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DILATION OF THE WEYL SYMBOL AND BALIAN-LOW THEOREM

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ABSTRACT. The key result of this paper describes the fact that for an important class of pseudodifferential operators the property of invertibility is preserved under minor dilations of their Weyl symbols. This observation has two implications in time-frequency analysis. First, it implies the stability of general Gabor frames under small dilations of the time-frequency set, previously known only for the case where the time-frequency set is a lattice. Secondly, it allows us to derive a new Balian-Low theorem (BLT) for Gabor systems with window in the standard window class and with general time-frequency families. In contrast to the classical versions of BLT the new BLT does not only exclude orthonormal bases and Riesz bases at critical density, but indeed it even excludes irregular Gabor frames at critical density.

1. INTRODUCTION AND MAIN RESULTS

Gabor frames are a key object in time-frequency analysis. They are families of functions obtained by applying a discrete family of time-frequency shift operators to a given function, called a Gabor atom resp. window. The standard settings are the so-called regular Gabor frames, where the time-frequency set is a lattice. The focus of this paper is on the general case of irregular Gabor frames, without restrictions on the (countable) set.

It is known that regular Gabor frames are robust under small dilations of the underlying set of points in the time-frequency plane [18]. This paper is motivated by the question of whether irregular Gabor frames also share this property. We were led to investigate whether the invertibility of a pseudodifferential operator

$$A_\sigma f(x) = \int_{\mathbb{R}^{2n}} \sigma\Big(\frac{x+y}{2}, \xi\Big) e^{2\pi i \xi^T (x-y)} f(y)\, dy\, d\xi, \quad f \in L^2(\mathbb{R}^n),$$

is preserved under small dilations of the Weyl symbol σ on $\mathbb{R}^{2n}$. Notice that dilations of Weyl symbols are not as well-behaved as translations or modulations [23, p. 91, "Warning"]. Note that standard perturbation results for linear operators do not apply in this case. In fact, even starting from a decent operator A_σ one does not have continuous dependence of the corresponding operators in the operator norm, even for some arbitrary small dilations.

Our first main theorem gives a positive result about dilation stability. It is formulated for operators A_σ in the Sjöstrand class, that is, we assume that the

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symbol σ belongs to the modulation space $M^{\infty,1}(\mathbb{R}^{2n})$. Meanwhile, the Sjöstrand class is recognized as a standard setting for pseudodifferential operator symbols in time-frequency analysis [28,29]. In Section 2 we include the definition of modulation spaces; for details we refer to [14,27].

Let $\mathrm{GL}(n,\mathbb{R})$ be the group of invertible $n \times n$ matrices with the usual topology, induced for example by the maximum norm

$$\|\rho\|_{\max} = \max_{j,k=1,\ldots,n} |\rho_{j,k}|, \qquad \rho \in \mathrm{GL}(n,\mathbb{R}).$$

The $n \times n$ identity matrix is denoted by I_n. For $\rho \in \mathrm{GL}(n,\mathbb{R})$, we denote the dilation operator applied to a function f on $\mathbb{R}^n$ by

$$D_\rho f(z) = f(\rho^{-1} z), \quad z \in \mathbb{R}^n.$$

The following theorem is the main result of this paper. The proof will be given in Section 3, based on a series of technical results, some of which are of independent interest.

Theorem 1.1. *Let A_σ denote the pseudodifferential operator with Weyl symbol $\sigma \in M^{\infty,1}(\mathbb{R}^{2n})$. For $\rho \in \mathrm{GL}(2n,\mathbb{R})$, let $A_{D_\rho \sigma}$ denote the operator with dilated symbol $D_\rho \sigma$. Let*

$$\Sigma = \{(\sigma,\rho) \in M^{\infty,1}(\mathbb{R}^{2n}) \times \mathrm{GL}(2n,\mathbb{R}) \colon A_{D_\rho\sigma} \text{ is invertible}\}.$$

Then Σ is an open set in $M^{\infty,1}(\mathbb{R}^{2n}) \times \mathrm{GL}(2n,\mathbb{R})$.

Remark 1.2. Let us rephrase the key implication of Theorem 1.1 for the case where σ is fixed. Suppose the operator A_σ with Weyl symbol $\sigma \in M^{\infty,1}(\mathbb{R}^{2n})$ is invertible on $L^2(\mathbb{R}^n)$. Then there exists $\delta > 0$ such that for $\|\rho - I_{2n}\|_{\max} < \delta$ the operator $A_{D_\rho\sigma}$ with dilated symbol $D_\rho\sigma(\lambda) = \sigma(\rho^{-1}\lambda)$ is also invertible on $L^2(\mathbb{R}^n)$.

Balian-Low theorem for Gabor frames. Theorem 1.1 has significant applications to Gabor frames, described next. A Gabor system is formed by the time-frequency shifts of some function $g \in L^2(\mathbb{R}^n)$ along a set Λ of time-frequency points in $\mathbb{R}^{2n}$,

$$\begin{aligned} &G(g,\Lambda) = \{g_\lambda : \lambda \in \Lambda\}, \\ &g_\lambda(t) = e^{2\pi i \omega^T t} g(t-x) = M_\omega T_x g(t), \qquad \lambda = (x,\omega) \in \mathbb{R}^{2n}, \end{aligned} \tag{1.1}$$

where M_ω and T_x are the modulation and translation operators respectively.

The system $G(g,\Lambda)$ is a Gabor frame if there exist $C_1, C_2 > 0$ such that

$$C_1 \|f\|_{L^2}^2 \le \sum_{\lambda\in\Lambda} |\langle f, g_\lambda\rangle|^2 \le C_2 \|f\|_{L^2}^2, \qquad \text{for all } f \in L^2(\mathbb{R}^n),$$

or, equivalently, if the frame operator, given by

$$S_{g,\Lambda} f = \sum_{\lambda\in\Lambda} \langle f, g_\lambda\rangle g_\lambda,$$

is bounded and invertible on $L^2(\mathbb{R}^n)$. See [27] for the details of Gabor analysis. We assume that g belongs to the standard window class, the modulation space $M^1(\mathbb{R}^n)$, also known as Feichtinger algebra $S_0(\mathbb{R}^n)$. For details on $M^1(\mathbb{R}^n)$, see [12,21,27] and Section 2. Note that the Schwartz class $\mathscr{S}(\mathbb{R}^n)$ is a dense subspace of $M^1(\mathbb{R}^n)$, so our results apply to arbitrary Schwartz atoms g in particular.

The first important consequence of Theorem 1.1 is the robustness of time-frequency sets generating irregular Gabor frames, under small dilations (including rotations, shearing and other operations close to the identity) described in the next theorem. In order to formulate it properly we write $\rho\Lambda = (\rho\lambda)_{\lambda\in\Lambda}$, for $\Lambda \subset \mathbb{R}^{2n}$ and a matrix $\rho \in \mathrm{GL}(2n, \mathbb{R})$.

For the regular case, i.e. for the case where Λ is a discrete lattice in $\mathbb{R}^{2n}$, this result has been given in [18] and it reads as follows: Assume that $G(g, \Lambda)$ is a Gabor frame, with $g \in M^1(\mathbb{R}^n)$ and a lattice Λ; then there exists $\delta > 0$ such that for any $\rho \in \mathrm{GL}(2n, \mathbb{R})$ with $\|\rho - I_{2n}\|_{\max} < \delta$ and any $g' \in M^1(\mathbb{R}^n)$ with $\|g' - g\|_{M^1} < \delta$, the system $G(g', \rho\Lambda)$ with dilated time-frequency lattice $\rho\Lambda$ is also a Gabor frame. The proof of this result in [18] relies heavily on the assumption that the time-frequency set Λ is a lattice and thus the arguments of [18] cannot be transfered directly to irregular Gabor frames. Our goal is an extension to general time-frequency sets, by providing the necessary tools applicable in this more general case. For recent developments on irregular Gabor frames see the summaries in [1, Section 1.1], [24, Section 1], [34, Section 5], or [19, 20, 36].

Theorem 1.3. *For any discrete set Λ in $\mathbb{R}^{2n}$ the set*

$$F_\Lambda = \{(g, \rho) \in M^1(\mathbb{R}^n) \times \mathrm{GL}(2n, \mathbb{R})\colon G(g, \rho\Lambda) \text{ is a Gabor frame}\}$$

is open in $M^1(\mathbb{R}^n) \times \mathrm{GL}(2n, \mathbb{R})$.

The proof of Theorem 1.3 will be given in Section 4.

Remark 1.4. We rephrase the key implications of Theorem 1.3 when g is fixed. Suppose that $G(g, \Lambda)$ is a Gabor frame, with $g \in M^1(\mathbb{R}^n)$. Then Theorem 1.3 implies that for some neighborhood U of I_{2n} in $\mathrm{GL}(2n, \mathbb{R})$, every family $G(g, \rho\Lambda)$ with $\rho \in U$ is also a Gabor frame.

Our next theorem is concerned with the density of the time-frequency set Λ. We will show that a fundamental phenomenon in time-frequency analysis, the Balian-Low theorem for Gabor orthonormal systems and Riesz bases, indeed holds for general Gabor frames, including the case of irregular time-frequency sets.

For a set X of points in $\mathbb{R}^n$, define its lower Beurling density $\mathscr{D}^-(X)$,

$$\mathscr{D}^-(X) = \liminf_{r\to\infty} \frac{1}{r^n} \min_{u\in\mathbb{R}^n} \#\big(X \cap (u + [0, r]^n)\big). \tag{1.2}$$

We define the upper Beurling density $\mathscr{D}^+(X)$ in an equivalent way using lim sup and sup instead of lim inf and inf.

It is known that if $G(g, \Lambda)$ is a frame, then Λ cannot be sparse, and the lower Beurling density of Λ in $\mathbb{R}^{2n}$ must satisfy

$$\mathscr{D}^-(\Lambda) \geq 1;$$

see [34]. Balian-Low theorems (BLTs) are results that draw the stronger conclusion

$$\mathscr{D}^-(\Lambda) > 1,$$

assuming that g is smooth and localized. Various variants of BLTs have been given in the literature (for example see [3–6, 17, 25, 26, 35]); we refer to [11] and [34, Section 3.7]. The original forms of BLTs were found for the case where Λ is a lattice. The BLT for Gabor systems with more general time-frequency sets Λ as in [31, Theorem 11] works for the case of orthonormal bases.

By our next result we obtain a BLT for Gabor frames, with general $g \in M^1$ and without restriction on the time-frequency set Λ.

Theorem 1.5. *For any Gabor frame $G(g, \Lambda)$ with $g \in M^1(\mathbb{R}^n)$ one has $\mathscr{D}^-(\Lambda) > 1$.*

Proof. We prove the theorem by contradiction. Assume that $\mathscr{D}^-(\Lambda) = 1$. Then by Theorem 1.3 there exists $\epsilon > 0$ such that the system $G(g, \Lambda')$ with expanded time-frequency set $\Lambda' = (1+\epsilon)\Lambda$ is also a Gabor frame. It implies

$$\mathscr{D}^-(\Lambda') = \mathscr{D}^-((1+\epsilon)\Lambda) = \frac{1}{(1+\epsilon)^{2n}}\mathscr{D}^-(\Lambda) < 1.$$

By [8], an extension of [43], this is impossible and we obtain a contradiction. □

Remark 1.6. (i) The implication $\mathscr{D}^-(\Lambda) > 1$ of Theorem 1.5 is sharp. Indeed the inequality cannot be improved even if some specially selected non-zero $g \in M^1(\mathbb{R}^n)$ is fixed. In fact, for any non-zero $g \in M^1(\mathbb{R}^n)$ and $\epsilon > 0$, by [2, Corollary 5.6] there exists a Gabor frame $G(g, \Lambda)$ whose time-frequency set Λ satisfies $\mathscr{D}^-(\Lambda) \leq 1 + \epsilon$.

(ii) We denote the Gaussian function in n variables by

$$\varphi_n(t) = 2^{\frac{n}{4}} e^{-\pi\|t\|_2^2}, \qquad t \in \mathbb{R}^n. \tag{1.3}$$

It is known in dimension one [41, 45, 46] that the density condition $\mathscr{D}^-(\Lambda) > 1$ on the time-frequency set $\Lambda \subset \mathbb{R}^2$ is both necessary and sufficient for $G(\varphi_1, \Lambda)$ to be a frame. For higher dimensions this description is no longer true in the general case. A complete description in the case of rectangular lattices has been found recently [38, Theorem 4.3], and for more general lattices the problem is studied in [30]. There are plenty of examples that show that it is not possible to extend the sufficiency part to general $g \in M^1(\mathbb{R}^n)$. By Theorem 1.5 we extend the necessity part to general $g \in M^1(\mathbb{R}^n)$.

Theorem 1.5 implies a new version for Gabor systems with window in the standard class M^1, of a recent non-existence result for time-frequency localized Riesz bases, found in [32]. In that paper, Gröchenig and Malinnikova proved the non-existence of Riesz bases when the functions used are well localized in time and frequency; we refer to [32] for more details. Applying this to the case of irregular Gabor frames we can deduce the non-existence of irregular Gabor Riesz bases with that localization property. Belonging to M^1 can be interpreted as a good time-frequency localization, but it differs slightly from the type of localization used in [32], and thus our next corollary is a new variant to this theme.

Corollary 1.7. *There is no Gabor Riesz basis for $L^2(\mathbb{R}^n)$, with atom $g \in M^1(\mathbb{R}^n)$.*

Proof. The corollary follows from Theorem 1.5, since it is known that Gabor Riesz bases have density exactly equal to $\mathscr{D}^-(\Lambda) = 1$; see [8, 43]. □

Sampling and interpolation in the Bargmann-Fock space. Our approach also leads to a new result concerned with sampling in the Bargmann-Fock space. The Bargmann-Fock space $\mathcal{F}(\mathbb{C}^n)$ consists of entire functions in $\mathbb{C}^n$ with the norm

$$\|F\|_{\mathcal{F}}^2 = \int_{\mathbb{C}^n} |F(z)|^2 e^{-\pi|z|^2}\, dz < \infty.$$

Recall that a discrete set $\Gamma \subset \mathbb{C}^n$ is a set of sampling for $\mathcal{F}(\mathbb{C}^n)$, if the following expression defines an equivalent norm on $\mathcal{F}(\mathbb{C}^n)$:

$$\sum_{z_i \in \Gamma} |F(z_i)|^2 e^{-\pi|z_i|^2}.$$

The Beurling density of a set of points in the complex plane is defined by the obvious modification of (1.2). It is known that the sets of sampling of $\mathcal{F}(\mathbb{C}^n)$ have lower density larger than or equal to 1. For the development of this fundamental density result, first obtained in one dimension, see [7, 41, 42, 44–46]. The generalization to several variables is due to Lindholm [40, Theorem 1]. Lindholm also conjectured that in fact a strict inequality, valid in one dimension, can also be obtained in several dimensions. By our next result we confirm his suggestion.

Corollary 1.8. *Any set of sampling $\Gamma \subset \mathbb{C}^n$ for the Bargmann-Fock space $\mathcal{F}(\mathbb{C}^n)$ satisfies $\mathscr{D}^-(\Gamma) > 1$.*

Proof. The proof follows from Theorem 1.5 by the known relation between Gabor frames and sets of sampling in the Bargmann-Fock space. The link between Gabor systems and the Bargmann-Fock space is the Bargmann transform [23, Section 1.6], [27, Section 3.4]; for the details we refer to [30, Proposition 4] and [40, Section 1]. One of the key equivalences in this relation is the fact that Γ is a set of sampling for $\mathcal{F}(\mathbb{C}^n)$ if and only if the Gabor system $G(g, \Lambda)$ is a frame, where $g = \varphi_n$ is the Gaussian function in n variables from (1.3) and

$$\Lambda = \{(x, \omega) \in \mathbb{R}^{2n} \colon x - i\omega \in \Gamma\}.$$

The corollary is thus obtained by applying Theorem 1.5 with $g = \varphi_n \in M^1(\mathbb{R}^n)$. $\square$

Remark 1.9. (i) In equivalence with the present knowledge of Gabor frames mentioned in Remark 1.6(ii), a complete description of the sampling sets for the Fock space is only known in one dimension (see [46]). For recent results in several dimensions we refer as before to the papers [30, 38]. Corollary 1.8 is a necessary condition for general sampling sets in several dimensions.

(ii) Corollary 1.8 also provides a new approach to a result in [32] on sampling and interpolation in the Bargmann-Fock space $\mathcal{F}(\mathbb{C}^n)$. Interpolation sets for the Fock space are those sets Γ for which the interpolation problem

$$F(z_i) = a_i e^{\pi |z_i|^2/2} \qquad \forall z_i \in \Gamma$$

has a solution F for every sequence $(a_i) \in l^2$. Interpolation sets of the Fock space correspond to Riesz sets of time-frequency translates of the Gaussian function. Consequently, sets that are both sampling and interpolation correspond to Riesz bases for the Fock space. It is known that an interpolation set Γ satisfies $\mathscr{D}^+(\Gamma) \leq 1$. For this fundamental density result for interpolation see [7, 41, 42, 44–46], for one dimension. The generalization to several variables is proved by Lindholm [40, Theorem 2]. Thus Corollary 1.8 implies [32, Theorem 2]: *There does not exist a set $\Gamma \subset \mathbb{C}^n$ that is both a sampling set and an interpolation set for $\mathcal{F}(\mathbb{C}^n)$ at the same time.*

Section 2 consists of preliminary results, Section 3 contains the proof of Theorem 1.1, and Section 4 contains the proof of Theorem 1.3.

2. Preliminary results

Let $\varphi = \varphi_n$ denote the Gaussian function on $\mathbb{R}^n$, defined in (1.3). Define the short-time Fourier transform $V_g f(\lambda) = \langle f, g_\lambda \rangle$, for $\lambda = (x, \omega) \in \mathbb{R}^{2n}$, where g_λ denotes the time-frequency shift of g as defined in (1.1). The usual description of modulation spaces used nowadays makes use of a mixed $L^{p,q}$ norm applied to the short-time Fourier transform $V_\varphi f$ of a function f,

$$\|f\|_{M^{p,q}} = \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |V_\varphi f(x,\omega)|^p dx \right)^{q/p} d\omega \right)^{1/q}, \qquad 1 \le p, q \le \infty\,, \tag{2.1}$$

with the usual modification if $p = \infty$ or $q = \infty$. We write $M^p = M^{p,p}$.

The next two lemmas are based on results of Sugimoto and Tomita [49], and Cordero and Nicola [10]. The first lemma states that the family of dilations D_ρ, for ρ in any compact subset of $\mathrm{GL}(n,\mathbb{R})$, is uniformly bounded on modulation spaces.

Lemma 2.1. *Let $1 \le p, q \le \infty$. Then for any compact subset $U \subset \mathrm{GL}(n,\mathbb{R})$, there exists $C_U > 0$ such that for all $\rho \in U$ and $f \in M^{p,q}$,*

$$\|D_\rho f\|_{M^{p,q}} \le C_U \|f\|_{M^{p,q}}\,.$$

Proof. By [10, Proposition 3.1] there exists a constant $c > 0$ such that for $\rho \in \mathrm{GL}(n,\mathbb{R})$ and $f \in M^{p,q}$,

$$\|D_\rho f\|_{M^{p,q}} \le c \cdot C_\rho \|f\|_{M^{p,q}}\,,$$

where

$$C_\rho = |\det \rho|^{-(1/p - 1/q + 1)} \big(\det(I_n + \rho^T \rho)\big)^{1/2}.$$

We let $C_U = c \cdot \max_{\rho \in U} C_\rho$. □

The next lemma is concerned with varying the dilation parameter.

Lemma 2.2. (i) *For $1 \le p, q \le \infty$, the mapping*

$$(h, \rho) \mapsto D_\rho h,$$
$$M^{p,q}(\mathbb{R}^n) \times \mathrm{GL}(n,\mathbb{R}) \to M^{p,q}(\mathbb{R}^n)$$

is continuous at all (h_0, ρ_0) such that h_0 belongs to the closure of $M^1(\mathbb{R}^n)$ in $M^{p,q}(\mathbb{R}^n)$ (which is all of $M^{p,q}(\mathbb{R}^n)$ when $p, q < \infty$) and any $\rho_0 \in \mathrm{GL}(n,\mathbb{R})$.

(ii) *For $p = \infty$ (and all q), the mapping is also continuous at (h_0, ρ_0), for any constant function $h_0 = \mathrm{const}$ and any $\rho_0 \in \mathrm{GL}(n,\mathbb{R})$.*

Proof. (i) Since $M^1(\mathbb{R}^n)$ (being the minimal reasonable time-frequency invariant Banach space [12]) is continuously embedded into $M^{p,q}$, for all choices of p, q [27, Corollary 12.1.10], and $D_\rho g$ depends continuously on the dilation parameter ρ according to [22, Corollary 3.4(iv)], the first claim is verified.

(ii) Since dilation acts trivially on any fixed constant function, the second claim follows from the uniform boundedness described in Lemma 2.1. □

Remark 2.3. In part (ii) of Lemma 2.2 the conclusion cannot be continuity on all of $M^{p,q}(\mathbb{R}^n) \times \mathrm{GL}(n,\mathbb{R})$. For example, in $M^{\infty,1}(\mathbb{R})$ the mapping is discontinuous at (h_0, ρ_0), for $h_0(t) = \cos(t)$ and $\rho_0 = 1$, since

$$\|\cos(t) - \cos(\rho t)\|_{M^{\infty,1}} \ge \|\cos(t) - \cos(\rho t)\|_{L^\infty} = 2, \qquad \text{for } \rho \ne 1\,.$$

The next lemma is a continuity result that we will use later, combining dilation and convolution in certain modulation spaces. The proof makes use of the Wiener amalgam spaces $W^{p,q}$, defined like modulation spaces but with the role of time and frequency interchanged:

$$\|f\|_{W^{p,q}} = \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |V_\varphi f(x,\omega)|^p d\omega \right)^{q/p} dx \right)^{1/q}, \qquad 1 \le p, q \le \infty.$$

The Wiener amalgam spaces are the isomorphic image of the modulation spaces under the Fourier transform. In particular, we have $\|f\|_{M^{p,q}} = \|\widehat{f}\|_{W^{p,q}}$; see [33]. We also notice (see [13, Theorem 3.2]) that $M^p = M^{p,p} = W^{p,p}$ is invariant under the Fourier transform, $1 \le p \le \infty$. Note that $W^{p_1,q_1} \subseteq W^{p_2,q_2}$ if and only if $p_1 \le p_2$ and $q_1 \le q_2$.

Results for modulation spaces often have a counterpart for Wiener amalgam spaces, obtained by applying the Fourier transform. For example, the claims formulated in Lemma 2.1 and Lemma 2.2 thus imply analogous results for Wiener amalgam spaces $W^{p,q}$. Also in the proof of the next lemma we make use of this correspondence.

Lemma 2.4. *The following mapping is continuous:*

$$\begin{gathered}(f,g,\rho) \mapsto f * D_\rho g,\\ M^{\infty,1}(\mathbb{R}^n) \times M^{1,\infty}(\mathbb{R}^n) \times \mathrm{GL}(n,\mathbb{R}) \to M^{\infty,1}(\mathbb{R}^n).\end{gathered}$$

Proof. We prove the equivalent claim obtained by applying the Fourier transform, that is, we show that the following mapping is continuous:

$$\begin{gathered}(f,g,\rho) \mapsto f \cdot D_\rho g,\\ W^{\infty,1}(\mathbb{R}^n) \times W^{1,\infty}(\mathbb{R}^n) \times \mathrm{GL}(n,\mathbb{R}) \to W^{\infty,1}(\mathbb{R}^n).\end{gathered} \tag{2.2}$$

There are various convenient multiplication relations of Wiener amalgam spaces; see [16, Theorem 2.11]. We will use the fact that for any $1 \le p, q \le \infty$, the following mapping is continuous:

$$\begin{gathered}(f,g) \mapsto f \cdot g,\\ W^{p,1}(\mathbb{R}^n) \times W^{1,q}(\mathbb{R}^n) \to W^{p,1}(\mathbb{R}^n).\end{gathered} \tag{2.3}$$

We also use the equivalent of Lemma 2.1 by applying the Fourier transform, that is, the uniform boundedness on compact sets of the mapping

$$\begin{gathered}(g,\rho) \mapsto D_\rho g,\\ W^{1,\infty}(\mathbb{R}^n) \times \mathrm{GL}(n,\mathbb{R}) \to W^{1,\infty}(\mathbb{R}^n).\end{gathered} \tag{2.4}$$

By combining (2.4) and the continuity of (2.3) with $p = \infty$, $q = \infty$ we observe that the mapping in (2.2) is uniformly bounded on compact sets. Therefore we only need to prove the continuity on dense subsets. To this end we assume that f is compactly supported, with $\operatorname{supp} f = \Omega$. Let $U \subset \mathrm{GL}(n,\mathbb{R})$ be an arbitrary compact set. Let $\Omega' = \bigcup_{\rho \in U} \rho^{-1}\Omega$, and let

$$h \in M^1 = W^{1,1} \quad \text{ such that } \ h|_{\Omega'} = 1\,.$$

Then $(D_\rho h)|_\Omega = 1$, for all $\rho \in U$, and we obtain

$$f = f \cdot D_\rho h, \qquad \text{for all } \rho \in U. \tag{2.5}$$

By the continuity of (2.3) with $p = 1$ and $q = \infty$, since $h \in M^1$ is fixed, the following mapping is continuous:

$$\begin{gathered}g \mapsto h \cdot g,\\ W^{1,\infty}(\mathbb{R}^n) \to M^1(\mathbb{R}^n).\end{gathered}$$

Hence, by using Lemma 2.2(i), the following mapping is continuous:

$$\begin{gathered}(g,\rho) \mapsto D_\rho(h \cdot g),\\ W^{1,\infty}(\mathbb{R}^n) \times \mathrm{GL}(n,\mathbb{R}) \to M^1(\mathbb{R}^n).\end{gathered}$$

Consequently, by the continuity of (2.3) with $p = \infty$ and $q = 1$, the following mapping is continuous:

$$\begin{gathered}(f, g, \rho) \mapsto f \cdot D_\rho(h \cdot g), \\ W^{\infty,1} \times W^{1,\infty}(\mathbb{R}^n) \times \mathrm{GL}(n, \mathbb{R}) \to W^{\infty,1}(\mathbb{R}^n).\end{gathered} \tag{2.6}$$

Since (2.5) implies

$$f \cdot D_\rho g = f \cdot D_\rho(h \cdot g), \qquad \text{for all } \rho \in U,$$

we conclude from (2.6) that the continuity on dense subsets is verified. □

In order to verify functorial properties for the modulation spaces $M^{\infty,1}$ defined over different domains, the following technical lemma will be useful. The lemma describes the Fourier domain analysis and synthesis of functions or distributions in $M^{\infty,1}(\mathbb{R}^n)$. We denote by $B(r, x) \subset \mathbb{R}^n$ the ball of radius $r > 0$ and center $x \in \mathbb{R}^n$.

Lemma 2.5. (i) *For $r > 0$, there exists $C_r > 0$ such that for any sequence of functions $f_1, f_2, \ldots \in L^\infty(\mathbb{R}^n)$ with $\operatorname{diam}\big(\operatorname{supp}(\widehat{f_k})\big) \le r$ and $\sum_{k=1}^\infty \|f_k\|_{L^\infty} < \infty$, we have that the series $f = \sum_{k=1}^\infty f_k$ converges in $M^{\infty,1}(\mathbb{R}^n)$ and*

$$\|f\|_{M^{\infty,1}} \le C_r \cdot \sum_{k=1}^{\infty} \|f_k\|_{L^\infty}.$$

(ii) *For $r > 0$, there exists $C'_r > 0$ such that for any $f \in M^{\infty,1}(\mathbb{R}^n)$, there exist $f_1, f_2, \ldots \in L^\infty(\mathbb{R}^n)$ with $\operatorname{diam}\big(\operatorname{supp}(\widehat{f_k})\big) \le r$ such that we have $f = \sum_{k=1}^\infty f_k$, with (absolute) convergence in $M^{\infty,1}(\mathbb{R}^n)$, and*

$$\sum_{k=1}^{\infty} \|f_k\|_{L^\infty(\mathbb{R}^n)} \le C'_r \cdot \|f\|_{M^{\infty,1}}.$$

Proof. (i) For distributions u having Fourier transform $\widehat{u}$ supported in a fixed compact subset $K \subset \mathbb{R}^n$, there exists $C_K > 0$ such that

$$C_K^{-1} \|u\|_{M^{p,q}} \le \|u\|_{M^{p,q}} \le C_K \|u\|_{M^{p,q}}.$$

Since $M^{\infty,1}(\mathbb{R}^n)$ is a Banach space and, due to the norm equivalence above, we can write for $u = f_k$: $\|f_k\|_{M^{\infty,1}} \asymp C_K \|f_k\|_{L^\infty}$, for every $k \in \mathbb{N}$, the estimate: $\|f\|_{M^{\infty,1}} \le C_K \sum_k \|f_k\|_{L^\infty}$ immediately follows.

(ii) Recall that the Fourier transform of $M^{\infty,1}$ equals the Wiener amalgam space $W^{\infty,1}$, resp. $W(\mathscr{F}L^\infty, l^1)$ in the notation of [15], consisting of functions locally with Fourier transform in L^∞ and globally in L^1. For given $r > 0$ one can choose a bounded uniform partition of unity $(\psi_m)_{m\in\mathbb{Z}^n} = \{T_{mr/2}\psi\}_{m\in\mathbb{Z}^n}$, with $\widehat{\psi} \in L^1$ and $\operatorname{supp}\psi \subseteq B(r, 0)$, e.g. a tensor product of triangular functions. Hence $\|\psi_m\|_{\mathscr{F}L^1} = \|\widehat{\psi}\|_{L^1}$, for all m.

Since $\mathscr{F}L^1$ is a Banach algebra acting pointwise on $\mathscr{F}L^\infty$, the equivalence theorem ([15, Theorem 2]) tells us that the splitting $\widehat{f} = \sum \widehat{f}\psi_m$ provides the required decomposition (after enumerating $\mathbb{Z}^n$) with $\widehat{f_m} = \widehat{f}\psi_m$ and $\sum_m \|f_m\|_{L^\infty}$ is an equivalent norm on $M^{\infty,1}$. □

We use the tensor product notation $f \otimes g(x, y) = f(x)g(y)$. The restriction to $\mathbb{R}^n$ of a function on $\mathbb{R}^{n+n'}$ is denoted by $Rf(x) = f(x, 0)$, $x \in \mathbb{R}^n$.

Lemma 2.6. (i) The following mapping is continuous:

$$(f, g) \mapsto f \otimes g,$$
$$M^{\infty,1}(\mathbb{R}^n) \times M^{\infty,1}(\mathbb{R}^n) \to M^{\infty,1}(\mathbb{R}^{2n}).$$

(ii) The following mapping is continuous:

$$f \mapsto Rf,$$
$$M^{\infty,1}(\mathbb{R}^{2n}) \to M^{\infty,1}(\mathbb{R}^n).$$

Proof. (i) The tensor product property follows from the identity ($1 \le p, q \le \infty$)

$$\|f \otimes g\|_{M^{p,q}} = \|f\|_{M^{p,q}} \|g\|_{M^{p,q}}.$$

This identity can be obtained directly from the definition of the $M^{p,q}$-norm by splitting the Gaussian window: $\varphi_{2n} = \varphi_n \otimes \varphi_n$.

(ii) Let $f \in M^{\infty,1}(\mathbb{R}^{n+n'})$ and $r > 0$. First, using Lemma 2.5(ii) there exist $F_1, F_2, \ldots \in L^\infty(\mathbb{R}^{n+n'})$ with $\operatorname{diam}\big(\operatorname{supp}(\widehat{F_k})\big) \le r$, $f = \sum F_k$, and

$$\sum_{k=1}^{\infty} \|F_k\|_{L^\infty(\mathbb{R}^{n+n'})} < \infty.$$

By Lemma 2.5(i) there exists $C_r > 0$ such that

$$\|f\|_{M^{\infty,1}(\mathbb{R}^{n+n'})} \le C_r \cdot \sum_{k=1}^{\infty} \|F_k\|_{L^\infty(\mathbb{R}^{n+n'})}.$$

Next, let $f_k = RF_k$, for $k = 1, 2, \ldots$. Then for $k = 1, 2, \ldots$, we have $f_k \in L^\infty(\mathbb{R}^n)$ and $\operatorname{diam}\big(\operatorname{supp}(\widehat{f_k})\big) \le r$. In fact, since F_k is bandlimited, and hence continuous, we have $\|f_k\|_{L^\infty(\mathbb{R}^n)} \le \|F_k\|_{L^\infty(\mathbb{R}^{n+n'})}$, for $k = 1, 2, \ldots$. Thus, finally, by Lemma 2.5(ii) there exists $C_r' > 0$ such that the restricted function $RF = \sum_{k=1}^\infty RF_k = \sum_{k=1}^\infty f_k$ converges in $M^{\infty,1}(\mathbb{R}^n)$, and indeed

$$\begin{aligned}
\|RF\|_{M^{\infty,1}(\mathbb{R}^n)} &\le C_r' \cdot \sum_{k=1}^{\infty} \|f_k\|_{L^\infty(\mathbb{R}^n)} \\
&\le C_r' \cdot \sum_{k=1}^{\infty} \|F_k\|_{L^\infty(\mathbb{R}^{n+n'})} \le C_r' \cdot C_r \cdot \|F\|_{M^{\infty,1}(\mathbb{R}^{n+n'})}.
\end{aligned}$$

□

Remark 2.7. Since translations and dilations (by general invertible matrices) are topological isomorphisms on $M^{\infty,1}$, we note that Lemma 2.6(ii) indeed holds for the restriction to an arbitrary affine subspace of $\mathbb{R}^n$. In particular, we can apply the lemma to the diagonal restriction operator R_Δ on $\mathbb{R}^{2n}$, defined by

$$R_\Delta f(t) = f(t, t), \qquad t \in \mathbb{R}^n.$$

3. Proof of Theorem 1.1

The proof of the theorem is based on the following lemma.

Lemma 3.1. *Let $M = (M, \odot, 1)$ be a unital Banach algebra and let $G = (G, \cdot, e)$ be a topological group acting on M by the family of bounded linear operators $(K_g)_{g \in G}$*

on M, with $K_{gg'} = K_g K_{g'}$ and $K_g^{-1} = K_{g^{-1}}$. Suppose that

(i) *the mapping $(\sigma, g) \mapsto K_g\sigma$, $M \times G \to M$ is continuous at the identity $1 \in M$, and*

(ii) *the mapping $(\sigma, \tau, g) \mapsto K_{g^{-1}}(K_g\sigma \odot K_g\tau)$, $M \times M \times G \to M$ is continuous at $e \in G$.*

Then the set $F = \{(\sigma, g) \in M \times G \colon K_g\sigma$ is invertible$\}$ is open in $M \times G$.

Proof. First, since M is a Banach algebra, the set $M_0 = \{\sigma \in M \colon \sigma$ invertible$\}$ is open in M, and the inversion $\sigma \mapsto \sigma^{-1}$, $M_0 \to M_0$, is continuous. Let

$$\left.\begin{aligned} u(\sigma, g) &= K_{g^{-1}}(K_g\sigma \odot K_g\sigma^{-1}), \\ v(\sigma, g) &= K_g\sigma \odot K_g\sigma^{-1}, \end{aligned}\right. \qquad \sigma \in M_0,\ g \in G.$$

Assumption (ii) and the continuity of inversion in M_0 imply continuity at $g = e$ of the mapping

$$(\sigma, g) \mapsto u(\sigma, g), \\ M_0 \times G \to M.$$

Since $K_e = id$, we have $u(\sigma, e) = \sigma \odot \sigma^{-1} = 1$, and thus (i) implies that the mapping

$$(\tau, g) \mapsto K_g\tau, \\ M \times G \to M$$

is continuous at $\tau = u(\sigma, e)$, for any $\sigma \in M_0$. Since $K_g u(\sigma, g) = v(\sigma, g)$, we conclude altogether that the mapping

$$(\sigma, g) \mapsto v(\sigma, g), \\ M_0 \times G \to M$$

is continuous at $g = e$. Moreover $v(\sigma, e) = \sigma \odot \sigma^{-1} = 1$ implies that there exists a neighborhood U of (σ, e) in $M_0 \times G$ such that

$$\|K_g\sigma \odot K_g\sigma^{-1} - 1\| < 1, \qquad \text{for } (\sigma, g) \in U.$$

Hence, $K_g\sigma \odot K_g\sigma^{-1}$ is invertible, for $(\sigma, g) \in U$, and this implies the existence of

$$(K_g\sigma)^{-1} = K_g\sigma^{-1} \odot (K_g\sigma \odot K_g\sigma^{-1})^{-1}, \qquad \text{for } (\sigma, g) \in U.$$

We have thus shown that the set F is open. □

We will apply Lemma 3.1 to the algebra $M^{\infty,1}$, endowed with the twisted product. To this aim we recall the following definitions and results. For $\Theta \in \mathrm{GL}(n, \mathbb{R})$, define

$$\sigma \natural_\Theta \tau(z) = \iint_{\mathbb{R}^n \times \mathbb{R}^n} \sigma(z')\tau(z'') e^{2\pi i (z - z'')^T \Theta (z - z')} dz' dz'', \qquad z \in \mathbb{R}^n.$$

By selecting $J = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$ for Θ (assuming dimension $2n$) we obtain the usual twisted product of Weyl symbols [23],

$$\sigma \natural \tau = \sigma \natural_J \tau.$$

Proposition 3.2. *The following mapping is continuous:*

$$(\sigma, \tau, \Theta) \mapsto \sigma \natural_\Theta \tau, \\ M^{\infty,1}(\mathbb{R}^n) \times M^{\infty,1}(\mathbb{R}^n) \times \mathrm{GL}(n, \mathbb{R}) \to M^{\infty,1}(\mathbb{R}^n).$$

Proof. Consider the bicharacter $B(x,y) = e^{2\pi i y^T x}$, for $x, y \in \mathbb{R}^n$. We point out that $B \in M^{1,\infty}(\mathbb{R}^{2n})$. Note that a dilation of B is the tensor product $B(x+y, x-y) = b_+(x)b_-(y)$ of the second degree characters $b_\pm(x) = e^{\pm 2\pi i x^T x}$. It is known that the Fourier transform of a second degree character is a multiple of another second degree character, and since by [10, Lemma 4.1] this class of functions belongs to $W^{1,\infty}(\mathbb{R}^n)$, we have $b_\pm \in M^{1,\infty}(\mathbb{R}^n)$. Hence we obtain $B \in M^{1,\infty}(\mathbb{R}^{2n})$.

Next, we use the notation for the restriction operator $R_\Delta f(z) = f(z,z)$ above and observe that

$$\begin{aligned}\sigma \sharp_\Theta \tau(z) &= \iint_{\mathbb{R}^{2n}} \sigma(z')\tau(z'')e^{2\pi i(z-z'')^T\Theta(z-z')}dz'dz'' \\ &= R_\Delta\big((\sigma\otimes\tau) * D_H B\big)(z), \qquad \text{for } H = \left(\begin{smallmatrix}\Theta & 0\\ 0 & I\end{smallmatrix}\right).\end{aligned}$$

By Lemma 2.6(i) the mapping $(\sigma,\tau) \mapsto \sigma\otimes\tau$ is continuous. Since the mapping $\Theta \mapsto H$ is continuous, we obtain by Lemma 2.4 that the mapping $(\sigma,\tau,\Theta) \mapsto (\sigma\otimes\tau) * D_H B$ is continuous. By Lemma 2.6(ii) and Remark 2.7 we conclude that the mapping $(\sigma,\tau,\Theta) \mapsto R\big((\sigma\otimes\tau) * D_H B\big)$ is continuous. ☐

Corollary 3.3. *The following mapping is continuous:*

$$(\sigma,\tau,\rho) \mapsto D_\rho(D_{\rho^{-1}}\sigma \sharp D_{\rho^{-1}}\tau),$$
$$M^{\infty,1}(\mathbb{R}^n) \times M^{\infty,1}(\mathbb{R}^n) \times \mathrm{GL}(n,\mathbb{R}) \to M^{\infty,1}(\mathbb{R}^n).$$

Proof. Since

$$D_\rho(D_{\rho^{-1}}\sigma \sharp D_{\rho^{-1}}\tau) = |\det\rho|^{-2}\cdot(\sigma \sharp_\Theta \tau), \qquad \text{for } \Theta = \rho^T J\rho,$$

we observe that the corollary is a consequence of Proposition 3.2. ☐

Proof of Theorem 1.1. Sjöstrand proved that $(M^{\infty,1},\sharp)$ is a Banach algebra [28, 29, 47, 48]. By Lemma 2.2(ii) for $M^{\infty,1}(\mathbb{R}^n)$ and Corollary 3.3 we have that the family of dilations $(D_\rho)_\rho$ on $(M^{\infty,1},\sharp)$ satisfies the assumptions of Lemma 3.1, which implies the theorem. ☐

4. Proof of Theorem 1.3

A set X of points in $\mathbb{R}^n$ is defined to be uniformly separated if there exists $a > 0$ such that $|x-y| \geq a$, for all $x, y \in X$ with $x \neq y$. A set X is called relatively separated if it is the finite union of uniformly separated subsets.

Lemma 4.1. *For every relatively separated set of points $X \subset \mathbb{R}^n$ the following mapping is continuous:*

$$h \mapsto \sum_{x\in X} T_x h,$$
$$M^1(\mathbb{R}^n) \to M^{\infty,1}(\mathbb{R}^n).$$

Proof. Since X is relatively separated, the functional $\delta_X\colon f \mapsto \sum_{x\in X} f(x)$ is an element of $M^\infty(\mathbb{R}^n) = (M^1(\mathbb{R}^n))'$, the dual space of $M^1(\mathbb{R}^n)$. Next, the mapping

$$(f,g) \mapsto f * g,$$
$$M^1(\mathbb{R}^n) \times M^\infty(\mathbb{R}^n) \to M^{\infty,1}(\mathbb{R}^n)$$

is continuous, which follows from applying the Fourier transform to the corresponding multiplication embedding of Wiener amalgam spaces in [16, Theorem 2.11] (cf. (2.3)) or from the general convolution properties in [9, 14, 50], observing that $\sum_{x\in X} T_x f = \delta_X * f$. For the norm of the mapping in the lemma we thus have

$$\Big\| \sum_{x\in X} T_x h \Big\|_{M^{\infty,1}} \le C\, \|\delta_X\|_{M^\infty} \|h\|_{M^1},$$

where C is a global constant that does not depend on X. $\square$

Let $G(g,\Lambda)$ be a Gabor system with $g\in M^1$ and a time-frequency set of translations Λ in $\mathbb{R}^{2n}$. The Weyl symbol of the (pre-)frame operator $S_{g,\Lambda}$ for $G(g,\Lambda)$ is of the form

$$\sigma_{g,\Lambda} = \sum_{\lambda\in\Lambda} T_\lambda W(g),$$

where $W(g)$ denotes the Wigner distribution of g, defined by

$$W(g)(x,\omega) = \int_{\mathbb{R}^n} g\Big(x+\frac{t}{2}\Big)\overline{g\Big(x-\frac{t}{2}\Big)} e^{-2\pi i\omega t}\, dt.$$

We will notice that $\sigma_{g,\Lambda}\in M^{\infty,1}$. In fact, while the mapping

$$\begin{gathered}(g,\rho)\mapsto \sigma_{g,\rho\Lambda},\\ M^1(\mathbb{R}^n)\times \mathrm{GL}(2n,\mathbb{R}) \to M^{\infty,1}(\mathbb{R}^{2n})\end{gathered}$$

is *not* continuous (except for $g=0$), we observe the following.

Lemma 4.2. *Suppose that Λ is a relatively separated set of points in $\mathbb{R}^{2n}$. Then the following mapping is continuous:*

$$\begin{gathered}(g,\rho)\mapsto D_\rho\sigma_{g,\rho\Lambda},\\ M^1(\mathbb{R}^n)\times \mathrm{GL}(2n,\mathbb{R}) \to M^{\infty,1}(\mathbb{R}^{2n}).\end{gathered}$$

Proof. The Wigner distribution $W(g)$ can be factorized into a tensor product, a dilation, and a partial Fourier transform [27, Lemma 4.3.3]. The functorial continuity properties of M^1 under these operators, shown in [12], allow us to conclude that the following mapping is continuous:

$$\begin{gathered}g\mapsto W(g),\\ M^1(\mathbb{R}^n)\to M^1(\mathbb{R}^{2n}).\end{gathered}$$

Therefore Lemma 2.2 implies that the following mapping is continuous:

$$\begin{gathered}g\mapsto D_\rho W(g),\\ M^1(\mathbb{R}^n)\to M^1(\mathbb{R}^{2n}).\end{gathered}$$

Hence, by Lemma 4.1 we conclude that the following mapping is continuous:

$$\begin{gathered}g\mapsto \sum_{\lambda\in\Lambda} T_\lambda D_\rho W(g),\\ M^1(\mathbb{R}^n)\to M^{\infty,1}(\mathbb{R}^{2n}).\end{gathered}$$

We conclude our argument using the identity

$$D_\rho\sigma_{g,\rho\Lambda} = D_\rho \sum_{\lambda\in\rho\Lambda} T_\lambda W(g) = \sum_{\lambda\in\Lambda} T_\lambda D_\rho W(g). \qquad \square$$

Using all these ingredients we can come to the proof of the main theorems.

Proof of Theorem 1.3. Suppose that $G(g_0,\Lambda)$ is a Gabor frame with $g_0 \in M^1$, hence a Bessel sequence. Consequently the set Λ in $\mathbb{R}^{2n}$ must be relatively separated [8, Theorem 3.1]; in fact, for $g_0 \in M^1(\mathbb{R}^n)$ the two properties are equivalent [34, Theorem 12].

Let $\sigma_{g_0,\Lambda} \in M^{\infty,1}(\mathbb{R}^n)$ denote the Weyl symbol constructed above, of the Gabor frame operator $S_{g_0,\Lambda} = A_{\sigma_{g_0,\Lambda}}$. Since $G(g_0,\Lambda)$ is a frame, the frame operator $S_{g_0,\Lambda}$ is invertible. Thus by Theorem 1.1 there is a neighborhood W of $\sigma_{g_0,\Lambda}$ in $M^{\infty,1}(\mathbb{R}^{2n})$ and a neighborhood V_1 of I in $\mathrm{GL}(2n,\mathbb{R})$ such that

$$A_{D_{\rho^{-1}\sigma}} \text{ is invertible,} \qquad \text{for all } \sigma \in W,\ \rho \in V_1\,. \tag{4.1}$$

Lemma 4.2 guarantees the existence of a neighborhood U of g_0 in $M^1(\mathbb{R}^n)$ and a neighborhood V_2 of I in $\mathrm{GL}(2n,\mathbb{R})$ such that

$$D_\rho \sigma_{g,\rho\Lambda} \in W, \qquad \text{for all } g \in U,\ \rho \in V_2\,. \tag{4.2}$$

Hence, setting $V = V_1 \cap V_2$, we have by (4.1) and (4.2) that

$$A_{D_{\rho^{-1}}(D_\rho \sigma_{g,\rho\Lambda})} \text{ is invertible,} \qquad \text{for all } g \in U,\ \rho \in V\,,$$

that is,

$$A_{\sigma_{g,\rho\Lambda}} \text{ is invertible,} \qquad \text{for all } g \in U,\ \rho \in V\,.$$

Since $A_{\sigma_{g,\rho\Lambda}} = S_{g,\rho\Lambda}$ is the Gabor frame operator for the Gabor system $G(g,\rho\Lambda)$, we conclude that $G(g,\rho\Lambda)$ is a Gabor frame, for all (g,ρ) in the neighborhood $U\times V$ of (g_0,I) in $M^1(\mathbb{R}^n)\times \mathrm{GL}(2n,\mathbb{R})$. □

5. Final comments

The results in this paper are formulated for Gabor frames $G(g,\Lambda)$ using a single window $g \in M^1(\mathbb{R}^n)$ and general relatively separated sets Λ. In fact, they also hold for multi-window Gabor frames $G(g_1,\Lambda_1)\cup\cdots\cup G(g_m,\Lambda_m)$ with $g_1,\ldots,g_m \in M^1(\mathbb{R}^n)$. For example, the multi-window version of Theorem 1.5 reads that if such a Gabor system is a frame, then $\mathscr{D}^-(\Lambda_1\cup\cdots\cup\Lambda_n) > 1$. Indeed the extension to the multi-window case is immediate, based on the fact that the Weyl symbol for the multi-window system is just $\sigma = \sigma_1 + \cdots + \sigma_m$, where σ_k is the Weyl symbol for the Gabor system $G(g_k,\Lambda_k)$, for $k = 1,\ldots,m$.

Note that single- and multi-window Gabor frames (especially with windows in Feichtinger's algebra) are closely related to projective modules over non-commutative tori, as pointed out by results of Luef and Manin [37–39]; cf. Remark 1.6(ii).

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