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FAKE WEDGES

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ABSTRACT. A *fake wedge* is a diagram of spaces $K \leftarrow A \to C$ whose double mapping cylinder is contractible. The terminology stems from the special case $A = K \vee C$ with maps given by the projections. In this paper, we study the homotopy type of the moduli space $\mathcal{D}(K, C)$ of fake wedges on K and C. We formulate two conjectures concerning this moduli space and verify that these conjectures hold after looping once. We show how embeddings of manifolds in Euclidean space provide a wealth of examples of non-trivial fake wedges. In an appendix, we recall discussions that the first author had with Greg Arone and Bob Thomason in early 1995 and explain how these are related to our conjectures.

CONTENTS

1. INTRODUCTION

There is a long history of constructing families of spaces which cannot be distinguished by homology. The purpose of this paper is to study a variant of this kind of problem for spaces that appear to be wedges from the viewpoint of every homology theory, but which are not the homotopy type of a wedge. In fact, our context is somewhat interesting in that the spaces we consider do split as wedges after a single suspension.

More precisely, suppose K and C are based spaces having the homotopy type of CW complexes. A *fake wedge* on K and C is a based space A equipped with maps $A \to K$ and $A \to C$ such that the homotopy colimit $\operatorname{hocolim}(K \leftarrow A \to C)$ is contractible.

The pair of maps in the definition can be amalgamated into a single map $A \to K \times C$ which we call a *structure map.* We usually suppress the structure map from the notation and specify a fake wedge by its underlying space. Two fake wedges

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A and A' on K and C are said to be *equivalent* if there is a finite chain of weak homotopy equivalences from A to A' each commuting with the structure map to $K \times C$.

For example, if $A = K \vee C$ and the maps are the projections, then the homotopy colimit is given by the wedge of reduced cones $C(K) \vee C(C)$ and is therefore contractible; we say in this case that A is a *trivial* wedge on K and C. More generally, we will also call a fake wedge A trivial on K and C if it is equivalent to the trivial one. If A is a fake wedge, then we can suspend to get maps $\Sigma A \to \Sigma K$ and $\Sigma A \to \Sigma C$ to obtain a fake wedge ΣA on ΣK and ΣC. However, this fake wedge is trivial, since we can use the comultiplication on ΣA to add the maps and get a trivialization.

A reservoir of non-trivial examples is provided by embedding theory. Suppose that $K^n \subset S^n$ is an orientable compact codimension zero submanifold whose boundary we will denote by E. Let C be the complement of K. Then the union $K \cup_E C$ is S^n. There are two methods of manufacturing fake wedges from this situation. The first is to remove a point x from the interior of C. This gives a space W such that the union $K \cup_E W$ is $\mathbb{R}^n$. Therefore E is a fake wedge on K and W. It is not necessarily trivial:

Example 1.1. Let $K = S^p \times D^q \subset S^{p+q}$ be standardly embedded. Then $E = S^p \times S^{q-1}$ and $C = D^{p+1} \times S^{q-1}$. Moreover, $W = C - x$ has the homotopy type of $S^p_+ \wedge S^{q-1} = S^{p+q-1} \vee S^{q-1}$ (where S^p_+ means S^p with a disjoint basepoint). Then E is a non-trivial fake wedge on K and W, since there are no non-trivial cup products $a \cup b$ in $K \vee W$ for a, b in positive degrees.

The second, and possibly more interesting way to manufacture examples of fake wedges from an embedding $K \subset S^n$ is to remove a point y from E. Let $A = E \setminus y$. Then the union $K \cup_A C$ is $\mathbb{R}^n$, so A is a fake wedge on K and C. In many instances A is non-trivial:

Theorem A. *Suppose $K \subset S^n$ is a compact tubular neighborhood of a closed connected orientable submanifold $M \subset S^n$ of codimension ≥ 3. Let E be the boundary of K and let $y \in E$ be any point. Assume M is not a homology sphere. Then $A = E \setminus y$ is a non-trivial fake wedge on K and C.*

(Our requirement that M is not a homology sphere is necessary, since the standard embedding $S^m \subset S^n$ gives rise to a trivial wedge on S^m and S^{n-m-1}).

Remark 1.2. Although we will not pursue the matter here, it is worth mentioning that the second construction suggests a new approach to finding codimension zero embeddings of K in S^n, where K is a compact smooth n-manifold with boundary:

- Step 1: find the possible complements C up to homotopy.
- Step 2: find all possible fake wedges A on K and C.
- Step 3: determine whether one can attach a cell to A to build a space E which maps to $K \times C$ extending the structure map from A such that the pairs (K, E) and (C, E) satisfy n-dimensional Poincare duality.
- Step 4: use surgery theory to smoothen the Poincaré duality data in Step 3.

Step 1 concerns the existence of Spanier-Whitehead n-duals of K and is a kind of desuspension question (cf. [K1]). Step 2 is related to the main results of this paper.

Step 3 is believed to be difficult and the known results are limited in scope [K3], [KR]. Step 4 is achieved by the application of the Browder-Casson-Sullivan-Wall theorem [Wl, Chap. 12].

The classifying space of fake wedges. Define a category $D(K,C)$ as follows: an *object* A is a fake wedge on K and C. A *morphism* $A \to A'$ is a weak equivalence of underlying spaces which commutes with the structure maps. We let $\mathcal{D}(K,C)$ denote the classifying space of $D(K,C)$, i.e., the realization of its nerve. It has a preferred basepoint given by the trivial wedge $K \vee C$. Then $\mathcal{D}(K,C)$ is a classifying space for fake wedges: if B is a finite complex, then up to homotopy, a map $B \to \mathcal{D}(K,C)$ is the same thing as specifying a fibration $E \to B$ together with a map $E \to K \times C$ such that for $b \in B$ the fiber $E_b \subset E$ has the structure of a fake wedge on K and C. Note that $\pi_0(\mathcal{D}(K,C))$ is the set of equivalence classes of fake wedges on K and C.

To formulate our main conjecture, suppose X is a based space. We let $X^{[j]}$ be the j-fold smash product of X and we let jX denote the wedge of j copies of X. For $n \geq 1$, let $p_n(x,y)$ be the polynomial given by

$$(x+y)^n - x^n - y^n = \sum_{i=1}^{n-1} \binom{n}{i} x^i y^{n-i} .$$

By analogy, let $p_n(K,C)$ be the space

$$\bigvee_{i=1}^{n-1} \binom{n}{i} K^{[i]} \wedge C^{[n-i]} .$$

Then $p_n(K,C)$ is a retract of $(K \vee C)^{[n]}$, and the action of the symmetric group Σ_n leaves the subspace $p_n(K,C)$ invariant. Hence $p_n(K,C)$ comes equipped with an action of Σ_n.

We let $\mathcal{W}_n$ denote the n-th coefficient spectrum of the identity functor from based spaces to based spaces in the sense of the calculus of homotopy functors [G, §8]. In particular, $\mathcal{W}_n$ is a spectrum with Σ_n-action whose underlying homotopy type is a wedge of $(n-1)!$ copies of the $(1-n)$-sphere spectrum. We will be considering $\mathcal{W}_n \wedge_{h\Sigma_n} p_n(K,C)$, the homotopy orbit spectrum of Σ_n acting diagonally on the smash product $\mathcal{W}_n \wedge p_n(K,C)$. If E is a spectrum and X is a based space, we let

$$F^{\mathrm{st}}(X,E) := \Omega^\infty F(X,E)$$

be the zeroth space of the function spectrum $F(X,E)$. This is the spectrum whose j-th space is the function space of based maps $X \to E_j$. As usual, if necessary we will replace E by an Ω-spectrum to ensure this delivers a well-defined homotopy type. When $E = \Sigma^\infty Y$ is a suspension spectrum, we abuse notation and write $F^{\mathrm{st}}(X,Y)$ in place of $F^{\mathrm{st}}(X,\Sigma^\infty Y)$; this is the function space of *stable maps* from X to Y.

Hypothesis 1.3. *Assume that K is r-connected and C is s-connected with $r,s \geq 1$. Assume K is homotopy equivalent to a CW complex of dimension $\leq k$ and that C is homotopy equivalent to a CW complex of dimension $\leq c$. Set*

$$\rho_n = \min(r+ns, nr+s) + 2 - \max(k,c) .$$

Conjecture B. *There is a tower of based spaces*

$$\cdots \to \mathcal{D}_n(K,C) \to \cdots \to \mathcal{D}_1(K,C)$$

and compatible maps $\mathcal{D}(K,C) \to \mathcal{D}_n(K,C)$ *such that*

- $\mathcal{D}_1(K,C)$ *is a point;*
- *the map* $\mathcal{D}(K,C) \to \mathcal{D}_n(K,C)$ *is* ρ_n*-connected. In particular, the map*
$$\mathcal{D}(K,C) \to \operatorname*{holim}_n \mathcal{D}_n(K,C)$$
is a weak equivalence;
- *each map of the tower sits in a homotopy fiber sequence*
$$\mathcal{D}_n(K,C) \to \mathcal{D}_{n-1}(K,C) \to F^{\mathrm{st}}(K \vee C, \Sigma^2 \mathcal{W}_n \wedge_{h\Sigma_n} p_n(K,C)) .$$

As evidence for the above conjecture we have

Theorem C. *Conjecture* B *holds after looping once, i.e., there is a tower of based spaces*
$$\cdots \to \delta_n(K,C) \to \cdots \to \delta_1(K,C)$$
and compatible maps $\Omega\mathcal{D}(K,C) \to \delta_n(K,C)$ *such that*

- $\delta_1(K,C)$ *is a point;*
- *the map* $\Omega\mathcal{D}(K,C) \to \delta_n(K,C)$ *is* $(\rho_n - 1)$*-connected. In particular, the map* $\Omega\mathcal{D}(K,C) \to \operatorname{holim}_n \delta_n(K,C)$ *is a weak equivalence;*
- *each map of the tower sits in a homotopy fiber sequence*
$$\delta_n(K,C) \to \delta_{n-1}(K,C) \to F^{\mathrm{st}}(K \vee C, \Sigma \mathcal{W}_n \wedge_{h\Sigma_n} p_n(K,C)) .$$

Remark 1.4. By [K2, th. A(2)] there is a function
$$\pi_0(\mathcal{D}(K,C)) \to \{K \vee C, K \wedge C\} ,$$
where the target is given by homotopy classes of stable maps $K \vee C \to K \wedge C$. It was shown that this function is surjective if $k, c \le 3\min(r,s)$ and a bijection if $k, c \le 3\min(r,s) - 1$. Based on what we do here, it seems plausible to us that these inequalities can be improved to $\rho_2 \ge 0$ and $\rho_2 \ge 1$ respectively.

The second coefficient spectrum $\mathcal{W}_2$ of the identity functor is the (-1)-sphere with trivial Σ_2-action. Furthermore, $p_2(K,C) = 2K \wedge C$. Hence $\Sigma\mathcal{W}_2 \wedge_{h\Sigma_2} p_2(K,C) \simeq K \wedge C$, so [K2, th. A(2)] is really a partial verification of Conjecture B on the level of path components.

Remark 1.5. The stable function space that appears in Theorem C can be simplified. Greg Arone has explained to us that for $i, n-i \ge 1$, the spectrum $\mathcal{W}_n$ splits $(\Sigma_i \times \Sigma_{n-i})$-equivariantly as a wedge of spectra of the form
$$(\Sigma_i \times \Sigma_{n-i})_+ \wedge_{\Sigma_d} \Sigma^{d-n} \mathcal{W}_d ,$$
where d ranges over the common divisors of i and $n-i$. The number of times that the displayed summand occurs is a certain number $B(i/d, (n-i)/d)$. This is the number of basic products in the free Lie algebra on two generators x_1, x_2, involving i/d copies of x_1 and $(n-i)/d$ copies of x_2 (for details, see [AK, th. 0.1]).

Equivariantly, the summand of $p_n(K,C)$ given by $\binom{n}{i} K^{[i]} \wedge C^{[n-i]}$ may also be rewritten as
$$(\Sigma_n)_+ \wedge_{\Sigma_i \times \Sigma_{n-i}} (K^{[i]} \wedge C^{[n-i]}) .$$
These two observations may be combined to show that the spectrum $\Sigma\mathcal{W}_n \wedge_{h\Sigma_n} p_n(K,C)$ splits into a wedge of spectra of the form
$$\Sigma^{d-n+1} \mathcal{W}_d \wedge_{h\Sigma_d} (K^{[i]} \wedge C^{[n-i]}) .$$

It follows that the stable function space $F^{\mathrm{st}}(K \vee C, \Sigma\mathcal{W}_n \wedge_{h\Sigma_n} p_n(K,C))$ splits into a product of terms of the form

$$F^{\mathrm{st}}(K \vee C, \Sigma^{d-n+1}\mathcal{W}_d \wedge_{h\Sigma_d} (K^{[i]} \wedge C^{[n-i]})). \tag{1}$$

In special cases we hope to be able to use this decomposition to obtain numerical information. For example, when K and C are spheres, the rational homotopy type of the stable function space (1) is contractible if $d > 2$ (and is also contractible for $d = 2$ in some cases; cf. [AM]).[1] More generally, for any K and C, the rational type of the stable function space (1) may be expressed in terms of the submodule $\mathcal{L}ie(d)$ of the free Lie algebra $\mathcal{L}ie(x_1, \dots, x_d)$ over the rational numbers that is given by all bracket monomials containing x_i exactly once (cf. [AK]). We plan to pursue this matter in another paper.

Stabilization. Define a functor $D(K,C) \to D(\Sigma K, C)$ by mapping an object A to $\Sigma_C A$ its fiberwise suspension over C:

$$\Sigma_C A \ := \mathrm{hocolim}(C \leftarrow A \to C),$$

where the homotopy colimit is taken in the category of based spaces and is given by the reduced double mapping cylinder $C \times 0 \cup A \wedge (I_+) \cup C \times 1$. There are evident maps from the latter to $\Sigma K = * \times 0 \cup K \wedge (I_+) \cup * \times 1$ and to C, so the assignment $A \mapsto \Sigma_C A$ is a functor.

Similarly, one has a functor $D(K,C) \to D(K, \Sigma C)$ which is defined by $A \mapsto \Sigma_K A$. Since homotopy colimits commute, we get a diagram of spaces

$$\begin{tikzcd} D(K,C) \arrow[r] \arrow[d] & D(\Sigma K, C) \arrow[d] \\ D(K, \Sigma C) \arrow[r] & D(\Sigma K, \Sigma C) \end{tikzcd}$$

which commutes up to canonical isomorphism.

Conjecture D. *The diagram of classifying spaces*

$$\begin{tikzcd} \mathcal{D}(K,C) \arrow[r] \arrow[d] & \mathcal{D}(\Sigma K, C) \arrow[d] \\ \mathcal{D}(K, \Sigma C) \arrow[r] & \mathcal{D}(\Sigma K, \Sigma C) \end{tikzcd}$$

is ρ_2-cartesian, where $\rho_2 = \min(r + 2s, 2r + s) + 2 - \max(k, c)$.

Remark 1.6. This conjecture is in part motivated by the second author's Ph.D. thesis where a similar result is proved for embedding spaces (cf. [P1], [P2]).

As evidence for Conjecture D, we have

Theorem E. *Conjecture* D *holds after looping once, i.e., the diagram*

$$\begin{tikzcd} \Omega\mathcal{D}(K,C) \arrow[r] \arrow[d] & \Omega\mathcal{D}(\Sigma K, C) \arrow[d] \\ \Omega\mathcal{D}(K, \Sigma C) \arrow[r] & \Omega\mathcal{D}(\Sigma K, \Sigma C) \end{tikzcd}$$

is $(\rho_2 - 1)$-cartesian.

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[1] We are indebted to the referee for pointing this out.

Conventions. We work in the Quillen model category of compactly generated spaces. Function spaces are topologized using the compact-open topology. A non-empty space X is r-connected if $\pi_j(X)$ is trivial for $j \leq r$. The empty space is (-2)-connected and every non-empty space is (-1)-connected. A map $X \to Y$ of spaces with Y non-empty is r-connected if every one of its homotopy fibers is $(r-1)$-connected.

The equivariant spectra appearing in this paper are *naive:* a spectrum with G-action E is a collection of based G-spaces $\{E_n\}_{n\geq 0}$ equipped with equivariant maps $\Sigma E_n \to E_{n+1}$, where G acts trivially on the suspension coordinate. Naive G-spectra have associated homotopy orbits E_{hG} and homotopy fixed points E^{hG}.

The category $D(K,C)$ is not small, so in order to take its classifying space one should replace it by an equivalent small category. One way to do this is to work in a Grothendieck universe. We choose not to address this matter any further.

2. Proof of Theorem A

We wish to show that there is no map $A \to K \vee C$ which covers the identity map of $K \times C$ up to homotopy. It will be sufficient to show that the composition

$$A \to K \times C \to K \wedge C$$

is non-trivial in reduced singular cohomology.

Case (1). Suppose that $E \to M$ admits a section $\zeta\colon M \to E$ such that the composite $M \to E \to C$ is null homotopic. The total space of the unit normal sphere bundle of M in S^n is identified with E. Let $\zeta\colon M \to E$ be a section; we can assume by transversality that the image of ζ is contained in $A \subset E$. We henceforth identify M with its image under ζ.

Note that E/M is identified with the Thom space M^ξ, where ξ denotes bundle given by taking the quotient of the normal bundle of M by the trivial line bundle defined by the section ζ. By choosing a null homotopy for the composite $M \to A \to C$, we obtain a map $A \cup CM \to C$, where $A \cup CM$ is the mapping cone of the inclusion $M \to A$. Identifying $A \cup CM$ with A/M, we obtain a map $A/M \to C$, and it is not difficult to see that this map is a homology isomorphism. The codimension ≥ 3 hypothesis guarantees that A/M and C are 1-connected. It follows that $A/M \to C$ is a homotopy equivalence.

The quotient map $q\colon A \to A/M$ is injective on cohomology in all degrees, since the inclusion $M \to A$ is a coretraction. Consider the reduced diagonal map

$$(d)\colon A/M \to M \wedge A/M$$

(this is the map of quotients induced by the map of pairs $(A, M) \to (M \times A, M \times M \cup * \times A)$ defined by $x \mapsto (p(x), x)$; here $* \in M$ is a choice of basepoint). With respect to our identifications, the composite

$$A \xrightarrow{q} A/M \xrightarrow{(d)} M \wedge A/M$$

is identified up to homotopy with the map $A \to K \wedge C$. Consequently, it will suffice to prove that the map (d) is non-trivial on reduced singular cohomology.

Consider the commutative diagram

$$\begin{tikzcd}
E/M \arrow[r, "(b)"] & M_+ \wedge E/M \arrow[dr, "(c)"] & \\
 & & M \wedge E/M \\
A/M \arrow[uu, "(a)"] \arrow[r, "(d)"'] & M \wedge A/M \arrow[ur, "(e)"'] &
\end{tikzcd}$$

where the maps (a) and (e) are inclusions, (b) and (d) are reduced diagonals, and (c) is given by sending the additional basepoint to the basepoint of M. If $U \in H^{n-m}(E/M)$ is the Thom class, where $m = \dim M$, and $x \in H^*(M_+)$, then the operation $x \mapsto (b)^*(x \times U) = x \cup U$ defines the Thom isomorphism (where $x \times U$ denotes the reduced external product of x and U). The map (a) is $(n-2)$-connected, so the Thom class pushes forward to a non-trivial cohomology class $U' \in H^{n-m}(A/M)$.

Let $z \in H^k(M)$ be any non-trivial cohomology class such that $1 < k < m$ (this is guaranteed by the assumption that M isn't a homology sphere). Then $z \cup U \in H^{n-m+k}(E/M)$ is non-trivial. Consequently, so is $z \cup U' \in H^{n-m+k}(A/M)$, by commutativity of the diagram. But $\cup U'$ is the effect of applying $(d)^*$ to the class represented by the reduced external product of z and U'. If follows that (d) is non-trivial in reduced singular cohomology. This completes Case (1).

Case (2): *the general case.* We argue by contradiction. Assume that there is a homotopy equivalence $A \simeq K \vee C$ covering the identity map of $K \times C$ up to homotopy. We can also think of A as a trivial wedge on M and C, using the homotopy equivalence $M \simeq K$.

Think of M as a submanifold of S^{n+1} (using the inclusion $S^n \subset S^{n+1}$). Then the complement of M is identified with ΣC. Let E' be the boundary of a tubular neighborhood of M in S^{n+1}. Then there is a preferred section $M \to E'$ such that the composite $M \to E' \to \Sigma C$ is null homotopic. To see this, note that E' is a double mapping cylinder of M with itself along E and that ΣC is a double mapping cylinder of a point with itself along C; the map $E' \to \Sigma C$ takes each copy of M to a point.

Hence we may apply Case (1) to obtain in this way a non-trivial fake wedge A' on K and ΣC, where A' is the effect of removing a point from E'. If A were trivial, then A' would also be trivial (this is because A' is obtained from A by taking the fiberwise suspension A over K). This gives the contradiction.

3. Proof of Theorem C

The moduli space of suspension spectra. For an Ω-spectrum E we recall from [K1] the moduli space of suspension spectra $\mathfrak{M}_E$ which was defined to be the realization of the category C_E whose objects are pairs (Y, h) in which Y is a based space and $h: \Sigma^\infty Y \to E$ is a weak (homotopy) equivalence. A morphism $(Y, h) \to (Y', h')$ is a map of based spaces $f: Y \to Y'$ such that $h' \circ \Sigma^\infty f = h$. Assume in what follows that C_E is non-empty. Let $y = (Y, h)$ be an object of C_E. This gives $\mathfrak{M}_E$ a basepoint.

Assume that E is r-connected and has the homotopy type of a CW spectrum of dimension $\le d$. We will also assume $r \ge 1$. In [K1, §6] we produced a tower of principal fibrations of based spaces

$$\cdots \to T_2(y) \to T_1(y)$$

and compatible maps $\Omega\mathfrak{M}_E \to T_j(y)$ such that

- $T_1(y)$ is contractible.
- The map $\Omega\mathfrak{M}_E \to T_j(y)$ is $((j+1)r+1-d)$-connected. In particular, the map $\Omega\mathfrak{M}_E \to \operatorname{holim}_j T_j(y)$ is a weak equivalence.
- For $j > 1$ there is a homotopy fiber sequence

$$T_j(y) \to T_{j-1}(y) \to F^{\mathrm{st}}(Y, \Sigma\mathcal{W}_j \wedge_{h\Sigma_j} Y^{[j]})\,. \tag{2}$$

We will reproduce the homotopy fiber sequence (2) below.

The space $T_j(y)$ is defined to be the space of all lifts filling in the dotted line in the diagram

$$\begin{tikzcd} & P_j(Y) \arrow[d] \\ Y \arrow[r] \arrow[ur, dashed] & P_1(Y) \simeq Q(Y)\,, \end{tikzcd}$$

where $P_j(Y)$ is the j-th stage of the Goodwillie tower of the identity functor. The map $T_j(y) \to T_{j-1}(y)$ is obtained by sending a lift $Y \to P_j(Y)$ to the composite $Y \to P_j(Y) \to P_{j-1}(Y)$. The fiber sequence (2) is a consequence of Goodwillie's observation [G, lem. 2.2] that $P_j(Y) \to P_{j-1}(Y)$ is a principal fibration that is classified by a map $P_{j-1}(Y) \to \Omega^\infty(\Sigma\mathcal{W}_j \wedge_{h\Sigma_j} Y^{[j]})$.

We also showed that $\Omega\mathfrak{M}_E$ is identified up to weak equivalence with the space of lifts $T_\infty(y)$ of the diagram

$$\begin{tikzcd} & \operatorname{holim}_j P_j(Y) \simeq Y \arrow[d] \\ Y \arrow[r] \arrow[ur, dashed] & P_1(Y) \end{tikzcd}$$

(this uses [Wd, 2.2.5], or alternatively [DK, §2]).

Remark 3.1. We digress to explain how [Wd, 2.2.5], [DK, §2] are used to give a model for $\Omega\mathfrak{M}_E$. Suppose $\mathcal{C}$ denotes a simplicial model category. Let $c \in \mathcal{C}$ be a fibrant and cofibrant object. Consider the category $h\mathcal{C}_{(c)}$ consisting of objects of $\mathcal{C}$ which are weak equivalent to C, where a morphism is a weak equivalence. Then the realization of this category, $|h\mathcal{C}_{(c)}|$, coincides with the classifying space $BG(c)$, where $G(c)$ is the simplicial monoid of homotopy automorphisms of c.

A special case of interest to us occurs when $C = \mathrm{Top}_{/B}$, the simplicial model category of spaces over B, where a morphism $X \to Y$ is a weak equivalence if and only if it is a weak homotopy equivalence of underlying spaces. In this case X is cofibrant if and only if its underlying space is a retract of a cell complex and X is fibrant if the structure map $X \to B$ is a fibration. If X is fibrant and cofibrant, it follows that $|h\mathrm{Top}_{/B,(X)}|$ is a model for the classifying space for the homotopy self-equivalences of X covering the identity of B.

In the case appearing in the proof, we may take $B = P_1(Y)$. Let $Y' \to P_1(Y)$ be a fibrant and cofibrant replacement. Then any endomorphism $Y' \to Y'$ is a homology equivalence of underlying spaces since the composite $Y' \to Y' \to Q(Y')$ is adjoint to the identity map of the suspension spectrum $\Sigma^\infty Y'$. Furthermore, such endomorphisms are identified with the space of lifts of the map $Y \to P_1(Y)$ through Y. In particular, when Y is 1-connected, $\Omega\mathrm{Top}_{/P_1(Y),(Y')}$ is identified up to homotopy with the space of such lifts. Even when Y is not 1-connected we can still obtain such an identification by replacing Y with its plus construction. For the details in this case see [K1, prop. 2.3].

In what follows we set $E = \Sigma^\infty K \times \Sigma^\infty C$. Define a functor

$$D(K,C) \to C_E$$

by $A \mapsto (A,h)$, where the weak equivalence $h\colon \Sigma^\infty A \to E$ is defined by applying Σ^∞ to the structure maps $A \to K$ and $A \to C$. The space $K \vee C$ together with the evident weak equivalence $\Sigma^\infty(K \vee C) \simeq E$ equips C_E with a basepoint y.

To simplify notation in what follows, we make an auxiliary definition.

Definition 3.2. Suppose $f\colon X \to B$ and $g\colon Y \to B$ are maps of spaces. By a slight abuse of notation, we let

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[ur, dashed] \arrow[r, "g"'] & B \end{tikzcd}$$

indicate the space of lifts obtained by converting f into a fibration $E \to B$, replacing f by this fibration and then taking lifts of the latter diagram. Equivalently, this is the space of pairs consisting of a map $h'\colon Y \to X$ and a choice of commuting homotopy from $f \circ h'$ to g.

Let $\epsilon(K,C)$ be the space

$$\begin{tikzcd} & K \times C \arrow[d] \\ K \vee C \arrow[ur, dashed] \arrow[r] & Q(K) \times Q(C)\,. \end{tikzcd}$$

Lemma 3.3. *There is a homotopy fiber sequence*

$$\Omega\mathcal{D}(K,C) \to \Omega\mathfrak{M}_E \to \epsilon(K,C).$$

Proof. We can identify $\Omega\mathcal{D}(K,C)$ up to weak equivalence with the space

$$\begin{tikzcd} & K \vee C \arrow[d] \\ K \vee C \arrow[ur, dashed] \arrow[r] & K \times C \end{tikzcd}$$

(cf. Remark 3.1; we are also using the observation that any such lift $K \vee C \to K \vee C$ appearing above is automatically a weak equivalence). Similarly, $\Omega\mathfrak{M}_E$ is identified with the space

$$\begin{tikzcd} & K \vee C \arrow[d] \\ K \vee C \arrow[ur, dashed] \arrow[r] & Q(K) \times Q(C) \end{tikzcd}$$

(cf. [K1, §5]).

The proof is completed by noticing that the two lifting spaces sit in a fibration sequence in which the base space is given by $\epsilon(K, C)$. $\square$

Definition 3.4. Let $\epsilon_j(K, C)$ be the space of lifts

$$\begin{tikzcd} & P_j(K) \times P_j(C) \arrow[d] \\ K \vee C \arrow[r] \arrow[ur, dashed] & Q(K) \times Q(C)\,. \end{tikzcd}$$

Proof of Theorem C. By naturality, we have an evident map

$$T_j(y) \to \epsilon_j(K, C)\,.$$

Define $\delta_j(K, C)$ to be the homotopy fiber of this map. Then we have a homotopy fiber sequence

$$\delta_j(K, C) \to T_j(y) \to \epsilon_j(K, C)\,.$$

More concretely, $\delta_j(K, C)$ may be identified with the space of lifts

$$\begin{tikzcd} & P_j(K \vee C) \arrow[d] \\ K \vee C \arrow[r] \arrow[ur, dashed] & P_j(K) \times P_j(C)\,. \end{tikzcd}$$

By properties of the Goodwillie tower of the identity functor, the map

$$\operatorname*{holim}_i \delta_i(K, C) \to \delta_j(K, C)$$

is $((j+1)r+1-d)$-connected, where r is the connectivity of $K \vee C$ and $d = \max(k, c)$ is the homotopy dimension of $K \vee C$. Furthermore, $\operatorname{holim}_i \delta_i(K, C)$ is identified with $\Omega\mathcal{D}(K, C)$.

By comparing this sequence with the corresponding one for $j - 1$, we obtain a commutative diagram whose rows and columns form homotopy fiber sequences

$$\begin{tikzcd} \delta_j(K,C) \arrow[r] \arrow[d] & T_j(y) \arrow[r] \arrow[d] & \epsilon_j(K,C) \arrow[d] \\ \delta_{j-1}(K,C) \arrow[r] \arrow[d] & T_{j-1}(y) \arrow[r] \arrow[d] & \epsilon_{j-1}(K,C) \arrow[d] \\ ? \arrow[r] & F^{\mathrm{st}}(K \vee C, \Sigma\mathcal{W}_j \wedge_{h\Sigma_j} (K \vee C)^{[j]}) \arrow[r, "f"'] & F^{\mathrm{st}}(K \vee C, \Sigma\mathcal{W}_j \wedge_{h\Sigma_j} (K^{[j]} \vee C^{[j]})) \end{tikzcd}$$

where the space ? is given by the homotopy fiber of the map labelled f that is induced by the projection map $(K \vee C)^{[j]} \to K^{[j]} \vee C^{[j]}$. It follows that ? is identified with the stable function space $F^{\mathrm{st}}(K \vee C, \Sigma\mathcal{W}_j \wedge_{h\Sigma_j} p_j(K, C))$. This establishes the homotopy fiber sequence

$$\delta_j(K, C) \to \delta_{j-1}(K, C) \to F^{\mathrm{st}}(K \vee C, \Sigma\mathcal{W}_j \wedge_{h\Sigma_j} p_j(K, C))\,. \tag{3}$$

A straightforward calculation we omit shows $F^{\mathrm{st}}(K \vee C, \Sigma\mathcal{W}_{j+1} \wedge_{h\Sigma_{j+1}} p_{j+1}(K, C))$ to be $(\rho_j - 1)$-connected. In particular, the map $\delta_{j+1}(K, C) \to \delta_j(K, C)$ is $(\rho_j - 1)$-connected. As ρ_j is an increasing function of j, it follows that the map $\Omega\mathcal{D}(K, C) \simeq \operatorname{holim}_i \delta_i(K, C) \to \delta_j(K, C)$ is $(\rho_j - 1)$-connected. $\square$

4. Proof of Theorem E

To prove Theorem E, by induction it will be enough to show that for each $j \geq 1$ a certain commutative diagram

$$\begin{tikzcd} \delta_j(K,C) \arrow[r] \arrow[d] & \delta_j(K,\Sigma C) \arrow[d] \\ \delta_j(\Sigma K,C) \arrow[r] & \delta_j(\Sigma K,\Sigma C) \end{tikzcd} \tag{4}$$

is $(\rho_2 - 1)$-cartesian (see below for a description of the maps of this diagram).

It will turn out that these diagrams are functorial in the index j, so by the homotopy fiber sequence (3), we are reduced to showing that the associated diagram

$$\begin{tikzcd} F^{\mathrm{st}}(K \vee C, \Sigma\mathcal{W}_j \wedge_{h\Sigma_j} p_j(K,C)) \arrow[r] \arrow[d] & F^{\mathrm{st}}(K \vee \Sigma C, \Sigma\mathcal{W}_j \wedge_{h\Sigma_j} p_j(K,\Sigma C)) \arrow[d] \\ F^{\mathrm{st}}(\Sigma K \vee C, \Sigma\mathcal{W}_j \wedge_{h\Sigma_j} p_j(\Sigma K,C)) \arrow[r] & F^{\mathrm{st}}(\Sigma K \vee \Sigma C, \Sigma\mathcal{W}_j \wedge_{h\Sigma_j} p_j(\Sigma K,\Sigma C)) \end{tikzcd}$$

is ρ_2-cartesian (again, we will describe the maps of the diagram below).

The proof will use the projection map

$$p_j(K,C) \to j((K^{[j-1]} \wedge C) \vee (K \wedge C^{[j-1]}))\,,$$

for $j \geq 3$. When $j = 2$, our convention will be that the target of the projection map is $2K \wedge C$. The projection map is $(2(r+s)+4)$-connected for all $j \geq 2$.

In what follows we set

$$q_j(K,C) = j((K^{[j-1]} \wedge C) \vee (K \wedge C^{[j-1]}))$$

for $j \geq 3$, and when $k = 2$ we set $q_2(K,C) = 2K \wedge C$. It is elementary to show that the induced map

$$F^{\mathrm{st}}(K \vee C, \Sigma\mathcal{W}_j \wedge_{h\Sigma_j} p_j(K,C)) \to F^{\mathrm{st}}(K \vee C, \Sigma\mathcal{W}_j \wedge_{h\Sigma_j} q_j(K,C))$$

is $(2(r+s)+2-d)$-connected, $d = \max(k,c)$. Note that the action of Σ_j on $q_j(K,C)$ is given by inducing up the evident Σ_{j-1}-actions on the summands $K^{[j-1]} \wedge C$ and $K \wedge C^{[j-1]}$.

Consequently, we are reduced to checking that the diagram

$$\begin{tikzcd} F^{\mathrm{st}}(K \vee C, \Sigma\mathcal{W}_j \wedge_{h\Sigma_j} q_j(K,C)) \arrow[r] \arrow[d] & F^{\mathrm{st}}(K \vee \Sigma C, \Sigma\mathcal{W}_j \wedge_{h\Sigma_j} q_j(K,\Sigma C)) \arrow[d] \\ F^{\mathrm{st}}(\Sigma K \vee C, \Sigma\mathcal{W}_j \wedge_{h\Sigma_j} q_j(\Sigma K,C)) \arrow[r] & F^{\mathrm{st}}(\Sigma K \vee \Sigma C, \Sigma\mathcal{W}_j \wedge_{h\Sigma_j} q_j(\Sigma K,\Sigma C)) \end{tikzcd} \tag{5}$$

is ρ_2-cartesian.

Set

$$u_j(K,C) = K \wedge C^{[j-1]} \quad \text{and} \quad v_j(K,C) = K^{[j-1]} \wedge C\,.$$

Then $q_j(K,C) = j(u_j(K,C) \vee v_j(K,C))$. The restriction to Σ_{j-1} of the action of Σ_j on $\mathcal{W}_j$ is identified with the permutation action on $(\Sigma_{j-1})_+ \wedge S^{1-j}$ (cf. Remark 1.5). Similarly, the action of Σ_j on $q_j(K,C)$ is given by inducing up from the permutation action of Σ_{j-1} on $u_j(K,C) \vee v_j(K,C)$. Hence,

$$F^{\mathrm{st}}(K \vee C, \Sigma\mathcal{W}_j \wedge_{h\Sigma_j} q_j(K,C)) \simeq F^{\mathrm{st}}(K \vee C, \Sigma^{2-j}(u_j(K,C) \vee v_j(K,C)))\,.$$

Observe that for $j \geq 3$, the inclusion map

$$\begin{aligned} &F^{\mathrm{st}}(K, \Sigma^{2-j} u_j(K,C)) \times F^{\mathrm{st}}(C, \Sigma^{2-j} v_j(K,C)) \\ &\qquad \to F^{\mathrm{st}}(K \vee C, \Sigma^{2-j}(u_j(K,C) \vee v_j(K,C))) \end{aligned}$$

is ρ_2-connected.

Then the above reduces us to showing that a certain commutative diagram

$$\begin{tikzcd} \begin{matrix} F^{\mathrm{st}}(K, \Sigma^{2-j} u_j(K,C)) \\ \times \\ F^{\mathrm{st}}(C, \Sigma^{2-j} v_j(K,C)) \end{matrix} \arrow[r] \arrow[d] & \begin{matrix} F^{\mathrm{st}}(K, \Sigma^{2-j} u_j(K,\Sigma C)) \\ \times \\ F^{\mathrm{st}}(\Sigma C, \Sigma^{2-j} v_j(K,\Sigma C)) \end{matrix} \arrow[d] \\ \begin{matrix} F^{\mathrm{st}}(\Sigma K, \Sigma^{2-j} u_j(\Sigma K,C)) \\ \times \\ F^{\mathrm{st}}(C, \Sigma^{2-j} v_j(\Sigma K,C)) \end{matrix} \arrow[r] & \begin{matrix} F^{\mathrm{st}}(\Sigma K, \Sigma^{2-j} u_j(\Sigma K,\Sigma C)) \\ \times \\ F^{\mathrm{st}}(\Sigma C, \Sigma^{2-j} v_j(\Sigma K,\Sigma C)) \end{matrix} \end{tikzcd} \tag{6}$$

is ρ_2-cartesian. We will explain the maps of this diagram below, and we will show that the diagram is in fact ∞-cartesian. This will complete the proof.

For $j \geq 2$, a point of $F^{\mathrm{st}}(K, \Sigma^{2-j} u_j(K,C)) \times F^{\mathrm{st}}(C, \Sigma^{2-j} v_j(K,C))$ is given by a pair of stable maps

$$a \colon K \to \Sigma^{2-j} K \wedge C^{[j-1]} \quad \text{and} \quad b \colon C \to \Sigma^{2-j} K^{[j-1]} \wedge C\,.$$

Then we claim that the horizontal arrows of diagram (4) are given by the operation $(a,b) \mapsto (*, \Sigma b)$. Similarly, we claim that the vertical arrows are given by the operation $(a,b) \mapsto (\Sigma a, *)$. From this description, and the fact that $F^{\mathrm{st}}(X,Y) = F^{\mathrm{st}}(\Sigma X, \Sigma Y)$, it is clear that diagram (4) is ∞-cartesian.

We now fill in some of the details in the above argument. We first need to define the stabilization maps appearing in diagram (4). Then we we need to show that the following diagram commutes:

$$\begin{tikzcd} \Omega\mathcal{D}(K,C) \arrow[r] \arrow[d] & \delta_{j-1}(K,C) \arrow[r] \arrow[d] & F^{\mathrm{st}}(K, \Sigma^{2-j} u_j(K,C)) \times F^{\mathrm{st}}(C, \Sigma^{2-j} v_j(K,C)) \arrow[d] \\ \Omega\mathcal{D}(K,\Sigma C) \arrow[r] & \delta_{j-1}(K,\Sigma C) \arrow[r] & F^{\mathrm{st}}(K, \Sigma^{2-j} u_j(K,\Sigma C)) \times F^{\mathrm{st}}(C, \Sigma^{2-j} v_j(K,\Sigma C)) \end{tikzcd} \tag{7}$$

where the left vertical map is the given stabilization map and the right vertical map is the one given above (i.e., $(a,b) \mapsto (*, \Sigma b)$). The middle vertical map is a map of diagram (4) with a change of index from j to $j-1$. We also need to show that a diagram analogous to diagram (7) that involves the other kind of stabilization map commutes. We will omit that argument since it is essentially the same.

We now define the map $\delta_j(K,C) \to \delta_j(K,\Sigma C)$. A point of $\delta_j(K,C)$ is represented by a map $g \colon K \vee C \to P_j(K \vee C)$ which projects to the canonical map $K \times C \to P_j K \times P_j C$ up to a choice of commuting homotopy, which for reasons of clutter will be ignored in our description. We map g to the composite

$$\Sigma_K(K \vee C) \to \Sigma_{P_j K} P_j(K \vee C) \to P_j(\Sigma_K(K \vee C))\,, \tag{8}$$

where the second map in the composite arises from applying the functor P_j to the homotopy pushout diagram

$$\begin{tikzcd} K \vee C \arrow[r] \arrow[d] & K \arrow[d] \\ K \arrow[r] & \Sigma_K(K \vee C) \end{tikzcd}$$

to get

$$\mathrm{hocolim}(P_jK \leftarrow P_j(K \vee C) \rightarrow P_jK) \rightarrow P_j(\Sigma_K(K \vee C))\,. \tag{9}$$

Note that $\Sigma_{P_jK}P_j(K \vee C)$ coincides with the domain of the map (9). We are also implicitly using the identification

$$\Sigma_K(K \vee C) = K \vee \Sigma C$$

to identify the map (8) with an element of $\delta_j(K, \Sigma C)$. A similar description defines the other stabilization map $\delta_j(K, C) \rightarrow \delta_j(\Sigma K, C)$. It is now straightforward to check that diagram (4) is commutative and is compatible with passage from j to $j - 1$. It is also not difficult to check that the left hand square of diagram (7) is commutative, and we will omit these details. It remains to verify the commutativity of the right hand square of diagram (7). This will involve a long and tedious diagram chase and we will be content providing an outline of the argument.

By construction, the map

$$\delta_{j-1}(K, C) \rightarrow F^{\mathrm{st}}(K, \Sigma^{2-j}u_j(K, C)) \times F^{\mathrm{st}}(C, \Sigma^{2-j}v_j(K, C))$$

factors through $F^{\mathrm{st}}(K \vee C, \Sigma\mathcal{W}_j \wedge_{h\Sigma_j} (K \vee C)^{[j]})$. We will first describe how the map

$$\delta_{j-1}(K, C) \rightarrow F^{\mathrm{st}}(K \vee C, \Sigma\mathcal{W}_j \wedge_{h\Sigma_j} (K \vee C)^{[j]})$$

behaves with respect to stabilization by constructing a stabilization map for the target.

Let $L_jX = \Sigma\mathcal{W}_j \wedge_{h\Sigma_j} X^{[j]}$. Let $g\colon K \vee C \rightarrow P_{j-1}(K \vee C)$ represent a point of $\delta_{j-1}(K, C)$. Then the chain of maps

$$K \vee C \xrightarrow{g} P_{j-1}(K \vee C) \rightarrow L_j(K \vee C)$$

gives rise to a commutative diagram

$$\begin{tikzcd} \Sigma_K(K \vee C) \arrow[r] \arrow[d, equal] & \Sigma_{P_{j-1}K}P_{j-1}(K \vee C) \arrow[r] \arrow[d] & \Sigma_{L_jK}L_j(K \vee C) \arrow[d] \\ \Sigma_K(K \vee C) \arrow[r] & P_{j-1}(\Sigma_K(K \vee C)) \arrow[r] & L_j(\Sigma_K(K \vee C))\,. \end{tikzcd}$$

The bottom left map of this diagram represents the stabilization of g lying in $\delta_{j-1}(K, \Sigma C)$. The bottom composite represents the image of this with respect to the map $\delta_{j-1}(K, \Sigma C) \rightarrow F^{\mathrm{st}}(K \vee \Sigma C, L_j(K \vee \Sigma C))$. The commutativity of this diagram implies the commutativity of the diagram

$$\begin{tikzcd} \delta_{j-1}(K, C) \arrow[r] \arrow[d] & F^{\mathrm{st}}(K \vee C, L_j(K \vee C)) \arrow[d] \\ \delta_{j-1}(K, \Sigma C) \arrow[r] & F^{\mathrm{st}}(K \vee \Sigma C, L_j(K \vee \Sigma C)) \end{tikzcd}$$

where the vertical maps are stabilization maps and the right vertical map is given by sending a stable map $K \vee C \to L_j(K \vee C)$ to the stable composite

$$\Sigma_K(K \vee C) \to \Sigma_{L_j K} L_j(K \vee C) \to L_j(\Sigma_K(K \vee C)) .$$

Finally, a straightforward check that we omit shows that the diagram

$$\begin{tikzcd}
F^{\mathrm{st}}(K \vee C, L_j(K \vee C)) \arrow[r] \arrow[d] & F^{\mathrm{st}}(K, \Sigma^{2-j} u_j(K,C)) \times F^{\mathrm{st}}(C, \Sigma^{2-j} v_j(K,C)) \arrow[d] \\
F^{\mathrm{st}}(K \vee \Sigma C, L_j(K \vee \Sigma C) \arrow[r] & F^{\mathrm{st}}(K, \Sigma^{2-j} u_j(K, \Sigma C)) \times F^{\mathrm{st}}(\Sigma C, \Sigma^{2-j} v_j(K, \Sigma C))
\end{tikzcd}$$

is commutative, where the top horizontal map is induced by the projection $L_j(K \vee C) \to \Sigma^{2-j} u_j(K,C) \times \Sigma^{2-j} u_j(K,C)$ and the right vertical map is given by $(a,b) \mapsto (*, \Sigma b)$.

5. Appendix: Discussions with Arone and Thomason in 1995

In Bielefeld in early 1995, the first author was involved in a series of discussions with Greg Arone and Bob Thomason about the possibility of developing a theory of E_∞-coalgebras over the sphere spectrum to serve as a recognition principle for deciding when a spectrum is weakly equivalent to a suspension spectrum. Unfortunately, Bob Thomason passed away in November, 1995 before we could get such a theory up and running. However, in [K1] the first author took the first step in this direction by constructing a theory of coalgebra spectra in the metastable range (cf. below). In [K4] we used this theory to give concrete results about embeddings. We will now try to outline some aspects of this project because it is relevant to the topic of this paper.

First observe that a based space X gives rise to a suspension spectrum $\Sigma^\infty X$. The diagonal map $\Delta\colon X \to X \wedge X$ induces a diagonal $\Delta\colon \Sigma^\infty X \to \Sigma^\infty X \wedge \Sigma^\infty X$ which is commutative up to all higher coherences. This gives $\Sigma^\infty X$ the structure of a coalgebra over the sphere spectrum (without co-unit) that is coherently homotopy commutative. Conversely, suppose we could define a category of E_∞-coalgebras over S^0. Then one might hope that the functor $X \to \Sigma^\infty X$ induces an equivalence of associated homotopy categories of 1-connected objects.

The vague conjectural idea is that whatever an E_∞-coalgebra is, it should amount in some way to a spectrum admitting a co-action by the E_∞-operad. The filtration of $E\Sigma_\infty$ by the $E\Sigma_n$ should then somehow correspond to the stages of the Goodwillie tower of the identity functor from spaces to spaces. More precisely, if we start with an E_∞-coalgebra structure on a spectrum E, then there should be a functorially associated tower of fibrations of based spaces $\cdots \to P_2 E \to P_1 E$ in which

- $P_1 E = \Omega^\infty E$.
- For $n \geq 2$, the n-th layer $L_n E = \mathrm{fiber}(P_n E \to P_{n-1} E)$ is given by the infinite loop space $\Omega^\infty(\mathcal{W}_n \wedge_{h\Sigma n} E^{[n]})$.
- If $X = \lim_n P_n E$, then the map $X \to \Omega^\infty E$ is adjoint to a weak equivalence when E is 1-connected.

Furthermore, the partial tower $P_n E \to \cdots \to P_1 E$ should functorially only depend on the co-action by the portion of the E_∞-operad that involves $E\Sigma_j$ for $j \leq n$. That is, the partial tower should depend on less structure: it should only require a choice of coaction on E of the *n-truncation* of the E_∞-operad in the sense of [F], [C, defn. 2.17]. We will be somewhat more precise about this below.

First consider the $n = 2$ case. We have a norm sequence

$$D_2E \to (E \wedge E)^{h\Sigma_2} \to (E \wedge E)^{t\Sigma_2}$$

in which D_2E is the quadratic construction, $(E \wedge E)^{h\Sigma_2}$ is the homotopy fixed points of Σ_2 acting on $E \wedge E$ and $(E \wedge E)^{t\Sigma_2}$ is the Tate construction of Σ_2 acting on $E \wedge E$. There is also a natural map $q_2 : E \to (E \wedge E)^{t\Sigma_2}$. We now define a 2-truncated E_∞-coalgebra structure on E to be a map $E \to (E \wedge E)^{h\Sigma_2}$ which factorizes q_2. What we have just defined is equivalent to the notion of a diagonal given in [K1] and this is a good enough notion in the metastable range: using this lift, we were able to construct a partial tower $P_2E \to \Omega^\infty E$ and we showed that the adjoint map $\Sigma^\infty P_2E \to E$ is an equivalence in the metastable range.

The above suggests that an n-truncated E_∞-coalgebra structure on a spectrum E might have an inductive definition. Given an $(n-1)$-truncated E_∞-coalgebra structure on E, we conjecture that one can associate to it a natural map $q_n : E \to (E^{[n]})^{t\Sigma_n}$ (this is easy to do when E is a suspension spectrum, but we do not yet know how to do this in general). Assuming this can be done, we define an n-truncated structure which extends the given $(n-1)$-truncated structure by choosing a factorization $E \to (E^{[n]})^{h\Sigma_n} \to (E^{[n]})^{t\Sigma_n}$. We surmise that this definition enables one to construct an n-stage tower $P_nE \to \cdots \to P_1E = \Omega^\infty E$ such that $\Sigma^\infty P_nE \to E$ is $(nr + r - 1)$-connected when E is r-connected.

If this idea works, then we should be able to study the moduli space of suspension structures $\mathfrak{M}_E$ by constructing a sequence of approximations $\cdots \to \mathfrak{M}_{E,n} \to \mathfrak{M}_{E,n-1}$ in which $\mathfrak{M}_E = \lim_n \mathfrak{M}_{E,n}$ and $\mathfrak{M}_{E,n}$ is a moduli space of n-truncated E_∞-structures on E (when $\mathfrak{M}_E$ is non-empty the space $\mathfrak{M}_{E,n}$ ought to be a nonconnective delooping of the space $T_n(y)$ appearing above). Moreover, there should be a homotopy fiber sequence

$$\Omega^\infty F(E, \Sigma\mathcal{W}_n \wedge_{h\Sigma_n} E^{[n]}) \to \mathfrak{M}_{E,n} \to \mathfrak{M}_{E,n-1}\,.$$

We could then use $\mathfrak{M}_{E,n}$ in place of $T_n(y)$ in the proof of Theorem C to obtain a delooping of the proof.

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