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# GLOBAL $L^p$ CONTINUITY OF FOURIER INTEGRAL OPERATORS

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ABSTRACT. In this paper we establish global $L^p(\mathbb{R}^n)$-regularity properties of Fourier integral operators. The orders of decay of the amplitude are determined for operators to be bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, as well as to be bounded from Hardy space $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. This extends local $L^p$-regularity properties of Fourier integral operators, as well as results of global $L^2(\mathbb{R}^n)$ boundedness, to the global setting of $L^p(\mathbb{R}^n)$. Global boundedness in weighted Sobolev spaces $W^{\sigma,p}_s(\mathbb{R}^n)$ is also established, and applications to hyperbolic partial differential equations are given.

## 1. Introduction

In this paper we investigate global $L^p(\mathbb{R}^n)$ continuity properties of non-degenerate Fourier integral operators. In particular, we are interested in the question of what decay properties of the amplitude guarantee the global boundedness of Fourier integral operators from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.

The analysis of the local $L^2$ boundedness of Fourier integral operators goes back to Eskin [16] and Hörmander [17], who showed that non-degenerate Fourier integral operators with amplitudes in the symbol class $S^0_{1,0}$ are locally bounded on $L^2(\mathbb{R}^n)$. A Fourier integral operator of class $I^\mu(X, Y; \mathcal{C})$ is called non-degenerate if its canonical relation $\mathcal{C}$ is locally a graph of a symplectic mapping from $T^*X\backslash 0$ to $T^*Y\backslash 0$. If the canonical relation of the operator degenerates, the local $L^2$ boundedness of zero order operators is known to fail; see e.g. Hörmander [19]. In this paper we will be concerned with non-degenerate operators only.

Since the 1970s this local $L^2$ boundedness result has been extended in different directions. On one hand, global $L^2(\mathbb{R}^n)$ boundedness has been studied, motivated by applications in microlocal analysis and hyperbolic partial differential equations. On the other hand, its extension to $L^p$ spaces with $p \neq 2$ has also been under study, motivated by applications in harmonic analysis.

The question of the global $L^2(\mathbb{R}^n)$ boundedness was first widely investigated in the case of pseudo-differential operators. The phase is trivial in this case, so the main question is to determine minimal assumptions on the amplitude which guarantees the global $L^2(\mathbb{R}^n)$ boundedness. For example, one wants to relax an assumption that the symbol of a pseudo-differential operator is in the symbol class $S^0_{0,0}$ for operators to be still bounded on $L^2(\mathbb{R}^n)$. There are different sets of assumptions; see e.g. Calderón and Vaillancourt [5], Childs [6], Coifman and Meyer

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[7], Cordes [10], Sugimoto [36], etc. The question of the global $L^2(\mathbb{R}^n)$ boundedness of Fourier integral operators is more subtle, and involves different sets of assumptions on both phase and amplitude. Operators arising in applications to hyperbolic equations and Feynman path integrals have been considered e.g. in Asada [1], Asada and Fujiwara [2], Kumano-go [20], Boulkhemair [4]. On the other hand, applications to smoothing estimates for evolution partial differential equations require less restrictive assumptions on the phase, and the required estimates have been established by Ruzhansky and Sugimoto [29, 30].

Local $L^p$ boundedness of Fourier integral operators has been under intensive study as well. In the case of $p \neq 2$ there is a loss of derivatives in $L^p$ spaces. For example, a loss of $(n-1)|1/p-1/2|$ derivatives has been established for operators appearing as solutions to the wave equations; see e.g. Beals [3], Peral [25], Miyachi [24]. Finally, Seeger, Sogge and Stein [33] showed that general non-degenerate Fourier integral operators in the class $I^\mu(\mathbb{R}^n,\mathbb{R}^n;\mathcal{C})$ are locally bounded in $L^p(\mathbb{R}^n)$, provided that their amplitudes are in the class $S^\mu_{1,0}$ with $\mu \le -(n-1)|1/p-1/2|$, $1<p<\infty$ (see also Sogge [34] and Stein [35]). In the case of $p=1$, they showed that operators of order $\mu=-(n-1)/2$ are locally bounded from the Hardy space $H^1$ to $L^1$, while Tao [38] showed that operators of the same order are also locally of weak type (1,1). Extensions of these results with smaller loss of regularity under additional geometric assumptions on the canonical relations have been studied by Ruzhansky [27, 28]. There is also a result by Sugimoto [37], establishing global $L^p$ estimates for translation invariant operators with phases with strictly convex level sets.

The aim of this paper is to establish global $L^p(\mathbb{R}^n)$ boundedness of Fourier integral operators, which depends on the growth/decay order of the amplitude in $x$ and $y$ variables. The results of this paper will extend the local $L^p$ results of Seeger, Sogge and Stein [33] as well as global $L^2$ results of Asada and Fujiwara [2], Coriasco [12], and Ruzhansky and Sugimoto [29], to the global setting of $L^p(\mathbb{R}^n)$. In fact, for $p \neq 2$, we will observe that there is a loss not only of derivatives but also of growth/decay dependent on the value of $p$. Both of these losses disappear in the case $p=2$. Consequently, using the global calculi of Fourier integral operators developed by Coriasco [12] and, more generally, by Ruzhansky and Sugimoto [31, 32], we can also obtain global weighted estimates in Sobolev spaces $W^{s,p}_\sigma(\mathbb{R}^n)$. The results of this paper have been partially announced in [14].

The $L^p$ continuity in Theorem 2.2 will be proved by interpolation. The endpoints will be the $L^2$ boundedness from Asada and Fujiwara [2] under assumption (2a) and from Ruzhansky and Sugimoto [29] under assumptions (2b) and (2c), and the $H^1-L^1$ boundedness which is the main task of this paper.

We will be initially concerned with operators $\mathcal{T}$ of the form

$$(\mathcal{T}u)(x)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i[\langle x,\xi\rangle-\varphi(y,\xi)]}\, b(x,y,\xi)\, u(y)\,dy\,d\xi, \tag{1.1}$$

where $\varphi$ is a real-valued phase function, positively homogeneous of order one in $\xi$, and $b$ is an amplitude. Local $L^p$ properties of such operators were considered by Seeger, Sogge and Stein [33] and their global $L^2$ properties were analysed by Ruzhansky and Sugimoto [29]. We note that a general Hörmander's Fourier integral operator can always be written in the form (1.1) microlocally, while there are in general topological obstructions globally. The microlocal qualitative properties of such operators are well known; see e.g. Hörmander [17, 19] or Duistermaat [15].

Since the aim of this paper is to investigate $L^p$ properties rather than trivialisations of the Maslov index, we will treat operators that can be written in the form (1.1) globally. We note that operators (1.1) and their adjoints appear as propagators to hyperbolic partial differential equations as well as canonical transforms in smoothing problems. Applications to hyperbolic Cauchy problems are given in Section 5.

Subsequently, we will deal with Fourier integral operators of the form

$$Au(x) = \int_{\mathbb{R}^n} e^{i\varphi(x,\xi)}\, a(x,\xi)\, \widehat{u}(\xi)\, d\xi, \tag{1.2}$$

where $\varphi$ is as above and the amplitude $a$ does not depend on $y$.

Finally, we mention that results on the local $L^p$ boundedness of Fourier integral operators with complex-valued phase functions have been established by Ruzhansky [28], extending previous local $L^2$ results by Melin and Sjöstrand [23] and Hörmander [18], and that there are also results in $(\mathcal{F}L^p)_{comp}$ spaces and in modulation spaces by Cordero, Nicola and Rodino [8].

Constants in this paper will be denoted by the letter $C$, and their values may vary even in the same formula. If the value of a constant is important and unchanged in a calculation, we will use sub-indices, denoting it e.g. by $C_1$, $C_2$, etc. We will denote $\langle x\rangle = (1+|x|^2)^{1/2}$. Occasionally, for functions $f(x,y,\xi,w), g(x,y,\xi,w)$, $x,y,\xi\in\mathbb{R}^n$, and $w$ varying in a suitable parameter space, we will write $f\prec g$, $f\succ g$, if there exist constants $A,B>0$ independent of $w$ such that, for arbitrary $x,y,\xi,w$, we have $|f(x,y,\xi,w)|\le A|g(x,y,\xi,w)|$, $|f(x,y,\xi,w)|\ge B|g(x,y,\xi,w)|$, respectively. If both $f\prec g$ and $f\succ g$ hold, we will write $f\sim g$. By $B_R(y)$ we will denote an open ball with radius $R$ centred at $y$.

## 2. Main results

Let the operator $\mathcal{T}$ be given by

$$(\mathcal{T}u)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i[\langle x,\xi\rangle - \varphi(y,\xi)]}\, b(x,y,\xi)\, u(y)\, dy\, d\xi, \tag{2.1}$$

with a real-valued phase $\varphi$ and an amplitude $b$ which are *admissible*, in the sense of Definition 2.1 below. In particular, the phase $\varphi$ must satisfy suitable smoothness conditions and

$$\begin{aligned} |\det\partial_y\partial_\xi\varphi(y,\xi)|\ge C>0,\quad \partial_y^\alpha\varphi(y,\xi)&\prec\langle y\rangle^{1-|\alpha|}|\xi| \text{ for all } \alpha,\\ \langle\nabla_\xi\varphi(y,\xi)\rangle\sim\langle y\rangle,\ &\langle d_y\varphi(y,\xi)\rangle\sim\langle\xi\rangle, \end{aligned}\tag{2.2}$$

either for all $x\in\mathbb{R}^n$ and $\xi\in\mathbb{R}^n\backslash 0$, or on $\operatorname{supp} b$, depending on the other properties which are assumed in Definition 2.1.

**Definition 2.1.** We say that the phase function $\varphi$ and the amplitude function $b$ are admissible if $|\xi|\ge\varepsilon$ on $\operatorname{supp} b$, for some $\varepsilon>0$, and, moreover, the following conditions hold:

(1) *Function $\varphi\in\mathcal{C}^\infty(\mathbb{R}^n\times(\mathbb{R}^n\backslash 0))$ is real-valued and positively homogeneous of order 1 in $\xi$ on* $\operatorname{supp} b$, *i.e.* $\varphi(y,\tau\xi)=\tau\varphi(y,\xi)$ *for all* $\tau>0$ *and* $\xi\neq 0$, *on* $\operatorname{supp} b$.

(2) *One of the following properties holds true:*

(2a) *let $\varphi$ satisfy (2.2) for all $y\in\mathbb{R}^n$ and $\xi\in\mathbb{R}^n\backslash 0$ and be such that*

$$\partial_y^\alpha\partial_\xi^\beta\varphi(y,\xi)\prec 1 \tag{2.3}$$

*for all $x, y, \xi$ on* $\operatorname{supp} b$ *and all multi-indices $\alpha, \beta$ such that $|\alpha+\beta| \geq 2$; in this part we also assume $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and let $b \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ satisfy*

$$\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma b(x, y, \xi) \prec \langle x\rangle^{m_1} \langle y\rangle^{m_2} \langle \xi\rangle^{\mu-|\gamma|} \tag{2.4}$$

*for all $x, y, \xi \in \mathbb{R}^n$ and all multi-indices $\alpha, \beta, \gamma$, with some $m_1, m_2 \in \mathbb{R}$ such that $m_1 + m_2 = m$;*

(2b) *let $\varphi$ satisfy* (2.2) *on* $\operatorname{supp} b$*, and*

$$\partial_y^\alpha \partial_\xi^\beta \varphi(y, \xi) \prec 1 \tag{2.5}$$

*for all $x, y, \xi$ on* $\operatorname{supp} b$ *and all $\alpha, \beta$ such that $|\alpha| \geq 1$ and $|\beta| \geq 1$; moreover, let $b \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ satisfy*

$$\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma b(x, y, \xi) \prec \langle x\rangle^{m_1-|\alpha|} \langle y\rangle^{m_2} \langle \xi\rangle^{\mu-|\gamma|} \tag{2.6}$$

*for all $x, y, \xi \in \mathbb{R}^n$ and all multi-indices $\alpha, \beta, \gamma$, with some $m_1, m_2 \in \mathbb{R}$ such that $m_1 + m_2 = m$;*

(2c) *let $\varphi$ satisfy* (2.2) *on* $\operatorname{supp} b$*, and*

$$\partial_y^\alpha \partial_\xi^\beta \varphi(y, \xi) \prec \langle y\rangle^{1-|\alpha|} \tag{2.7}$$

*for all $x, y, \xi$ on* $\operatorname{supp} b$ *and all $\alpha, \beta$ such that $|\beta| \geq 1$; moreover, let $b \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ satisfy*

$$\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma b(x, y, \xi) \prec \langle x\rangle^{m_1} \langle y\rangle^{m_2-|\beta|} \langle \xi\rangle^{\mu-|\gamma|} \tag{2.8}$$

*for all $x, y, \xi \in \mathbb{R}^n$ and all multi-indices $\alpha, \beta, \gamma$, with some $m_1, m_2 \in \mathbb{R}$ such that $m_1 + m_2 = m$.*

The main result of this paper is the following:

**Theorem 2.2.** *Let $1 < p < \infty$ and $m, \mu \in \mathbb{R}$. Let $\mathcal{T}$ be the operator* (2.1) *and assume that the phase function $\varphi$ and the amplitude $b$ are admissible, in the sense of Definition* 2.1*. Then $\mathcal{T}$ extends to a bounded operator from $L^p(\mathbb{R}^n)$ to itself, provided that*

$$m \leq -n \left| \frac{1}{p} - \frac{1}{2} \right| \text{ and } \mu \leq -(n-1) \left| \frac{1}{p} - \frac{1}{2} \right|. \tag{2.9}$$

Let us now discuss the assumptions of Theorem 2.2. First of all, we note that assumptions (2.2) are very natural in the sense that they ask that $\varphi$ be essentially of order one in both $y$ and $\xi$. Condition

$$|\det \partial_y \partial_\xi \varphi(y, \xi)| \geq C > 0, \tag{2.10}$$

for all $y \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n \backslash 0$, is simply a global version of the local graph condition of the non-degeneracy of the Fourier integral operator (2.1). Assumption (2.4) says that $b$ has a symbolic behaviour in $\xi$ and is of order $m_1 + m_2 = m$ jointly in $x$ and $y$.

We assume that $\xi \neq 0$ on the support of $b$ to avoid the singularity of the (homogeneous) phase at the origin. We note that this issue does not arise in local boundedness problems (as in [33]) since the corresponding part of the operator is locally smoothing. In our situation it is still smoothing but may, in principle, destroy the global behaviour with respect to $x$ and $y$. Some global results in $L^2(\mathbb{R}^n)$

for small frequencies have been established by Ruzhansky and Sugimoto in [29] using weighted estimates for multipliers of Kurtz and Wheeden [22], and we refer to [29] for a discussion of complications that arise in this situation.

Assumption (2b) is different from (2a) in that we do not assume the boundedness (2.3), but assume the boundedness only of mixed derivatives (i.e. with $|\alpha| \geq 1$ *and* $|\beta| \geq 1$), and in addition assume that derivatives of $b$ have some decay properties in (2.6) or in (2.8). In assumption (2c) we also allow non-mixed derivatives (i.e. $\partial_\xi^\beta$-derivatives when $\alpha = 0$) to grow in $y$. Moreover, in both (2b) and (2c) we assume (2.2) to hold only on the support of $b$.

We note that propagators for hyperbolic partial differential equations lead to operators (2.1) with $b(x, y, \xi) = b(y, \xi)$ independent of $x$, in which case assumption (2.6) becomes trivial if $\alpha \neq 0$. For these propagators the boundedness (2.3) is also satisfied under natural assumptions on the symbol of the hyperbolic equation. However, we do not always want to assume the boundedness (2.3) since it fails for non-mixed derivatives (i.e. when $\alpha = 0$ or $\beta = 0$), e.g. in applications to smoothing estimates for dispersive equations. For example, it is shown in [29, 30] that for canonical transforms appearing there condition (2.3) fails, but it is also shown that additional decay of derivatives as in (2.6) or (2.8) holds.

If the amplitude $b$ in Theorem 2.2 is compactly supported in $(x, y)$, Theorem 2.2 implies the local $L^p$ boundedness under the assumptions in Seeger, Sogge and Stein [33], implying, in particular, that the order $\mu$ in Theorem 2.2 cannot be improved in general. Let us now give some explanation about the order $m$. In [8], Cordero, Nicola and Rodino investigated the question of the boundedness of Fourier integral operators on $(\mathcal{F}L^p(\mathbb{R}^n))_{comp}$, the space of compactly supported distributions for which the Fourier transform is in $L^p(\mathbb{R}^n)$. They proved that if the amplitude of an operator is of order $-n\left|\frac{1}{p} - \frac{1}{2}\right|$ in $\xi$ (plus additional assumptions), then the operator is continuous on $(\mathcal{F}L^p(\mathbb{R}^n))_{comp}$. They also showed that this order of decay is sharp by constructing a counterexample for higher orders. Roughly speaking, the conjugation with the Fourier transform interchanges the roles of $x$ and $\xi$, so the orders in [8] correspond to orders $m = -n\left|\frac{1}{p} - \frac{1}{2}\right|$ and $\mu = -\infty$ for operators in the setting of Theorem 2.2 since the assumption of the compact support in $(\mathcal{F}L^p(\mathbb{R}^n))_{comp}$ corresponds to locally smoothing operators in $L^p(\mathbb{R}^n)$. From this point of view, our Theorem 2.2 also improves the result of [8] with respect to $\mu$ to the order $\mu = -(n-1)\left|\frac{1}{p} - \frac{1}{2}\right|$, which cannot be improved further in general. However, the order $m$ in Theorem 2.2 can still be improved if we restrict the size of the support while still allowing it to move to infinity. In this case a uniform estimate is possible for $m \leq -(n-1)\left|\frac{1}{p} - \frac{1}{2}\right|$ and it is given in Theorem 2.5. The same improved threshold for the order $m$ can be achieved for the Fourier integral operators (1.2), as stated in Theorem 2.6.

To prove Theorem 2.2 we use interpolation between the $L^2(\mathbb{R}^n)$ boundedness and boundedness from the Hardy space $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. The global $L^2(\mathbb{R}^n)$ boundedness under assumptions (2a) and (2b)–(2c) would follow from the results of Asada and Fujiwara [2] and Ruzhansky and Sugimoto [29], respectively. Thus, the main point is to prove the boundedness from the Hardy space $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. This can be achieved by using the atomic decomposition of $H^1(\mathbb{R}^n)$ and splitting the argument for atoms with large and small supports. However, there are a number of difficulties in this argument compared with that of [33]. For example, supports are no longer bounded and can become very large, and hence, while this case is simple for the local boundedness, it requires to be analysed further in the global setting. Another global feature is that even if the supports of atoms may be small, they may still move to infinity (while remaining small). We deal with this situation by introducing a dyadic decomposition in frequency which depends on $y$. The dyadic pieces that we work with are of the size $2^{-k}$ in the radial direction and of the size $2^{-\frac{k}{2}}\langle y\rangle^{\frac{1}{2}}$ in other directions (tangential to the sphere in the frequency space). Thus, we obtain the following theorem in the setting of the Hardy space $H^1(\mathbb{R}^n)$:

**Theorem 2.3.** *Let $\mathcal{T}$ be the Fourier integral operator* (2.1)*. Under the hypotheses of Theorem* 2.2*, operator $\mathcal{T}$ extends to a bounded operator from the Hardy space $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$, provided that $m \leq -n/2$ and $\mu \leq -(n-1)/2$.*

We can also establish a result in weighted Sobolev spaces. Let $W^{\sigma,p}_s(\mathbb{R}^n)$ denote the weighted Sobolev space, i.e. the space of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $\langle x\rangle^s(1-\Delta)^{\sigma/2}f(x)$ belongs to $L^p(\mathbb{R}^n)$.

**Theorem 2.4.** *Let $1 < p < \infty$ and let $\sigma, s \in \mathbb{R}$. Let $\mathcal{T}$ be the Fourier integral operator* (2.1) *as in Theorem* 2.2 *with orders $m, \mu \in \mathbb{R}$, and let $m_p = -n\left|\frac{1}{p}-\frac{1}{2}\right|$, $\mu_p = -(n-1)\left|\frac{1}{p}-\frac{1}{2}\right|$. Then operator $\mathcal{T}$ extends to a bounded operator from $W^{\sigma,p}_s(\mathbb{R}^n)$ to $W^{\sigma-\mu-\mu_p,p}_{s-m-m_p}(\mathbb{R}^n)$.*

Theorem 2.4 follows from Theorem 2.2 and composition formulae of Fourier integral operators with pseudo-differential operators as in [31] or in [32]. In fact, here we only need a special class of pseudo-differential operators, namely of operators with symbols $\pi_{s,\sigma}(x,\xi) = \langle x\rangle^s\langle\xi\rangle^\sigma$ for which we have $(\mathrm{Op}\ \pi_{s,\sigma})(W^{\sigma,p}_s(\mathbb{R}^n)) = L^p(\mathbb{R}^n)$. Global composition formulae of [31, 32] will also be used in the proof of Theorem 2.3.

The assumptions on the order of the amplitude in Theorem 2.2 can be relaxed if we work with functions with compact support. We will assume that the supports are uniformly bounded but will still allow them to move to infinity (while remaining bounded). In this situation the proof of Theorem 2.2 will also imply the following.

**Theorem 2.5.** *Let $1 < p < \infty$ and let $m, \mu \in \mathbb{R}$. Let $\mathcal{T}$ be the Fourier integral operator* (2.1) *as in Theorem* 2.2*. Let $R > 0$. Let $\mathcal{V}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ be the set of all functions $f \in L^p(\mathbb{R}^n)$ for which there exists $y \in \mathbb{R}^n$ such that $\operatorname{supp} f \subset B_R(y)$, and let $\mathcal{V}(\mathbb{R}^n)$ have the topology induced by $L^p(\mathbb{R}^n)$. Then operator $\mathcal{T}$ extends to a continuous operator from $\mathcal{V}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, provided that*

$$m \leq -(n-1)\left|\frac{1}{p}-\frac{1}{2}\right| \ \text{ and } \mu \leq -(n-1)\left|\frac{1}{p}-\frac{1}{2}\right|. \tag{2.11}$$

Theorem 2.5 will follow from Remarks 3.3 and 3.7. We also have natural counterparts of Theorem 2.5 for $H^1$ and $W^{\sigma,p}_s$ as in Theorems 2.3 and 2.4.

We note that for large atoms the cancellation condition is not used in our proof, so that the results extend to the boundedness from the local Hardy space $H^1_{loc}(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. The order in $x$ is improved in Theorem 2.5 and in Theorem 2.6.

By an argument similar to the one used in [9], it is also possible to prove the $L^p$ continuity of the classes of Fourier integral operators considered in [12], where the phase function is assumed positively homogeneous of order 1 in $\xi$ and satisfies (2.2):

**Theorem 2.6.** *Let $A = A_{\varphi,a}$ be a Fourier integral operator of the form*

$$Au(x) = \int_{\mathbb{R}^n} e^{i\varphi(x,\xi)}\ a(x,\xi)\ \widehat{u}(\xi)\ d\xi, \tag{2.12}$$

*with a real-valued phase function $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n \times (\mathbb{R}^n \backslash 0))$ such that $\varphi(x,\tau\xi) = \tau\varphi(x,\xi)$ for all $\tau > 0$ and $\xi \neq 0$, and assume that the condition* (2.2) *holds true for all $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n\backslash 0$. Moreover, assume that $|\xi| \geq \varepsilon$, for some $\varepsilon > 0$, on the support of the amplitude $a$, and that $a \in S^{m,\mu}$, i.e. that*

$$\partial_x^\alpha \partial_\xi^\beta a(x,\xi) \prec \langle x\rangle^{m-|\alpha|} \langle\xi\rangle^{\mu-|\beta|},$$

*for all $x, \xi \in \mathbb{R}^n$ and all multi-indices $\alpha, \beta$, with some $m, \mu \in \mathbb{R}$. Then $A$ extends to a bounded operator from $L^p(\mathbb{R}^n)$ to itself, provided that*

$$m \leq -(n-1)\left|\frac{1}{p} - \frac{1}{2}\right| \text{ and } \mu \leq -(n-1)\left|\frac{1}{p} - \frac{1}{2}\right|. \tag{2.13}$$

The thresholds (2.13) are sharp, by a modification of a counterexample described in [9]. The improvement in Theorem 2.6 compared to that in Theorem 2.2, under the assumption (2c) in Definition 2.1, comes from the independence of the amplitude of $A$ on the $y$-variable, if we write the adjoint $A^*$ in the form of an operator $\mathcal{T}$ in Theorem 2.2. The proof of Theorem 2.6 is given in Section 4. Finally, the composition formulae in [12] together with Theorem 2.6 imply the analog of Theorem 2.4 for the operator $A$:

**Theorem 2.7.** *Let $1 < p < \infty$ and let $\sigma, s \in \mathbb{R}$. Let $A$ be the Fourier integral operator* (1.2) *as in Theorem* 2.6 *with orders $m, \mu \in \mathbb{R}$, and let $m_p = -(n-1)\left|\frac{1}{p} - \frac{1}{2}\right|$. Then operator $A$ extends to a bounded operator from $W_s^{\sigma,p}(\mathbb{R}^n)$ to $W_{s-m-m_p}^{\sigma-\mu-m_p,p}(\mathbb{R}^n)$.*

We finish by briefly indicating the example for the sharpness of order $m$ in Theorem 2.6. Let $\psi : \mathbb{R} \to \mathbb{R}$ be a smooth diffeomorphism whose restriction to $(0,1)$ is a non-linear diffeomorphism on $(0,1)$, i.e. there exists an interval $I \subset (0,1)$ such that $|\psi''| > 0$ on the closure of $I$. Moreover, assume that $0 < c \leq |\psi'(t)| \leq C$ for all $t \in \mathbb{R}$. Define

$$\phi(\xi) := (\psi(\xi_1/\xi_n), \ldots, \psi(\xi_{n-1}/\xi_n))\, \xi_n$$

microlocally in a narrow cone around $(0,1)$, i.e. for $|\xi_n| \geq C|\xi'|$, where $\xi' = (\xi_1, \ldots, \xi_{n-1})$. Let $G \in C_0^\infty(\mathbb{R}^n)$ be such that $G \geq 0$, $G(\xi', \xi_n) = 1$ for $\epsilon \leq |\xi_n| \leq 1$, and such that the support of $G$ is contained in the set $(\xi', \xi_n)$ with $\epsilon/2 \leq |\xi_j| \leq M$, $j = 1, \ldots, n-1$, $M \geq |\xi_n| \geq C|\xi'|$, for some some sufficiently small $\epsilon > 0$ and some $M > 0$. Let us write $y = (y', y_n)$. Define the operator

$$\mathcal{T}f(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i[\langle x,\xi\rangle - \langle y',\phi(\xi)\rangle - y_n\xi_n]}\ \langle y\rangle^m\ G(\xi)\ \chi_{[1,2]}(y_n)\ f(y)\ dy\ d\xi,$$

where $\chi_{[1,2]} \in C_0^\infty(\mathbb{R})$ is a non-zero function with $\operatorname{supp} \chi_{[1,2]} \subset [1,2]$. We notice that $\mathcal{T}$ is a locally smoothing operator with phase and amplitude satisfying conditions of Theorem 2.2. After conjugating $\mathcal{T}$ with the Fourier transform and factoring out $\xi_n$, we observe that $y \sim y'$ on the support of the amplitude of $\mathcal{T}$. Therefore, if $\mathcal{T}$

is bounded on $L^p(\mathbb{R}^n)$, the operator $\mathcal{F}^{-1} \circ \mathcal{T} \circ \mathcal{F}$ with $\xi_n, y_n$ factored out would be bounded on $(\mathcal{F}L^p(\mathbb{R}^{n-1}))_{comp}$. Since this is an operator of the form considered in Section 6.2 in [8], we must have $m \leq -(n-1)|1/p - 1/2|$, $1 \leq p \leq \infty$.

## 3. Proof of Theorem 2.3

Since Theorem 2.2 follows by complex interpolation from Theorem 2.3 and $L^2$ boundedness results in [2] and [29] under assumptions (2a) and (2b)–(2c), respectively, we need to prove Theorem 2.3. This will be achieved through various subsequent steps. We also note that on several occasions we will use the composition formulae for globally defined Fourier integral operators and pseudo-differential operators. In order not to list the corresponding lengthy conditions for the calculus for such operators to work, we refer to Theorems 2.1, 2.5 and 2.8 in [32] for exact formulations.

Given $f \in H^1(\mathbb{R}^n)$, we can decompose (see e.g. [35]) the function

$$f = \sum_Q \lambda_Q a_Q,$$

where

$$\sum_Q |\lambda_Q| \simeq \|f\|_{H^1(\mathbb{R}^n)},$$

and the atoms $a_Q \in H^1(\mathbb{R}^n)$ have the following properties:

(1) $\operatorname{supp} a_Q \subset Q$, where $Q \subset \mathbb{R}^n$ is a cube of sidelength $q$;
(2) $\|a_Q\|_{L^\infty(\mathbb{R}^n)} \leq |Q|^{-1}$;
(3) $\displaystyle\int_Q a_Q(y)\, dy = 0$.

Theorem 2.3 would then follow if we show that

$$\|\mathcal{T} a_Q\|_{L^1(\mathbb{R}^n)} \leq C, \tag{3.1}$$

for a constant $C$ independent of $a_Q$.

Let $F = F(x,y)$ denote the distribution kernel of $\mathcal{T}$, given by the oscillatory integral

$$F(x,y) = \int_{\mathbb{R}^n} e^{i[\langle x,\xi\rangle - \varphi(y,\xi)]}\, b(x,y,\xi)\, d\xi. \tag{3.2}$$

We begin by showing that the amplitude function can be assumed to be supported only in a suitable neighbourhood of the wave front set of the distributional kernel of $\mathcal{T}$:

**Proposition 3.1.** *Let $\chi = \chi(x,y,\xi)$ be supported in*

$$E_k = \{(x,y,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n : |x - \nabla_\xi \varphi(y,\xi)| \leq k\langle x\rangle\},$$

*for $k \in (0,1)$ suitably small, and such that $\chi|_{E_{\frac{k}{2}}} \equiv 1$. Moreover,*[1] *let us assume that $\chi$ (is smooth and) satisfies $S^{0,0,0}$ estimates on* $\operatorname{supp} b$*, that is*

$$\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma \chi(x,y,\xi) \prec \langle x\rangle^{-|\alpha|} \langle y\rangle^{-|\beta|} \langle \xi\rangle^{-|\gamma|}.$$

[1]With $h \in \mathcal{C}^\infty(\mathbb{R})$ such that $h|_{(-\infty,\frac{1}{2})} \equiv 1$ and $h|_{(1,+\infty)} \equiv 0$, $k \in (0,1)$, set

$$\chi(x,y,\xi) = h\left(\frac{|x - \nabla_\xi \varphi(y,\xi)|}{k\langle x\rangle}\right).$$

*We set* $\widetilde{b} = (1-\chi)b$. *Then, defining*

$$\widetilde{F}(x,y) = \int_{\mathbb{R}^n} e^{i[\langle x,\xi\rangle - \varphi(y,\xi)]}\, \widetilde{b}(x,y,\xi)\, d\xi, \tag{3.3}$$

*it follows that* $\widetilde{F} \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$*, which implies, in particular, that*

$$\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \widetilde{F}(x,y)\, a_Q(y)\, dy \right| dx \leq C, \tag{3.4}$$

*with a constant* $C$ *independent of* $a_Q$.

*Proof.* We will show that kernel $\widetilde{F}$ satisfies

$$\partial_x^\alpha \partial_y^\beta \widetilde{F}(x,y) \prec (\langle x\rangle\langle y\rangle)^{-N}, \tag{3.5}$$

for all $N \in \mathbb{N}$, $x, y \in \mathbb{R}^n$ and all multi-indices $\alpha, \beta$. By the hypotheses on $b$ and $\varphi$, it is clear that it is enough to prove the estimate only for $\alpha = \beta = 0$ and arbitrary order in $x, y, \xi$ for $\widetilde{b}$.

Indeed, we have $|x - \nabla_\xi \varphi(y,\xi)| \succ \langle x\rangle$ on $\operatorname{supp} \widetilde{b}$, so that the operator $L_\xi$, acting on functions $v = v(x,y,\xi)$ with respect to $\xi$ as

$$(L_\xi v)(x,y,\xi) = \sum_{j=1}^n i\partial_{\xi_j} \left( \frac{x_j - \partial_{\xi_j}\varphi(y,\xi)}{|x - \nabla_\xi \varphi(y,\xi)|^2} v(x,y,\xi) \right),$$

is well defined on $\operatorname{supp} \widetilde{b}$. Moreover, on $\operatorname{supp} \widetilde{b}$, we have

$$|x - \nabla_\xi \varphi(y,\xi)| \succ \langle x\rangle.$$

Then

$$|\nabla_\xi \varphi(y,\xi)| \leq |x - \nabla_\xi \varphi(y,\xi)| + |x| \prec |x - \nabla_\xi \varphi(y,\xi)|,$$

and it follows that we also have

$$|x - \nabla_\xi \varphi(y,\xi)| \succ \langle \nabla_\xi \varphi(y,\xi)\rangle \succ \langle y\rangle.$$

Now (3.5) follows by integrating by parts in (3.3), observing that

$${}^tL_\xi e^{i[\langle x,\xi\rangle - \varphi(y,\xi)]} = e^{i[\langle x,\xi\rangle - \varphi(y,\xi)]}.$$

Then (3.4) holds, since, for all $N \in \mathbb{N}$, we have

$$\begin{aligned} \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \widetilde{F}(x,y)\, a_Q(y)\, dy \right| dx &\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\widetilde{F}(x,y)|\, |a_Q(y)|\, dy\, dx \\ &\leq \widetilde{C} \int_{\mathbb{R}^n} \langle x\rangle^{-N}\, dx \int_{\mathbb{R}^n} |a_Q(y)|\, dy \leq C\, |Q|\, |Q|^{-1} = C. \end{aligned}$$

□

Therefore, from now on we can then assume that for some $k \in (0,1)$ we have

$$\operatorname{supp} b \subseteq D = \{(x,y,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n : \, |x - \nabla_\xi \varphi(y,\xi)| \leq k\langle x\rangle\}. \tag{3.6}$$

This implies that on $\operatorname{supp} b$ we have

$$\langle x\rangle \sim \langle \nabla_\xi \varphi(y,\xi)\rangle \sim \langle y\rangle,$$

which in turn implies that $C_1\langle y\rangle \leq \langle x\rangle \leq C_2\langle y\rangle$, $x, y \in \mathbb{R}^n$, for suitable constants $C_1, C_2 > 0$.

**Proposition 3.2.** *Let $a_Q$ be an atom in $H^1(\mathbb{R}^n)$, supported in a cube $Q \subset \mathbb{R}^n$ centred at $y_0 \in \mathbb{R}^n$ and with sidelength $q \geq 1$ (hence also $|Q| \geq 1$). Then, estimate* (3.1) *holds with a constant $C$ independent of $a_Q$.*

*Proof.* Let us denote by $M_s$ the multiplication operator $(M_s v)(x) = \langle x\rangle^s v(x)$. From composition formulae with pseudo-differential operators (see [32]) it follows that operator $M_{\frac{n}{2}}\mathcal{T}$ is then a Fourier integral operator with amplitude bounded in $x$ and $y$, and of order $-\frac{n-1}{2}$ in $\xi$. Consequently, operator $M_{\frac{n}{2}}\mathcal{T}$ is bounded on $L^2(\mathbb{R}^n)$ by [2] under assumption (2a) and by [29] under assumptions (2b) and (2c). Applying Hölder's inequality and denoting

$$D_{q,y_0} = \{x \in \mathbb{R}^n \mid C_1\langle y\rangle \leq \langle x\rangle \leq C_2\langle y\rangle, y \in Q\},$$

we get

$$\begin{aligned}
\|\mathcal{T}a_Q\|_{L^1(\mathbb{R}^n)} &= \int\limits_{\substack{\langle x\rangle\sim\langle y\rangle \\ y\in Q}} |\langle x\rangle^{-\frac{n}{2}}(M_{\frac{n}{2}}\mathcal{T}a_Q)(x)|dx \\
&\leq \left(\int_{D_{q,y_0}} \langle x\rangle^{-n}\,dx\right)^{\frac12} \|(M_{\frac{n}{2}}\mathcal{T})a_Q\|_{L^2(\mathbb{R}^n)} \\
&\leq \widetilde{C}\|a_Q\|_{L^2(\mathbb{R}^n)}\left[\int_{D_{q,y_0}} (1+|x|^2)^{-\frac{n}{2}}\,dx\right]^{\frac12} \\
&= \widetilde{C}|Q|^{-1}|Q|^{\frac12}\left[\int_{D_{q,y_0}} (1+|x|^2)^{-\frac{n}{2}}\,dx\right]^{\frac12} \\
&= \widetilde{C}\left[|Q|^{-1}\int_{D_{q,y_0}} (1+|x|^2)^{-\frac{n}{2}}\,dx\right]^{\frac12} \leq C,
\end{aligned}$$

where $C \geq 0$ does not depend on $a_Q$. Indeed, let us prove the boundedness of the expression in the last line. Let us set $A = 1 + \dfrac{|C_1^2-1|^{\frac12}}{C_1}$. The required boundedness is a consequence of the following steps:

- Choose $\psi \in \mathcal{C}^\infty(\mathbb{R})$ supported in $(-\infty, 2]$, taking values in $[0,1]$, and such that $\psi(t) = 1$ for $t \in (-\infty, 1]$. Set $\chi(q, y_0) = \psi\left(\dfrac{|y_0|}{Aq\sqrt{n}}\right)$ and let

$$\begin{aligned}
I_1 &= \chi(q,y_0)\,|Q|^{-1}\int_{D_{q,y_0}} (1+|x|^2)^{-\frac{n}{2}}\,dx, \\
I_2 &= (1-\chi(q,y_0))\,|Q|^{-1}\int_{D_{q,y_0}} (1+|x|^2)^{-\frac{n}{2}}\,dx.
\end{aligned}$$

- On the support of $\chi(q, y_0)$ we have $|y_0| \leq 2Aq\sqrt{n}$, so, for $x \in D_{q,y_0}$,

$$\begin{aligned}
|x| < \langle x\rangle \leq C_2\langle y\rangle &\leq C_2\sqrt{(|y-y_0|+|y_0|)^2+1} \\
&\leq C_2\sqrt{\left(\frac{q\sqrt{n}}{2}+2Aq\sqrt{n}\right)^2+1} \leq Kq,
\end{aligned}$$

where $K > 0$ is independent of $q \geq 1$ and $y_0 \in \mathbb{R}^n$. Then, $D_{q,y_0} \subset B_{Kq}(0)$, where $B_{Kq}(0)$ is the ball centred at the origin with radius $Kq$, and we have

$$I_1 \leq |Q|^{-1}\,|B_{Kq}(0)| \leq K^n\,|B_1(0)| = B_1,$$

with $B_1 > 0$ independent of $q \geq 1$, $y_0 \in \mathbb{R}^n$.

- On the support of $1-\chi(q,y_0)$ we have $|y_0| \geq Aq\sqrt{n} > 1$ and, for $x \in D_{q,y_0}$, we have

$$\sqrt{C_1^2|y|^2 + C_1^2 - 1} \leq |x| \leq \sqrt{C_2^2|y|^2 + C_2^2 - 1},\ y \in Q.$$

Note also that, on the support of $1-\chi(q,y_0)$, for $y \in Q$ we have

$$\frac{|y-y_0|}{|y_0|} \leq \frac{q\sqrt{n}}{2}\,\frac{1}{Aq\sqrt{n}} = \frac{1}{2A} < \frac{1}{2}$$

and

$$|y| \geq |y_0| - |y-y_0| = |y_0|\left(1 - \frac{|y-y_0|}{|y_0|}\right) \geq |y_0|\left(1-\frac{1}{2A}\right) > \frac{|y_0|}{2} > \frac{1}{2}.$$

Hence we can estimate

$$\begin{aligned}
C_1^2|y|^2 + C_1^2 - 1 &\geq |y_0|^2\left[C_1^2\left(1-\frac{|y-y_0|}{|y_0|}\right)^2 + \frac{C_1^2-1}{|y_0|^2}\right] \\
&\geq |y_0|^2\left[C_1^2\left(1-\frac{1}{2A}\right)^2 - \frac{|C_1^2-1|}{|y_0|^2}\right] \\
&\geq |y_0|^2\left[\frac{C_1^2(2A-1)^2}{4A^2} - \frac{|C_1^2-1|}{A^2q^2n}\right] \\
&\geq |y_0|^2\frac{C_1^2q^2n\left(1+\dfrac{2|C_1^2-1|^{\frac{1}{2}}}{C_1}\right)^2 - 4|C_1^2-1|}{4A^2q^2n} \\
&\geq |y_0|^2\frac{q^2n(C_1^2+4C_1|C_1^2-1|^{\frac{1}{2}})}{4A^2q^2n} > 0,
\end{aligned}$$

from which we get that

$$r_1 := \min_{y\in Q}\sqrt{C_1^2|y|^2 + C_1^2 - 1} \geq K_1|y_0| > 0,$$

with $\dfrac{3C_1}{2} > K_1 > 0$ independent of $q \geq 1$, $y_0 \in \mathbb{R}^n$. Since $C_2 \geq C_1$, on the support of $1-\chi(q,y_0)$ we have $C_2^2|y|^2 + C_2^2 - 1 > 0$, and

$$\begin{aligned}
\sqrt{C_2^2|y|^2 + C_2^2 - 1} &\leq \sqrt{C_2^2(|y_0| + |y-y_0|)^2 + C_2^2} \\
&\leq C_2|y_0|\sqrt{\left(1+\frac{|y-y_0|}{|y_0|}\right)^2 + \frac{1}{|y_0|^2}} \\
&\leq 2C_2|y_0|,
\end{aligned}$$

so that $r_1 < r_2 := \max_{y\in Q}\sqrt{C_2^2|y|^2 + C_2^2 - 1} \leq K_2|y_0|$ with $K_2 > K_1 > 0$ independent of $q \geq 1$, $y_0 \in \mathbb{R}^n$. We have then proved that, on the support of $1-\chi(q,y_0)$, $D_{q,y_0}\subset \overline{B_{r_2}(0)\setminus B_{r_1}(0)}$, hence

$$I_2 \le (1-\chi(q,y_0))\,|Q|^{-1}\,|B_1(0)|\int_{r_1}^{r_2}\frac{r^{n-1}}{(1+r^2)^{\frac{n}{2}}}\,dr$$
$$\le |B_1(0)|\int_{r_1}^{r_2}\frac{dr}{r}\le |B_1(0)|\log\frac{K_2}{K_1}=B_2,$$

with $B_2>0$ independent of $q\ge 1$, $y_0\in\mathbb{R}^n$.

The proof is complete. □

*Remark* 3.3. Let the operator $\mathcal{T}$ be as in Theorem 2.2 with $\mu$ satisfying (2.9) but with any $m\le 0$. Let $R>0$. Let $a_Q$ be an atom in $H^1(\mathbb{R}^n)$, supported in a cube $Q\subset\mathbb{R}^n$ centred at $y_0\in\mathbb{R}^n$ and with sidelength $q$ such that $R\ge q\ge 1$. Then, estimate (3.1) holds with a constant $C$ independent of such $a_Q$.

This remark follows immediately from the proof of Proposition 3.2 if we observe that the boundedness of $I_1$ is actually independent of the order of $b$ in $x$, while the boundedness of $I_2$ is a consequence of the fact that the volume of $D_{q,y_0}$ is bounded by a uniform constant for all cubes $Q$ in Remark 3.3.

Of course, the argument in the proof of Proposition 3.2 still holds if the hypothesis $|Q|\ge 1$ is replaced by $|Q|\ge Q_0>0$, or, equivalently, by $q\ge q_0>0$. In the next steps of the proof we can then assume that $a_Q$ is supported in a cube $Q$ with sidelength $q=2^{-j}$, $j\ge j_0$, where $j_0$ is chosen so large that $\frac{q}{2}\sqrt{n}<1$. In this way,

$$y\in Q\Longrightarrow |y-y_0|\le\frac{q}{2}\sqrt{n}\Longrightarrow \langle y\rangle\sim\langle y_0\rangle,$$

with $y_0$ the centre of $Q$, so that we also have, on $\operatorname{supp} b$, that $\langle x\rangle\sim\langle y_0\rangle$.

We now define an "exceptional set" set $\mathcal{N}_Q$, which covers

$$\Sigma=\{x=\nabla_\xi\varphi(y,\xi)\text{ for some }y\in Q,\ \xi\in\mathbb{R}^n\},\tag{3.7}$$

and again use $L^2$ boundedness results, together with Hölder and Hardy-Littlewood-Sobolev inequalities, to estimate $\|\mathcal{T}a_Q\|_{L^1}$ on that set.

Choose unit vectors $\xi_k^\nu$, $\nu=1,\dots,N(k,y)$, $k\ge j_0$, $y\in\mathbb{R}^n$, such that:

- $|\xi_k^\nu-\xi_k^{\nu'}|\ge C_0 2^{-\frac{k}{2}}\langle y\rangle^{-\frac12}$, $\nu\ne\nu'$, for some fixed positive constant $C_0<1$;
- the unit sphere $\mathbb{S}^{n-1}$ is covered by the balls centred at $\xi_k^\nu$ with radius $2^{-\frac{k}{2}}\langle y\rangle^{-\frac12}$.

We then have

$$N(k,y)\approx 2^{k\frac{n-1}{2}}\langle y\rangle^{\frac{n-1}{2}}.$$

For $y\in Q$ and a constant $M$ to be fixed later, define

$$\mathcal{R}^y_{k\nu}=\left\{x\colon |\langle x-\nabla_\xi\varphi(y,\xi_k^\nu),\xi_k^\nu\rangle|\le M2^{-k}\text{ and }|\Pi^\perp_{k\nu}(x-\nabla_\xi\varphi(y,\xi_k^\nu))|\le M2^{-\frac{k}{2}}\langle y\rangle^{\frac12}\right\},\tag{3.8}$$

where $\Pi^\perp_{k\nu}$ is the projection onto the plane orthogonal to $\xi_k^\nu$. The set $\mathcal{R}^y_{k\nu}$ is then an $n$-rectangle with $n-1$ sides of length $M2^{-\frac{k}{2}}\langle y\rangle^{\frac12}$ and one side of length $M2^{-k}$. If $Q$ has sidelength $q=2^{-j}$, $j\ge j_0$, we define

$$\mathcal{N}_Q=\bigcup_{y\in Q}\bigcup_{\nu=1}^{N(j,y)}\mathcal{R}^y_{j\nu}.\tag{3.9}$$

Since $|\mathcal{R}^y_{j\nu}| \approx 2^{-j\frac{n+1}{2}}\langle y_0\rangle^{\frac{n-1}{2}}$ for $y \in Q$, it follows that

$$|\mathcal{N}_Q| \le C2^{j\frac{n-1}{2}}\langle y_0\rangle^{\frac{n-1}{2}}2^{-j\frac{n+1}{2}}\langle y_0\rangle^{\frac{n-1}{2}} = C2^{-j}\langle y_0\rangle^{n-1} = C|Q|^{\frac{1}{n}}\langle y_0\rangle^{n-1}, \tag{3.10}$$

for some constant $C \ge 0$ independent of $j \ge j_0$, $y_0 \in \mathbb{R}^n$.

**Lemma 3.4.** *If in* (3.8) *we take*

$$M = \sup_{\substack{|\alpha|=2,3\\(y,\xi)\in\mathbb{R}^n\times\mathbb{R}^n}} \langle y\rangle^{-1}\langle\xi\rangle^{-1+|\alpha|}|\partial^\alpha_\xi\varphi(y,\xi)|,$$

*the singular set* $\Sigma$ *defined in* (3.7) *is a subset of* $\mathcal{N}_Q$.

*Proof.* Let us denote

$$\mathrm{vers}(\xi) = \frac{\xi}{|\xi|}.$$

Since, for all $\xi \in \mathbb{R}^n$, we have

$$|\mathrm{vers}(\xi) - \xi^\nu_j| \le 2^{-\frac{j}{2}}\langle y\rangle^{-\frac{1}{2}}$$

for some $\nu = 1,\dots,N(j,y)$, then, with $M$ chosen as above, we have $\nabla_\xi\varphi(y,\xi) \in \mathcal{R}^y_{j\nu}$. Indeed, $\nabla_\xi\varphi(y,\xi)$ is homogeneous of order 0 in $\xi$ and $\Pi^\perp_{j\nu}$ is a projection, so that

$$\begin{aligned}|\Pi^\perp_{j\nu}(\nabla_\xi\varphi(y,\xi) - \nabla_\xi\varphi(y,\xi^\nu_j))| &\le |\nabla_\xi\varphi(y,\mathrm{vers}(\xi)) - \nabla_\xi\varphi(y,\xi^\nu_j)|\\ &\le M\langle y\rangle|\mathrm{vers}(\xi) - \xi^\nu_j| \le M2^{-\frac{j}{2}}\langle y\rangle^{\frac{1}{2}}.\end{aligned}$$

Moreover, again in view of the homogeneity of the phase function, if we set

$$h^\nu_j(y,\xi) = \langle\nabla_\xi\varphi(y,\xi),\xi^\nu_j\rangle - \langle\nabla_\xi\varphi(y,\xi^\nu_j),\xi^\nu_j\rangle = \langle\nabla_\xi\varphi(y,\xi),\xi^\nu_j\rangle - \varphi(y,\xi^\nu_j),$$

we have $h^\nu_j(y,\xi^\nu_j) = 0$ and $\nabla_\xi h^\nu_j(y,\xi) = \langle\varphi''_{\xi\xi}(y,\xi),\xi^\nu_j\rangle$. Therefore, we get

$$\nabla_\xi h^\nu_j(y,\xi^\nu_j) = 0$$

by Euler's formula. Writing the Taylor expansion of $h^\nu_j(y,\xi)$ with respect to $\xi$ at $\xi^\nu_j$, we obtain

$$|h^\nu_j(y,\xi)| \le M\langle y\rangle|\mathrm{vers}(\xi) - \xi^\nu_j|^2 \le M2^{-j},$$

as desired. □

**Proposition 3.5.** $\|\mathcal{T}a_Q\|_{L^1(\mathcal{N}_Q)} \le C$ *with* $C$ *independent of* $a_Q$.

*Proof.* First we observe that the operator $M_{\frac{n}{2}}\mathcal{T}(1-\Delta)^{\frac{n-1}{4}}$ is a Fourier integral operator with the same phase and same properties of the amplitude as those of $\mathcal{T}$ in view of the global calculus in [32]. Consequently, the operator $M_{\frac{n}{2}}\mathcal{T}(1-\Delta)^{\frac{n-1}{4}}$ is bounded on $L^2(\mathbb{R}^n)$ in view of the $L^2$ boundedness theorems in [2] under assumption (2a) and in [29] under assumptions (2b) and (2c). Writing $p_n = \dfrac{2n}{2n-1}$ and recalling

(3.10), we have

$$\begin{aligned}
\|\mathcal{T}a_Q\|_{L^1(\mathcal{N}_Q)} &= \left\| M_{-\frac{n}{2}} \left[ M_{\frac{n}{2}} \mathcal{T}(1-\Delta)^{\frac{n-1}{4}} \right] (1-\Delta)^{-\frac{n-1}{4}} a_Q \right\|_{L^1(\mathcal{N}_Q)} \\
&\le \left( \int\limits_{\substack{\mathcal{N}_Q \\ \langle x\rangle \sim \langle y\rangle}} \langle x\rangle^{-n}\, dx \right)^{\frac{1}{2}} \left\| \left[ M_{\frac{n}{2}} \mathcal{T}(1-\Delta)^{\frac{n-1}{4}} \right] \left[ (1-\Delta)^{-\frac{n-1}{4}} a_Q \right] \right\|_{L^2(\mathbb{R}^n)} \\
&\le C_1 \left( \langle y_0\rangle^{-n} |Q|^{\frac{1}{n}} \langle y_0\rangle^{n-1} \right)^{\frac{1}{2}} \left\| (1-\Delta)^{-\frac{n-1}{4}} a_Q \right\|_{L^2(\mathbb{R}^n)} \\
&\le C_2 |Q|^{\frac{1}{2n}} \|a_Q\|_{L^{p_n}(\mathbb{R}^n)} \le C\, |Q|^{\frac{1}{2n}} |Q|^{-\frac{1}{2n}} = C,
\end{aligned}$$

with a constant $C$ independent of $a_Q$, in view of the Hardy-Littlewood-Sobolev inequality

$$\left\| (1-\Delta)^{-\frac{n-1}{4}} a_Q \right\|_{L^2(\mathbb{R}^n)} \le \widetilde{C}\, \|a_Q\|_{L^{p_n}(\mathbb{R}^n)},$$

and since, obviously,

$$\|a_Q\|_{L^{p_n}(\mathbb{R}^n)} \le |Q|^{\frac{1}{p_n}-1} = |Q|^{-\frac{1}{2n}}.$$

□

We will now prove the estimate

$$\|\mathcal{T}a_Q\|_{L^1(\mathbb{R}^n \setminus \mathcal{N}_Q)} \le C \tag{3.11}$$

off the exceptional set. We first introduce a dyadic decomposition, choosing the function $\theta \in \mathcal{C}^\infty(\mathbb{R})$ such that $\operatorname{supp}\theta \subset \left(\frac{1}{4}, 4\right)$ and such that for all $s > 0$ we have $\sum_{k\in\mathbb{Z}} \theta(2^{-k}s) = 1$. We now set

$$F_k(x,y) = \int_{\mathbb{R}^n} e^{i[\langle x,\xi\rangle - \varphi(y,\xi)]} b(x,y,\xi)\, \theta_k(\xi)\, d\xi, \tag{3.12}$$

where $\theta_k(\xi) = \theta(2^{-k}|\xi|)$. We can assume without loss of generality that $b(x,y,\xi) = 0$ for $|\xi| < 8$. Defining $\theta_0 = 1 - \sum\limits_{k>0} \theta_k$, we have $F = \sum\limits_{k\ge 1} F_k$. Estimate (3.11) is then a consequence of the following proposition, where we recall that $j$ was introduced in a way that $2^{-j}$ is a sidelength of $Q$.

**Proposition 3.6.** *For all $y, y' \in Q$, $j, k \in \mathbb{N}$, $j \ge j_0$, we have*

$$\int_{\mathbb{R}^n \setminus \mathcal{N}_Q} |F_k(x,y)|\, dx \prec 2^{j-k} \text{ if } k > j, \tag{3.13}$$

$$\int_{\mathbb{R}^n} |F_k(x,y) - F_k(x,y')|\, dx \prec 2^{k-j} \text{ if } k \le j. \tag{3.14}$$

*Proof.* As introduced above, for each $k \in \mathbb{N}$, let $\{\chi_k^\nu\}$, $\nu = 1, \dots, N(y,k)$, be a homogeneous partition of unity associated with the covering of the unit sphere with the balls $B(\xi_k^\nu, c_0 2^{-\frac{k}{2}} \langle y\rangle^{-\frac{1}{2}})$. Explicitly, we choose $\mathcal{C}^\infty$ functions $\chi_k^\nu = \chi_k^\nu(y,\xi)$, homogeneous in $\xi$ of degree 0, such that, for all $y \in \mathbb{R}^n$, we have

- $\chi_k^\nu(y, \operatorname{vers}(\xi)) \equiv 1$ for $\operatorname{vers}(\xi)$ in a neighbourhood of $\xi_k^\nu$ in $\mathbb{S}^{n-1}$;
- $\chi_k^\nu(y,\xi) = 0$ if $|\operatorname{vers}(\xi) - \xi_k^\nu| \ge c_0 2^{-\frac{k}{2}} \langle y\rangle^{-\frac{1}{2}}$;

- $\sum_\nu \chi_k^\nu = 1$;
- $|\partial^\gamma \chi_k^\nu(y,\xi)| \prec |\xi|^{-|\gamma|}(2^k\langle y\rangle)^{\frac{|\gamma|}{2}}$ for all multi-indices $\gamma \in \mathbb{Z}_+^n$.

We now define

$$F_k^\nu(x,y) = \int_{\mathbb{R}^n} e^{i[\langle x,\xi\rangle - \varphi(y,\xi)]}\, b_k^\nu(x,y,\xi)\, d\xi,$$

where

$$b_k^\nu(x,y,\xi) = b(x,y,\xi)\,\theta_k(\xi)\,\chi_k^\nu(y,\xi).$$

Also set

$$r_k^\nu(y,\xi) = \varphi(y,\xi) - \langle \nabla_\xi\varphi(y,\xi_k^\nu),\xi\rangle$$

implying that

$$\nabla_\xi r_k^\nu(y,\xi) = \nabla_\xi\varphi(y,\xi) - \nabla_\xi\varphi(y,\xi_k^\nu),$$

and let $D_k^\nu = \langle \nabla_\xi, \xi_k^\nu\rangle$, $\nu = 1,\dots,N(k,y)$. Clearly, by the definition of $r_k^\nu$ and the homogeneity of $\varphi$, we have $r_k^\nu(y,\xi_k^\nu) = 0$ and $\nabla_\xi r_k^\nu(y,\xi_k^\nu) = 0$. Since, again by homogeneity,

$$(D_k^\nu r_k^\nu)(y,\xi) = D_k^\nu\varphi(y,\xi) - \varphi(y,\xi_k^\nu) \Longrightarrow (D_k^\nu r_k^\nu)(y,\xi_k^\nu) = 0,$$
$$(\nabla_\xi D_k^\nu r_k^\nu)(y,\xi) = D_k^\nu\nabla_\xi\varphi(y,\xi) \Longrightarrow (\nabla_\xi D_k^\nu r_k^\nu)(y,\xi_k^\nu) = 0,$$

by induction we also see that, for all $N \in \mathbb{N}$, we have

$$[(D_k^\nu)^N r_k^\nu](y,\xi_k^\nu) = 0, \qquad [\nabla_\xi (D_k^\nu)^N r_k^\nu](y,\xi_k^\nu) = 0. \tag{3.15}$$

Writing the Taylor expansion in $\xi$ of $r_k^\nu$ centred in $\xi_k^\nu$, (3.15) implies that, for all $N \in \mathbb{N}$, on $\operatorname{supp}(b_k^\nu)$ we have

$$[(D_k^\nu)^N r_k^\nu](y,\xi) \prec |\xi|^{1-N}\langle y\rangle|\operatorname{vers}(\xi) - \xi_k^\nu|^2 \prec 2^{k(1-N)}2^{-k} = 2^{-kN}. \tag{3.16}$$

On the other hand, for the "transversal derivatives" with $|\gamma| \geq 1$ we have, on $\operatorname{supp}(b_k^\nu)$,

$$D_\xi^\gamma r_k^\nu(y,\xi) \prec |\xi|^{1-|\gamma|}\langle y\rangle \prec 2^{-k(|\gamma|-1)}\langle y\rangle \prec 2^{-k\frac{|\gamma|}{2}}\langle y\rangle. \tag{3.17}$$

Indeed, first we recall that on $\operatorname{supp}(b_k^\nu)$, $|\xi|$ is equivalent to $2^k$. For $|\gamma| \geq 2$, we then have $|\xi|^{1-|\gamma|} \prec 2^{k(1-|\gamma|)} \leq 2^{-k\frac{|\gamma|}{2}}$ and hence also (3.17). For $|\gamma| = 1$, the first derivatives are actually bounded by $2^{-\frac{k}{2}}\langle y\rangle^{\frac{1}{2}}$, since by $\nabla_\xi r_k^\nu(y,\xi_k^\nu) = 0$ and the Taylor expansion we have

$$(\partial_{\xi_j} r_k^\nu)(y,\xi) \prec \langle y\rangle|\xi - \xi_k^\nu| \prec 2^{-\frac{k}{2}}\langle y\rangle^{\frac{1}{2}}.$$

Consequently, one can readily check that on $\operatorname{supp}(b_k^\nu)$, we have the estimate

$$D_\xi^\gamma e^{ir_k^\nu(y,\xi)} \prec 2^{-k\frac{|\gamma|}{2}}\langle y\rangle^{\frac{|\gamma|}{2}}. \tag{3.18}$$

Performing a rotation[2] $\xi = C\widetilde{\xi}$, we can simplify notation and assume $\xi_k^\nu = (1,0,\dots,0)$, $\Pi_k^\nu(\xi) = (0,\xi')$. Rewriting $F_k^\nu(x,y)$ as

$$F_k^\nu(x,y) = \int_{\mathbb{R}^n} e^{i\langle x - \nabla_\xi\varphi(y,\xi),\xi_k^\nu\rangle}\, \widetilde{b}_k^\nu(x,y,\xi)\, d\xi, \tag{3.19}$$

[2] Note that all the symbol estimates for $\theta_k$, $\chi_k^\nu$, $r_k^\nu$, $\varphi$, and $b$ hold unchanged for fixed $y$, since all the entries of $C$ are bounded, in view of $A \in O(n)$.

where $\widetilde{b}_k^\nu(x,y,\xi) = e^{ir_k^\nu(y,\xi)} b_k^\nu(x,y,\xi)$, we observe that the derivatives in the $\xi_1$ ("radial") direction of $\chi_k^\nu$ vanish identically, so that, defining the selfadjoint operator $L_k^\nu$ as

$$L_k^\nu = \left( I - 2^{2k} \frac{\partial^2}{\partial \xi_1^2} \right) \left( I - 2^k \langle y \rangle^{-1} \langle \nabla_{\xi'}, \nabla_{\xi'} \rangle \right),$$

(3.16), (3.18), the properties of $\chi_k^\nu$, the definition of $\theta_k$ and the hypotheses on $\varphi$ and $b$ imply, for all $N \in \mathbb{N}$, that we have

$$[(L_k^\nu)^N \widetilde{b}_k^\nu](x,y,\xi) \prec 2^{-k\frac{n-1}{2}} \langle y \rangle^{-\frac{n}{2}}. \tag{3.20}$$

Repeated integrations by parts allow us to write

$$F_k^\nu(x,y) = H_{k,N}^\nu(x,y) \int_{\mathbb{R}^n} e^{i\langle x - \nabla_\xi \varphi(y,\xi_k^\nu), \xi \rangle} [(L_k^\nu)^N \widetilde{b}_k^\nu](x,y,\xi)\, d\xi,$$

with

$$H_{k,N}^\nu(x,y) = \left(1 + |2^k (x - \nabla_\xi \varphi(y,\xi_k^\nu))_1|^2\right)^{-N} \left(1 + |2^{\frac{k}{2}} \langle y \rangle^{-\frac{1}{2}} (x - \nabla_\xi \varphi(y,\xi_k^\nu))'|^2\right)^{-N}.$$

Since

$$\mathrm{vol}_\xi(\mathrm{supp}(\widetilde{b}_k^\nu)) \prec 2^k\, (2^k \cdot 2^{-\frac{k}{2}} \langle y \rangle^{-\frac{1}{2}})^{n-1} = 2^{k\frac{n+1}{2}} \langle y \rangle^{-\frac{n-1}{2}},$$

by (3.20) it follows that

$$|F_k^\nu(x,y)| \prec H_{k,N}^\nu(x,y)\, 2^k \langle y \rangle^{-n+\frac{1}{2}}. \tag{3.21}$$

In $\mathbb{R}^n \setminus \mathcal{N}_Q$, we must have either $|2^k (x - \nabla_\xi \varphi(y,\xi_k^\nu))_1| \succ 2^{k-j}$ or $|2^{\frac{k}{2}} \langle y \rangle^{-\frac{1}{2}} (x - \nabla_\xi \varphi(y,\xi_k^\nu))'| \succ 2^{\frac{k-j}{2}}$. Since, obviously, $H_{k,N}^\nu = H_{k,N-N'}^\nu \cdot H_{k,N'}^\nu$ for any $N, N' \in \mathbb{N}$ such that $N > N'$, then, for any $k > j$, we can estimate

$$\int_{\mathbb{R}^n} H_{k,N}^\nu(x,y)\, dx \le C_{N-N'}\, 2^{-k}\, 2^{-k\frac{n-1}{2}} \langle y \rangle^{\frac{n-1}{2}}\, 2^{-N'(k-j)}, \tag{3.22}$$

which implies, together with (3.21), that

$$\int_{\mathbb{R}^n} |F_k^\nu(x,y)|\, dx \prec 2^{j-k}\, 2^{-k\frac{n-1}{2}} \langle y \rangle^{-\frac{n}{2}}. \tag{3.23}$$

Now (3.13) follows from (3.23), by summing over $\nu = 1, \ldots, N(y,k)$. Owing to

$$\begin{aligned} \int_{\mathbb{R}^n} |F_k(x,y) - F_k(x,y')|\, dx &\le \sum_\nu \int_{\mathbb{R}^n} |F_k^\nu(x,y) - F_k^\nu(x,y')|\, dx \\ &\le |y-y'| \sum_\nu \int_{\mathbb{R}^n} \sup_{y\in Q} |\nabla_y F_k^\nu(x,y)|\, dx \prec 2^{-j} \sum_\nu \int_{\mathbb{R}^n} \sup_{y \in Q} |\nabla_y F_k^\nu(x,y)|\, dx, \end{aligned}$$

estimate (3.14) would follow from

$$\int_{\mathbb{R}^n} \sup_{y\in Q} |\nabla_y F_k^\nu(x,y)|\, dx \prec 2^k \cdot 2^{-k\frac{n-1}{2}} \langle y_0 \rangle^{-\frac{n}{2}}. \tag{3.24}$$

Now, (3.24) indeed holds true, since $\nabla_y F_k^\nu(x,y)$ can be written in the form of (3.19) with

$$\widetilde{a}_k^\nu(x,y,\xi) = \nabla_y \widetilde{b}_k^\nu(x,y,\xi) - i\widetilde{b}_k^\nu(x,y,\xi) \cdot \nabla_y \varphi(y,\xi)$$

in place of $\widetilde{b}_k^\nu(x,y,\xi)$, and $\widetilde{a}_k^\nu(x,y,\xi)$ has the same properties of $\widetilde{b}_k^\nu(x,y,\xi)$ with order in $\xi$ increased by one unit. It is then possible to repeat the same argument used in the proof of (3.23) and to sum over $\nu = 1, \ldots, N(y,k)$, recalling that $\langle y \rangle \sim \langle y_0 \rangle$ for $y \in Q$. $\square$

*Conclusion of the proof of* (3.11). By properties (1), (2) and (3) of $a_Q$ and Proposition 3.6, denoting by $\mathcal{T}_k$ the operator with kernel $F_k$ defined in (3.12), we have

$$\begin{aligned}
\|\mathcal{T} a_Q\|_{L^1(\mathbb{R}^n\setminus\mathcal{N}_Q)} &\leq \sum_{k\geq 0} \|\mathcal{T}_k a_Q\|_{L^1(\mathbb{R}^n\setminus\mathcal{N}_Q)} \\
&\leq \sum_{0\leq k\leq j} \int_{\mathbb{R}^n} \left| \int_Q [F_k(x,y) - F_k(x,y')]\, a_Q(y)\, dy \right| dx \\
&+ \sum_{k>j} \int_{\mathbb{R}^n\setminus\mathcal{N}_Q} \left| \int_Q F_k(x,y)\, a_Q(y)\, dy \right| dx \\
&\leq \sum_{0\leq k\leq j} \int_Q \left[ \int_{\mathbb{R}^n} |F_k(x,y) - F_k(x,y')|\, dx \right] |a_Q(y)|\, dy \\
&+ \sum_{k>j} \int_Q \left[ \int_{\mathbb{R}^n\setminus\mathcal{N}_Q} |F_k(x,y)|\, dx \right] |a_Q(y)|\, dy \\
&\leq \frac{C}{3} \left( \sum_{0\leq k\leq j} 2^{k-j} + \sum_{k>j} 2^{j-k} \right) \leq C,
\end{aligned}$$

with $C$ independent of $a_Q$, as claimed.

*Remark* 3.7. We note that the statements of Propositions 3.5 and 3.6 remain true if the operator $\mathcal{T}$ satisfies the assumptions of Theorem 2.3 only with $m \leq -(n-1)/2$.

## 4. Proof of Theorem 2.6

A preliminary result to be proven is the following.

**Proposition 4.1** ($L^p(\mathbb{R}^n)$ boundedness of localised Fourier integral operators)**.** *Assume the hypotheses in Theorem* 2.6 *and let* $\widetilde{\psi} \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ *be supported in the shell* $2^{-2} \leq |x| \leq 2^2$. *Then we have, for* $k \geq 1$,

$$\|\widetilde{\psi}(2^{-k}x) A f\|_{L^p} \leq C \|f\|_{L^p},$$

*where the constant* $C$ *depends only on* $\widetilde{\psi}$, *on upper bounds for a finite number of the constants in the estimates satisfied by* $a$ *and* $\varphi$, *and on the lower bound* $\delta$ *for the determinant of the mixed Hessian of* $\varphi$.

*Proof.* We can write

$$\widetilde{\psi}(2^{-k}x) A = U_{2^{-k}} A_k' U_{2^k},$$

where $U_\lambda f(x) = f(\lambda x)$, $\lambda \neq 0$, is the dilation operator and

$$A_k' f(x) = \int_{\mathbb{R}^n} e^{i\varphi(2^k x, 2^{-k}\xi)} \widetilde{\psi}(x) a(2^k x, 2^{-k}\xi) \widehat{f}(\xi)\, d\xi.$$

Hence it suffices to prove the desired conclusion with $A_k'$ in place of $\widetilde{\psi}(2^{-k}x)A$. It follows from the estimates satisfied by $\varphi$ and the fact that $|x| \sim 1$ on the support of $\widetilde{\psi}$ that, there,

$$|\partial_x^\alpha \partial_\xi^\beta (\varphi(2^k x, 2^{-k}\xi))| \leq M_{\alpha,\beta} |\xi|^{1-|\beta|}$$

(in fact, $\langle 2^k x \rangle \sim 2^k$ on the support of $\widetilde{\psi}$). Moreover, we immediately have

$$\left| \det \left( \frac{\partial^2 (\varphi(2^k x, 2^{-k}\xi))}{\partial \xi_j \partial x_l} \right) \right| > \delta > 0. \tag{4.1}$$

Similarly, one sees that,[3] on the support of $\widetilde{\psi}$, we have

$$\begin{aligned}|\partial_x^\alpha\partial_\xi^\beta(a(2^k x, 2^{-k}\xi))| &= 2^{k(|\alpha|-|\beta|)}|(\partial_x^\alpha\partial_\xi^\beta a)(2^k x, 2^{-k}\xi)|\\ &\le C_{\alpha,\beta}2^{k(|\alpha|-|\beta|)}\langle 2^k x\rangle^{m-|\alpha|}\langle 2^{-k}\xi\rangle^{\mu-|\beta|}\\ &\le C_{\alpha,\beta}2^{k(|\alpha|-|\beta|)}\langle 2^k x\rangle^{m_p-|\alpha|}\langle 2^{-k}\xi\rangle^{m_p-|\beta|}\\ &\le C_{\alpha,\beta}2^{k(|\alpha|-|\beta|+m_p-|\alpha|-m_p+|\beta|)}\langle\xi\rangle^{m_p-|\beta|}\\ &= C_{\alpha,\beta}\langle\xi\rangle^{m_p-|\beta|},\end{aligned}$$

where we have set $m_p = -(n-1)\left|\dfrac{1}{p}-\dfrac{1}{2}\right| \ge m, \mu$.

We have then showed that the operators $A'_k$ satisfy the assumptions of Seeger-Sogge-Stein's theorem, uniformly with respect to $k \in \mathbb{N}$. An application of that theorem concludes the proof.[4] □

We then make use of a Littlewood–Paley partition of unity $\{\psi_k\}$, $k \in \mathbb{Z}_+$, such that $\psi_0 \in C_0^\infty(\mathbb{R}^n)$, $\psi_k(x) = \psi(2^{-k}x)$, $k \ge 1$, $\operatorname{supp}\psi \subset \{x \in \mathbb{R}^n : 2^{-1} \le |x| \le 2\}$, and write the operator $A$ of (2.12) as

$$A = \psi_0 A + \sum_{k=1}^{\infty}\psi_k A. \tag{4.2}$$

The operator $\psi_0 A$ is $L^p$-bounded by the Seeger-Sogge-Stein theorem [33], so we only treat the second term in (4.2), namely, the sum over $k \ge 1$, writing

$$\sum_{k=1}^{\infty}\psi_k A = \sum_{k=1}^{\infty}\sum_{k'=0}^{\infty}\psi_k A\psi_{k'}.$$

The functions $\psi_k$, $k \ge 1$, can be interpreted as SG pseudo-differential operators, so that it is possible to use the composition formulae of an SG Fourier integral operator with an SG pseudo-differential operator; see [12] or [31, 32]. Splitting the asymptotic expansion of the amplitude of the composed operator into the sum of the terms from order $(m,\mu)$ to order $(m-3,\mu-3)$ and of the corresponding remainder, we write

$$\psi_k A\psi_{k'} = A_{k,k'} + 2^{-k-k'}R_{k,k'}. \tag{4.3}$$

Actually, we can compose the operators in (4.3) on the left with the multiplication by $\widetilde{\psi}_k(x) := \widetilde{\psi}(2^{-k}x)$, and on the right with the multiplication by $\widetilde{\psi}_{k'}(x)$, for a suitable cutoff $\widetilde{\psi}$, so that $\widetilde{\psi}_k\psi_k = \psi_k$. This does not affect the left hand side and we find

$$\psi_k A\psi_{k'} = \widetilde{\psi}_k A_{k,k'}\widetilde{\psi}_{k'} + 2^{-k-k'}\widetilde{\psi}_k R_{k,k'}\widetilde{\psi}_{k'},$$

with Fourier integral operators $A_{k,k'}$ and $R_{k,k'}$, with amplitudes in $S^{m,\mu}$ and in $S^{m,\mu-2}$, respectively (*uniformly with respect to* $k,k'$). Note also that, in view of

[3] Precisely, to verify this last estimate, distinguish the case $|\xi| \le 2^k$ (which implies $\langle 2^{-k}\xi\rangle \sim 1$, $\langle\xi\rangle \prec 2^k \Rightarrow 1 \prec 2^{k(-m_p+|\beta|)}\langle\xi\rangle^{m_p-|\beta|}$) and the case $|\xi| \ge 2^k$ (which implies $\langle 2^{-k}\xi\rangle \sim |2^{-k}\xi|$, $\langle\xi\rangle \sim |\xi|$).

[4] Indeed, it suffices to observe that the amplitudes of the $A'_k$, $k \in \mathbb{N}$, are compactly supported and all the other requirements of the Seeger-Sogge-Stein theorem are fulfilled; moreover, the constant in the boundedness estimate of the aformentioned theorem depends only on upper bounds for a finite number of the constants in the estimates satisfied by the phase and amplitude functions, and a lower bound for the mixed Hessian of the phase.

the properties of the Littlewood-Paley partition of unity and the formula for the asymptotic expansion of the amplitude of the composition of a pseudo-differential operator and a Fourier integral operator, $|k-k'|>N$ implies $A_{k,k'}\equiv 0$, for some fixed $N>0$. Proposition 4.1 applied with $A_{k,k'}$ in place of $A$ and $\widetilde{\psi}_{k'}f$ in place of $f$, together with the properties of the dyadic decomposition $\{\psi_k\}$, $k\in\mathbb{Z}_+$, gives the desired estimate for the operator $\sum_{k=1}^{\infty}\sum_{k'=0}^{\infty}\widetilde{\psi}_k A_{k,k'}\widetilde{\psi}_{k'}$:

$$\begin{aligned}\Big\|\sum_{k=1}^{\infty}\sum_{k'\geq 0,|k'-k|\leq N}\widetilde{\psi}_k A_{k,k'}\widetilde{\psi}_{k'}f\Big\|_{L^p}^p &\prec \sum_{k=1}^{\infty}\Big\|\sum_{k'\geq 0,|k'-k|\leq N}\widetilde{\psi}_k A_{k,k'}\widetilde{\psi}_{k'}f\Big\|_{L^p}^p\\ &\prec \sum_{k=1}^{\infty}\sum_{k'\geq 0,|k'-k|\leq N}\|\widetilde{\psi}_k A_{k,k'}\widetilde{\psi}_{k'}f\|_{L^p}^p\\ &\prec \sum_{k=1}^{\infty}\sum_{k'\geq 0,|k'-k|\leq N}\|\widetilde{\psi}_{k'}f\|_{L^p}^p \leq (2N+1)\sum_{k'=0}^{\infty}\|\widetilde{\psi}_{k'}f\|_{L^p}^p \prec \|f\|_{L^p}^p,\end{aligned}$$

where we used $\sum_{k'=0}^{\infty}\|\widetilde{\psi}_{k'}f\|_{L^p}^p\prec\|f\|_{L^p}^p$, $\Big\|\sum_{k=1}^{\infty}\widetilde{\psi}_k u_k\Big\|_{L^p}^p\prec\sum_{k=1}^{\infty}\|\widetilde{\psi}_k u_k\|_{L^p}^p$, which hold for arbitrary $f,u_k\in L^p(\mathbb{R}^n)$, $k\geq 1$. A similar argument allows us to estimate

$$\|\sum_{k=1}^{\infty}\sum_{k'=0}^{\infty}2^{-k-k'}\widetilde{\psi}_k R_{k,k'}\widetilde{\psi}_{k'}f\|_{L^p}\leq\sum_{k=1}^{\infty}\sum_{k'=0}^{\infty}2^{-k-k'}\|\widetilde{\psi}_k R_{k,k'}\widetilde{\psi}_{k'}f\|_{L^p}.$$

Indeed, again by Proposition 4.1 applied with $R_{k,k'}$ in place of $A$, and $\widetilde{\psi}_{k'}f$ in place of $f$, we see that the right hand side is

$$\prec\sum_{k=1}^{\infty}\sum_{k'=0}^{\infty}2^{-k-k'}\|\widetilde{\psi}_{k'}f\|_{L^p}=\sum_{k'=0}^{\infty}2^{-k'}\|\widetilde{\psi}_{k'}f\|_{L^p},$$

and, by an application of Hölder's inequality, the last expression is dominated by

$$\prec\left(\sum_{k'=0}^{\infty}\|\widetilde{\psi}_{k'}f\|_{L^p}^p\right)^{1/p}\prec\|f\|_{L^p}.$$

## 5. Applications to hyperbolic partial differential equations

In this section we briefly give applications of the obtained results to the solutions of the Cauchy problem for strictly hyperbolic partial differential equations. Theorems 5.1 and 5.2 describe the loss of regularity and weight for solutions. We restrict the statements to large frequencies, also because it is known that different phenomena may occur for small frequencies.

Let us first look at the equation of the first order:

$$\begin{cases}(D_t+a(t,x,D_x))u(t,x)=0, & t\neq 0,\\ u|_{t=0}=f(x).\end{cases}\tag{5.1}$$

As usual, $D_t=-i\partial_t$ and $D_{x_j}=-i\partial_{x_j}$. We assume that the symbol $a(t,x,\xi)$ is a classical symbol of order one, depending smoothly on $t,x$ and $\xi$. The strict hyperbolicity means that the principal symbol $a_1(t,x,\xi)$ is real-valued.

The result of [33] states that if $f \in W^{p,\alpha+(n-1)|1/p-1/2|}(\mathbb{R}^n)$, for some $\alpha \in \mathbb{R}$, it follows that the solution satisfies $u(t,\cdot) \in W^{p,\alpha}(\mathbb{R}^n)$ locally, $1 < p < \infty$. Moreover, this order is sharp when for every $t$ in the complement of a discrete set in $\mathbb{R}$, $a$ is elliptic in $\xi$.

We now give a global version of this result. It follows from [21, Section 4, Ch. 10] that modulo a smooth bounded function, for sufficiently small times, the solution $u(t,x)$ to (5.1) can be constructed as a Fourier integral operator in the form (2.1). If we assume that $a$ is a classical symbol with real-valued principal part such that

$$|\partial_t^k \partial_x^\beta \partial_\xi^\alpha a(t,x,\xi)| \leq C_{k\alpha\beta} \langle \xi \rangle^{1-|\alpha|} \tag{5.2}$$

holds for all $x, \xi \in \mathbb{R}^n$, all $t \in [0,T]$ for some $T > 0$, and all $k, \alpha, \beta$, with constants $C_{k\alpha\beta}$ independent of $t, x, \xi$, then the phase and the amplitude of the propagator satisfy assumption (2a) of Definition 2.1. Thus, cutting off low frequencies, we obtain

**Theorem 5.1.** *Let the symbol $a(t,x,\xi)$ satisfy conditions (5.2). Let $1 < p < \infty$, and let $\chi \in C_0^\infty(\mathbb{R}^n)$ be such that $\chi(\xi) = 1$ for $|\xi| \leq \varepsilon$, for some sufficiently large $\varepsilon > 0$. If $f$ is such that $\langle x \rangle^{n|1/p-1/2|} f(x) \in W^{p,(n-1)|1/p-1/2|}(\mathbb{R}^n)$, then for each $t \in [0,T]$, the solution $u(t,x)$ of the Cauchy problem (5.1) satisfies $(1-\chi(D))u(t,\cdot) \in L^p(\mathbb{R}^n)$. Moreover, for every $\alpha \in \mathbb{R}$ and $m \in \mathbb{R}$, there is $C_T > 0$ such that we have the estimate*

$$\|\langle x \rangle^m (1-\chi(D))u(t,\cdot)\|_{W^{p,\alpha}(\mathbb{R}^n)} \leq C_T \|\langle x \rangle^{m+n|1/p-1/2|} f\|_{W^{p,\alpha+(n-1)|1/p-1/2|}(\mathbb{R}^n)},$$

*for all $t \in [0,T]$ and all $f$ such that the right hand side norm is finite.*

In the case when the symbol $a(t,x,\xi)$ has an SG-behaviour, we have an improvement. Assume that $a$ is a classical symbol with the real-valued principal part such that

$$|\partial_t^k \partial_x^\beta \partial_\xi^\alpha a(t,x,\xi)| \leq C_{k\alpha\beta} \langle x \rangle^{-|\beta|} \langle \xi \rangle^{1-|\alpha|} \tag{5.3}$$

holds for all $x, \xi \in \mathbb{R}^n$, all $t \in [0,T]$ for some $T > 0$, and all $k, \alpha, \beta$, with constants $C_{k\alpha\beta}$ independent of $t, x, \xi$. Then it was shown in [13] that the solution to the Cauchy problem (5.1) can be written in the form (2.1), with phase and amplitude satisfying the conditions of Theorem 2.6. Thus, we obtain

**Theorem 5.2.** *Let the symbol $a(t,x,\xi)$ satisfy conditions (5.3). Let $1 < p < \infty$, and let $\chi \in C_0^\infty(\mathbb{R}^n)$ be such that $\chi(\xi) = 1$ for $|\xi| \leq \varepsilon$, for some sufficiently large $\varepsilon > 0$. If $f$ is such that $\langle x \rangle^{(n-1)|1/p-1/2|} f(x) \in W^{p,(n-1)|1/p-1/2|}(\mathbb{R}^n)$, then for each $t \in [0,T]$, the solution $u(t,x)$ of the Cauchy problem (5.1) satisfies $(1-\chi(D))u(t,\cdot) \in L^p(\mathbb{R}^n)$. Moreover, for every $\alpha \in \mathbb{R}$ and $m \in \mathbb{R}$, there is $C_T > 0$ such that we have the estimate*

$$\|\langle x \rangle^m (1-\chi(D))u(t,\cdot)\|_{W^{p,\alpha}(\mathbb{R}^n)} \leq C_T \|\langle x \rangle^{m+(n-1)|1/p-1/2|} f\|_{W^{p,\alpha+(n-1)|1/p-1/2|}(\mathbb{R}^n)},$$

*for all $t \in [0,T]$ and all $f$ such that the right hand side norm is finite.*

We note that following [21] and [13], similar conclusions hold for higher order equations under appropriate conditions on lower order terms.

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## References


[1] Kenji Asada, *On the $L^2$ boundedness of Fourier integral operators in $\mathbf{R}^n$*, Proc. Japan Acad. Ser. A Math. Sci. **57** (1981), no. 5, 249–253. MR624035 (84c:47053)

[2] Kenji Asada and Daisuke Fujiwara, *On some oscillatory integral transformations in $L^2(\mathbf{R}^n)$*, Japan. J. Math. (N.S.) **4** (1978), no. 2, 299–361. MR528863 (80d:47076)

[3] M. Beals, *$L^p$ boundedness of Fourier integrals.* Mem. Amer. Math. Soc. **264** (1982).

[4] A. Boulkhemair, *Estimations $L^2$ précisées pour des intégrales oscillantes*, Comm. Partial Differential Equations **22** (1997), no. 1-2, 165–184, DOI 10.1080/03605309708821259 (French). MR1434142 (98m:42014)

[5] Alberto-P. Calderón and Rémi Vaillancourt, *On the boundedness of pseudo-differential operators*, J. Math. Soc. Japan **23** (1971), 374–378. MR0284872 (44 #2096)

[6] A. G. Childs, *On the $L^2$-boundedness of pseudo-differential operators*, Proc. Amer. Math. Soc. **61** (1976), no. 2, 252–254 (1977). MR0442755 (56 #1135)

[7] Ronald R. Coifman and Yves Meyer, *Au delà des opérateurs pseudo-différentiels*, Astérisque, vol. 57, Société Mathématique de France, Paris, 1978 (French). With an English summary. MR518170 (81b:47061)

[8] Elena Cordero, Fabio Nicola, and Luigi Rodino, *Boundedness of Fourier integral operators on $\mathcal{F}L^p$ spaces*, Trans. Amer. Math. Soc. **361** (2009), no. 11, 6049–6071, DOI 10.1090/S0002-9947-09-04848-X. MR2529924 (2010j:47066)

[9] Elena Cordero, Fabio Nicola, and Luigi Rodino, *On the global boundedness of Fourier integral operators*, Ann. Global Anal. Geom. **38** (2010), no. 4, 373–398, DOI 10.1007/s10455-010-9219-z. MR2733369 (2012d:35434)

[10] H. O. Cordes, *On compactness of commutators of multiplications and convolutions, and boundedness of pseudodifferential operators*, J. Funct. Anal. **18** (1975), 115–131. MR0377599 (51 #13770)

[11] H. O. Cordes, *The technique of pseudodifferential operators*, London Mathematical Society Lecture Note Series, vol. 202, Cambridge University Press, Cambridge, 1995. MR1314815 (96b:35001)

[12] S. Coriasco, *Fourier integral operators in SG classes. I. Composition theorems and action on SG Sobolev spaces*, Rend. Sem. Mat. Univ. Politec. Torino **57** (1999), no. 4, 249–302 (2002). MR1972487 (2004g:47069)

[13] Sandro Coriasco, *Fourier integral operators in SG classes. II. Application to SG hyperbolic Cauchy problems*, Ann. Univ. Ferrara Sez. VII (N.S.) **44** (1998), 81–122 (1999) (English, with English and Italian summaries). MR1744134 (2001m:47106)

[14] S. Coriasco and M. Ruzhansky, *On the boundedness of Fourier integral operators on $L^p(\mathbb{R}^n)$*, C. R. Math. Acad. Sci. Paris **348** (2010), no. 15-16, 847–851. MR2677978

[15] J. J. Duistermaat, *Fourier integral operators*, Progress in Mathematics, vol. 130, Birkhäuser Boston Inc., Boston, MA, 1996. MR1362544 (96m:58245)

[16] G. I. Èskin, *Degenerate elliptic pseudodifferential equations of principal type*, Mat. Sb. (N.S.) **82(124)** (1970), 585–628 (Russian). MR0510219 (58 #23202)

[17] Lars Hörmander, *Fourier integral operators. I*, Acta Math. **127** (1971), no. 1-2, 79–183. MR0388463 (52 #9299)

[18] Lars Hörmander, *$L^2$ estimates for Fourier integral operators with complex phase*, Ark. Mat. **21** (1983), no. 2, 283–307, DOI 10.1007/BF02384316. MR727350 (85h:47058)

[19] L. Hörmander, *The analysis of linear partial differential operators.* Vols. III–IV, Springer-Verlag, New York, Berlin, 1985.

[20] Hitoshi Kumano-go, *A calculus of Fourier integral operators on $R^n$ and the fundamental solution for an operator of hyperbolic type*, Comm. Partial Differential Equations **1** (1976), no. 1, 1–44. MR0397482 (53 #1341)

[21] Hitoshi Kumano-go, *Pseudodifferential operators*, MIT Press, Cambridge, Mass., 1981. Translated from the Japanese by the author, Rémi Vaillancourt and Michihiro Nagase. MR666870 (84c:35113)

[22] Douglas S. Kurtz and Richard L. Wheeden, *Results on weighted norm inequalities for multipliers*, Trans. Amer. Math. Soc. **255** (1979), 343–362, DOI 10.2307/1998180. MR542885 (81j:42021)

[23] Anders Melin and Johannes Sjöstrand, *Fourier integral operators with complex-valued phase functions*, Fourier integral operators and partial differential equations (Colloq. Internat.,

Univ. Nice, Nice, 1974), Springer, Berlin, 1975, pp. 120–223. Lecture Notes in Math., Vol. 459. MR0431289 (55 #4290)

[24] Akihiko Miyachi, *On some estimates for the wave equation in* $L^p$ *and* $H^p$, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), no. 2, 331–354. MR586454 (83g:35060)

[25] Juan C. Peral, $L^p$ *estimates for the wave equation*, J. Funct. Anal. **36** (1980), no. 1, 114–145, DOI 10.1016/0022-1236(80)90110-X. MR568979 (81k:35089)

[26] Michael Ruzhansky, *On the sharpness of Seeger-Sogge-Stein orders*, Hokkaido Math. J. **28** (1999), no. 2, 357–362. MR1698428 (2000e:35247)

[27] M. V. Ruzhanskiĭ, *Singularities of affine fibrations in the theory of regularity of Fourier integral operators*, Uspekhi Mat. Nauk **55** (2000), no. 1(331), 99–170, DOI 10.1070/rm2000v055n01ABEH000250 (Russian, with Russian summary); English transl., Russian Math. Surveys **55** (2000), no. 1, 93–161. MR1751819 (2001i:35317)

[28] M. Ruzhansky, *Regularity theory of Fourier integral operators with complex phases and singularities of affine fibrations*, CWI Tract, vol. 131, Stichting Mathematisch Centrum Centrum voor Wiskunde en Informatica, Amsterdam, 2001. MR1837457 (2002f:58048)

[29] Michael Ruzhansky and Mitsuru Sugimoto, *Global* $L^2$*-boundedness theorems for a class of Fourier integral operators*, Comm. Partial Differential Equations **31** (2006), no. 4-6, 547–569, DOI 10.1080/03605300500455958. MR2233032 (2007e:35310)

[30] Michael Ruzhansky and Mitsuru Sugimoto, *A smoothing property of Schrödinger equations in the critical case*, Math. Ann. **335** (2006), no. 3, 645–673, DOI 10.1007/s00208-006-0757-4. MR2221126 (2007e:35237)

[31] Michael Ruzhansky and Mitsuru Sugimoto, *Global calculus of Fourier integral operators, weighted estimates, and applications to global analysis of hyperbolic equations*, Pseudo-differential operators and related topics, Oper. Theory Adv. Appl., vol. 164, Birkhäuser, Basel, 2006, pp. 65–78, DOI 10.1007/3-7643-7514-0_5. MR2243967 (2007d:35299)

[32] Michael Ruzhansky and Mitsuru Sugimoto, *Weighted Sobolev* $L^2$ *estimates for a class of Fourier integral operators*, Math. Nachr. **284** (2011), no. 13, 1715–1738, DOI 10.1002/mana.200910080. MR2832678 (2012h:45012)

[33] Andreas Seeger, Christopher D. Sogge, and Elias M. Stein, *Regularity properties of Fourier integral operators*, Ann. of Math. (2) **134** (1991), no. 2, 231–251, DOI 10.2307/2944346. MR1127475 (92g:35252)

[34] Christopher D. Sogge, *Fourier integrals in classical analysis*, Cambridge Tracts in Mathematics, vol. 105, Cambridge University Press, Cambridge, 1993. MR1205579 (94c:35178)

[35] Elias M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy; Monographs in Harmonic Analysis, III. MR1232192 (95c:42002)

[36] Mitsuru Sugimoto, $L^p$*-boundedness of pseudodifferential operators satisfying Besov estimates. I*, J. Math. Soc. Japan **40** (1988), no. 1, 105–122, DOI 10.2969/jmsj/04010105. MR917398 (89g:47068)

[37] Mitsuru Sugimoto, *On some* $L^p$*-estimates for hyperbolic equations*, Ark. Mat. **30** (1992), no. 1, 149–163, DOI 10.1007/BF02384867. MR1171100 (94f:35079)

[38] Terence Tao, *The weak-type* $(1,1)$ *of Fourier integral operators of order* $-(n-1)/2$, J. Aust. Math. Soc. **76** (2004), no. 1, 1–21, DOI 10.1017/S1446788700008661. MR2029306 (2004j:42008)

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