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ACTIONS OF $K(\pi, n)$ SPACES ON K-THEORY AND UNIQUENESS OF TWISTED K-THEORY

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Abstract. We prove the uniqueness of twisted K-theory in both the real and complex cases using the computation of the K-theories of Eilenberg-MacLane spaces due to Anderson and Hodgkin. As an application of our method, we give some vanishing results for actions of Eilenberg-MacLane spaces on K-theory spectra.

1. Introduction

Twisted K-theory, due originally to Donovan and Karoubi [12], has become an important concept bridging the fields of analysis, geometry, topology and string theory. It is the home of many topological invariants which cannot be seen by untwisted K-theory in the same way that the fundamental class of a non-orientable manifold must live in twisted cohomology. For instance, an appropriate twisted K-theory is the receptor of a Thom isomorphism for non-Spin^c-vector bundles. Another example of the importance of twisted K-theory is the work of [13], which relates twisted (equivariant) K-theory of a compact Lie group G to the representation theory of the loop group $\mathscr{L}G$. Because of its importance and its place in many different fields, there are a wide variety of definitions appearing in the literature. In addition to [12], there are the accounts of [4], [7], [8], [10], [13], [17], [21], [25] and [26] to name just a few. All of the these definitions exploit different models available for K-theory, and it is a natural question to determine the relationship between all possible approaches. The goal of this article is to show that all reasonable definitions of twisted (real or complex) K-theory essentially agree.

Here is a non-rigorous idea of what "reasonable" should mean. Given a space X and some sort of twisting datum α, there should be twisted K-groups $K^n(X)_\alpha$, the homotopy groups of a cohomology theory (spectrum) $K(X)_\alpha$. This construction should be functorial and locally trivial, in the sense that for contractible open subsets U in X, the twisted K-theory spectra $K(U)_\alpha$ should reduce to the untwisted K-theory of a point $K(*)$. Moreover, we want there to be an external product $K^m(X) \times K^n(X)_\alpha \to K^{n+m}(X)_\alpha$, which is to say an action of the K-theory spectrum of X on the twisted K-theory spectrum. We explain below how to make these requirements rigorous in the setting of stable homotopy theory. In particular, we want the spectrum $K(X)_\alpha$ to be a "module" over $K(X)$ in a suitable homotopy coherent fashion. The key constructions are all close analogues of the idea of a line bundle on a space, and the technical homotopy theory language is simply required

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to make this idea precise when the sheaves are no longer sheaves of groups but rather "sheaves of cohomology theories".

In our context we take as a reasonable definition of (real or complex) K-theory as one arising from a map $K(\mathbb{Z},3) \to BGL_1K$ or $K(\mathbb{Z}/2,2) \to BGL_1KO$. In general, if R is an A_∞-spectrum we can twist the generalized cohomology represented by R over a space X. Let BGL_1R denote the classifying space of the space GL_1R of homotopy units in R. Given a map $f : X \to BGL_1R$ we have an induced map of ∞-categories $f_* : X \to \mathrm{Mod}_R$, where we regard X as an ∞-category via its fundamental ∞-groupoid (e.g. its singular complex) and Mod_R denotes the ∞-category of R-modules. We refer the reader to [19, Chapter 1] for an introduction to ∞-categories. This map factors through the full subgroupoid of Mod_R spanned by the free rank one R-module R, which is equivalent (as ∞-groupoids) to BGL_1R. One then constructs the f-twisted R-spectrum $R(X)_f$ as the colimit

$$R(X)_f = \operatorname*{colim}_X f_*.$$

This is the R-module which arises as the pushforward (left Kan extension), along the projection from X to the point, of the parametrized spectrum over X classified by the map $f : X \to BGL_1R$. The f-twisted R-cohomology groups of X are also obtained from this parametrized spectrum, but instead by taking sections (the right Kan extension along the projection from X to the point). These ideas are made precise in [5] and are outlined below in Section 3 (see also [4, Section 5]).

We are particularly interested in the case where $R = K$ or $R = KO$, the spectra representing real or complex K-theory. Thus, from the viewpoint of homotopy theory, there is only one definition of twisted K-theory: given a map $f : X \to BGL_1K$ one produces the f-twisted K-theory spectrum $K(X)_f$ over X. However, in applications, one typically wants to associate twists of K-theory arising from a geometrically accessible subspace of BGL_1K. In the case of complex K-theory for example, we have an inclusion $K(\mathbb{Z},3) \to BGL_1K$ and we are interested in twists of K-theory arising from maps $X \to BGL_1K$ that factor through $K(\mathbb{Z},3)$, at least up to homotopy. Such twists are classified by cohomology classes in $\mathrm{H}^3(X,\mathbb{Z})$. The point is that the classifying space $K(\mathbb{Z},3)$ has as its model the projective unitary group of Hilbert space $PU(\mathscr{H})$, which allows operator theory to be brought into play when studying these special twists. For instance, the definition of Donovan and Karoubi associates twisted K-theory spectra to the torsion classes in $\mathrm{H}^3(X,\mathbb{Z})$, while the definitions of Rosenberg and of Atiyah and Segal define twisted K-theory for all classes of $\mathrm{H}^3(X,\mathbb{Z})$. In these cases a map $j : K(\mathbb{Z},3) \to BGL_1K$ is fixed, and the α-twisted K-theory, where $\alpha \in \mathrm{H}^3(X,\mathbb{Z})$ is a cohomology class classifying a map $f : X \to K(\mathbb{Z},3)$, is the twisted K-theory corresponding to the composition

$$X \xrightarrow{f} K(\mathbb{Z},3) \xrightarrow{j} BGL_1K.$$

The above construction depends on the map $j : K(\mathbb{Z},3) \to BGL_1K$ chosen, and thus we face the problem classifying maps $K(\mathbb{Z},3) \to BGL_1K$. Similarly, in the real case, one twists by elements of $\mathrm{H}^2(X,\mathbb{Z}/2)$, and so desires a map $K(\mathbb{Z}/2) \to BGL_1KO$. Our computations lead to the following theorem.

Theorem 1.1. *There are natural isomorphisms of groups*

$$[K(\mathbb{Z},3), BGL_1K] \cong [K(\mathbb{Z},3), K(\mathbb{Z},3)] \cong \mathbb{Z},$$
$$[K(\mathbb{Z}/2,2), BGL_1KO] \cong [K(\mathbb{Z}/2,2), K(\mathbb{Z}/2,2)] \cong \mathbb{Z}/2.$$

Thus, any two maps from $K(\mathbb{Z},3)$ to BGL_1K differ by an endomorphism of $K(\mathbb{Z},3)$, up to homotopy, and similarly any two maps from $K(\mathbb{Z},2)$ to BGL_1KO differ by an endomorphism of $K(\mathbb{Z}/2,2)$, up to homotopy.

Next we outline our approach. By [20], there is a decomposition of infinite loop spaces

$$GL_1K \simeq K(\mathbb{Z}/2,0) \times K(\mathbb{Z},2) \times BSU_\otimes,$$

where $BSU_\otimes$ is the infinite loop space classifying virtual complex vector bundles of rank and determinant one, equipped with the tensor product structure. Since this splitting respects the infinite loop structures, it may be delooped, so that we obtain the splitting

$$BGL_1K \simeq K(\mathbb{Z}/2,1) \times K(\mathbb{Z},3) \times BBSU_\otimes.$$

Let $i : K(\mathbb{Z},3) \to BGL_1K$ be the canonical inclusion. We view a reasonable definition of K-theory as one arising from a map $j : K(\mathbb{Z},3) \to BGL_1K$, thus we wish to compare i and j.

Denote by $bsu_\otimes$ the connective spectrum such that $\Omega^\infty bsu_\otimes \simeq BSU_\otimes$. The main result of this work says in the complex case that

$$bsu^1_\otimes(K(\mathbb{Z},3)) = [K(\mathbb{Z},3), BBSU_\otimes] = 0.$$

More generally, in Section 2 we provide conditions on a finitely generated abelian group π and n that imply that $bsu^1_\otimes(K(\pi,n))$ vanishes. Our calculations rely on a computation of the K-theory of Eilenberg-MacLane spaces due to Anderson-Hodgkin [3]. In particular, it follows that any map $K(\mathbb{Z},3) \to BGL_1K$ is homotopic to a integer multiple of i. In practice, to figure out which integer, it suffices to compute a differential in the twisted Atiyah-Hirzebruch spectral sequence, as is done in [8]. All constructions appearing in the literature differ by a unit ± 1.

In the real case,

$$BGL_1KO \simeq K(\mathbb{Z}/2,1) \times K(\mathbb{Z}/2,2) \times BBSO_\otimes,$$

and we show that

$$bso^1_\otimes(K(\mathbb{Z}/2,2)) = [K(\mathbb{Z}/2,2), BBSO_\otimes] = 0,$$

where $BSO_\otimes$ is the infinite loop space classifying virtual real vector bundles of rank and determinant one, equipped with the tensor product structure. Here, $bso_\otimes$ denotes the associated connective spectrum.

Therefore, any map $j : K(\mathbb{Z}/2,2) \to BGL_1KO$ is either homotopically trivial or j is equivalent to the canonical inclusion, and therefore there is a unique non-trivial definition of twisted real K-theory. This is proven in a similar way to the complex case.

One calls twists associated to a map from X to $BBSU_\otimes$ (resp. to $BBSO_\otimes$) higher twists of K-theory on X. Thus our theorem amounts to saying that there are no higher twists of complex K-theory on $K(\mathbb{Z},3)$ or of real K-theory on $K(\mathbb{Z}/2,2)$. In fact, in Propositions 2.6, 2.7, and 2.8 we determine exactly when there are higher twists of K-theory on $K(\pi,n)$ for π finitely generated and $n \geq 2$ (or $n \geq 3$ if π is not torsion). The original result in this direction is due to the third-named author [14] who showed that there are no higher twists for complex K-theory on the classifying spaces of compact Lie groups G. The results in [14] imply in particular that there are no higher twists of complex K-theory over $K(\pi,1)$ when π is a finite group and also over $K(\mathbb{Z}^n,2)$ for any $n \geq 0$, and thus our computations generalize these facts.

One might also be interested in twists of complex K-theory coming from r-torsion classes in $\mathrm{H}^3(X,\mathbb{Z})$ for some fixed integer r as in [6]. We show that $bsu^1_{\otimes}(K(\mathbb{Z}/r,2)) = 0$ so that the only twists of K-theory by r-torsion classes come from composing the Bockstein map $\beta : K(\mathbb{Z}/r,2) \to K(\mathbb{Z},3)$ with a map $K(\mathbb{Z},3) \to BGL_1K$.

The actions of Eilenberg-MacLane spectra appear as follows. Given a map

$$K(\pi,n) \to BGL_1K,$$

one may pass to the level of loop spaces and obtain an A_∞-map

$$K(\pi,n-1) \to GL_1K.$$

For example, by looking at the map on loop spaces associated to $i : K(\mathbb{Z},3) \to BGL_1K$, we obtain an A_∞-map $\mathbb{CP}^\infty \simeq K(\mathbb{Z},2) \to GL_1K$ which classifies the action of $\mathbb{CP}^\infty$ on K given by tensoring with line bundles. We call an A_∞ map $K(\pi,n) \to GL_1K$ an action of $K(\pi,n)$ on the K-theory spectrum. If the map factors through $BSU_\otimes$, we call the action a higher action. As a corollary of our computations we obtain (Corollary 2.9) the classification of those $K(\pi,n)$ with finitely generated abelian group π and $n \geq 2$ (or $n \geq 3$ if π is not torsion) for which all higher actions are trivial.

Here is the outline of the paper. The technical engine of the paper is contained in Section 2 where we compute the generalized cohomology groups $bsu^1_\otimes$ and $bso^1_\otimes$ of Eilenberg-MacLane spaces. After this, Section 3 recalls the definition of twisted K-theory via the method of [5], and there we give the proof of the uniqueness theorem. Finally, in the appendix, we give a nice geometric model of K-theory in the spirit of Atiyah and Segal which has the advantage that it is a structured ring spectrum and so may be easily used to produce a map $K(\mathbb{Z},3) \to BGL_1K$.

Notation. We will denote by k the spectrum representing connective complex K-theory, by K the spectrum representing complex K-theory, and by KO the spectrum representing real K-theory. For a prime p we will denote by $\mathbb{Z}_p$ the ring of p-adic integers. Given a spectrum F and an abelian group G we can introduce G coefficients on F by considering the spectrum $F_G = F \wedge MG$, where MG is a Moore spectrum for the group G. Also, given an integer n, we denote the $(n-1)$-connected cover of F by $F\langle n\rangle$.

2. Cohomology computations

The goal of this section is to determine when the groups $bsu^1_\otimes(K(\pi,n))$ and $bso^1_\otimes(K(\pi,n))$ vanish (for $n \geq 2$). Here is an outline of this section. See the remainder of the section for precise statements of the results. To begin, we use results of Anderson-Hodgkin [3], as extended by Yosimura [28], to figure out when the reduced K-theory $\tilde{K}^*(K(\pi,n))$ of Eilenberg-MacLane spaces vanish. When reduced K-theory does vanish, we show that $k^5(K(\pi,n)) = 0$ and $k^5_{\mathbb{Z}_p}(K(\pi,n)) = 0$, where k denotes connective K-theory, and $k_{\mathbb{Z}_p}$ denotes connective K-theory completed at the prime p. To achieve our goal of showing that $bsu^1_\otimes(K(\pi,n))$ vanishes, we relate $k^5(K(\pi,n))$ to $bsu_\otimes$ in the following way. The homotopy groups of the 4th suspension of connective K-theory Σ^4k and $bsu_\otimes$ are the same, but the infinite loop structures are not. On the other hand, Adams and Priddy showed [2, Corollary 1.4] that the spectra become equivalent after localization or completion at any prime p. From the vanishing of $k^5_{\mathbb{Z}_p}(K(\pi,n)) \cong (\Sigma^4k_{\mathbb{Z}_p})^1(K(\pi,n))$, we deduce that $bsu^1_{\otimes,\mathbb{Z}_p}(K(\pi,n))$ vanishes for every prime p. Finally, we use these p-complete vanishing results to show that $bsu^1_{\otimes}(K(\pi,n))=0$, relying on an argument about limits of abelian groups due to Gómez [14]. When $\tilde{K}^*(K(\pi,n))$ does not vanish (with the restrictions below on π and n), we show that $bsu_{\otimes}(K(\pi,n))$ does not vanish. The outline in the real case is similar, and we only give the proofs when they differ significantly from the complex case.

Recall that an abelian group A is torsion if for every element $a\in A$ there exists an integer $n>0$ such that $na=0$.

Lemma 2.1. *Suppose that π is a torsion abelian group. Then,*

$$\tilde{K}^*(K(\pi,n))=0$$

if $n\geq 2$. Suppose that $n\geq 3$ and that π is non-torsion (not necessarily torsion-free). Then,

$$K^1(K(\pi,n))=0$$

if and only if $\pi\otimes_{\mathbb{Z}}\mathbb{Q}$ is a 1-dimensional $\mathbb{Q}$-vector space and n is odd.

Proof. To start note that by [3, Theorem I] and its extension [28, Theorem 3] we have

$$\tilde{K}^*(K(\pi,n))=0$$

when π is torsion and $n\geq 2$, since the reduced integral homology of $K(\pi,n)$ is torsion in this case. This handles the case of torsion groups. Now, suppose that π is non-torsion. Then, by [3, Theorem II] and its extension [28, Theorem 3], we have

$$K^1(K(\pi,n))=K^1(K(\pi\otimes_{\mathbb{Z}}\mathbb{Q},n))=\bigoplus_{p+q=1}\mathrm{H}^p\left(K(\pi\otimes_{\mathbb{Z}}\mathbb{Q},n),K^q(*)\right).$$

Therefore, it suffices to prove that if π is a free abelian group and $n\geq 3$, then $K(\pi\otimes_{\mathbb{Z}}\mathbb{Q},n)$ has integral cohomology concentrated in even degrees if and only if $\pi\otimes_{\mathbb{Z}}\mathbb{Q}$ is 1-dimensional and n is odd. We may as well assume that $\pi=\mathbb{Z}^I$ for some non-empty set I. Let $\{e_i\}_{i\in I}$ be a basis for $\mathbb{Z}^I$. Since homology commutes with direct limits, by the computation of Cartan [11, Théorème 1] of the integral homology of Eilenberg-MacLane spaces, if n is odd, then

$$\mathrm{H}_*(K(\mathbb{Z}^I,n),\mathbb{Q})\cong\Lambda_{\mathbb{Q}}[\sigma^n e_i],$$

the rational exterior algebra on symbols $\sigma^n e_i$ in degree n, and if n is even, then

$$\mathrm{H}_*(K(\mathbb{Z}^I,n),\mathbb{Q})\cong\mathbb{Q}[\sigma^n e_i],$$

a polynomial algebra. Since the reduced homology groups of $K(\mathbb{Q}^I,n)$ are $\mathbb{Q}$-vector spaces, by universal coefficients for homology,

$$\tilde{\mathrm{H}}_*(K(\mathbb{Q}^I,n),\mathbb{Z})\cong\tilde{\mathrm{H}}_*(K(\mathbb{Q}^I,n),\mathbb{Q}).$$

On the other hand, $K(\mathbb{Z}^I,n)\to K(\mathbb{Q}^I,n)$ is a rational homotopy equivalence (for instance, by [15, Corollary 7.6]), so

$$\tilde{\mathrm{H}}_*(K(\mathbb{Z}^I,n),\mathbb{Q})\cong\tilde{\mathrm{H}}_*(K(\mathbb{Q}^I,n),\mathbb{Q}).$$

Therefore, the reduced integral homology of $K(\mathbb{Q}^I,n)$ is concentrated in odd degrees if and only if n is odd and $|I|=1$. But, since the reduced integral cohomology of $K(\mathbb{Q}^I,n)$ consists of $\mathbb{Q}$-vector spaces, this implies that the reduced integral cohomology of $K(\mathbb{Q}^I,n)$ is concentrated in even degrees if and only if n is odd and $|I|=1$, by the universal coefficient theorem. □

The previous lemma has the following analogue in the real case.

Lemma 2.2. *Suppose that π is a torsion abelian group. Then,*

$$\widetilde{KO}^*(K(\pi,n)) = 0$$

if $n \geq 2$. Suppose that $n \geq 3$ and that π is non-torsion. Then,

$$\widetilde{KO}^1(K(\pi,n)) = 0$$

if and only if $\pi \otimes_{\mathbb{Z}} \mathbb{Q}$ is at most 3-dimensional as a $\mathbb{Q}$-vector space and n is odd.

Proof. If π is torsion, then by the previous lemma,

$$\widetilde{K}^*(K(\pi,n)) = 0$$

so that by [3, Appendix],

$$\widetilde{KO}^*(K(\pi,n)) = 0.$$

In general,

$$K(\pi,n) \to K(\pi \otimes_{\mathbb{Z}} \mathbb{Q}, n)$$

induces an isomorphism on K-homology by [28], and therefore also an isomorphism on KO-cohomology by [23, Corollary 1.13]. Therefore, we can assume that π is a free abelian group. Then, it is evidently sufficient to prove the statement for π a finitely generated free abelian group. Indeed if $\widetilde{KO}^1(K(\tau,n)) \neq 0$ for τ of rank at least 4, then choosing a splitting $\pi \cong \tau \oplus \sigma$ shows that $\widetilde{KO}^1(K(\pi,n)) \neq 0$ as well. Thus, let τ be a finitely generated free abelian group. We show that $\widetilde{KO}^1(K(\tau,n)) = 0$ if and only if $n \geq 3$ is odd and the rank of τ is at most 3. As we are in the finitely generated case, by [3, Appendix],

$$\widetilde{KO}^1(K(\tau,n)) \cong \bigoplus_{p+q=1} \mathrm{H}^p(K(\tau \otimes_{\mathbb{Z}} \mathbb{Q}, n), KO^q(*)).$$

Since the reduced cohomology of $K(\tau \otimes_{\mathbb{Z}} \mathbb{Q}, n)$ is a $\mathbb{Q}$-vector space, in the direct sum above,

$$\mathrm{H}^p(K(\tau \otimes_{\mathbb{Z}} \mathbb{Q}, n), KO^q(*))$$

can only be non-zero for $q = 0 \mod 4$. Therefore, $\widetilde{KO}^1(K(\tau,n)) = 0$ if and only if $K(\tau \otimes_{\mathbb{Z}} \mathbb{Q}, n)$ has no integral cohomology in degrees equal to $1 \mod 4$. If n is even, then $K(\tau \otimes_{\mathbb{Z}} \mathbb{Q})$ has integral cohomology in degrees $1 \mod 4$. Thus, $\widetilde{KO}^1(K(\tau \otimes_{\mathbb{Z}} \mathbb{Q}, n)) \neq 0$. If n is odd, supposing that $\tau = \mathbb{Z}^m$, the Cartan calculation we saw in the proof of the previous lemma says that $K(\tau \otimes_{\mathbb{Z}} \mathbb{Q}, n)$ has integral cohomology in degrees

$$n+1, 2n+1, \ldots, mn+1.$$

If $m \leq 3$, then these degrees are $n+1, 2n+1, 3n+1$. Since n is odd, none of these numbers are equal to $1 \mod 4$. If $m \geq 4$, $4n+1 = 1 \mod 4$. This completes the proof. ∎

Now we turn to the vanishing of $bsu^1_{\otimes}(K(\pi,n))$. Let $\Sigma^4 k \simeq K\langle 4\rangle$ denote the 3-connected cover of K-theory. This is a connective spectrum with infinite loop space $BSU_{\oplus} \simeq \Omega^\infty \Sigma^4 k$, though here the infinite loop space structure is additive and does not agree with the multiplicative one on $BSU_{\otimes}$. The main result of [2, Corollary 1.4], however, asserts that the infinite loop structures become equivalent after localization or completion at any prime p. This implies in particular that

$\Sigma^4 k \wedge M\mathbb{Z}_p \simeq bsu_\otimes \wedge M\mathbb{Z}_p$ for every prime p. We are going to use this fact to show the triviality of $bsu^1_\otimes(K(\pi,n))$ for various π and n. For this we need the following lemma.

Lemma 2.3. *If (1) π is a finite abelian group and $n \geq 2$ or (2) π is a finitely generated abelian group with $\dim_\mathbb{Q} \pi \otimes_\mathbb{Z} \mathbb{Q} = 1$ and $n \geq 3$ is odd, then $k^5_{\mathbb{Z}_p}(K(\pi,n)) = 0$ for every prime p and $k^5(K(\pi,n)) = 0$.*

Proof. By Lemma 2.1 $K^1(K(\pi,n)) = 0$ in cases (1) and (2). Note also that

$$\tilde{\mathrm{H}}^r(K(\pi,n),\mathbb{Z}) = 0 \text{ for } 0 \leq r \leq 2 \tag{1}$$

under the hypotheses.

Let $K(\pi,n)$ be endowed with a CW-complex structure with m-skeleton F_m, a finite CW-complex. Note that, since F_m has all cells of $K(\pi,n)$ up through dimension m, $\tilde{\mathrm{H}}^r(F_m,\mathbb{Z}) = \tilde{\mathrm{H}}^r(K(\pi,n),\mathbb{Z}) = 0$ for $0 \leq r \leq 2$ and $m > 2$. On the other hand, since $\{F_m\}_{m\geq 0}$ is the skeleton filtration of $K(\pi,n)$, then, by [24], there are exact sequences

$$0 \to \lim_{m\to\infty}{}^1 K^4(F_m) \to K^5(K(\pi,n)) \to \lim_{m\to\infty} K^5(F_m) \to 0,$$
$$0 \to \lim_{m\to\infty}{}^1 k^4(F_m) \to k^5(K(\pi,n)) \to \lim_{m\to\infty} k^5(F_m) \to 0.$$

We will prove separately that $\lim^1_{m\to\infty} k^4(F_m) = 0$ and $\lim_{m\to\infty} k^5(F_m) = 0$.

Let's show first that $\lim^1_{m\to\infty} k^4(F_m) = 0$. Since $K^5(K(\pi,n)) = 0$ we have $\lim^1_{m\to\infty} K^4(F_m) = 0$, and it is easy to see that this implies that $\lim^1_{m\to\infty} \tilde{K}^4(F_m) = 0$. Fix m large enough and consider the Atiyah-Hirzebruch spectral sequences computing $K^*(F_m)$ and $k^*(F_m)$:

$$\mathrm{E}_2^{r,s} = \mathrm{H}^r(F_m, k^s(*)) \Longrightarrow k^{r+s}(F_m), \tag{2}$$

$$\tilde{\mathrm{E}}_2^{r,s} = \mathrm{H}^r(F_m, K^s(*)) \Longrightarrow K^{r+s}(F_m). \tag{3}$$

Both of these spectral sequences converge strongly as F_m is a finite CW-complex. We have a map of spectra $k \to K$ inducing an isomorphism on homotopy groups in non-negative degrees. This provides a map of spectral sequences $f^{r,s}_* : \mathrm{E}^{r,s}_* \to \tilde{\mathrm{E}}^{r,s}_*$ such that $f^{r,s}_2$ is an isomorphism whenever $s \leq 0$. Moreover, since $\tilde{\mathrm{H}}^r(F_m,\mathbb{Z}) = 0$ for $0 \leq r \leq 2$ we have that $f^{r,s}_2$ is an isomorphism whenever $r+s = 4$ and $r > 0$. Also note that there are no differentials that kill elements in total degree 4 in the case of K that fail to do so in the case of k. This is because the only possible such differentials must have source $\tilde{\mathrm{E}}^{1,2}_*$, but this is trivial as $\tilde{\mathrm{E}}^{1,2}_2 = \mathrm{H}^1(F_m,\mathbb{Z}) = 0$. This proves that $f^{r,s}_*$ induces an isomorphism $f^{r,s}_\infty : \mathrm{E}^{r,s}_\infty \to \tilde{\mathrm{E}}^{r,s}_\infty$ whenever $r+s = 4$ and $r > 0$. Also $f^{0,4}_\infty : \mathrm{E}^{0,4}_\infty = 0 \to \tilde{\mathrm{E}}^{0,4}_\infty \cong \mathbb{Z}$ since $\tilde{\mathrm{E}}^{0,4}_2 = \mathrm{H}^0(F_m,\mathbb{Z}) \cong \mathbb{Z}$, and any differential with source $\tilde{\mathrm{E}}^{0,4}_* \cong \mathbb{Z}$ is trivial as one sees by comparing $\tilde{\mathrm{E}}^{r,s}_*$ with the Atiyah-Hirzebruch spectral sequence computing $K(*)$. This in turn proves that the map of spectra $k \to K$ induces a short exact sequence

$$0 \to k^4(F_m) \to K^4(F_m) \to \mathbb{Z} \to 0.$$

Note that in fact $k^4(F_m) \subset \tilde{K}^4(F_m)$. We conclude that the map $k \to K$ induces an isomorphism $k^4(F_m) \cong \tilde{K}^4(F_m)$. Since $\lim^1_{m\to\infty} \tilde{K}^4(F_m) = 0$ we conclude that

$$\lim_{m\to\infty}{}^1 k^4(F_m) = 0.$$

Let's prove now that $\lim_{m\to\infty} k^5(F_m) = 0$. To prove this we again compare the spectral sequences (2) $\mathrm{E}_*^{r,s}$ and $\tilde{\mathrm{E}}_*^{r,s}$ for F_m. In total degree 5 the map $f_*^{r,s}$ is such that $f_2^{r,s}$ is an isomorphism whenever $s \le 0$. A similar argument as before also shows in this case there are no differentials that kill elements in total degree 5 in the case of K that fail to do so in the case of k. Therefore $f_\infty^{r,s} : \mathrm{E}_*^{r,s} \to \tilde{\mathrm{E}}_*^{r,s}$ is an isomorphism whenever $r+s=5$ and $s \le 0$. Also note that $\mathrm{E}_\infty^{r,s} = 0$ whenever $r+s=5$ and $s>0$. These facts show that the map of spectra $k \to K$ induces an injective map $k^5(F_m) \to K^5(F_m)$ for m large enough. Indeed, a map of filtered abelian groups with finite decreasing filtrations is injective if the map on each slice is injective by an iterated use of the snake lemma. Given the commutative diagram

$$\begin{array}{ccc} k^5(F_{m+1}) & \xrightarrow{i_*} & k^5(F_m) \\ \downarrow & & \downarrow \\ K^5(F_{m+1}) & \xrightarrow{i_*} & K^5(F_m) \end{array} \tag{4}$$

and the fact that $\lim_{m\to\infty} K^5(F_m) = 0$, it follows that $\lim_{m\to\infty} k^5(F_m) = 0$ since lim is left-exact. The fact that $k^5_{\mathbb{Z}_p}(K(\pi,n)) = 0$ is proved in the same way once we know that $K^5_{\mathbb{Z}_p}(K(\pi,n)) = 0$. To see this note that

$$K_{\mathbb{Z}_p} = \operatorname*{holim}_{k\to\infty} K_{\mathbb{Z}/(p^k)},$$

with structure maps coming from the maps $\mathbb{Z}/(p^{k+1}) \to \mathbb{Z}/(p^k)$. Because of this, we have a short exact sequence

$$0 \to \lim_{k\to\infty}{}^1 K^4_{\mathbb{Z}/(p^k)}(K(\pi,n)) \to K^5_{\mathbb{Z}_p}(K(\pi,n)) \to \lim_{k\to\infty} K^5_{\mathbb{Z}/(p^k)}(K(\pi,n)) \to 0. \tag{5}$$

On the one hand, by [1, Proposition 6.6] we have a short exact sequence

(6)

$$0 \to K^5(K(\pi,n)) \otimes_{\mathbb{Z}} \mathbb{Z}/(p^k) \to K^5_{\mathbb{Z}/(p^k)}(K(\pi,n)) \to \mathrm{Tor}_1^{\mathbb{Z}}(K^6(K(\pi,n)), \mathbb{Z}/(p^k)) \to 0.$$

Under the given hypothesis $K^5(K(\pi,n)) = 0$ by Lemma 2.1. By [3, Theorem I] we have $\tilde{K}^*(K(\pi,n)) = 0$ when π is as in (1), and by [3, Theorem II] we have that $\tilde{K}^*(K(\pi,n)) = H^{**}(K(\pi \otimes_{\mathbb{Z}} \mathbb{Q}, n), \mathbb{Z}) = 0$ when π satisfies (2). In particular $\tilde{K}^6(K(\pi,n))$ is a vector space over $\mathbb{Q}$ in this case. In either case it follows that $\mathrm{Tor}_1^{\mathbb{Z}}(K^6(K(\pi,n)), \mathbb{Z}/(p^k)) = 0$ and we conclude that $K^5_{\mathbb{Z}/(p^k)}(K(\pi,n)) = 0$. This proves that the right hand side in the short exact sequence (5) vanishes. We are left to prove that

$$\lim_{k\to\infty}{}^1 K^4_{\mathbb{Z}/(p^k)}(K(\pi,n)) = 0.$$

To show this we use the exact sequence

(7)

$$0 \to K^4(K(\pi,n)) \otimes \mathbb{Z}/(p^k) \to K^4_{\mathbb{Z}/(p^k)}(K(\pi,n)) \to \mathrm{Tor}_1^{\mathbb{Z}}(K^5(K(\pi,n)), \mathbb{Z}/(p^k)) \to 0.$$

Since $K^5(K(\pi,n)) = 0$, we conclude from (7) that

$$K^4_{\mathbb{Z}/(p^k)}(K(\pi,n)) = K^4(K(\pi,n)) \otimes_{\mathbb{Z}} \mathbb{Z}/(p^k).$$

From here we can see that the maps $K^4_{\mathbb{Z}/(p^{k+1})}(K(\pi,n)) \to K^4_{\mathbb{Z}/(p^k)}(K(\pi,n))$ are surjective and thus the $\lim^1$ term in the short exact sequence (5) vanishes. This proves that $K^5_{\mathbb{Z}_p}(K(\pi,n)) = 0$. ∎

A similar computation can be done in the real case. Consider $KO\langle 2\rangle$, the 1-connected cover of KO. Then $KO\langle 2\rangle$ is a connective spectrum with $\Omega^\infty KO\langle 2\rangle \simeq BSO_\oplus$. By [2, Corollary 1.4] it follows that $KO\langle 2\rangle \wedge M\mathbb{Z}_p \simeq bso_\otimes \wedge M\mathbb{Z}_p$ for every prime p.

Lemma 2.4. *If (1) π is a finite abelian group and $n \geq 2$ or (2) π is a finitely generated abelian group with $1 \leq \dim_{\mathbb{Q}} \pi \otimes_{\mathbb{Z}} \mathbb{Q} \leq 3$ and $n \geq 3$ is odd, then $KO\langle 2\rangle^1_{\mathbb{Z}_p}(K(\pi,n)) = 0$ for every prime p and $KO\langle 2\rangle^1(K(\pi,n)) = 0$.*

Proof. Let $\{F_m\}_{m\geq 0}$ be the skeleton filtration of CW-complex structure on $K(\pi,n)$ in such a way that F_m is a finite CW-complex. Note that in these cases we also have for large m

$$\tilde{\mathrm{H}}^r(F_m,\mathbb{Z}) = 0 \text{ for } 0 \leq r \leq 2.$$

Also, $\widetilde{KO}^*(K(\pi,n)) = 0$ when π is finite abelian and $n \geq 2$, and $\widetilde{KO}^1(K(\pi,n)) = 0$ for π finitely generated and $n \geq 3$ odd, as proved above. We argue in a similar way as in the previous lemma. We can compare the Atiyah-Hirzebruch spectral sequences computing $KO\langle 2\rangle^*(F_m)$ and $KO^*(F_m)$. By doing so we prove that

$$\lim_{m\to\infty}{}^1 KO\langle 2\rangle^0(F_m) = 0 \text{ and } \lim_{m\to\infty} KO\langle 2\rangle^1(F_m) = 0.$$

The lemma follows using the $\lim^1$ exact sequence in $KO\langle 2\rangle^1$ associated to the filtration $\{F_m\}_{m\geq 0}$. The argument for the p-completed KO-theory of $K(\pi,n)$ follows along the same lines as the complex case using the fact that

$$KO\langle 2\rangle_{\mathbb{Z}_p} = \operatorname*{holim}_{k\to\infty} KO\langle 2\rangle_{\mathbb{Z}/(p^k)}.$$

□

Definition 2.5. An inverse system of groups $\{G_n\}$, i.e., a diagram of the form

$$\cdots \to G_{n+1} \to G_n \to \cdots \to G_2 \to G_1,$$

is said to satisfy the Mittag-Leffler condition if for every i we can find a $j > i$ such that for every $k > j$,

$$\mathrm{im}(G_k \to G_i) = \mathrm{im}(G_j \to G_i).$$

It is well known [27, Proposition 3.5.7] that if $\{G_n\}$ satisfies the Mittag-Leffler condition, then

$$\lim_{k\to\infty}{}^1 G_k = 0.$$

On the other hand, if each G_k is a countable group and $\lim_{m\to\infty}^1 G_m = 0$, then by [22, Theorem 2] we have that the system $\{G_n\}$ must satisfy the Mittag-Leffler condition.

Using the previous propositions we can show that $bsu^1_\otimes(K(\pi,n))$ vanishes for certain vales of n and some abelian groups π. This computation is central for our treatment of twisted K-theory.

Proposition 2.6. *If (1) π is a finite abelian group and $n \geq 2$ or (2) π is a finitely generated abelian group with $\dim_{\mathbb{Q}} \pi \otimes_{\mathbb{Z}} \mathbb{Q} = 1$ and $n \geq 3$ is odd, then we have*

$$bsu^1_\otimes(K(\pi,n)) = 0.$$

In particular, there are no higher twists of complex K-theory on $K(\pi,n)$ in either case.

Proof. As above, since π is finitely generated, we can give $K(\pi,n)$ a CW-complex structure in such a way that the m-skeleton F_m is a finite CW-complex. The filtration $\{F_m\}_{m\geq 0}$ induces a short exact sequence

$$0 \to \lim_{m\to\infty}{}^1\, bsu^0_{\otimes}(F_m) \to bsu^1_{\otimes}(K(\pi,n)) \to \lim_{m\to\infty} bsu^1_{\otimes}(F_m) \to 0.$$

Below we prove that the sequence of groups $\{bsu^0_{\otimes}(F_m)\}_{m\geq 0}$ satisfies the Mittag-Leffler condition. Thus the $\lim^1$ in the previous sequence vanishes yielding

$$bsu^1_{\otimes}(K(\pi,n)) = \lim_{m\to\infty} bsu^1_{\otimes}(F_m). \tag{8}$$

We are going to show that $\lim\limits_{m\to\infty}(bsu^1_{\otimes}(F_m)\otimes \mathbb{Z}_p) = 0$ for all prime numbers p. Because each $bsu^1_{\otimes}(F_m)$ is finitely generated group, then by [14, Lemma 7] $\lim\limits_{m\to\infty} bsu^1_{\otimes}(F_m) = 0$, and thus $bsu^1_{\otimes}(K(\pi,n)) = 0$ by (8).

Consider the short exact sequence

$$\begin{aligned} 0 &\to \lim_{m\to\infty}{}^1\, (bsu_{\otimes}\wedge M\mathbb{Z}_p)^0(F_m) \to (bsu_{\otimes}\wedge M\mathbb{Z}_p)^1(K(\pi,n)) \\ &\to \lim_{m\to\infty} (bsu_{\otimes}\wedge M\mathbb{Z}_p)^1(F_m) \to 0. \end{aligned} \tag{9}$$

By [2, Corollary 1.4] we have $\Sigma^4 k\wedge M\mathbb{Z}_p \simeq bsu_{\otimes}\wedge M\mathbb{Z}_p$. This together with Lemma 2.3 gives

$$(bsu_{\otimes}\wedge M\mathbb{Z}_p)^1(K(\pi,n)) \cong k^5_{\mathbb{Z}_p}(K(\pi,n)) = 0.$$

We conclude that the middle term in (9) vanishes, and hence we see that

$$\lim_{m\to\infty} (bsu_{\otimes}\wedge M\mathbb{Z}_p)^1(F_m) = 0.$$

By [1, Proposition III.6.6] there is a short exact sequence

$$0 \to bsu^1_{\otimes}(F_m)\otimes \mathbb{Z}_p \to (bsu_{\otimes}\wedge M\mathbb{Z}_p)^1(F_m) \to \mathrm{Tor}^1_{\mathbb{Z}}(bsu^2_{\otimes}(F_m),\mathbb{Z}_p) \to 0.$$

The Tor term in this sequence vanishes as $\mathbb{Z}_p$ is flat as a $\mathbb{Z}$-module. Therefore

$$(bsu_{\otimes}\wedge M\mathbb{Z}_p)^1(F_m) = bsu^1_{\otimes}(F_m)\otimes_{\mathbb{Z}} \mathbb{Z}_p,$$

and this in turn shows that for every prime p

$$\lim_{m\to\infty} (bsu^1_{\otimes}(F_m)\otimes_{\mathbb{Z}} \mathbb{Z}_p) = 0.$$

We are left to prove that the system of groups $B_m := bsu^0_{\otimes}(F_m)$ satisfies the Mittag-Leffler condition. For every $m\geq 0$ let $A_m := k^4(F_m)$. We saw in the proof of Lemma 2.3 that

$$\lim_{m\to\infty}{}^1\, A_m = 0.$$

Also since F_m is a finite CW-complex, we have that A_m and B_m are finitely generated abelian groups for every $m\geq 0$, in particular they are countable. Therefore the system $\{A_m\}_{m\geq 0}$ satisfies the Mittag-Leffler condition.

On the other hand, as pointed out above, $\Sigma^4 k\wedge M\mathbb{Z}_p \simeq bsu_{\otimes}\wedge M\mathbb{Z}_p$; thus for every $m\geq 0$ and any prime number p,

$$A_m\otimes_{\mathbb{Z}} \mathbb{Z}_p = k^4_{\mathbb{Z}_p}(F_m) \xrightarrow{\cong} (bsu_{\otimes}\wedge M\mathbb{Z}_p)^0(F_m) = B_m\otimes \mathbb{Z}_p. \tag{10}$$

The outer equalities follow by [1, Proposition III.6.6] since the Tor terms also vanish here. This yields a commutative diagram in which the vertical arrows are isomorphisms:

$$\begin{array}{ccccccc} \to & A_m \otimes_{\mathbb{Z}} \mathbb{Z}_p & \cdots & \to & A_2 \otimes_{\mathbb{Z}} \mathbb{Z}_p & \to & A_1 \otimes_{\mathbb{Z}} \mathbb{Z}_p \\ & \downarrow & & & \downarrow & & \downarrow \\ \to & B_m \otimes_{\mathbb{Z}} \mathbb{Z}_p & \cdots & \to & B_2 \otimes_{\mathbb{Z}} \mathbb{Z}_p & \to & B_1 \otimes_{\mathbb{Z}} \mathbb{Z}_p. \end{array}$$

Using this diagram and the fact that $\{A_m\}_{m\geq 0}$ satisfies the Mittag-Leffler property, it can be seen that the system $\{B_m\}_{m\geq 0}$ also satisfies the Mittag-Leffler condition using an argument similar to that in [14, Theorem 5]. $\square$

The previous proposition has the following real analogue that can be proved in the same way using the fact that $KO\langle 2\rangle \wedge M\mathbb{Z}_p \simeq bso_{\otimes} \wedge M\mathbb{Z}_p$ for every prime p.

Proposition 2.7. *If (1) π is a finite abelian group and $n \geq 2$ or (2) π is a finitely generated abelian group with $1 \leq \dim_{\mathbb{Q}} \pi \otimes_{\mathbb{Z}} \mathbb{Q} \leq 3$ and $n \geq 3$ is odd, then we have*

$$bso^1_{\otimes}(K(\pi, n)) = 0.$$

In particular, there are no higher twists of real K-theory on $K(\pi, n)$ in either case.

On the other hand, Proposition 2.6 is sharp as we show next. A real analogue can be proved in a similar way.

Proposition 2.8. *If (1) π is a non-torsion (not necessarily torsion-free) finitely generated abelian group and $n > 3$ is even or (2) π is a finitely generated abelian group with $\dim_{\mathbb{Q}} \pi \otimes_{\mathbb{Z}} \mathbb{Q} > 1$ and $n \geq 3$ is odd, then $bsu^1_{\otimes}(K(\pi, n)) \neq 0$.*

Proof. By Lemma 2.1 we know that $K^5(K(\pi, n)) \neq 0$ in these cases. Let's first show that $k^5(K(\pi, n)) \neq 0$. Assume by contradiction that $k^5(K(\pi, n)) = 0$. As before give $K(\pi, n)$ a structure of a CW-complex such that F_k, the k-skeleton of $K(\pi, n)$, is a finite CW-complex. Since we are assuming that $k^5(K(\pi, n)) = 0$ we have that $\lim^1_{k\to\infty} k^4(F_k) = 0$ and $\lim_{k\to\infty} k^5(F_k) = 0$. By comparing the Atiyah-Hirzebruch spectral sequences computing $K^*(F_k)$ and $k^*(F_k)$ as in Lemma 2.3 we can see that $\lim^1_{k\to\infty} K^4(F_k) = 0$ and $\lim_{k\to\infty} K^5(F_k) = 0$. This in turn proves that $K^5(K(\pi, n)) = 0$, which is a contradiction. Let's now show that $bsu^1_{\otimes}(K(\pi, n)) \neq 0$. Reasoning by contradiction again assume that $bsu^1_{\otimes}(K(\pi, n)) = 0$. The short exact sequence

$$0 \to \lim^1_{k\to\infty} bsu^0_{\otimes}(F_k) \to bsu^1_{\otimes}(K(\pi, n)) \to \lim_{k\to\infty} bsu^1_{\otimes}(F_k) \to 0$$

shows that $\lim_{k\to\infty} bsu^1_{\otimes}(F_k) = 0$ and $\lim^1_{k\to\infty} bsu^0_{\otimes}(F_k) = 0$. Since F_k is a finite CW-complex we have that $B_k := bsu^0_{\otimes}(F_k)$ is finitely generated for every $k \geq 0$. In particular we conclude that the system $\{B_k\}_{k\geq 0}$ satisfies the Mittag-Leffler condition. Let $A_k = k^4(F_k)$. As in the proof of the previous proposition we have a commutative diagram in which the vertical arrows are isomorphisms:

$$\begin{array}{ccccccc} \cdots \to & A_n \otimes_{\mathbb{Z}} \mathbb{Z}_p & \cdots & \to & A_2 \otimes_{\mathbb{Z}} \mathbb{Z}_p & \to & A_1 \otimes_{\mathbb{Z}} \mathbb{Z}_p \\ & \downarrow & & & \downarrow & & \downarrow \\ \cdots \to & B_n \otimes_{\mathbb{Z}} \mathbb{Z}_p & \cdots & \to & B_2 \otimes_{\mathbb{Z}} \mathbb{Z}_p & \to & B_1 \otimes_{\mathbb{Z}} \mathbb{Z}_p. \end{array}$$

This diagram, the fact that $\{B_k\}_{k\geq 0}$ satisfies the Mittag-Leffler property, and an argument similar to the one provided in [14, Theorem 5] prove that $\{A_k\}_{k\geq 0}$ also satisfies the Mittag-Leffler property and in particular

$$\lim^1_{k\to\infty} A_k = \lim^1_{k\to\infty} k^4(F_k) = 0.$$

On the other hand, by [3, Theorem III] we have $\lim_{k\to\infty} K^5(F_k) = 0$. Comparing the Atiyah-Hirzebruch spectral sequences computing $K^*(F_k)$ and $k^*(F_k)$ we can see that $K^5(F_k) \cong k^5(F_k)$; in particular we obtain $\lim_{k\to\infty} k^5(F_k) = 0$. Finally, the short exact sequence

$$0 \to \lim_{k\to\infty}{}^1 k^4(F_k) \to k^5(K(\pi,n)) \to \lim_{k\to\infty} k^5(F_k) \to 0$$

shows that $k^5(K(\pi,n)) = 0$, which is a contradiction. $\square$

Next we consider actions of Eilenberg-MacLane spaces on the K-theory spectrum. We call an A_∞-map $K(\pi, n-1) \to GL_1K$ an action of $K(\pi, n-1)$ on K. Given given an A_∞-action of $K(\pi, n-1)$ on K, it can be de-looped to obtain a map $K(\pi,n) \to BGL_1K$. Conversely, given a map $K(\pi,n) \to BGL_1K$, we obtain an A_∞-map $K(\pi, n-1) \to GL_1K$ by passing to the level of loop space. In fact, actions of $K(\pi, n-1)$ on the K-theory spectrum are in one to one correspondence with maps $K(\pi,n) \to BGL_1K$. We call an action of $K(\pi, n-1)$ on K a higher action if the corresponding map $K(\pi,n) \to BGL_1K$ factors through $BBSU_\otimes$. The above work can be rephrased as follows.

Corollary 2.9. *Let π be a finitely generated abelian group and $n \geq 2$ an integer.*

(1) *There are no higher actions of $K(\pi,n)$ on K if and only if π is torsion or n is even and $\dim \pi \otimes_{\mathbb{Z}} \mathbb{Q} = 1$.*
(2) *There are no higher actions of $K(\pi,n)$ on KO if and only if π is torsion or n is even and $1 \leq \dim \pi \otimes_{\mathbb{Z}} \mathbb{Q} \leq 3$.*

Corollary 2.10. *Let π be a finite abelian group. Then, there are no higher actions of $K(\pi, 1)$ on either K or KO.*

Corollary 2.10 was obtained by Gómez [14].

3. Uniqueness of twisted K-theory

In this section we use the computations of $bsu_\otimes$- and $bso_\otimes$-cohomology in the previous section to establish a uniqueness statement for definitions of twisted K-theory for both the real and complex cases.

Let R denote an A_∞-ring spectrum. We can twist the generalized cohomology represented by R over a space X. Let GL_1R be the space of homotopy units of R. This space is defined as the homotopy pullback in the diagram

$$\begin{array}{ccc} GL_1R & \longrightarrow & \Omega^\infty R \\ \big\downarrow & & \big\downarrow \\ (\pi_0\Omega R)^\times & \longrightarrow & \pi_0\Omega^\infty R. \end{array}$$

The space GL_1R is a group-like A_∞-space that acts on the spectrum R through R-module automorphisms. Just as in the classical theory of principal homogeneous spaces, twists of the theory R over a space X are classified by homotopy classes of maps $X \to BGL_1R$, where BGL_1R is the classifying space of the topological group GL_1R. Specifically, given a map $f : X \to BGL_1R$, we obtain, by associating to a space X its fundamental ∞-groupoid (one model of which is its singular complex, as in [19, Chapter 1]), which we will simply denote X, an induced map of ∞-categories

$$X \xrightarrow{f} BGL_1R \simeq B\mathrm{Aut}_R(R) \xrightarrow{i} \mathrm{Mod}_R.$$

Here $\mathrm{Aut}_R(R)$ denotes the group-like A_∞-space of automorphisms of R in Mod_R, $B\mathrm{Aut}_R(R)$ its delooping, and $B\mathrm{Aut}_R(R) \to \mathrm{Mod}_R$ the inclusion of the full subgroupoid of Mod_R spanned by the free rank one R-module R.

The R-module spectrum $R(X)_f$, or the f-twisted R-theory spectrum of X, is the resulting "Thom spectrum"

$$R(X)_f = \operatorname*{colim}_X i \circ f,$$

the colimit in Mod_R of the composite map $i \circ f : X \to \mathrm{Mod}_R$. The colimit exists since Mod_R admits colimits indexed by an arbitrary small ∞-category. See [19, Chapter 4] for an account of the ∞-categorical theory of colimits. The f-twisted R-cohomology groups of X are

$$R^n(X)_f := \pi_0 F_R(R(X)_f, \Sigma^n R),$$

where $F_R(R(X)_f, \Sigma^n R)$ is the function spectrum of R-module maps $R(X)_f \to \Sigma^n R$.

We can use this method of twisting in the particular cases of twisted (real or complex) K-theory. In the complex case the decomposition

$$GL_1K \simeq K(\mathbb{Z}/2, 0) \times K(\mathbb{Z}, 2) \times BSU_\otimes$$

is compatible with the evident A_∞-structures so it deloops to a decomposition

$$BGL_1K \simeq K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}, 3) \times BBSU_\otimes.$$

Therefore, twists of complex K-theory over X are classified by homotopy classes of maps

$$X \to BGL_1K \simeq K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}, 3) \times BBSU_\otimes.$$

The twisted cohomology groups $K^n(X)_f$ depend only on the homotopy class of f, through non-canonical isomorphisms, and thus by the decomposition

$$BGL_1K \simeq K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}, 3) \times BBSU_\otimes$$

we have twists of complex K-theory associated to elements in

$$\mathrm{H}^1(X, \mathbb{Z}/2) \times \mathrm{H}^3(X, \mathbb{Z}) \times bsu^1_\otimes(X).$$

In the geometric applications however, one specializes to twists of K-theory associated to maps f representing a cohomology class in $\mathrm{H}^3(X, \mathbb{Z})$. More precisely, let

$$i : K(\mathbb{Z}, 3) \longrightarrow BGL_1K$$

be the inclusion map and suppose that $f : X \to K(\mathbb{Z}, 3)$ is a map representing a cohomology class in $\mathrm{H}^3(X, \mathbb{Z})$. Then the f-twisted K-theory spectrum of X is defined as $K(X)_{i\circ f}$.

Different constructions or models of twisted K-theory associated to cohomology classes in $\mathrm{H}^3(X, \mathbb{Z})$ can be constructed by specifying a map

$$j : K(\mathbb{Z}, 3) \longrightarrow BGL_1K.$$

Given such a map we can define the twisted K-groups as above.

Thus a particular model or definition of twisted (complex) K-theory associated to cohomology classes in $\mathrm{H}^3(X, \mathbb{Z})$ amounts to producing a particular map $K(\mathbb{Z}, 3) \longrightarrow BGL_1K$. We discuss here how the construction given in [8] by Atiyah and Segal fits into this framework. Let $\mathcal{H}$ be a fixed infinite-dimensional separable Hilbert space. The space of Fredholm operators $Fred(\mathcal{H})$ with the norm topology is then a classifying space for complex K-theory. The space of unitary operators $U(\mathcal{H})$ acts by conjugation on $Fred(\mathcal{H})$. This induces an action of the projective unitary group $PU(\mathcal{H})$ on $Fred(\mathcal{H})$. The space $PU(\mathcal{H})$ is a $K(\mathbb{Z},2)$, and given a map

$$f : X \to BPU(\mathcal{H}) \simeq K(\mathbb{Z},3)$$

there is an associated principal $PU(\mathcal{H})$-bundle $P \to X$. We can then form the bundle $\xi := P \times_{PU(\mathcal{H})} Fred(\mathcal{H}) \to X$, and Atiyah and Segal define

$$K^0(X)_{f,AS} := \pi_0\Gamma(\xi \to X),$$

the group of homotopy classes of sections of $\xi \to X$. In the appendix, we give details on how to use the symmetric spectrum model of K-theory due to Joachim [16] to obtain a map $K(\mathbb{Z},3) \to BGL_1K$ which is very much in the spirit of Atiyah and Segal.

The case of twisted real K-theory can be handled in the same way. As in the complex case, we are interested in those twists associated to a map f representing a cohomology class $H^2(X;\mathbb{Z}/2)$. Thus given a map $f : X \to K(\mathbb{Z}/2,2)$ representing a cohomology class in $H^2(X;\mathbb{Z}/2)$ we can define the f-twisted real K-theory as $KO(X)_{i\circ f}$, where

$$i : K(\mathbb{Z}/2,2) \to BGL_1KO$$

is the inclusion map.

Therefore a construction of twisted complex K-theory associated to integral cohomology classes in $\mathrm{H}^3(X,\mathbb{Z})$ amounts to a pointed map

$$j : K(\mathbb{Z},3) \to BGL_1K.$$

Similarly, a construction of twisted real K-theory associated to cohomology classes in $\mathrm{H}^2(X,\mathbb{Z}/2)$ amounts to a pointed map

$$j : K(\mathbb{Z}/2,2) \to BGL_1KO.$$

On the other hand, a construction for r-torsion integral classes in the complex case is determined by a pointed map

$$j_r : K(\mathbb{Z}/r,2) \to BGL_1K.$$

A construction of all integral classes yields one for the r-torsion ones by composition with the Bockstein $\beta : K(\mathbb{Z}/r,2) \to K(\mathbb{Z},3)$.

Our main theorem says there are no higher twists of complex K-theory on $K(\mathbb{Z},3)$ or $K(\mathbb{Z}/r,2)$. In the real case we establish the non-existence of higher twists of real K-theory on $K(\mathbb{Z}/2,2)$. Let $p : BGL_1K \to K(\mathbb{Z},3)$ and $q : BGL_1KO \to K(\mathbb{Z},2)$ be the projection maps.

Theorem 3.1. *The map p induces isomorphisms*

$$\sigma : [K(\mathbb{Z},3), BGL_1K] \to [K(\mathbb{Z},3), K(\mathbb{Z},3)] \cong \mathbb{Z},$$
$$\sigma_r : [K(\mathbb{Z}/r,2), BGL_1K] \to [K(\mathbb{Z}/r,2), K(\mathbb{Z},3)] \cong \mathbb{Z}/r.$$

The map q induces an isomorphism

$$\tau : [K(\mathbb{Z}/2,2), BGL_1KO] \to [K(\mathbb{Z}/2,2), K(\mathbb{Z}/2,2)] \cong \mathbb{Z}/2.$$

Proof. The inclusion of the $K(\mathbb{Z},3)$ component into BGL_1K gives surjectivity. So, it suffices to prove that σ is injective. In other words, we wish to show that if $\sigma(j) = 0$, then j is null-homotopic. But, if $\sigma(j) = 0$, then the map

$$K(\mathbb{Z},3) \to K(\mathbb{Z}/2,1) \times K(\mathbb{Z},3) \times BBSU_{\otimes}$$

is homotopically trivial on the first two components. By Proposition 2.6, it is also trivial on the third component. The statement for σ_r is similar. The statement for τ is proved in the same way by using Proposition 2.7. $\square$

The previous theorem shows that in particular, any definition of complex twisted K-theory arising through a map

$$K(\mathbb{Z},3) \to BGL_1K$$

agrees, up to multiplication of an integer, with the definition given above. For a given definition this integer can be obtained by determining the differential d_3 in the Atiyah-Hirzebruch spectral sequence computing the twisted equivariant K-groups. Equivalently, we can determine this integer by computing the twisted K-groups on the sphere S^3 for a generator $\alpha \in \mathrm{H}^3(S^3,\mathbb{Z}) \cong \mathbb{Z}$. A similar situation occurs for twisted K-theory associated to r-torsion integral classes. For the case of twisted real K-theory the situation simplifies. In this case, by Theorem 3.1 we have $[K(\mathbb{Z}/2,2), BGL_1KO] \cong \mathbb{Z}/2$. In particular, any two non-trivial definitions of twisted real K-theory arising through a map

$$K(\mathbb{Z}/2,2) \to BGL_1KO$$

must coincide.

Corollary 3.2. *Let $j : K(\mathbb{Z},3) \to BGL_1K$ be a pointed map. Then,*

$$\Omega j : K(\mathbb{Z},2) \to GL_1K$$

sends a line bundle L on X to the equivalence $K(X) \xrightarrow{\sim} K(X)$ given by tensoring with $L^{\otimes\sigma(j)}$. Similarly, if $j : K(\mathbb{Z}/2,2) \to BGL_1KO$ is a pointed map, then Ωj is the auto-equivalence of KO given by tensoring with the $\tau(j)$-th power of real line bundles.

Proof. This is true by construction for the morphism $K(\mathbb{Z},3) \to BGL_1K$, which has $\sigma(j) = 1$, constructed in [4]. Thus, it is true for all other definitions. $\square$

4. Appendix: A geometric model of twisted K-theory

We outline how the symmetric spectrum model of K-theory (or KO-theory) due to Joachim [16] may be used to twist K-theory in a fashion that is nice from both the geometric and homotopical perspectives. The original model for twisted K-theory spectra is due to Atiyah and Segal [7], but it is not easy to see directly how it fits into the homotopical framework of twists in the sense of [5]. On the other hand, in [5] and [4] it is hard to see the concrete analysis and geometry which were the original foundation for twisted K-theory. Joachim's spectrum provides a vantage where both views may be appreciated.

In his paper, Joachim works with real periodic K-theory, but no alterations except replacing 8 by 2 at various places are required to apply the same arguments to complex periodic K-theory. For simplicity, we present the complex case.

Let $\mathcal{H}$ be a fixed infinite-dimensional separable Hilbert space, and let $\mathcal{H}_* = \mathcal{H}_0 \oplus \mathcal{H}_1$ be the $\mathbb{Z}/2$-graded Hilbert space with $\mathcal{H}_0 = \mathcal{H}_1 = \mathcal{H}$. The space $U(\mathcal{H})$ is the group of all unitary operators on $\mathcal{H}$, equipped with the norm topology; it is a contractible space by Kuiper's theorem. The quotient of $U(\mathcal{H})$ by the subgroup $U(1)$ of diagonal operators is $PU(\mathcal{H}) \simeq K(\mathbb{Z},3)$. If $F : \mathcal{H} \to \mathcal{H}$ is an operator and $P \in PU(\mathcal{H})$, then $P^{-1}FP$ is another operator. Let $\mathcal{F}^1(\mathcal{H}_*)$ be the space of self-adjoint odd Fredholm operators on $\mathcal{H}_*$. Thus, any element of F of $\mathcal{F}^1(\mathcal{H}_*)$ can be represented by a matrix

$$\begin{pmatrix} 0 & \tilde{F} \\ \tilde{F}^* & 0 \end{pmatrix},$$

where $\tilde{F}$ is a Fredholm operator on $\mathcal{H}$. The group $PU(\mathcal{H})$ acts continuously on $\mathcal{F}^1(\mathcal{H}_*)$ by

$$P \cdot \begin{pmatrix} 0 & \tilde{F} \\ \tilde{F}^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & P^{-1}\tilde{F}P \\ P^{-1}\tilde{F}^*P & 0 \end{pmatrix},$$

where we purposely confuse P with any operator in $U(\mathcal{H})$ representing P.

If one is only interested in twisted K-groups, then this is already enough setup to do so. If P is a principal $PU(\mathcal{H})_{\mathrm{norm}}$-bundle on X, Atiyah and Segal define $K^0_P(X)$ as the group of homotopy classes of sections of the associated bundle

$$P \times_{PU(\mathcal{H})} \mathcal{F}^1(\mathcal{H}_*) \to X$$

with fiber $\mathcal{F}^1(\mathcal{H}_*)$. However, we are of course interested in an entire spectrum.

To describe the twisted K-theory spectrum of Joachim, we recall the Clifford algebra $Cl(n)$, the complex Clifford algebra of $\mathbb{C}^n$ equipped with the quadratic form

$$q((z_1, \dots, z_n)) = -\sum_{i=1}^{n} z_i^2.$$

There are canonical isomorphisms

$$Cl(p)\widehat{\otimes}Cl(q) \to Cl(p+q)$$

for $p, q \geq 1$, where $\widehat{\otimes}$ denotes the $\mathbb{Z}/2$-graded tensor product. For details on Clifford algebras, consult [18].

If $\mathcal{J}_*$ is a $\mathbb{Z}/2$-graded Hilbert space module for $Cl(n)$, we let $\mathcal{F}^1_{Cl(n)}(\mathcal{J}_*)$ denote the space of odd self-adjoint Fredholm operators on $\mathcal{J}_*$ which are $Cl(n)$-module morphisms when n is even or the complement of the two contractible components of this space identified in [9] when n is odd.

Let

$$\mathcal{H}(n)_* = (Cl(1)\widehat{\otimes}\mathcal{H}_*)^{\hat{\otimes}n}.$$

Then, $\mathcal{H}(n)_*$ is naturally a graded $Cl(n)$-module. Joachim shows that the multiplication maps

$$\mu_{p,q} : \mathcal{F}^1_{Cl(p)}(\mathcal{H}(p)_*) \times \mathcal{F}^1_{Cl(q)}(\mathcal{H}(q)_*) \to \mathcal{F}^1_{Cl(p+q)}(\mathcal{H}(p+q)_*),$$
$$(F, G) \mapsto F \star G = F\widehat{\otimes}\mathrm{Id} + \mathrm{Id}\widehat{\otimes}G$$

are continuous. The maps $\mu_{p,q}$ are $\Sigma_p \times \Sigma_q$-equivariant, where Σ_n acts naturally on $\mathcal{F}^1_{Cl(n)}(\mathcal{H}(n)_*)$.

To create based maps, let $K_n = \mathcal{F}^1_{Cl(n)}(\mathcal{H}(n))_+$, topologized as in [16, Section 3] so that $\mathcal{F}^1_{Cl(n)}(\mathcal{H}(n)) \to K_n$ is a homotopy equivalence. Then, the $\mu_{p,q}$ induce continuous maps $K_p \wedge K_q \to K_{p+q}$ for $p, q \geq 1$.

Let $PU(\mathcal{H})$ act on $\mathcal{F}^1_{Cl(n)}(\mathcal{H}(n)_*)$ in the natural way, through the diagonal action of $U(\mathcal{H})$ on $\mathcal{H}_*^{\widehat{\otimes}n}$. This extends to a continuous action of $PU(\mathcal{H})$ on K_n, where $PU(\mathcal{H})$ fixes the basepoint.

Proposition 4.1. *The natural $PU(\mathcal{H})$ actions on the spaces of Joachim's spectrum K make K into a $PU(\mathcal{H})$-spectrum in the sense that the actions are compatible with the multiplication maps and the symmetric group actions.*

Proof. This is clear as $PU(\mathcal{H})$ acts diagonally on $\mathcal{H}_*^{\widehat{\otimes} n}$. $\square$

Corollary 4.2. *The conjugation action of $PU(\mathcal{H})$ on Joachim's spectrum K determines an A_∞-map $PU(\mathcal{H}) \to GL_1K$ which deloops to a map $K(\mathbb{Z}, 3) \simeq BPU(\mathcal{H}) \to BGL_1K$.*

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