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HARDY SPACES ASSOCIATED TO THE DISCRETE LAPLACIANS ON GRAPHS AND BOUNDEDNESS OF SINGULAR INTEGRALS

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ABSTRACT. Let Γ be a graph with a weight σ. Let d and μ be the distance and the measure associated with σ such that (Γ, d, μ) is a doubling space. Let p be the natural reversible Markov kernel associated with σ and μ and P be the associated operator defined by $Pf(x) = \sum_y p(x, y)f(y)$. Denote by $L = I - P$ the discrete Laplacian on Γ. In this paper we develop the theory of Hardy spaces associated to the discrete Laplacian H^p_L for $0 < p \leq 1$. We obtain square function characterization and atomic decompositions for functions in the Hardy spaces H^p_L, then establish the dual spaces of the Hardy spaces $H^p_L, 0 < p \leq 1$. Without the assumption of Poincaré inequality, we show the boundedness of certain singular integrals on Γ such as square functions, spectral multipliers and Riesz transforms on the Hardy spaces H^p_L, $0 < p \leq 1$.

CONTENTS

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1. Introduction

Analysis on graphs has been an interesting topic in mathematics which has had a long history and was studied extensively with its wide applications ranging from modeling in physics, chemistry, potential theory, and electrical networks to mathematics in biology. We refer the reader to [W1, W2] and the references therein for the background and applications of graph analysis. Recently, harmonic analysis on graphs has attracted lots of interest such as the study of the Laplace operator on graphs and its spectral theory [BR, BkR, JP]. Boundedness of singular integral operators on graphs is a natural question in harmonic analysis. Recent works include [BR] in which the authors proved boundedness of square functions associated to the Laplace operators, [KM] which studied boundedness of spectral multipliers on graph setting and [R2, R3] which showed boundedness of the Riesz transforms associated to the Laplace operator on L^p spaces and on Hardy spaces (in the sense of Coifman and Weiss) on graphs.

The aim of this article is twofold. First, we study the theory of Hardy spaces associated to the discrete Laplacians on weighted graphs with doubling property. This includes obtaining square function characterization of the Hardy spaces as well as the atomic Hardy spaces, and characterizing the dual spaces of the Hardy spaces and proving interpolation property between the Hardy spaces and L^p spaces. Second, we show the boundedness of certain singular integral operators such as the square functions, spectral multipliers and Riesz transforms associated with the discrete Laplacians on these Hardy spaces.

Let us now recall the definition of spaces with doubling property. Assume that X is a metric space equipped with a distance d and a nonnegative Borel measure μ. Denote by $B(x,r)$ the open ball of radius $r>0$ and center $x\in M$, and by $V(x,r)$ its measure $\mu(B(x,r))$. We say that (X,d,μ) is a doubling space if the measure μ satisfies the doubling property, i.e. there exists a constant $C>0$ so that

$$V(x,2r)\le CV(x,r)$$

for all $x\in X$ and $r>0$. Note that on a weighted graph, an appropriate distance and a suitable measure can be defined so that the graph is a doubling space.

The theory of a Hardy space on doubling spaces has been a central part of modern harmonic analysis, especially because of its role as the natural replacement of L^1 space in interpolation theory concerning Calderón-Zygmund singular integral operators. See for example [S] and the references therein for a good treatment of standard Hardy spaces and their applications.

For the background of our work, we recall the definition of the Coifman and Weiss Hardy spaces $H^p_{CW}(X)$ as in [CW]. Let $0<p\le 1$. We say that a function a is a $(2,p)$ atom if there exists a ball B such that

(i) supp $a\subset B$;
(ii) $\|a\|_{L^2}\le V(B)^{1/2-1/p}$;
(iii) $\int a(x)\mu(x)=0$.

The atomic Hardy space H^1_{CW} is defined as follows. We say that a function $f \in H^1_{CW}(X)$, if $f \in L^1$ and there exist a sequence $(\lambda_j)_{j\in\mathbb{N}} \in l^1$ and a sequence of $(2,1)$ atoms $(a_j)_{j\in\mathbb{N}}$ such that $f = \sum_j \lambda_j a_j$. We set

$$\|f\|_{H^1_{CW}} = \inf\{\sum_j |\lambda_j| : f = \sum_j \lambda_j a_j\}.$$

To define the Hardy space H^p_{CW} for $0 < p < 1$, we need to introduce the Lipschitz space $\mathfrak{L}_\alpha$. We say that the function $f \in \mathfrak{L}_\alpha$ if there exists a constant $c > 0$, such that

$$|f(x) - f(y)| \leq c|B|^\alpha$$

for all balls B and $x, y \in B$, and the best constant c can be taken to be the norm of f and is denoted by $\|f\|_{\mathfrak{L}_\alpha}$.

Let $0 < p < 1$ and $\alpha = 1/p - 1$. We say that a function $f \in H^p_{CW}(X)$, if $f \in (\mathfrak{L}_\alpha)^*$ and there are a sequence $(\lambda_j)_{j\in\mathbb{N}} \in l^p$ and a sequence of $(2,p)$ atoms $(a_j)_{j\in\mathbb{N}}$ such that $f = \sum_j \lambda_j a_j$. We set

$$\|f\|_{H^p_{CW}} = \inf\{(\sum_j |\lambda_j|^p)^{1/p} : f = \sum_j \lambda_j a_j\}.$$

Note that when $0 < p < 1$, $\|\cdot\|_{H^p_{CW}}$ is not the norm but $d(f,g) = \|f - g\|_{H^p_{CW}}$ is a metric.

The theory of Hardy spaces H^p_{CW}, $0 < p \leq 1$, works well for singular integral operators which possess enough smoothness on their associated kernels so that they belong to the class of Calderón-Zygmund operators, i.e. their associated kernels satisfy at least the Hörmander condition. However, when the singular integral operators have rough associated kernels so that they do not belong to the class of Calderón-Zygmund operators, the Hardy spaces H^p_{CW} are no longer the suitable underlying spaces for the study of boundedness of these operators. In the last decade, there has been extensive work on new Hardy spaces which are defined as being associated to operators which possess appropriate estimates on the heat kernels. See for example [ADM, DY2, HM, HLMMY, JY, DL, AMR] and the references therein. This recent theory of Hardy spaces associated with operators has been a success for the study of boundedness of singular integral operators with rough kernels.

Since the weighted graphs in this article are assumed to possess the doubling property, Hardy spaces H^p_{CW} on graphs can be defined and Calderón-Zygmund operators can be shown to be bounded on these spaces for appropriate values of p; see for example [BR, KM, R2, R3]. However, we will study the boundedness of certain singular integrals which may have rough kernels and do not belong to the class of Calderón-Zygmund operators. The new results of this article are the following:

(i) We introduce the Hardy spaces $H^p_L(\Gamma), 0 < p \leq 1$, associated to the discrete Laplacian L on doubling graphs via atomic decomposition as well as square function characterizations (Section 3).

(ii) We study the Lipschitz spaces $\mathfrak{L}_L(\alpha, M, \Gamma)$ associated to the discrete Laplacian L (where M is a suitable constant) and show the connection between these Lipschitz spaces and Carleson measures (Section 4).

(iii) We prove that the dual spaces of $H^p_L(\Gamma)$ are $\mathfrak{L}_L(\alpha, M, \Gamma)$ for appropriate values p and α. It then follows that the new Hardy spaces $H^p_L(\Gamma)$ have the important property of interpolation property with L^p spaces (Section 5).

(iv) We show the boundedness of square functions, spectral multipliers and the Riesz transforms associated to the discrete Laplacians on the Hardy spaces $H^p_L(\Gamma)$ (Section 6). See also Section 2.3 about these singular integrals.

We would like to give a brief comment on the technical elements used in this paper. While we use a similar approach to those in [ADM, DY2, HLMMY], the theory of Hardy spaces in [ADM, DY2, HLMMY] is not directly applicable to our graph setting due to the lack of suitable estimates related to the heat semigroups. For example, one of the key ingredients in obtaining some important properties on Hardy spaces $H^p_L(\Gamma)$ such as the atomic decomposition in the articles mentioned above is the use the Calderón's reproducing formula related to the semigroup e^{-tL}, but this kind of formula does not work in our graph setting. We overcome this problem by establishing a "discrete" version of Calderón's reproducing formula (see Theorem 3.7).

Remark on notation. We will often write B for $B(x_B, r_B)$ and $V(E)$ for $\mu(E)$ for any measurable subset $E \subset \Gamma$. Also given $\lambda > 0$, we write λB for the λ-dilated ball, which is the ball with the same center as B and with radius $r_{\lambda B} = \lambda r_B$. For each ball $B \subset \Gamma$ we set

$$S_0(B) = B \text{ and } S_j(B) = 2^j B \backslash 2^{j-1} B \text{ for } j \in \mathbb{N}.$$

We also write the sum $\displaystyle\sum_{k=\alpha}^{\beta}$ with real values $\alpha \geq 0$, $\beta > 0$ for the sum over $k = [\alpha], \cdots, [\beta]$, where $[\alpha], [\beta]$ denote the integer parts of α, β and use the convention that the constants C, c can change from each of their appearances.

2. Weighted graphs

2.1. **Graphs.** We now recall basic definitions and properties about graphs; see [C, CS, BR, KM].

Assume that Γ is a countably infinite set and that $\sigma(x, y)$ is a weight on Γ satisfying $\sigma(x, y) = \sigma(y, x) \geq 0$ for all $x, y \in \Gamma$. The weight $\sigma(x, y)$ then gives a graph structure on Γ. The vertices x and y are said to be neighbors, denoted by $x \sim y$ when $\sigma(x, y) > 0$. On Γ, one considers the discrete measure μ defined by

$$\mu(x) = \sum_{y \sim x} \sigma(x, y), \quad x \in \Gamma.$$

For a subset $E \subset \Gamma$, define $\mu(E) = \displaystyle\sum_{x \in E} \mu(x)$. The space L^p then means $L^p(\Gamma, \mu)$ for $0 < p < \infty$.

A path of length n joining the vertices x and y is a sequence of vertices $x = x_0, x_1, \ldots, x_n = y$ such that $x_i \sim x_{i-1}, i = 1, \ldots, n$. In this article, we also assume that Γ is connected, i.e. for any $x, y \in \Gamma$, there exists a path joining x and y. The distance $d(x, y)$ is then defined as the minimum of the lengths of paths joining x and y.

Let $B(x, r) = \{y \in \Gamma : d(x, y) \leq r\}$ denote the ball of center x and radius r. In the sequel, we assume that Γ satisfies the locally uniformly finite property, i.e.

there exists $N \geq 1$ so that for any $x \in \Gamma$, $\sharp B(x,1) \leq N$, where $\sharp E$ denotes the cardinal of any subset $E \subset \Gamma$.

We consider the transition kernel $p(x,y)$ defined by

$$p(x,y) = \frac{\sigma(x,y)}{\mu(x)}, \quad x,y \in \Gamma.$$

It is easy to check that for all $x \in \Gamma$, $\sum_{y\in\Gamma} p(x,y) = 1$ and that $p(x,y) = 0$ if $d(x,y) \geq 2$.

One can also see that $p(x,y)$ may not be symmetric, but the identity

$$p(x,y)\mu(x) = p(y,x)\mu(y)$$

holds for all $x,y \in \Gamma$.

Associated to the kernel $p(\cdot,\cdot)$, we consider the operator P defined on $L^2(\Gamma)$ by

$$Pf(x) = \sum_{y\in\Gamma} p(x,y)f(y).$$

The operator $L = I - P$ is called the discrete Laplacian on Γ. Let us set

$$p_0(x,y) = \delta_x(y), \quad p_1(x,y) = p(x,y),$$

where δ_x is the Dirac mass at x. Let $p_n(x,y), n \in \mathbb{N}$, be the n^{th} convolution power of $p(x,y)$ defined by

$$p_n(x,y) = \sum_{z\in\Gamma} p(x,z)p_{n-1}(z,y).$$

Then we have the operator P^n given by

$$P^n f(x) = \sum_{y\in\Gamma} p_n(x,y)f(y)$$

for $x \in \Gamma$ and $n \in \mathbb{N}$.

Remark 2.1. It can be verified that if supp $\varphi \subset B(x,k)$, then $(a_0 I + a_1 P + \ldots + a_n P^n)\varphi \subset B(x,k+n)$.

We define the operator "length of the gradient" ∇ by

$$\nabla f(x) = \Big(\frac{1}{2}\sum_{y\in\Gamma} p(x,y)|f(x) - f(y)|^2\Big)^{1/2}$$

for any L^2 function f on Γ and $x \in \Gamma$. It is easy to check that

$$\langle (I-P)f, f\rangle_{L^2} = \|\nabla f\|^2_{L^2}$$

(see, for example, [BR] Section 1).

2.2. **Assumptions.** Throughout this paper we assume the following conditions:

(D) Doubling property: The weighted graph (Γ,μ) satisfies the doubling property (D), i.e., there exists a constant $C > 0$ such that

$$V(x,2r) \leq CV(x,r), \quad x \in \Gamma, r > 0, \tag{1}$$

where $V(x,r) = \mu(B(x,r))$. Note that this implies that there exist constants $c, D > 0$ such that

$$V(x,r) \leq c\Big(\frac{r}{s}\Big)^D V(x,s), \quad \text{for } r > s > 0. \tag{2}$$

It follows from the doubling assumption (D) that the graph Γ is a space of homogeneous type in the sense of Coifman and Weiss [CW].

(S_α) Uniform lower bound condition for $p(x,y)$: Given $\alpha > 0$, we say that (Γ, μ) satisfies the condition (S_α) if

$$x \sim y \ \Rightarrow \sigma(x,y) \geq \alpha\mu(x) \quad \text{and} \quad x \sim x, \forall x \in \Gamma.$$

(UE) Upper estimate for $p_n(x,y)$: We say that (Γ, μ) satisfies the condition (UE) if there exist $C, c > 0$ such that

$$p_n(x,y) \leq C \frac{\mu(y)}{V(x, \sqrt{n})} \exp\Big(- c\frac{d(x,y)^2}{n}\Big)$$

for all $n \in \mathbb{N}$ and $x, y \in \Gamma$.

Remark 2.2. (i) Note that the condition (S_α) implies that $\sigma(x,x) > 0$. This condition was used in [CS] to prove the analyticity of the Markov operator P which plays an important role in obtaining the regularity in time of the Markov kernel $p_n(x,y)$; see for example [Ch, B, R2].

(ii) It was proved in [R2] that assumption (S_α) implies that 0 does not belong to the spectrum of the discrete Laplacian L on $L^2(\Gamma)$. Hence, L is one-to-one on L^2. Since $L^2(\Gamma) := \overline{\mathcal{R}(L)} \oplus \mathcal{N}(L)$, $\mathcal{R}(L)$ stands for the range $\mathcal{R}(L) := \{Lu : u \in L^2(\Gamma)\}$ and $\mathcal{N}(L)$ stands for the nullspace of L. Therefore

$$L^2(\Gamma) = \overline{\mathcal{R}(L)}.$$

(iii) For $n \in \mathbb{N}_+, k \in \mathbb{N}$, let us denote by $\widetilde{p}_{n,k}$ the associated kernel to $(I-P)^k P^n$. Note that in the continuous setting, Gaussian upper bounds for the time derivatives of the heat kernels follow from the Gaussian upper bounds (UE) on the heat kernels via the analyticity of the semigroup in time. In the discrete setting, the Gaussian upper bound for $\widetilde{p}_{n,k}$ as in (3) below was shown by S. Blunck in [B] (see also [Ch, D]) in the case of polynomial volume growth and exponential volume growth. However, this proof still works in the setting of doubling graphs and it follows from conditions (D), (S_α) and (UE) that

$$|\widetilde{p}_{n,k}(x,y)| \leq C \frac{\mu(y)}{n^k V(x, \sqrt{n})} \exp\Big(- c\frac{d(x,y)^2}{n}\Big). \tag{3}$$

2.3. **Anterior results of singular integrals on graphs and our results.** In this section, we remind the reader of known results concerning the boundedness of several singular integrals on graphs in [BR, KM, R2, R3] and explain our results on the boundedness of singular integrals on the new Hardy spaces associated with the discrete Laplacian.

(i) Square functions: Consider the square function defined by, for $M \geq 1$,

$$\mathcal{G}_{L,M} f(x) = \Big(\sum_{k=1}^\infty \frac{|k^M (I-P)^M P^k f(x)|}{k} \Big)^{1/2}.$$

Under the assumptions of (D), (UE) and (S_α) and $M = 1$, it was proved in [BR] that $\mathcal{G}_{L,M}$ is bounded on L^p for all $1 < p < \infty$. Under the same three assumptions, we will show the boundedness of $\mathcal{G}_{L,M}$ from the Hardy spaces H^p_L into L^p for the range $0 < p \leq 1$ in Theorem 6.1. Consequently, we also obtain the boundedness of $\mathcal{G}_{L,M}$ on L^p for all $1 < p < \infty$ by interpolation.

(ii) It was proved recently in [KM] that under the assumptions of (D), (UE), (S_α) and Poincaré inequality, the spectral multiplier of the discrete Laplacian L is bounded on H^p_{CW} for $p \in (p_0, 1]$ for some $p_0 > \frac{D}{D+1}$, where H^p_{CW} is Coifman and Weiss's Hardy spaces and D is the dimension in (2). Recall that Γ is said to satisfy the Poincaré inequality if there exists a constant $c > 0$ such that for every function f, $x_0 \in \Gamma$ and $r > 0$

$$\sum_{x \in B(x_0,r)} |f(x) - f_B|^2 \mu(x) \le cr^2 \times \sum_{x,y \in B(x_0,2r)} |f(y) - f(x)|^2 \sigma(x,y), \tag{4}$$

where

$$f_B = \frac{1}{V(B)} \sum_{x \in B} f(x)\mu(x).$$

Under the assumptions of (D), (UE), and (S_α) but without the assumption of Poincaré inequality, we will show the boudedness of the spectral multiplier of the discrete Laplacian L on H^p_L for the range $0 < p \le 1$.

(iii) Riesz transforms: It was proved in [R2] that under the conditions (D), (UE) and (S_α), the Riesz transform $\nabla(I-P)^{-1/2}$ is bounded on L^p for all $1 < p \le 2$. The boundedness from H^1_{CW} to L^1 for the Riesz transform $\nabla(I-P)^{-1/2}$ was obtained in [R3] under the extra condition of the Poincaré inequality. The L^p-boundedness of the Riesz transform for the range $p > 2$ was studied in [BR], also under the assumption of Poincaré inequality. Moreover, the example of two copies of $\mathbb{Z}^2$ linked between by an edge shows that boundedness of Riesz transforms for $p > 2$ would imply Poincaré inequality on balls; see [R2], Section 4.

Under the assumptions of (D), (UE) and (S_α), again in the absence of Poincaré inequality, we will show that the Riesz transform is bounded from H^p_L into L^p for the range $0 < p \le 1$; see Theorem 6.5.

3. Hardy spaces associated to the discrete Laplacian

3.1. **Tent spaces on graphs.** In this section, we study the concept of "discrete tent spaces", which is the main tool to introduce Hardy spaces associated to operators on graphs by adapting the ideas of tent spaces in [CMS].

For any $x \in \Gamma$, denote by $\Upsilon(x)$ the cone with vertex x, i.e.

$$\Upsilon(x) = \{(y,k) \in \Gamma \times \mathbb{N}_+ : d(y,x) < k\}.$$

Given the ball $B = B(x_B, r_B)$ in Γ, the tent over B is defined by

$$\widehat{B} := \{(x,k) \in \Gamma \times \mathbb{N}_+ : d(x_B, x) \le r_B - k\}.$$

For any function on f on $\Gamma \times \mathbb{N}_+$ and any $x \in \Gamma$, define

$$(\mathcal{A}f)(x) = \Big(\sum_{d(y,x)<k} \sum_{k=1}^{\infty} \frac{|f(y,k)|^2}{kV(x,k)} \mu(y) \Big)^{1/2}$$

and

$$(\mathcal{C}_p f)(x) = \sup_{B \ni x} \frac{1}{V(B)^{1/p-1/2}} \Big(\sum_{d(y,x) \le r_B - k} \sum_{k=1}^{r_B} \frac{|f(y,k)|^2}{k} \mu(y) \Big)^{1/2}, \quad 0 < p \le 1.$$

For all $0 < p < \infty$, we say that $f \in T^p(\Gamma)$, briefly $f \in T^p$, if

$$\|f\|_{T^p} := \|\mathcal{A}f\|_{L^p} < +\infty.$$

For $0 < p \le 1$, we put

$$\|f\|_{T^{p,\infty}} := \|\mathcal{C}_p f\|_{L^\infty} < +\infty.$$

Let $p \in (0,\infty)$. A measurable function a on $\Gamma \times \mathbb{N}_+$ is said to be a T^p atom if there exists a ball $B \subset X$ such that a is supported in $\widehat{B}$ and

$$\Big(\frac{1}{V(B)}\sum_{d(y,x)\le r_B}\sum_{k=1}^{r_B}\frac{|a(y,k)|^2}{k}\mu(y)\Big)^{1/2} \le \frac{1}{V(B)^{1/p-1/2}}.$$

One can check that a T^p atom is also a T^p function. Conversely, any T^p function with $0<p\le 1$ can be decomposed into a sum of T^p atoms.

Proposition 3.1. *Let $0<p\le 1$. Then for any $f \in T^p$, there exist a sequence $(\lambda_n)_{n\in\mathbb{N}} \in l^p$ and a sequence of T^p atoms $(a_n)_{n\in\mathbb{N}}$ such that*

$$f = \sum_{n=1}^{\infty}\lambda_n a_n \tag{5}$$

and

$$\sum_{n=1}^{\infty}|\lambda_n|^p \le C\|f\|_{T^p}, \tag{6}$$

where the series in (5) *converges in T^p.*

For the proof of Proposition 3.1, we refer the reader to [R1].

Proposition 3.2. *(a) Let $1<p<\infty$. For $f \in T^p$ and $g \in T^{p'}$ we have*

$$\sum_{x\in\Gamma}\sum_{k=1}^{\infty}\frac{|f(x,k)g(x,k)|}{k}\mu(x) \le \sum_{x\in\Gamma}\mathcal{A}f(x)\mathcal{A}g(x)\mu(x).$$

(b) The dual space of T^p is $T^{p,\infty}$ for $0<p\le 1$. More precisely, the pairing

$$\langle f,g\rangle \to \sum_{x\in\Gamma}\sum_{k=1}^{\infty}\frac{f(x,k)g(x,k)}{k}\mu(x)$$

realizes $T^{p,\infty}$ as being equivalent to the dual space of T^p.

The proof of Proposition 3.2 is similar to that of [CMS, Theorem 1] (see also [Y]) and we omit the details here.

3.2. **Atomic Hardy spaces.** Recently, the theory of atomic Hardy spaces associated to operators was studied carefully; see for example [ADM, DY2, HM, HLMMY, DL, JY]. We use similar definitions (with minor modifications) of $(2,p,M)$ atoms, $(2,p,M,\epsilon)$ molecules and atomic Hardy spaces as those used in these works.

Definition 3.3. Given $M \in \mathbb{N}_+$ and $0<p\le 1$, a function $a \in L^2(\Gamma)$ is said to be a $(2,p,M)$ atom if there exist a function $b \in L^2$ and a ball B with radius $r_B \ge 1$ so that

(i) $a = L^M b$;
(ii) supp $L^k b \subset B$ for all $k = 0,\dots,M$;
(iii) $\|L^k b\|_{L^2} \le (r_B)^{M-k}V(B)^{1/2-1/p}$ for all $k=0,\dots,M$.

Remark 3.4. (i) Every $(2,p,M)$ atom a satisfies the cancelation condition, i.e.,

$$\sum_{x\in\Gamma}a(x)\mu(x) = 0.$$

Indeed, we have

$$\begin{aligned}\sum_{x\in\Gamma} a(x)\mu(x) &= \sum_{x\in\Gamma}(I-P)(L^{M-1}b)(x)\mu(x)\\ &= \sum_{x\in\Gamma}(L^{M-1}b)(x)\mu(x) - \sum_{x\in\Gamma}\sum_{y\in\Gamma} p(x,y)(L^{M-1}b)(y)\mu(x)\\ &= \sum_{x\in\Gamma}(L^{M-1}b)(x)\mu(x) - \sum_{x\in\Gamma}\sum_{y\in\Gamma} p(y,x)(L^{M-1}b)(y)\mu(y).\end{aligned}$$

Using $\sum_{x\in\Gamma} p(y,x) = 1$, we obtain

$$\sum_{x\in\Gamma} a(x)\mu(x) = \sum_{x\in\Gamma}(L^{M-1}b)(x)\mu(x) - \sum_{y\in\Gamma}(L^{M-1}b)(y)\mu(y) = 0.$$

(ii) Due to (i), for $0 < p \leq 1$ and $M \geq 1$, one can check that every $(2,p,M)$ atom is also a $(2,p)$ atom of the Hardy space H^1_{CW}, hence every $(2,p,M)$ atom belongs to H^p_{CW}. It is not clear whether a $(2,p)$ atom belongs to the space $H^p_{L,M,at}$ as defined in Definition 3.5.

(iii) Note that $L^M = (I-P)^M = a_0 I + a_1 P + \ldots + a_M P^M$; hence if supp$\varphi \subset B(x,k)$, then $L^M\varphi \subset B(x,k+M)$. Therefore $L^M b$ has compact support if b has compact support.

The atomic Hardy space is defined as follows:

Definition 3.5. We say that $f = \sum_j \lambda_j a_j$ is an atomic $(2,p,M)$-representation if $\{\lambda_j\}_{j=0}^{\infty} \in l^p$, each a_j is a $(2,p,M)$ atom, and the sum converges in $L^2(\Gamma)$. Define

$$\mathbb{H}^p_{L,M,at}(\Gamma) := \{f : f \text{ has an atomic } (2,p,M)\text{-representation}\}$$

with the norm

$$\|f\|_{\mathbb{H}^p_{L,M,at}(\Gamma)} = \inf\left\{\Big(\sum_{j=0}^{\infty}|\lambda_j|^p\Big)^{1/p} : f = \sum_j \lambda_j a_j \text{ is an atomic } (2,p,M)\text{-representation}\right\}.$$

The space $H^p_{L,M,at}(\Gamma)$ is then defined as the completion of $\mathbb{H}^p_{L,M,at}(\Gamma)$ with respect to this norm.

We now introduce the concept of $(2,p,M,\epsilon)$ molecules.

Definition 3.6. Given $M \in \mathbb{N}_+$, $0 < p \leq 1$ and $\epsilon > 0$, a function $m \in L^2(\Gamma)$ is said to be a $(2,p,M,\epsilon)$ molecule if there exist a function $b \in L^2$ and a ball B with radius $r_B \geq 1$ such that

(i) $m = L^M b$;

(ii) $\|L^k b\|_{L^2(S_j(B))} \leq (r_B)^{M-k} 2^{-j\epsilon} V(2^j B)^{1/2-1/p}$ for all $k = 0,\ldots,M$ and $j = 0,1,2,\ldots$.

We remark that $(2,p,M,\epsilon)$ molecules will have an important role in showing the boundedness of the spectral multipliers in Section 6.

3.3. **Hardy spaces via square functions.** We consider the following discrete square functions:

$$\mathcal{G}_{L,M}f(x) = \Big(\sum_{k=1}^{\infty}\frac{|k^M(I-P)^M P^k f|^2}{k}\Big)^{1/2} \quad \text{for given integers } M \geq 1,$$

$$S_h f(x) = \Big(\sum_{k=1}^{\infty} k|(I-P)P^{[\frac{k}{2}]}f(x)|^2\Big)^{1/2}$$

and

$$S_L f(x) = \Big(\sum_{d(y,x)<k}\sum_{k=1}^{\infty}\frac{|k(I-P)P^{[\frac{k}{2}]}f(y)|^2}{kV(y,k)}\mu(y)\Big)^{1/2},$$

where $[k/2]$ denotes the integer part of the real number $k/2$.

It is not difficult to check that $|S_h(f)| \leq S_h(|f|) \leq C(|f| + \mathcal{G}_{L,1}(|f|))$. Moreover, by using an argument similar to that of [BR] p. 281, one can show that S_h is bounded on L^2. Hence S_L is also bounded on L^2:

$$\|S_L f\|_{L^2} \leq C\|S_h f\|_{L^2} \leq C\|f\|_{L^2}.$$

For $0 < p < \infty$, the Hardy space $H^p_L(\Gamma)$ is defined as the completion of

$$\{f \in L^2(\Gamma) : S_L f \in L^p(\Gamma)\}$$

in the norm $\|f\|_{H^p_L} = \|S_L f\|_{L^p}$.

3.4. **Characterizations of Hardy spaces H^p_L.** The following identity will play a role as the main vehicle to obtain the atomic decomposition for Hardy spaces $f \in H^p_L$.

Theorem 3.7. *Let $M \geq 0$. Then for any $f \in L^2(\Gamma)$, we have*

$$f = \sum_{k=0}^{\infty} c_{k,M}(I-P)^M P^k f \tag{7}$$

on $L^2(\Gamma)$, where the coefficients $c_{k,M}$ are defined as follows:

(i) $c_{k,1} = 1$ for all $k = 0, 1, 2, \ldots$;

(ii) $c_{k,M+1} = \sum_{j=0}^{k} c_{j,M}$ for all $k = 0, 1, 2, \ldots$.

Remark 3.8. (i) By a simple calculation, we obtain $c_{k,M} \leq k^{M-1}$ for all k and M.

(ii) Formula (7) gives an expansion of $f \in L^2(\Gamma)$ in terms of $(I-P)^M P^k, k \geq 0$. If $\|P\|_{L^2\to L^2} < 1$, we can write $I = (I-P)^M(I-P)^{-M}$ and then, using the expansion $(I-P)^{-1} = \sum_{k=0}^{\infty} P^k$, we will obtain the desired estimate. However, one may have $\|P\|_{L^2\to L^2} = 1$, hence in general, the expansion $(I-P)^{-1} = \sum_{k=0}^{\infty} P^k$ may not hold. This subtle problem needs a careful examination.

Proof of Theorem 3.7. We will prove Theorem 3.7 by induction for $f \in \mathcal{R}(L)$.

For $M = 1$, we have

$$\|\sum_{k=0}^{N}(I-P)P^k f - f\|_{L^2} = \|P^N f\|_{L^2}.$$

Since $f \in \mathcal{R}(L)$, we can pick an L^2-function g so that $f = Lg = (I-P)g$. This together with the Gaussian upper bound (3) for $(I-P)P^N$ implies that

$$\|P^N f\|_{L^2} = \|(I-P)P^N g\|_{L^2} \leq \frac{C}{N}\|g\|_{L^2}.$$

Hence,

$$\lim_{N\to\infty} \|\sum_{k=0}^{N}(I-P)P^k f - f\|_{L^2} = 0.$$

Therefore (7) holds for $M=1$.

We now assume that (7) holds for $M = \widetilde{M}$, that is,

$$\lim_{N\to\infty} \|\sum_{k=0}^{N} c_{k,\widetilde{M}}(I-P)^{\widetilde{M}} P^k f - f\|_{L^2} = 0. \tag{8}$$

We will show that (7) holds for $M = \widetilde{M}+1$. Indeed, we have, for any $N \in \mathbb{N}_+$,

$$\sum_{k=0}^{N} c_{k,\widetilde{M}+1}(I-P)^{\widetilde{M}+1}P^k f - f = \sum_{k=0}^{N} c_{k,\widetilde{M}+1}(I-P)^{\widetilde{M}}(P^k - P^{k+1})f - f.$$

Hence

$$\begin{aligned}
&\sum_{k=0}^{N} c_{k,\widetilde{M}+1}(I-P)^{\widetilde{M}+1}P^k f - f \\
&\quad = (I-P)^{\widetilde{M}} + \sum_{k=0}^{N}(c_{k+1,\widetilde{M}+1} - c_{k,\widetilde{M}} + 1)(I-P)^{\widetilde{M}}P^{k+1}f \\
&\qquad - c_{N,\widetilde{M}+1}(I-P)^{\widetilde{M}}P^{N+1}f - f \\
&\quad = ((I-P)^{\widetilde{M}} + \sum_{k=0}^{N} c_{k+1,\widetilde{M}}(I-P)^{\widetilde{M}}P^{k+1}f - f) - c_{N,\widetilde{M}+1}(I-P)^{\widetilde{M}}P^{N+1}f \\
&\quad = (\sum_{k=0}^{N} c_{k,\widetilde{M}}(I-P)^{\widetilde{M}}P^k f - f) + \frac{c_{N,\widetilde{M}+1}}{(N+1)^{\widetilde{M}+1}}(N+1)^{\widetilde{M}+1}(I-P)^{\widetilde{M}+1}P^{N+1}g,
\end{aligned}$$

where $f = (I-P)g$.

Due to (8),

$$\lim_{N\to\infty} \|\sum_{k=0}^{N} c_{k,\widetilde{M}}(I-P)^{\widetilde{M}}P^k f - f\|_{L^2} = 0.$$

Using (3) and the fact that $c_{N,\widetilde{M}+1} \leq N^{\widetilde{M}}$, one obtains

$$\lim_{N\to\infty} \left\| \frac{c_{N,\widetilde{M}+1}}{(N+1)^{\widetilde{M}+1}}(N+1)^{\widetilde{M}+1}(I-P)^{\widetilde{M}+1}P^{N+1}g \right\|_{L^2} = 0.$$

In general, for $f \in L^2(\Gamma)$, there exist $f_n \in \mathcal{R}(L)$ so that $\lim_{n\to\infty} \|f_n - f\|_{L^2} = 0$. Then we write, for any $n \in \mathbb{N}_+$,

$$f - \sum_{k=0}^{\infty} c_{k,M}(I-P)^M P^k f = (f_n - \sum_{k=0}^{\infty} c_{k,M}(I-P)^M P^k f_n) + (f - f_n)$$
$$+ (\sum_{k=0}^{\infty} c_{k,M}(I-P)^M P^k (f_n - f)).$$

Therefore,

$$\|f - \sum_{k=0}^{\infty} c_{k,M}(I-P)^M P^k f\|_{L^2}$$
$$\leq \|f - f_n\| + \lim_{N\to\infty} \Big\| \sum_{k=0}^{N} c_{k,M}(I-P)^M P^k \Big\|_{L^2\to L^2} \|f_n - f\|_{L^2}$$

for any $n \in \mathbb{N}_+$.

To estimate $\Big\| \sum_{k=0}^{N} c_{k,M}(I-P)^M P^k \Big\|_{L^2\to L^2}$, we write

$$\sum_{k=0}^{N} c_{k,M}(I-P)^M P^k = \sum_{k=0}^{N} c_{k,M}(I-P)^{M-1}(P^k - P^{k+1}).$$

This gives

$$\begin{aligned}
&\sum_{k=0}^{N} c_{k,M}(I-P)^M P^k \\
&\quad = (I-P)^{M-1} + \sum_{k=0}^{N-1} (c_{k+1,M} - c_{k,M})(I-P)^{M-1} P^{k+1} - c_{N,M}(I-P)^{M-1} P^{N+1} \\
&\quad = (I-P)^{M-1} + \sum_{k=0}^{N-1} c_{k+1,M-1}(I-P)^{M-1} P^{k+1} - c_{N,M}(I-P)^{M-1} P^{N+1} \\
&\quad = \sum_{k=0}^{N} c_{k,M-1}(I-P)^{M-1} P^k - c_{N,M}(I-P)^{M-1} P^{N+1}.
\end{aligned}$$

Repeating the argument above $(M-1)$ times, we obtain

$$\sum_{k=0}^{N} c_{k,M}(I-P)^M P^k = I - \sum_{k=0}^{M-1} c_{N,k+1}(I-P)^k P^{N+1}. \tag{9}$$

Due to (3) and the fact that $c_{N,k+1} \leq N^{k+1}$, there exists a constant $C_M > 0$ so that

$$\|c_{N,k+1}(I-P)^k P^{N+1}\|_{L^2\to L^2} \leq C.$$

This together with (9) gives

$$\Big\| \sum_{k=0}^{N} c_{k,M}(I-P)^M P^k \Big\|_{L^2\to L^2} \leq 1 + MC_M.$$

Hence,

$$\|f - \sum_{k=0}^{\infty} c_{k,M}(I-P)^M P^k f\|_{L^2} \le \|f - f_n\|_{L^2} + (1 + MC_M)\|f_n - f\|_{L^2}$$

for all $n \in \mathbb{N}_+$.

Letting $n \to \infty$, we obtain the desired estimate. This completes our proof. $\square$

We are now in a position to establish the atomic decomposition for Hardy spaces H^p_L.

Proposition 3.9. *Let $f \in H^p_L \cap L^2(\Gamma)$, with $0 < p \le 1$. Then there exist a sequence $(\lambda_j)_{j\in\mathbb{N}}$ and a sequence of $(2, p, M)$ atoms $(a_j)_{j\in\mathbb{N}}$ so that*

$$f = \sum_j \lambda_j a_j \quad \text{on } L^2(\Gamma),$$

and

$$\sum_j |\lambda_j|^p \le C\|f\|^p_{H^p_L}.$$

Proof. Since $f \in H^p_L \cap L^2$, by definition, we have $F(\cdot,\cdot) \in T^p$, where $F(y,k) = k(I-P)P^{[\frac{k}{2}]}f(y)$ for all $(y,k) \in \Gamma \times \mathbb{N}_+$. Hence by Proposition 3.1, there exist a sequence $(\lambda_n)_{n\in\mathbb{N}} \in l^p$ and a sequence of T^p atoms $(A_n)_{n\in\mathbb{N}}$ so that

$$k(I-P)P^{[\frac{k}{2}]}f(x) = \sum_{j\ge 0} \lambda_j A_j(x,k), \quad (x,k) \in \Gamma \times \mathbb{N}_+, \tag{10}$$

and

$$\sum_{j\ge 0} |\lambda_j|^p \le C\|F\|^p_{T^p}.$$

Now for any $f \in L^2$, by Theorem 3.7,

$$f = \sum_{k=0}^{\infty} c_{k,M+1}(I-P)^{M+1}P^k f = \sum_{k=0}^{\infty} \frac{c_{k,M+1}}{k+1}(I-P)^M P^{k-[\frac{k+1}{2}]}\Big((k+1)(I-P)P^{[\frac{k+1}{2}]}f\Big)$$

on $L^2(\Gamma)$, where the coefficients $c_{k,M+1}$ are defined as in Theorem 3.7.

Assume that each T^p atom A_j is supported in $\widehat{B}_j$ for some ball B_j, $j \in \mathbb{N}$. Using (10), we have

$$f = \sum_{j\ge 0} \lambda_j \sum_{k=0}^{r_{B_j}-1} \frac{c_{k,M+1}}{k+1}(I-P)^M P^{k-[\frac{k+1}{2}]}A_j(\cdot, k+1).$$

Let $a_j = L^M b_j$, where

$$b_j = \sum_{k=0}^{r_{B_j}-1} \frac{c_{k,M+1}}{k+1} P^{k-[\frac{k+1}{2}]}A_j(\cdot, k+1).$$

For $k = 0, \ldots, M$, by Remark 2.1, we have supp $L^k b_j \subset B(x_{B_j}, M + 2r_{B_j}) := \widetilde{B}_j$.

Fix $0 \leq N \leq M$. Let $h \in L^2(\Gamma)$ with $\|h\|_{L^2} = 1$. We have, by Hölder's inequality,

$$
\begin{aligned}
&\sum_x L^N b_j(x)h(x)\mu(x)\\
&\quad = \Big(\sum_x \sum_{k=0}^{r_{B_j}-1} \frac{c_{k,M+1}}{k+1}(I-P)^N P^{k-[\frac{k+1}{2}]} A_j(x,k+1)h(x)\mu(x)\Big)\\
&\quad \leq \Big(\sum_x \sum_{k=0}^{r_{B_j}-1} \frac{c^2_{k,M+1}}{(k+1)^{2N}} \frac{|A_j(x,k+1)|^2}{k+1}\mu(x)\Big)^{1/2}\\
&\quad \times \Big(\sum_x \sum_{k=0}^{r_{B_j}-1} \frac{|(k+1)^N(I-P)^N P^{k-[\frac{k+1}{2}]} h(x)|^2}{k+1}\mu(x)\Big)^{1/2}\\
&\quad \leq r_{B_j}^{M-N}\Big(\sum_x \sum_{k=0}^{r_{B_j}-1} \frac{|A_j(x,k+1)|^2}{k+1}\mu(x)\Big)^{1/2} \quad (\text{since } c_{k,M+1} \leq (k+1)^M)\\
&\quad \times \Big(\sum_x \sum_{k=0}^{r_{B_j}-1} \frac{|(k+1)^N(I-P)^N P^{k-[\frac{k+1}{2}]} h(x)|^2}{k+1}\mu(x)\Big)^{1/2}.
\end{aligned}
$$

We have

$$
\begin{aligned}
\Big(\sum_x \sum_{k=0}^{r_{B_j}-1} \frac{|(k+1)^N(I-P)^N P^{k-[\frac{k+1}{2}]} h(x)|^2}{k+1}\mu(x)\Big)^{1/2} &\leq C(\|h\|_{L^2} + \|\mathcal{G}_{L,N}h\|_{L^2})\\
&\leq C\|h\|_{L^2}.
\end{aligned}
$$

Therefore

$$
\sum_x L^N b_j(x)h(x)\mu(x) \leq C r_{B_j}^{M-N} V(B_j)^{1/2-1/p}\|h\|_{L^2}.
$$

This implies $\|L^N b_j\| \leq C r_{\widetilde{B}_j}^{M-N} V(\widetilde{B}_j)^{1/2-1/p}$. Hence, the a_j's are, up to a harmless multiplicative constant, $(2,p,M)$ atoms associated to the balls $\widetilde{B}_j$. $\square$

Proposition 3.10. *Let $0 < p \leq 1$ and $M > D(1/p - 1/2)$. Assume that $f = \sum_{j\geq 0} \lambda_j a_j$, where $\{a_j\}_{j\geq 0}$ is a family of $(2,p,M)$ atoms and $\sum_{j\geq} |\lambda_j|^p < \infty$. Then the series $\sum_{j\geq 0}\lambda_j a_j$ converges in H^p_L and*

$$
\Big\| \sum_{j\geq 0} \lambda_j a_j \Big\|_{H^p_L} \leq C\Big(\sum_{j\geq 0} |\lambda_j|^p\Big)^{1/p}.
$$

In particular, the inequality

$$
\|f\|_{H^p_L} \leq C\|f\|_{\mathbb{H}^p_{L,M,at}}
$$

holds.

For the proof of Proposition 3.10, we need the following technical lemma whose proof is completely analogous to Lemma [HM, 4.1].

Lemma 3.11. *Let T be a L^2 bounded linear (or nonnegative sublinear) operator. Then the following holds:*

(i) Assume that for every $(2,p,M)$ atom a, $M \in \mathbb{N}_+$ and $0 < p \leq 1$,

$$
\|Ta\|_{L^p} \leq C.
$$

Then T extends to a bounded operator from $H^p_{L,M,at}$ to L^p.

(ii) Assume that for every $(2,p,M)$ atom a, $M > D(1/p-1/2)$ and $0 < p \leq 1$,

$$\|Ta\|_{\mathbb{H}^p_{L,M,at}} \leq C.$$

Then T extends to a bounded operator on $H^p_{L,M,at}$.

Proof of Proposition 3.10. By Lemma 3.11, it suffices to show that there exists a constant C so that for any $(2,p,M)$ atom $a = L^M b$ associated to the ball B, we have

$$\|S_L a\|_{L^p} \leq C.$$

We write

$$\|S_L a\|_{L^p} \leq C(\|S_L a\|_{L^p(8B)} + \|S_L a\|_{L^p(\Gamma\backslash\widetilde{B})}) := I + II,$$

where $\widetilde{B} = 8B$.

For the term I, using Hölder's inequality and the L^2-boundedness of S_L we have

$$\|S_L a\|_{L^p(\widetilde{B})} \leq C\|S_L a\|_{L^2(\widetilde{B})} V(B)^{1/p-1/2} \leq C\|a\|_{L^2} V(B)^{1/p-1/2} \leq C.$$

To estimate the term II, we write, for $x \in \Gamma\backslash\widetilde{B}$ which is outside the support of a,

$$\begin{aligned}
(S_L a(x)) = &\Big(\sum_{d(y,x)<1} \frac{|(I-P)a(y)|^2}{V(y,1)}\mu(y)\Big)^{1/2} \\
&+ \Big(\sum_{d(y,x)<k} \sum_{k=2}^{d(x_B,x)/4} \frac{|k(I-P)P^{[\frac{k}{2}]}a(y)|^2}{kV(y,k)}\mu(y)\Big)^{1/2} \\
&+ \Big(\sum_{d(y,x)<k} \sum_{k=d(x_B,x)/4}^{\infty} \frac{|k(I-P)^{M+1}P^{[\frac{k}{2}]}b(y)|^2}{kV(y,k)}\mu(y)\Big)^{1/2} \\
&:= II_1(x) + II_2(x) + II_3(x).
\end{aligned}$$

Therefore,

$$\|S_L a\|_{L^p(\Gamma\backslash\widetilde{B})} \leq \|II_1(\cdot)\|_{L^p(\Gamma\backslash\widetilde{B})} + \|II_2(\cdot)\|_{L^p(\Gamma\backslash\widetilde{B})} + \|II_3(\cdot)\|_{L^p(\Gamma\backslash\widetilde{B})}.$$

It is easy to see that, for $x \in \Gamma\backslash\widetilde{B}$,

$$II_1(x) \leq C|(I-P)a(x)| = |Pa(x)|.$$

Since the kernel $p(x,y)$ satisfies (UE) and $r_B \geq 1$, we have, for any $\kappa > 0$,

$$\begin{aligned}
\|II_1(\cdot)\|^p_{L^p(\Gamma\setminus\widetilde{B})} &= \|Pa\|^p_{L^p(\Gamma\setminus\widetilde{B})} \\
&\leq \sum_{j>3} \|Pa\|^p_{L^p(2^jB)} \leq C\sum_{j>3} \|Pa\|^p_{L^2(S_j(B))} V(2^jB)^{1-p/2} \\
&\leq C\sum_{j>3} \Big[\sum_{x\in S_j(B)} \Big|\sum_{y\in B} \frac{1}{V(x,1)} e^{-d(x,y)^2} a(y)\mu(y)\Big|^2 \mu(x)\Big]^{p/2} \\
&\quad \times V(2^jB)^{1-p/2} \\
&\leq C\sum_{j>3} \Big[\sum_{x\in S_j(B)} \frac{1}{V(2^jB)^2}(2^jr_B)^{-\kappa}\|a\|^2_{L^1}\mu(x)\Big]^{p/2} V(2^jB)^{1-p/2} \\
&\leq C\sum_{j>3} 2^{-j\kappa}V(2^jB)^{1-p}\|a\|^p_{L^2}V(B)^{p/2} \\
&\leq C\sum_{j>3} 2^{-j(\kappa-(1-p)D)} \leq C;
\end{aligned}$$

as long as $\kappa > D(1-p)$.

For $II_2(x)$, using (3), we write

$$(II_2(x))^2 \leq C\sum_{d(y,x)<k} \sum_{k=2}^{d(x_B,x)/4} \Big|\sum_{z\in B} \frac{1}{V(z,\sqrt{k})} \exp\Big(-c\frac{d(y,z)^2}{k}\Big)a(z)\Big|^2 \frac{\mu(y)}{kV(y,k)}.$$

It can be verified that

$$\frac{1}{V(z,\sqrt{k})} \leq \frac{C}{V(x_B,d(x,x_B))}\Big(\frac{d(x_B,x)}{\sqrt{k}}\Big)^D$$

and $d(y,z) \approx d(x_B,x)$. Therefore, for any $\kappa \in \mathbb{N}_+$, we have

$$\begin{aligned}
&(II_2(x))^2 \\
&\quad\leq C\sum_{d(y,x)<k} \sum_{k=2}^{d(x_B,x)/4} \Big|\sum_{z\in B} \frac{1}{V(x_B,d(x,x_B))}\Big(\frac{d(x_B,x)}{\sqrt{k}}\Big)^D \\
&\quad\times \exp\Big(-c\frac{d(x_B,x)^2}{k}\Big)a(z)\Big|^2 \frac{\mu(y)}{kV(y,k)} \\
&\quad\leq C\sum_{d(y,x)<k} \sum_{k=2}^{d(x_B,x)/4} \Big|\sum_{z\in B} \frac{1}{V(x_B,d(x,x_B))} \exp\Big(-c'\frac{d(x_B,x)^2}{k}\Big)a(z)\Big|^2 \frac{\mu(y)}{kV(y,k)} \\
&\quad\leq C\sum_{k=2}^{d(x_B,x)/4} \frac{1}{V(x_B,d(x,x_B))^2}\frac{k^{\kappa-1}}{d(x_B,x)^{2\kappa}}\|a\|^2_{L^1} \\
&\quad leq \frac{C}{V(x_B,d(x,x_B))^2}\frac{1}{d(x_B,x)^{\kappa}}\|a\|^2_{L^2}V(B) \\
&\quad\leq C\frac{V(B)^{2-2/p}}{V(x_B,d(x,x_B))^2}\Big(\frac{r_B}{d(x_B,x)}\Big)^{\kappa}.
\end{aligned}$$

Hence, $\|II_2(\cdot)\|_{L^p(\Gamma\setminus\widetilde{B})} \leq C$.

For the last term $II_3(x)$, we write

$$(II_3(x))^2 \le C \sum_{d(y,x)<k} \sum_{k=d(x_B,x)/4}^{\infty} \frac{|k^{M+1}(I-P)^{M+1}P^{[\frac{k}{2}]}b(y)|^2}{k^{2M+1}V(y,k)}\mu(y). \tag{11}$$

For $j \ge 4$, since the kernel of $k^{M+1}(I-P)^{M+1}P^{[\frac{k}{2}]}$ satisfies the Gaussian estimate (3), we have

$$\begin{aligned}
\|II_3(\cdot)\|^2_{L^2(S_j(B))} &\le C \sum_{y\in\Gamma} \sum_{x:d(y,x)<k} \frac{1}{V(x,k)} \sum_{k\ge 2^{j-3}r_B}^{\infty} \\
&\quad \times \frac{|k^{M+1}(I-P)^{M+1}P^{[\frac{k}{2}]}b(y)|^2}{k^{2M+1}}\mu(x)\mu(y) \\
&\le C \sum_{k\ge 2^{j-3}r_B}^{\infty} \frac{\|k^{M+1}(I-P)^{M+1}P^{[\frac{k}{2}]}b\|^2_{L^2}}{k^{2M+1}} \\
&\le C \sum_{k\ge 2^{j-3}r_B}^{\infty} \frac{\|b(y)\|^2_{L^2}}{k^{2M+1}} \le C2^{-2Mj}V(B)^{1-2/p} \\
&\le C2^{-2j(M-D(1/p-1/2))}V(2^jB)^{1-2/p}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\|II_3(\cdot)\|^p_{L^p(\Gamma\setminus\widetilde{B})} &\le \sum_{j\ge 4} \|A_3(\cdot)\|^p_{L^p(S_j(B))} \\
&\le C\sum_{j\ge 4} \|A_3(\cdot)\|^p_{L^2(S_j(B))} V(2^jB)^{1-p/2} \\
&\le C\sum_{j\ge 4} 2^{-jp(M-D(1/p-1/2))}V(2^jB)^{p/2-1}V(2^jB)^{1-p/2} \le C,
\end{aligned}$$

provided $M > D(1/p-1/2)$.

This completes our proof. □

Similarly, we also obtain that each $(2,p,M,\epsilon)$ molecule belongs to the Hardy spaces H^p_L for $0 < p \le 1$.

Proposition 3.12. *Let $0 < p \le 1$, $M > D(1/p-1/2)$ and $\epsilon > 0$. There exists $C > 0$ so that for any $(2,p,M,\epsilon)$ molecule m, we have $\|m\|_{H^p_L} \le C$.*

The proof of Proposition 3.12 is analogous to that of Proposition 3.10 with minor modifications. Indeed, we decompose $m = \sum_{j\ge 0} m\chi_{S_j(B)}$ and then repeat the argument in Proposition 3.10 for each $m\chi_{S_j(B)}$. Hence we omit the details here.

As a consequence of Propositions 3.10 and 3.12, we obtain the following result.

Corollary 3.13. *Let $0 < p \le 1$ and $M > n(1/p-1/2)$. Then the spaces H^p_L and $H^p_{L,M,at}$ coincide and their norms are equivalent.*

Let T^2_c be the set of all $f \in T^2$ with bounded support in $\Gamma \times \mathbb{N}_+$. For $M \in \mathbb{N}_+$, we define the operator $\pi_{L,M}$ by setting

$$\pi_{L,M}f(x) = \sum_{k=1}^{\infty} \frac{c_{k-1,M+1}(I-P)^M P^{k-1-[\frac{k}{2}]}f(x,k)}{k}$$

for $f \in T^2_c$.

For the boundedness of π_L, we have the following result.

Proposition 3.14. *The operator π_L, initially defined on T^2_c, extends to a bounded linear operator from*
(i) T^2 into L^2;
(ii) T^p into H^p_L for $0 < p \leq 1$.

Proof. (i) Let $g \in L^2$. Then we have by using Proposition 3.2

$$\begin{aligned}\sum_x \pi_{L,M} f(x) g(x)\mu(x) &= \sum_x \sum_{k=1}^{\infty} f(x,k)\frac{c_{k-1,M+1}(I-P)^M P^{k-1-[\frac{k}{2}]}g(x)}{k}\mu(x)\\ &\leq C\sum_x \mathcal{A}(f)(x)\mathcal{A}(G)(x)\mu(x)\\ &\leq C\|\mathcal{A}(f)\|_{L^2}\|\mathcal{A}(G)\|_{L^2} = C\|f\|_{T^2}\|\mathcal{A}(G)\|_{L^2},\end{aligned}$$

where $G(x,k) = c_{k-1,M+1}(I-P)^M P^{k-1-[\frac{k}{2}]}g(x)$ for all $(x,k) \in \Gamma \times \mathbb{N}_+$.

It can be verified that

$$\|\mathcal{A}(G)\|_{L^2} \leq C(\|g\|_{L^2} + \|S_{h,M}g\|_{L^2}) \leq C\|g\|_{L^2}.$$

Hence,

$$\sum_x \pi_{L,M} f(x)g(x)\mu(x) \leq C\|f\|_{T^2}\|g\|_{L^2}.$$

This completes the proof of (i).

(ii) Since $\pi_{L,M}$ is bounded on L^2, it is sufficient to show that for each T^p atom a associated to $\widehat{B}$, one has $\pi_{L,M}(a) \in H^p_L$.

Indeed, we write

$$\pi_{L,M} a(x) = \sum_{k=1}^{\infty} \frac{c_{k-1,M+1}(I-P)^M P^{k-1-[\frac{k}{2}]}a(x,k)}{k} = L^M b,$$

where

$$b = \sum_{k=1}^{\infty} \frac{c_{k-1,M+1}P^{k-1-[\frac{k}{2}]}a(x,k)}{k}.$$

Therefore, by an argument similar to that in the proof of Proposition 3.9, we obtain that $\pi_{L,M} a$ is a $(2,p,M)$ atom associated to the ball $B = (x_B, Mx_B)$, and hence by Proposition 3.10, we conclude that $\pi_L a \in H^p_L$. This finishes our proof. □

4. The space of functions of Lipschitz type associated to operators

4.1. **Definition of $\mathfrak{L}_L(\alpha, M, \Gamma)$ spaces.** By adapting the idea in [DY1], we define the class of functions that the operators P^n act on.

We say that the function $f \in L^2_{\rm loc}$ is in $\mathcal{M}(x_0,\beta)$ with $\beta > 0$ and $x_0 \in \Gamma$ if it satisfies

$$\|f\|_{\mathcal{M}(x_0,\beta)} := \Big(\sum_{x\in\Gamma} \frac{|f(x)|^2}{(1+d(x,x_0))^\beta V(x_0, 1+d(x,x_0))}\mu(x)\Big)^{1/2} < \infty.$$

It can be verified that $\mathcal{M}(x_0,\beta) = \mathcal{M}(x_1,\beta)$ with equivalent norms for any $x_0, x_1 \in \Gamma$. We set

$$\mathcal{M} = \bigcup_{x_0\in\Gamma}\bigcup_{\beta>0}\mathcal{M}(x_0,\beta).$$

Remark 4.1. By Hölder's inequality, if $f \in \mathcal{M}$, then there exists $\beta > 0$ so that for any x_0

$$\|f\|_{\mathcal{M}(x_0,\beta)} := \sum_{x\in\Gamma} \frac{|f(x)|}{(1+d(x,x_0))^\beta V(x_0, 1+d(x,x_0))}\mu(x) < \infty.$$

The Lipschitz spaces associated to operators satisfying the Gaussian estimates was introduced in [DY1] (see also [T, DL]). The following Lipschitz spaces can be considered to be the discrete version of those in [DY1, DL, T].

Definition 4.2. Let $\alpha > 0$ and $M \in \mathbb{N}_+$. We say that $f \in L^2(\Gamma)$ is in $\mathfrak{L}_L(\alpha, M, \Gamma)$, if there exists a constant $c > 0$ so that for any ball B,

$$\frac{1}{V(B)^{1+\alpha}} \sum_{x\in B} |(I - P^{[r_B^2]})^M f(x)|\mu(x) \le c. \tag{12}$$

The smallest c in (12) is taken to be the norm of f and denoted by $\|f\|_{\mathfrak{L}_L(\alpha,\Gamma)}$.

We would like to state an important result concerning the spaces $\mathfrak{L}_L(\alpha, M, \Gamma)$.

Proposition 4.3. *Let $f \in L^2(\Gamma)$ and $p \in [1, \infty)$. Then $f \in \mathfrak{L}_L(\alpha, M, \Gamma)$ if and only if*

$$\frac{1}{V(B)^\alpha}\Big(\frac{1}{V(B)} \sum_{x\in B} |(I - P^{[r_B^2]})^M f(x)|^p\mu(x)\Big)^{1/p} \le c.$$

Moreover, the smallest constant c in (4.3) *is comparable to* $\|f\|_{\mathfrak{L}_L(\alpha,M,\Gamma)}$.

The proof of Proposition 4.3 is similar to that of Theorem 3.4 in [DY1], and we omit the details here.

The following simple inequality will be used in the sequel.

Let $0 < p \le 1$ and $k \ge 1$. If $f \in \mathfrak{L}_L(\alpha, M, \Gamma)$, then for each $j \ge 0$ and all balls B, we have

$$\Big(\sum_{x\in kB} |(I - P^{[r_B^2]})^M f(x)|^p\mu(x)\Big)^{1/p} \le V(kB)^{\alpha+1/p}\|f\|_{\mathfrak{L}_L(\alpha,M,\Gamma)}. \tag{13}$$

Proof. First, we have $kB = \bigcup_{x\in kB} B(x, r_B)$. By Vitali's covering lemma, we can pick a subfamily of balls $\{B(x_i, r_B)\}_{i\in I}$ so that $kB = \bigcup_{i\in I} B(x_i, r_B)$ and $B(x_i, r_B/3) \cap B(x_j, r_B/3) = \emptyset$ for all $i \ne j$ and $i, j \in I$. Therefore,

$$\begin{aligned}
\sum_{x\in kB} |(I - P^{[r_B^2]})^M f(x)|^p\mu(x) &\le \sum_{i\in I}\sum_{x\in B(x_i,r_B)} |(I - P^{[r_B^2]})^M f(x)|^p\mu(x)\\
&\le \sum_{i\in I} V(x_i, r_B)^{1+\alpha p}\|f\|_{\mathfrak{L}_L(\alpha,M,\Gamma)}\\
&\le C\Big(\sum_{i\in I} V(x_i, r_B/3)\Big)^{1+\alpha p}\|f\|_{\mathfrak{L}_L(\alpha,M,\Gamma)}
\end{aligned}$$

in which the last inequality follows from the doubling property.

Since $x_i \in kB$, $B(x_i, r_B/3) \subset 2kB$, one has

$$\begin{aligned}
\sum_{x\in kB} |(I - P^{[r_B^2]})^M f(x)|^p\mu(x) &\le CV(2kB)^{1+\alpha p}\|f\|_{\mathfrak{L}_L(\alpha,M,\Gamma)}\\
&\approx V(kB)^{1+\alpha p}\|f\|_{\mathfrak{L}_L(\alpha,M,\Gamma)}.
\end{aligned}$$

This completes our proof. $\square$

Now we consider the case when $\alpha = 0$. In this case, the Lipschitz space $\mathfrak{L}_L(\alpha, M, \Gamma)$ becomes $\mathrm{BMO}_{L,M}(\Gamma)$, the generalized BMO space associated to the operator L on Γ. It is interesting to note that the new space $\mathrm{BMO}_{L,M}(\Gamma)$ is a good substitution to the space $L^\infty(\Gamma)$ in the role of end-point interpolation with $L^p(\Gamma)$ spaces. This makes the space $\mathrm{BMO}_{L,M}(\Gamma)$ have a useful role in the study of boundedness of singular integrals associated to the operator L.

The following result is Theorem 5.2 of [DY1].

Theorem 4.4. *Let T be a linear operator or a nonnegative sublinear operator on Γ. If T is bounded on $L^p(\Gamma)$ for $1 < p < \infty$ and from L^∞ into $BMO_{L,M}(\Gamma)$ for any $M \in \mathbb{N}_+$, then T is bounded on L^q for all $p < q < \infty$.*

We note that the interpolation between the Hardy spaces $H^1_L(\Gamma)$ and $L^p(\Gamma)$ will be proved later in Theorem 5.3 of Section 5.

4.2. **The spaces $\mathfrak{L}_L(\alpha, M, \Gamma)$ and Carleson measures.** Similar to the classical BMO spaces, the spaces $\mathfrak{L}_L(\alpha, M, \Gamma)$ are related to Carleson measures. The following result will play an important role in the proof that the dual spaces of the Hardy spaces $H^p_L(\Gamma)$ are the Lipschitz spaces $\mathfrak{L}_L(\alpha, M, \Gamma)$.

Proposition 4.5. *Let $f \in \mathfrak{L}_L(\alpha, M, \Gamma)$. Then we have*

(14)
$$\Big(\sum_{(x,k)\in\widehat{B}} \frac{|c_{k-1,2M+1}(I-P)^{2M}P^{k-1-[\frac{k}{2}]}f(x)|^2}{k}\mu(x)\Big)^{1/2} \leq CV(B^{\alpha+1/2})\|f\|_{\mathfrak{L}_L(\alpha,M,\Gamma)}$$

for all balls $B \subset \Gamma$.

Proof of Proposition 4.5. We write

$$\begin{aligned}
&\Big(\sum_{(x,k)\in\widehat{B}} \frac{|c_{k-1,2M+1}(I-P)^{2M}P^{k-1-[\frac{k}{2}]}f(x)|^2}{k}\mu(x)\Big)^{1/2}\\
&\quad\leq \Big(\sum_{(x,k)\in\widehat{B}} \frac{|c_{k-1,2M+1}(I-P)^{2M}P^{k-1-[\frac{k}{2}]}(I-P^{[r_B^2]})^M f(x)|^2}{k}\mu(x)\Big)^{1/2}\\
&\quad\quad+ \Big(\sum_{(x,k)\in\widehat{B}} \frac{|c_{k-1,2M+1}(I-P)^{2M}P^{k-1-[\frac{k}{2}]}[I-(I-P^{[r_B^2]})^M]f(x)|^2}{k}\mu(x)\Big)^{1/2}\\
&\quad := I + II.
\end{aligned}$$

We break the term I into

$$\begin{aligned}
I^2 &= \sum_{x\in B(x_B, r_B-k)} \sum_{k=1,2} \frac{|c_{k-1,2M+1}(I-P)^{2M}P^{k-1-[\frac{k}{2}]}(I-P^{[r_B^2]})^M f(x)|^2}{k}\mu(x)\\
&\quad+ \sum_{x\in B(x_B, r_B-k)} \sum_{k\geq 3} \frac{|c_{k-1,2M+1}(I-P)^{2M}P^{k-1-[\frac{k}{2}]}(I-P^{[r_B^2]})^M f(x)|^2}{k}\mu(x)\\
&= I_1 + I_2, \text{ respectively.}
\end{aligned}$$

To estimate I_1, we write

$$
\begin{aligned}
I_1 &\le C \sum_{x\in B(x_B,r_B-k)} \sum_{k=1,2} |c_{k-1,2M+1}(I-P)^{2M}(I-P^{[r_B^2]})^M f(x)|^2\mu(x)\\
&\le C\sum_{x\in B}|(I-P^{[r_B^2]})^M f(x)|^2\mu(x) + C\sum_{x\in B}\sum_{k=1}^{2M} c_k|P^k(I-P^{[r_B^2]})^M f(x)|^2\mu(x)\\
&:= I_{11}+I_{12}.
\end{aligned}
$$

Using Proposition 4.3 and (13), we have

$$
I_{11}\le cV(B)^{1+2\alpha}\|f\|^2_{\mathfrak{L}_L(\alpha,M,\Gamma)}.
$$

For the term I_{12}, we decompose $(I-P^{[r_B^2]})^M f = g_0+\sum_{j=3}^{\infty} g_j$, where $g_0=(I-P^{[r_B^2]})^M(f)\chi_{4B}$ and $g_j=(I-P^{[r_B^2]})^M(f)\chi_{S_j(B)}$ for $j\ge 3$. Then we have, for $1\le k\le M-1$,

$$
\Big(\sum_{x\in B}|P^k(I-P^{[r_B^2]})^M f|^2\mu(x)\Big)^{1/2}\le \|P^k g_0\|_{L^2(B)}+\sum_{j=3}^{\infty}\|P^k g_j\|_{L^2(B)}.
$$

The L^2-boundedness of P^k implies that

$$
\|P^k g_0\|_{L^2(B)}\le \|g_0\|_{L^2} = \|(I-P^{[r_B^2]})^M f\|_{L^2(4B)}\le cV(B)^{\alpha+1/2}\|f\|_{\mathfrak{L}_L(\alpha,M,\Gamma)}.
$$

Since the associated kernel $p_k(x,y)$ to P satisfies the Gaussian estimate (UE), for each $j\ge 3$, the following inequality holds:

$$
\|P^k g_j\|_{L^2(B)}\le Ce^{-c2^{2j}}\|(I-P^{[r_B^2]})^M f\|_{L^2(S_j(B))}.
$$

This together with Proposition 4.3 gives

$$
\begin{aligned}
\sum_{j=3}^{\infty}\|P^k g_j\|_{L^2(B)} &\le C\sum_{j\ge 3} e^{-c2^{2j}}2^{jD\alpha}V(B)^{\alpha+1/2}\|f\|_{\mathfrak{L}_L(\alpha,M,\Gamma)}\\
&\le CV(B)^{\alpha+1/2}\|f\|_{\mathfrak{L}_L(\alpha,M,\Gamma)}.
\end{aligned}
$$

It then follows that $I_1\le CV(B)^{1+2\alpha}\|f\|^2_{\mathfrak{L}_L(\alpha,M,\Gamma)}$.

For the term I_2, one has

$$
\begin{aligned}
&\Big(\sum_{x\in B(x_B,r_B-k)}\sum_{k\ge 3}\frac{|c_{k-1,2M+1}(I-P)^{2M}P^{k-1-[\frac{k}{2}]}(I-P^{[r_B^2]})^M f(x)|^2}{k}\mu(x)\Big)^{1/2}\\
&\quad\le \Big(\sum_{x\in B(x_B,r_B-k)}\sum_{k\ge 3}\frac{|c_{k-1,2M+1}(I-P)^{2M}P^{k-1-[\frac{k}{2}]}g_0|^2}{k}\mu(x)\Big)^{1/2}\\
&\quad+\sum_{j\ge 3}\Big(\sum_{x\in B(x_B,r_B-k)}\sum_{k\ge 3}\frac{|c_{k-1,2M+1}(I-P)^{2M}P^{k-1-[\frac{k}{2}]}g_j(x)|^2}{k}\mu(x)\Big)^{1/2}.
\end{aligned}
$$

We have

$$\begin{aligned}
&\Big(\sum_{x\in B(x_B,r_B-k)}\sum_{k\geq 3}\frac{|c_{k-1,2M+1}(I-P)^{2M}P^{k-1-[\frac{k}{2}]}g_0(x)|^2}{k}\mu(x)\Big)^{1/2}\\
&\quad\leq \Big(\sum_{x\in\Gamma}\sum_{k=1}^{\infty}\frac{|c_{k-1,2M+1}(I-P)^{2M}P^{k-1-[\frac{k}{2}]}g_0|^2}{k}\mu(x)\Big)^{1/2}\\
&\quad\leq C(\|g_0\|_{L^2}+\|\mathcal{G}_{L,2M}g_0\|_{L^2})\leq \|g_0\|_{L^2}=C\|(I-P^{[r_B^2]})^M f\|_{L^2(4B)}\\
&\quad\leq cV(B)^{\alpha+1/2}\|f\|_{\mathfrak{L}_L(\alpha,M,\Gamma)}.
\end{aligned}$$

For each $j\geq 3$ and $x\in B$, by (3), Proposition 4.3 and (13),

$$\begin{aligned}
&|c_{k-1,2M+1}(I-P)^{2M}P^{k-1-[\frac{k}{2}]}g_j(x)|\\
&\quad\leq \sum_{y\in S_j(B)}C\frac{c_{k-1,2M+1}}{k^{2M}V(y,d(x,y))}\exp\Big(-c\frac{d(x,y)^2}{k}\Big)|g_j(y)|\mu(y)\\
&\quad\leq \frac{C}{V(2^jB)}\exp\Big(-c\frac{2^{2j}r_B^2}{r_B}\Big)\sum_{y\in S_j(B)}|(I-P^{[r_B^2]})^M f(y)|\mu(y)\\
&\quad\leq \frac{C}{V(2^jB)}\exp\Big(-c\frac{2^{2j}r_B^2}{r_B}\Big)V(2^jB)^{1+\alpha}\|f\|_{\mathfrak{L}_L(\alpha,M,\Gamma)}\\
&\quad\leq Ce^{-c2^{2j}r_B}V(2^jB)^{\alpha}\|f\|_{\mathfrak{L}_L(\alpha,M,\Gamma)}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\sum_{j\geq 3}\Big(\sum_{x\in B(x_B,r_B-k)}\sum_{k\geq 3}\frac{|c_{k-1,2M+1}(I-P)^{2M}P^{k-1-[\frac{k}{2}]}g_j(x)|^2}{k}\mu(x)\Big)^{1/2}\\
&\quad\leq C\sum_{j\geq 3}\Big(\sum_{k=3}^{r_B}\sum_{x\in B}\Big|e^{-c2^{2j}r_B}2^{jD\alpha}V(B)^{\alpha}\|f\|_{\mathfrak{L}_L(\alpha,M,\Gamma)}\Big|^2\mu(x)\Big)^{1/2}\\
&\quad\leq CV(B)^{\alpha+1/2}\|f\|_{\mathfrak{L}_L(\alpha,M,\Gamma)}.
\end{aligned}$$

Hence, $I\leq CV(B)^{\alpha+1/2}\|f\|_{\mathfrak{L}_L(\alpha,M,\Gamma)}$.

It remains to estimate II. In this case $r_B\geq 3$. We write

$$(I-P)^{2M}P^{k-1-[\frac{k}{2}]}(I-(I-P^{[r_B^2]})^M)=\sum_{l=1}^{M}c_l(I-P)^{2M}P^{k-1-[\frac{k}{2}]+l[r_B^2]}(I-P)^M.$$

Note that $k-1-[\frac{k}{2}]+l[r_B^2]\approx r_B^2$. Applying (3) and (13), we obtain that, for any $\kappa>0$,

$$\begin{aligned}
&|c_{k-1,2M+1}(I-P)^{2M}P^{k-1-[\frac{k}{2}]}(I-(I-P^{[r_B^2]})^M)f(x)|\\
&\qquad\leq C\frac{c_{k-1,2M+1}}{r_B^{2M}V(B)}\sum_{y\in\Gamma}\exp\Big(-c\frac{d(x,y)^2}{r_B^2}\Big)(I-P)^M f(y)\mu(y).
\end{aligned}$$

We then decompose Γ into the annuli associated to the ball $B(x,r_B)$ and have

$$\begin{aligned}
&|c_{k-1,2M+1}(I-P)^{2M}P^{k-1-[\frac{k}{2}]}(I-(I-P^{[r_B^2]})^M f(x)| \\
&\quad\leq C\frac{c_{k-1,2M+1}}{r_B^{2M}V(B)}\sum_{j\geq 0}\sum_{y\in S_j(B(x,r_B))}\exp\Big(-c\frac{d(x,y)^2}{r_B^2}\Big)(I-P)^M f(y)\mu(y) \\
&\quad\leq C\frac{c_{k-1,2M+1}}{r_B^{2M}V(B)}\sum_{j\geq 0}2^{-j\kappa}V(2^jB)^{1+\alpha}\|f\|_{\mathfrak{L}_L(\alpha,M,\Gamma)} \\
&\quad\leq C\frac{c_{k-1,2M+1}}{r_B^{2M}V(B)}\sum_{j\geq 0}2^{-j\kappa}V(2^jB)^{1+\alpha}\|f\|_{\mathfrak{L}_L(\alpha,M,\Gamma)} \\
&\quad\leq C\frac{k^{2M}}{r_B^{2M}}V(2^jB)^{\alpha}\|f\|_{\mathfrak{L}_L(\alpha,M,\Gamma)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
II &\leq C\Big(\sum_{x\in B}\sum_{k=1}^{r_B}\frac{k^3}{r_B^{4M}}\mu(x)\Big)^{1/2}V(B)^{\alpha}\|f\|_{\mathfrak{L}_L(\alpha,M,\Gamma)} \\
&\leq CV(B)^{\alpha+1/2}\|f\|_{\mathfrak{L}_L(\alpha,M,\Gamma)}.
\end{aligned}$$

This completes our proof. □

5. Duality of Hardy spaces

Let $f\in\mathcal{M}$ satisfy (14) and $g\in H_L^p$ which can be represented as a finite linear combination of $(2,p,M)$ atoms. We set

(15)
$$F(x,k)=c_{k-1,2M+1}(I-P)^{2M}P^{k-1-[\frac{k}{2}]}f(x)\quad\text{and}\quad G(x,k)=k(I-P)P^{[\frac{k}{2}]}g(x)$$

for all $(x,k)\in\Gamma\times\mathbb{N}_+$.

The following proposition will play an important role in the sequel.

Proposition 5.1. *Let F and G be defined as in* (15)*. Then the following holds:*

$$\sum_{x\in\Gamma}f(x)g(x)\mu(x)=\frac{1}{2}\sum_{x\in\Gamma}\sum_{k=1}^{\infty}\frac{F(x,k)G(x,k)}{k}\mu(x) \tag{16}$$

with $\alpha=1-1/p, 0<p\leq 1$.

Proof. We can assume that $g=a$ is a $(2,p,M)$ atom associated to some ball B with $r_B\geq 1$.

We have

$$\begin{aligned}
&\sum_{x\in\Gamma}\sum_{k=1}^{\infty}\frac{F(x,k)G(x,k)}{k}\mu(x) \\
&\qquad=\lim_{N\to\infty}\sum_{x\in\Gamma}\sum_{k=1}^{N}\frac{F(x,k)G(x,k)}{k}\mu(x) \\
&\qquad=\lim_{N\to\infty}\sum_{x\in\Gamma}\sum_{k=1}^{N}f(x)[c_{k-1,2M+1}(I-P)^{2M+1}P^{k-1}a(x)]\mu(x).
\end{aligned}$$

We now break $f_1 = f\chi_{B(x_B,16Mr_B)}$, and set $\widetilde{B} = B(x_B, 16Mr_B) f_2 = f - f_1$ to obtain

$$
\begin{aligned}
&\sum_{x\in\Gamma}\sum_{k=1}^{\infty}\frac{F(x,k)G(x,k)}{k}\mu(x)\\
&\quad = \lim_{N\to\infty}\sum_{x\in\Gamma}\sum_{k=1}^{N} f_1(x)[c_{k-1,2M+1}(I-P)^{2M+1}P^{k-1}a(x)]\mu(x)\\
&\quad + \lim_{N\to\infty}\sum_{x\in\Gamma}\sum_{k=1}^{N} f_2(x)[c_{k-1,2M+1}(I-P)^{2M+1}P^{k-1}a(x)]\mu(x)\\
&\quad := I_1 + I_2.
\end{aligned}
$$

Since $f \in \mathcal{M}$, $f_1 \in L^2$. By Theorem 3.7, we get that

$$I_1 = 2\sum_{x\in\Gamma} f_1(x)g(x)\mu(x).$$

For the term I_2, note that supp $a \subset B$, supp $(I-P)^{2M}P^{k-1}a(x) \subset \widetilde{B}$ for $k = 1, 2$. Hence,

$$
\begin{aligned}
I_2 &= \lim_{N\to\infty}\sum_{x\in\Gamma}\sum_{k=3}^{N} f_2(x)[c_{k-1,2M+1}(I-P)^{2M+1}P^{k-1}a(x)]\mu(x)\\
&= \lim_{N\to\infty}\sum_{x\in\Gamma}\sum_{k=3}^{N} f_2(x)[c_{k-1,2M+1}(I-P)^{2M+1}P^{k-1}b(x)]\mu(x).
\end{aligned}
$$

For $k \geq 3$, using (3), we have for $x \in \Gamma\backslash\widetilde{B}$,

$$
\begin{aligned}
&\Big|\sum_{k=3}^{N}[c_{k-1,2M+1}(I-P)^{2M+1}P^{k-1}a(x)]\Big|\\
&\qquad \leq \sum_{k=3}^{N}\sum_{y\in B}\frac{c_{k-1,2M+1}}{k^{2M+1}V(x,d(x,y))}\exp\Big(-c\frac{d(x,y)^2}{k}\Big)b(y)\mu(y).
\end{aligned}
$$

In this situation, we have $d(x,y) \approx d(x_B,x)$ and $V(x,d(x,y)) \approx V(x_B,d(x_B,x))$. Hence,

$$
\begin{aligned}
&\Big|\sum_{k=3}^{N}[c_{k-1,2M+1}(I-P)^{2M+1}P^{k-1}a(x)]\Big|\\
&\qquad \leq \sum_{k=3}^{N}\sum_{y\in B}\frac{C}{kV(x_B,d(x_B,x))}\exp\Big(-c\frac{d(x_B,x)^2}{k}\Big)b(y)\mu(y)\\
&\qquad \leq \sum_{k=3}^{N}\frac{C}{kV(x_B,d(x_B,x))}\Big(\frac{k}{d(x_B,x)^2}\Big)^{\epsilon}\|b\|_{L^1}\\
&\qquad \leq \sum_{k=3}^{N}\frac{C}{V(x_B,d(x_B,x))d(x_B,x)^{2\epsilon}}\|b\|_{L^1}.
\end{aligned}
$$

This together with Remark 4.1 and Theorem 3.7 gives

$$I_2 = 2\sum_{x\in\Gamma} f_2(x)g(x)\mu(x).$$

This completes our proof. □

We are ready to state the main result of this section.

Theorem 5.2. *Let $0 < p \le 1$ and $M \ge \max\{D(1/p-1/2), 2\}$. Then the following statements hold:*

(i) For any $f \in \mathfrak{L}_L(\frac{1}{p}-1, M, \Gamma)$, the linear functional given by $\ell(g) = \langle f, g\rangle$, initially defined on the dense subspace of $H^p_{L,at,fin}$ consisting of finite linear combinations of $(2, p, M)$ atoms, has a unique bounded extension to H^p_L with

$$\|\ell\|_{(H^p_L)^*} \le C\|f\|_{\mathfrak{L}_L(\frac{1}{p}-1,M,\Gamma)}.$$

(ii) Conversely, if $\ell \in (H^p_L)^*$, then $\ell \in \mathfrak{L}_L(\frac{1}{p}-1, M, \Gamma)$ and

$$\|\ell\|_{\mathfrak{L}_L(\frac{1}{p}-1,M,\Gamma)} \le C\|\ell\|_{(H^p_L)^*}.$$

Proof. (i) Let $g \in H^p_{L,at,fin}$ and F and G be as in (15); then $g \in L^2$ and $G(\cdot,\cdot) \in T^p \cap T^2$. Hence, by Proposition 3.1, we can pick a sequence $\{\lambda_j\}_{j\in\mathbb{N}}$ and the sequence of T^p atoms $\{a_j\}_{j\in\mathbb{N}}$ associated to the family of cones $\{\widehat{B}_j\}_{j\in\mathbb{N}}$ so that (5) and (6) hold. This together with the proposition of (16) gives

$$\begin{aligned}
\langle f, g\rangle &= \frac{1}{2}\sum_{x\in\Gamma}\sum_{k=1}^{\infty}\frac{F(x,k)G(x,k)}{k}\mu(x)\\
&\le \frac{1}{2}\sum|\lambda_j|\sum_{x\in\Gamma}\sum_{k=1}^{\infty}\frac{F(x,k)a_j(x,k)}{k}\mu(x)\\
&\le \frac{1}{2}\sum|\lambda_j|\|a_j\|_{T^2}\Big(\sum_{\widehat{B}_j}^{\infty}\frac{|F(x,k)|^2}{k}\mu(x)\Big)^{1/2}.
\end{aligned}$$

Since a_j is a T^p atom, $\|a_j\|_{T^2} \le CV(B)^{1/2-1/p}$. Moreover, (14) gives

$$\Big(\sum_{\widehat{B}_j}^{\infty}\frac{|F(x,k)|^2}{k}\mu(x)\Big)^{1/2} \le CV(B)^{1/p-1/2}\|f\|_{\mathfrak{L}_L(\frac{1}{p}-1,M,\Gamma)}.$$

Hence

$$\begin{aligned}
\langle f, g\rangle &\le C\sum|\lambda_j|\|f\|_{\mathfrak{L}_L(\frac{1}{p}-1,M,\Gamma)}\\
&\le C(\sum|\lambda_j^p|)^{1/p}\|f\|_{\mathfrak{L}_L(\frac{1}{p}-1,M,\Gamma)}\\
&\le C\|G\|_{T^p}|f\|_{\mathfrak{L}_L(\frac{1}{p}-1,M,\Gamma)} \approx C\|g\|_{H^p_L}|f\|_{\mathfrak{L}_L(\frac{1}{p}-1,M,\Gamma)}.
\end{aligned}$$

This completes the proof of (i).

(ii) We adapt the argument in [DY1] to our present situation. Since $\pi_{L,M}$ is a bounded linear operator from T^p into H^p_L, for each $\ell \in (H^p_L)^*$, $\ell\circ\pi_{L,M}$ is a continuous linear operator on T^p. Therefore, by Proposition 3.2, there exists a function $z(\cdot,\cdot) \in T^{p,\infty}$ so that

$$(\ell\circ\pi_{L,M})(h)(x) = \sum_{x}\sum_{k=1}^{\infty}\frac{z(x,k)h(x,k)}{k}\mu(x).$$

Moreover, Theorem 3.7 tells us that for $g\in L^2\cap H^p_L$, we have

$$g(x)=\pi_{L,M}(Q_k(g))(x),\quad Q_k(g)(x):=k(I-P)P^{[\frac{k}{2}]}g(x) \text{ for } (x,k)\in\Gamma\times\mathbb{N}_+.$$

This in combination with the fact that $L^2\cap H^p_L$ is dense in H^p_L gives

$$\begin{aligned}\ell(g)&=\ell\circ\pi_{L,M}\circ Q_k(g)\\&=\sum_x\sum_{k=1}^\infty\frac{z(x,k)Q_kg(x)}{k}\mu(x)=\sum_x g(x)\Big(\sum_{k=1}^\infty\frac{Q_k(z(\cdot,k))(x)}{k}\Big)\mu(x)\\&=\sum_x f(x)g(x)\mu(x),\end{aligned}$$

where $f(x)=\sum_{k=1}^\infty\frac{Q_k(z(\cdot,k))(x)}{k}$.

To complete the proof, we need to show that $f\in\mathfrak{L}_L(\frac{1}{p}-1,M,\Gamma)$. To do this, let $\varphi\in L^2(B)$ of the norm 1 for some ball B with $r_B\geq 1$. We will show that

$$m=V(B)^{1/2-1/p}(I-P^{[r_B^2]})^M\varphi$$

is a multiple of a $(2,p,M,\epsilon)$ molecule for some $\epsilon>0$.

Indeed, we can write $m=L^Mb$, where

$$b=V(B)^{1/2-1/p}(I+P+\ldots+P^{[r_B^2]-1})^M\varphi=\sum_{k=0}^{Mr_B^2}a_kP^k\varphi.$$

Now for each $0\leq N\leq M$ and $j\geq 2$, we have

$$\begin{aligned}\|L^Nb\|_{L^2(S_j(B))}&\leq C\sum_{k=0}^{Mr_B^2}\|L^NP^k\varphi\|_{L^2(S_j(B))}V(B)^{1/2-1/p}\\&\leq C\sum_{k=0}^{Mr_B^2}\frac{1}{k^N}\exp\Big(-c\frac{2^{2j}r_B^2}{k}\Big)\|\varphi\|_{L^2}V(B)^{1/2-1/p}\\&\leq C\sum_{k=0}^{Mr_B^2}\frac{1}{k^N}\Big(\frac{k}{2^{2j}r_B^2}\Big)^{n(1/p-1/2)/2+N+\epsilon/2}2^{jD(1/p-1/2)}V(2^jB)^{1/2-1/p}\\&\leq\frac{C}{r_B^{2N-2}}2^{-j\epsilon}V(2^jB)^{1/2-1/p}\leq r_B^{M-N}2^{-j\epsilon}V(2^jB)^{1/2-1/p}.\end{aligned}$$

For $j=0,1$, we have

$$\begin{aligned}\|L^Nb\|_{L^2(S_j(B))}&\leq C\sum_{k=0}^{Mr_B^2}\|L^NP^k\varphi\|_{L^2(S_j(B))}V(B)^{1/2-1/p}\\&\leq C\sum_{k=0}^{Mr_B^2}\frac{1}{k^N}\|\varphi\|_{L^2}V(B)^{1/2-1/p}\leq c2^{-j\epsilon}V(2^jB)^{1/2-1/p}.\end{aligned}$$

Hence, m is a multiple of a $(2, p, M, \epsilon)$-molecule. This implies $m \in H^p_L$. Therefore, we have

$$\begin{aligned}
&\frac{1}{V(B)^{1/p-1}}\Big(\frac{1}{V(B)}\sum_{x\in B}|(I-P^{[r_B]})f(x)|^2\mu(x)\Big)^{1/2}\\
&\quad= \sup_{\|\varphi\|_{L^2}=1}\frac{1}{V(B)^{1/p-1/2}}\langle (I-P^{[r_B]})f,\varphi\rangle\\
&\quad= \sup_{\|\varphi\|_{L^2}=1}\langle f, V(B)^{1/2-1/p}(I-P^{[r_B]})\varphi\rangle\\
&\quad\le \|f\|_{(H^p_L)^*}\|a\|_{H^p_L}\le C\|f\|_{(H^p_L)^*}.
\end{aligned}$$

This completes our proof. □

As a consequence of Theorem 5.2, the Lipschitz spaces $\mathfrak{L}_L(\alpha, M, \Gamma)$ coincide for all $M > \max(D(1/p-1/2), 2)$. Moreover, by combining Theorem 5.2 and Theorem 4.4, we obtain the following result.

Theorem 5.3. *Let T be a linear operator which is bounded on $L^p(\Gamma)$ for some $1 < p < \infty$ and from $H^1_L(\Gamma)$ into $L^1(\Gamma)$. Then T is bounded on L^q for all $1 < q < p$.*

6. Boundedness of some singular integrals on graphs

In this section, we will consider the boundedness of singular integrals on graphs such as square functions, spectral multipliers and the Riesz transforms. It will be shown that the Hardy spaces $H^p_L, 0 < p \le 1$, are the suitable spaces for the study of these singular integrals.

6.1. **Square functions.** In this section, we consider the square function defined by, for $M \ge 1$,

$$\mathcal{G}_{L,M}f(x) = \Big(\sum_{k=1}^{\infty}\frac{|k^M(I-P)^M P^k f(x)|^2}{k}\Big)^{1/2}.$$

In the particular case when $M = 1$, it was proved in [BR] that $\mathcal{G}_{L,M}$ is bounded on L^p for all $1 < p < \infty$. The aim of this section is to study the boundedness of $\mathcal{G}_{L,M}$ on the Hardy spaces H^p_L for $0 < p \le 1$.

Theorem 6.1. *The square function $\mathcal{G}_{L,M}$ is bounded on $L^2(\Gamma)$ and bounded from H^p_L to L^p for all $0 < p \le 1$.*

Proof. The L^2-boundedness of $\mathcal{G}_{L,M}$ can be obtained by using an argument similar to that of [BR, p.10]; hence we omit the details here.

To show $\mathcal{G}_{L,M}$ is bounded from H^p_L to L^p for all $0 < p \le 1$, due to Lemma 3.11, it suffices to show that there exists $C > 0$ so that

$$\|\mathcal{G}_{L,M}a\|_{L^p} \le C$$

for any atom $a = L^{\widetilde{M}}b$ associated to the ball B.

Indeed, we write

$$\begin{aligned}
\|\mathcal{G}_{L,M}a\|^p_{L^p} &\le \sum_{x\in 4B}|\mathcal{G}_{L,M}a(x)|^p\mu(x) + \sum_{j\ge 3}\Big(\sum_{x\in S_j(B)}|\mathcal{G}_{L,M}a(x)|^p\mu(x)\Big)\\
&= I_0 + \sum_{j\ge 3} I_j.
\end{aligned}$$

For the first term, using Hölder's inequality and the L^2-boundedness of $\mathcal{G}_{L,M}$ we have

$$I_0 \le \|\mathcal{G}_{L,M} a\|^p_{L^2(4B)} V(4B)^{1-p/2} \le c\|a\|^p_{L^2} V(B)^{1-p/2} \le C.$$

Similarly, Hölder's inequality also gives for $j \ge 3$

$$\begin{aligned} I_j &\le \|\mathcal{G}_{L,M} a\|^p_{L^2(S_j(B))} V(2^jB)^{1-p/2} \\ &\le \Big(\sum_{x\in S_j(B)} \sum_{k=1}^{\infty} \frac{|k^M(I-P)^M P^k a(x)|^2}{k} \mu(x)\Big)^{p/2} V(2^jB)^{1-p/2}. \end{aligned}$$

For $x \in S_j(B), j \ge 3$, we decompose

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{|k^M(I-P)^M P^k a(x)|^2}{k} &\le \sum_{k=1}^{r_B^2} \frac{|k^M(I-P)^M P^k a(x)|^2}{k} \\ &\quad + \sum_{k=r_B^2}^{\infty} \frac{|k^M(I-P)^M P^{\widetilde{M}+k} b(x)|^2}{k} \\ &= I_{j1} + I_{j2}. \end{aligned}$$

Using (3), we have, for $k \le r_B^2$,

$$\begin{aligned} |k^M(I-P)^M P^k a(x)| &\le \sum_{y\in B} \frac{C}{V(2^jB)} \exp\Big(-c\frac{2^{2j}r_B^2}{k}\Big) a(y)\mu(y) \\ &\le \frac{C}{V(2^jB)} \Big(\frac{k}{2^{2j}r_B^2}\Big)^{(1+D(1-1/p))/2} \|a\|_{L^1} \\ &\le \frac{C}{V(2^jB)} \Big(\frac{k}{2^{2j}r_B^2}\Big)^{(1+D(1-1/p))/2} V(B)^{1-1/p} \\ &\le C\Big(\frac{k}{2^{2j}r_B^2}\Big)^{(1+D(1-1/p))/2} 2^{jD(1-1/p)} V(2^jB)^{-1/p} \\ &\le C\frac{\sqrt{k}}{2^j r_B} V(2^jB)^{-1/p}. \end{aligned}$$

Hence

$$I_{j1} \le C2^{-2j} V(2^jB)^{-2/p}.$$

We now estimate I_{j2}. In this case, $k \ge r_B^2$. Using (3) again, we have

$$\begin{aligned} |k^M(I-P)^{M+\widetilde{M}} P^k b(x)| &\le \sum_{y\in B} \frac{C}{k^{\widetilde{M}} V(2^jB)} \exp\Big(-c\frac{2^{2j}r_B^2}{k}\Big) b(y)\mu(y) \\ &\le \frac{C}{k^{\widetilde{M}} V(2^jB)} \Big(\frac{k}{2^{2j}r_B^2}\Big)^{(1+D(1-1/p))/2} \|b\|_{L^1}. \end{aligned}$$

Since $\|b\|_{L^1} \le r_B^{\widetilde{M}} V(B)^{1-1/p} \le r_B^{\widetilde{M}} 2^{jD(1-1/p)} V(2^jB)^{1-1/p}$, we have

$$|k^M(I-P)^{M+\widetilde{M}} P^k b(x)| \le C\frac{2^{-j}}{k^{\widetilde{M}} V(2^jB)} \Big(\frac{k}{r_B^2}\Big)^{(1+D(1-1/p))/2} r_B^{\widetilde{M}} V(2^jB)^{1-1/p}.$$

If we choose $\widetilde{M} > (1+D(1-1/p))/2 + 1$, then we have

$$I_{j2} \le C2^{-2j} V(2^jB)^{-2/p}.$$

From the estimates of I_{j1} and I_{j2}, we obtain

$$I_j \leq C2^{-jp}V(2^j(B))^{p/2-1}V(2^jB)^{1-p/2} = C2^{-jp}.$$

Hence,

$$\|\mathcal{G}_{L,M}a\|_{L^p} \leq C$$

for any $(2, p, \widetilde{M})$ atom $a = L^{\widetilde{M}}b$. This completes our proof. □

6.2. **Spectral multipliers of the discrete Laplacian.** Since $\|L\|_{L^2\to L^2} = \|I - P\|_{L^2\to L^2} \leq 2$, it admits the spectral resolution $L = \int_0^2 \lambda dE_L(\lambda)$. Let $F : [0, 2] \to \mathbb{C}$ be a bounded Borel measurable function; we define the operator

$$F(L) = \int_0^2 F(\lambda)dE_L(\lambda)$$

which is bounded on $L^2(\Gamma)$.

Let $s > 0$. For each function f, we define

$$\|f\|_{C^s} = \sum_{k=0}^{[s]} \|f^{(k)}\|_\infty + M_s(f),$$

where

$$M_s(f) = \sup\Big\{\frac{|f^{([s])}(x+t) - f^{([s])}(x)|}{t^{s-[s]}} : t > 0, x \in \mathbb{R}\Big\}.$$

We set

$$C^s(\mathbb{R}) := \{f : \|f\|_{C^s} < \infty\}.$$

In this section, we will study the boundedness of $F(L)$ on Hardy spaces H_L^p for $0 < p \leq 1$. The main result is the following theorem.

Theorem 6.2. *Let $0 < p \leq 1$ and F be a bounded Borel measurable function. Assume that $s > D(1/p - 1/2)$ and the function F satisfies the following condition:*

$$\sup_{t>0} \|\eta(\lambda)F(t\lambda)\|_{C^s} < \infty, \tag{17}$$

where $\eta \in C_c^\infty(0,\infty)$ is a fixed function not identically zero.

Then $F(L)$ is bounded on H_L^p.

Proof. Due to Lemma 3.11 and Proposition 3.12, it is sufficient to show that $F(L)a$ is a multiple of a $(2, p, M, \epsilon)$ molecule with $\epsilon = (s - D(\frac{1}{p} - \frac{1}{2}))/2$, for any $(2, p, 2M)$ atom $a = L^{2M}b$ associated to some ball B. We write $F(L)a = L^M(F(L)L^Mb)$. Hence we need only to show that, for all $k = 0, \dots, M$,

$$\|L^k(F(L)L^Mb)\|_{L^2(S_j(B))} \leq 2^{-j\epsilon}r_B^{M-k}V(2^jB)^{1/2-1/p} \quad \text{for all } j = 0, 1, 2, \dots. \tag{18}$$

Using the L^2-boundedness of $F(L)$, it is easy to check that (18) holds for $j = 0, 1, 2, 3$. To prove (18) for $j \geq 4$, we need the following approximation result; see for example [A].

Lemma 6.3. *Let $s > 0$ and $f \in C^s(\mathbb{R})$ with* supp $f \subset [-4, 4]$. *Then there exists $c > 0$ such that for all $k \in \mathbb{N}_+$, there is a polynomial Q with* $\deg(Q) \leq k$ *such that*

$$\|f - Q\|_{L^\infty([-4,4])} \leq c\frac{M_s(f)}{k^s}.$$

We are now ready to establish the proof of (18) for $j \geq 4$. Following a standard argument, see for example [A, KM], we can decompose

$$F(\lambda) = \sum_{\ell\geq 1} F(\lambda)\theta(\lambda)\varphi(2^\ell\lambda) + (1-\theta(\lambda))F(\lambda) := \sum_{\ell\geq 1} F_\ell(\lambda) + F_0(\lambda),$$

where $0 \leq \theta \in C_c^\infty(\mathbb{R})$ and $\varphi \in C_c^\infty$ satisfy

$$\theta(\lambda) = 1 \text{ for } \lambda \in [-1/4, 1/4] \text{ and } \theta(\lambda) = 0 \text{ for } \lambda \notin [-1/2, 1/2]$$

and

$$\text{supp}\varphi \subset (1/2, 1), \sum_{\ell\geq 1} \varphi(2^\ell\lambda) = 1, \lambda \in (0, 2].$$

Then we can write

$$F(L)a = \sum_{\ell\geq 0} F_\ell(L)a = L^M\widetilde{b},$$

where $\widetilde{b} = \sum_{\ell\geq 0} F_\ell(L)L^M b$.

It is easy to check that supp $F_0 \subset [1/2, 2]$ and supp $F_\ell \subset [2^{-(\ell+1)}, 2^{-\ell}]$ for $\ell \geq 1$.

For $\ell = 0$ and $j \geq 4$, we have, by Minkowski's inequality,

$$\begin{aligned} \|L^k(F_0(L)L^M b)\|_{L^2(S_j(B))} &= \|F_0(L)(L^{M+k}b)\|_{L^2(S_j(B))} \\ &= \|\sum_{y\in B} K_{F_0(L)}(\cdot, y)L^{M+k}b(y)\|_{L^2(S_j(B))} \\ &\leq \|L^{M+k}b(y)\|_{L^1} \sup_{y\in B} \|K_{F_0(L)}(\cdot, y)\|_{L^2(S_j(B))} \\ &\leq r_B^{M-k}V(B)^{1-1/p} \sup_{y\in B} \|K_{F_0(L)}(\cdot, y)\|_{L^2(S_j(B))}. \end{aligned} \tag{19}$$

Note that $K_{F_0(\sqrt{L})}(\cdot, y) = (F_0(L)p_0(\cdot, y))(x)$. Lemma 6.3 tells us that we can pick a polynomial Q so that

$$\deg(Q) \leq [2^{j-2}r_B] \quad \text{and} \quad \|F_0 - Q\|_{L^\infty([1/2,2])} \leq cM_s(F_0)2[2^{j-2}r_B]^{-s} \approx 2^{-sj}r_B^{-s}.$$

Since $x \in S_j(B), y \in B$, by Remark 2.1,

$$\begin{aligned} \|K_{F_0(L)}(\cdot, y)\|_{L^2(S_j(B))} &= \|[(F_0(L) - Q(L))p_0(\cdot, y)](x)\|_{L^2(S_j(B))} \\ &\leq C\|F_0 - Q\|_{L^\infty([1/2,2])}\|p_0(\cdot, y)\|_{L^2} \\ &\leq c\frac{2^{-sj}r_B^{-s}}{V(y,1)^{1/2}} \leq c\frac{2^{-sj}r_B^{-(s-D/2)}}{V(B)^{1/2}}. \end{aligned}$$

This together with (19) gives

$$\begin{aligned} \|L^k(F_0(L)L^M b)\|_{L^2(S_j(B))} &\leq cr_B^{M-k}2^{-sj}r_B^{-(s-D/2)}V(B)^{1/2-1/p} \\ &\leq cr_B^{M-k}2^{-j(s-D(\frac{1}{p}-\frac{1}{2}))}V(2^jB)^{1/2-1/p}. \end{aligned}$$

Hence, for $k = 0, 1, \dots, M$ and $j \geq 4$,

$$\|L^k(F_0(L)L^M b)\|_{L^2(S_j(B))} \leq cr_B^{M-k}2^{-j\epsilon}V(2^jB)^{1/2-1/p}. \tag{20}$$

For $\ell \geq 1$, set

$$\widetilde{F}_\ell(\lambda) = (1-\lambda)^{-2^\ell}F_\ell(\lambda).$$

Then we have

$$\|\widetilde{F}_\ell\|_\infty \leq c\|F_\ell\|_\infty \quad \text{and} \quad F_\ell(L) = \widetilde{F}_\ell(L)P^{2^\ell}.$$

We write, for $k = 0, 1, \ldots, M$ and $j \geq 4$,

$$\begin{aligned}\sum_{\ell\geq 1}\|L^k(F_\ell(L)L^M b)\|_{L^2(S_j(B))} &\leq \sum_{\ell:2^\ell\geq 2^j r_B}\|\widetilde{F}_\ell(L)(L^{M+k}P^{2^\ell}b)\|_{L^2(S_j(B))}\\ &\quad+\sum_{\ell:2^\ell<2^j r_B}\|F_\ell(L)L^{M+k}b)\|_{L^2(S_j(B))}\\ &:= I + II.\end{aligned}$$

Due to (3), we have

$$\begin{aligned}I &\leq c\sum_{\ell:2^\ell\geq 2^j r_B}\|L^{M+k}P^{2^\ell}b\|_{L^2} \leq \sum_{\ell:2^\ell\geq 2^j r_B}\frac{c}{2^{\ell(M+k)}}\|b\|_{L^2}\\ &\leq c\sum_{\ell:2^\ell\geq 2^j r_B}\Big(\frac{2^j r_B}{2^l}\Big)^{M+k}2^{-j(M+k)}r_B^{M-k}V(B)^{1/2-1/p}\\ &\leq c2^{-j\epsilon}r_B^{M-k}V(2^jB)^{1/2-1/p}.\end{aligned}$$

Let us estimate the term II. For each ℓ with $2^\ell < 2^j r_B$, we can pick a polynomial Q of degree at most $2^j r_B - r_B - 2$ so that

$$\|F_\ell - Q\|_{L^\infty([2^{-(\ell+1)},2^{-\ell}])} \leq c(2^j r_B)^{-s} \quad (\text{since } 2^j r_B - r_B - 2 \approx 2^j r_B).$$

In light of Remark 2.1, one has, for $s > s' > D(1/p-1/2)$ with $s' - D(1/p-1/2) > \epsilon$,

$$\begin{aligned}II &\leq c\sum_{\ell:2^\ell<2^j r_B}\|(F_\ell - Q)L^{M+k}b\|_{L^2}\\ &\leq c\sum_{\ell:2^\ell<2^j r_B}\|F_\ell - Q\|_{L^\infty([2^{-(\ell+1)},2^{-\ell}])}\|L^{M+k}b\|_{L^2}\\ &\leq c\sum_{\ell:2^\ell<2^j r_B}(2^j r_B)^{-s}r_B^{M-k}V(B)^{1/2-1/p}\\ &\leq c\sum_{\ell:2^\ell<2^j r_B}(2^j r_B)^{-s}2^{jD(1/p-1/2)}r_B^{M-k}V(2^jB)^{1/2-1/p}\\ &\leq c[(\log_2(2^j r_B))(2^j r_B)^{-s+s'}](2^j r_B)^{-s'}2^{jD(1/p-1/2)}r_B^{M-k}V(2^jB)^{1/2-1/p}\\ &\leq c2^{-js'}2^{jD(1/p-1/2)}r_B^{M-k}V(2^jB)^{1/2-1/p}\\ &\leq c2^{-j\epsilon}r_B^{M-k}V(2^jB)^{1/2-1/p}.\end{aligned}$$

The combination of these estimates I, II and (20) shows that $F(L)a$ is a multiple of a $(2, p, M, \epsilon)$ molecule. Hence our proof is complete. □

6.3. **Riesz transforms.** The following result concerning Riesz transforms $\nabla(I - P)^{-1/2}$ was proved in [R2].

Theorem 6.4. *Under the assumptions (D), (S_α) and (UE), the Riesz transform* $\nabla(I - P)^{-1/2}$ *is bounded on* $L^p(\Gamma)$ *for* $1 < p \leq 2$. *Moreover, the Riesz transform is of weak type* $(1, 1)$.

Our aim is to establish the boundedness of the Riesz transform on the Hardy spaces for the range $0 < p \leq 1$.

Theorem 6.5. *The Riesz transform* $\nabla(I - P)^{-1/2}$ *is bounded from* H^p_L *to* L^p *for all* $0 < p \leq 1$.

Proof. Invoking Lemma 3.11 again, we need to show that there exists a constant $C > 0$ so that

$$\|\nabla(I-P)^{-1/2}a\|_{L^p} \leq C$$

for any $(2, p, M)$ atom associated to some ball B.

Denote $T = \nabla(I-P)^{-1/2}$. Then we write, by Hölder inequality,

$$\|Ta\|^p_{L^p} \leq \|Ta\|^p_{L^p(8B)} + \sum_{j\geq 4}\|Ta\|^p_{L^p(S_j(B))}.$$

Using L^2-boundedness of T and Hölder inequality, we obtain

$$\begin{aligned}\|Ta\|^p_{L^p(8B)} &\leq \|Ta\|^p_{L^2(8B)}V(8B)^{1/p-1/2} \leq C\|a\|^p_{L^2}V(B)^{1-p/2}\\ &\leq CV(B)^{p/2-1}V(B)^{1-p/2} = C.\end{aligned}$$

For $j \geq 4$, Hölder inequality tells us that

$$\|Ta\|^p_{L^p(S_j(B))} \leq \|Ta\|^p_{L^2(2^jB)}V(2^jB)^{1/p-1/2}.$$

Now, as in [BR], the following identity holds:

$$\nabla(I-P)^{-1/2}f = \nabla\Big(\sum_{k\geq 0}a_kP^k\Big)f \quad \text{in } L^2(\Gamma)$$

for all $f \in E := \{f \in L^2 : f = L^{1/2}g \text{ for some } g \in L^2\}$, where $0 \leq a_k \leq \frac{1}{\sqrt{k}},\ k \geq 1$.

Therefore, if a is an atom, then by definition, $a = L^Mb, M \geq 1$, where $b \in L^2$. Hence we can write $a = L^{1/2}g$ with $g = L^{M-1/2}b$. Since the spectrum of L is contained in [0,2] and L is nonnegative selfadjoint, by the spectral theory, $g = L^{M-1/2}a \in L^2$. This concludes that $a \in E$. For this reason, we can write, for $x \in S_j(B), j \geq 4$,

$$Ta(x) = \sum_{k\geq 0}a_k\nabla P^k a(x).$$

Due to Remark 2.1,

$$\begin{aligned}Ta(x) &= \sum_{k\geq 3}a_k\nabla P^k a(x) = \sum_{k=3}^{r_B^2}a_k\nabla P^ka(x) + \sum_{k\geq r_B^2}a_k\nabla(I-P)^MP^ka(x)\\ &= \sum_{k=3}^{r_B^2}a_k(\sum_{y\in B}\nabla_x p_k(x,y)a(y)) + \sum_{k\geq r_B^2}a_k(\sum_{y\in B}\nabla_x\widetilde{p}_{M,k}(x,y)a(y))\\ &= I(x) + II(x),\end{aligned}$$

where $\widetilde{p}_{M,k}(x,y)$ is the associated kernel to $(I-P)^MP^k$.

To estimate $\|Ta\|^p_{L^2(2^jB)}$ for $j \geq 4$, we need the following lemma.

Lemma 6.6. *For any $n \in \mathbb{N}$, there exists a constant $\gamma > 0$ so that*

$$\Big(\sum_{x\in\Gamma}|\nabla_x\widetilde{p}_{n,k}(x,y)|^2e^{\frac{\gamma d(x,y)^2}{k}}\mu(x)\Big)^{1/2} \leq \frac{C\mu(y)}{k^{n+1/2}V(y,\sqrt{k})^{1/2}}$$

for all $k \in \mathbb{N}_+$ and $y \in \Gamma$.

In the case $n = 0$, the estimate in Lemma 6.6 was proved in [R2, Lemma 7] whose proof may be adapted to this situation with minor modifications. Hence, we leave the proof to the interested reader.

Returning to the proof of Theorem 6.5, to control $\|Ta\|^p_{L^2(2^jB)}$ for $j \geq 4$, we use the estimate in Lemma 6.6 and Minkowski's inequality to obtain that

$$\begin{aligned}\|I(\cdot)\|_{L^2(2^jB)} &\leq \sum_{k=3}^{r_B^2} a_k \Big\|\big(\sum_{y\in B} \nabla_x p_k(x,y)a(y)\big)\Big\|_{L^2(\mu(x),S_j(B))} \\ &\leq \sum_{k=3}^{r_B^2} a_k \sum_{y\in B} \|\nabla_x p_k(x,y)\|_{L^2(\mu(x),S_j(B))} a(y) \\ &\leq \sum_{k=3}^{r_B^2} a_k \frac{C}{\sqrt{k}V(y,\sqrt{k})^{1/2}} \exp\Big(-c\gamma\frac{2^{2j}r_B^2}{k}\Big)\|a\|_{L^1} \\ &\leq \sum_{k=3}^{r_B^2} \frac{C}{kV(2^jB)^{1/2}} \exp\Big(-c\gamma\frac{2^{2j}r_B^2}{k}\Big)V(B)^{1-1/p}.\end{aligned}$$

It is easy to check that

$$\exp\Big(-\gamma\frac{2^{2j}r_B^2}{k}\Big)V(B)^{1-1/p} \leq \frac{k}{2^{2j}r_B^2}V(2^jB)^{1-1/p}.$$

Hence

$$\|I(\cdot)\|_{L^2(2^jB)} \sum_{k=3}^{r_B^2} \frac{c}{2^{2j}r_B^2}V(2^jB)^{1/2-2/p} \leq c2^{-2j}V(2^jB)^{1/2-2/p}.$$

By a similar argument, we obtain

$$\begin{aligned}\|II(\cdot)\|_{L^2(2^jB)} &\leq \sum_{k>r_B^2} a_k \Big\|\big(\sum_{y\in B} \nabla_x \widetilde{p}_{M,k}(x,y)b(y)\big)\Big\|_{L^2(\mu(x),S_j(B))} \\ &\leq \sum_{k>r_B^2} a_k \frac{C}{k^{M+1/2}V(y,\sqrt{k})^{1/2}} \exp\Big(-c\gamma\frac{2^{2j}r_B^2}{k}\Big)\|b\|_{L^1} \\ &\leq \sum_{k>r_B^2} \frac{C}{k^{M+1}V(2^jB)^{1/2}} \exp\Big(-c\gamma\frac{2^{2j}r_B^2}{k}\Big)r_B^M V(B)^{1-1/p} \\ &\leq \sum_{k>r_B^2} \frac{C}{k^{M+1}}\Big(\frac{k}{2^{2j}r_B^2}\Big)^{(1+n(1/p-1))/2} r_B^M 2^{jD(1/p-1)}V(2^jB)^{1/2-1/p} \\ &\leq 2^{-2j}V(2^jB)^{1/2-2/p}.\end{aligned}$$

This together with the estimate of $\|I(\cdot)\|_{L^2(S_j(G))}$ gives

$$\|Ta\|^p_{L^2(2^jB)}V(2^jB)^{1/p-1/2} \leq c2^{-jp}.$$

Hence

$$\|Ta\|_{L^p} \leq c$$

for all $(2, p, M)$ atoms a. This completes our proof. $\square$

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