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QUOTIENTS OF THE CROWN DOMAIN BY A PROPER ACTION OF A CYCLIC GROUP

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ABSTRACT. Let G/K be an irreducible Riemannian symmetric space of the non-compact type and denote by Ξ the associated crown domain. We show that for any proper action of a cyclic group Γ the quotient Ξ/Γ is Stein. An analogous statement holds true for discrete nilpotent subgroups of a maximal split-solvable subgroup of G. We also show that Ξ is taut.

INTRODUCTION

Let G/K be an irreducible Riemannian symmetric space of the non-compact type, where G is assumed to be embedded in its universal complexification $G^{\mathbb{C}}$. In [AkGi90] D. N. Akhiezer and S. G. Gindikin pointed out a distinguished invariant domain Ξ of $G^{\mathbb{C}}/K^{\mathbb{C}}$ containing G/K (as a maximal totally-real submanifold) such that the extended (left) G-action on Ξ is proper. The domain Ξ, which is usually referred to as the crown domain or the Akhiezer-Gindikin domain, is Stein and Kobayashi hyperbolic by a result of D. Burns, S. Hind and S. Halverscheid ([BHH03]; cf. [Bar03], [KrSt05]). In fact G. Fels and A.T. Huckleberry ([FeHu05]) have shown that with respect to these properties it is the maximal G-invariant complexification of G/K in $G^{\mathbb{C}}/K^{\mathbb{C}}$. By using the characterization of the G-invariant, plurisubharmonic functions on Ξ given in [BHH03], here we also note that Ξ is taut (Proposition 3.4). It seems not to be known whether all crown domains are complete Kobayashi hyperbolic.

The crown domain can also be regarded as the maximal domain in the tangent bundle of G/K admitting an adapted complex structure (see [BHH03] for more details). Recently it has been intensively investigated in connection with the harmonic analysis of G/K (see, e.g. [GiKr02b], [KrSt04], [KrSt05], [KrOp08]). Here we consider particular complex manifolds associated to Ξ. Namely, given a proper action on Ξ of a discrete group Γ of biholomorphisms, we are interested in complex-geometric properties of the quotient Ξ/Γ. If Γ is finite one knows that Ξ/Γ is Stein by a classical result of H. Grauert and R. Remmert ([GrRe79, Thm. 1, Ch. V]; cf. [Hei91]). Moreover if Γ is a discrete, cocompact, torsion-free subgroup of G, then the quotient Ξ/Γ was shown to be Stein in [BHH03, Proof of Cor. 11] (cf. the proof of Proposition 3.4). It is natural to ask whether Ξ/Γ is also Stein in the case when Γ is not cocompact. Our main goal is to give an affirmative answer in the simplest interesting case of an infinite cyclic group $\Gamma \cong \mathbb{Z}$.

One should observe that Ξ is biholomorphic to a simply connected domain in $\mathbb{C}^n$. For this, consider an Iwasawa decomposition NAK of G. Then Ξ can be realized as an NA-invariant domain in the universal complexification of NA, which

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is biholomorphic to a complex affine space (cf. section 5). In this setting one may ask the following more general question: given proper $\mathbb{Z}$-action on a simply connected Stein domain X of $\mathbb{C}^n$, is the quotient $X/\mathbb{Z}$ Stein? This is not always the case; for instance J. Winkelmann ([Win90]) has given an example of a free and properly discontinuous action on $\mathbb{C}^5$ such that the quotient is not holomorphically separable. On the other hand, if X is a simply connected, bounded, Stein domain of $\mathbb{C}^2$ and the $\mathbb{Z}$-action is induced by a (proper) $\mathbb{R}$-action, then $X/\mathbb{Z}$ is Stein by a result of C. Miebach and K. Oeljeklaus ([MiOe09]). Furthermore, C. Miebach ([Mie10]) has shown that if X is a homogeneous bounded domain, then $X/\mathbb{Z}$ is Stein for any proper $\mathbb{Z}$-action. For $n \geq 2$ we are not aware of any example of a simply connected, bounded, Stein domain of $\mathbb{C}^n$, with a proper $\mathbb{Z}$-action such that the quotient is not Stein.

The crown domain is either a Hermitian symmetric space of a larger group or it is rigid, i.e. the automorphism group of Ξ coincides with the group of isometries of G/K ([BHH03]). Hence, the latter case is a source of interesting examples of simply connected, non-homogenous, Stein domains of $\mathbb{C}^n$ with a large automorphism group. Our main result is

Theorem. *Let G/K be an irreducible Riemannian symmetric space of the non-compact type and let Ξ be the associated crown domain. Then $\Xi/\mathbb{Z}$ is a Stein manifold for every proper $\mathbb{Z}$-action.*

An important ingredient in the proof is the above mentioned realization of Ξ as an NA-invariant domain in the universal complexification of NA. For this we recall that the crown domain is given by $\Xi = G\exp(i\omega)K^{\mathbb{C}}/K^{\mathbb{C}}$, where ω is the cell in the Lie algebra $\mathfrak{a}$ of A defined by

$$\omega := \{X \in \mathfrak{a} : |\alpha(X)| < \pi/2, \text{ for every restricted root } \alpha\}.$$

By a result of S. Gindikin and B. Krötz ([GiKr02a, Thm. 1.3]), the crown Ξ is contained in $N^{\mathbb{C}}A\exp(i\omega)K^{\mathbb{C}}/K^{\mathbb{C}}$. Thus, in order to realize Ξ as an NA-invariant domain in the universal complexification of NA, it is sufficient to note that the multiplication map $N^{\mathbb{C}} \times A\exp(i\omega) \to G^{\mathbb{C}}/K^{\mathbb{C}}$, given by $(n, a) \to naK^{\mathbb{C}}$, is an open embedding (Lemma 3.5). Then by following a strategy carried out in [Mie10], one can reduce to the case of $\mathbb{Z}$ contained in NA. Since NA is a connected, split-solvable Lie group, the quotient of the universal complexification of NA by $\mathbb{Z}$ is Stein (Proposition 2.6). Finally, $\Xi/\mathbb{Z}$ is locally Stein in such a Stein quotient and the proof of the above theorem follows by applying a classical result of F. Docquier and H. Grauert ([DoGr60]).

As pointed out to us by C. Miebach, the above arguments also apply to show the following proposition (Proposition 4.4)

Proposition. *Let Γ be any discrete, nilpotent subgroup of NA. Then Ξ/Γ is Stein.*

The paper is organized as follows. In the first section we recall basic results on semisimple Lie groups and on the Iwasawa decomposition. In section 2 we discuss complex-geometric properties of quotients of complex Lie groups by a proper action of a discrete subgroup. In section 3 we recall basic properties of the crown domain and we show that Ξ is taut. The main result is proved in section 4.

1. Preliminaries

Here we introduce the notation and we recall some basic facts on Riemannian symmetric spaces and semisimple Lie groups.

Definition 1.1. Let G be a real Lie group. A complex Lie group $G^{\mathbb{C}}$ together with a Lie group homomorphism $\gamma : G \to G^{\mathbb{C}}$ is the *universal complexification* of G if it satisfies the following universal property: for every complex Lie group H and every Lie group homomorphism $\phi : G \to H$ there exists a unique morphism $\phi^{\mathbb{C}} : G^{\mathbb{C}} \to H$ such that $\phi = \phi^{\mathbb{C}} \circ \gamma$.

By the above universal property $G^{\mathbb{C}}$ is unique up to isomorphisms. For the existence and the construction of the universal complexification and its fundamental properties we refer to [Hoc65, XVII.5].

Let G be a connected, non-compact, simple[1] Lie group which is assumed to be embedded in its universal complexification $G^{\mathbb{C}}$. Choose a maximal compact subgroup K of G and note that K is connected. The quotient space $M = G/K$ is an irreducible, Riemannian symmetric space of the non-compact type. The universal complexification $K^{\mathbb{C}}$ of K coincides with the complexification of K in $G^{\mathbb{C}}$, i.e. with the connected Lie subgroup of $G^{\mathbb{C}}$ associated to the complexification of the Lie algebra of K. In the sequel we will be interested in the G-orbit structure of $G^{\mathbb{C}}/K^{\mathbb{C}}$. One has (cf. [GeIa08, Rem. 4.1])

Remark 1.2. Let G/K and G'/K' be two different Klein representations of the same irreducible, Riemannian symmetric space of the non-compact type M, with $\mathrm{Lie}(G) = \mathrm{Lie}(G')$ a simple Lie algebra. Then the complexifications $G^{\mathbb{C}}/K^{\mathbb{C}}$ and $(G')^{\mathbb{C}}/(K')^{\mathbb{C}}$ are biholomorphic and they have the same orbit structure with respect to the actions of G and of G', respectively.

Let $\mathfrak{k}$ be the Lie algebra of K and consider the Cartan decomposition $\mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g}$ associated to $\mathfrak{k}$. For $\mathfrak{a}$ a maximal abelian subalgebra of $\mathfrak{p}$, consider the corresponding restricted root system $\Sigma \subset \mathfrak{a} \backslash \{0\}$ and the root space decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in \Sigma} \mathfrak{g}^{\alpha} \oplus \mathfrak{a} \oplus \mathfrak{m} \,,$$

where $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$ and the root spaces are defined by $\mathfrak{g}^{\alpha} = \{X \in \mathfrak{g} : [H, X] = \alpha(H)X, \text{ for every } H \in \mathfrak{a}\}$. One has

$$[\mathfrak{g}^{\alpha}, \mathfrak{g}^{\beta}] \subset \mathfrak{g}^{\alpha+\beta} \,.$$

Fix a system of positive roots $\Sigma^+ \subset \Sigma$. One has the associated Iwasawa decomposition at the Lie algebra level

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k},$$

where $\mathfrak{n} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}^{\alpha}$.

By construction $\mathfrak{a}$ normalizes $\mathfrak{n}$, therefore $\mathfrak{n} \oplus \mathfrak{a}$ is a semidirect product of a nilpotent and an abelian algebra. In particular $\mathfrak{n} \oplus \mathfrak{a}$ is a solvable Lie algebra. Let A and N be the analytic subgroups of G corresponding to $\mathfrak{a}$ and $\mathfrak{n}$, respectively. The Iwasawa decomposition at the group level is given by $G = NAK$, meaning that the multiplication map $N \times A \times K \to G$ is an analytic diffeomorphism (see [Hel01, Thm. 5.1, Ch. VI]). Moreover, NA is a (closed) solvable subgroup of G

[1] Here a Lie group G is simple if its Lie algebra is simple. Therefore G may have a non-trivial discrete center.

isomorphic to the semidirect product $N \rtimes A$. The following facts are well known and scattered in the literature. For a proof see e.g. [Vit12, Prop. 1.3, Rem. 1.4].

Proposition 1.3. *Let G be a connected, real, simple Lie group and let NAK be an Iwasawa decomposition of G. Then*

(i) *The complexification*[2] *$A^{\mathbb{C}}$ of A in $G^{\mathbb{C}}$ is closed and is isomorphic to $(\mathbb{C}^*)^r$, with $r = \dim_{\mathbb{R}} \mathfrak{a}$.*
(ii) *The complexification $N^{\mathbb{C}}$ of N in $G^{\mathbb{C}}$ is closed and simply connected.*
(iii) *The group $A^{\mathbb{C}}$ normalizes $N^{\mathbb{C}}$ and $N^{\mathbb{C}} \cap A^{\mathbb{C}} = \{e\}$. In particular $N^{\mathbb{C}}A^{\mathbb{C}}$ is isomorphic to a semidirect product $N^{\mathbb{C}} \rtimes A^{\mathbb{C}}$.*
(iv) *The map $N^{\mathbb{C}} \times A^{\mathbb{C}} \times N^{\mathbb{C}} \longrightarrow G^{\mathbb{C}}$ given by $(n, a, n') \mapsto na\theta(n')$ is injective, where θ denotes the holomorphic extension to $G^{\mathbb{C}}$ of the Cartan involution of G with respect to K.*

We also recall some general facts regarding the multiplicative Jordan-Chevalley decomposition. Let G be a real, simple Lie group. An element $g \in G$ is said to be *unipotent* (resp. *hyperbolic*) if it is of the form $g = \exp(X)$, where $\mathrm{ad}(X)$ is nilpotent (resp. $\mathrm{ad}(X)$ is diagonalizable over $\mathbb{R}$) and *elliptic* if $\mathrm{Ad}(g)$ is diagonalizable over $\mathbb{C}$ with eigenvalues of norm 1. Then one has

Proposition 1.4 (see [Kos73, Prop. 2.1])**.** *Let G be a real simple Lie group. Then every element g of G admits a unique decomposition $g_u g_h g_e$, where g_u is unipotent, g_h is hyperbolic, g_e is elliptic and every pair of elements in $\{g_u, g_h, g_e\}$ commute.*

It is easy to check that if nak is an Iwasawa decomposition of an element g of G, then the elements n, a and k are unipotent, hyperbolic and elliptic, respectively. The relation between the two decompositions is clarified by the following proposition (cf. [Kos73, Prop. 2.3-2.5]).

Proposition 1.5. *Let G be a real, non-compact, simple Lie group and let NAK be an Iwasawa decomposition of G. Then an element of G*

(i) *is unipotent if and only if it is conjugate to an element of N,*
(ii) *is hyperbolic if and only if it is conjugate to an element of A,*
(iii) *is elliptic if and only if it is conjugate to an element of K,*
(iv) *has a trivial elliptic part if and only if it is conjugate to an element of NA.*

2. Discrete group actions on complex Lie groups

Let G be a connected non-compact real Lie group embedded in its universal complexification $G^{\mathbb{C}}$ and let Γ be a discrete subgroup of G. Then Γ acts freely and properly discontinuously on $G^{\mathbb{C}}$ and $G^{\mathbb{C}}/\Gamma$ is a complex manifold. We recall that the universal complexification $G^{\mathbb{C}}$ of a real Lie group is Stein (see [Hei93, p. 147]). It is of interest to know when the quotient $G^{\mathbb{C}}/\Gamma$ is Stein in terms of sufficient and/or necessary conditions on G and Γ. For instance if G is nilpotent, by a result of B. Gilligan and A. T. Huckleberry (see the proof of Thm. 7 in [GiHu78]), it follows that $G^{\mathbb{C}}/\Gamma$ is Stein. For Lie groups with simply connected complexification one has the following result of J. J. Loeb.

Theorem 2.1 ([Loe85, Thm. 1, Lemma 1])**.** *Let G be a real connected Lie group with simply connected universal complexification $G^{\mathbb{C}}$ and let Γ be a discrete, cocompact subgroup of G. Then $G^{\mathbb{C}}/\Gamma$ is Stein if and only if G has a purely imaginary spectrum, i.e. for every X in $\mathfrak{g}$ the eigenvalues of $\mathrm{ad}(X)$ are purely imaginary.*

[2]Note that the complexification $A^{\mathbb{C}}$ of A in $G^{\mathbb{C}}$ is not the universal complexification of A.

In the sequel we will be interested in the universal complexification of a solvable Lie group. We point out that the solvability of the group G is not sufficient in order to satisfy Loeb's condition of Theorem 2.1. Indeed one can give an example (see [Loe85, p. 76]; cf. [CIT00, Ex. 3.2]) of a solvable Lie group G with simply connected complexification $G^{\mathbb{C}}$, admitting a discrete cocompact subgroup Γ and such that $\mathrm{ad}(X)$ has eigenvalues with non-trivial real part, for some X in $\mathfrak{g}$. Thus Theorem 2.1 implies that $G^{\mathbb{C}}/\Gamma$ is not Stein. In the sequel we will be interested in the following class of solvable Lie groups.

Definition 2.2. A real Lie algebra $\mathfrak{s}$ is *split-solvable* if it is solvable and the eigenvalues of $\mathrm{ad}(X)$ are real for every $X \in \mathfrak{s}$. A real Lie group S is *split-solvable* if it is simply connected and its Lie algebra $\mathfrak{s}$ is split-solvable.

Remark 2.3. Let G be a real simple Lie group and let $G = NAK$ be an Iwasawa decomposition of G. Then NA is a maximal split-solvable subgroup of G. Indeed, it is easy to check that NA is simply connected and maximal solvable. Moreover one can choose a suitable basis of $\mathfrak{g}$ such that $\mathrm{ad}(X)$ is represented by upper triangular matrices for every X in $\mathfrak{n} \oplus \mathfrak{a}$ (cf. [Vit12, Prop. 1.3]). Thus NA is a maximal split-solvable subgroup of G.

Remark 2.4. If S is a split-solvable Lie group, then the exponential map is a diffeomorphism (see, e.g. [Vin94, Thm. 6.4, Ch. 2]). In particular, every connected subgroup of S is closed and simply connected. In fact, the latter property holds true for every simply connected solvable Lie group (cf. [Var84, Thm. 3.18.12]).

Definition 2.5. A discrete subgroup Γ of a real split-solvable Lie group S is *nilpotent* if equivalently

(i) it is contained in a connected, nilpotent subgroup of S;
(ii) it admits a finite central series $\Gamma \rhd \Gamma^{(1)} \rhd \ldots \rhd \Gamma^{(m)} = \{e\}$, where $\Gamma^{(1)} := [\Gamma, \Gamma]$ and $\Gamma^{(i)} := [\Gamma, \Gamma^{(i-1)}]$.

Proposition 2.6. *Let Γ be a discrete subgroup of a connected, split-solvable Lie group S and let $S^{\mathbb{C}}$ be the universal complexification of S. Then $S^{\mathbb{C}}/\Gamma$ is Stein if and only if Γ is nilpotent.*

Proof. Since S is split-solvable, by [Wit02, Cor. 3.4] there exists a unique connected subgroup S_Γ of S such that S_Γ/Γ is compact. The subgroup S_Γ being unique, it coincides with the connected subgroup associated to the Lie subalgebra $\mathfrak{s}_\Gamma$ of $\mathfrak{s} := \mathrm{Lie}(S)$ generated by $\exp^{-1}(\Gamma)$. In particular S_Γ is the smallest connected (closed) Lie subgroup of S containing Γ. Note that since S is simply connected, so is its universal complexification $S^{\mathbb{C}}$. As a consequence the connected Lie subgroup $S_\Gamma^{\mathbb{C}}$ of $S^{\mathbb{C}}$ associated to the complexified Lie algebra $\mathfrak{s}_\Gamma^{\mathbb{C}}$ is closed and simply connected (see [Var84, Thm. 3.18.12]). Moreover, by [HuOe81, Thm. 1] the quotient $S^{\mathbb{C}}/S_\Gamma^{\mathbb{C}}$ is biholomorphic to $\mathbb{C}^k$. Then a result of Grauert ([Gra58]) implies that the fibration $S^{\mathbb{C}}/\Gamma \to S^{\mathbb{C}}/S_\Gamma^{\mathbb{C}}$ is holomorphically trivial, i.e. $S^{\mathbb{C}}/\Gamma \cong S_\Gamma^{\mathbb{C}}/\Gamma \times \mathbb{C}^k$. Thus, $S^{\mathbb{C}}/\Gamma$ is Stein if and only if so is $S_\Gamma^{\mathbb{C}}/\Gamma$.

Since S_Γ is also split-solvable, from Theorem 2.1 it follows that $S_\Gamma^{\mathbb{C}}/\Gamma$ is Stein if and only if $\mathrm{ad}(X)$ has no non-zero eigenvalues for all $X \in \mathfrak{s}_\Gamma$. By Engel's theorem (see [Var84, Thm. 3.5.4]), this is equivalent to saying that S_Γ is nilpotent. Since S_Γ is the smallest (closed), connected, real subgroup of S containing Γ, it follows that $S_\Gamma^{\mathbb{C}}/\Gamma$ is Stein and is nilpotent if and only if Γ is nilpotent, which implies the statement. ∎

3. The crown domain

Let G/K be an irreducible, non-compact Riemannian symmetric space, with G a connected, non-compact, simple Lie group embedded in its universal complexification $G^{\mathbb{C}}$. Let $G^{\mathbb{C}}/K^{\mathbb{C}}$ be its Lie group complexification. By construction the left action of G on G/K extends to a holomorphic action on $G^{\mathbb{C}}/K^{\mathbb{C}}$. However, such an extended action turns out not to be proper (cf. [AkGi90]). In particular the Riemannian metric on G/K does not extend to a G-invariant metric on $G^{\mathbb{C}}/K^{\mathbb{C}}$. Then it is natural to look for G-invariant domains in $G^{\mathbb{C}}/K^{\mathbb{C}}$ on which the restriction of the G-action is proper.

In [AkGi90] D. N. Akhiezer and S. G. Gindikin pointed out a natural candidate, which turns out to be canonical from several points of view. Let $\mathfrak{k}\oplus\mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ induced by K and let $\Sigma = \Sigma(\mathfrak{a},\mathfrak{g})$ be the restricted root system associated to a chosen maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$. Consider the convex polyhedron

$$\omega := \Big\{X \in \mathfrak{a} : \big|\alpha(X)\big| < \frac{\pi}{2}, \;\; \text{for every } \alpha \in \Sigma\Big\}.$$

Definition 3.1. The *crown domain* associated to the Riemannian symmetric space of the non-compact type G/K is defined by

$$\Xi := G\exp(i\omega)\cdot p_0\ ,$$

where $p_0 := eK^{\mathbb{C}}$ is the base point in $G^{\mathbb{C}}/K^{\mathbb{C}}$.

It is easy to check that Ξ does not depend on the choice of $\mathfrak{a}$, and it was proved in [AkGi90] that the G-action on Ξ is proper. For examples of crown domains see the table in [BHH03, p. 8].

In [AkGi90] it was also conjectured that the crown domain is Stein, giving evidence of this fact in several examples. The conjecture was positively solved in [BHH03] (cf. [Bar03], [KrSt05]). For this an important tool is the characterization of plurisubharmonic G-invariant functions on Ξ as those functions whose restriction on the G-slice $\exp(i\omega)\cdot p_0$ is W-invariant and convex. Here W denotes the Weyl group with respect to the maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$.

We are going to use such a characterization in order to show that the crown domain is taut. Let us first recall the following definitions.

Definition 3.2. A complex manifold is *taut* if the family of holomorphic discs $\mathcal{O}(\Delta, X)$ is normal. That is, for a sequence of holomorphic discs $\{f_j : \Delta \longrightarrow X\}_j$ there are two possibilities:

(1) it admits a subsequence $\{f_{j_k}\}$ which converges uniformly on compact subsets to a holomorphic disc in $\mathcal{O}(\Delta, X)$, or
(2) it is compactly divergent, i.e. given two compact subsets $K \subset \Delta$ and $L \subset X$ there exists $\nu \in \mathbb{N}$ such that $f_j(K) \cap L = \varnothing$ for every $j > \nu$.

Definition 3.3 (cf. [Ste75])**.** A Stein manifold is *hyperconvex* if it admits a negative, continuous plurisubharmonic exhaustion.

Hyperconvex manifolds are taut (see [Sib81, Cor. 5]) and taut manifolds are Kobayashi hyperbolic (see [Kob98]). Here we show

Proposition 3.4. *Let G/K be a Riemannian symmetric space of the non-compact type. Then the associated crown domain is taut.*

Proof. By Theorem B in [Bor63] there exists a discrete, cocompact, torsion-free subgroup Γ of G. Since the G-action on Ξ is proper, it follows that Γ acts freely on Ξ and the canonical projection $\Xi \to \Xi/\Gamma$ is a covering map.

Consider the negative, strictly convex, W-invariant exhaustion of ω defined by

$$u(\xi) = \sum_{\alpha \in \Sigma} \left(\alpha^2(\xi) - \left(\frac{\pi}{2} \right)^2 \right), \qquad \text{for } \xi \in \omega.$$

Since one has an isomorphism of orbit spaces $\Xi/G \cong \omega/W$ (see [AkGi90, Prop. 8]), the function u extends to a negative, G-invariant function on Ξ, also denoted by u. By [BHH03, Thm. 10] the function u is strictly plurisubharmonic.

Also note that u pushes down to a negative, continuous, strictly plurisubharmonic function $\tilde{u}$ of Ξ/Γ. Since G/Γ is compact, the preimage $\tilde{u}^{-1}(C)$ of any compact subset $C \subset (-\infty, 0)$ is compact in Ξ/Γ. In particular Ξ/Γ is hyperconvex and [Sib81, Cor. 5] implies that Ξ/Γ is taut. Since a covering of a taut manifold is taut by [ThHu93, Cor. 4], it follows that Ξ is taut as well. $\square$

Now we recall a result of S. Gindikin and B. Krötz which will be used in the sequel in order to realize the crown domain as a Stein invariant domain in the universal complexification of a maximal, split-solvable subgroup of G. For this let us consider an Iwasawa decomposition NAK of G. Let $N^{\mathbb{C}}$, $A^{\mathbb{C}}$, and $K^{\mathbb{C}}$ denote the complexifications of N, A and K in $G^{\mathbb{C}}$, as in Proposition 1.3. One can show that $N^{\mathbb{C}}A^{\mathbb{C}}K^{\mathbb{C}}$ is a proper, Zariski open subset of $G^{\mathbb{C}}$ (see [SiWo02]) and in general $A^{\mathbb{C}} \cap K^{\mathbb{C}} \neq \{e\}$. Hence $N^{\mathbb{C}}A^{\mathbb{C}}K^{\mathbb{C}}$ is not a decomposition of $G^{\mathbb{C}}$.

However, by considering the A-invariant domain T_ω of $A^{\mathbb{C}}$ defined by the cell ω of the crown domain, i.e. $T_\omega := A\exp(i\omega)$, one obtains a tubular neighborhood $N^{\mathbb{C}}T_\omega K^{\mathbb{C}}$ of G which can be regarded as a local complexification of the Iwasawa decomposition of G. Let $H^{\mathbb{C}}$ be the complexification of a real Lie group H. In the sequel we will refer to a *tube* domain in $H^{\mathbb{C}}$ as an H-invariant domain of $H^{\mathbb{C}}$. One has

Lemma 3.5. *The multiplication map*

$$\phi : N^{\mathbb{C}} \times T_\omega \times K^{\mathbb{C}} \longrightarrow G^{\mathbb{C}},$$

defined by $\phi(n, a, k) := nak$, is an open analytic biholomorphism onto its image $N^{\mathbb{C}}T_\omega K^{\mathbb{C}} \subset G^{\mathbb{C}}$. In particular, the map $N^{\mathbb{C}} \times T_\omega \longrightarrow G^{\mathbb{C}}/K^{\mathbb{C}}$, given by $(n, a) \mapsto na \cdot p_0$, with $p_0 = eK^{\mathbb{C}}$, defines an equivariant biholomorphism between a tube of $N^{\mathbb{C}} \rtimes A^{\mathbb{C}}$ and a Stein, NA-invariant domain of $G^{\mathbb{C}}/K^{\mathbb{C}}$.

Proof. The multiplication map $N^{\mathbb{C}} \times A^{\mathbb{C}} \times K^{\mathbb{C}} \longrightarrow G^{\mathbb{C}}$, given by $(n, a, k) \to nak$, can be obtained as the composition of the two maps

$$N^{\mathbb{C}} \times A^{\mathbb{C}} \times K^{\mathbb{C}} \longrightarrow N^{\mathbb{C}}A^{\mathbb{C}} \times K^{\mathbb{C}}, \quad (n, a, k) \to (na, k)$$

and

$$N^{\mathbb{C}}A^{\mathbb{C}} \times K^{\mathbb{C}} \longrightarrow G^{\mathbb{C}}, \quad (na, k) \to nak\,.$$

The first map is a biholomorphism by (iii) of Proposition 1.3. The second map is a local biholomorphism, since it is left $N^{\mathbb{C}}A^{\mathbb{C}}$-equivariant, right $K^{\mathbb{C}}$-equivariant and, if one identifies $\mathfrak{g}^{\mathbb{C}}$ with $\mathfrak{n}^{\mathbb{C}} \oplus \mathfrak{a}^{\mathbb{C}} \oplus \mathfrak{k}^{\mathbb{C}}$, its differential at (e, e) is the identity. Thus ϕ is a local biholomorphism.

It remains to check that ϕ is injective. Suppose that $nak = n'a'k'$ with $n, n' \in N^{\mathbb{C}}$, $a, a' \in T_\omega$ and $k, k' \in K^{\mathbb{C}}$. Let θ be the holomorphic extension to $G^{\mathbb{C}}$ of the Cartan involution of G with respect to K. Then $nak\,\theta(nak)^{-1} = n'a'k'\theta(n'a'k')^{-1}$, and consequently

$$na^2\theta(n^{-1}) = n'(a')^2\theta((n')^{-1}).$$

Since by (iv) of Proposition 1.3 the map $N^{\mathbb{C}} \times A^{\mathbb{C}} \times \theta(N^{\mathbb{C}}) \longrightarrow N^{\mathbb{C}}A^{\mathbb{C}}\theta(N^{\mathbb{C}})$ given by $(n, a, \theta(n)) \longmapsto na\theta(n)$ is injective, it follows that $n = n'$ and $a^2 = (a')^2$. As a consequence, for $a = t\exp(iX)$ and $a' = t\exp(iX')$, with $t, t' \in A$ and $X, X' \in \omega$, one has $t = t'$ and $\exp(i2(X - X')) = e$.

Thus in order to show that $a = a'$, it is enough to check that if $\exp(iY) = e$ for some $Y \in 4\omega$, then $Y = 0$. For this assume that $\exp(iY) = e$ and note that given $\alpha \in \Sigma$ and $Z \in \mathfrak{g}^\alpha$ one has $Z = \mathrm{Ad}_{\exp(iY)}Z = e^{ad(iY)}Z = e^{i\alpha(Y)}Z$. Therefore $e^{i\alpha(Y)} = 1$ which, by the definition of ω, implies that $\alpha(Y) = 0$ for all $\alpha \in \Sigma$. Hence $Y = 0$.

For the last statement, note that the tube T_ω of $T^{\mathbb{C}} \cong \mathbb{C}^r$ has a convex base, therefore it is Stein. As a consequence $N^{\mathbb{C}} \times T_\omega$ is a Stein tube in $N^{\mathbb{C}} \rtimes A^{\mathbb{C}}$. Finally, we proved above that the map

$$N^{\mathbb{C}} \times T_\omega \longrightarrow N^{\mathbb{C}}T_\omega \cdot p_0\,,$$

defined by $(n, a) \longmapsto na \cdot p_0$, is a biholomorphism. Moreover, for (n', a') in $N \rtimes A \cong NA$ one has $(n', a')(n, a) \cdot p_0 = n'(a'n(a')^{-1})a'a \cdot p_0 = n'a' \cdot (na \cdot p_0)$, implying that the map is equivariant with respect to $N \rtimes A \cong NA$. $\square$

By [GiKr02a, Thm. 1.3], one has

Theorem 3.6. *The crown domain Ξ is the connected component containing G/K of the intersection $\bigcap_{g\in G} gN^{\mathbb{C}}T_\omega \cdot p_0$.*

Then, as a consequence of Lemma 3.5 and the above theorem, Ξ is an *NA*-invariant domain of $N^{\mathbb{C}}T_\omega$. Therefore one has

Corollary 3.7. *The crown domain is biholomorphic to a Stein tube in $N^{\mathbb{C}} \rtimes A^{\mathbb{C}}$ contained in $N^{\mathbb{C}} \times T_\omega$.*

4. Quotients of crown domains by cyclic groups

Let X be a Stein manifold endowed with a proper action of a discrete subgroup Γ of $\mathrm{Aut}(X)$. In general one cannot expect the quotient X/Γ to be Stein (cf. examples in [Win90, Sect. 3], [Oel92], [MiOe09, Sect. 5]). In some special cases it is possible to give necessary and/or sufficient conditions so that X/Γ has nice properties, e.g. it is Kähler (see [HuOe86], [Loe85]) and Stein (cf. [GiHu78], [Loe85], [HuOe86], [OeRi88], [MiOe09], [Mie10]). Of course if Γ is finite, then the space X/Γ is Stein by [GrRe79, Thm. 1, Ch. V] (cf. [Hei91]).

Here we consider the case of an infinite cyclic group $\Gamma \cong \mathbb{Z}$ acting properly by biholomorphisms on a Stein domain X of $\mathbb{C}^n$. In this situation there exist positive results. For instance, we already recalled C. Miebach's result ([Mie10]), stating that if X is a bounded, homogeneous domain of $\mathbb{C}^n$, the quotient $X/\mathbb{Z}$ is Stein.

The quotient is also Stein when X is a simply connected domain of $\mathbb{C}^2$ and the $\mathbb{Z}$-action is induced by a proper $\mathbb{R}$-action (see [MiOe09]). In fact, under the latter assumption, we are not aware of any example of a domain X of $\mathbb{C}^n$ such that $X/\mathbb{Z}$ is not Stein for $n > 2$ (cf. [MiOe09, Sect. 5]).

Recall that every crown domain Ξ is biholomorphic to a (taut, by Proposition 3.4) Stein domain of $\mathbb{C}^n$, since it can be regarded as a Stein tube in the universal covering $N^{\mathbb{C}} \rtimes \widetilde{A^{\mathbb{C}}}$ of $N^{\mathbb{C}} \rtimes A^{\mathbb{C}}$ and $N^{\mathbb{C}} \rtimes \widetilde{A^{\mathbb{C}}}$ is biholomorphic to $\mathbb{C}^l \times \mathbb{C}^r$ (see the proof of Theorem 4.3 below and cf. Corollary 3.7 for more details). In this section we will show that $\Xi/\mathbb{Z}$ is Stein for any proper $\mathbb{Z}$-action. For those crown domains which are Hermitian symmetric spaces of a larger group this fact follows directly from C. Miebach's result. Indeed, Hermitian symmetric spaces are biholomorphic to bounded symmetric domains via the Harish-Chandra embedding (see [Ha-Ch56]). For rigid crown domains the result yields new interesting examples of non-homogenous, Stein, taut domains of $\mathbb{C}^n$ with a large group of automorphisms such that the quotient by a proper action of a cyclic group is Stein.

4.1. **Reduction to the case of automorphisms in a maximal split-solvable subgroup of G.** Let G/K be an irreducible Riemannian symmetric space of the non-compact type with rigid crown domain Ξ. In order to prove that the quotient Ξ/Γ is Stein for every proper action of an infinite cyclic group $\Gamma \cong \mathbb{Z}$ of biholomorphisms, we apply the argument pointed out by C. Miebach in [Mie10, Sect. 3] showing that one can reduce to the case of Γ contained in a maximal split-solvable subgroup of G. For the sake of completeness we carry out the details in our situation.

First note that it is enough to consider the case of Γ lying in the connected component of the identity in $\mathrm{Iso}(G/K)$, which is given by G/Z, where Z is the (finite) center of G (cf. [Hel01, Thm. 4.1, Ch. V]). By Remark 1.2 the center Z plays no role in the geometry of Ξ, so we will assume that it is trivial. Since G/K is a Riemannian manifold, every isotropy subgroup of $\mathrm{Iso}(G/K)$ is compact ([Hel01, Thm. 2.4, Ch. IV]). In particular all isotropy subgroups have a finite number of connected components. As a consequence so does $\mathrm{Iso}(G/K)$, being G/K homogeneous with respect to the connected component of the identity in $\mathrm{Iso}(G/K)$. Thus $\Gamma/(\Gamma \cap G)$ is finite and consequently Ξ/Γ is Stein if and only if so is $\Xi/(\Gamma \cap G)$.

So let Γ be contained in the connected component G of the neutral element in $\mathrm{Iso}(G/K)$. Consider an Iwasawa decomposition NAK of G. We want to reduce to the case of Γ lying in the split-solvable subgroup NA of G (cf. Remark 2.3). For this let $\gamma \in G$ be a generator of Γ and consider its Jordan-Chevalley decomposition $\gamma = \gamma_u \gamma_h \gamma_e$, where γ_u is unipotent, γ_h is hyperbolic and γ_e is elliptic (cf. Proposition 1.4). Set $\gamma' := \gamma_u \gamma_h$ and let $\Gamma' := \langle \gamma' \rangle$ be the subgroup generated by γ'. By (iv) of Proposition 1.5, we may assume that Γ' is contained in NA. Moreover the exponential map of NA is a diffeomorphism (see Remark 2.4), implying that Γ' is closed in NA, and consequently in G. Thus it acts properly and freely on the domain Ξ. In particular the canonical projections $p : \Xi \to \Xi/\Gamma$ and $p' : \Xi \to \Xi/\Gamma'$ are coverings.

Since γ_e is conjugated to an element of K, the closure T of the cyclic subgroup generated by γ_e is a compact abelian subgroup of G. Note that γ_e and γ' commute, therefore $T\Gamma'$ is a subgroup of G. In fact, since all the elements in the Jordan-Chevalley decomposition of Γ commute, one has $T \times \Gamma \cong T\Gamma = T\Gamma' \cong T \times \Gamma'$. Therefore the compact group T acts (properly) on both Ξ/Γ and Ξ/Γ' and one has a commutative diagram

$$\begin{tikzcd} & \Xi \arrow[dl, "p"'] \arrow[dr, "p'"] & \\ \Xi/\Gamma \arrow[dr, "q"'] & & \Xi/\Gamma' \arrow[dl, "q'"] \\ & Y & \end{tikzcd}$$

where $Y := (\Xi/\Gamma)/T = (\Xi/\Gamma')/T$ and the canonical projections q and q' are proper. One has ([Mie10, Prop. 3.6])

Proposition 4.1. *Let Γ be a cyclic discrete group acting properly by biholomorphisms on Ξ and let Γ' be the above defined infinite cyclic subgroup of NA. The complex manifold Ξ/Γ is Stein if and only if so is Ξ/Γ'.*

Proof. The proof is carried out by noting that given a smooth, T-invariant function f of Ξ/Γ', there exists a unique smooth, T-invariant function f' on Ξ/Γ such that $f(p(z)) = f'(p'(z))$ for all z in Ξ. Since p and p' are coverings, the function f' is strictly plurisubharmonic if and only if so is f. Moreover, by using the commutativity of the above diagram and the properness of the projections q and q', one checks that f is an exhaustion if and only if so is f'. As a consequence Ξ/Γ admits a smooth, T-invariant, strictly plurisubharmonic exhaustion if and only if so does Ξ/Γ'. Since by [Hör66, Thm. 5.2.10] and by integration over T one has that Ξ/Γ is Stein if and only if it admits a smooth, T-invariant, strictly plurisubharmonic exhaustion, this implies the statement. For further details we refer to the original proof in [Mie10, Prop. 3.6]. $\square$

4.2. **The main result.** We first need the following lemma.

Lemma 4.2. *Let Γ be a discrete group which acts freely and properly discontinuously on a Stein manifold X and let D be a Γ-invariant, Stein subdomain of X. If X/Γ is Stein, then so is D/Γ.*

Proof. First we shall recall that by a classical result of F. Docquier and H. Grauert ([DoGr60]) a domain O in a Stein manifold Z is Stein if and only if it is locally Stein, i.e. for every element z of the boundary of O there exists a neighborhood V of z in Z such that $V \cap O$ is Stein. Hence, it is enough to prove that D/Γ is locally Stein in X/Γ.

For this note that the canonical projection $\pi : X \longrightarrow X/\Gamma$ is a covering map and for $z \in \partial(D/\Gamma)$ choose $x \in \pi^{-1}(z)$. Since D is Γ-invariant, $\pi^{-1}(\overline{D/\Gamma}) = \overline{D}$, therefore $x \in \partial D$. Let U be a Stein neighborhood of x such that $gU \cap U = \emptyset$ for every $g \in \Gamma \backslash \{e\}$. Then $\pi : U \longrightarrow \pi(U)$ is a biholomorphism and $\pi(U) \cap (D/\Gamma) = \pi(U \cap D)$ is Stein. Therefore D/Γ is locally Stein in X/Γ, implying the statement. $\square$

Our main result is

Theorem 4.3. *Let G/K be an irreducible Riemannian symmetric space of the non-compact type and let Ξ be the associated crown domain. Then $\Xi/\mathbb{Z}$ is a Stein manifold for every proper $\mathbb{Z}$-action.*

Proof. If the crown domain Ξ is not rigid, then it is biholomorphic to a bounded symmetric domain and the statement follows from C. Miebach's result ([Mie10]). Thus we may assume that Ξ is rigid. Let NAK be an Iwasawa decomposition of G.

By the argument in section 4.1, one may assume that $\mathbb{Z}$ is contained in the split-solvable subgroup $NA \cong N \rtimes A$ of G. Consider the universal coverings $\widetilde{A^{\mathbb{C}}}$ of $A^{\mathbb{C}}$ and $N^{\mathbb{C}} \rtimes \widetilde{A^{\mathbb{C}}}$ of $N^{\mathbb{C}} \rtimes A^{\mathbb{C}}$, respectively. Since the Lie group $N \rtimes A$ is simply connected, it lifts to a real form of $N^{\mathbb{C}} \rtimes \widetilde{A^{\mathbb{C}}}$ (in fact $N^{\mathbb{C}} \rtimes \widetilde{A^{\mathbb{C}}}$ is the universal complexification of $N \rtimes A$). Moreover, the crown domain Ξ is simply connected and by Corollary 3.7 it is biholomorphic to a Stein, $N \rtimes A$-invariant domain of $N^{\mathbb{C}} \rtimes A^{\mathbb{C}}$. Thus it lifts to a Stein tube in $N^{\mathbb{C}} \rtimes \widetilde{A^{\mathbb{C}}}$. One has a commutative diagram

$$\begin{tikzcd} \Xi \arrow[r, hook] \arrow[d] & N^{\mathbb{C}} \rtimes \widetilde{A^{\mathbb{C}}} \arrow[d] \\ \Xi/\mathbb{Z} \arrow[r, hook] & (N^{\mathbb{C}} \rtimes \widetilde{A^{\mathbb{C}}})/\mathbb{Z} \end{tikzcd} ,$$

where the horizontal arrows are the natural inclusions and the vertical arrows are the canonical covering maps. By Remark 2.3 the group $N \rtimes A$ is split-solvable and from Proposition 2.6 it follows that $(N^{\mathbb{C}} \rtimes \widetilde{A^{\mathbb{C}}})/\mathbb{Z}$ is Stein. Finally, since the crown domain Ξ is a Stein tube in $N^{\mathbb{C}} \rtimes \widetilde{A^{\mathbb{C}}}$, Lemma 4.2 applies to show that $\Xi/\mathbb{Z}$ is Stein, as wished. $\square$

As suggested to us by C. Miebach, a similar argument as in the above proof applies to the case of discrete nilpotent subgroups of NA.

Proposition 4.4. *Let Γ be any discrete, nilpotent subgroup of NA. Then Ξ/Γ is Stein.*

Proof. By Proposition 2.6, the quotient $(N^{\mathbb{C}} \times \widetilde{A^{\mathbb{C}}})/\Gamma$ is a Stein manifold. Then Lemma 4.2 implies that Ξ/Γ is Stein, as wished. $\square$

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