be collinear to $\mathbb{J}d\bar{\varphi}(e_1)$. Hence it is a $\mathbb{G}$-null vector as well.

4.4.2. $\mathbb{G}'$*-marginally trapped Lagrangian surfaces.* We start from the expression of the mean curvature vector of $\bar{\mathcal{S}}$ with respect to $\mathbb{G}'$ obtained in Section 4.3.2:

$$\vec{H}' = \frac{-\epsilon}{2(\kappa_2 - \kappa_1)^2}\Big(\epsilon_1 e_1(\kappa_2)\mathbb{J}' d\bar{\varphi}(e_1) + \epsilon_2 e_2(\kappa_1)\mathbb{J}' d\bar{\varphi}(e_2)\Big).$$

Since $\bar{g}'(e_1, e_1)$ and $\bar{g}'(e_1, e_1)$ vanish, the pair $(\mathbb{J}' d\bar{\varphi}(e_1), \mathbb{J}' d\bar{\varphi}(e_2))$ is a $\mathbb{G}$-null basis of the normal space $N\bar{\mathcal{S}}$. Therefore, the mean curvature vector $\vec{H}'$ is $\mathbb{G}'$-null if and only if it is collinear to one of the two vectors $\mathbb{J}' d\bar{\varphi}(e_i)$, i.e. if and only if either $e_1(\kappa_2)$ or $e_2(\kappa_1)$ vanishes. This occurs in at least the following two cases:

- If $\mathcal{S}$ is a tube, i.e. the set of equidistant points to a given curve of $\mathbb{S}^3_{p,1}$, then one of its principal curvatures is constant;
- If $\mathcal{S}$ is a surface of revolution, i.e. a surface invariant by the action of a subgroup $SO(2)$ or $SO(1,1)$ of $SO(4-p,p)$, then both principal curvatures are constant along the orbits of the action, which are in addition tangent to one of the principal directions (cf. [3]). Therefore, $e_1(\kappa_2)$ or $e_2(\kappa_1)$ vanishes.

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References

1. D. V. Alekseevsky, B. Guilfoyle and W. Klingenberg, *On the Geometry of Spaces of Oriented Geodesics*, Diff. Geom. and its Applications **28** (2010) no. 4, 454–468.
2. D. V. Alekseevsky, C. Medori and A. Tomassini, *Homogeneous para-Kähler Einstein manifolds*, Russian Mathematical Surveys **64** (2009) 1–43. MR2503094 (2010k:53068)
3. H. Anciaux, *Minimal submanifolds in Pseudo-Riemannian geometry*, World Scientific, 2010. MR2722116 (2012g:53110)
4. H. Anciaux, B. Guilfoyle and P. Romon, *Minimal Lagrangian surfaces in the tangent bundle of a Riemannian surface*, J. of Geom. and Physics **61** (2011) 237–247. MR2746995 (2012a:53152)
5. I. Castro and F. Urbano, *Minimal Lagrangian surfaces in* $\mathbb{S}^2 \times \mathbb{S}^2$, Communications in Anal. and Geom. **15** (2007) 217–248. MR2344322 (2008j:53107)
6. V. Cruceanu, *Almost product bi-complex structures on manifolds,* An. Stiint. Univ. Al. I. Cuza Iasi **51** (2005), no. 1, 99–118. MR2187361 (2006h:53020)
7. V. Cruceanu, P. Fortuny and P. M. Gadea, *A survey on paracomplex geometry*, Rocky Mountain J. **26** (1996), 83–115. MR1386154 (97c:53112)
8. N. Georgiou, *On maximal surfaces in the space of oriented geodesics of hyperbolic 3-space*,Math. Scand. **111** (2012), no. 2, 187–209.
9. N. Georgiou and B. Guilfoyle, *On the space of oriented geodesics of hyperbolic 3-space*, Rocky Mountain J. Math. **40** (2010) no. 4, 1183–1219. MR2718810 (2011i:53116)
10. N. Georgiou and B. Guilfoyle, *A characterization of Weingarten surfaces in hyperbolic 3-space,* Abh. Math. Semin. Univ. Hambg. **80** (2010) no. 2, 233–253. MR2734689 (2011i:51017)
11. B. Guilfoyle and W. Klingenberg, *An indefinite Kähler metric on the space of oriented lines*, J. London Math. Soc. **72** (2005) 497–509. MR2156666 (2006e:53125)
12. B. Guilfoyle and W. Klingenberg, *On Weingarten surfaces in Euclidean and Lorentzian 3-space,* Diff. Geom. and its Applications **28** (2010) no. 4, 454–468. MR2651535 (2011d:53173)
13. F. Hélein and P. Romon (2002), *Hamiltonian stationary Lagrangian surfaces in Hermitian symmetric spaces,* in Differential geometry and integrable systems (Tokyo, 2000), M. Guest, R. Miyaoka and Y. Ohnita, Eds., Contemp. Math. **308**, Amer. Math. Soc., Providence, RI, 2002, 161–178. MR1955633 (2004j:53103)
14. N. Hitchin, *Monopoles and geodesics,* Comm. Math. Phys. **83** (1982), 579–602. MR649818 (84i:53071)

15. A. Honda, *Isometric Immersions of the Hyperbolic Plane into the Hyperbolic Space,* Tohoku Math. J. **64** (2012) no. 2, 171–193.
16. B. Khesin and S. Tabachnikov, *Pseudo-Riemannian geodesics and billiards,* Adv. Math. **221** (2009), 1364–1396. MR2518642 (2010g:53163)
17. M. Kimura, *Space of geodesics in hyperbolic spaces and Lorentz numbers,* Mem. Fac. Sci. Eng. Shimane Univ. Ser. B Math. Sci. **36** (2003), 61–67. MR1976998 (2004a:53045)
18. M. Kriele, *Spacetime, Foundations of General Relativity and Differential Geometry*, Springer (1999). MR1726656 (2001g:53126)
19. K. Leichtweiss, *Zur Riemannschen geometrie in Grassmannschen manigfaltigkeiten,* Math. Zeit. **76** (1961) 334–366. MR0126808 (23:A4102)
20. M. A. Magid, *Lorentzian Isothermic Surfaces in* $\mathbb{R}^n_j$, Rocky Mountain J. Math. **35** (2005) 627–640. MR2135589 (2006e:53031)
21. R. Osserman, *A survey of minimal surfaces,* Van Nostrand Reinhold Co., New York-London-Melbourne (1969). MR0256278 (41:934)
22. B. Palmer, *Hamiltonian minimality and Hamiltonian stability of Gauss maps,* Diff. Geom. Appl. **7** (1997) no. 1, 51–58. MR1441918 (97m:53106)
23. M. Salvai, *On the geometry of the space of oriented lines of Euclidean space,* Manuscripta Math. **118** (2005) no. 2, 181–189. MR2177684 (2006h:53043)
24. M. Salvai, *On the geometry of the space of oriented lines of the hyperbolic space,* Glasg. Math. J. **49** (2007) no. 2, 357–366. MR2347266 (2008h:53062)

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