TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 366, Number 7, July 2014, Pages 3693–3707
S 0002-9947(2014)05964-3
Article electronically published on March 20, 2014

A CLASS OF II_1 FACTORS WITH AN EXOTIC ABELIAN MAXIMAL AMENABLE SUBALGEBRA

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ABSTRACT. We show that for every mixing orthogonal representation $\pi : \mathbf{Z} \to \mathcal{O}(H_{\mathbf{R}})$, the abelian subalgebra $\mathrm{L}(\mathbf{Z})$ is maximal amenable in the crossed product II_1 factor $\Gamma(H_{\mathbf{R}})'' \rtimes_\pi \mathbf{Z}$ associated with the free Bogoljubov action of the representation π. This provides uncountably many non-isomorphic A-A-bimodules which are disjoint from the coarse A-A-bimodule and of the form $\mathrm{L}^2(M \ominus A)$ where $A \subset M$ is a maximal amenable masa in a II_1 factor.

1. Introduction

A (separable) finite von Neumann algebra P is *amenable* if there exists a norm one projection $E : \mathbf{B}(\mathrm{L}^2(P)) \to P$. Connes' celebrated result [4] shows that all finite amenable von Neumann algebras are hyperfinite [14].

As the amenable von Neumann algebras form a monotone class, any von Neumann algebra has maximal amenable von Neumann subalgebras. Popa exhibited in [18] the first concrete examples of maximal amenable von Neumann subalgebras in II_1 factors by showing that the *generator* maximal abelian subalgebra (masa) in free group factors is maximal amenable. In fact, Popa showed [18, Lemma 2.1] that the generator masa in a free group factor satisfies the *asymptotic orthogonality property* (see Definition 3.1). He then used this property to deduce that the generator masa is maximal amenable (see [18, Corollary 3.3]).

Subsequently in [3, Theorem 6.2], the *radial* masa in free group factors was shown to satisfy Popa's asymptotic orthogonality property. Since the radial masa is moreover singular by [20, Theorem 7], it follows maximal amenable by [3, Corollary 2.3]. The *cup* masa in the II_1 factors associated with a planar algebra subfactor [1] gives another example of maximal amenable masa.

In this paper we provide new examples of maximal amenable masas in II_1 factors. Our construction is natural and consists of looking at $\mathrm{L}(\mathbf{Z})$ as a masa inside the crossed product $\mathrm{L}(\mathbf{F}_\infty) \rtimes \mathbf{Z}$ where the action $\mathbf{Z} \curvearrowright \mathrm{L}(\mathbf{F}_\infty)$ is a free Bogoljubov action obtained via Voiculescu's *free Gaussian functor* [24]. Recall from [24, Chapter 2] that to any separable real Hilbert space $H_{\mathbf{R}}$, one can associate a finite von Neumann algebra $\Gamma(H_{\mathbf{R}})''$ which is $*$-isomorphic to the free group factor $\mathrm{L}(\mathbf{F}_{\dim H_{\mathbf{R}}})$. To any orthogonal representation $\pi : \mathbf{Z} \to \mathcal{O}(H_{\mathbf{R}})$ there corresponds a trace-preserving action $\sigma^\pi : \mathbf{Z} \curvearrowright \Gamma(H_{\mathbf{R}})''$ called the *free Bogoljubov action* associated with the orthogonal representation π. Our main result is the following.

Received by the editors April 30, 2012 and, in revised form, September 21, 2012.
2010 *Mathematics Subject Classification.* Primary 46L10, 46L54, 46L55, 22D25.
Key words and phrases. Free Gaussian functor, maximal amenable subalgebras, asymptotic orthogonality property, Rajchman measures.
The author's research was partially supported by ANR grants AGORA NT09-461407 and NEUMANN.

Theorem. *Let G be a countable infinite abelian group and $\pi : G \to \mathcal{O}(H_{\mathbf{R}})$ a faithful mixing orthogonal representation. Denote by $\Gamma(H_{\mathbf{R}})'' \rtimes_\pi G$ the crossed product* II_1 *factor associated with the free Bogoljubov action of π. Then $\mathrm{L}(G)$ is maximal amenable in $\Gamma(H_{\mathbf{R}})'' \rtimes_\pi G$.*

To prove the theorem, we actually show that $\mathrm{L}(G) \subset \Gamma(H_{\mathbf{R}})'' \rtimes_\pi G$ satisfies Popa's asymptotic orthogonality property (see Theorem 3.2). Observe that when $(\pi, H_{\mathbf{R}}) = (\lambda_G, \ell^2_{\mathbf{R}}(G))$ is the left regular representation, we have

$$\left(\mathrm{L}(G) \subset \Gamma(\ell^2_{\mathbf{R}}(G))'' \rtimes_{\lambda_G} G\right) \cong \left(\mathrm{L}(G) \subset \mathrm{L}(\mathbf{Z}) * \mathrm{L}(G)\right),$$

and so our theorem recovers Popa's original result [18]. The interesting feature of our theorem is that we are able to prove maximal amenability for *any* mixing orthogonal representation. This, in turn, will allow us to obtain new examples of maximal amenable masas.

Let $A \subset M$ be a (diffuse) masa in a separable II_1 factor M. Write $A = \mathrm{L}^\infty(Y, \nu)$, where Y is a second countable compact space and $\tau|A$ is given by integration against ν. Denote by

$$\Theta : \mathrm{C}(Y) \otimes \mathrm{C}(Y) \to \mathbf{B}(\mathrm{L}^2(M \ominus A)) : \Theta(a \otimes b) = a\, Jb^*J$$

the $*$-representation that encodes the A-A-bimodule structure of $\mathrm{L}^2(M \ominus A)$. One can then associate to Θ a unique measure class $[\eta]$ on the Borel subsets of Y^2 and a multiplicity function $m : Y^2 \to \{1, \dots, \infty\}$. (We can always assume that η is a Borel probability measure on Y^2 quasi-invariant under the flip $\sigma : Y^2 \to Y^2 : \sigma(x, y) = (y, x)$.) The triple $(Y, [\eta], m)$ is a *conjugacy* invariant for the masa $A \subset M$ in the following sense (see [15, Section 3]). Let $A \subset M$ and $B \subset N$ be masas in II_1 factors. Then there exists a unitary $U : \mathrm{L}^2(M) \to \mathrm{L}^2(N)$ such that $UAU^* = B$ and $UJ_MU^* = J_N$ if and only if there exists a surjective Borel isomorphism $\theta : Y_A \to Y_B$ such that $\theta_*[\nu_A] = [\nu_B]$, $(\theta \times \theta)_*[\eta_A] = [\eta_B]$ and $m_B \circ (\theta \times \theta) = m_A$ (η_A-almost everywhere). From now on, since A is diffuse, we will always assume $(Y, \nu) = (\mathbf{T}, \mathrm{Haar})$, that is, $A = \mathrm{L}^\infty(\mathbf{T}, \mathrm{Haar})$.

For the three aforementioned examples, generator [18], radial [7] and cup [1] masas, the corresponding A-A-bimodule $\mathrm{L}^2(M \ominus A)$ is always isomorphic to an infinite direct sum of coarse A-A-bimodules. So, in that case, the measure class $[\eta]$ is simply the class of the Haar measure on $\mathbf{T}^2$ and the multiplicity function m equals ∞ Haar-almost everywhere. It is then natural to ask which and how many measure classes $[\eta]$ on $\mathbf{T}^2$ can be concretely realized as the measure class of a maximal amenable masa $A \subset M$ in a II_1 factor.

In order to answer this question, first recall that two Borel measures μ and ν on a standard Borel space X are *singular* if there exists a Borel subset $\mathcal{U} \subset X$ such that $\mu(\mathcal{U}) = 0$ and $\nu(X \setminus \mathcal{U}) = 0$. We say that an inclusion of a masa in a II_1 factor $A \subset M$ is *exotic* if for the disintegration of η with respect to the factor map $p : (\mathbf{T}^2, [\eta]) \to (\mathbf{T}, [\mathrm{Haar}]) : (z, t) \to t$, that we write $\eta = \int_{\mathbf{T}} \eta_t \mathrm{d}t$, almost every Borel measure η_t is atomless and singular with respect to the Haar measure. When $A \subset M$ is exotic, the A-A-bimodule $\mathrm{L}^2(M \ominus A)$ is *disjoint* from the coarse A-A-bimodule $\mathrm{L}^2(A) \otimes \mathrm{L}^2(A)$ in the sense that non-zero A-A-sub-bimodules of $\mathrm{L}^2(M \ominus A)$ are never isomorphic to A-A-sub-bimodules of $\mathrm{L}^2(A) \otimes \mathrm{L}^2(A)$.

A Borel measure μ on $\mathbf{T}$ is *symmetric* if $\mu(\mathcal{U}) = \mu(\overline{\mathcal{U}})$ for all Borel subsets $\mathcal{U} \subset \mathbf{T}$. To any symmetric Borel probability measure μ on $\mathbf{T}$, one can associate a real Hilbert space

$$H^\mu_{\mathbf{R}} = \left\{\zeta \in \mathrm{L}^2(\mathbf{T}, \mu) : \overline{\zeta(z)} = \zeta(\overline{z}) \ \mu\text{-almost everywhere}\right\}$$

together with an orthogonal representation

$$\pi^\mu : \mathbf{Z} \to \mathcal{O}(H^\mu_{\mathbf{R}}) : (\pi^\mu(n)\zeta)\,(z) = z^n \zeta(z).$$

A symmetric *Rajchman* measure μ on $\mathbf{T}$ is a symmetric Borel probability measure whose Fourier-Stieltjes coefficients $\widehat{\mu}(n) = \int_{\mathbf{T}} z^n \mathrm{d}\mu(z)$ converge to 0 as $|n| \to +\infty$. Equivalently, the corresponding orthogonal representation $\pi^\mu : \mathbf{Z} \to \mathcal{O}(H^\mu_{\mathbf{R}})$ is *mixing* in the sense that $\langle \pi^\mu(n)\zeta_1, \zeta_2\rangle \to 0$ as $|n| \to +\infty$ for all $\zeta_1, \zeta_2 \in H^\mu_{\mathbf{R}}$ (see e.g. [5, Chapter 14, pp. 369-371]).

By the Theorem, $\mathrm{L}(\mathbf{Z}) \subset \Gamma(H^\mu_{\mathbf{R}})'' \rtimes_{\pi^\mu} \mathbf{Z}$ is maximal amenable for all symmetric Rajchman measures μ on $\mathbf{T}$. By considering the $\mathrm{L}(\mathbf{Z})$-$\mathrm{L}(\mathbf{Z})$-bimodules $\mathrm{L}^2\left((\Gamma(H^\mu_{\mathbf{R}})'' \rtimes_{\pi^\mu} \mathbf{Z}) \ominus \mathrm{L}(\mathbf{Z})\right)$ and using a combination of results in [11], [13], we construct in Section 4 a Borel map $\eta : 2^{\mathbf{N}} \to \mathrm{Prob}(\mathbf{T}^2) : x \mapsto \eta_x$ such that:

- The Borel probability measures $(\eta_x)_{x\in 2^{\mathbf{N}}}$ are pairwise singular and all singular with respect to the Haar measure on $\mathbf{T}^2$.
- The measure class $[\eta_x]$ corresponds to an A-A-bimodule of the form $\mathrm{L}^2(M \ominus A)$ with $A \subset M$ maximal amenable masa in a II_1 factor.

In particular, we obtain the following.

Corollary. *There exists an explicit continuum $(\mathcal{H}(x))_{x\in 2^{\mathbf{N}}}$ of pairwise non-isomorphic A-A-bimodules of the form $\mathrm{L}^2(M \ominus A)$ where $A \subset M$ is an exotic maximal amenable masa in a II_1 factor.*

By Voiculescu's celebrated result [25, Corollary 7.6], the II_1 factors arising in the corollary are not $*$-isomorphic to interpolated free group factors in the sense of [6], [21]. Moreover, by [9, Theorem B], these II_1 factors are also *strongly solid* in the sense of [17, Section 4], that is, the normalizer of any diffuse amenable subalgebra generates an amenable subalgebra.

2. Preliminaries

2.1. Elementary facts on ε-orthogonality.

Definition 2.1. Let H be a Hilbert space, $K, L \subset H$ closed subspaces and $\varepsilon \geq 0$. We say that K and L are ε-*orthogonal* and write $K \perp_\varepsilon L$ if

$$|\langle \xi, \eta\rangle| \leq \varepsilon \|\xi\| \|\eta\|, \forall \xi \in K, \forall \eta \in L.$$

Observe that when $K \perp_\varepsilon L$ with $\varepsilon < 1$, we have that $K + L$ is closed. Let H be a Hilbert space and $p, q \in \mathbf{B}(H)$ be projections. We have that $pH \perp_\varepsilon qH$ if and only if $\|pq\|_\infty \leq \varepsilon$. Therefore, whenever $pH \perp_\varepsilon qH$, for all $\xi \in H$ we get

$$\begin{aligned}
\|p\xi\|^2 + \|q\xi\|^2 &= \|p(q\xi + (p \vee q - q)\xi)\|^2 + \|q\xi\|^2 \\
&= \|pq\xi + p(p \vee q - q)\xi\|^2 + \|q\xi\|^2 \\
&\leq \|pq\xi\|^2 + \|(p \vee q - q)\xi\|^2 + 2\|pq\xi\| \|(p \vee q - q)\xi\| + \|q\xi\|^2 \\
&\leq (1 + \varepsilon)^2 \|(p \vee q)\xi\|^2.
\end{aligned}$$

Lemma 2.2. *Let $0 \leq \varepsilon < \frac{1}{2}$. Let $p_1, p_2, p_3, p_4 \in \mathbf{B}(H)$ be projections which satisfy $p_i H \perp_\varepsilon p_j H$ for all $i, j \in \{1, 2, 3, 4\}$ such that $i \neq j$. We have*

$$(p_1 \vee p_2)H \perp_{\delta(\varepsilon)} (p_3 \vee p_4)H$$

with $\delta(\varepsilon) = \frac{2\varepsilon}{\sqrt{1-\varepsilon-\sqrt{2}\,\varepsilon\sqrt{1-\varepsilon}}}$.

Proof. We first prove the following easy fact: whenever $0 \leq \varepsilon < 1$ and $q_1, q_2, q_3 \in \mathbf{B}(H)$ are projections which satisfy $q_1 H \perp_\varepsilon q_2 H$, $q_2 H \perp_\varepsilon q_3 H$, $q_3 H \perp_\varepsilon q_1 H$, we have $(q_1 \vee q_2)H \perp_{\varepsilon'} q_3 H$ with $\varepsilon' = \frac{\sqrt{2}\,\varepsilon}{\sqrt{1-\varepsilon}}$. Indeed, let $\xi_i \in q_i H$ for $i = 1, 2$. We have

$$\|q_3(\xi_1 + \xi_2)\|^2 \leq 2(\|q_3\xi_1\|^2 + \|q_3\xi_2\|^2) \leq 2\varepsilon^2(\|\xi_1\|^2 + \|\xi_2\|^2).$$

Moreover, we have

$$\|\xi_1 + \xi_2\|^2 \geq \|\xi_1\|^2 + \|\xi_2\|^2 - 2\varepsilon\|\xi_1\|\|\xi_2\| \geq (1-\varepsilon)(\|\xi_1\|^2 + \|\xi_2\|^2).$$

Altogether, we get

$$\|q_3(\xi_1 + \xi_2)\|^2 \leq \frac{2\varepsilon^2}{1-\varepsilon}\|\xi_1 + \xi_2\|^2.$$

Now let $0 \leq \varepsilon < \frac{1}{2}$. Applying the fact, we get $(p_1 \vee p_2)H \perp_{\varepsilon'} p_3 H$ and $(p_1 \vee p_2)H \perp_{\varepsilon'} p_4 H$ with $\varepsilon' = \frac{\sqrt{2}\,\varepsilon}{\sqrt{1-\varepsilon}} < 1$. Applying the fact once more, we get $(p_1 \vee p_2)H \perp_{\varepsilon''} (p_3 \vee p_4)H$ with $\varepsilon'' = \frac{\sqrt{2}\,\varepsilon'}{\sqrt{1-\varepsilon'}} = \frac{2\varepsilon}{\sqrt{1-\varepsilon-\sqrt{2}\,\varepsilon\sqrt{1-\varepsilon}}}$. □

Write $\delta : [0, \frac{1}{2}) \to \mathbf{R}_+ : \varepsilon \mapsto \frac{2\varepsilon}{\sqrt{1-\varepsilon-\sqrt{2}\,\varepsilon\sqrt{1-\varepsilon}}}$ for the function which appears in Lemma 2.2.

Proposition 2.3. *Let $k \geq 1$. Let $0 \leq \varepsilon < 1$ such that $\delta^{\circ(k-1)}(\varepsilon) < 1$. For $1 \leq i \leq 2^k$, let $p_i \in \mathbf{B}(H)$ be projections such that $p_i H \perp_\varepsilon p_j H$ for all $i, j \in \{1, \dots, 2^k\}$ such that $i \neq j$. Write $P_\ell = \bigvee_{i=1}^{2^\ell} p_i$ for $1 \leq \ell \leq k$. Then for all $1 \leq \ell \leq k$ and all $\xi \in H$, we have*

$$\sum_{i=1}^{2^\ell} \|p_i\xi\|^2 \leq \prod_{j=0}^{\ell-1} \left(1 + \delta^{\circ j}(\varepsilon)\right)^2 \|P_\ell\xi\|^2.$$

Proof. We prove the result by induction on $k \geq 1$. It is clear for $k = 1$ as we observed above. Assume it is true for $k - 1 \geq 1$. Write $q_i = p_{2i-1} \vee p_{2i}$ for all $i \in \{1, \dots, 2^{k-1}\}$. By Lemma 2.2, we have $q_i H \perp_{\delta(\varepsilon)} q_j H$ for all $i, j \in \{1, \dots, 2^{k-1}\}$ such that $i \neq j$. Observe that $\bigvee_{i=1}^{2^{k-1}} q_i = P_k$. Since $\delta^{\circ(k-2)}(\delta(\varepsilon)) = \delta^{\circ(k-1)}(\varepsilon) < 1$, the induction hypothesis yields

$$\sum_{i=1}^{2^{k-1}} \|q_i\xi\|^2 \leq \prod_{j=0}^{k-2} \left(1 + \delta^{\circ j}(\delta(\varepsilon))\right)^2 \|P_k\xi\|^2 = \prod_{j=1}^{k-1} \left(1 + \delta^{\circ j}(\varepsilon)\right)^2 \|P_k\xi\|^2$$

for all $\xi \in H$. Since, moreover, we have

$$\|p_{2i-1}\xi\|^2 + \|p_{2i}\xi\|^2 \leq (1+\varepsilon)^2\|q_i\xi\|^2$$

for all $i \in \{1, \dots, 2^{k-1}\}$ and all $\xi \in H$, it follows that

$$\sum_{i=1}^{2^k} \|p_i\xi\|^2 \leq (1+\varepsilon)^2 \sum_{i=1}^{2^{k-1}} \|q_i\xi\|^2 \leq \prod_{j=0}^{k-1} \left(1 + \delta^{\circ j}(\varepsilon)\right)^2 \|P_k\xi\|^2, \forall \xi \in H.$$

□

2.2. **Voiculescu's free Gaussian functor** [23], [24]**.** Let $H_{\mathbf{R}}$ be a real separable Hilbert space. Let $H = H_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C} = H_{\mathbf{R}} \oplus \mathrm{i}H_{\mathbf{R}}$ be the corresponding complexified Hilbert space. The canonical complex conjugation on H will be simply denoted by $\overline{e + \mathrm{i}f} = e - \mathrm{i}f$ for all $e, f \in H_{\mathbf{R}}$. The *full Fock space* of H is defined by

$$\mathcal{F}(H) = \mathbf{C}\Omega \oplus \bigoplus_{n \geq 1} H^{\otimes n}.$$

The unit vector Ω is called the *vacuum vector.* For all $e \in H$, we define the *left creation operator*

$$\ell(e) : \mathcal{F}(H) \to \mathcal{F}(H) : \begin{cases} \ell(e)\Omega = e, \\ \ell(e)(e_1 \otimes \cdots \otimes e_n) = e \otimes e_1 \otimes \cdots \otimes e_n. \end{cases}$$

We have $\ell(e)^*\ell(f) = \langle e, f\rangle$ for all $e, h \in H$. In particular, $\ell(e)$ is an isometry for all unit vectors $e \in H$.

For all $e \in H_{\mathbf{R}}$, put $W(e) := \ell(e) + \ell(e)^*$. Voiculescu's result [24, Lemma 2.6.3] shows that the distribution of the selfadjoint operator $W(e)$ with respect to the vacuum vector state $\langle \cdot\Omega, \Omega\rangle$ is the semicircular law supported by the interval $[-2\|e\|, 2\|e\|]$. Moreover, [24, Lemma 2.6.6] shows that for any subset $\Xi \subset H_{\mathbf{R}}$ of pairwise orthogonal vectors, the family $\{W(e) : e \in \Xi\}$ is freely independent.

We denote by $\Gamma(H_{\mathbf{R}})$ the C*-algebra generated by $\{W(e) : e \in H_{\mathbf{R}}\}$ and $\Gamma(H_{\mathbf{R}})''$ the von Neumann algebra generated by $\Gamma(H_{\mathbf{R}})$. The vector state $\tau = \langle \cdot\Omega, \Omega\rangle$ is a faithful normal trace on $\Gamma(H_{\mathbf{R}})''$, and we have that $\Gamma(H_{\mathbf{R}})''$ is $*$-isomorphic to the free group factor on $\dim H_{\mathbf{R}}$ generators, that is, $\Gamma(H_{\mathbf{R}})'' \cong \mathrm{L}(\mathbf{F}_{\dim H_{\mathbf{R}}})$.

Since the vacuum vector Ω is separating and cyclic for $\Gamma(H_{\mathbf{R}})''$, any $x \in \Gamma(H_{\mathbf{R}})''$ is uniquely determined by $\xi = x\Omega \in \mathcal{F}(H)$. Thus we will write $x = W(\xi)$. Note that for $e \in H_{\mathbf{R}}$, we recover the semicircular random variables $W(e) = \ell(e) + \ell(e)^*$ generating $\Gamma(H_{\mathbf{R}})''$. More generally we have $W(e) = \ell(e) + \ell(\overline{e})^*$ for all $e \in H$. Given any vectors $e_i \in H$, it is easy to check that $e_1 \otimes \cdots \otimes e_n$ lies in $\Gamma(H_{\mathbf{R}})''\Omega$. The corresponding words $W(e_1 \otimes \cdots \otimes e_n) \in \Gamma(H_{\mathbf{R}})''$ enjoy useful properties that are summarized in the following.

Proposition 2.4. *Let $e_i \in H$, for $i \geq 1$. The following are true:*

(1) *We have the* Wick formula*:*

$$W(e_1 \otimes \cdots \otimes e_n) = \sum_{k=0}^{n} \ell(e_1) \cdots \ell(e_k)\ell(\overline{e}_{k+1})^* \cdots \ell(\overline{e}_n)^*.$$

(2) *If $\langle \overline{e}_r, e_{r+1}\rangle = 0$, then we have*

$$W(e_1 \otimes \cdots \otimes e_r)W(e_{r+1} \otimes \cdots \otimes e_n) = W(e_1 \otimes \cdots \otimes e_r \otimes e_{r+1} \otimes \cdots \otimes e_n).$$

(3) *We have $W(e_1 \otimes \cdots \otimes e_n)^* = W(\overline{e}_n \otimes \cdots \otimes \overline{e}_1)$.*

(4) *The linear span of $\{1, W(e_1 \otimes \cdots \otimes e_n) : n \geq 1, e_i \in H\}$ forms a unital weakly dense $*$-subalgebra of $\Gamma(H_{\mathbf{R}})''$.*

Proof. The proof of (1) is borrowed from [8, Lemma 3.2]. We prove the formula by induction on n. For $n \in \{0, 1\}$, we have $W(\Omega) = 1$, and we already observed that $W(e_i) = \ell(e_i) + \ell(\overline{e}_i)^*$.

Next, for $e_0 \in H$, we have

$$\begin{aligned} W(e_0)W(e_1 \otimes \cdots \otimes e_n)\Omega &= W(e_0)(e_1 \otimes \cdots \otimes e_n) \\ &= (\ell(e_0) + \ell(\overline{e}_0)^*)e_1 \otimes \cdots \otimes e_n \\ &= e_0 \otimes e_1 \otimes \cdots \otimes e_n + \langle \overline{e}_0, e_1\rangle\, e_2 \otimes \cdots \otimes e_n. \end{aligned}$$

So, we obtain

$$\begin{aligned}W(e_0 \otimes \cdots \otimes e_n) &= W(e_0)W(e_1 \otimes \cdots \otimes e_n) - \langle \overline{e}_0, e_1\rangle W(e_2 \otimes \cdots \otimes e_n)\\ &= \ell(\overline{e}_0)^* W(e_1 \otimes \cdots \otimes e_n) - \langle \overline{e}_0, e_1\rangle W(e_2 \otimes \cdots \otimes e_n)\\ &\quad + \ell(e_0)W(e_1 \otimes \cdots \otimes e_n).\end{aligned}$$

Using the assumption for n and $n-1$ and the relation $\ell(\overline{e}_0)^*\ell(e_1) = \langle \overline{e}_0, e_1\rangle$, we obtain

$$\ell(\overline{e}_0)^* W(e_1 \otimes \cdots \otimes e_n) = \langle \overline{e}_0, e_1\rangle W(e_2 \otimes \cdots \otimes e_n) + \ell(\overline{e}_0)^*\ell(\overline{e}_1)^* \cdots \ell(\overline{e}_n)^*.$$

Since $\ell(e_0)W(e_1 \otimes \cdots \otimes e_n)$ gives the last $n+1$ terms in the Wick formula at order $n+1$ and $\ell(\overline{e}_0)^*\ell(\overline{e}_1)^* \cdots \ell(\overline{e}_n)^*$ gives the first term, we are done.

(2) By the Wick formula, we have that $W(e_1 \otimes \cdots \otimes e_r)W(e_{r+1} \otimes \cdots \otimes e_n)$ is equal to

$$\sum_{0 \leq j \leq r \leq k \leq n} \ell(e_1) \cdots \ell(e_j)\ell(\overline{e}_{j+1})^* \cdots \ell(\overline{e}_r)^*\ell(e_{r+1}) \cdots \ell(e_k)\ell(\overline{e}_{k+1})^* \cdots \ell(\overline{e}_n)^*.$$

Whenever $j \leq r-1$ and $k \geq r+1$, since $\langle \overline{e}_r, e_{r+1}\rangle = 0$, we have

$$\ell(e_1) \cdots \ell(e_j)\ell(\overline{e}_{j+1})^* \cdots \ell(\overline{e}_r)^*\ell(e_{r+1}) \cdots \ell(e_k)\ell(\overline{e}_{k+1})^* \cdots \ell(\overline{e}_n)^* = 0.$$

Therefore the above sum simply equals

$$\sum_{0 \leq j \leq r-1} \ell(e_1) \cdots \ell(e_j)\ell(\overline{e}_{j+1})^* \cdots \ell(\overline{e}_n)^* + \sum_{r \leq k \leq n} \ell(e_1) \cdots \ell(e_k)\ell(\overline{e}_{k+1})^* \cdots \ell(\overline{e}_n)^*,$$

and so $W(e_1 \otimes \cdots \otimes e_r)W(e_{r+1} \otimes \cdots \otimes e_n) = W(e_1 \otimes \cdots \otimes e_r \otimes e_{r+1} \otimes \cdots \otimes e_n)$.

(3) It is a straightforward consequence of (1).

(4) Denote by $\mathcal{W}$ the linear span of $\{1, W(e_1 \otimes \cdots \otimes e_n) : n \geq 1, e_i \in H\}$. We only have to show that $\mathcal{W}$ is stable under taking products. Let $e_0, \ldots, e_m, f_1, \ldots, f_n \in H$. We prove by induction on m that $W(e_0 \otimes \cdots \otimes e_m)W(f_1 \otimes \cdots \otimes f_n) \in \mathcal{W}$. As we observed above, we have

$$W(e_0)W(f_1 \otimes \cdots \otimes f_n) = W(e_0 \otimes f_1 \otimes \cdots \otimes f_n) + \langle \overline{e}_0, f_1\rangle W(f_2 \otimes \cdots \otimes f_n) \in \mathcal{W},$$

so the result is true for $m = 0$. Assume it is true for all $0 \leq k \leq m-1$. We can write $W(e_0 \otimes \cdots \otimes e_m)W(f_1 \otimes \cdots \otimes f_n)$ as

$$W(e_0)W(e_1 \otimes \cdots \otimes e_m)W(f_1 \otimes \cdots \otimes f_n) - \langle \overline{e}_0, e_1\rangle W(e_2 \otimes \cdots \otimes e_m)W(f_1 \otimes \cdots \otimes f_n).$$

Using the induction hypothesis, we get that $W(e_0 \otimes \cdots \otimes e_m)W(f_1 \otimes \cdots \otimes f_n) \in \mathcal{W}$. This shows that $\mathcal{W}$ is a unital weakly dense $*$-subalgebra of $\Gamma(H_{\mathbf{R}})''$. □

Let G be a countable group together with an orthogonal representation $\pi : G \to \mathcal{O}(H_{\mathbf{R}})$. We shall still denote by $\pi : G \to \mathcal{U}(H)$ the corresponding unitary representation on the complexified Hilbert space $H = H_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$. The *free Bogoljubov action* $\sigma^\pi : G \curvearrowright (\Gamma(H_{\mathbf{R}})'', \tau)$ associated with the representation π is defined by

$$\sigma^\pi_g = \operatorname{Ad}(\mathcal{F}(\pi(g))), \forall g \in G,$$

where $\mathcal{F}(\pi(g)) = \mathrm{id}_{\mathbf{C}\Omega} \oplus \bigoplus_{n \geq 1} \pi(g)^{\otimes n} \in \mathcal{U}(\mathcal{F}(H))$.

Example 2.5. If $(\pi, H_{\mathbf{R}}) = (\lambda_G, \ell^2_{\mathbf{R}}(G))$ is the left regular orthogonal representation of G, then the action $\sigma^{\lambda_G} : G \curvearrowright \Gamma(\ell^2_{\mathbf{R}}(G))''$ is the free Bernoulli shift, and in that case we have

$$\left(\mathrm{L}(G) \subset \Gamma(\ell^2_{\mathbf{R}}(G))'' \rtimes_{\lambda_G} G\right) \cong \left(\mathrm{L}(G) \subset \mathrm{L}(\mathbf{Z}) * \mathrm{L}(G)\right).$$

Recall that an orthogonal representation $\pi : G \to \mathcal{O}(H_{\mathbf{R}})$ is *mixing* if

$$\lim_{g\to\infty} \langle \pi(g)\xi, \eta\rangle = 0$$

for all $\xi, \eta \in H_{\mathbf{R}}$.

Proposition 2.6. *Let G be a countable group together with an orthogonal representation $\pi : G \to \mathcal{O}(H_{\mathbf{R}})$. The following are equivalent:*

(1) *The representation $\pi : G \to \mathcal{O}(H_{\mathbf{R}})$ is mixing.*
(2) *The τ-preserving action $\sigma^\pi : G \curvearrowright \Gamma(H_{\mathbf{R}})''$ is mixing, that is,*

$$\lim_{g\to\infty} \tau(\sigma_g^\pi(x)y) = 0, \forall x, y \in \Gamma(H_{\mathbf{R}})'' \ominus \mathbf{C}.$$

Proof. (1) $\Rightarrow$ (2). Observe that since the linear span of

$$\{1, W(e_1 \otimes \cdots \otimes e_n) : n \geq 1, e_i \in H\}$$

is a unital weakly dense $*$-subalgebra of $\Gamma(H_{\mathbf{R}})''$, it suffices to show that $\tau(\sigma_g^\pi(x)y) \to 0$ as $g \to \infty$ for $x = W(e_1 \otimes \cdots \otimes e_m)$, $y = W(f_1 \otimes \cdots \otimes f_n)$. Using Proposition 2.4 (2), for all $g \in G$, we get

$$\tau(\sigma_g^\pi(x)y) = \frac{\langle \pi(g)e_m, \overline{f}_1\rangle}{\|\overline{f}_1\|^2} \tau\left(W(\pi(g)e_1 \otimes \cdots \otimes \pi(g)e_{m-1} \otimes \overline{f}_1)y\right).$$

Since π is mixing, we obtain $\lim_{g\to\infty} \tau(\sigma_g^\pi(x)y)$.

(2) $\Rightarrow$ (1). Let $e, f \in H_{\mathbf{R}}$. Using Proposition 2.4 (2), for all $g \in G$, we get

$$\lim_{g\to\infty} \langle \pi(g)e, f\rangle = \lim_{g\to\infty} \tau\left(\sigma_g^\pi(W(e))W(f)\right) = 0.$$

□

As a consequence of the previous proposition and [19, Theorem 3.1], we obtain that whenever $\pi : G \to \mathcal{O}(H_{\mathbf{R}})$ is a mixing representation of an abelian group G, $\mathrm{L}(G)$ is a *singular* masa in $\Gamma(H_{\mathbf{R}})'' \rtimes_\pi G$, that is,

$$\{u \in \mathcal{U}(\Gamma(H_{\mathbf{R}})'' \rtimes_\pi G) : u\,\mathrm{L}(G)u^* = \mathrm{L}(G)\} = \mathcal{U}(\mathrm{L}(G)).$$

Finally, recall from [9, Theorem 5.1] that whenever the orthogonal representation $\pi : G \to \mathcal{O}(H_{\mathbf{R}})$ is faithful, the associated free Bogoljubov action $\sigma^\pi : G \curvearrowright \Gamma(H_{\mathbf{R}})''$ is *properly outer*, that is, $\sigma_g^\pi \notin \mathrm{Inn}(\Gamma(H_{\mathbf{R}})'')$ for all $g \in G \setminus \{1\}$. In that case we have

$$\Gamma(H_{\mathbf{R}})' \cap (\Gamma(H_{\mathbf{R}})'' \rtimes_\pi G) = \Gamma(H_{\mathbf{R}})' \cap \Gamma(H_{\mathbf{R}})'' = \mathbf{C},$$

and so $\Gamma(H_{\mathbf{R}})'' \rtimes_\pi G$ is a II_1 factor.

3. The asymptotic orthogonality property

We refer to [2, Appendix A] for a brief account on ultrafilters and ultraproducts of tracial von Neumann algebras.

Definition 3.1 ([18])**.** Let (M, τ) be a tracial von Neumann algebra. We say that a von Neumann subalgebra $A \subset M$ has the *asymptotic orthogonality property* if there exists a free ultrafilter ω on $\mathbf{N}$ such that for all $x, y \in (M^\omega \ominus A^\omega) \cap A'$ and all $a, b \in M \ominus A$, the vectors ax and yb are orthogonal in $\mathrm{L}^2(M^\omega, \tau_\omega)$.

Popa proved in [18, Lemma 2.1] that the generator masa in free group factors satisfies the asymptotic orthogonality property. The main result of this section is the following.

Theorem 3.2. *Let G be an infinite countable abelian group and $\pi : G \to \mathcal{O}(H_{\mathbf{R}})$ a mixing orthogonal representation. Denote by $\Gamma(H_{\mathbf{R}})'' \rtimes_\pi G$ the crossed product von Neumann algebra. Then $\mathrm{L}(G) \subset \Gamma(H_{\mathbf{R}})'' \rtimes_\pi G$ has the asymptotic orthogonality property.*

Proof. We denote by $H = H_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$ the complexification of $H_{\mathbf{R}}$ and $\mathcal{H} = \mathcal{F}(H)$ the full Fock space of H. The conjugation on H is simply denoted by $e \mapsto \overline{e}$. We still denote by $\pi : G \to \mathcal{U}(H)$ the corresponding unitary representation. Observe that $\pi(g)\overline{e} = \overline{\pi(g)e}$ for all $g \in G$ and all $e \in H$. Let $\sigma : G \curvearrowright \Gamma(H_{\mathbf{R}})''$ be the free Bogoljubov action associated with π and $\rho : G \to \mathcal{U}(\mathcal{H})$ the Koopman representation of the action σ. Observe that

$$\rho(g) = \mathrm{id}_{\mathbf{C}\Omega} \oplus \bigoplus_{n \geq 1} \pi(g)^{\otimes n}, \forall g \in G.$$

Put $Q = \Gamma(H_{\mathbf{R}})''$ and $M = \Gamma(H_{\mathbf{R}})'' \rtimes_\pi G$. We will always identify the GNS-Hilbert space $\mathrm{L}^2(Q)$ with the full Fock space $\mathcal{H}$ via the unitary operator

$$U : \mathrm{L}^2(Q) \to \mathcal{H} : \begin{cases} 1 \mapsto \Omega, \\ W(e_1 \otimes \cdots \otimes e_n) \mapsto e_1 \otimes \cdots \otimes e_n. \end{cases}$$

We denote by $J : \mathcal{H} \to \mathcal{H}$ the canonical conjugation

$$J\Omega = \Omega \quad \text{and} \quad J(e_1 \otimes \cdots \otimes e_n) = \overline{e}_n \otimes \cdots \otimes \overline{e}_1.$$

We will identify $\mathrm{L}^2(M)$ with $\mathcal{H} \otimes \ell^2(G)$ via the unitary operator $\mathrm{L}^2(M) \ni au_h \mapsto U(a) \otimes \delta_h \in \mathcal{H} \otimes \ell^2(G)$. We denote by $\mathcal{J} : \mathcal{H} \otimes \ell^2(G) \to \mathcal{H} \otimes \ell^2(G)$ the conjugation $\mathcal{J}(\xi \otimes \delta_g) = J\rho(g^{-1})\xi \otimes \delta_{g^{-1}}$.

With a proof that is very similar to [18, Lemma 2.1], we will reach the conclusion of Theorem 3.2. We fix once and for all a free ultrafilter ω on $\mathbf{N}$. We want to show that $ax \perp yb$ for all $x, y \in (M^\omega \ominus \mathrm{L}(G)^\omega) \cap \mathrm{L}(G)'$ and all $a, b \in M \ominus \mathrm{L}(G)$. Note that since G is abelian, we have $xu_g \in (M^\omega \ominus \mathrm{L}(G)^\omega) \cap \mathrm{L}(G)'$ whenever $x \in (M^\omega \ominus \mathrm{L}(G)^\omega) \cap \mathrm{L}(G)'$. So, using the Kaplansky density theorem, it suffices to show that $ax \perp yb$ for all $x, y \in (M^\omega \ominus \mathrm{L}(G)^\omega) \cap \mathrm{L}(G)'$ and a, b of the form $a = W(\xi_1 \otimes \cdots \otimes \xi_s)$, $b = W(\eta_1 \otimes \cdots \otimes \eta_t)$.

From now on, we fix $a = W(\xi_1 \otimes \cdots \otimes \xi_s)$ and $b = W(\eta_1 \otimes \cdots \otimes \eta_t)$. Denote by $K \subset H$ the finite dimensional subspace generated by ξ_i and η_j and write $r = \dim K$. We will further assume that $\|\xi_i\| = \|\eta_j\| = 1$ for all i, j and that $K = \overline{K}$.

Fix $h \in G$. Denote by $\mathcal{X}_h \subset \mathcal{H} \ominus (\mathbf{C}\Omega \oplus H)$ the closed linear span of all the words $e_1 \otimes \cdots \otimes e_n$ where $n \geq 2$ and such that the first letter e_1 belongs to K or the last letter e_n belongs to $\pi(h)K$. Using the above identification, we can then split $\mathcal{X}_h$ as an orthogonal sum $\mathcal{X}_h = \mathcal{X}_h^1 \oplus \mathcal{X}_h^2 \oplus \mathcal{X}_h^3$ such that

$$\begin{aligned} \mathcal{X}_h^1 &= K \otimes \mathcal{H} \otimes \pi(h)K, \\ \mathcal{X}_h^2 &= K \otimes \mathcal{H} \otimes (\pi(h)K)^\perp, \\ \mathcal{X}_h^3 &= K^\perp \otimes \mathcal{H} \otimes \pi(h)K. \end{aligned}$$

Likewise, denote by $\mathcal{Y}_h \subset \mathcal{H} \ominus (\mathbf{C}\Omega \oplus H)$ the closed linear span of all the words $e_1 \otimes \cdots \otimes e_n$ where $n \geq 2$ and such that the first letter e_1 belongs to $K^\perp$ and the last letter e_n belongs to $(\pi(h)K)^\perp$, that is, $\mathcal{Y}_h = K^\perp \otimes \mathcal{H} \otimes (\pi(h)K)^\perp$. Therefore we have

$$\mathcal{H} \ominus \mathbf{C}\Omega = K \oplus K^\perp \oplus \mathcal{X}_h \oplus \mathcal{Y}_h = K \oplus K^\perp \oplus \mathcal{X}_h^1 \oplus \mathcal{X}_h^2 \oplus \mathcal{X}_h^3 \oplus \mathcal{Y}_h.$$

Claim. For every $\varepsilon > 0$, there exists a finite subset $\mathcal{F}_\varepsilon \subset G$ such that

$$\rho(g)K \perp_\varepsilon K \text{ and } \rho(g)\mathcal{X}_h^i \perp_\varepsilon \mathcal{X}_h^i$$

for all $g \in G \setminus \mathcal{F}_\varepsilon$, all $i \in \{1, 2, 3\}$ and all $h \in G$.

Proof of the Claim. Fix $\varepsilon > 0$. Let $\zeta_1, \dots, \zeta_r$ be an orthonormal basis of K. Since π is a mixing representation, there exists a finite subset $\mathcal{F}_\varepsilon \subset G$ such that $|\langle \pi(g)\zeta_i, \zeta_j\rangle| \leq \varepsilon/r$ for all $g \in G \setminus \mathcal{F}_\varepsilon$ and all $i, j \in \{1, \dots, r\}$. Observe that since G is abelian, we also have

$$(1) \qquad |\langle \pi(g)\pi(h)\zeta_i, \pi(h)\zeta_j\rangle| \leq \frac{\varepsilon}{r}, \forall g \in G \setminus \mathcal{F}_\varepsilon, \forall h \in G, \forall i, j \in \{1, \dots, r\}.$$

Let $\xi, \eta \in K \otimes \mathcal{H}$ that we write $\xi = \sum_{i=1}^r \zeta_i \otimes e_i$ and $\eta = \sum_{j=1}^r \zeta_j \otimes f_j$, with $e_i, f_j \in \mathcal{H}$. Note that $\|\xi\|^2 = \sum_{i=1}^r \|e_i\|^2$ and $\|\eta\|^2 = \sum_{j=1}^r \|f_j\|^2$. We have $\rho(g)\xi = \sum_{i=1}^r \pi(g)\zeta_i \otimes \rho(g)e_i$. Thus, for all $g \in G \setminus \mathcal{F}_\varepsilon$, using the Cauchy-Schwarz inequality, we get

$$|\langle \rho(g)\xi, \eta\rangle| \leq \sum_{i,j=1}^r |\langle \pi(g)\zeta_i, \zeta_j\rangle||\langle \rho(g)e_i, f_j\rangle| \leq \frac{\varepsilon}{r}\sum_{i,j=1}^r \|e_i\|\|f_j\| \leq \varepsilon\|\xi\|\|\eta\|.$$

So, we obtain $\rho(g)(K \otimes \mathcal{H}) \perp_\varepsilon K \otimes \mathcal{H}$ and, in particular, $\rho(g)K \perp_\varepsilon K$, $\rho(g)\mathcal{X}_h^1 \perp_\varepsilon \mathcal{X}_h^1$, $\rho(g)\mathcal{X}_h^2 \perp_\varepsilon \mathcal{X}_h^2$ for all $g \in G \setminus \mathcal{F}_\varepsilon$ and all $h \in G$.

Likewise, using inequality (1), we obtain $\rho(g)(\mathcal{H} \otimes \pi(h)K) \perp_\varepsilon \mathcal{H} \otimes \pi(h)K$ and, in particular, $\rho(g)\mathcal{X}_h^3 \perp_\varepsilon \mathcal{X}_h^3$ for all $g \in G \setminus \mathcal{F}_\varepsilon$ and all $h \in G$. □

Let $x \in (M^\omega \ominus \mathrm{L}(G)^\omega) \cap \mathrm{L}(G)'$. We may and will always represent x by a sequence (x_n) such that $\sup_n \|x_n\|_\infty \leq 1$, $x_n \in M \ominus \mathrm{L}(G)$, and $\lim_{n\to\omega} \|u_g x_n u_g^* - x_n\|_2 = 0$ for all $g \in G$. Write $x_n = \sum_{h \in G}(x_n)^h u_h$ for the Fourier expansion of x_n in M with respect to the crossed product decomposition $M = Q \rtimes G$. Observe that $(x_n)^h \in Q \ominus \mathbf{C}$ for all $n \in \mathbf{N}$ and all $h \in G$. Define subspaces of $(\mathcal{H} \ominus \mathbf{C}\Omega) \otimes \ell^2(G)$ by $\mathscr{X} = \bigoplus_{h\in G}(\mathcal{X}_h \otimes \mathbf{C}\delta_h)$ and $\mathscr{Y} = \bigoplus_{h \in G}(\mathcal{Y}_h \otimes \mathbf{C}\delta_h)$. Under the previous identification, we then have

$$(2) \qquad \mathrm{L}^2(M \ominus \mathrm{L}(G)) = (K \otimes \ell^2(G)) \oplus (K^\perp \otimes \ell^2(G)) \oplus \mathscr{X} \oplus \mathscr{Y}.$$

Step 1. For all $x = (x_n) \in (M^\omega \ominus \mathrm{L}(G)^\omega) \cap \mathrm{L}(G)'$, we have

$$\lim_{n\to\omega} \|P_{(K\otimes\ell^2(G))\oplus\mathscr{X}}(x_n)\|_2 = 0.$$

Proof of Step 1. We will be using the notation $\mathcal{X}_h^0 := K$ for all $h \in G$. For all $g, h \in G$, all $i \in \{0, 1, 2, 3\}$ and all $n \in \mathbf{N}$, we have

$$\begin{aligned}
\|P_{\mathcal{X}_h^i}((x_n)^h)\|_2^2 &= \|\rho(g)P_{\mathcal{X}_h^i}((x_n)^h)\|_2^2 \\
&= \|\rho(g)P_{\mathcal{X}_h^i}((x_n)^h) - P_{\rho(g)\mathcal{X}_h^i}((x_n)^h) + P_{\rho(g)\mathcal{X}_h^i}((x_n)^h)\|_2^2 \\
&\leq 2\|\rho(g)P_{\mathcal{X}_h^i}((x_n)^h) - P_{\rho(g)\mathcal{X}_h^i}((x_n)^h)\|_2^2 + 2\|P_{\rho(g)\mathcal{X}_h^i}((x_n)^h)\|_2^2 \\
&= 2\|P_{\rho(g)\mathcal{X}_h^i}(u_g(x_n)^h u_g^* - (x_n)^h)\|_2^2 + 2\|P_{\rho(g)\mathcal{X}_h^i}((x_n)^h)\|_2^2 \\
&\leq 2\|\sigma_g((x_n)^h) - (x_n)^h\|_2^2 + 2\|P_{\rho(g)\mathcal{X}_h^i}((x_n)^h)\|_2^2.
\end{aligned}$$

Fix $k \geq 1$. Choose $\varepsilon > 0$ very small such that $\prod_{\ell=0}^{k-1}(1 + \delta^{\circ\ell}(\varepsilon))^2 \leq 2$, where $\delta : [0, \frac{1}{2}) \to \mathbf{R}$ is the function which appeared in Lemma 2.2. Then choose a finite subset $\mathcal{F}_\varepsilon \subset G$ according to the Claim. Finally, choose a subset $\mathcal{G} \subset G$ of cardinality 2^k with the property that $s^{-1}t \in G \setminus \mathcal{F}_\varepsilon$ whenever $s, t \in \mathcal{G}$ such that $s \neq t$. So, we have that $\rho(s)\mathcal{X}_h^i \perp_\varepsilon \rho(t)\mathcal{X}_h^i$ for all $s, t \in \mathcal{G}$ such that $s \neq t$, all $i \in \{0, 1, 2, 3\}$ and all $h \in G$. Therefore, using Proposition 2.3 and the above inequality, we get

$$\begin{aligned}
2^k \|P_{\mathcal{X}_h^i}((x_n)^h)\|_2^2 &= \sum_{g\in\mathcal{G}} \|\rho(g)P_{\mathcal{X}_h^i}((x_n)^h)\|_2^2 \\
&\leq \sum_{g\in\mathcal{G}} \left(2\|\sigma_g((x_n)^h) - (x_n)^h\|_2^2 + 2\|P_{\rho(g)\mathcal{X}_h^i}((x_n)^h)\|_2^2\right) \\
&= 2\sum_{g\in\mathcal{G}} \|\sigma_g((x_n)^h) - (x_n)^h\|_2^2 + 2\sum_{g\in\mathcal{G}} \|P_{\rho(g)\mathcal{X}_h^i}((x_n)^h)\|_2^2 \\
&\leq 2\sum_{g\in\mathcal{G}} \|\sigma_g((x_n)^h) - (x_n)^h\|_2^2 + 2\prod_{\ell=0}^{k-1}(1+\delta^{\circ\ell}(\varepsilon))^2\|(x_n)^h\|_2^2 \\
&\leq 2\sum_{g\in\mathcal{G}} \|\sigma_g((x_n)^h) - (x_n)^h\|_2^2 + 4\|(x_n)^h\|_2^2.
\end{aligned}$$

Finally, since G is abelian, summing up over all $h \in G$ and all $i \in \{0, 1, 2, 3\}$, we get

$$2^k \|P_{(K\otimes\ell^2(G))\oplus\mathscr{X}}(x_n)\|_2^2 \leq 8\sum_{g\in\mathcal{G}} \|u_g x_n u_g^* - x_n\|_2^2 + 16\|x_n\|_2^2.$$

This yields $\lim_{n\to\omega} \|P_{(K\otimes\ell^2(G))\oplus\mathscr{X}}(x_n)\|_2^2 \leq 2^{4-k}$. Since this is true for every $k \geq 1$, we finally get $\lim_{n\to\omega} \|P_{(K\otimes\ell^2(G))\oplus\mathscr{X}}(x_n)\|_2 = 0$. ◻

Step 2. We have

$$a\left((K^\perp \otimes \ell^2(G)) \oplus \mathscr{Y}\right) \perp \mathcal{J}b^*\mathcal{J}\mathscr{Y} \text{ and } a\mathscr{Y} \perp \mathcal{J}b^*\mathcal{J}\left((K^\perp \otimes \ell^2(G)) \oplus \mathscr{Y}\right)$$

in the Hilbert space $\mathcal{H} \otimes \ell^2(G)$.

Proof of Step 2. We first prove that $a\left((K^\perp \otimes \ell^2(G)) \oplus \mathscr{Y}\right) \perp \mathcal{J}b^*\mathcal{J}\mathscr{Y}$. Recall that $a = W(\xi_1 \otimes \cdots \otimes \xi_s)$ and $b = W(\eta_1 \otimes \cdots \otimes \eta_t)$ with $\xi_i, \eta_j \in K$. Using the Fourier decomposition, it suffices to show that for all $h \in G$, $a(K^\perp \oplus \mathcal{Y}_h) \perp J\sigma_h(b)^*J\mathcal{Y}_h$ in the Hilbert space $\mathcal{H}$.

Let $e_1 \otimes \cdots \otimes e_m$ be an elementary word in $K^\perp \oplus \mathcal{Y}_h$ with $e_1 \in K^\perp$ (possibly $m = 1$). Let $f_1 \otimes \cdots \otimes f_n$ be an elementary word in $\mathcal{Y}_h$ with $n \geq 2$, $f_1 \in K^\perp$ and $f_n \in (\pi(h)K)^\perp$. Proposition 2.4 yields

$$\begin{aligned}
a(e_1 \otimes \cdots \otimes e_m) &= \xi_1 \otimes \cdots \otimes \xi_s \otimes e_1 \otimes \cdots \otimes e_m, \\
J\sigma_h(b)^*J(f_1 \otimes \cdots \otimes f_n) &= f_1 \otimes \cdots \otimes f_n \otimes \pi(h)\eta_1 \otimes \cdots \otimes \pi(h)\eta_t.
\end{aligned}$$

Since $\xi_1 \in K$ and $f_1 \in K^\perp$, we get $a(e_1 \otimes \cdots \otimes e_m) \perp J\sigma_h(b)^*J(f_1 \otimes \cdots \otimes f_n)$. This shows that $a(K^\perp \oplus \mathcal{Y}_h) \perp J\sigma_h(b)^*J\mathcal{Y}_h$ in the Hilbert space $\mathcal{H}$.

Since $K = \overline{K}$, a^* and b^* have all their letters in K and the above proof shows that $\mathcal{J}b\mathcal{J}\mathscr{Y} \perp a^*\left((K^\perp \otimes \ell^2(G)) \oplus \mathscr{Y}\right)$. Since a and $\mathcal{J}b\mathcal{J}$ commute, we finally obtain that $a\mathscr{Y} \perp \mathcal{J}b^*\mathcal{J}\left((K^\perp \otimes \ell^2(G)) \oplus \mathscr{Y}\right)$. ◻

Step 3. Let $x, y \in (M^\omega \ominus \mathrm{L}(G)^\omega) \cap \mathrm{L}(G)'$. Then we have

$$\lim_{n\to\omega} \langle aP_{K^\perp\otimes\ell^2(G)}(x_n), \mathcal{J}b^*\mathcal{J}P_{K^\perp\otimes\ell^2(G)}(y_n)\rangle = 0.$$

Proof of Step 3. Let $x, y \in (M^\omega \ominus \mathrm{L}(G)^\omega) \cap \mathrm{L}(G)'$. Write $P_{K^\perp\otimes\ell^2(G)}(x_n) = \sum_{h\in G} W(e_n^h)\Omega \otimes \delta_h$ and $P_{K^\perp\otimes\ell^2(G)}(y_n) = \sum_{h\in G} W(f_n^h)\Omega \otimes \delta_h$ with $e_n^h, f_n^h \in K^\perp$.

Note that $\sum_{h\in G} \|e_n^h\|^2 \leq \|x_n\|_2^2$ and $\sum_{h\in G} \|f_n^h\|^2 \leq \|y_n\|_2^2$. Using Proposition 2.4, a simple calculation shows that

$$aP_{K^\perp\otimes\ell^2(G)}(x_n) = \sum_{h\in G} W(\xi_1 \otimes \cdots \otimes \xi_s \otimes e_n^h)\Omega \otimes \delta_h,$$
$$\mathcal{J}b^*\mathcal{J}P_{K^\perp\otimes\ell^2(G)}(y_n) = \sum_{h\in G} W(f_n^h)W(\pi(h)\eta_1 \otimes \cdots \otimes \pi(h)\eta_t)\Omega \otimes \delta_h.$$

Put $A_n := \langle aP_{K^\perp\otimes\ell^2(G)}(x_n), \mathcal{J}b^*\mathcal{J}P_{K^\perp\otimes\ell^2(G)}(y_n)\rangle$. Therefore, again using Proposition 2.4, we obtain

$$\begin{aligned}
A_n &= \sum_{h\in G} \langle W(\xi_1 \otimes \cdots \otimes \xi_s \otimes e_n^h)\Omega, W(f_n^h)W(\pi(h)\eta_1 \otimes \cdots \otimes \pi(h)\eta_t)\Omega\rangle \\
&= \sum_{h\in G} \langle W(f_n^h)^* W(\xi_1 \otimes \cdots \otimes \xi_s \otimes e_n^h)\Omega, W(\pi(h)\eta_1 \otimes \cdots \otimes \pi(h)\eta_t)\Omega\rangle \\
&= \sum_{h\in G} \langle W(\overline{f}_n^h \otimes \xi_1 \otimes \cdots \otimes \xi_s \otimes e_n^h)\Omega, W(\pi(h)\eta_1 \otimes \cdots \otimes \pi(h)\eta_t)\Omega\rangle \\
&= \sum_{h\in G} \langle \overline{f}_n^h \otimes \xi_1 \otimes \cdots \otimes \xi_s \otimes e_n^h, \pi(h)\eta_1 \otimes \cdots \otimes \pi(h)\eta_t\rangle.
\end{aligned}$$

Note that if $t \neq s+2$, then $A_n = 0$ for all $n \in \mathbf{N}$, whence $\lim_{n\to\omega} A_n = 0$.

Next, assume that $t = s+2$ and fix $\varepsilon > 0$. Since π is mixing, there exists a finite subset $\mathcal{F} \subset G$ such that for all $h \in G \setminus \mathcal{F}$, we have $|\langle \xi_1, \pi(h)\eta_2\rangle| \leq \varepsilon$. So, for all $h \in G \setminus \mathcal{F}$, we have

$$|\langle \overline{f}_n^h \otimes \xi_1 \otimes \cdots \otimes \xi_s \otimes e_n^h, \pi(h)\eta_1 \otimes \cdots \otimes \pi(h)\eta_t\rangle| \leq \varepsilon \|e_n^h\| \|f_n^h\|.$$

By the Cauchy-Schwarz inequality, for all $n \in \mathbf{N}$, we have

$$\sum_{h\in G\setminus\mathcal{F}} |\langle \overline{f}_n^h \otimes \xi_1 \otimes \cdots \otimes \xi_s \otimes e_n^h, \pi(h)\eta_1 \otimes \cdots \otimes \pi(h)\eta_t\rangle| \leq \varepsilon \|x_n\|_2 \|y_n\|_2 \leq \varepsilon.$$

For all $g \in G$, since $\rho(g)(H \otimes \ell^2(G)) = H \otimes \ell^2(G)$, we have

$$\begin{aligned}
\sum_{h\in G} (\pi(g)e_n^h - e_n^h) \otimes \delta_h &= \rho(g)P_{K^\perp\otimes\ell^2(G)}(x_n) - P_{K^\perp\otimes\ell^2(G)}(x_n) \\
&= P_{H\otimes\ell^2(G)}(u_g x_n u_g^* - x_n) + (1-\rho(g))P_{K\otimes\ell^2(G)}(x_n).
\end{aligned}$$

Using the fact that $x, y \in (M^\omega \ominus \mathrm{L}(G)^\omega) \cap \mathrm{L}(G)'$ together with Step 1, we get that for all $g, h \in G$, $\lim_{n\to\omega} \|\pi(g)e_n^h - e_n^h\| = 0$ and $\lim_{n\to\omega} \|\pi(g)f_n^h - f_n^h\| = 0$. Since π is mixing and thus ergodic, we get that $e_n^h \to 0$ and $f_n^h \to 0$ weakly in H as $n \to \omega$ for all $h \in G$. Since $\mathcal{F}$ is finite, this implies

$$\lim_{n\to\omega} \sum_{h\in\mathcal{F}} |\langle \overline{f}_n^h \otimes \xi_1 \otimes \cdots \otimes \xi_s \otimes e_n^h, \pi(h)\eta_1 \otimes \cdots \otimes \pi(h)\eta_t\rangle| = 0.$$

Thus we have $\lim_{n\to\omega} |A_n| \leq \varepsilon$. Since this is true for every $\varepsilon > 0$, we get $\lim_{n\to\omega} A_n = 0$. ☐

Let $x, y \in (M^\omega \ominus \mathrm{L}(G)^\omega) \cap \mathrm{L}(G)'$. By combining Steps 2 and 3, we obtain

$$\lim_{n\to\omega} \langle aP_{(K^\perp\otimes\ell^2(G))\oplus\mathscr{Y}}(x_n), \mathcal{J}b^*\mathcal{J}P_{(K^\perp\otimes\ell^2(G))\oplus\mathscr{Y}}(y_n)\rangle = 0.$$

Moreover, Step 1 yields

$$\lim_{n\to\omega} \|P_{(K\otimes\ell^2(G))\oplus\mathscr{X}}(x_n)\|_2 = 0 \text{ and } \lim_{n\to\omega} \|P_{(K\otimes\ell^2(G))\oplus\mathscr{X}}(y_n)\|_2 = 0.$$

Therefore, thanks to equality (2), we finally get

$$\langle ax, yb\rangle_{\mathrm{L}^2(M^\omega)} = \lim_{n\to\omega}\langle ax_n, y_n b\rangle_{\mathrm{L}^2(M)} = \lim_{n\to\omega}\langle ax_n, \mathcal{J}b^*\mathcal{J}y_n\rangle_{\mathrm{L}^2(M)} = 0.$$

As we mentioned before, this finishes the proof. □

4. Proof of the Theorem and the Corollary

We prove a stronger version of the main Theorem.

Theorem 4.1. *Let G be a countable infinite abelian group and $\pi : G \to \mathcal{O}(H_{\mathbf{R}})$ a faithful mixing orthogonal representation. Then for any intermediate von Neumann subalgebra $\mathrm{L}(G) \subset P \subset \Gamma(H_{\mathbf{R}})'' \rtimes_\pi G$, there exist pairwise orthogonal projections $p_n \in \mathcal{Z}(P)$ with $\sum_{n\geq 0} p_n = 1$ such that*

- *$Pp_0 = \mathrm{L}(G)p_0$ and*
- *Pp_n is a non-Gamma II_1 factor for all $n \geq 1$.*

Proof. Put $A = \mathrm{L}(G)$ and $M = \Gamma(H_{\mathbf{R}})'' \rtimes G$. Since $A \subset M$ is a masa, we have $\mathcal{Z}(P) \subset A$. Denote by $p \in \mathcal{Z}(P)$ the maximal projection such that Pp is amenable. Then $P(1-p)$ has no amenable direct summand.

By [9, Theorem 3.10 and Theorem 5.1], M is a strongly solid II_1 factor and in particular *solid* in the sense of [16]; that is, the relative commutant of any diffuse subalgebra of M must be amenable. (The strong solidity result [9, Theorem 3.10] is only stated for mixing orthogonal representations of $\mathbf{Z}$, but the same proof works for any countable infinite abelian group G as well.) Since $P(1-p)$ has no amenable direct summand and is solid, we get that its center $\mathcal{Z}(P(1-p)) = \mathcal{Z}(P)(1-p)$ is purely atomic. Denote by p_n, $n \geq 1$, the minimal projections of $\mathcal{Z}(P)(1-p)$. For every $n \geq 1$, since the II_1 factor Pp_n is solid and non-amenable, it does not have property Gamma by [16, Proposition 7].

It remains to prove that $Ap = Pp$. The rest of the proof is now identical to the one of [3, Corollary 2.3], but we nevertheless give a detailed proof for the sake of completeness. We first show that Pp is of type I. Indeed, assume by contradiction that there exists a non-zero projection $q \in \mathcal{Z}(P)p$ such that Pq is of type II_1. Since Pq is hyperfinite by Connes' result [4], we may find an increasing sequence $Q_k \subset Pq$ of finite dimensional unital $*$-subalgebras such that $\bigvee_{k\geq 1} Q_k = Pq$. Since $Q_k' \cap Pq$ is of type II_1 and A is abelian, [19, Corollary 2.3] yields a unitary $u_k \in \mathcal{U}(Q_k' \cap Pq)$ such that $\|E_A(u_k)\|_2 \leq \frac{1}{k}$ for all $k \geq 1$. Therefore, the sequence (u_k) represents a unitary $u \in \mathcal{U}((Pq)' \cap (Pq)^\omega)$ such that $E_{A^\omega}(u) = 0$. Since $Aq \subset Pq$ is a masa in a type II_1 von Neumann algebra, we may find a unitary $v \in \mathcal{U}(Pq)$ such that $E_A(v) = E_{Aq}(v) = 0$. By Theorem 3.2, we get that vu and uv are orthogonal in $\mathrm{L}^2(M^\omega, \tau_\omega)$. Since moreover $vu = uv$, we obtain $uv = 0$, whence $q = (uv)^*(uv) = 0$. This is a contradiction. Therefore Pp is of type I. Since $A \subset Pp \oplus A(1-p)$ is a masa in a finite type I von Neumann algebra, we have that A is regular inside $Pp \oplus A(1-p)$ by [10, Theorem 3.19]. By the singularity of A, we get $A = Pp \oplus A(1-p)$, and so $Ap = Pp$. □

Proof of the Corollary. The proof is very similar to the one of [12, Theorem 5.7]. For $\mu \in \mathrm{Prob}(\mathbf{T})$ a Borel probability measure on $\mathbf{T}$, we use the notation $\mu^\infty = \sum_{n\geq 1} \frac{1}{2^n}\mu^{*n}$. Write $\mathrm{supp}(\mu)$ for the *topological support* of μ, that is,

$$\mathrm{supp}(\mu) = \bigcap \{F \subset \mathbf{T} \text{ closed subset} : \mu(F) = 1\}.$$

We have $\operatorname{supp}(\mu * \nu) \subset \operatorname{supp}(\mu)\operatorname{supp}(\nu)$ for all $\mu, \nu \in \operatorname{Prob}(\mathbf{T})$. Define the real Hilbert space

$$H^\mu_{\mathbf{R}} = \{\zeta \in \mathrm{L}^2(\mathbf{T}, \mu) : \overline{\zeta(z)} = \zeta(\overline{z}) \ \mu\text{-almost everywhere}\}$$

and the orthogonal representation

$$\pi^\mu : \mathbf{Z} \to \mathcal{O}(H^\mu_{\mathbf{R}}) : (\pi^\mu(n)\zeta)(z) = z^n \zeta(z).$$

Observe that the complexification of $H^\mu_{\mathbf{R}}$ is simply $\mathrm{L}^2(\mathbf{T}, \mu)$. The corresponding unitary representation on $\mathrm{L}^2(\mathbf{T}, \mu)$ will still be denoted by π^μ.

Using a combination of [11, VIII, 3, Théorème II] and [13, VII, 1, Theorem 7], there exists a closed independent[1] set $\Lambda \subset \mathbf{T}$ and a Borel map $2^{\mathbf{N}} \ni x \mapsto \mu_x \in \operatorname{Prob}(\mathbf{T})$ such that:

- For all $x \in 2^{\mathbf{N}}$, μ_x is a symmetric Rajchman measure such that $\operatorname{supp}(\mu_x) \subset \Lambda \cup \overline{\Lambda}$.
- For all $x, y \in 2^{\mathbf{N}}$ such that $x \neq y$, we have $\operatorname{supp}(\mu_x) \cap \operatorname{supp}(\mu_y) = \emptyset$.

Since μ_x is a Rajchman measure, $\mathrm{L}(\mathbf{Z}) \subset \Gamma(H^{\mu_x}_{\mathbf{R}})'' \rtimes_{\pi^{\mu_x}} \mathbf{Z}$ is maximal amenable by the theorem. Put

$$A = \mathrm{L}(\mathbf{Z}) \text{ and } {}_A\mathcal{H}(x)_A = {}_A\,\mathrm{L}^2((\Gamma(H^{\mu_x}_{\mathbf{R}})'' \rtimes_{\pi^{\mu_x}} \mathbf{Z}) \ominus \mathrm{L}(\mathbf{Z}))_A.$$

We have $\operatorname{supp}(\mu_x^{*n}) \subset (\Lambda \cup \overline{\Lambda})^n$ for all $x \in 2^{\mathbf{N}}$ and all $n \geq 1$. Since the measures $(\mu_x)_{x \in 2^{\mathbf{N}}}$ are atomless with pairwise disjoint supports and Λ is a closed independent set, we obtain that the measures μ_x^{*n} for $x \in 2^{\mathbf{N}}$ and $n \geq 1$ are pairwise singular by [22, Theorem 5.3.2]. In the language of spectral theory, this shows that the maximal spectral type of the unitary representation $\bigoplus_{n \geq 1} (\pi^{\mu_x})^{\otimes n}$ is equal to μ_x^∞ and that the measures $(\mu_x^\infty)_{x \in 2^{\mathbf{N}}}$ are moreover pairwise singular. Since Λ is a closed independent set, the subgroup $H(\Lambda) \subset \mathbf{T}$ generated by Λ has Haar measure zero by [22, Theorem 5.3.6]. In particular, the measures μ_x^∞ are singular with respect to the Haar measure for all $x \in 2^{\mathbf{N}}$.

Write $\mathbf{\Psi} : \mathbf{T}^2 \to \mathbf{T}^2$ for the group homomorphism $\mathbf{\Psi}(z_1, z_2) = (z_1 \overline{z}_2, z_2)$. Then the map

$$\eta : 2^{\mathbf{N}} \to \operatorname{Prob}(\mathbf{T}^2) : x \mapsto \eta_x = \mathbf{\Psi}_*(\mu_x^\infty \times \mathrm{Haar})$$

is Borel and the measure class of the A-A-bimodule $\mathcal{H}(x)$ is the class of η_x. Since the measures $(\mu_x^\infty)_{x \in 2^{\mathbf{N}}}$ are pairwise singular and all singular with respect to the Haar measure on $\mathbf{T}$, the measures $(\eta_x)_{x \in 2^{\mathbf{N}}}$ are pairwise singular and all singular with respect to the Haar measure on $\mathbf{T}^2$. Therefore the A-A-bimodules $\mathcal{H}(x)$ are pairwise non-isomorphic and all disjoint from the coarse A-A-bimodule. This finishes the proof. □

Acknowledgments

The author is grateful to Rémi Boutonnet and Sven Raum for their useful comments and careful reading of a first draft of this paper.

[1]For all distinct elements $z_1, \dots, z_k \in \Lambda$ and all $n_1, \dots, n_k \in \mathbf{Z}$, if $z_1^{n_1} \cdots z_k^{n_k} = 1$, then $n_1 = \cdots = n_k = 0$.

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