TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 366, Number 6, June 2014, Pages 3275–3293
S 0002-9947(2013)06018-7
Article electronically published on November 5, 2013

WEIGHTED INVERSION OF GENERAL DIRICHLET SERIES

HELGE GLÖCKNER AND LUTZ G. LUCHT

Abstract. Inversion theorems of Wiener type are essential tools in analysis and number theory. We derive a weighted version of an inversion theorem of Wiener type for general Dirichlet series from that of Edwards from 1957, and we outline an alternative proof based on the duality theory of convex cones and extension techniques for characters of semigroups. Variants and arithmetical applications are described, including the case of multidimensional weighted generalized Dirichlet series.

1. Introduction

By $\mathcal{A} := \mathcal{A}(\Lambda) := \{a\colon \Lambda \to \mathbb{C} \text{ with } \|a\| < \infty\}$ we denote the class of complex-valued functions defined on an additive semigroup $\Lambda \subseteq [0,\infty)$ with $0 \in \Lambda$ and satisfying

$$\|a\| := \sum_{\lambda\in\Lambda} |a(\lambda)| < \infty.$$

Then, under the usual linear operations and the convolution defined by

$$c(\lambda) := (a * b)(\lambda) := \sum_{\substack{\lambda',\lambda''\in\Lambda\\ \lambda'+\lambda''=\lambda}} a(\lambda')\, b(\lambda'') \qquad (\lambda \in \Lambda), \tag{1}$$

$\mathcal{A}$ forms a commutative unitary Banach algebra. In fact, the convolution is well defined by (1) because of $\|c\| \le \|a\|\cdot\|b\| < \infty$ for $a, b \in \mathcal{A}$ with the above norm $\|\,.\,\|$, and the function ε given by $\varepsilon(\lambda) = \delta_{0,\lambda}$ with Kronecker's symbol $\delta\colon \Lambda^2 \to \{0,1\}$ serves as unity (cf. [16]).

Let $\mathbb{H} := \{s \in \mathbb{C} : \operatorname{Re} s > 0\}$ denote the open right half plane. Then the Banach algebra $\mathcal{A}$ is isomorphic to the Banach algebra $\widetilde{\mathcal{A}} = \widetilde{\mathcal{A}}(\Lambda)$ of absolutely convergent *general Dirichlet series*

$$\widetilde{a}(s) = \sum_{\lambda\in\Lambda} a(\lambda)\, e^{-\lambda s} \qquad (s \in \overline{\mathbb{H}})$$

associated with $a \in \mathcal{A}$, under the usual linear operations, pointwise multiplication and norm $\|\widetilde{a}\| := \|a\|$. This allows one to switch between complex sequences and Dirichlet series.

Received by the editors December 3, 2011 and, in revised form, August 3, 2012 and November 13, 2012.

2010 *Mathematics Subject Classification.* Primary 11M41; Secondary 30B50, 30J99, 46H99.

Key words and phrases. General Dirichlet series, weighted inversion, Banach algebra, dual cone, rational vector space, separation theorem, Hahn-Banach theorem, rational polytope, semigroup algebra.

The first author was supported by Deutsche Forschungsgemeinschaft, GZ: GL 357/5–2.

For instance, inversion in $\widetilde{\mathcal{A}}$ is equivalent to that in $\mathcal{A}$, and the inversion theorem of Edwards [5] may be formulated as

Theorem of Edwards (1957). *The multiplicative group of* $\mathcal{A}$ *is* $\mathcal{A}^* = \{a \in \mathcal{A} : 0 \notin \overline{\widetilde{a}(\mathbb{H})}\}$.

Remark 1. Special cases of Edwards' theorem trace back to Wiener [35] for Fourier and power series and to Hewitt and Williamson [15] for ordinary Dirichlet series. Elementary proofs of the latter were given by Spilker and Schwarz [32], and by Goodman and Newman [10].

We aim to investigate the convergence quality of $\widetilde{a}$ on $\overline{\mathbb{H}}$ and refine the above norm with the help of *weight functions* on Λ, i.e. functions $w\colon \Lambda \to (0,\infty)$ satisfying $w(0) = 1$ and $w(\lambda' + \lambda'') \le w(\lambda')\, w(\lambda'')$ for all $\lambda', \lambda'' \in \Lambda$. We let $\mathcal{W} = \mathcal{W}(\Lambda)$ be the set of all weight functions w on Λ which are *admissible* in the sense that

a) $w(\lambda) \ge 1$ for all $\lambda \in \Lambda$, and
b) $\lim_{k\to\infty} \sqrt[k]{w(k\lambda)} = 1$ for all $\lambda \in \Lambda$.[1]

Note that $\mathcal{W}$ is closed under pointwise multiplication. If $w \in \mathcal{W}$, then the functions $a \in \mathcal{A}$ with the weighted norm

$$\|a\|_w := \sum_{\lambda\in\Lambda} |a(\lambda)|\, w(\lambda) < \infty$$

form a commutative unitary Banach algebra $\mathcal{A}_w = \mathcal{A}_w(\Lambda) \subseteq \mathcal{A} = \mathcal{A}_1$ (see Section 3 for a more detailed discussion of weighted semigroup algebras, even for arbitrary weights).

Since weight functions are submultiplicative, $\|\,.\,\|_w$ is also submultiplicative. Conditions a) and b) ensure that the spectrum of $\mathcal{A}_w$ can be identified with the set of all bounded characters of Λ (see Section 4). They also restrict the growth of admissible weight functions w such that in relevant cases, e.g. for non-decreasing weight functions, we obtain

$$w(\lambda) \ll e^{\vartheta\lambda} \quad \text{for every } \vartheta > 0 \tag{2}$$

(see Remark 3 below).[2] Whenever (2) holds, the abscissa of absolute convergence of $\widetilde{a}$ equals that of $(aw)^{\sim}$ for any $a\colon \Lambda \to \mathbb{C}$. Moreover, for non-decreasing $w \in \mathcal{W}$, the remainder term estimate

$$\sum_{\substack{\lambda\in\Lambda\\ \lambda\ge\ell}} |a(\lambda)|\, e^{-\lambda s} = \mathrm{o}\Big(\frac{1}{w(\ell)}\Big) \qquad (\ell\in\Lambda,\ \ell\to\infty)$$

holds uniformly for $s \in \overline{\mathbb{H}}$.[3] [4] For instance, $w(\lambda) = (\lambda+1)^c$ defines an admissible ascending weight function for any $c \ge 0$. Note that for $a \in \mathcal{A} = \mathcal{A}_1$ the derivative

$$\widetilde{a}\,'(s) = -\sum_{\lambda\in\Lambda} \lambda\, a(\lambda)\, e^{-\lambda s}$$

[1] The sum $\lambda + \cdots + \lambda$ with $k \in \mathbb{N}$ summands $\lambda \in \Lambda$ is written as $k\lambda$.

[2] For $f\colon \Lambda \to \mathbb{C}$ and $g\colon \Lambda \to \mathbb{R}_+$ we write $f \ll g$ if $\sup_{\lambda\in\Lambda} \frac{|f(\lambda)|}{g(\lambda)} < \infty$ (*Vinogradov* $\ll$ *symbol*).

[3] For $f\colon \Lambda \to \mathbb{C}$ and $g\colon \Lambda \to \mathbb{R}_+$ we write $f(\lambda) = \mathrm{o}(g(\lambda))$ for $\lambda \to \infty$ if $\lim_{\lambda\to\infty} \frac{f(\lambda)}{g(\lambda)} = 0$ (*Landau* o *symbol*).

[4] The series is dominated by $\frac{1}{w(\ell)} \sum_{\lambda\ge\ell} |a(\lambda)|\, w(\lambda)$.

of $\widetilde{a}$ converges absolutely only for $s \in \mathbb{H}$, in general. But for $w(\lambda) = (1+\lambda)^k$, $k \in \mathbb{N}$, the Dirichlet series $\widetilde{a}$ of $a \in \mathcal{A}_w$ *and its derivatives up to order* k converge absolutely for $s \in \overline{\mathbb{H}}$, and the quality of convergence is described by

$$\sum_{\substack{\lambda\in\Lambda \\ \lambda\geq\ell}} \lambda^j \, |a(\lambda)| \, e^{-\lambda s} = \mathrm{o}\Big((\ell+1)^{j-k}\Big) \qquad (\ell \in \Lambda,\ \ell \to \infty),$$

uniformly for $s \in \overline{\mathbb{H}}$ and $j = 0, \ldots, k$.

Remark 2. Condition b) is equivalent to $\inf \big\{ \sqrt[k]{w(k\lambda)} : k \in \mathbb{N} \big\} = 1$.

Obviously b) yields the infimum condition. Conversely, the infimum condition implies that for every $\epsilon > 0$ there exists k_0 such that $\sqrt[k_0]{w(k_0\lambda)} < 1+\epsilon$. Since every $k \geq k_0$ can be written as $k = n_k k_0 + r_k$ with $0 \leq r_k < k_0$, we have

$$\sqrt[k]{w(k\lambda)} \leq \sqrt[k]{w(r_k\lambda)} \, \sqrt[k]{w(k_0\lambda)^{n_k}}.$$

Here the first term on the right tends to 1 as $k \to \infty$ (as there are only finitely many choices for r_k), and the second term can be estimated via

$$\sqrt[k]{w(k_0\lambda)^{n_k}} < \sqrt[k]{(1+\epsilon)^{k_0 n_k}} = (1+\epsilon)^{k_0 n_k/k} \leq 1+\epsilon.$$

Remark 3. If $w \in \mathcal{W}(\Lambda)$ is *slowly decreasing*,[5] then $w(\lambda) \ll e^{\vartheta\lambda}$ for every $\vartheta > 0$.

To see this, fix some $\lambda_1 \in \Lambda \setminus \{0\}$. Then, for $\vartheta > 0$ and $\lambda \in \Lambda$ sufficiently large, there exists $k \in \mathbb{N}$ such that $(k-1)\lambda_1 < \lambda \leq k\lambda_1$ and

$$w(\lambda) = w(k\lambda_1) - (w(k\lambda_1) - w(\lambda)) \leq w(k\lambda_1) + 1 \ll e^{\vartheta\lambda_1 k} + 1 \ll e^{\vartheta\lambda}.$$

Remark 4. In general, conditions a) and b) do *not* imply that $w(\lambda) \ll e^{\vartheta\lambda}$ for any $\vartheta > 0$.[6]

Prototypes of general Dirichlet series are power series for $\Lambda = \mathbb{N}_0$ and ordinary Dirichlet series for $\Lambda = \log \mathbb{N}$. Both of these additive semigroups admit *free generating sets*,[7] namely $\{1\}$ and $\log \mathbb{P}$, respectively, where $\mathbb{P} = \{2, 3, 5, \ldots\}$ denotes the set of primes.

The purpose of this note is to outline two different proofs of the following

Theorem 1. *Let* $\Lambda \subseteq [0, \infty)$ *be an additive semigroup with* $0 \in \Lambda$. *Then, for* $w \in \mathcal{W}$, *the Banach algebra* $\mathcal{A}_w = \mathcal{A}_w(\Lambda)$ *has the multiplicative group*

$$\mathcal{A}_w^* = \mathcal{A}_w^*(\Lambda) = \{a \in \mathcal{A}_w : 0 \notin \overline{\widetilde{a}(\mathbb{H})}\}.$$

Theorem 1 is a weighted inversion theorem of Wiener type for general Dirichlet series. The necessity of the inversion condition is immediate, for if $1/\widetilde{a}(s)$ can be represented by an absolutely convergent Dirichlet series, then it is bounded on $\mathbb{H}$, and hence $\widetilde{a}(s)$ must be bounded away from zero on $\mathbb{H}$. Note that the inversion condition does not depend on the weight function $w \in \mathcal{W}$.

In Section 4 we derive the proof of Theorem 1 from the Theorem of Edwards and provide an alternative proof based on an approximation of multiplicative linear

---

[5]A function $f\colon \Lambda \to [0, \infty)$ is called *slowly decreasing* if $\liminf\,(f(\lambda) - f(\lambda')) \geq 0$ whenever $1 \leq \lambda/\lambda' \to 1$ and $\lambda \to \infty$.

[6]This was erroneously stated in [22].

[7]A subset $\emptyset \neq \mathrm{B} \subseteq \Lambda$ is called a *free generating set* of an additive semigroup $\Lambda \subseteq [0, \infty)$, if every $\lambda \in \Lambda$ has a unique representation of the form $\lambda = \sum_{\beta\in\mathrm{B}} \nu_\beta \beta$ with $\nu_\beta \in \mathbb{N}_0$ and $\nu_\beta \neq 0$ for only finitely many $\beta \in \mathrm{B}$. Then B is $\mathbb{Q}$-linearly independent in $\mathbb{R}$.

functionals on $\mathcal{A}_w$ (taking $w = 1$, this also gives a new proof for the Theorem of Edwards). The required Density Lemma is established step by step in Sections 5 to 7. For discrete additive semigroups $\Lambda \subseteq [0, \infty)$, the above Theorem 1 and the strategy of its proof are described in a preliminary (incomplete) paper of Lucht and Reifenrath [24]. The Lévy generalization and a multidimensional Lévy version of Theorem 1, Theorems 3 and 5, are established in Section 8, which also deals with an arithmetical application (Proposition 3).

For further recent studies of weighted convolution algebras of subsemigroups of $\mathbb{R}$ (with a different thrust), the reader is referred to [3]. In the case of groups (rather than semigroups), the admissibility condition b) is related to symmetry properties of weighted group algebras, and has been attributed to Gelfand, Naimark, Raikov and Šhilov in some recent works (see [6, p. 796], [7, Definition 1.2(a)], [18, Definition 3.1(a)]).

2. Tools from Gelfand's theory

With any commutative Banach algebra A Gelfand's theory associates the space $\Delta(A)$ of homomorphisms of A to the complex field, i.e. the non-trivial multiplicative linear functionals $h\colon A \to \mathbb{C}$. The supremum norm on $\Delta(A)$ is related to the norm on A. This helps to characterize the invertible elements of A (cf., for instance, [28, Theorems 18.3 and 18.17]):

Theorem 2. *In a commutative Banach algebra with unity u the following assertions hold:*

a) *If $a \in A$ satisfies $\|a - u\| < 1$, then a is invertible in A.*
b) *If $a \in A$ and $h \in \Delta(A)$, then $|h(a)| \le \|a\|$.*
c) *If $a \in A$ is invertible in A, then $h(a) \neq 0$ for all $h \in \Delta(A)$, and vice versa.*

From b) we infer that every $h \in \Delta(A)$ is continuous. The *spectrum* $\sigma(a)$ of $a \in A$ denotes the set of all $\lambda \in \mathbb{C}$ such that $a - \lambda u$ is *not* invertible. Then c) relates the spectrum $\sigma(a)$ to the space $\Delta(A)$ of all non-trivial multiplicative linear functionals $h\colon A \to \mathbb{C}$ by

$$\sigma(a) = \{h(a) : h \in \Delta(A)\}.$$

In particular, $a \in A$ is invertible if and only if $0 \notin \sigma(a)$. This suggests identifying the invertible elements of A by determining all non-trivial multiplicative linear functionals $h \in \Delta(A)$.

The following lemma taken from Edwards [5, 11.4.5] (see also Rudin [29, Exercise 11.5]) generalizes Theorem 2 c) by replacing the inversion map with a holomorphic function.

Composition Lemma. *Let $a \in A$, and let f be a holomorphic function defined on a region $G \subseteq \mathbb{C}$ such that $\sigma(a) \subseteq G$. Then there exists an element $b \in A$ such that $h(b) = f(h(a))$ for every $h \in \Delta(A)$.*

3. Weighted semigroup algebras and their spectra

Let $(\Lambda, \cdot)$ be an arbitrary semigroup with unit element e. For the moment, multiplicative notation is used, as Λ need not be commutative. We call a function $w\colon \Lambda \to (0, \infty)$ a *weight function* if $w(e) = 1$ and w is submultiplicative, i.e. $w(\lambda\lambda') \le w(\lambda)w(\lambda')$ for all $\lambda, \lambda' \in \Lambda$. Then a Banach algebra $\mathcal{A}_w(\Lambda)$ can be defined, which is a weighted analogue of the Banach algebra $\ell_1(\Lambda)$ introduced by Hewitt and Zuckerman [16] (see [3]; cf. [25, p. 70] for the case of semigroups with involution). In particular, the construction applies to additive subsemigroups of $[0, \infty)$ (as discussed in the introduction), or additive subsemigroups of $[0, \infty)^r$ (as needed in connection with multidimensional Dirichlet series).

Recall that the sum $\sum_{i \in I} a_i$ of a family $(a_i)_{i \in I}$ of numbers $a_i \in [0, \infty) \cup \{\infty\}$ is defined as the supremum of the sums $\sum_{i \in F} a_i$ over finite index sets $F \subseteq I$. A family $(a_i)_{i \in I}$ of complex numbers is called *absolutely summable* if $\sum_{i \in I} |a_i| < \infty$. Then $a_i \neq 0$ for only countably many $i \in I$, and the net of finite partial sums $\sum_{i \in F} a_i$ converges. Its limit is denoted by $\sum_{i \in I} a_i$ (cf. [4, Chapter V, §3], [30, Chapter III, Exercise 23]). Now the weighted semigroup algebras are obtained as follows ([3]; cf. [25]):

Proposition 1. *Let $(\Lambda, \cdot)$ be a semigroup with a unit element and w be a weight function on Λ. Let $\mathcal{A}_w(\Lambda)$ be the set of all families $a = (a(\lambda))_{\lambda \in \Lambda}$ of complex numbers such that $\|a\|_w := \sum_{\lambda \in \Lambda} |a(\lambda)| w(\lambda) < \infty$. If $a, b \in \mathcal{A}_w(\Lambda)$, then the following holds:*

a) *For each $\lambda \in \Lambda$ the numbers $a(\lambda')b(\lambda'')$ for $(\lambda', \lambda'') \in \Lambda \times \Lambda$ with $\lambda'\lambda'' = \lambda$ form an absolutely summable family. Thus $c(\lambda) := \sum_{\lambda'\lambda''=\lambda} a(\lambda')b(\lambda'')$ is defined.*

b) *The family $a * b := c$ is in $\mathcal{A}_w(\Lambda)$.*

The multiplication $*$ makes $(\mathcal{A}_w(\Lambda), \|.\|_w)$ a unital Banach algebra.

For the remainder of this section we return to the additive notation and let $(\Lambda, +)$ be a commutative semigroup with neutral element 0. By a *character* of Λ we mean a homomorphism ψ of $(\Lambda, +)$ to $(\mathbb{C}, \cdot)$ such that $\psi(0) = 1$. If w is a weight function on Λ, we let $\widehat{\Lambda}_w$ be the set of all characters ψ of Λ which are *w-bounded* in the sense that

$$|\psi(\lambda)| \leq w(\lambda) \quad \text{for all } \lambda \in \Lambda$$

(cf. [1] and [25] for the case of semigroups with involution).

Complex homomorphisms of $\mathcal{A}_w = \mathcal{A}_w(\Lambda)$ and w-bounded characters of Λ are closely related. To see this, we again use Kronecker's δ to define an element $\delta_\lambda \colon \Lambda \to \mathbb{C}$ with $\mu \mapsto \delta_{\lambda,\mu}$ in $\mathcal{A}_w$, such that $\|\delta_\lambda\|_w = w(\lambda)$. If $h \in \Delta(\mathcal{A}_w)$, then

$$\psi_h \colon (\Lambda, +) \to (\mathbb{C}, \cdot), \quad \psi_h(\lambda) := h(\delta_\lambda)$$

is a homomorphism, for $\delta_0 = \varepsilon$ and $\delta_{\lambda+\lambda'} = \delta_\lambda * \delta_{\lambda'}$. Since $|\psi_h(\lambda)| = |h(\delta_\lambda)| \leq \|\delta_\lambda\|_w = w(\lambda)$, the character ψ_h is w-bounded. Conversely, let $\psi \in \widehat{\Lambda}_w$. Then the family $(a(\lambda)\psi(\lambda))_{\lambda \in \Lambda}$ is absolutely summable for each $a \in \mathcal{A}_w$, as

$$\sum_{\lambda \in \Lambda} |a(\lambda)\, \psi(\lambda)| \leq \sum_{\lambda \in \Lambda} |a(\lambda)|\, w(\lambda) = \|a\|_w < \infty. \tag{3}$$

We can therefore define a function $h_\psi \colon \mathcal{A}_w \to \mathbb{C}$ via

$$h_\psi(a) := \sum_{\lambda \in \Lambda} a(\lambda)\, \psi(\lambda) \quad \text{for all } a \in \mathcal{A}_w. \tag{4}$$

Then h_ψ is linear and of operator norm ≤ 1 by (3), and hence continuous. In fact, $h_\psi \in \Delta(\mathcal{A}_w)$. To see this, it remains to show that $h_\psi(a * b) = h_\psi(a)h_\psi(b)$ for all $a, b \in \mathcal{A}_w$. It suffices to assume that $a = \delta_\lambda$ and $b = \delta_{\lambda'}$ with $\lambda, \lambda' \in \Lambda$ (as such elements span a dense vector subspace of $\mathcal{A}_w$). But then $h_\psi(a * b) = \psi(\lambda\lambda') = \psi(\lambda)\psi(\lambda') = h_\psi(a)h_\psi(b)$ indeed. We readily deduce:

Proposition 2. *Let $(\Lambda, +)$ be a commutative semigroup with neutral element 0, and w be a weight function on Λ. Let $\mathcal{A}_w = \mathcal{A}_w(\Lambda)$. Then the map*

$$\Delta(\mathcal{A}_w) \to \widehat{\Lambda}_w, \quad h \mapsto \psi_h \tag{5}$$

is a bijection, with inverse $\psi \mapsto h_\psi$.

We say that a weight function w on a commutative semigroup $(\Lambda, +)$ is *admissible* if it satisfies the conditions a) and b) described in the introduction. We write $\mathcal{W}(\Lambda)$ (or simply $\mathcal{W}$) for the set of all admissible weight functions on Λ.

Lemma 1. *Let w be an admissible weight function on a commutative semigroup $(\Lambda, +)$ with neutral element 0. Let ψ be a character of Λ. Then ψ is w-bounded if and only if ψ is a bounded function.*

Proof. If ψ is w-bounded, then $|\psi(\lambda)| = \sqrt[k]{|\psi(kn)|} \leq \sqrt[k]{w(k\lambda)} \to 1$ as $k \to \infty$, and thus $|\psi(\lambda)| \leq 1$ for each $\lambda \in \Lambda$. Conversely, assume that ψ is bounded, say $|\psi(\lambda)| \leq C$ with $C > 0$. Then $|\psi(\lambda)| = \sqrt[k]{|\psi(k\lambda)|} \leq \sqrt[k]{C}$ for each $k \in \mathbb{N}$, and thus $|\psi(\lambda)| \leq 1 \leq w(\lambda)$, using the fact that $\sqrt[k]{C} \to 1$. □

For future use, we write $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$. If ψ is a bounded character, then ψ only takes values in the closed unit disk $\overline{\mathbb{U}}$.

4. Two proofs of Theorem 1

According to Theorem 2 and the definition of the spectrum it suffices to show that $0 \notin \sigma(a)$ for $a \in \mathcal{A}_w$ such that $0 \notin \overline{\widetilde{a}(\mathbb{H})}$ or, equivalently, $h(a) \neq 0$ for all $h \in \Delta(\mathcal{A}_w)$. To enable this, it is useful to have a description of functionals $h \in \Delta(\mathcal{A}_w)$. Combining Proposition 2 and Lemma 1 we get the following lemma (to be found in Lucht and Reifenrath [24] for discrete $\Lambda \subseteq [0, \infty)$):

Representation Lemma. *Let $(\Lambda, +)$ be a commutative semigroup with identity 0, and w be an admissible weight function on Λ. Then to every $h \in \Delta(\mathcal{A}_w)$ there corresponds a bounded character ψ of Λ such that*

$$h(a) = \sum_{\lambda \in \Lambda} a(\lambda)\,\psi(\lambda) \quad \textit{for all } a \in \mathcal{A}_w. \tag{6}$$

Conversely, every bounded character $\psi\colon \Lambda \to \mathbb{C}$ determines a unique $h \in \Delta(\mathcal{A}_w)$ with (6).

First we derive Theorem 1 from the special case $w = 1$, due to Edwards [5].

First proof of Theorem 1. Since $\mathcal{A}_w$ is a Banach subalgebra of $\mathcal{A}_1$ and the characterization (6) is independent of w, the Representation Lemma shows that

$$\mathcal{A}_w^* = \Big\{a \in \mathcal{A}_w : \sum_{\lambda \in \Lambda} a(\lambda)\,\psi(\lambda) \neq 0 \ \text{ for all characters } \psi \text{ of } \Lambda\Big\}.$$

Thus $\mathcal{A}_w^* = \mathcal{A}_1^* \cap \mathcal{A}_w$. By Edwards' theorem [5],

$$\mathcal{A}_1^* = \big\{a \in \mathcal{A}_1 : 0 \notin \overline{\widetilde{a}(\mathbb{H})}\big\},$$

and the fact that $\widetilde{a}$ does not depend on w, the assertion follows. □

A different option for proving Theorem 1 without recourse to the Theorem of Edwards is based on a topological linkage of the image set $\widetilde{a}(\mathbb{H})$ and the spectrum $\sigma(a)$ of functions $a \in \mathcal{A}_w$, namely an approximation of the functions $h \in \Delta(\mathcal{A}_w)$ by the functions $h_s \in \Delta(\mathcal{A}_w)$ associated with the specific characters $\lambda \mapsto \psi_s(\lambda) = e^{-\lambda s}$ for $s \in \overline{\mathbb{H}}$.

Density Lemma. *Let $\Lambda \subseteq [0, \infty)$ be an additive semigroup with $0 \in \Lambda$, and let $w \in \mathcal{W}$. Then for any $a \in \mathcal{A}_w$ the set $\widetilde{a}(\mathbb{H})$ is dense in $\sigma(a)$.*

The Density Lemma yields the announced alternative proof of Theorem 1.

Second proof of Theorem 1. According to Theorem 2 c) the invertibility of a in $\mathcal{A}_w$ follows from $\sigma(a) \subseteq \overline{\widetilde{a}(\mathbb{H})}$ and $0 \notin \overline{\widetilde{a}(\mathbb{H})}$. □

It remains to verify the Density Lemma.

5. Proof of the Density Lemma, Part 1

First we establish a special case of the Density Lemma, assuming, in addition, that the semigroup Λ is *free*, i.e. Λ has a free generating set B.

Let $a \in \mathcal{A}_w$, $\vartheta > 0$, and $h \in \Delta(\mathcal{A}_w)$. Denote by ψ the bounded character of Λ associated with h, as in the Representation Lemma. Then $\psi(\Lambda) \subseteq \overline{\mathbb{U}}$. In order to show that there exists an $s \in \mathbb{H}$ satisfying

$$|h_s(a) - h(a)| < 3\vartheta,$$

it suffices to verify that, for any finite subset $\Gamma \subseteq \Lambda$, the estimate

$$\left| \sum_{\lambda \in \Gamma} a(\lambda)\, e^{-\lambda s} - \sum_{\lambda \in \Gamma} a(\lambda)\, \psi(\lambda) \right| < \vartheta \tag{7}$$

holds with suitably chosen $s \in \mathbb{H}$. In fact, there exists a finite subset $\Gamma \subseteq \Lambda$ such that $\sum_{\lambda \in \Lambda \setminus \Gamma} w(\lambda) |a(\lambda)| < \vartheta$ so that

$$\left| \sum_{\lambda \in \Lambda \setminus \Gamma} a(\lambda)\, e^{-\lambda s} - \sum_{\lambda \in \Lambda \setminus \Gamma} a(\lambda)\, \psi(\lambda) \right| \leq 2 \sum_{\lambda \in \Lambda \setminus \Gamma} |a(\lambda)| \leq 2\vartheta$$

for each $s \in \mathbb{H}$, from which the assertion follows.

Let $\mathfrak{b} = (\beta_1, \ldots, \beta_k)$ consist of generators $0 < \beta_1, \ldots, \beta_k \in$ B such that every $\lambda \in \Gamma$ can be expressed in the form

$$\lambda = \sum_{\kappa} \nu_\kappa \beta_\kappa \tag{8}$$

with $\nu_\kappa \in \mathbb{N}_0$ for $1 \leq \kappa \leq k$. Then

$$P(\mathfrak{z}) = \sum_{\lambda \in \Gamma} a(\lambda)\, e^{-\lambda s}$$

can be regarded as a polynomial of $\mathfrak{z} = (z_1, \ldots, z_k) \in \overline{\mathbb{U}}^k$ with variables $z_\kappa = e^{-\beta_\kappa s}$ for $1 \leq \kappa \leq k$. Now (8) leads to

$$\psi(\lambda) = \prod_{1 \leq \kappa \leq k} \psi(\beta_\kappa)^{\nu_\kappa}$$

and

$$\sum_{\lambda \in \Gamma} a(\lambda)\, \psi(\lambda) = P\big(\psi(\beta_1), \ldots, \psi(\beta_k)\big).$$

Since $|\psi(\beta_\kappa)| \le 1$ for all κ, the following generalized version of a lemma of Spilker and Schwarz [32, Hilfssatz 5.1] (see also Hewitt and Williamson [15, Lemma 2]) yields the existence of $s \in \mathbb{H}$ with (7), which establishes the Density Lemma as well as Theorem 1 in the special case of free semigroups Λ.

Lemma 2. *For $k \in \mathbb{N}$ let $\mathfrak{v}\colon \overline{\mathbb{H}} \to \overline{\mathbb{U}}^k$ be defined by*

$$\mathfrak{v}(s) = e^{-\mathfrak{b}s} := (e^{-\beta_1 s}, \dots, e^{-\beta_k s})$$

with $\mathbb{Q}$-linearly independent numbers $\beta_\kappa > 0$ for $1 \le \kappa \le k$. If $f\colon \overline{\mathbb{U}}^k \to \mathbb{C}$ is continuous and holomorphic on $\mathbb{U}^k$, then $f(\mathfrak{v}(\overline{\mathbb{H}}))$ is a dense subset of $f(\overline{\mathbb{U}}^k)$.

Proof. The assertion is trivial for constant functions $f\colon \overline{\mathbb{U}}^k \to \mathbb{C}$.

Let f be non-constant. It suffices to show that for any $\vartheta > 0$ there is some $s \in \overline{\mathbb{H}}$ such that $|g(\mathfrak{v}(s))| < \vartheta$, where $g(\mathfrak{z}) = f(\mathfrak{z}) - c$ for $\mathfrak{z} = (z_1, \dots, z_k) \in \overline{\mathbb{U}}^k$ with an arbitrary $c \in f(\overline{\mathbb{U}}^k)$. Suppose, to the contrary, that there is some $\vartheta > 0$ such that

$$|g(\mathfrak{z})| \ge \vartheta \quad \text{for all } \mathfrak{z} \in \mathfrak{v}(\overline{\mathbb{H}}). \tag{9}$$

First we show that for any $\vartheta > 0$ there is some $t \in \mathbb{R}$ such that

$$\left| z_\kappa - e^{-\beta_\kappa(\sigma + it)} \right| < \vartheta \qquad (\kappa = 1, \dots, k), \tag{10}$$

if $|z_\kappa| = e^{-\beta_\kappa \sigma}$ for $\kappa = 1, \dots, k$. It suffices to verify (10) for $\sigma = 0$ (i.e., $|z_k| = 1$): The Kronecker approximation theorem (cf. Hardy and Wright [12, Theorem 444]) applied to the $\mathbb{Q}$-linearly independent set $\{\beta_1, \dots, \beta_k\}$ entails that for any $\vartheta > 0$ there exist numbers $t \in \mathbb{R}$ and $m_1, \dots, m_k \in \mathbb{Z}$ satisfying

$$\left| \frac{t}{2\pi}\, \beta_\kappa - m_\kappa - \frac{\arg z_\kappa}{2\pi} \right| < \frac{\vartheta}{2} \qquad (\kappa = 1, \dots, k).$$

For $\vartheta > 0$ sufficiently small, it follows that

$$\left| e^{i \arg z_\kappa} - e^{it\beta_\kappa} \right| < \vartheta \qquad (\kappa = 1, \dots, k),$$

as asserted. Hence, for any $\sigma \ge 0$, $\mathfrak{v}(\sigma + i\mathbb{R})$ is a dense subset of the poly-circle

$$e^{-\mathfrak{b}\sigma} \partial\mathbb{U} := \{\mathfrak{z} \in \mathbb{C}^k : |z_\kappa| = e^{-\beta_\kappa \sigma} \ \text{ for } \ 1 \le \kappa \le k\},$$

and

$$|g(\mathfrak{z})| \ge \vartheta \quad \text{for all } \mathfrak{z} \in e^{-\mathfrak{b}\sigma} \partial\mathbb{U}. \tag{11}$$

As $\mathfrak{o} = (0, \dots, 0) \in \overline{\mathbb{U}}^k$ is a limit point of $\mathfrak{v}(\overline{\mathbb{H}})$, we have $|g(\mathfrak{o})| \ge \vartheta$. Therefore we may define

$$\sigma_0 = \inf\left\{\sigma \ge 0 : |g(\mathfrak{z})| \ge \tfrac{1}{2}\,\vartheta \text{ for all } \mathfrak{z} \in e^{-\mathfrak{b}\sigma}\overline{\mathbb{U}}\right\},$$

where $e^{-\mathfrak{b}\sigma}\overline{\mathbb{U}} := e^{-\beta_1\sigma}\overline{\mathbb{U}} \times \cdots \times e^{-\beta_k\sigma}\overline{\mathbb{U}}$. Then g has no zero in the compact poly-disc $e^{-\mathfrak{b}\sigma_0}\overline{\mathbb{U}}$, and $1/g$ represents a continuous function on $e^{-\mathfrak{b}\sigma_0}\overline{\mathbb{U}}$ that is holomorphic on $e^{-\mathfrak{b}\sigma_0}\mathbb{U}$. By applying the multidimensional maximum modulus principle[8] we obtain from (11) that

$$\max\left\{\left|\frac{1}{g(\mathfrak{z})}\right| : \mathfrak{z} \in e^{-\mathfrak{b}\sigma_0}\overline{\mathbb{U}}\right\} = \max\left\{\left|\frac{1}{g(\mathfrak{z})}\right| : \mathfrak{z} \in e^{-\mathfrak{b}\sigma_0}\partial\mathbb{U}\right\} \leq \frac{1}{\vartheta}\,.$$

Hence $|g(\mathfrak{z})| \geq \vartheta$ for all $\mathfrak{z} \in e^{-\mathfrak{b}\sigma_0}\overline{\mathbb{U}}$.

This gives the desired contradiction, as $e^{-\mathfrak{b}\sigma_0}\overline{\mathbb{U}}$ contains some point $\mathfrak{z}$ with $|g(\mathfrak{z})| < \vartheta$, in case of $\sigma_0 > 0$ by the definition of σ_0 and in case of $\sigma_0 = 0$ by $c \in f(\overline{\mathbb{U}}^k)$. □

6. Proof of the Density Lemma, Part 2

To complete the proof of the Density Lemma we have to remove the assumption on Λ to be free. As in the preceding section, for $\vartheta > 0$ and a finite subset $\Gamma \subseteq \Lambda$, we merely need to find $s \in \mathbb{H}$ with (7). Since only the values of ψ on Γ enter (7), the next lemma allows Λ to be replaced with a free semigroup [B], to which the special case from Section 5 applies.

We shall use the following terminology and facts concerning convex cones: A subset C of a finite-dimensional real vector space W is called a *convex cone* if C is convex and $[0,\infty)\cdot C \subseteq C$. Then C is a semigroup under addition. If $C \neq \emptyset$, then $C - C$ is a vector subspace of W and C has non-empty interior in $C - C$ (cf. [25, Proposition V.1.4 (ii)]). The *dimension of* C is defined as the dimension of $C - C$. A convex cone $F \subseteq C$ is called a *face of* C if $x + y \in F$ for elements $x, y \in C$ implies $x, y \in F$. A face of the form $[0,\infty)x$ with $x \neq 0$ is called an *extreme ray* of C. A convex cone $C \subseteq W$ is called *polyhedral* if it is generated by a finite set $E \subseteq W$ (i.e., C is of the form (12) below).[9] Then every face $F \subseteq C$ is generated by $F \cap E$, as is well known (cf. [2, Theorems 7.2 and 7.3]).[10] In particular, every extreme ray of C is of the form $[0,\infty)x$ with some $x \in E$.

Extension Lemma. *Let $\psi\colon \Lambda \to \mathbb{C}$ be a bounded character of an additive semigroup $\Lambda \subseteq [0,\infty)$ with $0 \in \Lambda$, and let $\Gamma \subseteq \Lambda$ be a finite subset. Then there exists a $\mathbb{Q}$-linearly independent set $\mathrm{B} = \{\beta_1, \ldots, \beta_k\} \subseteq (0,\infty)$ such that*

$$\Gamma \subseteq [\mathrm{B}] := \mathbb{N}_0\beta_1 + \cdots + \mathbb{N}_0\beta_k,$$

and a bounded character $\phi\colon [\mathrm{B}] \to \mathbb{C}$ which coincides with ψ on Γ.

The proof requires some notional arrangements.

Let $\mathbb{Q}^+ = \{q \in \mathbb{Q} : q > 0\}$ and $\mathbb{Q}_0^+ = \mathbb{Q}^+ \cup \{0\}$. Given a $\mathbb{Q}$-vector space V, we let $V_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Q}} V$ be the $\mathbb{R}$-vector space[11] associated with the $\mathbb{Q}$-vector space V (cf. [19, Chapt. XVI, § 4]). If $E = \{x_1, \ldots, x_k\}$ is a finite subset of V, we write $[E] := \mathbb{N}_0 x_1 + \cdots + \mathbb{N}_0 x_k$ for the submonoid of $(V, +)$ generated by E,

$$\mathrm{conv}_{\mathbb{Q}}(E) := \left\{q_1x_1 + \cdots + q_kx_k : q_1, \ldots, q_k \in \mathbb{Q}_0^+,\ q_1 + \cdots + q_k = 1\right\}$$

[8] *Let $G = G_1 \times \cdots \times G_k$ with non-empty bounded regions $G_1, \ldots, G_k \subset \mathbb{C}$. If $f\colon \overline{G} \to \mathbb{C}$ is continuous and holomorphic on G, then $|f|$ takes its maximum value on $\partial G_1 \times \cdots \times \partial G_k$.* A short inductive proof uses the ordinary maximum modulus principle combined with the Weierstraß convergence theorem.

[9] See [25, Chapter V.1] for basic facts on polyhedral cones, as first spelled out in [34].

[10] Omitting only a trivial case, consider a non-zero element $x \in F$. Then $x = r_1x_1 + \cdots + r_kx_k$ with elements $x_1, \ldots, x_k \in E$ and $r_1, \ldots, r_k > 0$. Since F is a face, it follows that $r_1x_1, \ldots, r_kx_k \in F$, and hence $x_1, \ldots, x_k \in F$.

[11] With the $\mathbb{R}$-basis $\{1\} \otimes B$, where B is a $\mathbb{Q}$-basis of V.

for its rational convex hull, $\mathrm{conv}_{\mathbb{R}}(E)$ for its usual real convex hull, and

$$\mathrm{cone}_{\mathbb{R}}(E) := [0,\infty)\cdot \mathrm{conv}_{\mathbb{R}}(E) \tag{12}$$

for the convex cone generated by E.

The dual space V^* of a $\mathbb{Q}$-vector space V consists of all $\mathbb{Q}$-linear functionals from V to $\mathbb{Q}$. Given a subset $T \subseteq V$, we define its *rational dual cone* by

$$T^\star := \{\rho \in V^* : \rho(T) \subseteq [0,\infty)\}.$$

Then, by identifying V with $(V^*)^*$, $T \subseteq (T^\star)^\star$. If $T \subseteq U$, then $T^\star \supseteq U^\star$. If $v_1,\ldots,v_k \in V$ is a basis and $v_1^*,\ldots,v_k^* \in V^*$ its dual basis, $v_i^*(v_j) = \delta_{ij}$, then

$$\{v_1,\ldots,v_k\}^\star = \mathbb{Q}_0^+ v_1^* + \cdots + \mathbb{Q}_0^+ v_k^*. \tag{13}$$

We let $V_{\mathbb{R}}^*$ be the real dual space of $V_{\mathbb{R}}$ and define the *real dual cone*

$$T_{\mathbb{R}}^\star := \{\rho \in V_{\mathbb{R}}^* : \rho(T) \subseteq [0,\infty)\}$$

of a subset $T \subseteq V_{\mathbb{R}}$. As usual, we identify $V_{\mathbb{R}}^*$ with $(V^*)_{\mathbb{R}}$. For T finite and $\rho \in V_{\mathbb{R}}^*$, ρ is in the interior of $T_{\mathbb{R}}^\star$ if and only if $\rho(t) > 0$ for all $t \in T$, $t \neq 0$.

Proof of the Extension Lemma. We may assume that $0 \notin \Gamma \neq \emptyset$ and, after shrinking Λ to $[\Gamma]$, that $\Lambda = [\Gamma]$. Let $V = \mathrm{span}_{\mathbb{Q}}(\Gamma)$ denote the $\mathbb{Q}$-linear space spanned by Γ, and define

$$\Gamma' := \{\gamma \in \Gamma : \psi(\gamma) \neq 0\}, \quad \Gamma_0 := \{\gamma \in \Gamma : \psi(\gamma) = 0\},$$

and $V' := \mathrm{span}_{\mathbb{Q}}(\Gamma')$.

Step 1. Write $\psi_1 := |\psi|$ and $\psi_2(\xi) := \psi(\xi)/\psi_1(\xi)$ for $\xi \in [\Gamma']$. By the theorem in Ross [27], ψ_2 extends to a character $\varphi_2\colon (V,+) \to (\mathbb{C},\cdot)$ with values in the circle group $\partial\mathbb{U}$. If we can extend ψ_1 to a bounded character φ_1 on $[B]$ for suitable B, then $\varphi_1\varphi_2$ extends ψ. Hence $\psi(\Gamma) \subseteq [0,\infty)$ without loss of generality.

Step 2. Since $(0,\infty)$ is a divisible, torsion-free abelian group under multiplication, the homomorphism $\psi|_{[\Gamma']}\colon [\Gamma'] \to (0,\infty)$ extends uniquely to a homomorphism of groups $\vartheta\colon V' \to (0,\infty)$ (cf. [14, A7]).[12] Then $-\ln\circ\vartheta\colon V' \to \mathbb{R}$ is a $\mathbb{Q}$-linear map and thus extends uniquely to an $\mathbb{R}$-linear functional $\rho\colon V'_{\mathbb{R}} \to \mathbb{R}$. Note that

$$\vartheta(\xi) = e^{-\rho(\xi)} \quad \text{for all } \xi \in V'$$

by construction of ρ. Since $|\vartheta(\xi)| = |\psi(\xi)| \leq 1$, we have $\rho(\xi) \geq 0$ for all $\xi \in \Gamma'$, and thus $\rho \in (\Gamma')_{\mathbb{R}}^\star$.

Step 3. We claim that

$$\mathrm{conv}_{\mathbb{Q}}(\Gamma_0) \cap V' = \emptyset. \tag{14}$$

Suppose, to the contrary, that $\eta \in \mathrm{conv}_{\mathbb{Q}}(\Gamma_0) \cap V' \neq \emptyset$. Then there exist numbers $k,\ell,t \in \mathbb{N}$ with $k \leq \ell$, elements $\eta_1,\ldots,\eta_t \in \Gamma_0$, $\xi_1,\ldots,\xi_\ell \in \Gamma'$ and coefficients $q_1,\ldots,q_t,r_1,\ldots,r_\ell \in \mathbb{Q}^+$ such that

$$0 \neq \eta = \sum_{1\leq\tau\leq t} q_\tau\,\eta_\tau = \sum_{1\leq\lambda\leq k} r_\lambda\,\xi_\lambda - \sum_{k<\lambda\leq\ell} r_\lambda\,\xi_\lambda.$$

[12]In a first step, extend $\psi|_{[B']}$ to a group homomorphism $[B'] - [B'] \to (0,\infty)$ via $\xi_1 - \xi_2 \mapsto \psi(\xi_1)/\psi(\xi_2)$.

By multiplying the equation with the common denominator $a \in \mathbb{N}$ of the rational coefficients q_τ, r_λ for $\tau \leq t$, $\lambda \leq \ell$, we obtain

$$a\,\eta + \sum_{k<\lambda\leq\ell} b_\lambda\,\xi_\lambda = \sum_{1\leq\lambda\leq k} b_\lambda\,\xi_\lambda$$

with certain coefficients $b_\lambda \in \mathbb{N}$ and $0 \neq a\eta \in [\Gamma_0]$. Then

$$\psi(a\eta)\cdot\psi(\xi_{k+1})^{b_{k+1}}\cdots\psi(\xi_\ell)^{b_\ell} = \psi(\xi_1)^{b_1}\cdots\psi(\xi_k)^{b_k},$$

which is a contradiction since $\psi(a\eta) = 0$ but $\psi(\xi_\lambda) \neq 0$ for $1 \leq \lambda \leq k$.

Step 4. Let $U := V/V'$ and $\pi\colon V \to U$ be the quotient map. From Step 3 we know that $0 \notin \pi(\mathrm{conv}_{\mathbb{Q}}(\Gamma_0)) = \mathrm{conv}_{\mathbb{Q}}(\pi(\Gamma_0))$. Hence the Separation Lemma (see Section 7) provides a $\mathbb{Q}$-linear functional $\chi \in U^*$ such that $\chi(\alpha) > 0$ for each $\alpha \in \pi(\Gamma_0)$. Then, with $\theta := \chi \circ \pi \in V^*$, we obtain

$$\theta(\alpha) > 0 \ \text{ for each } \ \alpha \in \Gamma_0 \ \text{ and } \ \theta|_{\Gamma'} = 0, \tag{15}$$

whence $\theta \in \Gamma^\star$ in particular.

Step 5. Let $\rho' \in V_{\mathbb{R}}^*$ be any real functional such that $\rho'|_{V'_{\mathbb{R}}} = \rho$. We choose $c > 0$ so large that $\zeta := \rho' + c\theta \in \Gamma_{\mathbb{R}}^\star$. This is possible since

$$\zeta|_{\Gamma'} = \rho'|_{\Gamma'} \geq 0 \tag{16}$$

(as θ vanishes on Γ'), and furthermore

$$\zeta(\alpha) = \rho'(\alpha) + c\theta(\alpha)$$

for $\alpha \in \Gamma_0$, which can be made arbitrarily large since $\theta(\alpha) > 0$.

Step 6. We have $0 \notin \mathrm{conv}_{\mathbb{R}}(\Gamma)$ in $V_{\mathbb{R}}$, because Γ is a subset of the convex set $(0,\infty)$. Hence $P := \mathrm{cone}_{\mathbb{R}}(\Gamma)$ is a pointed[13] polyhedral cone in $V_{\mathbb{R}}$ whose extreme rays are of the form $[0,\infty)\alpha$ for certain $\alpha \in \Gamma$ (as recalled above). Since $\mathrm{span}_{\mathbb{R}}(\Gamma) = V_{\mathbb{R}}$, the cone P has non-empty interior. Hence also $P_{\mathbb{R}}^\star = \Gamma_{\mathbb{R}}^\star \subseteq V_{\mathbb{R}}^*$ is a pointed polyhedral cone with non-empty interior (see Neeb [25, Propositions V.1.5 (ii) and V.1.21]). It is known from the theory of polyhedral cones in real vector spaces that the extreme rays of $P_{\mathbb{R}}^\star$ are of the form $[0,\infty)\alpha$ for a functional $\alpha \in V_{\mathbb{R}}^*$ such that $F := \ker\alpha \cap P$ is a codimension 1 face of P (see, e.g. [33, Theorem 3]). We claim that α *can be chosen in* V^**.*

To see this, recall first that $F = \mathrm{cone}_{\mathbb{R}}(\Gamma'')$ for some subset $\Gamma'' \subseteq \Gamma$, and F has non-empty interior in $\ker\alpha$. Hence $\ker\alpha = \mathrm{span}_{\mathbb{R}}(F) = (\mathrm{span}_{\mathbb{Q}}(\Gamma''))_{\mathbb{R}}$ is defined over $\mathbb{Q}$. After replacing α with a positive real multiple to ensure that $\alpha(V) \subseteq \mathbb{Q}$, we have $\alpha \in V^*$, as desired.

Consequently, $F \cap V^*$ is dense in F for each face F of $P_{\mathbb{R}}^\star$. Furthermore, $\mathrm{span}_{\mathbb{Q}}(\Gamma^\star) = V^*$.

Step 7. Let $F \subseteq \Gamma_{\mathbb{R}}^\star$ be a face of dimension $\ell \geq 1$, and $\mathrm{algint}(F)$ be its interior relative to $\mathrm{aff}_{\mathbb{R}}(F) = \mathrm{span}_{\mathbb{R}}(F)$. We show: *For every* $\eta \in \mathrm{algint}(F)$ *there exists a* $\mathbb{Q}$*-basis* $b_1,\ldots,b_\ell \in F \cap V^*$ *of* $\mathrm{span}_{\mathbb{Q}}(F \cap V^*)$ *with* $\eta \in \mathrm{cone}_{\mathbb{R}}(b_1,\ldots,b_\ell)$*. Moreover,* b_1 *can be chosen as an arbitrary non-zero vector in* $F \cap V^*$*.*

In fact, if $\eta = 0$, we can simply select a $\mathbb{Q}$-basis from generators $\alpha \in V^*$ for extreme rays of F, which exist by Step 6 (or extend a given vector b_1 by such vectors to a basis). If $\eta \neq 0$ (which we assume now), we proceed by induction on ℓ:

[13] That means, P does not contain lines.

If $\ell = 1$, then $F = [0, \infty)\alpha$ for some $\alpha \in \Gamma^\star \subseteq V^*$ (see Step 6). We can now take $b_1 := \alpha$ (or any prescribed non-zero vector in $F \cap V^*$).

Induction step. There exists an x in the interior P^0, such that $\eta(x) > 0$. Since V is dense in $V_{\mathbb{R}}$, we may assume that $x \in V$. After passage to a positive multiple of η, we may also assume that $\eta(x) \in \mathbb{Q}$. Then $\gamma(x) > 0$ for all $\gamma \in \Gamma^\star_{\mathbb{R}} \setminus \{0\}$. Hence $K := \{\gamma \in \Gamma^\star_{\mathbb{R}} : \gamma(x) = \eta(x)\}$ is a closed convex set such that $\Gamma^\star_{\mathbb{R}} = [0, \infty)K$ and $\eta \in K$. Choose $\alpha_1, \ldots, \alpha_n \in \Gamma^\star_{\mathbb{R}}$ such that $[0, \infty)\alpha_j$, $j = 1, \ldots, n$, are the extreme rays of $\Gamma^\star_{\mathbb{R}}$; after passage to positive multiples, we may assume that $\alpha_j(x) = \eta(x)$ for all j. If $r_1, \ldots, r_n \geq 0$ and $\gamma := r_1\alpha_1 + \cdots + r_n\alpha_n$ satisfies $\gamma(x) = \eta(x)$, then $\sum_{j=1}^n r_j = 1$. Hence $K = \mathrm{conv}_{\mathbb{R}}(\alpha_1, \ldots, \alpha_n)$, and thus K is compact. If a candidate for b_1 is given, after passing to a positive rational multiple we may assume that this b_1 lies in K. Otherwise we choose any non-zero element $b_1 \in V^* \cap F \cap K$. If $\eta \in [0, \infty)b_1$, we can use Step 6 to extend b_1 (using generators of some extreme rays) to a $\mathbb{Q}$-basis with the desired properties. Otherwise we find $t > 1$ such that $d := b_1 + t(\eta - b_1)$ lies in the boundary of F relative to $\mathrm{span}_{\mathbb{R}}(F)$ (using the fact that K is compact). Then $d \in \mathrm{algint}(F')$ for some face F' of F of dimension $0 < m < \ell$. By induction we find a $\mathbb{Q}$-basis $b_2, \ldots, b_{m+1}$ of $\mathrm{span}_{\mathbb{Q}}(F' \cap V^*)$ in $F' \cap V^*$, such that $d \in \mathrm{cone}_{\mathbb{R}}(b_2, \ldots, b_{m+1})$. Note that $b_1 \notin F'$ (otherwise the convex combination η of b_1 and d would lie in the proper face F' of F, contradicting the assumption that $\eta \in \mathrm{algint}(F)$). Hence $b_1 \notin F' - F' = \mathrm{span}_{\mathbb{R}}(F')$ (using the fact that F' is a face). Thus $b_1, b_2, \ldots, b_{m+1}$ are $\mathbb{Q}$-linearly independent and can be extended to a $\mathbb{Q}$-basis $b_1, \ldots, b_\ell \in F \cap V^*$ of $\mathrm{span}_{\mathbb{Q}}(F \cap V^*)$, using Step 6. It remains to observe that $\eta = \frac{1}{t}d + (1 - \frac{1}{t})b_1 \in \mathrm{cone}_{\mathbb{R}}(b_1, \ldots, b_{m+1}) \subseteq \mathrm{cone}_{\mathbb{R}}(b_1, \ldots, b_\ell)$.

Step 8. If $\zeta = 0$, we choose a $\mathbb{Q}$-basis $\beta_1^*, \ldots, \beta_k^* \in \Gamma^\star$ of V^* such that

$$\theta \in \mathbb{Q}_0^+\beta_1^* + \cdots + \mathbb{Q}_0^+\beta_k^*, \tag{17}$$

which is trivial for $\theta = 0$, and can be achieved by taking β_1^* as a positive rational multiple of θ otherwise. If $\zeta \neq 0$, let F be the minimal face of $\Gamma^\star_{\mathbb{R}}$ containing ζ. Then ζ is in the interior of F relative to $\mathrm{span}_{\mathbb{R}}(F)$, and $F \cap V^*$ is dense in F. If $\theta \in F$, we let $\beta_1^* \in F \cap V^*$ be a non-zero vector such that θ is a non-negative rational multiple of β_1^*. By Step 7, we can extend β_1^* to a $\mathbb{Q}$-basis $\beta_1^*, \ldots, \beta_\ell^* \in F \cap V^*$ of $\mathrm{span}_{\mathbb{Q}}(F \cap V^*)$ such that

$$\zeta \in \mathrm{cone}_{\mathbb{R}}(\beta_1^*, \ldots, \beta_\ell^*), \tag{18}$$

which in turn we extend to a $\mathbb{Q}$-basis $\beta_1^*, \ldots, \beta_k^* \in \Gamma^\star$ of V^*. If $\theta \notin F$, we first find a $\mathbb{Q}$-basis $\beta_1^*, \ldots, \beta_\ell^* \in F \cap V^*$ of $\mathrm{span}_{\mathbb{Q}}(F \cap V^*)$ such that (18) holds (using Step 7), and then extend it to a $\mathbb{Q}$-basis $\beta_1^*, \ldots, \beta_k^* \in \Gamma^\star$ of V^* such that $\beta_{\ell+1}^*$ is a positive rational multiple of θ. This is possible since $\theta \notin F - F = \mathrm{span}_{\mathbb{R}}(F)$ (as $\theta \in \Gamma^\star_{\mathbb{R}} \setminus F$ and F is a face). In either case, $\zeta \in \mathbb{R}_0^+\beta_1^* + \cdots + \mathbb{R}_0^+\beta_k^*$, and (17) holds. Let $\beta_1, \ldots, \beta_k \in V$ be the basis dual to $\beta_1^*, \ldots, \beta_k^*$, and write $\mathrm{B} := \{\beta_1, \ldots, \beta_k\}$. Then $\{\beta_1^*, \ldots, \beta_k^*\} \subseteq \Gamma^\star$ entails that

$$\mathbb{Q}_0^+\beta_1 + \cdots + \mathbb{Q}_0^+\beta_k = \{\beta_1^*, \ldots, \beta_k^*\}^\star \supseteq \Gamma^{\star\star} \supseteq \Gamma.$$

After replacing each β_κ by a positive rational multiple, we may assume that $[\mathrm{B}] \supseteq \Gamma$.

Step 9. Using Kronecker's δ, we define

$$\phi\colon [\mathrm{B}] \to [0, \infty), \quad \phi(\xi) := e^{-\zeta(\xi)} \cdot \delta_{0,\theta(\xi)}.$$

Then ϕ is a homomorphism, since $\theta(\mathrm{B}) \subseteq [0,\infty)$ by (17) and $\delta_{0,\cdot}\colon ([0,\infty),+) \to ([0,\infty),\cdot)$ is a homomorphism of monoids. If $\xi \in \Gamma'$, then $\phi(\xi) = e^{-\rho(\xi)} = \psi(\xi)$ by (15) and (16). If $\xi \in \Gamma_0$, then $\theta(\xi) > 0$ by (15), and thus $\phi(\xi) = 0 = \psi(\xi)$. Finally, ϕ is bounded, since $\{\zeta\} \subseteq \mathbb{R}_0^+\beta_1^* + \cdots + \mathbb{R}_0^+\beta_k^*$, and thus $[\mathrm{B}] \subseteq \{\beta_1^*, \dots, \beta_k^*\}^\star \subseteq \{\zeta\}_{\mathbb{R}}^\star$.

This completes the proof of the Density Lemma. □

7. A Hahn-Banach separation theorem for rational polytopes

The possibility of separation theorems for polytopes in vector spaces over ordered fields is already mentioned in [11, p. 287] (without proof). The proof of the Extension Lemma, Step 4 of Section 6, required the following

Separation Lemma. *Let V be a finite-dimensional $\mathbb{Q}$-vector space and let $E = \{x_1, \dots, x_m\} \subseteq V$ be a non-empty finite subset such that $0 \notin C := \mathrm{conv}_{\mathbb{Q}}(E)$. Then there exists a $\mathbb{Q}$-linear functional $\rho\colon V \to \mathbb{Q}$ such that $\rho(E) \subseteq (0,\infty)$.*

Proof. Let $W \subseteq V$ be the affine subspace generated by E. If $0 \notin W$, then there exists $\rho \in V^\star$ such that $\rho|_W = 1$ and hence $\rho|_E = 1$. Now assume that $0 \in W$. After replacing V with W, we may assume that $V = \mathrm{aff}_{\mathbb{Q}}(E)$. We may also assume $V = \mathbb{Q}^n$ for some n. Then $\mathrm{aff}_{\mathbb{R}}(E) = \mathbb{R}^n$ in $\mathbb{R}^n$, whence $C_{\mathbb{R}} := \mathrm{conv}_{\mathbb{R}}(E)$ has non-empty interior in $V_{\mathbb{R}} := \mathbb{R}^n$.

We claim that $0 \notin C_{\mathbb{R}}$.

If this is true, then there is $y \in \mathbb{R}^n$ such that $\langle y, C_{\mathbb{R}}\rangle \subseteq (0,\infty)$ by the Hahn-Banach Separation Theorem. Then $\langle y, x_j\rangle > 0$ for $j = 1, \dots, m$. By continuity, we find $w \in \mathbb{Q}^n$ close to y such that $\langle w, x_j\rangle > 0$ for $j = 1, \dots, m$. Then $\rho := \langle w, \cdot\rangle \in (\mathbb{Q}^n)^\star$ is as desired.

We now prove the claim by induction on $\dim_{\mathbb{Q}}(V)$. If $\dim_{\mathbb{Q}}(V) = 1$, then E is a finite subset of $\mathbb{Q}$ and $C = [x_\star, x^\star] \cap \mathbb{Q}$, where $x_\star$ and $x^\star$ are the minimum and maximum of E, respectively. Since $0 \notin C$, we deduce that $\{x_\star, x^\star\} \subseteq (0,\infty)$ or $\{x_\star, x^\star\} \subseteq (-\infty, 0)$, entailing that also $0 \notin C_{\mathbb{R}} = [x_\star, x^\star]$. Induction step: If $0 \in \partial C_{\mathbb{R}}$ is in the boundary, then 0 is contained in a face $\Phi \neq C_{\mathbb{R}}$ of the polytope $C_{\mathbb{R}}$ (see [2, Theorem 5.6]). We have $\Phi = \mathrm{conv}_{\mathbb{R}}(E')$ with $E' := \Phi \cap E$, by [2, Theorems 7.2 and 7.3]. Then $\mathrm{aff}_{\mathbb{R}}(E')$ is a proper vector subspace of $V_{\mathbb{R}}$ (see [2, Corollary 5.5]), and hence $\mathrm{aff}_{\mathbb{Q}}(E')$ is a proper vector subspace of V. By induction, $0 \notin \mathrm{conv}_{\mathbb{R}}(E') = \Phi$. We have reached a contradiction.

It remains to discuss the case where 0 is in the interior of $C_{\mathbb{R}}$. For some $\epsilon > 0$ we then have $w \in C_{\mathbb{R}}$ for all $w = (w_1, \dots, w_n) \in \{-\epsilon, \epsilon\}^n$. Since C is dense in $C_{\mathbb{R}}$, for each w (as before) we find an element $u \in C$ such that $\|w - u\|_\infty < \frac{\epsilon}{2}$. For each $j \in \{1, \dots, n\}$, the jth component u_j of u is then non-zero and has the same sign as w_j. Let U be the set of all u as before, for w ranging through $\{-\epsilon, \epsilon\}^n$. Now the next lemma shows that $0 \in \mathrm{conv}_{\mathbb{Q}}(U) \subseteq C$, which is a contradiction. □

Here, we used

Lemma 3. *Let $U \subseteq \mathbb{Q}^n$ be such that for all signs $\sigma_1, \dots, \sigma_n \in \{-1, 1\}$, there exists $u = (u_1, \dots, u_n) \in U$ with $\mathrm{sgn}\, u_j = \sigma_j$ for each $j \in \{1, \dots, n\}$. Then $0 \in \mathrm{conv}_{\mathbb{Q}}(U)$.*

Proof. By induction on n. The case $n = 1$ is trivial. Given $U \subseteq \mathbb{Q}^n$ and signs $\sigma_1, \dots, \sigma_{n-1}$, we find $u \in U$ with signs $\sigma_1, \dots, \sigma_{n-1}, 1$ and $v \in U$ with signs $\sigma_1, \dots, \sigma_{n-1}, -1$. By the case $n = 1$, there is $w = (w_1, \dots, w_n) \in \mathrm{conv}_{\mathbb{Q}}\{u, v\}$

such that $w_n = 0$. Thus $w = (w_1, \ldots, w_{n-1}, 0)$, where $w_1, \ldots, w_{n-1}$ have signs $\sigma_1, \ldots, \sigma_{n-1}$. Consider the first $n-1$ coordinates. Then the induction hypothesis yields $0 \in \mathrm{conv}_{\mathbb{Q}}(U)$. □

8. Extensions and applications

The Lévy type extension of Theorem 1 (cf. Lévy [20] for the case of Fourier series) is obtained by applying the Composition Lemma.

Theorem 3. *Let $\Lambda \subseteq [0,\infty)$ be an additive semigroup with $0 \in \Lambda$, let f be a holomorphic function defined on a region $G \subseteq \mathbb{C}$, and let $w \in \mathcal{W}$. Suppose that $a \in \mathcal{A}_w$ satisfies $\overline{\widetilde{a}(\mathbb{H})} \subseteq G$. Then there exists a function $c \in \mathcal{A}_w$ such that $f \circ \widetilde{a} = \widetilde{c}$.*

Proof. For $a \in \mathcal{A}_w$ we have

$$\{h(a) : h \in \Delta(\mathcal{A}_w)\} \subseteq \overline{\{h_s(a) : s \in \overline{\mathbb{H}}\,\}} = \overline{\widetilde{a}(\mathbb{H})} \subseteq G,$$

and the assertion follows from Theorem 1 and the Composition Lemma. □

In particular, Theorems 1 and 3 apply to $\Lambda = \mathbb{N}_0$. We write $z = e^{-s}$ (a transformation which maps $\mathbb{H}$ onto $\mathbb{U} \setminus \{0\}$) and associate with $a \in \mathcal{A}_w(\mathbb{N}_0)$ the power series

$$\widetilde{a}(z) = \widetilde{a}(e^{-s}) = \sum_{n=0}^{\infty} a(n)\, z^n.$$

Since $\widetilde{a}$ is continuous on the compact set $\overline{\mathbb{U}}$, we have $\overline{\widetilde{a}(\mathbb{U})} = \widetilde{a}(\overline{\mathbb{U}})$.

Then the weighted version of Wiener's inversion theorem for power series reads as follows.

Corollary 1. *For $w \in \mathcal{W}(\mathbb{N}_0)$ the multiplicative group of the Banach algebra $\mathcal{A}_w(\mathbb{N}_0)$ is*

$$\mathcal{A}_w^*(\mathbb{N}_0) = \big\{a \in \mathcal{A}_w(\mathbb{N}_0) : 0 \notin \widetilde{a}(\overline{\mathbb{U}})\big\}.$$

The lemmas needed for the proof of Theorems 1 and 3 easily extend to additive semigroups Λ of *product type*,[14] by which we understand semigroups $\Lambda = \Lambda_1 \times \cdots \times \Lambda_r$ with additive semigroups $\Lambda_\varrho \subseteq [0,\infty)$ and $0 \in \Lambda_\varrho$ for $\varrho = 1, \ldots, r$. Then the multidimensional versions of Theorems 1 and 3 cover arithmetic functions of r variables. We put $\lambda \cdot s = \lambda_1 s_1 + \cdots + \lambda_r s_r$ for $\lambda = (\lambda_1, \ldots, \lambda_r) \in \Lambda$ and $s = (s_1, \ldots, s_r) \in \mathbb{C}^r$. The r-dimensional Dirichlet series associated with $a\colon \Lambda \to \mathbb{C}$ is

$$\widetilde{a}(s) = \sum_{\lambda \in \Lambda} a(\lambda)\, e^{-\lambda \cdot s} \qquad (s \in \mathbb{C}^r). \tag{19}$$

Repeating the reasoning leading to Theorem 1, we see that the assertion of Theorem 1 remains true for additive semigroups $\Lambda \subseteq [0,\infty)^r$ of product type. Indeed, we can dispense with Λ being of product type.

Theorem 4. *Let $L \subseteq [0,\infty)^r$ be an additive semigroup with $0 \in L$ and $w \in \mathcal{W}(L)$. If $a \in \mathcal{A}_w(L)$ satisfies $0 \notin \overline{\widetilde{a}(\mathbb{H}^r)}$, then a is invertible in $\mathcal{A}_w(L)$.*

---

[14]In fact, every bounded character ψ of Λ is of the form $\psi(\lambda_1, \ldots, \lambda_r) = \prod_{\varrho=1}^{r} \psi_\varrho(\lambda_\varrho)$ with bounded characters ψ_ϱ of Λ_ϱ, which can be approximated by $e^{-\lambda_\varrho s_\varrho}$ on a finite set $\Gamma_\varrho \subseteq \Lambda_\varrho$. Thus $\psi(\lambda)$ can be approximated by $e^{-\lambda \cdot s}$ on $\Gamma_1 \times \cdots \times \Gamma_r$.

Proof. According to the Representation Lemma $\mathcal{A}_w^*(L) = \mathcal{A}_1^*(L) \cap \mathcal{A}_w(L)$ holds for any additive semigroup L with $0 \in L$ and any admissible weight function w. Therefore it suffices to show that $a \in \mathcal{A}_1^*(L)$.

Denote by Λ_ϱ the set of ϱth components of L. Then $\Lambda_\varrho \subseteq [0,\infty)$ is an additive semigroup with $0 \in \Lambda_\varrho$, L is a subsemigroup of $\Lambda = \Lambda_1 \times \cdots \times \Lambda_r$, and $\mathcal{A}_1(L) \subseteq \mathcal{A}_1(\Lambda)$. By the previous remark we have $a \in \mathcal{A}_1^*(\Lambda)$ and, in particular, $a(0) \neq 0$.

To obtain $a \in \mathcal{A}_1^*(L)$, we have to show that $a^{-1}(\lambda) = 0$ for all $\lambda \in \Lambda \setminus L$. As a tool, let us consider the weight functions w_ρ on L (and Λ) for $\rho \geq 0$ defined via $w_\rho(\lambda) := e^{-\rho(\lambda_1+\cdots+\lambda_r)}$ (which are not admissible if $\rho > 0$, but define the weighted algebra $\mathcal{A}_{w_\rho}$). Choose $\rho > 0$ so large that[15]

$$\frac{1}{|a(0)|} \sum_{\lambda \in L \setminus \{0\}} |a(\lambda)| e^{-\rho(\lambda_1+\cdots+\lambda_r)} < 1. \tag{20}$$

Then $a \in \mathcal{A}_{w_\rho}^*(L)$, since $a = a(0)\big(\varepsilon + \frac{1}{a(0)}(a - a(0)\varepsilon)\big)$, where $\big\|\frac{1}{a(0)}(a-a(0)\varepsilon)\big\|_{w_\rho} < 1$ by (20). By Theorem 2 a) the inverse b of a in $\mathcal{A}_{w_\rho}^*(L)$ exists. Then both $a^{-1} \in \mathcal{A}_1(\Lambda)$ and b are inverses of a in the algebra $\mathcal{A}_{w_\rho}(\Lambda)$ (which contains both $\mathcal{A}_1(\Lambda)$ and $\mathcal{A}_{w_\rho}(L)$ as unital subalgebras), and hence coincide. Thus $a^{-1} = b \in \mathcal{A}_1(\Lambda) \cap \mathcal{A}_{w_\rho}(L) = \mathcal{A}_1(L)$. □

Similarly we obtain a multidimensional weighted Wiener-Lévy type theorem by using the Composition Lemma and the Density Lemma in the version $\sigma(a) \subseteq \overline{\widetilde{a}(\mathbb{H}^r)}$ for $a \in \mathcal{A}_w(L)$ for additive semigroups $L \subseteq [0,\infty)^r$ with $0 \in L$.

Theorem 5. *Let $L \subseteq [0,\infty)^r$ be an additive semigroup with $0 \in L$, f a holomorphic function defined on a region $G \subseteq \mathbb{C}$, and $w \in \mathcal{W}(L)$. If $a \in \mathcal{A}_w(L)$ satisfies $\overline{\widetilde{a}(\mathbb{H}^r)} \subseteq G$, then there exists a function $c \in \mathcal{A}_w(L)$ such that $f \circ \widetilde{a} = \widetilde{c}$.*

Let $\Lambda \subset [0,\infty)$ be a free additive semigroup with $0 \in \Lambda$ and finite or countable generating set B. Arithmetical applications are usually based on the associated free *multiplicative* semigroup $\mathcal{N} := e^\Lambda \in [1,\infty)$ with $1 \in \mathcal{N}$ and the finite or countable generating set $\mathcal{P} := e^{\mathrm{B}}$ of *prime elements*. By definition each $n \in \mathcal{N}$ has a unique factorization of the form $n = p_1^{\nu_1} \cdots p_r^{\nu_r}$ with distinct $p_1, \ldots, p_r \in \mathcal{P}$ and positive integer exponents $\nu_1, \ldots, \nu_r$, apart from the order of prime element powers. As usual, the empty product has the value 1, and elements $m, n \in \mathcal{N}$ not having any common prime divisors are called *coprime*. The prototype of such multiplicative semigroups is $\mathbb{N}$ generated by the set $\mathbb{P}$ of primes.

The functions $a\colon \mathcal{N} \to \mathbb{C}$ now correspond to the Dirichlet series

$$\widetilde{a}(s) := \sum_{n \in \mathcal{N}} \frac{a(n)}{n^s} \qquad (s \in \mathbb{C}),$$

and the functions $\omega\colon \mathcal{N} \to [1,\infty)$ with $\omega(n) \geq \omega(1) = 1$ that are submultiplicative, i.e. $\omega(mn) \leq \omega(n)\,\omega(m)$ for all $m, n \in \mathcal{N}$, and satisfy $\lim_{k\to\infty} \sqrt[k]{\omega(n^k)} = 1$ for all $n \in \mathcal{N}$ form the class $\mathcal{W}_1 := \mathcal{W}_1(\mathcal{N})$ of admissible weight functions. By Theorem 1 each $\omega \in \mathcal{W}_1$ yields a Banach algebra $\mathcal{A}_\omega := \mathcal{A}_\omega(\mathcal{N})$ of functions $a\colon \mathcal{N} \to \mathbb{C}$, endowed with the linear operations and convolution

$$(a * b)(n) := \sum_{\substack{\ell, m \in \mathcal{N} \\ \ell\, m = n}} a(\ell)\, b(m) \qquad (n \in \mathcal{N}) \tag{21}$$

[15] This is possible by an elementary argument. Alternatively, one can use [9, Lemma 8].

and with bounded ω-norm $\|a\|_\omega := \sum_{n\in\mathcal{N}} |a(n)|\,\omega(n)$. The unity in $\mathcal{A}_\omega$ is $\varepsilon := \delta_1$, and $\mathcal{A}_\omega$ has the multiplicative group

$$\mathcal{A}_\omega^*(\mathcal{N}) = \{a \in \mathcal{A}_\omega : 0 \notin \overline{\widetilde{a}(\mathbb{H})}\}.$$

We consider the class $\mathcal{M}$ of *multiplicative functions* $a\colon \mathcal{N} \to \mathbb{C}$, i.e. $a(mn) = a(m)\,a(n)$ for all coprime $m, n \in \mathcal{N}$ and $a(1) = 1$. Since $\mathcal{M}$ is closed under convolution and inversion, $\mathcal{M}$ is a group. For each $\omega \in \mathcal{W}_1$, the Dirichlet series $\widetilde{a}(s)$ of $a \in \mathcal{M}_\omega := \mathcal{M} \cap \mathcal{A}_\omega(\mathcal{N})$ converges absolutely for $s \in \overline{\mathbb{H}}$ and has the Euler product representation

$$\widetilde{a}(s) = \prod_{p\in\mathcal{P}} \Big(1 + \frac{a(p)}{p^s} + \frac{a(p^2)}{p^{2s}} + \cdots\Big) =: \prod_{p\in\mathcal{P}} \widetilde{a}_p(s).$$

Since each factor $\widetilde{a}_p(s)$ represents an absolutely convergent power series in $z = p^{-s} \in \overline{\mathbb{U}}$ and thus a continuous function on the compact disc $\overline{\mathbb{U}}$, its weighted inversion according to Corollary 1 with $w(k) = \omega(p^k)$ for $k \in \mathbb{N}_0$ only requires $0 \notin \widetilde{a}_p(\overline{\mathbb{H}})$, instead of $0 \notin \overline{\widetilde{a}_p(\mathbb{H})}$.

Corollary 2. *For $\omega \in \mathcal{W}_1$ the class $\mathcal{M}_\omega$ is a unital subsemigroup of $\mathcal{M}$ under the convolution* (21) *with the multiplicative group*

$$\mathcal{M}_\omega^* = \{a \in \mathcal{M}_\omega : \widetilde{a}_p(s) \neq 0 \text{ for all } s \in \overline{\mathbb{H}} \text{ and } p \in \mathcal{P}\}. \tag{22}$$

For an infinite generating set $\mathcal{P}$, there are many arithmetically interesting multiplicative functions a that do not belong to $\mathcal{M}_\omega$, particularly those for which the series $\sum_{p\in\mathcal{P}} a(p)\,\omega(p)$ does not converge absolutely. Therefore we extend $\mathcal{M}_\omega$ by partly replacing the ω-norm with the mean square ω-norm (for $\mathcal{N} = \mathbb{N}$ cf. Lucht [21] and, with $\omega = 1$, Heppner and Schwarz [13]):

Proposition 3. *Let $\mathcal{N} \subset [1,\infty)$ be a free multiplicative semigroup with $1 \in \mathcal{N}$ and countable generating set $\mathcal{P}$. Then, for $\omega \in \mathcal{W}_1$, the class*

$$\mathcal{G}_\omega = \Big\{a \in \mathcal{M} : \sum_{p\in\mathcal{P}} |a(p)|^2\,\omega^2(p) < \infty \text{ and } \sum_{\substack{p\in\mathcal{P}\\ k\geq 2}} |a(p^k)|\,\omega(p^k) < \infty\Big\}$$

is a unital subsemigroup of $\mathcal{M}_\omega$ under the convolution (21) *with the multiplicative group*

$$\mathcal{G}_\omega^* = \big\{a \in \mathcal{G}_\omega : \widetilde{a}_p(s) \neq 0 \text{ for all } s \in \overline{\mathbb{H}} \text{ and } p \in \mathcal{P}\big\}.$$

Note that $\mathcal{G}_\omega = \mathcal{M}_\omega$ for finite $\mathcal{P}$, whereas $\mathcal{M}_\omega \subsetneq \mathcal{G}_\omega$ for infinite $\mathcal{P}$. Further, for $a \in \mathcal{G}_\omega$, the series $\sum_{p\in\mathcal{P}} a(p)$ even might diverge.

Proof of Proposition 3. The submultiplicativity of the ω-norm combined with the Cauchy-Schwarz inequality entails that $\mathcal{G}_\omega$ is closed under $*$, and trivially $\varepsilon \in \mathcal{G}_\omega$. For $a, b \in \mathcal{G}_\omega^*$ and $p \in \mathcal{P}$ we have $a_p, b_p \in \mathcal{G}_\omega^*$ and $(a*b)_p\widetilde{\ }(s) = \widetilde{a}_p(s)\,\widetilde{b}_p(s) \neq 0$ for $s \in \overline{\mathbb{H}}$. Hence $\mathcal{G}_\omega^*$ is also closed under $*$. It remains to verify that $a \in \mathcal{G}_\omega^*$ implies $a^{-1} \in \mathcal{G}_\omega$.

From Corollary 1 applied to $\widetilde{a}_p(s)$ with $w(k) = \omega(p^k)$ and $z = p^{-s}$ for $p \in \mathcal{P}$ fixed and $k \in \mathbb{N}_0$ we infer that $a_p^{-1} \in \mathcal{G}_\omega$ for each $p \in \mathcal{P}$. To transfer this invertibility property from all factors a_p to a we consider the Euler product representation of $\widetilde{a}(s)$ written as

$$\widetilde{a}(s) = \prod_{p\leq p_0} \widetilde{a}_p(s) \cdot \prod_{p>p_0} \Big(1 - \frac{a(p)}{p^s}\Big)^{-1} \cdot \prod_{p>p_0} \Big(1 - \frac{a(p)}{p^s}\Big)\,\widetilde{a}_p(s).$$

It corresponds to the decomposition

$$a = \Big(\mathop{\ast}_{p \le p_0} a_p \Big) \ast b \ast h \tag{23}$$

with $p_0 \in \mathcal{P}$ suitably large, and $b, h \in \mathcal{M}$ defined by

$$b(p^k) = \begin{cases} a^k(p) & \text{for } p > p_0,\ k \in \mathbb{N}_0, \\ 0 & \text{otherwise,} \end{cases}$$

$$h(p^k) = \begin{cases} a(p^k) - a(p^{k-1})\, a(p) & \text{for } p > p_0,\ k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously $h(p) = 0$ for $p \in \mathcal{P}$ and $b(p) = a(p)$ for all $p > p_0$. We choose $p_0 \in \mathcal{P}$ sufficiently large such that both estimates

$$|a(p)|\, \omega(p) \le \frac{1}{2} \text{ for } p > p_0 \quad \text{and} \quad \sum_{\substack{p > p_0 \\ k \ge 2}} |h(p^k)|\, \omega(p^k) \le \frac{1}{2}$$

hold. Then $b \in \mathcal{G}_\omega^*$ and $b^{-1} = \mu b \in \mathcal{G}_\omega$, as b is completely multiplicative and $|\mu(n)| \le 1$ for $n \in \mathbb{N}$. Further, the submultiplicativity of ω together with the Cauchy-Schwarz inequality yields $h \in \mathcal{G}_\omega$.

In order to verify $h \in \mathcal{G}_\omega^*$ we conclude from $h^{-1} \ast h = \varepsilon$ that $h^{-1}(p) = h(p) = 0$ for all $p \in \mathcal{P}$, $h(p^k) = 0$ for all $p \le p_0$ and $k \in \mathbb{N}$, and

$$h^{-1}(p^k) = - \sum_{0 \le j \le k-2} h^{-1}(p^j)\, h(p^{k-j}) \qquad (p > p_0,\ k \ge 2).$$

From this combined with the submultiplicativity of ω we obtain

$$\begin{aligned} \Sigma := &\sum_{\substack{p^k \le x \\ k \ge 2}} |h^{-1}(p^k)|\, \omega(p^k) \\ \le &\sum_{\substack{p^k \le x \\ k \ge 2}} \sum_{0 \le j \le k-2} |h^{-1}(p^j)|\, w(p^j) \cdot |h(p^{k-j})|\, \omega(p^{k-j}) \\ = &\sum_{\substack{p^k \le x \\ k \ge 2}} |h(p^k)|\, \omega(p^k) + \sum_{\substack{p^{j+\ell} \le x \\ j, \ell \ge 2}} |h^{-1}(p^j)|\, \omega(p^j) \cdot |h(p^\ell)|\, \omega(p^\ell) \\ \le &\big(1 + \Sigma\big) \sum_{\substack{p^\ell \le x \\ \ell \ge 2}} |h(p^\ell)|\, \omega(p^\ell) \le \frac{1}{2} \big(1 + \Sigma\big). \end{aligned}$$

Hence $\Sigma \le 1$ and $h^{-1} \in \mathcal{G}_\omega$. Now (23) entails that $a \in \mathcal{G}_\omega$ is a convolution of finitely many elements of $\mathcal{G}_\omega^*$, and thus $a^{-1} \in \mathcal{G}_\omega$. □

An important arithmetical application of weight functions is based on Proposition 3 and the notion of related arithmetical functions of $\mathcal{A} := \mathcal{A}(\mathcal{N})$. For $\omega \in \mathcal{W}_1$, we say that $a \in \mathcal{A}$ is *ω-related* to $b \in \mathcal{A}^*$, if $h := a \ast b^{-1} \in \mathcal{A}_\omega$. Via Proposition 3 the ω-relationship of functions $a \in \mathcal{G}_\omega$, $b \in \mathcal{G}_\omega^*$ from their values at prime elements can easily be verified. This leads, for instance, to a new concept demanded for by Knopfmacher [17, Ch. 7, § 5] of Ramanujan expansions of arithmetic functions (cf. Lucht [23]).

Acknowledgement

The authors are greatly indebted to the referee for suggesting several improvements to this paper.

References

[1] Christian Berg, Jens Peter Reus Christensen, and Paul Ressel, *Harmonic analysis on semigroups*, Graduate Texts in Mathematics, vol. 100, Springer-Verlag, New York, 1984. Theory of positive definite and related functions. MR747302 (86b:43001)

[2] Arne Brøndsted, *An introduction to convex polytopes*, Graduate Texts in Mathematics, vol. 90, Springer-Verlag, New York, 1983. MR683612 (84d:52009)

[3] H. G. Dales and H. V. Dedania, *Weighted convolution algebras on subsemigroups of the real line*, Dissertationes Math. (Rozprawy Mat.) **459** (2009), 60, DOI 10.4064/dm459-0-1. MR2477218 (2010d:46059)

[4] J. Dieudonné, *Foundations of modern analysis*, Pure and Applied Mathematics, Vol. X, Academic Press, New York, 1960. MR0120319 (22 #11074)

[5] D. A. Edwards, *On absolutely convergent Dirichlet series*, Proc. Amer. Math. Soc. **8** (1957), 1067–1074. MR0096086 (20 #2583)

[6] G. Fendler, K. Gröchenig, M. Leinert, J. Ludwig, and C. Molitor-Braun, *Weighted group algebras on groups of polynomial growth*, Math. Z. **245** (2003), no. 4, 791–821, DOI 10.1007/s00209-003-0571-6. MR2020712 (2004k:43008)

[7] Gero Fendler, Karlheinz Gröchenig, and Michael Leinert, *Symmetry of weighted L^1-algebras and the GRS-condition*, Bull. London Math. Soc. **38** (2006), no. 4, 625–635, DOI 10.1112/S0024609306018777. MR2250755 (2007f:43002)

[8] H. Glöckner, *Remarks concerning a lemma of Lucht and Reifenrath*, Draft of December 9, 2005; unpublished.

[9] Helge Glöckner, Lutz G. Lucht, and Štefan Porubský, *General Dirichlet series, arithmetic convolution equations and Laplace transforms*, Studia Math. **193** (2009), no. 2, 109–129, DOI 10.4064/sm193-2-2. MR2515515 (2010f:46075)

[10] Arthur Goodman and D. J. Newman, *A Wiener type theorem for Dirichlet series*, Proc. Amer. Math. Soc. **92** (1984), no. 4, 521–527, DOI 10.2307/2045419. MR760938 (86a:30005)

[11] Peter Gritzmann and Victor Klee, *Separation by hyperplanes in finite-dimensional vector spaces over Archimedean ordered fields*, J. Convex Anal. **5** (1998), no. 2, 279–301. MR1670356 (99k:46012)

[12] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, 6th ed., Oxford University Press, Oxford, 2008. Revised by D. R. Heath-Brown and J. H. Silverman; With a foreword by Andrew Wiles. MR2445243 (2009i:11001)

[13] E. Heppner and W. Schwarz, *Benachbarte multiplikative Funktionen*, Studies in pure mathematics, Birkhäuser, Basel, 1983, pp. 323–336 (German). MR820232 (87h:11097)

[14] Edwin Hewitt and Kenneth A. Ross, *Abstract harmonic analysis. Vol. I*, 2nd ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 115, Springer-Verlag, Berlin, 1979. Structure of topological groups, integration theory, group representations. MR551496 (81k:43001)

[15] Edwin Hewitt and J. H. Williamson, *Note on absolutely convergent Dirichlet series*, Proc. Amer. Math. Soc. **8** (1957), 863–868. MR0090680 (19,851b)

[16] Edwin Hewitt and Herbert S. Zuckerman, *The l_1-algebra of a commutative semigroup*, Trans. Amer. Math. Soc. **83** (1956), 70–97. MR0081908 (18,465b)

[17] John Knopfmacher, *Abstract analytic number theory*, 2nd ed., Dover Books on Advanced Mathematics, Dover Publications Inc., New York, 1990. MR1068138 (91d:11110)

[18] Yulia Kuznetsova and Carine Molitor-Braun, *Harmonic analysis of weighted L^p-algebras*, Expo. Math. **30** (2012), no. 2, 124–153, DOI 10.1016/j.exmath.2012.01.002. MR2928197

[19] Serge Lang, *Algebra*, 3rd ed., Graduate Texts in Mathematics, vol. 211, Springer-Verlag, New York, 2002. MR1878556 (2003e:00003)

[20] P. Lévy, *Sur la convergence absolue des séries de Fourier*, C.R. Acad. Sci. Paris **196** (1933), 463–464; Compositio Math. **1** (1934), 1–14.

[21] Lutz Lucht, *An application of Banach algebra techniques to multiplicative functions*, Math. Z. **214** (1993), no. 2, 287–295, DOI 10.1007/BF02572405. MR1240890 (94g:11072)

[22] Lutz G. Lucht, *Banach algebra techniques in the theory of arithmetic functions*, Acta Math. Univ. Ostrav. **16** (2008), no. 1, 45–56. MR2498636 (2009k:11150)

[23] Lutz G. Lucht, *A survey of Ramanujan expansions*, Int. J. Number Theory **6** (2010), no. 8, 1785–1799, DOI 10.1142/S1793042110003800. MR2755471 (2011m:11195)

[24] L.G. Lucht and K. Reifenrath, *Weighted Wiener-Lévy theorems*, in: Analytic Number Theory. Proceedings of a Conference in Honor of Heini Halberstam, Urbana-Champaign, 1995, Vol. 2, Birkhäuser, Boston 1996, pp. 607–619.

[25] Karl-Hermann Neeb, *Holomorphy and convexity in Lie theory*, de Gruyter Expositions in Mathematics, vol. 28, Walter de Gruyter & Co., Berlin, 2000. MR1740617 (2001j:32020)

[26] K. Reifenrath, *Gewichtete Wiener-Lévy-Sätze und arithmetische Halbgruppen*, Dissertation, TU Clausthal 1995.

[27] Kenneth A. Ross, *Extending characters on semigroups*, Proc. Amer. Math. Soc. **12** (1961), 988–990. MR0132793 (24 #A2629)

[28] W. Rudin, *Real and Complex Analysis,* McGraw-Hill, London, 1970.

[29] Walter Rudin, *Functional analysis*, 2nd ed., International Series in Pure and Applied Mathematics, McGraw-Hill Inc., New York, 1991. MR0365062

[30] H. H. Schaefer and M. P. Wolff, *Topological vector spaces*, 2nd ed., Graduate Texts in Mathematics, vol. 3, Springer-Verlag, New York, 1999. MR1741419 (2000j:46001)

[31] Wolfgang Schwarz and Jürgen Spilker, *Arithmetical functions*, London Mathematical Society Lecture Note Series, vol. 184, Cambridge University Press, Cambridge, 1994. An introduction to elementary and analytic properties of arithmetic functions and to some of their almost-periodic properties. MR1274248 (96b:11001)

[32] Jürgen Spilker and Wolfgang Schwarz, *Wiener-Lévy-Sätze für absolut konvergente Reihen*, Arch. Math. (Basel) **32** (1979), no. 3, 267–275, DOI 10.1007/BF01238499 (German). MR541625 (81d:32002)

[33] Bit Shun Tam, *A note on polyhedral cones*, J. Austral. Math. Soc. Ser. A **22** (1976), no. 4, 456–461. MR0429954 (55 #2962)

[34] H. Weyl, *Elementare Theorie der konvexen Polyeder*, Comment. Math. Helv. **7** (1934), no. 1, 290–306, DOI 10.1007/BF01292722 (German). MR1509514

[35] Norbert Wiener, *Tauberian theorems*, Ann. of Math. (2) **33** (1932), no. 1, 1–100, DOI 10.2307/1968102. MR1503035

Institut für Mathematik, Universität Paderborn, Warburger Str. 100, D-33098 Paderborn, Germany

E-mail address: glockner@math.upb.de

Institut für Mathematik, Technische Universität Clausthal, Erzstr. 1, D-38678 Clausthal-Zellerfeld, Germany

E-mail address: lg.lucht@cintech.de