HIGHER ORDER EXTENSION OF LÖwner’s THEORY:
OPERATOR k-TONE FUNCTIONS

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Abstract. The new notion of operator/matrix k-tone functions is introduced, which is a higher order extension of operator/matrix monotone and convex functions. Differential properties of matrix k-tone functions are shown. Characterizations, properties, and examples of operator k-tone functions are presented. In particular, integral representations of operator k-tone functions are given, generalizing familiar representations of operator monotone and convex functions.

Introduction

The theory of operator/matrix monotone functions was initiated by the celebrated paper of L"owner [23], which was soon followed by Kraus [22] on operator/matrix convex functions. After further developments due to some authors (for instance, Bendat and Sherman [6] and Koranyi [21]), in their seminal paper [15] Hansen and Pedersen established a modern treatment of operator monotone and convex functions. In [3,8,13] (also [18]) are found comprehensive expositions on the subject matter. A remarkable feature of L"owner’s theory is that we have several characterizations of operator monotone and convex functions from several different points of view. The importance of complex analysis in studying operator monotone functions is well understood from their characterization in terms of analytic continuation as Pick functions. Integral representations for operator monotone and convex functions are essential ingredients of the theory from both theoretical and application sides. The notion of divided differences has played a vital role in the theory from its very beginning.

The operator/matrix-valued differential calculus of the form

\[
\frac{d^k}{dt^k} f(A + tX) \bigg|_{t=0}
\]

is quite relevant to operator/matrix monotone and convex functions, where A is a self-adjoint operator with spectrum in the domain interval of \( f \). For matrices the above \( k \)th derivative exists whenever \( f \) is \( C^k \) on \((a,b)\) (see [9,15]), and its formula in terms of divided differences due to Dalecki and Krein [11] is very useful as is clearly mentioned in the survey paper [12]. In the infinite-dimensional setting, the existence of the \( k \)th derivative is rather subtle but it exists in the operator norm when \( f \) is analytic (see [24] for the \( k \)th derivative under a weaker assumption). It

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is well known \[9\] that the operator/matrix monotonicity of \(f\) is characterized by the positivity of derivative \(\frac{d}{d\alpha} f(\alpha)\) for \(k = 1\), and the operator/matrix convexity is similarly characterized by the positivity of \(\frac{d^2}{d\alpha^2} f(\alpha)\) for \(k = 2\). Differential methods were also adopted in \[4,16,17\] for some analysis of matrix convex functions. Our motivation for the present paper came from the naïve question of what is a higher order extension of the operator monotonicity related to the higher order derivative in \(\frac{d^k}{d\alpha^k} f(\alpha)\) for \(k > 2\). Although the idea seems very natural, this kind of higher order extension of Löwner’s theory is new, while some other types of extensions have been discussed (see \[2\] for instance).

In Section 1 of the paper we first define, for a real function on \((a, b)\), operator/matrix-valued divided differences of \(f\) by generalizing the usual divided differences. By using these generalized divided differences we introduce, for \(k, n \in \mathbb{N}\), the notions of operator \(k\)-tone functions and matrix \(k\)-tone functions of order \(n\), which are higher order extensions of operator/matrix monotone and convex functions. In fact, when \(k = 1\) and \(k = 2\), operator \(k\)-tone functions are operator monotone and convex functions, respectively. In the last part of Section 1 slightly refined forms of the standard integral representations of operator monotone functions on \((-1,1)\) and on \((0,\infty)\) are provided for later use. The main theorem of Section 2 gives differentiability properties of matrix \(k\)-tone functions of order \(n\); such a function must be of class \(C^{2n+2k-4}\). In particular, an \(n\)-convex function is \(C^{2n-2}\), extending the classical result of Kraus \[22\] when \(n = 2\). Section 3 presents several characterizations of operator \(k\)-tone functions on \((a, b)\), e.g., in terms of derivatives or divided differences of \(f\). It turns out that \(f\) is operator \(k\)-tone on \((a, b)\) if and only if for any \(\alpha \in (a, b)\) there exists an operator monotone function \(g\) on \((a, b)\) such that

\[
(0.2) \quad f(x) = \sum_{l=0}^{k-2} \frac{f^{(l)}(\alpha)}{l!} (x-\alpha)^l + (x-\alpha)^{k-1} g(x), \quad x \in (a, b),
\]

that is, the \((k-1)\)th remainder term of the Taylor series of \(f\) at \(\alpha\) is given as \((x-\alpha)^{k-1}\) times an operator monotone function. This shows that operator \(k\)-tone functions have rather simple structure with only additive and multiplicative polynomial factors beyond operator monotone functions. Sections 4 and 5 contain further properties of operator \(k\)-tone functions on \((-1,1)\) and on \((0,\infty)\), respectively. In particular, we present integral expressions of such functions, generalizing the well-known versions for operator monotone functions (when \(k = 1\)). Furthermore, in the last parts of Sections 4 and 5 we clarify what are the operator versions of absolutely monotone and completely monotone functions on \((-1,1)\) and on \((0,\infty)\). Examples of operator \(k\)-tone functions on \((0,\infty)\) are provided in Section 6. Finally, it is worth noting that the operator \(k\)-tonicity condition gets weaker and weaker as \(k\) gets bigger and bigger (see Propositions 3.9 and 5.3 for precise statements), unlike the usual differentiability property of numerical functions.

1. Definitions and preliminaries

1.1. Notation. For each \(n \in \mathbb{N}\), \(M_n\) is the set of \(n \times n\) matrix algebra, \(M_n^sa\) the set of \(n \times n\) Hermitian matrices, and \(M_n^+\) the set of \(n \times n\) positive semidefinite matrices. Throughout the paper \(\mathcal{H}\) is a fixed infinite-dimensional separable Hilbert space, \(B(\mathcal{H})\) is the set of all bounded operators on \(\mathcal{H}\), \(B(\mathcal{H})^sa\) the set of self-adjoint operators on \(\mathcal{H}\), and \(B(\mathcal{H})^+\) the set of positive operators on \(\mathcal{H}\). The symbol \(I\)
denotes the identity matrix or the identity operator. For an open interval \((a, b)\) in the real line \(\mathbb{R}\), we write \(M_n^{sa}(a, b)\) and \(B(H)^{sa}(a, b)\) for the sets of all elements of \(M_n^{sa}\) and of \(B(H)^{sa}\), respectively, with spectra in \((a, b)\), which are convex and open (in the norm topology) in \(M_n^{sa}\) and \(B(H)^{sa}\), respectively. In particular, \(M_n^{sa}(0, \infty)\) is the set of \(n \times n\) invertible positive semidefinite matrices. When \(f\) is a real function on \((a, b)\) and \(A \in M_n^{sa}(a, b)\), \(f(A)\) is the usual functional calculus of \(A\) by \(f\), and \(f(A)\) for \(A \in B(H)^{sa}(a, b)\) is similar when \(f\) is continuous.

1.2. Operator/matrix-valued divided differences. Let \(f\) be a real function on an open interval \((a, b)\), where \(-\infty \leq a < b \leq \infty\). For distinct \(x_0, x_1, x_2, \ldots\) in \((a, b)\), the divided differences of \(f\) are recursively defined as

\[
f[1](x_0, x_1) := \frac{f(x_0) - f(x_1)}{x_0 - x_1},
\]

and for \(k = 2, 3, \ldots\),

\[
f[k](x_0, x_1, \ldots, x_k) := \frac{f[k-1](x_0, x_1, \ldots, x_{k-1}) - f[k-1](x_1, \ldots, x_{k-1}, x_k)}{x_0 - x_k}.
\]

For each \(k \in \mathbb{N}\) the \(k\)th divided difference \(f[k](x_0, x_1, \ldots, x_k)\) can be extended by continuity to a continuous function on \((a, b)^{k+1}\) whenever \(f\) is \(C^k\) on \((a, b)\). See [13] pp. 1–7 and [18] Sect. 2.2 for properties of divided differences.

To introduce the key notion of operator or matrix \(k\)-tone functions, we need to extend the above divided differences to the operator-valued or matrix-valued version. Of course it does not make sense to replace the real variables \(x_0, x_1, \ldots\) in the above with self-adjoint operators or matrices. To extend the divided differences to operators, we fix two \(A, B\) in \(B(H)^{sa}(a, b)\) or in \(M_n^{sa}(a, b)\) and distinct \(t_0, t_1, t_2, \ldots\) in \([0, 1]\). Let \(X_k := (1-t_k)A+t_kB\) for \(k = 0, 1, 2, \ldots\), and define the operator-valued or matrix-valued divided difference of \(f\) as follows:

\[
f[1](A, B; t_0, t_1) := \frac{f(X_0) - f(X_1)}{t_0 - t_1},
\]

and for \(k = 2, 3, \ldots\),

\[
f[k](A, B; t_0, t_1, \ldots, t_k)
\]

\[:= \frac{f[k-1](A, B; t_0, t_1, \ldots, t_{k-1}) - f[k-1](A, B; t_1, \ldots, t_{k-1}, t_k)}{t_0 - t_k}.
\]

In particular, for \(\alpha, \beta \in (a, b)\) with \(\alpha < \beta\) and for distinct \(t_0, t_1, \ldots \in [0, 1]\), let \(x_k := (1-t_k)\alpha + t_k\beta\). We then notice that

\[
f[k](x_0, x_1, \ldots, x_k)I = \frac{1}{(\beta - \alpha)^k} f[k](\alpha I, \beta I; t_0, t_1, \ldots, t_k),
\]

from which we can consider \(f[k](A, B; t_0, t_1, \ldots, t_k)\) as a natural operator or matrix version of the usual \(k\)th divided difference. (A further generalization of the divided difference for functions on vector spaces was recently proposed in [7].)
Lemma 1.1. Let \( A, B \) and \( t_k, X_k \) for \( k = 0, 1, \ldots \) be as above. For every \( k \in \mathbb{N} \),
\[
   f^{[k]}(A, B; t_0, t_1, \ldots, t_k) = \sum_{l=0}^{k} \frac{f(X_l)}{\prod_{0 \leq j < k, j \neq l} (t_i - t_j)}
   = \sum_{l=0}^{k} (-1)^{k-l} \frac{\prod_{0 \leq i < j \leq k, i, j \neq l} (t_j - t_i) f(X_l)}{\prod_{0 \leq i < j \leq k} (t_j - t_i)}.
\]

Hence, \( f^{[k]}(A, B; t_0, t_1, \ldots, t_k) \) is symmetric in the variables \( t_0, t_1, \ldots, t_k \).

Proof. The first equality is easy to prove by induction on \( k \) and the second is a simple rewriting. \qed

Example 1.2. Let \( m \in \mathbb{N} \), \( f(x) = x^m \), and let \( A, B \) be in \( B(\mathcal{H})^{sa} \) or in \( M_n^{sa} \). Using an induction on \( k \) one can easily verify that
\[
   f^{[k]}(A, B; t_0, t_1, \ldots, t_k)
   = F_{k, m-k}(B - A, A) + \sum_{l=k+1}^{m} \left( \sum_{j_0, j_1, \ldots, j_k \geq 0, j_0 + j_1 + \cdots + j_k = l-k} j_0 j_1 \cdots j_k \right) F_{l, m-l}(B - A, A)
\]
for every \( k \in \mathbb{N} \), where \( F_{l, m-l}(X, Y) \) denotes the sum of all products of \( l \) \( X \)'s and \( m-l \) \( Y \)'s for \( X, Y \in B(\mathcal{H})^{sa} \) (this is defined to be zero unless \( 0 \leq l \leq m \)). In particular,
\[
   f^{[m]}(A, B; t_0, t_1, \ldots, t_m) = F_{m, 0}(B - A, A) = (B - A)^m,
   f^{[k]}(A, B; t_0, t_1, \ldots, t_k) = 0 \quad \text{for all } k > m.
\]

It is known [9, Theorem 2.1] (also [18, Theorem 2.3.1]) that if \( f \) is \( C^k \) on \((a, b)\), then the matrix functional calculus \( f(A) \) is \( k \) times Fréchet differentiable at every \( A \in M_n^{sa}(a, b) \) and the \( k \)th Fréchet derivative \( D^k f(A) \), a multi-linear map from \((M_n^{sa})^k\) into \( M_n^{sa} \), is continuous in \( A \). Consequently, the \( k \)th derivative
\[
   \left. \frac{d^k}{dt^k} f(A + tX) \right|_{t=0} = D^k f(A) (X, \ldots, X)_k
\]
exists for every \( A \in M_n^{sa}(a, b) \) and \( X \in M_n^{sa} \) and is continuous in \( A \) and \( X \). For infinite-dimensional Hilbert space operators, it was shown in [11] that \( t \mapsto f(A + tX) \) is differentiable and expressed as a double operator integral under the \( C^2 \) assumption of \( f \), and later Birman and Solomyak developed the general theory of double operator integrals (its concise account is found in [19]). However, the situation in the infinite-dimensional case is rather subtle; indeed the \( C^1 \) assumption of \( f \) is not sufficient for the differentiability of \( f(A + tX) \) as mentioned in [24]. Taking this into account, we restrict ourselves to the matrix-valued case for the following continuous extendability property of \( f^{[k]}(A, B; t_0, t_1, \ldots, t_n) \).

Proposition 1.3. Assume that \( f \) is \( C^k \) on \((a, b)\). Then for every \( n \in \mathbb{N} \) and every \( A, B \in M_n^{sa}(a, b) \), \( f^{[k]}(A, B; t_0, t_1, \ldots, t_k) \) for distinct \( t_0, t_1, \ldots, t_k \in [0, 1] \) can be extended by continuity to a function on the whole \([0, 1]^{k+1}\) so that \( f^{[k]}(A, B; t_0, t_1, \ldots, t_k) \) is continuous in \((A, B; t_0, t_1, \ldots, t_k) \in (M_n^{sa}(a, b))^2 \times [0, 1]^{k+1}\).
Proof. Let $A, B \in \mathbb{M}^{sa}_n(a, b)$ and choose a $\delta > 0$ so that $(1 - t)A + tB \in \mathbb{M}^{sa}_n(a, b)$ for all $t \in (-\delta, 1 + \delta)$. For any state $\omega$ on $\mathbb{M}_n$, define
\[ \phi_\omega(t) := \omega((1 - t)A + tB), \quad t \in (-\delta, 1 + \delta). \]
For any distinct $t_0, t_1, \ldots, t_k$ in $[0, 1]$ it is obvious by definition that
\[ \phi_\omega^{[k]}(t_0, t_1, \ldots, t_k) = \omega( (A, B; t_0, t_1, \ldots, t_k)). \]
From the $C^k$ assumption of $f$, the matrix-valued function $t \mapsto f((1 - t)A + tB)$ is $C^k$ on $(-\delta, 1 + \delta)$ as remarked before the proposition. Hence $\phi_\omega$ is $C^k$ on $(-\delta, 1 + \delta)$ so that $\phi_\omega^{[k]}(t_0, t_1, \ldots, t_k)$ can extend to a continuous function on $(-\delta, 1 + \delta)^{k+1}$. Since this is the case for every state $\omega$, we see that $f^{[k]}(A, B; t_0, t_1, \ldots, t_k)$ extends to a continuous function on $[0, 1]^{k+1}$. Furthermore, the stronger continuity of $f^{[k]}$ in the whole variables $(A, B; t_0, t_1, \ldots, t_k)$ can be shown in a way similar to the proof of [13] Lemma 2.2.4 by using the continuity of $\frac{d}{dt} f((1 - t)A + tB), 0 \leq t \leq k$, in $(A, B, t)$. We omit the details for this. \hfill \Box

1.3. Definition of operator/matrix $k$-tone functions.

Definition 1.4. Let $k \in \mathbb{N}$ and $f$ be a real continuous function on $(a, b)$. We say that $f$ is operator $k$-tone on $(a, b)$ if, for every $A, B \in B(\mathcal{H})^{sa}(a, b)$ with $A \leq B$ and for any $0 = t_0 < t_1 < \cdots < t_k = 1$, we have
\[ f^{[k]}(A, B; t_0, t_1, \ldots, t_k) \geq 0, \tag{1.2} \]
or equivalently, due to Lemma 1.1,
\[ \sum_{l=0}^{k} (-1)^{k-l} \prod_{0 \leq i < j \leq k} (t_j - t_i) f(X_l) \geq 0, \tag{1.3} \]
where $X_l := (1 - t_l)A + t_lB, 0 \leq l \leq k$. Moreover, for each $n \in \mathbb{N}$, if a real (not necessarily continuous) function $f$ on $(a, b)$ satisfies (1.2), or equivalently (1.3), for every $A, B \in \mathbb{M}^{sa}_n(a, b)$ with $A \leq B$ and for any $t_0, t_1, \ldots, t_k$ as above, then we say that $f$ is matrix $k$-tone of order $n$ on $(a, b)$. A matrix $k$-tone function of order 1 (i.e., a $k$-tone function in the numerical sense) is said to be $k$-tone for short.

When $k = 1$, inequality (1.3) is nothing but $-f(A) + f(B) \geq 0$ so that $f$ is operator 1-tone on $(a, b)$ if and only if it is operator monotone on $(a, b)$. When $k = 2$, (1.3) is
\[ (1 - t_1)f(A) - f((1 - t_1)A + t_1B) + t_1f(B) \geq 0, \quad 0 < t_1 < 1. \]
Hence $f$ is matrix 2-tone of order $n$ on $(a, b)$ if and only if $f$ is conditionally $n$-convex there (the conditional matrix convexity here means the matrix convexity under condition $A \leq B$). Note [22] (also [13] Theorem 2.4.4) that the conditional $n$-convexity is equivalent to the usual $n$-convexity. Thus, Definition 1.4 may be considered as a natural higher order extension of operator/matrix monotone or convex functions.

By Example 1.2 note that any polynomial function $\sum_{l=0}^{k} \alpha_l x^l$ of real coefficients with $\alpha_k \geq 0$ is operator $k$-tone on the whole $\mathbb{R}$.

Lemma 1.5. Let $k, n \in \mathbb{N}$. A real function $f$ on $(a, b)$ is matrix $k$-tone of order $n$ if and only if $f$ satisfies (1.2) for every $A, B \in \mathbb{M}^{sa}_n(a, b)$ and for any distinct $t_0, t_1, \ldots, t_k \in [0, 1]$. The same is true for the operator $k$-tonicity of a real continuous function $f$. 

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The lemma is easily shown by using Lemma 1.1 and the following proposition is shown by a standard convergence argument (as will also be used in the proof of Corollary 3.5(c)). Proofs of these may be omitted here.

Proposition 1.6. Let \( k \in \mathbb{N} \) and \( f \) be a real continuous function on \((a, b)\). Then \( f \) is operator \( k \)-tone if and only if it is matrix \( k \)-tone of every order \( n \).

1.4. Integral representations of operator monotone functions. In this subsection we show integral representations of operator monotone functions on \((-1, 1)\) and on \((0, \infty)\), which will be useful in our later discussions. Although such integral representations are well known as described in [8, Sect. V.4] (also [13, Sect. 2.7]), our representations below are slight modifications of the standard ones, including a new insight on certain universality of the representing measure. The representations will indeed be extended to operator \( k \)-tone functions in Theorems 4.1 and 5.1.

Theorem 1.7. Let \( f \) be an operator monotone function on \((-1, 1)\). Then there exists a unique finite positive measure \( \mu \) on \([-1, 1] \) such that for any choice of \( \alpha \in (-1, 1) \),

\[
\tag{1.4} f(x) = f(\alpha) + \int_{[-1,1]} \frac{x - \alpha}{(1 - \lambda x)(1 - \lambda \alpha)} \, d\mu(\lambda), \quad x \in (-1, 1).
\]

Proof. For each fixed \( \alpha \in (-1, 1) \) consider the transformation \( \varphi_\alpha \) from \([-1, 1] \) onto itself defined by

\[
\varphi_\alpha(t) := \frac{t + \alpha}{1 + \alpha t}, \quad t \in [-1, 1],
\]

which is an operator monotone function with \( \varphi_\alpha(0) = \alpha \). Apply [8, V.4.5] to obtain a unique finite positive measure \( m_\alpha \) on \([-1, 1] \) such that

\[
\tag{1.5} (f \circ \varphi_\alpha)(t) = f(\alpha) + \int_{[-1,1]} \frac{t}{1 - \kappa t} \, dm_\alpha(\kappa), \quad t \in (-1, 1).
\]

Defining a finite positive measure \( \mu_\alpha \) on \([-1, 1] \) by

\[
d\mu_\alpha(\lambda) := \frac{1 - \lambda \alpha}{1 + \alpha \varphi_\alpha^{-1}(\lambda)} \, dm_\alpha(\varphi_\alpha^{-1}(\lambda)),
\]

we have

\[
\tag{1.6} f(x) = f(\alpha) + \int_{[-1,1]} \frac{x - \alpha}{(1 - \lambda x)(1 - \lambda \alpha)} \, d\mu_\alpha(\lambda), \quad x \in (-1, 1),
\]

which is representation (1.4) while \( \mu = \mu_\alpha \) is depending on \( \alpha \in (-1, 1) \) at the moment. Moreover, since expression (1.6) can conversely be converted into (1.5), it is seen that a representing measure \( \mu_\alpha \) in (1.6) is unique.

Now, we prove that \( \mu_\alpha \) is independent of the parameter \( \alpha \in (-1, 1) \). For any \( \alpha, \beta \in (-1, 1) \), inserting

\[
\frac{x - \alpha}{(1 - \lambda x)(1 - \lambda \alpha)} = \frac{\beta - \alpha}{(1 - \lambda \alpha)(1 - \lambda \beta)} + \frac{x - \beta}{(1 - \lambda x)(1 - \lambda \beta)}
\]

into (1.6) we have

\[
f(x) = f(\alpha) + \int_{[-1,1]} \frac{\beta - \alpha}{(1 - \lambda \alpha)(1 - \lambda \beta)} \, d\mu_\alpha(\lambda) + \int_{[-1,1]} \frac{x - \beta}{(1 - \lambda x)(1 - \lambda \beta)} \, d\mu_\alpha(\lambda).
\]
Letting $x = \beta$ gives
\[
f(\beta) = f(\alpha) + \int_{[-1,1]} \frac{\beta - \alpha}{(1 - \lambda \alpha)(1 - \lambda \beta)} \, d\mu_\alpha(\lambda),
\]
and $\mu_\alpha = \mu_\beta$ follows from the uniqueness of $\mu_\beta$ representing $f$ in (1.6) with $\beta$ in place of $\alpha$, so the theorem has been proved. \hfill \Box

Note that from (1.4) we have
\[
f'(\alpha) = \int_{[-1,1]} \frac{1}{(1 - \lambda \alpha)^2} \, d\mu(\lambda), \quad \alpha \in (-1,1).
\]
In particular, $\mu([-1,1]) = f'(0)$.

The theorem has the following corollary, which will play an essential role to prove the main theorem of Section 3.

**Corollary 1.8.** Let $f$ be an operator monotone function on $(-1,1)$ with the representing measure $\mu$ as in Theorem 1.7. For every $\alpha \in (-1,1)$ and every $m, k \in \mathbb{N}$ with $m \geq k$,
\[
((x - \alpha)^{k-1} f)[m](x_1, x_2, \ldots, x_{m+1}) = \int_{[-1,1]} \frac{\lambda^{m-k}(1 - \lambda \alpha)^{k-1}}{(1 - \lambda x_1)(1 - \lambda x_2) \cdots (1 - \lambda x_{m+1})} \, d\mu(\lambda)
\]
for all $x_1, x_2, \ldots, x_{m+1} \in (-1,1)$, where $\lambda^{m-k} \equiv 1$ on $[-1,1]$ if $m = k$.

**Proof.** For every $\lambda \in [-1,1] \setminus \{0\}$ we have
\[
\frac{(x - \alpha)^k}{(1 - \lambda x)(1 - \lambda \alpha)^k} = \frac{(1 - \lambda \alpha) - (1 - \lambda x)}{\lambda^k(1 - \lambda x)(1 - \lambda \alpha)^k} = \frac{1}{\lambda^k(1 - \lambda x)} + \text{(a polynomial of degree } k - 1).\]

Since $m \geq k$, we hence have
\[
\left(\frac{(x - \alpha)^k}{(1 - \lambda x)(1 - \lambda \alpha)^k}\right)[m](x_1, x_2, \ldots, x_{m+1}) = \frac{1}{\lambda^k} \left(\frac{1}{1 - \lambda x}\right)[m](x_1, x_2, \ldots, x_{m+1}) = \frac{\lambda^{m-k}}{(1 - \lambda x_1)(1 - \lambda x_2) \cdots (1 - \lambda x_{m+1})}
\]
for all $x_1, x_2, \ldots, x_{k+1} \in (-1,1)$. The above certainly holds for $\lambda = 0$ as well. Therefore, integrating against the measure $(1 - \lambda \alpha)^{k-1} \, d\mu$ gives the result, as we can take the $k$th divided difference inside the integral in (1.7). \hfill \Box

**Theorem 1.9.** Let $f$ be an operator monotone function on $(0,\infty)$. Then there exists a unique $\gamma \geq 0$ and a unique positive measure $\mu$ on $[0,\infty)$ such that
\[
\int_{[0,\infty)} \frac{1}{(1 + \lambda)^2} \, d\mu(\lambda) < +\infty,
\]
and for any choice of $\alpha \in (0,\infty)$,
\[
f(x) = f(\alpha) + \gamma(x - \alpha) + \int_{[0,\infty)} \frac{x - \alpha}{(x + \lambda)(\alpha + \lambda)} \, d\mu(\lambda), \quad x \in (0,\infty).
\]
Proof. For each $\alpha \in (0, \infty)$ consider the transformation $\psi_\alpha$ from $[-1, 1)$ onto $[0, \infty)$ defined by

$$\psi_\alpha(t) := \frac{\alpha(1 + t)}{1 - t}, \quad t \in [-1, 1),$$

which is operator monotone on $[-1, 1)$. Representing $f \circ \psi_\alpha$ as in (1.5) and defining $\gamma_\alpha \geq 0$ as a positive measure $\mu_\alpha$ on $[0, \infty)$ by

$$\gamma_\alpha := m_\alpha([1]) \quad d\mu_\alpha(\lambda) := \frac{\alpha + \lambda}{1 - \psi_\alpha^{-1}(\lambda)} d\mu_\alpha(\psi_\alpha^{-1}(\lambda)),$$

we have the integral expression of $f$. The remaining proof is similar to that of Theorem 1.7, so the details are omitted. \hfill \Box

From (1.8) we have

$$f'(\alpha) = \gamma + \int_{[0, \infty)} \frac{1}{(\alpha + \lambda)^2} d\mu(\lambda), \quad \alpha \in (0, \infty),$$

and hence $\gamma = \lim_{\alpha \to \infty} f'(\alpha)$.

2. Differentiability properties of matrix $k$-tone functions

It is well known [13, 23] that if $f$ is $n$-monotone (or matrix 1-tone of order $n$ in our terminology) on $(a, b)$, then it is $C^{2n-3}$ on $(a, b)$ and $f^{(2n-3)}$ is convex there. Also, a primary result of [22] is that if $f$ is conditionally 2-convex (or matrix 2-tone of order 2), then it is $C^2$ there. The main aim of this section is to prove the next theorem extending the above results to matrix $k$-tone functions of order $n$ for general $k$ and $n$. In particular, when $k = 2$, the theorem shows differentiability results for $n$-convex functions. It seems that assertion (a) is new even in this particular case where $k = 2$ and $n > 2$. Results in [22, 23] say that when $k = 1$ and $k = 2$ property (c) is not only necessary but also sufficient for $f$ to be matrix $k$-tone of order $n$. Also, see [13, 16, 17] for property (d) when $k = 1, 2$.

Theorem 2.1. Let $k, n \in \mathbb{N}$ and assume that a real function $f$ on $(a, b)$ is matrix $k$-tone of order $n$. Then the following properties hold:

(a) $f$ is $C^{2n+k-4}$ on $(a, b)$ if $k \geq 2$ or $n \geq 2$.

(b) The following functions are convex on $(a, b)$:

$$\begin{cases} f', f^{(3)}, \ldots, f^{(2n-3)} & \text{if } k = 1 \text{ and } n \geq 2, \\ f^{(k-2)}, f^{(k)}, \ldots, f^{(2n+k-4)} & \text{if } k \geq 2 \text{ and } n \geq 1. \end{cases}$$

(c) The matrix

$$\left[f^{[k]}(x_i, x_j, x_1, \ldots, x_1)\right]_{i,j=1}^{n}$$

is positive semidefinite for any choice of $x_1, \ldots, x_n$ from $(a, b)$ if $n \geq 2$.

(d) The matrix

$$\begin{bmatrix} \frac{f^{(i+j+k)}(x)}{(i+j+k)!} \end{bmatrix}_{i,j=0}^{n-1}$$

exists and is positive semidefinite for almost every $x \in (a, b)$. 

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Lemma 2.2. Let \( f \) be a \( C^k \) function on \( (a, b) \) satisfying the assumption of Theorem 2.1. Then
\[
\frac{d^k}{dt^k} f(A + tX)\bigg|_{t=0} \geq 0
\]
for every \( A \in \mathbb{M}_n^{sa}(a, b) \) and every \( X \in \mathbb{M}_n^+ \).

Proof. Let \( A \in \mathbb{M}_n^{sa}(a, b) \) and \( X \in \mathbb{M}_n^+ \). We may assume that \( A + X \in \mathbb{M}_n^{sa}(a, b) \); then a \( \delta > 0 \) is chosen so that \( A + tX \in \mathbb{M}_n^{sa}(a, b) \) for all \( t \in (-\delta, 1) \). For any state \( \omega \) on \( \mathbb{M}_n \) define
\[
\phi_{\omega}(t) := \omega(f(A + tX)), \quad t \in (-\delta, 1),
\]
which is \( C^k \) on \((-\delta, 1)\) due to the \( C^k \) assumption on \( f \). When \( 0 = t_0 < t_1 < \cdots < t_k < 1 \), we have as [13],
\[
\phi_{\omega}^{[k]}(t_0, t_1, \ldots, t_k) = \omega(f^{[k]}(A, A + X; t_0, t_1, \ldots, t_k)) \geq 0
\]
by Lemma [15]. Letting \( t_l \searrow 0 \) for \( 1 \leq l \leq k \) gives
\[
0 \leq \phi_{\omega}^{[k]}(0, 0, \ldots, 0) = \frac{1}{k!} \phi_{\omega}^{(k)}(0) = \frac{1}{k!} \omega\left( \frac{d^k}{dt^k} f(A + tX)\bigg|_{t=0} \right)
\]
by [13, p. 6] (also [18, Lemma 2.2.4]). The conclusion follows since the state \( \omega \) is arbitrary.

We next prove properties (c) and (d) of the theorem under the additional assumption of \( f \) being \( C^\infty \).

Lemma 2.3. Let \( f \) be a \( C^\infty \) function on \( (a, b) \) satisfying the assumption of Theorem 2.1. Then (c) holds including the case \( n = 1 \) and (d) holds for every \( x \in (a, b) \).

Proof. (c) For every \( x_1, \ldots, x_n \in (a, b) \) let \( A := \text{Diag}(x_1, \ldots, x_n) \), the diagonal matrix with diagonal entries \( x_1, \ldots, x_n \). According to Daleckii and Krein’s derivative formula in the matrix case (see [18, Theorem 2.3.1]), for every \( X \in \mathbb{M}_n^{sa} \) we have
\[
\frac{d^k}{dt^k} f(A + tX)\bigg|_{t=0} = \left[ \sum_{r_1, \ldots, r_{k-1}=1}^n k! f^{[k]}(x_i, x_{r_1}, \ldots, x_{r_{k-1}}, x_j) X_{i r_1} X_{r_1 r_2} \cdots X_{r_{k-1} j} \right]_{i,j=1}^n.
\]
(2.1)

For any \( \xi_1, \ldots, \xi_n \in \mathbb{C} \) let \( X := \left[ \bar{\xi}_i \xi_j \right]_{i,j=1}^n \in \mathbb{M}_n^+ \). Lemma 2.2 then implies that for any \( \zeta_1, \ldots, \zeta_n \in \mathbb{C} \),
\[
\sum_{i,j=1}^n \sum_{r_1, \ldots, r_{k-1}=1}^n f^{[k]}(x_i, x_{r_1}, \ldots, x_{r_{k-1}}, x_j) \bar{\xi}_i |\xi_{r_1}|^2 \cdots |\xi_{r_{k-1}}|^2 \bar{\xi}_j \zeta_j \geq 0.
\]

We may replace \( \zeta \) above with \( \zeta / \xi \) under the assumption that \( \xi \neq 0 \) for all \( i \). Now let \( \xi_1 = 1 \) and \( \xi_r \to 0 \) for \( r \neq 1 \) to obtain
\[
\sum_{i,j=1}^n f^{[k]}(x_i, x_{r_1}, \ldots, x_{r_{k-1}}, x_j) \bar{\xi}_i \zeta_j \geq 0.
\]

(d) For any fixed \( x \in (a, b) \) define a \( C^\infty \) function \( g \) on \( (a, b) \) by
\[
g(t) := f^{[k-1]}(t, x, \ldots, x), \quad t \in (a, b).
\]
It is plain to notice that
\begin{equation}
(2.2) \quad g(t) = \frac{1}{(t-x)^{k-1}} \left\{ f(t) - \sum_{l=0}^{k-2} \frac{f^{(l)}(x)}{l!} (t-x)^l \right\}, \quad t \in (a,b),
\end{equation}
where \( g(t) = f(t) \) for \( k = 1 \). Set \( \delta := \min\{x-a, b-x\}/n \). For every \( h \in (0, \delta) \) we then have
\[
G_h := \left[ g^{[1]}(x + ih, x + jh) \right]_{i,j=0}^{n-1} = \left[ f^{[k]}(x + ih, x + jh, x, \ldots, x) \right]_{i,j=0}^{n-1} \geq 0
\]
thanks to (c) proved above. By Taylor’s theorem we expand \( g^{[1]}(x + ih, x + jh) \) as
\[
g^{[1]}(x + ih, x + jh) = \sum_{m=0}^{2n-2} \frac{g^{(m+1)}(x)}{(m+1)!} \cdot \sum_{i+j = m+1} \frac{i^{m+1} - j^{m+1}}{i-j} h^m + O(h^{2n-1}) \quad \text{as } h \searrow 0
\]
with the convention \( (i^{m+1} - j^{m+1})/(i-j) = m+1 \) if \( i = j \). Therefore,
\[
G_h = \sum_{m=0}^{2n-2} \frac{g^{(m+1)}(x)}{(m+1)!} \left( \sum_{i=0}^{m} u_i \otimes u_{m-i} \right) h^m + O(h^{2n-1}),
\]
where \( u_l := (0^l, 1^l, \ldots, (n-1)^l) \in \mathbb{C}^n \) and \( u_l \otimes u_{m-l} := [i^l j^{m-l}]_{i,j=0}^{n-1} \) for \( 0 \leq l \leq m \leq 2n-2 \) (with \( 0^0 := 1 \)). For every \( \zeta_0, \ldots, \zeta_{n-1} \in \mathbb{C} \), since \( u_0, \ldots, u_{n-1} \) are linearly independent, there exists a \( v \in \mathbb{C}^n \) such that \( \langle u_l, v \rangle = \zeta_l h^{-l} \) for \( l = 0, \ldots, n-1 \). Since \( \langle u_l, v \rangle = O(h^{-(n-l)}) \) if \( l \geq n \), one can easily verify that
\[
\langle v, u_l \otimes u_{m-l}, v \rangle h^m = \langle u_l, v \rangle \langle u_{m-l}, v \rangle h^m = O(h)
\]
if \( l \geq n \) or \( m-l \geq n \). Therefore,
\[
0 \leq \langle v, G_h v \rangle = \sum_{m=0}^{2n-2} \frac{g^{(m+1)}(x)}{(m+1)!} \left( \sum_{0 \leq l \leq n-1, 0 \leq m-l \leq n-1} \zeta_l \zeta_{m-l} \right) + O(h)
\]
\[
= \sum_{i,j=0}^{n-1} \frac{g^{(i+j+1)}(x)}{(i+j+1)!} \zeta_i \zeta_j + O(h),
\]
and letting \( h \searrow 0 \) yields that \( \left[ g^{(i+j+1)}(x)/(i+j+1)! \right]_{i,j=0}^{n-1} \geq 0 \). It immediately follows from (2.2) that
\[
g^{(l+1)}(x) = \frac{(l+1)! f^{(l+k)}(x)}{(l+k)!}, \quad l = 0, 1, \ldots,
\]
and we have the conclusion. \( \square \)

**Lemma 2.4.** Let \( k \in \mathbb{N} \) with \( k \geq 2 \) and assume that \( f \) is \( k \)-tone on \((a,b)\). Then \( f \) is \( C^{k-2} \) and \( f^{(k-2)} \) is convex on \((a,b)\). That is, (a) and (b) of Theorem 2.1 hold when \( n = 1 \).

**Proof.** The proof is by induction on \( k \). The case \( k = 2 \) is obvious since 2-tonicity means convexity. Assume that \( f \) is \((k+1)\)-tone on \((a,b)\). For any \( c \in (a,b) \), since \( f^{[1]}(x,c) \) is \( k \)-tone on \((a,c)\), it follows from induction hypothesis that \( f^{[1]}(x,c) \) is \( C^{k-2} \) on \((a,c)\). Hence \( f \) is \( C^{k-2} \) on \((a,b)\). If \( x_1, \ldots, x_{k+1}, y_1, \ldots, y_{k+1} \) are distinct in \((a,b)\) and \( x_i < y_i \) for all \( i = 1, \ldots, k+1 \), then the \((k+1)\)-tone property of \( f \) implies that \( f^{[k]}(x_1, \ldots, x_{k+1}) \leq f^{[k]}(y_1, \ldots, y_{k+1}) \). For any \( a' < b' \) in \((a,b)\) choose
\(\alpha_1 < \cdots < \alpha_{k+1}\) in \((a, a')\) and \(\beta_1 < \cdots < \beta_{k+1}\) in \((b', b)\). The above inequality then implies that

\[
\begin{align*}
 f^{[k]}(\alpha_1, \ldots, \alpha_{k+1}) &\leq f^{[k]}(x_1, \ldots, x_{k+1}) \\
 & \leq f^{[k]}(\beta_1, \ldots, \beta_{k+1})
\end{align*}
\]

for every distinct \(x_1, \ldots, x_{k+1} \in (a', b')\). Hence there exists a \(K > 0\) (depending on \(a', b'\)) such that \(|f^{[k]}(x_1, \ldots, x_{k+1})| \leq K\) for all distinct \(x_1, \ldots, x_{k+1} \in (a', b')\). This in turn implies that if \(x_1, \ldots, x_k, y_1, \ldots, y_k\) are distinct in \((a', b')\), then

\[
(2.3) \quad |f^{[k-1]}(x_1, \ldots, x_k) - f^{[k-1]}(y_1, \ldots, y_k)| \leq K \sum_{i=1}^{k} |x_i - y_i|.
\]

For every \(\alpha, \beta, x, y \in (a', b')\) such that \(x \neq \alpha\) and \(y \neq \beta\), let \(x_1 \rightarrow x\), \(y_1 \rightarrow y\), \(x_2, \ldots, x_k \rightarrow \alpha\), and \(y_2, \ldots, y_k \rightarrow \beta\) in \((2.3)\) to obtain

\[
(2.4) \quad |f^{[k-1]}(x, \alpha, \ldots, \alpha) - f^{[k-1]}(y, \beta, \ldots, \beta)| \leq K \{ |x - y| + (k - 1)|\alpha - \beta| \},
\]

where \(f^{[k-1]}(x, \alpha, \ldots, \alpha)\) is well defined as

\[
f^{[k-1]}(x, \alpha, \ldots, \alpha) = \frac{1}{(x - \alpha)^{k-1}} \left\{ f(x) - \sum_{l=0}^{k-2} \frac{f^{(l)}(\alpha)}{l!} (x - \alpha)^l \right\}
\]

due to the \(C^{k-2}\) of \(f\). By \((2.4)\) with \(\beta = \alpha\) we have the limit

\[
\theta(\alpha) := \lim_{x \rightarrow \alpha} f^{[k-1]}(x, \alpha, \ldots, \alpha), \quad \alpha \in (a', b').
\]

For every \(\alpha, \beta \in (a', b')\), letting \(x \rightarrow \alpha\) and \(y \rightarrow \beta\) in \((2.4)\), we have \(|\theta(\alpha) - \theta(\beta)| \leq K k|\alpha - \beta|\) so that \(\theta\) is continuous on \((a', b')\). Furthermore, by \((2.4)\) we have

\[
|f^{[k-1]}(x, \alpha, \ldots, \alpha) - \theta(\alpha)| \leq K |x - \alpha|, \quad x, \alpha \in (a', b'), \quad x \neq \alpha.
\]

Letting \(r(x, \alpha) := f^{[k-1]}(x, \alpha, \ldots, \alpha) - \theta(\alpha)\) with \(r(\alpha, \alpha) = 0\) we write

\[
(2.5) \quad f(x) = \sum_{l=0}^{k-2} \frac{f^{(l)}(\alpha)}{l!} (x - \alpha)^l + \theta(\alpha)(x - \alpha)^{k-1} + r(x, \alpha)(x - \alpha)^{k-1}, \quad x, \alpha \in (a', b').
\]

Since \(f^{(l)}(\alpha)\) for \(0 \leq l \leq k - 2\) and \(\theta(\alpha)\) are continuous in \(\alpha \in (a', b')\) and \(|r(x, \alpha)| \leq K |x - \alpha|\) for \(x, \alpha \in (a', b')\), expression \((2.5)\) shows \([1]\) pp. 6–9\) (also \([18]\) Lemma A.1.1) that \(f\) is \(C^k\) on \((a', b')\) with \(f^{(k-1)}(\alpha) = (k - 1)! \theta(\alpha)\). The \(C^{k-1}\) of \(f\) on \((a, b)\) follows since \(a', b'\) are arbitrary.

To complete the induction procedure, it remains to prove that \(f^{(k-1)}\) is convex on \((a, b)\). To do so, we adopt a standard regularization technique (see \([13]\) pp. 11–13\) for example). Let \(\phi(t)\) be a non-negative \(C^\infty\) function on \(\mathbb{R}\) supported in \([-1, 1]\) such that \(\int \phi(s) \, ds = 1\). For every \(\varepsilon > 0\) sufficiently small define

\[
f_\varepsilon(x) := \frac{1}{\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} \phi\left(\frac{x-s}{\varepsilon}\right) f(s) \, ds = \int_{-1}^{1} \phi(s) f(x - \varepsilon s) \, ds, \quad x \in (a + \varepsilon, b - \varepsilon),
\]

which is a \(C^\infty\) function on \((a + \varepsilon, b - \varepsilon)\). For every distinct \(x_1, \ldots, x_{k+2} \in (a + \varepsilon, b - \varepsilon)\) one can easily see that

\[
f_\varepsilon^{[k+1]}(x_1, \ldots, x_{k+2}) = \int_{-1}^{1} \phi(s) f^{[k+1]}(x_1 - \varepsilon s, \ldots, x_{k+2} - \varepsilon s) \, ds \geq 0.
\]
Hence \( f^{(k+1)}_\varepsilon(x) \geq 0 \) for all \( x \in (a+\varepsilon, b-\varepsilon) \) so that \( f^{(k-1)}_\varepsilon \) is convex on \((a+\varepsilon, b-\varepsilon)\).

Since \( f \) is \( C^{k-1} \) on \((a, b)\) as already proved, \( f^{(k-1)}_\varepsilon(x) \to f^{(k-1)}(x) \) as \( \varepsilon \searrow 0 \) for all \( x \in (a, b) \) and the convexity of \( f^{(k-1)} \) follows. \( \square \)

**Lemma 2.5.** Let \( \{f_m\} \) be a sequence of \( C^1 \) functions on \([a, b]\) such that the finite limits \( \lim_{m \to \infty} f_m(a) \) and \( \lim_{m \to \infty} f_m(b) \) exist. Assume that \( f'_m \) is convex on \([a, b]\) for every \( m \) and there exists a \( K > 0 \) such that \( f'_m(x) \leq K \) for all \( x \in [a, b] \) and all \( m \). Then \( \{f'_m\} \) is uniformly equicontinuous on \([a, b]\).

**Proof.** We can assume that \( K = 0 \). Since \( f'_m \leq 0 \) is convex on \([a, b]\), we have

\[
    f_m(b) - f_m(a) = \int_a^b f'_m(t) \, dt \leq \left( \frac{b - a}{2} \right) f'_m(x)
\]

for all \( x \in [a, b] \). Hence \( \{f'_m\} \) is also uniformly bounded below on \([a, b]\), and so \( \{f_m\} \) is uniformly equicontinuous there. \( \square \)

We are now in a position to complete the proof of the theorem.

**Proof of Theorem 2.1** The theorem when \( k = 1 \) and \( n \geq 2 \) is Löwner’s result [13, p. 76], so we may assume that \( k \geq 2 \). Let us prove (a) and (b) for all \( k \geq 2 \) by induction on \( n \). The initial case \( n = 1 \) is Lemma 2.3. Let \( n \in \mathbb{N} \) and assume that \( f \) is matrix \( k \)-tone of order \( n + 1 \) on \((a, b)\). Since \( f \) is matrix \( k \)-tone of order \( n \), it follows from induction hypothesis that \( f \) is \( C^{2n+k-4} \) and \( f^{(k-2)} \), \( f^k \), \ldots, \( f^{2n+k-4} \) are convex on \((a, b)\). Define regularizations \( f_\varepsilon \) of \( f \) for small \( \varepsilon > 0 \) as in the last part of the proof of Lemma 2.4. For every \( A, B \in M_{n+1}(a+\varepsilon, b-\varepsilon) \) with \( A \leq B \) one can easily see that if \( 0 = \lambda_0 < \lambda_1 < \cdots < \lambda_k = 1 \), then

\[
    f^{[k]}_\varepsilon(A, B; \lambda_0, \ldots, \lambda_k) = \int_{-1}^1 \phi(s) f^{[k]}(A - \varepsilon s I, B - \varepsilon s I; \lambda_0, \ldots, \lambda_k) \, ds \geq 0.
\]

This means that \( f_\varepsilon \) is matrix \( k \)-tone of order \( n + 1 \) on \((a+\varepsilon, b-\varepsilon)\). So one can apply (d) proved in Lemma 2.3 to \( f_\varepsilon \) with \( n + 1 \) in place of \( n \) to see that \( f^{(2n+k-4)}_\varepsilon(x) \geq 0 \) for all \( x \in (a+\varepsilon, b-\varepsilon) \) and hence \( f^{(2n+k-2)}_\varepsilon \) is convex on \((a+\varepsilon, b-\varepsilon)\).

Since \( f^{(2n+k-4)}_\varepsilon \) is convex on \((a, b)\) as already mentioned, \( f^{(2n+k-3)}_\varepsilon(x) \) exists and hence \( f^{(2n+k-3)}_\varepsilon(x) \to f^{(2n+k-3)}(x) \) as \( \varepsilon \searrow 0 \) for all \( x \in (a, b) \) except for at most countable points. Choose \( a' < b' \) in \((a, b)\), arbitrarily near \( a, b \) respectively, at which \( f^{(2n+k-4)} \) is differentiable. Since \( \{f^{(2n+k-4)}_\varepsilon\} \) (for \( \varepsilon > 0 \) sufficiently small) is uniformly bounded above on \([a', b']\), we see by Lemma 2.5 that \( \{f^{(2n+k-3)}_\varepsilon\} \) (for small \( \varepsilon > 0 \)) is uniformly equicontinuous on \([a', b']\). Hence there exists a continuous function \( \varphi \) on \([a', b']\) such that \( f^{(2n+k-3)}_\varepsilon(x) \to \varphi(x) \) uniformly on \([a', b']\). Note that \( f^{(2n+k-4)}_\varepsilon(x) \to f^{(2n+k-4)}(x) \) for all \( x \in (a, b) \). Hence \( f^{(2n+k-4)} \) is differentiable with \( f^{(2n+k-3)}(x) = \varphi(x) \) on \((a', b')\). Therefore, \( f \) is \( C^{2n+k-3} \) on \((a, b)\).

Since \( f^{(2n+k-4)} \) is convex on \((a, b)\), \( f^{(2n+k-3)} \) is non-decreasing on \((a, b)\) and so differentiable almost everywhere on \((a, b)\) by Lebesgue’s theorem. Let \( D \) be the set of \( x \in (a, b) \) at which \( f^{(2n+k-3)} \) is differentiable. Note that \( f^{(2n+k-3)}_\varepsilon(x) \to f^{(2n+k-3)}(x) \) as \( \varepsilon \searrow 0 \) for all \( x \in (a, b) \) and \( f^{(2n+k-2)}_\varepsilon(x) \to f^{(2n+k-2)}(x) \) for all \( x \in D \). Choose \( \alpha_1 < \alpha_2 < a' < b' < \beta_1 < \beta_2 \) from \( D \). Since \( f^{(2n+k-2)}_\varepsilon \) is convex on
(a + \varepsilon, b - \varepsilon), we have
\[
\frac{f^{(2n+k-2)}_\varepsilon(\alpha_1) - f^{(2n+k-2)}_\varepsilon(\alpha_2)}{\alpha_1 - \alpha_2} \leq f^{(2n+k-1)}_\varepsilon(x) \leq \frac{f^{(2n+k-2)}_\varepsilon(\beta_1) - f^{(2n+k-2)}_\varepsilon(\beta_2)}{\beta_1 - \beta_2}
\]
for all \(x \in [a', b']\), which implies that \(\{f^{(2n+k-2)}_\varepsilon\}\) is uniformly equicontinuous on \([a', b']\). Hence there exists a continuous function \(\psi\) on \([a', b']\) such that \(f^{(2n+k-2)}_\varepsilon(x) \to \psi(x)\) uniformly on \([a', b']\). This shows that \(f^{(2n+k-3)}_\varepsilon\) is differentiable with \(f^{(2n+k-2)}_\varepsilon(x) = \psi(x)\) on \((a', b')\). Therefore, \(f\) is \(C^{2n+k-2}\) on \((a, b)\). Moreover, \(f^{(2n+k-2)}_\varepsilon\) is convex on \((a, b)\) since \(f^{(2n+k-2)}_\varepsilon(x) \to f^{(2n+k-2)}_\varepsilon(x)\) for all \(x \in (a, b)\).

Next, we prove (c) and (d). Let \(f\) be matrix \(k\)-tone of order \(n\) on \((a, b)\) and \(f^\varepsilon\) be the regularization of \(f\) as above. Then by Lemma 2.3 (c) and (d) hold for \(f^\varepsilon\) on \((a + \varepsilon, b - \varepsilon)\). When \(n \geq 2\), since \(k \leq 2n + k - 4\), \(f\) is \(C^k\) on \((a, b)\) by assertion (a), so (c) for \(f\) follows by taking the limit of (c) for \(f^\varepsilon\). Since \(f^{(2n+k-4)}\) is convex on \((a, b)\), the same argument as above shows that \(f^{(2n+k-2)}_\varepsilon(x)\) exists for almost every \(x \in (a, b)\) and that \(f^{(2n+k-3)}_\varepsilon(x) \to f^{(2n+k-3)}(x)\) and \(f^{(2n+k-2)}_\varepsilon(x) \to f^{(2n+k-2)}(x)\) for almost every \(x \in (a, b)\). Hence (d) for \(f\) follows as the almost everywhere limit of (d) for \(f^\varepsilon\).

As immediately seen from [13, p. 6], a real \(C^k\) function on \((a, b)\) is \(k\)-tone if and only if \(f^{(k)}(x) \geq 0\) for all \(x \in (a, b)\). This can be extended to the differential calculus in \(n \times n\) matrices as follows:

**Proposition 2.6.** Let \(k, n \in \mathbb{N}\) with \(n \geq 2\) and \(f\) be a real function on \((a, b)\). Then the following conditions are equivalent:

(i) \(f\) is matrix \(k\)-tone of order \(n\) on \((a, b)\);

(ii) \(f\) is \(C^k\) on \((a, b)\) and
\[
\frac{d^k}{dt^k} f(A + tX) \bigg|_{t=0} \geq 0
\]
for every \(A \in \mathbb{M}^{sa}_n(a, b)\) and every \(X \in \mathbb{M}^+_n\);

(iii) \(f\) is \(C^k\) on \((a, b)\) and
\[
f(A + X) \geq \sum_{l=0}^{k-1} \frac{1}{l!} D^l f(A) (X, \ldots, X)
\]
for every \(A \in \mathbb{M}^{sa}_n(a, b)\) and every \(X \in \mathbb{M}^+_n\) such that \(A + X \in \mathbb{M}^{sa}_n(a, b)\), where \(D^l f(A)\) is the \(l\)th Fréchet derivative of \(A \mapsto f(A)\) on \(\mathbb{M}^{sa}_n(a, b)\).

**Proof.** (i) \(\Rightarrow\) (ii). Since \(n \geq 2\), assertion (i) implies the \(C^k\) of \(f\) due to Theorem 2.1(a), and hence the implication was already proved in Lemma 2.2

(ii) \(\Rightarrow\) (i). Let \(A, B \in \mathbb{M}^{sa}_n(a, b)\) with \(A \leq B\). As in the proof of Proposition 1.3 choose a \(\delta > 0\) and define \(\phi_\omega\) for any state \(\omega\) on \(\mathbb{M}_n\). For any \(0 = t_0 < t_1 < \cdots < t_k = 1\), there exists a \(\xi \in [0, 1]\) such that
\[
\phi^{[k]}_\omega(t_0, t_1, \ldots, t_k) = \frac{1}{k!} \phi^{[k]}_\omega(\xi) = \frac{1}{k!} \omega \left( \frac{d^k}{dt^k} \left[ f(A + t(B - A)) \right] \bigg|_{t=\xi} \right),
\]
which is non-negative due to assumption (ii). Hence by (1.1),
\[
\omega(f^{[k]}(A, B; t_0, t_1, \ldots, t_k)) \geq 0
\]
for all states \(\omega\), so (i) follows.
Proof. By Propositions 1.6, 2.6 and Theorem 2.1, we know that $\phi(t) = \omega(f(A + tX))$ for $t \in (-\delta, 1 + \delta)$. By Taylor’s theorem there exists a $\theta \in (0, 1)$ such that

$$
\phi(t) = \phi(0) + \frac{\phi'(t)}{1!} + \frac{\phi''(t)}{2!} + \cdots + \frac{\phi^{(k)}(t)}{k!}.
$$

(ii) $\Rightarrow$ (iii). Let $A, X$ be as stated in (iii); there is a $\delta > 0$ such that $A + tX \in M_n^a(a, b)$ for all $t \in (-\delta, 1 + \delta)$. For any state $\omega$ on $M_n$ define $\phi_\omega(t) := \omega(f(A + tX))$ for $t \in (-\delta, 1 + \delta)$. By Taylor’s theorem there exists a $\theta \in (0, 1)$ such that

$$
\phi(1) = \sum_{l=0}^{k-1} \frac{\phi^{(l)}(0)}{l!} + \frac{\phi^{(k)}(\theta)}{k!}.
$$

Notice that

$$
\phi^{(l)}(0) = \omega \left( \frac{d^l}{dt^l} f(A + tX) \bigg|_{t=0} \right) = \omega(D^l f(A)(X, \ldots, X)), \quad 0 \leq l \leq k - 1,
$$

and

$$
\phi^{(k)}(\theta) = \omega \left( \frac{d^k}{dt^k} f(A + tX) \bigg|_{t=\theta} \right) = \omega \left( \frac{d^k}{dt^k} f((A + \theta X) + tX) \bigg|_{t=0} \right) \geq 0
$$

due to (ii). Hence

$$
\omega \left( f(A + X) - \sum_{l=0}^{k-1} \frac{1}{l!} D^l f(A)(X, \ldots, X) \right) \geq 0
$$

for all states $\omega$, so (iii) follows.

(iii) $\Rightarrow$ (ii). Let $A, X$ be as in (ii); we may assume that $A + X \in M_n^a(a, b)$. For each $t \in (0, 1)$, as in (2.6) there exists a $\theta_t \in (0, 1)$ such that

$$
\frac{1}{k!} \omega(D^k f(A + \theta_t tX)(X, \ldots, X)) = \omega \left( f(A + tX) - \sum_{l=0}^{k-1} \frac{1}{l!} D^l f(A)(tX, \ldots, tX) \right) \geq 0.
$$

Since $D^k f(B)$ is continuous in $B \in M_n^a(a, b)$, letting $t \searrow 0$ we have

$$
\omega(D^k f(A)(X, \ldots, X)) \geq 0
$$

for all states $\omega$. Hence $\frac{d^k}{dt^k} f(A + tX) \bigg|_{t=0} = D^k f(A)(X, \ldots, X) \geq 0$. \hfill \Box

3. Characterizations of operator $k$-tone functions

The aim of this section is to present general characterizations of operator $k$-tone functions on $(a, b)$. A well-known theorem of Bernstein is stated in [26, Chapter IV, Theorem 3a] as follows: If $f$ is absolutely monotone (i.e., all $f^{(k)}$, $k = 0, 1, 2, \ldots$, are non-negative) on $[a, b]$, then $f$ can be analytically continued into $\{z \in \mathbb{C} : |z - a| < b - a\}$. Its variation due to Valiron given in [10, pp. 160–161] says that if a function $f$ on $(a, b)$ has non-negative even derivatives $f, f', f'', f^{(4)}, \ldots$, then it is analytic on $(a, b)$. We will need a little bit more, as given in the following:

Lemma 3.1. Let $f$ be operator $k$-tone on $(a, b)$. Then $f$ is analytic on $(a, b)$ and moreover the radius of convergence of the Taylor expansion of $f$ at $x \in (a, b)$ is at least $\delta_x := \min\{|x - a|, b - a\}$.

Proof. By Propositions 1.6, 2.6 and Theorem 2.1 we know that $g := f^{(k)}$ is $C^\infty$ on $(a, b)$ and moreover $g, g'', g^{(4)}, \ldots$ are non-negative. Take $x \in (a, b)$, and for $|h| < \delta_x$ let $\tilde{g}(h) := g(x + h) + g(x - h)$. Clearly $\tilde{g}$ is absolutely monotone on $(-\delta_x, \delta_x)$. By Bernstein’s theorem mentioned above, $\tilde{g}$ is analytic and its Taylor
expansion at $x$, which is $\sum_{n=0}^{\infty} \left(\frac{g^{(2n)}(x)}{2n!}\right) h^{2n}$, has radius of convergence at least $\delta_x$. On the other hand, by Theorem 2.1(d), for every $n \geq 0$,
\[
\left( \frac{g^{(2n+1)}(x)}{(2n+1+k)!} \right)^2 \leq \frac{g^{(2n)}(x)}{(2n+k)!} \cdot \frac{g^{(2n+2)}(x)}{(2n+2+k)!},
\]
which shows that $\sum_{n=0}^{\infty} \left(\frac{g^{(n)}(x)}{n!}\right) h^n$ also has radius of convergence at least $\delta_x$. From these estimates one can easily see, using any Taylor formula, that $g(x+h) = \sum_{n=0}^{\infty} \left(\frac{g^{(n)}(x)}{n!}\right) h^n$ for $h \in (\delta_x/2, \delta_x/2)$. Thus $g$ is analytic. But it coincides on $(x-\delta_x/2, x+\delta_x/2)$ with its Taylor expansion at $x$, which is known to have radius of convergence at least $\delta_x$. Thus by the analytic continuation principle, they have to agree on $(x-\delta_x, x+\delta_x)$. The conclusion on $f$ now follows. 

\begin{proof}
Let $g$ be an operator monotone function on $(a, b)$ and $\alpha \in (a, b)$. Then $(x-\alpha)^{(k-1)}g$ is operator $(k+2l)$-tone for any $k \in \mathbb{N}$ and any $l \in \mathbb{N} \cup \{0\}$.
\end{proof}

\section*{Lemma 3.2.}
Let $g$ be an operator monotone function on $(a, b)$ and $\alpha \in (a, b)$. Then $(x-\alpha)^{(k-1)}g$ is operator $(k+2l)$-tone for any $k \in \mathbb{N}$ and any $l \in \mathbb{N} \cup \{0\}$.

\begin{proof}
Set $m := k+2l$. By Lemma 3.1, $f := (x-\alpha)^{(k-1)}g$ is analytic. By Propositions 1.6 and 2.6 we need to check that $\frac{d^m}{dt^m} f(A + tX)|_{t=0} \geq 0$ for all $A \in \mathbb{M}_{a,b}^{sa}(a, b)$, $X \in \mathbb{M}_{a,b}^{\mathbb{R}^+}$ and $n \in \mathbb{N}$. Since this is a local estimate, using a restriction, a translation and a dilation, we can always assume that actually $f$ is operator monotone on $(-1, 1)$, $\alpha \in (-1, 1)$, and the spectrum of $A$ sits in $(-1, 1)$.

It suffices to assume that $A$ is diagonal so that $A = \text{Diag}(a_1, \ldots, a_n)$. Using Dalecki and Krein’s derivative formula in (2.1) and the formula for divided differences given in Corollary 1.8 we have
\begin{equation}
\frac{d^m}{dt^m} f(A + tX)|_{t=0} = m! \int_{[-1,1]} D(\lambda)^{1/2}(D(\lambda)^{1/2} XD(\lambda)^{1/2})^m D(\lambda)^{1/2} \lambda^{2l}(1 - \lambda \alpha)^{(k-1)} d\mu(\lambda) \geq 0,
\end{equation}
where
\[
D(\lambda) := \text{Diag}\left(\frac{1}{1 - \lambda a_1}, \ldots, \frac{1}{1 - \lambda a_n}\right), \quad \lambda \in [-1,1].
\]
\end{proof}

\section*{Theorem 3.3.}
Let $f$ be a real function on $(a, b)$, where $-\infty \leq a < b \leq \infty$. Let $k \in \mathbb{N}$. Then the following conditions are equivalent:

(i) $f$ is operator $k$-tone on $(a, b)$;
(ii) $f$ is matrix $k$-tone of order $n$ on $(a, b)$ for every $n \in \mathbb{N}$;
(iii) $f$ is $C^k$ on $(a, b)$ and
\[
\frac{d^k}{dt^k} f(A + tX)|_{t=0} \geq 0
\]
for every $A \in \mathbb{M}_{a,b}^{sa}(a, b)$, $X \in \mathbb{M}_{a,b}^{\mathbb{R}^+}$ and $n \in \mathbb{N}$;
(iv) $f$ is analytic on $(a, b)$ and
\[
\frac{d^k}{dt^k} f(A + tX)|_{t=0} \geq 0
\]
for every $A \in B(\mathcal{H})^{sa}(a, b)$ and every $X \in B(\mathcal{H})^{\mathbb{R}^+}$, where the above derivative of order $k$ is well defined in the operator norm;
(v) \( f \) is \( C^{k-2} \) on \((a,b)\) (this is void for \(k = 1\)) and \(f^{[k-1]}(x,\alpha,\ldots,\alpha)\) is operator monotone on \((a,b)\) for some (equivalently, any) \(\alpha \in (a,b)\) (with continuation of value at \(x = \alpha\));

(vi) \( f \) is \( C^{k-1} \) on \((a,b)\) and \(f^{[k-1]}(x,\alpha_1,\ldots,\alpha_{k-1})\) (this is \(f(x)\) for \(k = 1\)) is operator monotone on \((a,b)\) for some (equivalently, any) choice of \(\alpha_1,\ldots,\alpha_n\) from \((a,b)\);

(vii) \( f \) is \( C^\infty \) on \((a,b)\) and

\[
\left[ \frac{f(i+j+k)(x)}{(i+j+k)!} \right]_{i,j=0}^{n-1}
\]

is positive semidefinite for every \(x \in (a,b)\) and every \(n \in \mathbb{N}\).

**Proof.** The equivalence of (i), (ii) and (iii) is included in Propositions 1.6 and 2.6.

(i) \(\Rightarrow\) (iv). By Lemma 3.1, \(f\) is analytic. It remains to justify that \(A \mapsto f(A)\) is \(C^k\) on \(B(\mathcal{H})^{sa}(a,b)\) and the inequality in (iv). Let \(A \in B(\mathcal{H})^{sa}(a,b)\) with spectrum included in \([x_0 - h, x_0 + h] \subset (a,b), h > 0\). By Lemma 3.1 \(f\) is equal to its Taylor expansion at \(x_0\) on \((x_0 - \delta x_0, x_0 + \delta x_0)\). Thus for any \(B\) with spectrum in \((x_0 - \delta x_0, x_0 + \delta x_0)\) we have \(f(B) = \sum_{m=0}^{\infty} c_m (B - x_0 I)^m\) with \(c_m := f^{(m)}(x_0)/m!\), and so \(B \mapsto f(B)\) is \(C^\infty\) on \(B(\mathcal{H})^{sa}(x_0 - \delta x_0, x_0 + \delta x_0)\), a neighborhood of \(A\) as \(h < \delta x_0\). For every \(X \in B(\mathcal{H})^+\) we have

\[
\frac{d^k}{dt^k} f(A + tX)\bigg|_{t=0} = \sum_{m=0}^{\infty} c_m \frac{d^k}{dt^k} (A - x_0 I + tX)^m\bigg|_{t=0} = \sum_{m=k}^{\infty} c_m F_{k,m-k}(X, A - x_0 I),
\]

where \(F_{k,m-k}(X, Y)\) was introduced in Example 1.2. On the other hand, for \(0 = t_0 < t_1 < \cdots < t_k < 1\), by assumption (i) we have

\[
0 \leq f^{[k]}(A, A + X; t_0, t_1, \ldots, t_k) = \sum_{m=0}^{\infty} c_m (x^m)^{[k]} (A - x_0 I, A - x_0 I + X; t_0, t_1, \ldots, t_k).
\]

Now apply the formula of Example 1.2 to each term of the above expansion and then let \(t_i \searrow 0\) for \(1 \leq i \leq k\) to obtain \(\sum_{m=k}^{\infty} c_m F_{k,m-k}(X, A - x_0 I) \geq 0\).

(iv) \(\Rightarrow\) (iii) is obvious, and so (i)–(iv) are equivalent.

Next, we prove that (i) \(\Rightarrow\) (v) for any \(\alpha \in (a,b)\). Assume (i); then \(f\) is analytic. For every \(n \in \mathbb{N}\) and for any choice of \(\alpha, x_1, \ldots, x_n\) from \((-1,1)\), apply (e) proved in Lemma 2.3 to \(n + 1\) points \(\alpha, x_1, \ldots, x_n\) and take the \(n \times n\) submatrix deleting the first row and the first column to obtain

\[
\left[ \frac{f^{[k-1]}(x_i, \alpha, \ldots, \alpha) - f^{[k-1]}(x_j, \alpha, \ldots, \alpha)}{x_i - x_j} \right]_{i,j=1}^{n} = \left[ f^{[k]}(x_i, \alpha, \ldots, x_j) \right]_{i,j=1}^{n} \geq 0,
\]

which gives (v) due to Löwner’s theorem 23 (or 8 V.3.4]). (Note also that Löwner’s theorem easily follows from Proposition 2.6 and (2.1) for \(k = 1\).)

Conversely, assume (v) for some \(\alpha \in (a,b)\); so \(f\) is \(C^{k-2}\) and

\[
g(x) := f^{[k-1]}(x, \alpha, \ldots, \alpha)
\]
is operator monotone on $(a, b)$ (with continuation $g(\alpha) = \beta$ at $x = \alpha$). As in \eqref{2.2} we have
\begin{equation}
(3.3) \quad f(x) = \sum_{l=0}^{k-2} \frac{f^{(l)}(\alpha)}{l!}(x - \alpha)^l + (x - \alpha)^{k-1}g(x), \quad x \in (a, b),
\end{equation}
where $f(x) = g(x)$ for $k = 1$. Hence Lemma 3.2 yields (i).

Before going to (vi), let us prove that (i) implies that $(x - \alpha)f$ is operator $(k+1)$-tonic for any $\alpha \in (a, b)$. By (i) $\Rightarrow$ (v) and \eqref{3.3} we have a polynomial $P$ of degree at most $k - 2$ and an operator monotone function $g$ so that $f = P + (x - \alpha)^{k-1}g$. Hence $(x - \alpha)f = (x - \alpha)P + (x - \alpha)^k g$. By Lemma 3.2 this yields the operator $(k + 1)$-tonicity of $(x - \alpha)f$.

Now, assume that (vi) holds for some choice of $\alpha_1, \ldots, \alpha_n$ from $(a, b)$. As is easily verified, $g(x) = f^{[k-1]}(x, \alpha_1, \ldots, \alpha_{k-1})$ is of the form
\begin{equation}
(3.4) \quad f(x) = P(x) + \left\{ \prod_{l=1}^{k-1} (x - \alpha_l) \right\} g(x),
\end{equation}
where $P$ is a polynomial of degree at most $k - 2$ and $g$ is operator monotone on $(a, b)$. Hence (i) follows by applying, $k - 1$ times, the result we have proved above.

Conversely, let us prove that (i) $\Rightarrow$ (vi) for all possible choices of $\alpha_1, \ldots, \alpha_{k-1}$. Assume (i), so $f$ is analytic on $(a, b)$. Let $\alpha_1, \ldots, \alpha_{k-1}$ be arbitrary in $(a, b)$. By (i) $\Rightarrow$ (v), $f^{[k-1]}(x, \alpha_1, \ldots, \alpha_1)$ is operator monotone on $(a, b)$. Let $g_1(x) := f^{[1]}(x, \alpha_1)$; then we have $g_1^{[k-2]}(x, \alpha_1, \ldots, \alpha_1) = f^{[k-1]}(x, \alpha_1, \ldots, \alpha_1)$. Hence by (v) $\Rightarrow$ (i) with some $\alpha$ and $k - 1$ in place of $k$, $g_1$ is operator $(k - 1)$-tone on $(a, b)$ and hence $g_1^{[k-2]}(x, \alpha_2, \ldots, \alpha_2)$ is operator monotone on $(a, b)$. Repeat this argument to $g_2(x) := g_1^{[1]}(x, \alpha_2)$, ..., $g_{k-1}(x) := g_1^{[1]}(x, \alpha_{k-1})$ and notice that $g_{k-1}(x) = f^{[1]}(x, \alpha_1, \ldots, \alpha_{k-1})$. Hence (vi) follows.

Theorem 2.4 shows that (i) $\Rightarrow$ (vii). Conversely, assume (vii). By the same argument as in the proof of Lemma 3.2 we can assume that $(a, b) = (-1, 1)$. The proof of Lemma 3.1 can be performed under assumption (vii) so that the Taylor expansion $\sum_{n=0}^{\infty} \left( f^{(n)}(0)/n! \right) x^n$ is convergent on $(-1, 1)$. Thus we can write
\begin{equation*}
f(x) = \sum_{l=0}^{k-1} \frac{f^{(l)}(0)}{l!} x^l + x^{k-1}g(x), \quad x \in (-1, 1),
\end{equation*}
where $g(x) := \sum_{n=0}^{\infty} \left( f^{(n+k)}(0)/(n + k)! \right) x^{n+1}$. Now, the same proof as that of \cite[Theorem 2.8]{6} appealing to the Hamburger moment problem can be done due to (vii) to see that $g$ is operator monotone on $(-1, 1)$. Hence Lemma 3.2 yields (i).

In the rest of the section we will point out some interesting consequences of the above theorem and its proof. The first corollary is seen from the connections between $f$ and its divided differences in \eqref{3.3} and \eqref{3.4}. \qed
Corollary 3.4. Let $f$ be a real function on $(a, b)$, where $-\infty \leq a < b \leq \infty$. Let $k \in \mathbb{N}$. Then the following conditions are equivalent:

(i) $f$ is operator $k$-tone on $(a, b)$;
(ii) $f$ is $C^{k-2}$ on $(a, b)$ (this is void for $k = 1$), and for some (equivalently, any) $\alpha \in (a, b)$ there exists an operator monotone function $g$ on $(a, b)$ such that

$$f(x) = \sum_{l=0}^{k-2} \frac{f^{(l)}(\alpha)}{l!} (x - \alpha)^l + (x - \alpha)^{k-1} g(x);$$

(iii) for some (equivalently, any) $\alpha_1, \ldots, \alpha_{k-1}$ in $(a, b)$ there exist a polynomial $P(x)$ of degree less than or equal to $k - 2$ and an operator monotone function $g$ on $(a, b)$ such that

$$f(x) = P(x) + \left\{ \prod_{l=1}^{k-1} (x - \alpha_l) \right\} g(x).$$

The corollary tells us that the structure of operator $k$-tone functions is rather simple with additive and multiplicative polynomial factors beyond operator monotone functions. With $\alpha_l = \alpha \in (a, b)$ fixed in (iii), there is a one-to-one correspondence, up to polynomials of degree less than or equal to $k - 2$, between operator $k$-tone functions on $(a, b)$ and operator monotone functions on $(a, b)$.

It is also worthwhile to observe

Corollary 3.5. Let $f$ be an operator $k$-tone function on $(a, b)$, where $-\infty \leq a < b \leq \infty$. Then

(a) For any $\alpha \in (a, b)$, $(x - \alpha) f$ is operator $(k + 1)$-tone on $(a, b)$.
(b) For any $l \in \mathbb{N}$, $f$ is operator $(k + 2l)$-tone on $(a, b)$.
(c) If $k$ is even, then $\frac{d^k}{dt^k} f(A + tX)|_{t=0} \geq 0$ for all $A \in B(\mathcal{H})^{sa}(a, b)$ and $X \in B(\mathcal{H})^{sa}$.

Proof. (a) has already been shown in the proof of Theorem 3.3 (see the paragraph after 3.3), and (b) is similarly shown by using Lemma 3.2. Let us prove (c). For every $A \in \mathcal{M}_n^{sa}(a, b)$, $X \in \mathcal{M}_n^{sa}$ and $n \in \mathbb{N}$, this is immediately seen from Corollary 3.4(iii) and 3.1 in the proof of Lemma 3.2. Next, let $A \in B(\mathcal{H})^{sa}(a, b)$ and $X \in B(\mathcal{H})^{sa}$. Choose an orthogonal basis $\{e_n\}_{n=1}^{\infty}$ of $\mathcal{H}$, and let $\mathcal{H}_n$ be the linear span of $e_1, \ldots, e_n$ and $P_n$ the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_n$. Set $\hat{A}_n := P_n A P_n|_{\mathcal{H}_n}$ and $\hat{X}_n := P_n X P_n|_{\mathcal{H}_n}$, which are considered as elements of $\mathcal{M}_n^{sa}(a, b)$ and of $\mathcal{M}_n^{sa}$, respectively. By the matrix case shown above we have $\frac{d^k}{dt^k} f(\hat{A}_n + t\hat{X}_n)|_{t=0} \geq 0$. Using 3.2 one can easily verify that $P_n(\frac{d^k}{dt^k} f(\hat{A}_n + t\hat{X}_n)|_{t=0}) P_n$ converges strongly to $\frac{d^k}{dt^k} f(A + tX)|_{t=0}$ as $n \to \infty$. Hence the latter $k$th derivative is non-negative.

Remark 3.6. The phenomenon in the above (c) was observed in the paper [22] for $k = 2$. For the proof one can alternatively use the higher derivative formula for infinite-dimensional operators in [24] to verify the same expression as 3.2.

It is well known that any operator monotone function on the whole line $(-\infty, \infty)$ is a linear function $\alpha + \beta x$ with $\beta \geq 0$; any operator convex function on $(-\infty, \infty)$ is a quadratic function $\alpha + \beta x + \gamma x^2$ with $\gamma \geq 0$. The following higher order extension is immediate from (iii) of Corollary 3.4.
Corollary 3.7. Let \( k \in \mathbb{N} \). A real function on \((-\infty, \infty)\) is operator \( k \)-tone if and only if it is a polynomial of degree less than or equal to \( k \) with non-negative coefficient of \( x^k \).

The following corollary is also immediate from Theorem \[3.3\]

Corollary 3.8. Let \( f \) be a real function on \((a, b)\), where \(-\infty \leq a < b \leq \infty\). Let \( k, m \in \mathbb{N} \) with \( m < k \). Then the following conditions are equivalent:

(i) \( f \) is operator \( k \)-tone on \((a, b)\);
(ii) \( f \) is \( C^m \) on \((a, b)\) and \( f^{(m)}(x, \alpha_1, \ldots, \alpha_m) \) is operator \((k - m)\)-tone on \((a, b)\) for every \( \alpha_1, \ldots, \alpha_m \in (a, b) \);
(iii) \( f \) is \( C^{m-1} \) on \((a, b)\) and \( f^{(m)}(x, \alpha, \ldots, \alpha) \) is operator \((k - m)\)-tone on \((a, b)\) for some \( \alpha \in (a, b) \) (with continuation of value at \( x = \alpha \)).

Let \( \mathcal{F}(a, b) \) denote the space of all real functions on \((a, b)\), which is a locally convex topological vector space equipped with the pointwise convergence topology. For each \( k \in \mathbb{N} \) we denote by \( \mathcal{P}^{(k)}(a, b) \) the set of all operator \( k \)-tone functions on \((a, b)\).

Proposition 3.9. The set \( \mathcal{P}^{(k)}(a, b) \) is a closed convex cone in \( \mathcal{F}(a, b) \) for every \( k \in \mathbb{N} \) and

\[
\mathcal{P}^{(1)}(a, b) \subseteq \mathcal{P}^{(3)}(a, b) \subseteq \mathcal{P}^{(5)}(a, b) \subseteq \cdots,
\]

\[
\mathcal{P}^{(2)}(a, b) \subseteq \mathcal{P}^{(4)}(a, b) \subseteq \mathcal{P}^{(6)}(a, b) \subseteq \cdots.
\]

Proof. Lemma \[1.1\] clearly implies that the set \( \mathcal{P}^{(n)}(a, b) \) of all matrix \( k \)-tone functions of order \( n \) on \((a, b)\) is a positive cone and is closed for the pointwise convergence. Hence \( \mathcal{P}^{(k)}(a, b) = \bigcap_{n \geq 0} \mathcal{P}^{(n)}(a, b) \) has the same properties.

The inclusions have already been seen in Corollary \[3.5\](b). Moreover, for each \( k \in \mathbb{N} \) with \( k \geq 2 \), let \( f(x) = (x - \alpha)^k \) with \( \alpha \in (a, b) \). Then

\[
f^{[k-1]}(x_1, \ldots, x_k) = \sum_{i=1}^{k} (x_i - \alpha), \quad f^{[k+1]}(x_1, \ldots, x_{k+2}) \equiv 0.
\]

Hence \( f \in \mathcal{P}^{(k+1)}(a, b) \), but \( f \not\in \mathcal{P}^{(k-1)}(a, b) \) since \( f^{[k-1]}(x_1, \ldots, x_k) \) is negative for some \( x_1, \ldots, x_k \in (a, b) \). (More intrinsic differences among \( \mathcal{P}^{(k)}(a, b) \)'s will be seen from Proposition \[4.3\] and examples in Section \[4\].)

Recall \[13\] p. 131] that \( n \)-monotone functions have the following local property: Let \( a < c < b < d \) and \( f \) be a real function on \((a, d)\). If \( f|_{(a, b)} \) and \( f|_{(c, d)} \) are \( n \)-monotone, then so is \( f \) on \((a, d)\). This property is essential in the proof \[13\] pp. 83–84] of the fact that (d) of Theorem \[2.1\] with \( k = 1 \) is a sufficient condition for \( n \)-monotone functions. This property for \( n \)-convex functions is still open, while that for 2-convex functions was proved in \[16\]. It is immediate from (vii) or (v) of Theorem \[3.3\] that operator \( k \)-tone functions have a similar local property for every \( k \in \mathbb{N} \).

4. Operator \( k \)-tone functions on \((-1, 1)\)

In this section we discuss operator \( k \)-tone functions restricted on the domain interval \((-1, 1)\). In the next theorem we show further characterizations of such functions by using Theorem \[3.3\] and the integral representation of operator monotone functions in Theorem \[1.7\] (in fact, \[1.4\] is \[1.1\] when \( k = 1 \)).
Theorem 4.1. Let $f$ be a real function on $(-1,1)$, and let $k \in \mathbb{N}$. Then the following conditions (i)-(iii) are equivalent:

(i) $f$ is operator $k$-tone on $(-1,1)$;

(ii) $f$ is $C^{k-1}$ on $(-1,1)$ and there exists a finite positive measure $\mu$ on $[-1,1]$ such that for any choice of $\alpha \in (-1,1)$,

$$
(4.1) \quad f(x) = \sum_{l=0}^{k-1} \frac{f^{(l)}(\alpha)}{l!} (x-\alpha)^l + \int_{[-1,1]} \frac{(x-\alpha)^k}{(1-\lambda x)(1-\lambda \alpha)^k} d\mu(\lambda), \quad x \in (-1,1);
$$

(iii) $f$ is $C^k$ on $(-1,1)$ and there exists a finite positive measure $\mu$ on $[-1,1]$ such that

$$
(4.2) \quad f^{[k]}(x_1,x_2,\ldots,x_{k+1}) = \int_{[-1,1]} \frac{1}{(1-\lambda x_1)(1-\lambda x_2)\cdots(1-\lambda x_{k+1})} d\mu(\lambda)
$$
for every $x_1,x_2,\ldots,x_{k+1} \in (-1,1)$.

Moreover, in the above situation, the measures $\mu$ in (ii) and in (iii) are unique and the same, and the following hold for every $m > k$:

(a) For any choice of $\alpha \in (-1,1)$,

$$
(4.3) \quad f(x) = \sum_{l=0}^{m-1} \frac{f^{(l)}(\alpha)}{l!} (x-\alpha)^l + \int_{[-1,1]} \frac{(x-\alpha)^m \lambda^{m-k}}{(1-\lambda x)(1-\lambda \alpha)^m} d\mu(\lambda), \quad x \in (-1,1).
$$

(b) For every $x_1,x_2,\ldots,x_{m+1} \in (-1,1)$,

$$
(4.4) \quad f^{[m]}(x_1,x_2,\ldots,x_{m+1}) = \int_{[-1,1]} \frac{\lambda^{m-k}}{(1-\lambda x_1)(1-\lambda x_2)\cdots(1-\lambda x_{m+1})} d\mu(\lambda).
$$

Proof. (i) $\Rightarrow$ (ii). For each $\alpha \in (-1,1)$ use Corollary 3.4(iii) to have an operator monotone function $g$ on $(-1,1)$, which is represented by Theorem 1.7 as in (4.4) with a representing measure $\mu$. Replacing $(1-\lambda \alpha)^{k-1} d\mu(\lambda)$ with $d\mu(\lambda)$ we have expression (4.1). We next prove that the measure $\mu$ does not depend on $\alpha$. Let $\tilde{\alpha} \in (-1,1)$ be arbitrary. Since

$$
\frac{(x-\alpha)^k}{(1-\lambda x)(1-\lambda \alpha)^k} - \frac{(x-\tilde{\alpha})^k}{(1-\lambda x)(1-\lambda \tilde{\alpha})^k} = \frac{(x-\alpha)^k(1-\lambda \tilde{\alpha})^k - (x-\tilde{\alpha})^k(1-\lambda \alpha)^k}{(1-\lambda x)(1-\lambda \alpha)^k(1-\lambda \tilde{\alpha})^k}
$$

and the numerator of the above right-hand side is zero when $x = 1/\lambda$, we notice that the above expression is written in the form

$$
P_{k-1}(x) := \sum_{l=0}^{k-1} a_l(\lambda)x^l,
$$

where $a_l(\lambda)$, $0 \leq l \leq k-1$, are functions of $\lambda$ ($\alpha, \tilde{\alpha}$ being constants). Since $P_{k-1}(0), P'_{k-1}(0), \ldots, P^{(k-1)}_{k-1}(0)$ are functions of $\lambda$ on $[-1,1]$ integrable with respect to $\mu$, we notice that $a_l(\lambda)$, $0 \leq l \leq k-1$, are integrable with respect to $\mu$. Therefore, we have

$$
f(x) = \text{(a polynomial of at most degree } k-1) + \int_{[-1,1]} \frac{(x-\tilde{\alpha})^k}{(1-\lambda x)(1-\lambda \tilde{\alpha})^k} d\mu(\lambda).
$$

Since the above integral term is $o((x-\tilde{\alpha})^{k-1})$, the first polynomial term must be given by the Taylor formula. Thus (4.1) is valid with $\tilde{\alpha}$ in place of $\alpha$. 


The proof of (ii) ⇒ (iii), as well as that of (b) from (4.1), is included in the proof of Corollary 4.8 and (iii) ⇒ (i) was actually shown in the proof of Lemma 3.2. Moreover, (a) follows from (b) by letting \( x_1 = x \) and \( x_2 = \cdots = x_{m+1} = \alpha \).

It remains to prove the uniqueness of \( \mu \) in (iii). Recall that the linear span of functions \( h_x(\lambda) := 1/(1-\lambda x) \), where \( x \in (-1,1) \), is dense in \( C([-1,1]) \), the space of continuous functions on \([-1,1]\). So, letting \( x_1 = x \) and \( x_2 = \cdots = x_{n+2} = 0 \) in (1.2) one can easily see that \( \mu \) is unique. \( \square \)

Of course the integral term of (1.3) is the \( m \)th remainder term of the Taylor series of \( f \) at \( \alpha \). This remainder term converges to 0 as \( m \to \infty \) for \( |x-\alpha| < 1-|\alpha| \) by Lemma 3.1 (this follows also by the Lebesgue convergence theorem).

We call the finite measure \( \mu \) on \([-1,1]\) in (ii) and (iii) of Theorem 4.1 the representing measure of \( f \). Theorem 4.1(a) says that if \( f \in \mathcal{P}(k)(-1,1) \) with the representing measure \( \mu \) and if \( m > k \) and \( m-k \) is even, then \( f \in \mathcal{P}(m)(-1,1) \) with the representing measure \( \lambda^{m-k} d\mu(\lambda) \). In this connection we show

**Proposition 4.2.** Let \( f \in \mathcal{P}(k)(-1,1) \) with the representing measure \( \mu \). Then

1. \( f \in \mathcal{P}(k+1)(-1,1) \) if and only if \( \mu \) is supported in \([0,1]\). In this case, \( f \in \mathcal{P}(m)(-1,1) \) for all \( m > k \).
2. \( \lambda f \in \mathcal{P}(k+1)(-1,1) \) if and only if \( \mu \) is supported in \([-1,0]\). In this case, \( (1)^{-m-k} f \in \mathcal{P}(m)(-1,1) \) for all \( m > k \).

**Proof.** (1) Assume that \( f \in \mathcal{P}(k+1)(-1,1) \). Theorem 4.1 implies that there exists a finite positive measure \( \mu' \) on \([-1,1]\) such that

\[
\int_{[-1,1]} \frac{1}{1-\lambda x_1} \cdots \frac{1}{1-\lambda x_{k+2}} \, d\mu'(\lambda)
\]

for every \( x_1, \ldots, x_{k+2} \in (-1,1) \). On the other hand, by Theorem 4.1(b) we have

\[
\int_{[-1,1]} \frac{\lambda}{1-\lambda x_1} \cdots \frac{\lambda}{1-\lambda x_{k+2}} \, d\mu(\lambda)
\]

for every \( x_1, \ldots, x_{k+2} \in (-1,1) \). As in the last part of the proof of Theorem 4.1 letting \( x_1 = x \) and \( x_2 = \cdots = x_{n+2} = 0 \), we must have \( d\mu'(\lambda) = \lambda d\mu(\lambda) \) on \([-1,1]\). This means that \( \mu \) is supported in \([0,1]\). Conversely, if \( \mu \) is supported in \([0,1]\), then \( f \in \mathcal{P}(m)(-1,1) \) for all \( m \geq k \) thanks to Theorem 4.1.

(2) is similarly shown. \( \square \)

In the case of \( m = 1 \) (or \( m = 2 \)) the next proposition shows when \( f \in \mathcal{P}(k)(-1,1) \) is a sum of a polynomial of degree \( k \) and an operator monotone (or operator convex) function on \((-1,1)\).

**Proposition 4.3.** Let \( k, m \in \mathbb{N} \) with \( m < k \). Let \( f \in \mathcal{P}(k)(-1,1) \) with the representing measure \( \mu \).

1. Assume that \( k-m \) is even. Then there exist \( a_0, \ldots, a_k \in \mathbb{R} \) with \( a_k \geq 0 \) and \( g \in \mathcal{P}(m)(-1,1) \) such that

\[
f(x) = \sum_{l=0}^{k} a_l x^l + g(x), \quad x \in (-1,1),
\]

if and only if

\[
\int_{[-1,1}\setminus\{0\}} \frac{1}{\lambda^{k-m}} \, d\mu(\lambda) < +\infty.
\]
(2) Assume that \( k - m \) is odd and let \( \varepsilon = \pm 1 \). Then there exist \( a_0, \ldots, a_k \in \mathbb{R} \) with \( a_k \geq 0 \) and \( g \in \mathcal{P}^{(m)}(-1, 1) \) such that

\[
f(x) = \sum_{l=0}^{k} a_l x^l + \varepsilon g(x), \quad x \in (-1, 1),
\]

if and only if \( \mu \) is supported in \( \varepsilon [0, 1] \) and

\[
\int_{\varepsilon[0,1]} \frac{1}{|\lambda|^{k-m}} \, d\mu(\lambda) < +\infty.
\]

**Proof.** (1) Assume that \( f \) is of the form in (1). Let \( \mu' \) be the representing measure of \( g \). By Theorem 4.1(b) we then have

\[
f^{[k]}(x_1, x_2, \ldots, x_{k+1}) = a_k + \int_{[-1,1]} \frac{\lambda^{k-m}}{(1-\lambda x_1)(1-\lambda x_2)\cdots(1-\lambda x_{k+1})} \, d\mu'(\lambda)
\]

\[
= \int_{[-1,1]} \frac{1}{(1-\lambda x_1)(1-\lambda x_2)\cdots(1-\lambda x_{k+1})} \, d\mu(\lambda).
\]

As in the proof of Proposition 4.2 we have

\[
d\mu(\lambda) = a_k \, d\delta_0(\lambda) + \lambda^{k-m} \, d\mu'(\lambda)
\]

so that

\[
\int_{[-1,1]\backslash\{0\}} \frac{1}{\lambda^{k-m}} \, d\mu(\lambda) = \int_{[-1,1]\backslash\{0\}} \, d\mu'(\lambda) < +\infty.
\]

Conversely, assume that \( \int_{[-1,1]\backslash\{0\}} 1/\lambda^{k-m} \, d\mu(\lambda) < +\infty \). Since

\[
\frac{x^k}{1-\lambda x} = \frac{x^m \{1-(1-\lambda x)\}^{k-m}}{(1-\lambda x)^{k-m}}
\]

\[
= x^m \sum_{l=1}^{k-m} \binom{k-m}{l} (-1)^l \frac{(1-\lambda x)^{l-1}}{\lambda^{k-m}} + \frac{x^m}{(1-\lambda x)^{k-m}}, \quad \lambda \neq 0,
\]

we have

\[
f(x) = \sum_{l=0}^{k-1} \frac{f^{(l)}(0)}{l!} x^l + \int_{[-1,1]} \frac{x^k}{1-\lambda x} \, d\mu(\lambda)
\]

\[
= (\text{a polynomial of at most degree } k) + \int_{[-1,1]\backslash\{0\}} \frac{x^m}{(1-\lambda x)^{k-m}} \, d\mu(\lambda).
\]

This is the desired form since

\[
g(x) := \int_{[-1,1]\backslash\{0\}} \frac{x^m}{(1-\lambda x)^{k-m}} \, d\mu(\lambda)
\]

belongs to \( \mathcal{P}^{(m)}(-1, 1) \), where \( g(x) \) is well defined by the integrability assumption.

(2) We do it only for \( \varepsilon = 1 \). Assume that \( f \) is of the form in (2). Let \( \mu' \) be the representing measure of \( g \). Then we have (4.4) as in the proof of (1). Since \( k - m \) is odd, \( \mu \) and \( \mu' \) are supported in \([0, 1]\) and

\[
\int_{(0,1)} \frac{1}{\lambda^{k-m}} \, d\mu(\lambda) = \int_{[0,1]} \, d\mu'(\lambda) < +\infty.
\]

The proof of the converse implication is also similar to that of (1) by replacing the integral region \([-1,1] \backslash \{0\}\) with \((0,1)\). \( \square \)
We say that a smooth real function \( f \) on \((a,b)\) is operator absolutely monotone if
\[
(4.5) \quad \frac{d^k}{dt^k} f(A + tX)\big|_{t=0} \geq 0, \quad k = 0, 1, 2, \ldots,
\]
for every \( A \in \mathbb{M}_n^sa(a,b) \), \( X \in \mathbb{M}_n^+ \) and \( n \in \mathbb{N} \), and operator completely monotone if
\[
(4.6) \quad (-1)^k \frac{d^k}{dt^k} f(A + tX)\big|_{t=0} \geq 0, \quad k = 0, 1, 2, \ldots,
\]
for every \( A \in \mathbb{M}_n^sa(a,b) \), \( X \in \mathbb{M}_n^+ \) and \( n \in \mathbb{N} \). From Proposition 4.2 one can characterize these operator versions of absolutely/completely monotone functions on \((-1,1)\) as follows:

**Proposition 4.4.** The following conditions for a real function on \((-1,1)\) are equivalent:

(i) \( f \) is operator absolutely monotone;
(ii) \( f \) is a non-negative operator monotone function whose representing measure is supported in \([0,1]\);
(iii) \( f \) admits the integral expression
\[
 f(x) = \beta + \int_{[0,1]} \frac{1 + x}{1 - \lambda x} \, d\mu(\lambda)
\]
with \( \beta \geq 0 \) and a (unique) finite positive measure \( \mu \) on \([0,1]\).

**Proof.** (i) \(\iff\) (ii) is obvious from Proposition 4.2(1) for \( k = 1 \). (ii) \(\Rightarrow\) (iii) follows by letting \( \alpha \searrow -1 \) in (1.4) of Theorem 1.7 and then replacing \( (1 + \lambda)^{-1} d\mu(\lambda) \) with \( d\mu(\lambda) \). (iii) \(\Rightarrow\) (ii) is immediate since \((1 + x)/(1 - \lambda x)\) is non-negative and operator monotone on \((-1,1)\) for \( \lambda \in [0,1] \).

**Proposition 4.5.** The following conditions for a real function on \((-1,1)\) are equivalent:

(i) \( f \) is operator completely monotone;
(ii) \(-f\) is a non-positive operator monotone function whose representing measure is supported in \([-1,0]\);
(iii) \( f \) admits the integral expression
\[
 f(x) = \beta + \int_{[-1,0]} \frac{1 - x}{1 - \lambda x} \, d\mu(\lambda)
\]
with \( \beta \geq 0 \) and a (unique) finite positive measure \( \mu \) on \([-1,0]\).

**Proof.** The proof is similar to that of Proposition 4.4. Alternatively, this proposition immediately follows from Proposition 4.4 since \( f \) is operator completely monotone on \((-1,1)\) if and only if \( f(-x) \) is operator absolutely monotone on \((-1,1)\).

Furthermore, one can see from Theorem 3.3 that if \( f \) is operator absolutely monotone (resp., operator completely monotone) on \((-1,1)\), then (4.5) (resp., 4.6) holds for every \( A \in B(\mathcal{H})^sa(-1,1) \) and \( X \in B(\mathcal{H})^+ \). This justifies our terminology.
5. OPERATOR $k$-TONE FUNCTIONS ON $(0, \infty)$

In addition to general characterizations in Theorem 3.3, the next theorem gives integral representation of operator $k$-tone functions on the unbounded interval $(0, \infty)$. Note that (1.8) is (5.2) when $k = 1$.

**Theorem 5.1.** Let $f$ be a real function on $(0, \infty)$, and let $k \in \mathbb{N}$. Then the following conditions (i)–(iii) are equivalent:

(i) $f$ is operator $k$-tone on $(0, \infty)$;

(ii) $f$ is $C^{k-1}$ on $(0, \infty)$ and there exist a $\gamma \geq 0$ and a positive measure $\mu$ on $[0, \infty)$ such that

$$
\int_{[0, \infty)} \frac{1}{(1 + \lambda)^{k+1}} d\mu(\lambda) < +\infty,
$$

and for any choice of $\alpha \in (0, \infty)$,

$$
f(x) = \sum_{l=0}^{k-1} \frac{f^{(l)}(\alpha)}{l!} (x - \alpha)^l + \gamma (x - \alpha)^k + \int_{[0, \infty)} \frac{(x - \alpha)^k}{(x + \lambda)(\alpha + \lambda)^k} d\mu(\lambda), \quad x \in (0, \infty);
$$

(iii) $f$ is $C^k$ on $(0, \infty)$ and there exist a $\gamma \geq 0$ and a positive measure $\mu$ on $[0, \infty)$ such that (5.1) holds and

$$
f^{[k]}(x_1, x_2, \ldots, x_{k+1}) = \gamma + \int_{[0, \infty)} \frac{1}{(x_1 + \lambda)(x_2 + \lambda) \cdots (x_{k+1} + \lambda)} d\mu(\lambda)
$$

for every $x_1, x_2, \ldots, x_{k+1} \in (0, \infty)$.

Moreover, in the above situation, $\gamma$ and $\mu$ in (ii) and in (iii) are unique and the same, and the following hold for every $m > k$:

(a) For any choice of $\alpha \in (0, \infty)$,

$$
f(x) = \sum_{l=0}^{m-1} \frac{f^{(l)}(\alpha)}{l!} (x - \alpha)^l + (-1)^{m-k} \int_{[0, \infty)} \frac{(x - \alpha)^m}{(x + \lambda)(\alpha + \lambda)^m} d\mu(\lambda), \quad x \in (0, \infty),
$$

and hence $(-1)^{m-k} f$ is operator $m$-tone on $(0, \infty)$.

(b) For every $x_1, x_2, \ldots, x_{m+1} \in (0, \infty)$,

$$
f^{[m]}(x_1, x_2, \ldots, x_{m+1}) = (-1)^{m-k} \int_{[0, \infty)} \frac{1}{(x_1 + \lambda)(x_2 + \lambda) \cdots (x_{m+1} + \lambda)} d\mu(\lambda).
$$

**Proof.** (i) $\Rightarrow$ (ii). Assume (i); by Theorem 3.3, $f$ is analytic and $g(x) := f^{[k-1]}(x, \alpha, \ldots, \alpha)$ is operator monotone on $(0, \infty)$ for any fixed $\alpha \in (0, \infty)$. Apply Theorem 1.9 to $g$ and replace $(\alpha + \lambda)^{k-1} d\mu(\lambda)$ with $d\mu(\lambda)$. Then, thanks to (3.3) with $(a, b) = (0, \infty)$, there exist a $\beta \in \mathbb{R}$, a $\gamma \geq 0$ and a positive measure $\mu$ on $[0, \infty)$ satisfying (5.1) such that

$$
f(x) = \sum_{l=0}^{k-1} \frac{f^{(l)}(\alpha)}{l!} (x - \alpha)^l + \beta (x - \alpha)^{k-1} + \gamma (x - \alpha)^k
$$

$$
+ \int_{[0, \infty)} \frac{(x - \alpha)^k}{(x + \lambda)(\alpha + \lambda)^k} d\mu(\lambda), \quad x \in (0, \infty).$$
Since the above integral term is $o((x - \alpha)^{k-1})$, we have $\beta = f^{(k-1)}(\alpha)/(k-1)!$ so that expression (5.2) holds for $\alpha$ fixed above. Let $\tilde{\alpha} \in (0, \infty)$ be arbitrary. Since

\[
\frac{(x - \alpha)^k}{(x + \lambda)(\alpha + \lambda)^k} - \frac{(x - \tilde{\alpha})^k}{(x + \lambda)(\tilde{\alpha} + \lambda)^k}
\]

is a polynomial in $x$ (with coefficients depending on $\lambda$) of at most degree $k - 1$, one can show as in the proof of (i) $\Rightarrow$ (ii) of Theorem 4.1 that the measure $\mu$ does not depend on $\alpha$.

(ii) $\Rightarrow$ (iii). For every $\lambda \in [0, \infty)$, similarly to (1.7) we have

\[
\frac{(x - \alpha)^k}{(x + \lambda)(\alpha + \lambda)^k} = (\text{a polynomial of degree } k - 1) + \frac{(-1)^k}{x + \lambda}
\]

so that

\[
\left(\frac{(x - \alpha)^k}{(x + \lambda)(\alpha + \lambda)^k}\right)^{[k]}(x_1, x_2, \ldots, x_{k+1}) = \frac{1}{(x_1 + \lambda)(x_2 + \lambda)\cdots(x_{k+1} + \lambda)}
\]

for every $x_1, x_2, \ldots, x_{k+1} \in (0, \infty)$. Hence (iii) follows from (ii) by taking the $k$th divided differences of both sides of (5.2).

(iii) $\Rightarrow$ (i). Let $A = \text{Diag}(a_1, \ldots, a_n) \in M_n^{sa}(0, \infty)$ and $X \in M_n^{+}$. Similarly to the proof of Lemma 3.2 based on Daleckii and Krein’s derivative formula, one can show from (iii) that

\[
\frac{d^k}{dt^k} f(A + tX) \bigg|_{t=0} = k! \int_{[0, \infty)} D(\lambda)^{1/2}(D(\lambda)^{1/2}XD(\lambda)^{1/2})^kD(\lambda)^{1/2}d\mu(\lambda) \geq 0
\]

with

\[
D(\lambda) := \text{Diag}\left(\frac{1}{a_1 + \lambda}, \ldots, \frac{1}{a_n + \lambda}\right), \quad \lambda \in [0, \infty).
\]

This yields (i) by Theorem 3.3.

Next, we prove the uniqueness of $\gamma$ and $\mu$ in (ii) or in (iii). It suffices to show the uniqueness of $\gamma$ and $\mu$ in (iii). Let $x_1 = x$ and $x_2 = \cdots = x_{k+1} = 1$ in (5.3).

Then

\[
g(x) := f^{[k]}(x, 1, \ldots, 1) = \gamma + \int_{[0, \infty)} \frac{1}{x + \lambda} \, d\nu(\lambda),
\]

where

\[
d\nu(\lambda) := \frac{1}{(1 + \lambda)^k} \, d\mu(\lambda), \quad \text{hence } \int_{[0, \infty)} \frac{1}{1 + \lambda} \, d\nu(\lambda) < +\infty.
\]

This says that $g$ is a non-negative operator monotone decreasing function on $(0, \infty)$. It is well known that a $\gamma \geq 0$ and a measure $\nu$ on $[0, \infty)$ representing $g$ in (5.5) are unique (in fact, $\gamma = \lim_{x \to +\infty} g(x)$). So $\gamma$ and $\mu$ in (iii) are unique.

Finally, we prove (a) and (b). Assertion (b) immediately follows by computing higher order divided differences from (5.3). Moreover, (5.4) follows from (b) by letting $x_1 = x$ and $x_2 = \cdots = x_{m+1} = \alpha$. The operator $m$-tonicity of $(-1)^{m-k}f$ now follows from expression (5.4) due to condition (ii).

One can understand the integral term of (5.4) as the $m$th remainder term of the Taylor series of $f$ at $\alpha$, which converges to 0 as $m \to \infty$ for $x \in (0, 2\alpha)$. □
Remark 5.2. By using condition (iii) of Theorem 5.1 and Theorem 3.1 (see also Proposition 5.4 and the examples in Section 6). The first inclusion (5.6). Similarly to the proof of Proposition 4.3(1) we have

Assertion (a) of Theorem 5.1 gives the following inclusion property of the \( P^{(k)}(0,\infty) \)'s, where the strict inclusions are seen as in the proof of Proposition 3.9 (also [14]) we see that a real function \( f \) is operator monotone (resp., operator convex) if and only if \( f^{[1]}(x,\alpha) \) (resp., \( f^{[2]}(x,\alpha,\alpha) \) with continuation at \( x=\alpha \)). It is also known ([25, Lemma 2.1] or [18, Corollary 2.7.8]) that a real function \( f \in P^{(k)}(0,\infty) \) (with continuation at \( x=\alpha \)).

Proposition 5.3. The closed convex cones \( P^{(k)}(0,\infty) \), \( k \in \mathbb{N} \), of \( F(0,\infty) \) satisfy

\[
P^{(1)}(0,\infty) \subseteq P^{(2)}(0,\infty) \subseteq P^{(3)}(0,\infty) \subseteq \cdots,
\]

where \( P^{(k)}(0,\infty) := \{ -f : f \in P^{(k)}(0,\infty) \} \).

Theorem 5.1 says that a function \( f \in P^{(k)}(0,\infty) \) admits a unique integral expression given in (5.2) with a constant \( \gamma \geq 0 \) and a measure \( \mu \) on \([0,\infty)\) satisfying (5.1). We call \( \gamma \) the coefficient of the kth degree of \( f \) and \( \mu \) the representing measure of \( f \).

In the case of \( m = 1 \) (or \( m = 2 \)) the next proposition characterizes when \( f \in P^{(k)}(0,\infty) \) is a sum or difference of a polynomial of degree \( k \) and an operator monotone (or operator convex) function on \((0,\infty)\).

Proposition 5.4. Let \( k, m \in \mathbb{N} \) with \( m < k \). For a real function \( f \) on \((0,\infty)\) the following are equivalent:

(i) there exist \( a_0, \ldots, a_k \in \mathbb{R} \) with \( a_k \geq 0 \) and \( g \in P^{(m)}(0,\infty) \) such that

\[
f(x) = \sum_{l=0}^{k} a_l x^l + (-1)^{k-m} g(x), \quad x \in (0,\infty);
\]

(ii) \( f \in P^{(k)}(0,\infty) \) and the representing measure \( \mu \) of \( f \) satisfies

\[
\int_{[0,\infty)} \frac{1}{(1 + \lambda)^{m+1}} d\mu(\lambda) < +\infty.
\]

Proof. (i) \( \Rightarrow \) (ii). Assume that \( f \) is of the form in (i). Let \( \mu \) be the representing measure of \( g \), which satisfies (5.6). Then by (b) of Theorem 5.1 we have

\[
f^{[k]}(x_1, x_2, \ldots, x_{k+1}) = a_k + \int_{[0,\infty)} \frac{1}{(x_1 + \lambda)(x_2 + \lambda) \cdots (x_{k+1} + \lambda)} d\mu(\lambda).
\]

Hence Theorem 5.1 implies that \( f \in P^{(k)}(0,\infty) \) with the coefficient of the kth degree \( a_k \) and the representing measure \( \mu \).

(ii) \( \Rightarrow \) (i). Assume that \( f \) is represented as (5.2) with the measure \( \mu \) satisfying (5.6). Similarly to the proof of Proposition 4.3(1) we have

\[
f(x) = (\text{a polynomial of at most degree } k) + (-1)^{k-m} g(x)
\]
with
\[ g(x) := \int_{[0, \infty)} \frac{(x - \alpha)^m}{(x + \lambda)(\alpha + \lambda)^m} \, d\mu(\lambda) \]
belonging to \( P^{(m)}(0, \infty) \).

The case \( m = 0 \) version of Proposition 5.2 can be stated as follows: A function \( f \in P^{(k)}(0, \infty) \) with the representing measure \( \mu \) is of the form \( f(x) = a_0 + (-1)^k g(x) \) with \( a_0 \geq 0 \) and a non-negative operator monotone decreasing function \( g \) on \((0, \infty)\) if and only if
\[ \int_{[0, \infty)} \frac{1}{1 + \lambda} \, d\mu(\lambda) < +\infty. \]
The proof is similar to the above. This suggests us to define \( P^{(0)}(0, \infty) \) as the set of all non-negative operator monotone decreasing functions on \((0, \infty)\); then \( P^{(0)}(0, \infty) \subset -P^{(1)}(0, \infty) \subset P^{(2)}(0, \infty) \).

**Lemma 5.5.** Let \( k \in \mathbb{N} \) and \( f \in P^{(k)}(0, \infty) \). Then \( \lim_{x \searrow 0} xf(x) \) and \( \lim_{x \to \infty} f(x) / x^k \) exist and
\[ \lim_{x \searrow 0} xf(x) \in \begin{cases} (-\infty, 0] & \text{if } k \text{ is odd}, \\ (0, \infty) & \text{if } k \text{ is even}, \end{cases} \quad \lim_{x \to \infty} \frac{f(x)}{x^k} \in [0, \infty). \]

**Proof.** First, assume that \( g \) is an operator monotone function on \((0, \infty)\). From the integral representation of \( g \), it is easy to show that \( \lim_{x \searrow 0} xg(x) \) and \( \lim_{x \to \infty} g(x) / x \) exist and
\[ \lim_{x \searrow 0} xg(x) \in (-\infty, 0], \quad \lim_{x \to \infty} \frac{g(x)}{x} \in [0, \infty); \]
see also [20 Corollary 2.7]. Next, assume that \( f \in P^{(k)}(0, \infty) \). By Corollary 3.4 there is an operator monotone function \( g \) on \((0, \infty)\) such that \( f \) is a sum of a polynomial of degree less than or equal to \( k - 2 \) and \( (x - 1)^{k-1} g(x) \). This together with (5.7) yields the conclusion. (In fact, by using the Lebesgue convergence theorem, one can easily see from (5.2) that \( \gamma = \lim_{x \to \infty} f(x) / x^k \), the coefficient of the \( k \)-th degree of \( f \).)

**Proposition 5.6.** Let \( k, m \in \mathbb{N} \) with \( k < m \). For a real function \( f \) on \((0, \infty)\) the following are equivalent:

(i) \( f \in P^{(k)}(0, \infty) \);
(ii) \( (-1)^{m-k} f \in P^{(m)}(0, \infty) \) and \( \lim_{x \to \infty} f(x) / x^k \in [0, \infty) \).

**Proof.** (i) \( \Rightarrow \) (ii). Assume that \( f \in P^{(k)}(0, \infty) \), and let \( \gamma \geq 0 \) be the coefficient of the \( k \)-th degree and \( \mu \) the representing measure of \( f \). Then \( (-1)^{m-k} f \in P^{(m)}(0, \infty) \) by Proposition 5.3 and Lemma 5.5 implies that \( \lim_{x \to \infty} f(x) / x^k \in [0, \infty) \).

(ii) \( \Rightarrow \) (i). It suffices to prove the case where \( k = m - 1 \), that is,
\[ (*) \text{ if } - f \in P^{(m)}(0, \infty) \text{ and } \lim_{x \to \infty} f(x) / x^{m-1} \in [0, \infty), \text{ then } f \in P^{(m-1)}(0, \infty). \]
Indeed, assume that (ii) holds for \( k < m \). Since \( \lim_{x \to \infty} (-1)^{m-k-1} f(x) / x^{m-1} \in [0, \infty) \) (in fact, \( \lim_{x \to \infty} f(x) / x^{m-1} = 0 \) if \( k < m - 1 \)), we apply (*) to have \((-1)^{m-k-1} f \in P^{(m-1)}(0, \infty) \). If \( k < m - 1 \), then we apply (*) again to have \((-1)^{m-k-2} f \in P^{(m-2)}(0, \infty) \). Repeating this procedure yields that \( f \in P^{(k)}(0, \infty) \).
To prove (*), assume that \(-f \in \mathcal{P}(m)(0, \infty)\) and \(\lim_{x \to \infty} f(x)/x^{m-1} \in [0, \infty)\). Let \(\gamma_0 \geq 0\) and \(\mu\) be the coefficient of the \(m\)th degree and the representing measure of \(-f\), respectively. With \(\alpha \in (0, \infty)\) we have

\[
f(x) = \sum_{l=0}^{m-1} \frac{f^{(l)}(\alpha)}{l!} (x-\alpha)^l - \gamma_0 (x-\alpha)^m - \int_{[0, \infty)} \frac{(x-\alpha)^m}{(x+\lambda)(\alpha+\lambda)^m} \, d\mu(\lambda), \quad x \in (0, \infty).
\]

Since

\[
\lim_{x \to \infty} \frac{1}{x^{m-1}} \int_{[0, \infty)} \frac{(x-\alpha)^m}{(x+\lambda)(\alpha+\lambda)^m} \, d\mu(\lambda) = \int_{[0, \infty)} \frac{1}{(\alpha+\lambda)^m} \, d\mu(\lambda)
\]

by the monotone convergence theorem, the assumption \(\lim_{x \to \infty} f(x)/x^{m-1} \in [0, \infty)\) implies that \(\gamma_0 = 0\) and

\[
(5.8) \quad \gamma_1 := \frac{f^{(m-1)}(\alpha)}{(m-1)!} - \int_{[0, \infty)} \frac{1}{(\alpha+\lambda)^m} \, d\mu(\lambda) \in [0, \infty)
\]

so that

\[
(5.9) \quad \int_{[0, \infty)} \frac{1}{(\alpha+\lambda)^m} \, d\mu(\lambda) < +\infty.
\]

Since

\[
\frac{(x-\alpha)^m}{(x+\lambda)(\alpha+\lambda)^m} = \frac{(x-\alpha)^{m-1}}{(\alpha+\lambda)^{m-1}} - \frac{(x-\alpha)^{m-1}}{(x+\lambda)(\alpha+\lambda)^{m-1}},
\]

one can write

\[
f(x) = \sum_{l=0}^{m-2} \frac{f^{(l)}(\alpha)}{l!} (x-\alpha)^l + \gamma_1 (x-\alpha)^{m-1} + \int_{[0, \infty)} \frac{(x-\alpha)^{m-1}}{(x+\lambda)(\alpha+\lambda)^{m-1}} \, d\mu(\lambda).
\]

Thanks to (5.8) and (5.9) this yields that \(f \in \mathcal{P}^{(m-1)}(0, \infty)\). \(\square\)

By using Proposition 5.3 and Lemma 5.5 it is not difficult to prove the following characterizations of operator absolutely/completely monotone functions on \((0, \infty)\) (introduced in the previous section).

**Proposition 5.7.** Let \(f\) be a smooth real function on \((0, \infty)\). Then \(f\) is operator absolutely monotone if and only if \(f(x) = \alpha x + \beta\) with \(\alpha, \beta \geq 0\). Also, \(f\) is operator completely monotone if and only if \(f\) is a non-negative operator monotone decreasing function on \((0, \infty)\).

6. Examples

In this section we present several examples of operator \(k\)-tone functions on \((0, \infty)\). First, recall a convenient way to obtain operator \(k\)-tone functions on \((0, \infty)\). Let \(g\) be an operator monotone function on \((0, \infty)\), \(k, m \in \mathbb{N}\) with \(m < k\), and let \(\alpha_1, \ldots, \alpha_m \in (0, \infty)\). By Corollary 3.2 and Proposition 5.3 we obtain

\[
(6.1) \quad (-1)^{k-m-1} \left\{ \prod_{l=1}^{m} (x-\alpha_l) \right\} g(x) \in \mathcal{P}^{(k)}(0, \infty).
\]

Furthermore, by taking the limit as \(\alpha_l \downarrow 0\), (6.1) is valid for any \(\alpha_1, \ldots, \alpha_m \in [0, \infty)\) with any \(m < k\).
Example 6.1. Note that $-x^r$ and $x^r(x-\alpha)$ are operator monotone on $(0,\infty)$ for every $r \in [-1,0]$ and $\alpha \in [0,\infty)$. For each $k \in \mathbb{N}$ and every $\alpha_1, \ldots, \alpha_k \in [0,\infty)$, by (6.1) we have

$$(-1)^{k-m}x^r \prod_{i=1}^{m} (x - \alpha_i) \in \mathcal{P}(k)(0,\infty)$$

if $r \in [-1,0]$ and $m = 0, 1, \ldots, k$.

Concerning the power functions $\pm x^p$ on $(0,\infty)$ with $p \in \mathbb{R}$, let us prove that, for each $k \in \mathbb{N}$, $x^p \in \mathcal{P}(k)(0,\infty)$ if and only if

$$p \in [0,1] \cup [2,3] \cup \cdots \cup [k-1,k] \quad \text{if } k \text{ is odd},$$

and $-x^p \in \mathcal{P}(k)(0,\infty)$ if and only if

$$p \in [-1,0] \cup [1,2] \cup \cdots \cup [k-1,k] \quad \text{if } k \text{ is even}.$$

We need to prove the “only if” parts. By Lemma 5.5 note that $p \in [-1,k]$ is necessary for $\pm x^p$ to belong to $\mathcal{P}(k)(0,\infty)$. Since

$$\frac{d^k}{dx^k} x^p = p(p-1) \cdots (p-k+1)x^{p-k}, \quad x \in (0,\infty),$$

the condition $p(p-1) \cdots (p-k+1) \geq 0$ (resp., $p(p-1) \cdots (p-k+1) \leq 0$) is necessary for $x^p$ (resp., $-x^p$) to belong to $\mathcal{P}(k)(0,\infty)$. Therefore, (6.2) is necessary for $x^p$ to belong to $\mathcal{P}(k)(0,\infty)$, and (6.3) is necessary for $-x^p$ to belong to $\mathcal{P}(k)(0,\infty)$.

Example 6.2. Since log $x$ is operator monotone on $(0,\infty)$, for each $k \in \mathbb{N}$ and every $\alpha_1, \ldots, \alpha_{k-1} \in [0,\infty)$, by (6.1) we have

$$(-1)^{k-m-1} \prod_{i=1}^{m} (x - \alpha_i) \log x \in \mathcal{P}(k)(0,\infty), \quad m = 0, 1, \ldots, k - 1.$$
Hence, if \( p \not\in \{0, 1, \ldots, k - 1\} \), then \( \frac{d^k}{dx^k}(x^p \log x) \) takes both positive and negative values on \((0, \infty)\), so neither \( x^p \log x \) nor \(-x^p \log x \) belongs to \( \mathcal{P}^{(k)}(0, \infty) \). Moreover, if both \( \pm x^p \log x \) belong to \( \mathcal{P}^{(k)}(0, \infty) \), then \( x^p \log x \) must be a polynomial of at most degree \( k - 1 \), which is impossible. Combining these facts shows the assertions.

**Example 6.3.** Note that \(-1/(x + 1) \) and \( x/(x + 1) \) are operator monotone on \((0, \infty)\). By (6.1), for each \( k \in \mathbb{N} \) and every \( \alpha_1, \ldots, \alpha_k \in [0, \infty) \),

\[
\frac{(-1)^{k-m}}{x + 1} \prod_{l=1}^{m}(x - \alpha_l) \in \mathcal{P}^{(k)}(0, \infty), \quad m = 0, 1, \ldots, k.
\]

Concerning the functions \( \pm x^p/(x + 1) \) on \((0, \infty)\) with \( p \in \mathbb{R} \), let us show that, for each \( k \in \mathbb{N} \), \( x^p/(x + 1) \in \mathcal{P}^{(k)}(0, \infty) \) if and only if

\[
\begin{cases}
p \in \{1, 3, \ldots, k\} & \text{if } k \text{ is odd,} \\
p \in \{0, 2, \ldots, k\} & \text{if } k \text{ is even,}
\end{cases}
\]

and \(-x^p/(x + 1) \in \mathcal{P}^{(k)}(0, \infty) \) if and only if

\[
\begin{cases}
p \in \{0, 2, \ldots, k - 1\} & \text{if } k \text{ is odd,} \\
p \in \{1, 3, \ldots, k - 1\} & \text{if } k \text{ is even.}
\end{cases}
\]

By induction one can compute the \( k \)th derivative,

\[
\frac{d^k}{dx^k}(x^p/(x + 1)) = x^{p-k}(x+1)^{-(k+1)}Q_k(x), \quad k = 1, 2, \ldots,
\]

where \( Q_k(x) \) is a polynomial given as

\[
Q_k(x) = (p-1)(p-2) \cdots (p-k)x^k + \alpha^{(k)}_{k-1}x^{k-1} + \cdots + \alpha^{(k)}_{1}x + p(p-1) \cdots (p-k+1),
\]

that is, \( Q_k(x) \) is a polynomial of at most degree \( k \) with the coefficient \((p-1)(p-2) \cdots (p-k)\) of \( x^k \) and the constant term \( p(p-1) \cdots (p-k+1) \). If \( x^p/(x + 1) \in \mathcal{P}^{(k)}(0, \infty) \), then \( Q_k(x) \geq 0 \) for all \( x \in (0, \infty) \) and we must have

\[
(p-1)(p-2) \cdots (p-k) \geq 0, \quad p(p-1) \cdots (p-k+1) \geq 0.
\]

These imply that

\[
\begin{cases}
p \in \{1, 2, 3, \ldots, k - 1\} \cup [k, \infty) & \text{if } k \text{ is odd,} \\
p \in (-\infty, 0] \cup \{1, 2, 3, \ldots, k - 1\} \cup [k, \infty) & \text{if } k \text{ is even.}
\end{cases}
\]

On the other hand, if \(-x^p/(x + 1) \in \mathcal{P}^{(k)}(0, \infty) \), then we must similarly have

\[
\begin{cases}
p \in (-\infty, 0] \cup \{1, 2, 3, \ldots, k - 1\} & \text{if } k \text{ is odd,} \\
p \in \{1, 2, 3, \ldots, k - 1\} & \text{if } k \text{ is even.}
\end{cases}
\]

Here \( \pm x^p/(x + 1) \) cannot belong to \( \mathcal{P}^{(k)}(0, \infty) \) at the same time. Hence it remains to show that \( x^p/(x + 1) \) does not belong to \( \mathcal{P}^{(k)}(0, \infty) \) if \( p \in (k, \infty) \) and that \( \pm x^p/(x + 1) \) do not belong to \( \mathcal{P}^{(k)}(0, \infty) \) if \( p \in (-\infty, 0) \). In the case \( k = 1 \), these can be shown by appealing to the analytic continuation property (as Pick functions) of operator monotone functions. Then one can use an induction argument based on a characterization of \( f \in \mathcal{P}^{(k)}(0, \infty) \) in terms of \( f^{[k-1]}(x, \alpha, \ldots, \alpha) \) given in Theorem 3.3 while the details are omitted.
Example 6.4. It is well known that \((x - 1)/\log x\) is operator monotone on \((0, \infty)\). By (6.1), for each \(k \in \mathbb{N}\) and every \(\alpha_1, \ldots, \alpha_{k-1} \in [0, \infty)\),

\[
(-1)^{k-m-1} \left\{ \prod_{l=1}^{m} (x - \alpha_l) \right\} \frac{x - 1}{\log x} \in \mathcal{P}^{(k)}(0, \infty), \quad m = 0, 1, \ldots, k - 1.
\]

Concerning the functions \(\pm x^p(1-x)/\log x\) on \((0, \infty)\) with \(p \in \mathbb{R}\), we notice that, for each \(k \in \mathbb{N}\), \(x^p(1-x)/\log x \in \mathcal{P}^{(k)}(0, \infty)\) if and only if

\[
\begin{cases} 
  p \in \{0, 2, \ldots, k - 1\} & \text{if } k \text{ is odd,} \\ 
  p \in \{1, 3, \ldots, k - 1\} & \text{if } k \text{ is even,}
\end{cases}
\]

and \(-x^p/\log x \in \mathcal{P}^{(k)}(0, \infty)\) if and only if

\[
\begin{cases} 
  p \in \{1, 3, \ldots, k - 2\} & \text{if } k \text{ is odd (empty if } k = 1), \\ 
  p \in \{0, 2, \ldots, k - 2\} & \text{if } k \text{ is even.}
\end{cases}
\]

The proof of these assertions is similar to those of the above examples. We omit the details.

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