

COMPACT KÄHLER MANIFOLDS WITH AUTOMORPHISM GROUPS OF MAXIMAL RANK

DE-QI ZHANG

ABSTRACT. For an automorphism group G on an n -dimensional ($n \geq 3$) normal projective variety or a compact Kähler manifold X so that G modulo its subgroup $N(G)$ of null entropy elements is an abelian group of maximal rank $n - 1$, we show that $N(G)$ is virtually contained in $\text{Aut}_0(X)$, the X is a quotient of a complex torus T and G is mostly descended from the symmetries on the torus T , provided that both X and the pair (X, G) are minimal.

1. INTRODUCTION

We work over the field \mathbb{C} of complex numbers. For a linear transformation L on a finite-dimensional vector space V over \mathbb{C} or its subfields, its *spectral radius* is defined as

$$\rho(L) := \max\{|\lambda|; \lambda \in \mathbb{C} \text{ is an eigenvalue of } L\}.$$

Let X be a compact complex Kähler manifold and Y a normal projective variety, and let $g \in \text{Aut}(X)$ and $f \in \text{Aut}(Y)$. Define the (topological) *entropy* $h(*)$ and *first dynamical degrees* $d_1(*)$ as

$$\begin{aligned} h(g) &:= \log \rho(g^* | \bigoplus_{i \geq 0} H^i(X, \mathbb{C})), \\ d_1(g) &:= \rho(g^* | H^2(X, \mathbb{C})) (= \rho(g^* | H^{1,1}(X))), \\ d_1(f) &:= \rho(f^* | \text{NS}_{\mathbb{C}}(Y)), \end{aligned}$$

where $\text{NS}_{\mathbb{C}}(Y) := \text{NS}(Y) \otimes_{\mathbb{Z}} \mathbb{C}$ is the *complexified Neron-Severi group*. By the fundamental work of Gromov and Yomdin, the above definition of entropy is equivalent to its original definition (cf. [6, §2.2] and the references therein). Further, when Y is smooth, the above two definitions of $d_1(*)$ coincide; for \mathbb{Q} -factorial Y (cf. [10, 0.4(1)]), we have $d_1(f) = d_1(\tilde{f})$, where \tilde{f} is the lifting of f to the one on an $\text{Aut}(Y)$ -equivariant resolution of Y . We say that $\tau := g$ or f is of *positive entropy* (resp. *null entropy*) if $d_1(\tau) > 1$ (resp. $d_1(\tau) = 1$), or equivalently $h(\tau) > 0$ (resp. $h(\tau) = 0$) in the case of a compact Kähler manifold.

We say that the induced action $G | H^{1,1}(X)$ is *Z-connected* if its Zariski-closure in $\text{GL}(H^{1,1}(X))$ is connected with respect to the Zariski topology; in this case, the *null set*

$$N(G) := \{g \in G \mid g \text{ is of null entropy}\}$$

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is a (necessarily normal) subgroup of G (cf. [19, Theorem 1.2]). In [19], we have proved:

Theorem 1.1. *Let X be an n -dimensional ($n \geq 2$) compact complex Kähler manifold and G a subgroup of $\text{Aut}(X)$. Then one of the following two assertions holds:*

- (1) G contains a subgroup isomorphic to the non-abelian free group $\mathbb{Z} * \mathbb{Z}$, and hence G contains subgroups isomorphic to non-abelian free groups of all countable ranks.
- (2) There is a finite-index subgroup G_1 of G such that the induced action $G_1 | H^{1,1}(X)$ is solvable and Z -connected. Further, the subset

$$N(G_1) := \{g \in G_1 \mid g \text{ is of null entropy}\}$$

of G_1 is a normal subgroup of G_1 and the quotient group $G_1/N(G_1)$ is a free abelian group of rank $r \leq n - 1$. We call this r **the rank of G_1 and denote it as $r = r(G_1)$** .

Therefore, we are interested in the group $G \leq \text{Aut}(X)$, where $G | H^{1,1}(X)$ is solvable and Z -connected and where the rank $r(G) = \dim X - 1$ (maximal value). In the following, denote by $\text{Aut}_0(X)$ the *identity connected component* of $\text{Aut}(X)$. A group *virtually has a property (P)* if a finite-index subgroup of it has the property (P).

A complex torus has lots of symmetries. Conversely, our main result, Theorem 1.2 (see also Theorem 2.1 for non-algebraic manifolds), says that the maximality $r(G) = \dim X - 1$ occurs only when X is a quotient of a complex torus T and G is mostly descended from the symmetries on the torus T .

The statement of Theorem 1.2 involves minimal varieties and canonical singularities, but our method uses only well-known and precisely referred facts in algebraic geometry rather than the technical part of the Minimal Model Program, and hence is accessible.

Below is our main result and the final one of our series of papers [19], [22] and [23].

Theorem 1.2. *Let X be an n -dimensional ($n \geq 3$) normal projective variety and $G \leq \text{Aut}(X)$ a subgroup such that the induced action $G | \text{NS}_{\mathbb{C}}(X)$ is solvable and Z -connected and that the rank $r(G) = n - 1$ (i.e., $G/N(G) = \mathbb{Z}^{\oplus n-1}$). Assume the following three conditions:*

- (i) X has at worst canonical, quotient singularities¹ (cf. [10, Definition 2.34]).
- (ii) X is a minimal variety, i.e., the canonical divisor K_X is nef (cf. [10, 0.4(3)]).
- (iii) The pair (X, G) is minimal in the sense of Section 2.5.

Then the following four assertions hold:

- (1) The induced action $N(G) | \text{NS}_{\mathbb{C}}(X)$ is a finite group.
- (2) $G | \text{NS}_{\mathbb{C}}(X)$ is a virtually free abelian group of rank $n - 1$.

¹In Theorem 1.2, the condition (i) can be weakened to: (i)' X has at worst klt singularities and is smooth in codimension two. This is by the same proof of ours, taking the global index-1 cover, and using Theorem 1.16 in the recent paper of D. Greb, S. Kebekus and T. Peternell: *Étale fundamental groups of Kawamata log terminal spaces, flat sheaves, and quotients of Abelian varieties*, arXiv:1307.5718.

- (3) *Either $N(G)$ is a finite subgroup of G and hence G is a virtually free abelian group of rank $n-1$, or X is an abelian variety and the group $N(G) \cap \text{Aut}_0(X)$ has finite-index in $N(G)$ and is Zariski-dense in $\text{Aut}_0(X) (\cong X)$.*
- (4) *We have $X \cong T/F$ for a finite group F acting freely outside a finite set of an abelian variety T . Further, for some finite-index subgroup G_1 of G , the action of G_1 on X lifts to an action of G_1 on T .*

For non-algebraic manifolds, a parallel result is given in Theorem 2.1.

In [23], we assumed that (i) G is abelian and (ii) the absence of pointwise G -fixed subvarieties of positive dimension or G -periodic rational curves or Q -tori. In the current paper, these two restrictions are replaced by the natural minimality condition on X and the pair (X, G) , and that $G \mid \text{NS}_{\mathbb{C}}(X)$ is solvable, the latter of which is natural in view of Theorem 1.1. The quotient singularities assumption in Theorem 1.2 is necessary because an effective characterization of the torus quotient is only available in dimension three by [16], where the bulk of the argument is to show that the variety has only quotient singularities.

The lack of the abelian-ness assumption on G makes our argument much harder, for instance we cannot simultaneously diagonalize $G \mid \text{NS}_{\mathbb{C}}(X)$ or find enough number of linearly independent common nef eigenvectors of G as required in [6] for abelian groups.

Theorems 1.2 and 2.1 answer [19, Question 2.17], assuming the conditions here. When G is abelian, the finiteness of $N(G)$ is proved in the inspiring paper of Dinh-Sibony [6, Theorem 1] (cf. also [22]), assuming only $r(G) = n - 1$. For non-abelian G , the finiteness of $N(G)$ is not true and we can at best expect that $N(G)$ is virtually included in $\text{Aut}_0(X)$ (as done in Theorems 1.2 and 2.1), since a larger group $\tilde{G} := \text{Aut}_0(X)G$ satisfies

$$\begin{aligned} \tilde{G} \mid \text{NS}_{\mathbb{C}}(X) &= G \mid \text{NS}_{\mathbb{C}}(X), \quad N(\tilde{G}) = \text{Aut}_0(X).N(G) \geq \text{Aut}_0(X), \\ \tilde{G}/N(\tilde{G}) &\cong G/N(G). \end{aligned}$$

There are examples (X, G) with $\text{rank } r(G) = \dim X - 1$ and X complex tori or their quotients (cf. [6, Example 4.5], [22, Example 1.7]).

The proof of Theorem 1.2 is much harder than that of Theorem 2.1 because of the presence of singularities on X .

The conditions (i) - (iii) in Theorem 1.2 are quite necessary in deducing $X \cong T/F$ as in Theorem 1.2(4). Indeed, if $X \cong T/F$ as in Theorem 1.2(4), then X has only quotient singularities and $dK_X \sim 0$ (linear equivalence) with $d = |F|$, and we may even assume that X has only canonical singularities if we replace X by its global index-1 cover; thus X is a minimal variety. If the pair (X, G) is not minimal so that there is a non-isomorphic G_1 -equivariant birational morphism $X \rightarrow Y$ as in Section 2.5, then the exceptional locus of this morphism is G_1 - and hence G -periodic, contradicting the fact that the rank $r(G) = n - 1$ (cf. the proof of Claim 2.20).

The *first key step* in proving Theorem 1.2 is the analysis of our quasi-nef sequence $L_1 \dots L_k$ ($0 < k < n$) (cf. [19, §2.2]) and we are able to show that L_i can actually be taken to be nef, when the rank $r(G) = n - 1$ (cf. Lemma 2.9). The *second key step* is Theorem 2.2, where we split G as $N(G)H$ such that $H \mid H^{1,1}(X)$ is free abelian. Thus we have a nef and big class A as the sum of nef common eigenvectors of H , leading to the vanishing of $A^{n-i}.c_i(X)$, where $c_i(X)$ ($i = 1, 2$) are Chern classes (cf. [16, pages 265-267]). The *third key step* is to use the minimality of (X, G) and

Kawamata’s base point freeness result for the \mathbb{R} -divisor (cf. [2, Theorem 3.9.1], available only for the projective variety at the moment) to deduce the vanishing of $c_2(X)$ as a linear form. This vanishing does not directly follow from the vanishing of $A^{n-2}.c_2(X)$ because A may not be ample. Now Theorem 1.2(4) follows from the vanishing of $c_i(X)$ ($i = 1, 2$) and the characterization of the torus quotient originally deduced from Yau’s deep result (cf. [1])

Remark 1.3.

- (1) When $\dim X = 3$, condition (i) in Theorem 1.2 can be replaced by: (i)’ X has at worst canonical singularities (cf. the Proof of Lemma 2.19 and [16, Corollary on p. 266]).
- (2) Theorems 1.2 and 2.1 are not true when $n := \dim X = 2$. We used $n \geq 3$ to deduce the vanishing of $c_2(X).A^{n-2}$ as commented above.

With Theorem 2.2 in mind, we ask:

Question 1.4. Suppose a group G acts on a compact complex Kähler manifold (say a complex torus) such that the null set $N(G)$ is a subgroup of (and hence normal in) G and the quotient $G/N(G)$ is a free abelian group. Under what condition can we write G (or its finite-index subgroup) as $G = N(G) \rtimes H$ with $H \leq G$ a free abelian subgroup of G ?

2. PROOF OF THE THEOREMS

In this section, we prove Theorem 1.2 in the introduction and the three results below.

When the X below is non-algebraic, we don’t require the minimality of the pair (X, G) .

Theorem 2.1. *Let X be an n -dimensional ($n \geq 3$) compact complex Kähler manifold which is not algebraic. Let $G \leq \text{Aut}(X)$ be a subgroup such that the induced action $G | H^{1,1}(X) \leq \text{Aut}(H^{1,1}(X))$ is solvable and Z -connected and that the rank $r(G) = n - 1$ (i.e., $G/N(G) = \mathbb{Z}^{\oplus n-1}$). Assume that X is minimal, i.e., the canonical divisor K_X is contained in the closure of the Kähler cone of X . Then the following four assertions hold:*

- (1) *The induced action $N(G) | H^{1,1}(X)$ is a finite group.*
- (2) *$G | H^{1,1}(X)$ is a virtually free abelian group of rank $n - 1$.*
- (3) *Either $N(G)$ is a finite subgroup of G and hence G is a virtually free abelian group of rank $n - 1$, or X is a complex torus and the group $N(G) \cap \text{Aut}_0(X)$ has finite-index in $N(G)$ and is Zariski-dense in $\text{Aut}_0(X) (\cong X)$.*
- (4) *We have $X \cong T/F$ for a finite group F acting freely outside a finite set of a complex torus T . Further, for some finite-index subgroup G_1 of G , the action of G_1 on X lifts to an action of G_1 on T .*

The X or the pair (X, G) below is not assumed to be minimal.

Theorem 2.2. *Let X be an n -dimensional ($n \geq 2$) compact complex Kähler manifold and $G \leq \text{Aut}(X)$ a subgroup such that the induced action $G | H^{1,1}(X) \leq \text{Aut}(H^{1,1}(X))$ is solvable and Z -connected and that the rank $r(G) = n - 1$ (i.e., $G/N(G) = \mathbb{Z}^{\oplus n-1}$). Then, replacing G by its finite-index subgroup, we can find a*

subgroup $H \leq G$ such that:

- (1) $G = N(G)H$;
- (2) $G | H^{1,1}(X) = (N(G) | H^{1,1}(X)) \rtimes (H | H^{1,1}(X))$;
- (3) $N(G) | H^{1,1}(X)$ is unipotent; and
- (4) the induced action $H | H^{1,1}(X) \leq \text{Aut}(H^{1,1}(X))$ is a free abelian group of rank $n - 1$, i.e., $H | H^{1,1}(X) \cong \mathbb{Z}^{\oplus n-1}$.

In the process of proving Theorem 2.2, we also deduce:

Proposition 2.3. *For the X and G in Theorem 2.2, replacing G by its finite-index subgroup, we can find some $g_0 \in G \setminus N(G)$, such that the first dynamical degrees satisfy*

$$d_1(g) = d_1(g_0)^t$$

for every $g \in G$ with $t \in \mathbb{Z}_{\geq 0}$ depending on g .

Remark 2.4. (1) In Theorem 2.2 and Proposition 2.3, if X is a normal projective variety, then a similar proof implies the same conclusions but with all the $H^{1,1}(X)$ in Theorem 2.2 replaced by $\text{NS}_{\mathbb{C}}(X)$.

(2) To allow singularities of X in Theorem 2.1, we need the Kähler version of the birational contraction theorem [2, Theorem 3.9.1] and Miyaoka’s pseudo-effectivity of $c_2(X)$ for minimal variety X , both of which seem to be very hard to confirm, since the Minimal Model Program has not been fully developed for Kähler manifolds.

2.5. s -cycles and minimal pairs. Let X be an n -dimensional compact Kähler manifold or a normal projective variety. When X is projective, set $\text{NS}_{\mathbb{R}}(X) := \text{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. A codimension- s (i.e., dimension- $(n - s)$) cycle D is an element in $H^{s,s}(X, \mathbb{R}) := H^{s,s}(X) \cap H^{2s}(X, \mathbb{R})$ (resp. a linear combination of $(n - s)$ -dimensional subvarieties with coefficients in \mathbb{R}) when X is Kähler (resp. projective). Two codimension- s cycles D_i are numerically equivalent, denoted as $D_1 \equiv D_2$, if $(D_1 - D_2) \cdot L_1 \dots L_{n-s} = 0$ for all L_i in $H^{1,1}(X, \mathbb{R})$ (resp. in $\text{NS}_{\mathbb{R}}(X)$), where we use $D \cdot L$ to denote the cup product (resp. intersection) for Kähler (resp. projective) X . Denote by $[D]$ the numerical equivalence class containing D and

$$N^s(X) := \{[D]; D \text{ is a codimension-}s \text{ cycle}\}$$

which is a finite-dimensional \mathbb{R} -vector space. We will loosely write $D \in N^s(X)$ by abuse of notation. Note that $N^1(X) = \text{NS}_{\mathbb{R}}(X)$ when X is projective. Denote by $K(X)$ (resp. $\text{Amp}(X)$) the open Kähler (resp. ample) cone and $\overline{K(X)}$ (resp. $\text{Nef}(X)$) its closure in $H^{1,1}(X, \mathbb{R})$ (resp. in $\text{NS}_{\mathbb{R}}(X)$). Elements in $\overline{K(X)}$ and $\text{Nef}(X)$ are called *nef*.

Let X be a normal projective variety with at worst canonical singularities (cf. [10, Definition 2.34]) and $G \leq \text{Aut}(X)$ a subgroup such that the null set $N(G)$ is a subgroup (and hence normal in G). The pair (X, G) is *non-minimal* if: there are a finite-index subgroup G_1 of G and a *non-isomorphic* G_1 -equivariant birational morphism $X \rightarrow Y$ onto a normal projective variety Y with at worst *isolated* canonical singularities. The pair (X, G) is *minimal* if it is not non-minimal.

Lemma 2.6. *Let G be a group and $H \triangleleft G$ a finite normal subgroup. Suppose that*

$$G/H = \langle \bar{g}_1 \rangle \times \dots \times \langle \bar{g}_r \rangle \cong \mathbb{Z}^{\oplus r}$$

for some $r \geq 1$ and $g_i \in G$. Then there is an integer $s > 0$ such that $G_1 := \langle g_1^s, \dots, g_r^s \rangle$ satisfies

$$G_1 = \langle g_1^s \rangle \times \dots \times \langle g_r^s \rangle \cong \mathbb{Z}^{\oplus r}$$

and it is a finite-index subgroup of G ; further, the quotient map $\gamma : G \rightarrow G/H$ restricts to an isomorphism $\gamma|_{G_1} : G_1 \rightarrow \gamma(G_1)$ onto a finite-index subgroup of G/H .

Proof. We only need to find $s > 0$ such that g_i^s and g_j^s are commutative to each other for all i, j . Since G/H is abelian, the commutator subgroup $[G, G] \leq H$. Thus the commutators $[g_1^t, g_2]$ ($t > 0$) all belong to H . The finiteness of H implies that $[g_1^{t_1}, g_2] = [g_1^{t_2}, g_2]$ for some $t_2 > t_1$, which implies that $g_1^{s_{12}}$ commutes with g_2 , where $s_{12} := t_2 - t_1$. Similarly, we can find an integer $s_{1j} > 0$ such that $g_1^{s_{1j}}$ commutes with g_j . Set $s_1 := s_{12} \times \dots \times s_{1r} > 0$. Then $g_1^{s_1}$ commutes with every g_j . Similarly, for each i , we can find an integer $s_i > 0$ such that $g_i^{s_i}$ commutes with g_j for all j . Now $s := s_1 \times \dots \times s_r > 0$ will do the job. This proves the lemma. \square

From now on until Section 2.14, we prove Theorem 2.2 and Proposition 2.3.

2.7. By [19, Proof of Theorem 1.2, §2.2], there is a quasi-nef sequence $L_1 \dots L_k \in H^{k,k}(X)$ ($1 \leq k \leq n - 1$) which is non-zero in $N^k(X)$, such that

$$g^*(L_1 \dots L_k) = \chi_1(g) \dots \chi_k(g)(L_1 \dots L_k)$$

for all $k = 1, \dots, n - 1$ with characters $\chi_i : G \rightarrow (\mathbb{R}_{>0}, \times)$, and that

$$\begin{aligned} \varphi : G &\rightarrow (\mathbb{R}^{n-1}, +) \\ g &\mapsto (\log \chi_1(g), \dots, \log \chi_{n-1}(g)) \end{aligned}$$

is a homomorphism having

$$\text{Ker}(\varphi) = N(G)$$

and the following discrete subgroup of $\mathbb{R}^{\oplus n-1}$ as its image (now with $r(G) = n - 1$):

$$\text{Im}(\varphi) = \mathbb{Z}^{\oplus r(G)}.$$

Lemma 2.8. *Each projection $\text{Im}(\log \chi_i) \subset \mathbb{R}$ ($1 \leq i \leq n - 1$) is a free abelian group of rank one.*

Proof. It suffices to show that $\text{Im}(\log \chi_i) \cong \mathbb{Z}$ since $\text{rank Im}(\varphi) = n - 1$. If $\text{rank Im}(\log \chi_i) \geq 2$, then Kronecker’s theorem implies the existence of a sequence $f_t \in G$ with $\chi_i(f_t) > 1$ so that $\lim_{t \rightarrow \infty} \chi_i(f_t) = 1$ (cf. [18, proof of Theorem 3.1]), which contradicts the discreteness of $\text{Im}(\varphi)$ as proved in [19, Claim 2.9]. This proves the lemma. \square

By the above result, $\text{Im}(\varphi)$ is the direct product of $n - 1$ cyclic groups $\text{Im}(\log \chi_i)$ which are the φ -images of $\langle g_i \rangle$ say, with

$$\lambda_i := \chi_i(g_i) > 1, \quad \chi_j(g_i) = 1 \quad (j \neq i).$$

Hence

$$G/N(G) = \langle \overline{g_1}, \dots, \overline{g_{n-1}} \rangle = \langle \overline{g_1} \rangle \oplus \dots \oplus \langle \overline{g_{n-1}} \rangle \cong \mathbb{Z}^{\oplus n-1}$$

and

$$\text{Im}(\varphi) = \langle \varphi(g_1), \dots, \varphi(g_{n-1}) \rangle = \langle \varphi(g_1) \rangle \oplus \dots \oplus \langle \varphi(g_{n-1}) \rangle \cong \mathbb{Z}^{\oplus n-1}.$$

By the generalization of the Perron-Frobenius theorem [3] applied to the action of $g_i^{\pm 1}$ on the closure $\overline{K(X)}$ of the Kähler cone $K(X)$ (which spans $H^{1,1}(X, \mathbb{R})$),

there are non-zero nef $L_{g_i^{\pm 1}} \in \overline{K(X)}$ (which we now fix) such that

$$(g_i^{\pm 1})^* L_{g_i^{\pm 1}} = d_1(g_i^{\pm 1}) L_{g_i^{\pm 1}}.$$

Note that the first dynamical degree $d_1(g_i^{\pm 1}) > 1$ since $g_i^{\pm 1} \notin N(G)$.

Lemma 2.9. *Let $k \in \{1, \dots, n - 1\}$. Let $M \in \overline{K(X)}$ be a non-zero nef element such that $g_k^* M = \lambda M$ for some $\lambda \neq 1$ (e.g., we can take $M = L_{g_k}, L_{g_k^{-1}}$).*

- (0) *For D (e.g., $D = L_1 \dots L_k$) in $\overline{P^k(X)}$ (the closure of Kähler (k, k) -forms as defined in [12, before Lemma A.3]), $D = 0$ in $N^k(X)$ if and only if $D = 0$ in $H^{k,k}(X)$.*
- (1) *Suppose that $\lambda \neq 1/\lambda_k$ (e.g., we can take $M = L_{g_k}$). Then $L_1 \dots L_k \cdot M = 0$ in $H^{k+1}(X)$.*
- (2) *Suppose that $k \geq 2$. Then $L_1 \dots L_{k-1} \cdot M$ is non-zero in $N^k(X)$.*
- (3) *Suppose that $k = 1$ and $\lambda > 1$ (e.g., $M = L_{g_1}$). Then M is parallel to L_1 in $H^{1,1}(X)$ and hence $\lambda = \lambda_1$. In particular, L_{g_1} is parallel to L_1 in $H^{1,1}(X)$.*
- (4) *Suppose that $k \geq 2$ and $\lambda \neq 1/\lambda_k$. Then $L_1 \dots L_k$ and $L_1 \dots L_{k-1} \cdot M$ are parallel in $N^k(X)$. In particular, $\lambda = \lambda_k$.*
- (5) *For all $1 \leq s \leq n - 1$, $L_1 \dots L_s$ and $L_{g_1} \dots L_{g_s}$ are parallel in $N^s(X)$.*
- (6) *λ equals either $\lambda_k > 1$ or $1/\lambda_k < 1$.*
- (7) *For all $1 \leq s \leq n - 1$, we have*

$$d_1(g_s) = \lambda_s (= \chi_s(g_s)), \quad d_1(g_s^{-1}) = d_1(g_s) (> 1).$$

- (8) *For all $1 \leq s \leq n - 1$, $L_1 \dots L_{n-1} \cdot L_{g_s^{-1}}$ is a non-zero (positive) scalar in $N^n(X) = H^{n,n}(X, \mathbb{R}) \cong \mathbb{R}$.*

Proof. The assertion (0) is well known; see e.g. [12, Lemma A.4].

(1) Suppose to the contrary that $L_1 \dots L_k \cdot M$ is non-zero in $H^{k,k}(X)$ (i.e., in $N^{k+1}(X)$ by the assertion (0)). Since

$$g_k^*(L_1 \dots L_k \cdot L_{k+1}) = \lambda_k(L_1 \dots L_k \cdot L_{k+1}), \quad g_k^*(L_1 \dots L_k \cdot M) = \lambda_k \lambda(L_1 \dots L_k \cdot M)$$

with $\lambda_k \neq \lambda_k \lambda$, we have $L_1 \dots L_{k+1} \cdot L_{g_k} \neq 0$ in $N^{k+1}(X)$ (cf. [19, Lemma 2.3], [6, Lemma 4.4]). Inductively, the same g_k^* action and [ibid.] imply that $L_1 \dots L_t \cdot M$ is non-zero in $N^{t+1}(X)$ for all $k \leq t \leq n - 1$. In particular, $L_1 \dots L_{n-1} \cdot M$ equals a positive scalar b_n in $N^n(X) \cong \mathbb{R}$. Thus $b_n = g_k^* b_n = g_k^*(L_1 \dots L_{n-1} \cdot M) = \lambda_k \lambda b_n \neq b_n$, by the condition on λ . This is a contradiction.

(2) Since $g_k^* M = \lambda M$ and $g_k^* L_1 = L_1$ with $\lambda \neq 1$, we have $L_1 \cdot M \neq 0$ in $N^2(X)$ by [6, Lemma 4.4].

Let $k \geq 3$ and let $t \geq 1$ be the largest integer such that $L_1 \dots L_t \cdot M \neq 0$ in $N^{t+1}(X)$. If $t \geq k - 1$, then (2) is true. Suppose to the contrary that $t \leq k - 2$. Since

$$g_k^*(L_1 \dots L_t \cdot L_{t+1}) = L_1 \dots L_t \cdot L_{t+1}, \quad g_k^*(L_1 \dots L_t \cdot M) = \lambda(L_1 \dots L_t \cdot M)$$

with $\lambda \neq 1$, we have $L_1 \dots L_t \cdot L_{t+1} \cdot M \neq 0$ in $N^{t+2}(X)$ (cf. [19, Lemma 2.3], [6, Lemma 4.4]). This contradicts the maximality of t . So $t \geq k - 1$. Hence (2) is true.

(3) By (1), $L_1 \cdot L_{g_1} = 0$ in $H^{1,1}(X)$, so L_1 and M are parallel in $H^{1,1}(X)$ (cf. [6, Corollary 3.2]).

(4) We have $L_1 \dots L_k \cdot M = 0$ by (1). Then (4) follows (cf. [19, Lemma 2.3], [6, Corollary 3.5]). Indeed, for the last part, just apply g_k^* to the equality of the first part (cf. (2)).

(5) follows from (3) and (4) with $M = L_{g_k}$, by induction on k .

(6) By (2), $L_1 \dots L_{k-1}.M \neq 0$ in $N^k(X)$. Suppose to the contrary that $\lambda \neq \lambda_k^\pm$. Since

$$g_k^*(L_1 \dots L_{k-1}.L_k) = \lambda_k(L_1 \dots L_k), \quad g_k^*(L_1 \dots L_{k-1}.M) = \lambda(L_1 \dots L_{k-1}.M)$$

with $\lambda_k \neq \lambda$, we have $L_1 \dots L_k.M \neq 0$ in $N^{k+1}(X)$ (cf. [19, Lemma 2.3], [6, Lemma 4.4]), contradicting (1).

(7) follows from (6) with $M = L_{g_k}, L_{g_k^{-1}}$.

(8) Set $M := L_{g_k^{-1}}$. By (2), we have the non-vanishing of $L_1 \dots L_{k-1}.M$ in $N^k(X)$. As in the proof of (1), inductively, the action of g_k^* implies the non-vanishing of $L_1 \dots L_t.M$ in $N^{t+1}(X)$ for all $k-1 \leq t \leq n-1$. \square

By Lemma 2.9, we may and will take

$$L_i = L_{g_i}.$$

Lemma 2.10. *For all $1 \leq k \leq n-1$, as elements in $H^{2,2}(X)$, we have*

$$L_{g_k}^2 = 0, \quad L_{g_k^{-1}}^2 = 0.$$

*In particular, in $H^{1,1}(X)$, when $g_k^*M = \lambda M$ for some non-zero nef M , then $\lambda = d_1(g_k)$ and M is parallel to L_{g_k} if $\lambda > 1$, and $\lambda^{-1} = d_1(g_k) = d_1(g_k^{-1})$ and M is parallel to $L_{g_k^{-1}}$ if $\lambda < 1$. So the choice of nef L_{g_k} (resp. $L_{g_k^{-1}}$) is unique, up to a positive multiple.*

Proof. We take $L_i = L_{g_i}$. Set $M = L_{g_k}$ or $L_{g_k^{-1}}$.

We claim that $L_1 \dots L_{k-1}.M^2$ vanishes in $N^{k+1}(X)$ (i.e., in $H^{k+1,k+1}(X)$, by Lemma 2.9(0)). When $M = L_{g_k}$, this is true by Lemma 2.9(1). When $M = L_{g_k^{-1}}$, suppose to the contrary that $L_1 \dots L_{k-1}.M^2$ is non-vanishing in $N^{k+1}(X)$. Note that $L_1 \dots L_k.M \neq 0$ in $N^{k+1}(X)$ by Lemma 2.9(8). Since

$$\begin{aligned} g_k^*(L_1 \dots L_{k-1}.M.M) &= (1/\lambda_k)^2(L_1 \dots L_{k-1}.M.M), \\ g_k^*(L_1 \dots L_{k-1}.M.L_k) &= (1/\lambda_k)\lambda_k(L_1 \dots L_{k-1}.L_k.L_t) \end{aligned}$$

with $(1/\lambda_k)^2 \neq (1/\lambda_k)\lambda_k$, we have the non-vanishing of $L_1 \dots L_k.M^2$ in $N^{k+2}(X)$ (cf. [19, Lemma 2.3], [6, Lemma 4.4]). Inductively, the action of g_k^* and [ibid.] imply the non-vanishing of $b_{t+2} := L_1 \dots L_t.M^2$ in $N^{t+2}(X)$, for all $k-1 \leq t \leq n-2$. Thus $b_n = g_k^*b_n = \alpha(1/\lambda_k)^2b_n \neq b_n$, where $\alpha = 1$ (when $k = n-1$) or $\alpha = \lambda_k$ (when $k \leq n-2$), a contradiction! Hence the claim is true. In particular, the lemma is true when $k = 1$.

Let $k \geq 2$. Suppose to the contrary that M^2 is non-zero in $H^{2,2}(X)$ (i.e., in $N^2(X)$ by Lemma 2.9(0)). By the claim above, we can choose $1 \leq t \leq k-1$ to be the smallest integer such that $L_1 \dots L_t.M^2 = 0$ in $N^{t+2}(X)$. Thus $L_1 \dots L_{t-1}.M^2 \neq 0$ in $N^{t+1}(X)$. Note that $L_1 \dots L_t.M \neq 0$ in $N^{t+1}(X)$, by Lemma 2.9(8) (for $M = L_{g_k^{-1}}$) and by the non-vanishing of $L_1 \dots L_{n-1}$ in $N^{n-1}(X)$ (for $M = L_k$). Since

$$\begin{aligned} g_k^*(L_1 \dots L_{t-1}.M.M) &= \lambda^2(L_1 \dots L_{t-1}.M.M), \\ g_k^*(L_1 \dots L_{t-1}.M.L_t) &= \lambda(L_1 \dots L_{t-1}.M.L_t) \end{aligned}$$

with $\lambda^2 \neq \lambda$ (because $\lambda = \lambda_k$ or λ_k^{-1} for $M = L_{g_k}$ or $L_{g_k^{-1}}$), we have the non-vanishing of $L_1 \dots L_t.M^2$ in $N^{t+2}(X)$ (cf. [19, Lemma 2.3], [6, Lemma 4.4]). This contradicts the choice of t .

For the final part, we may suppose that $\lambda \neq 1$. By Lemma 2.9, if $\lambda > 1$ (resp. $\lambda < 1$), we have $\lambda = d_1(g_k) = d_1(g_k^{-1})$ (resp. $\lambda^{-1} = d_1(g_k) = d_1(g_k^{-1})$). Taking M or $M + L_{g_k}$ (resp. M or $M + L_{g_k^{-1}}$) as new L_{g_k} (resp. $L_{g_k^{-1}}$), the first part shows that, in $H^{2,2}(X)$, we have $M^2 = L_{g_k}^2 = (M + L_{g_k})^2 = 0$ (resp. $M^2 = L_{g_k^{-1}}^2 = (M + L_{g_k^{-1}})^2 = 0$), hence $M.L_{g_k} = 0$ (resp. $M.L_{g_k^{-1}} = 0$). Then M and L_{g_k} (resp. M and $L_{g_k^{-1}}$) are parallel in $H^{1,1}(X)$, by [6, Corollary 3.2]. \square

Set $L_n = L_{g_k^{-1}}$ for some $1 \leq k \leq n - 1$. Then

$$A := L_1 + \dots + L_n$$

is a nef and big class because $H^n \geq L_1 \dots L_{n-1}.L_n > 0$ by Lemma 2.9(8). We may also write $g^*(L_1 \dots L_n) = \chi_1(g) \dots \chi_n(g)(L_1 \dots L_n)$. Then

$$\chi_1 \dots \chi_n = 1$$

since $L_1 \dots L_n$ is a non-zero (positive) scalar in $N^n(X) \cong \mathbb{R}$.

Lemma 2.11. $C := (Z(G)|H^{1,1}(X)) \cap (N(G)|H^{1,1}(X))$ is a finite subgroup of $\text{Aut}(H^{1,1}(X))$.

Proof. Take any $z|H^{1,1}(X) \in C$. Then $z|H^{1,1}(X) \in Z(G)|H^{1,1}(X)$, and hence $z|H^{1,1}(X)$ commutes with every $g_i|H^{1,1}(X)$. Thus, $g_i^*z^*L_{g_i} = z^*g_i^*L_{g_i} = d_1(g_i)z^*L_{g_i}$, so $z^*L_{g_i}$ and L_{g_i} are parallel by Lemma 2.10, and hence are equal, since $z \in N(G)$ has the first dynamical degree $d_1(z) = 1$. By the same reasoning, $z^*L_{g_i^{-1}} = L_{g_i^{-1}}$. Thus z^* fixes the nef and big class $A = \sum_{i=1}^n L_i$ with $L_i = L_{g_i}$ and $L_n = L_{g_1^{-1}}$. Hence $z^s \in \text{Aut}_0(X)$ for some $s > 0$ by [11, Proposition 2.2] (cf. [20, Lemma 2.23] and note that a big class is the sum of a Kähler class and a positive real current, according to Demailly-Paun). Thus z^s acts trivially on the lattice $H^2(X, \mathbb{Z})$ and hence also on $H^{1,1}(X)$. So C is a periodic group and defined over \mathbb{Z} , and hence is a finite group by Burnside’s theorem (cf. [14, Proof of Proposition 2.2]). This proves the lemma. \square

Now Theorem 2.2 follows from Proposition 2.12:

Proposition 2.12. Replacing G by its finite-index subgroup and g_i by some element in $g_iN(G)$, the group $H := \langle g_1, \dots, g_{n-1} \rangle$ has its image $H|H^{1,1}(X)$ in $\text{Aut}(H^{1,1}(X))$, a free abelian group of rank $n - 1$, so that

$$G = N(G)H, N(G)|H^{1,1}(X)$$

is unipotent, and

$$G|H^{1,1}(X) = (N(G)|H^{1,1}(X)) \times (H|H^{1,1}(X)).$$

Proof. By [17, Proposition 4.1] and Selberg’s lemma, replacing G by its finite-index subgroup, we may assume that $G|H^{1,1}(X)$ is torsion free. Note that the set $U(G) := \{u \in G; u|H^{1,1}(X) \text{ is unipotent}\}$ is a finite-index subgroup of $N(G)$ and a characteristic subgroup of G (cf. [14, Proof of Prop. 2.2], [4, Theorem 3.1]). Replacing G by a finite-index subgroup of $G_1 := U(G)\langle g_1, \dots, g_{n-1} \rangle$, we may assume that $N(G) = U(G)$ (applying Lemma 2.6 to the group $G_1/U(G)$). Since $\text{Ker}(G \rightarrow G|H^{1,1}(X)) \leq N(G)$, we get

$$(*) \quad (G|H^{1,1}(X))/(N(G)|H^{1,1}(X)) \cong G/N(G) = \langle \overline{g_1}, \dots, \overline{g_{n-1}} \rangle \cong \mathbb{Z}^{\oplus n-1},$$

which is abelian and hence nilpotent. By [15, Thm. 3, p. 48], there is a subgroup H of G such that $H \mid H^{1,1}(X)$ is nilpotent and $G = N(G)H$, after replacing G by its finite-index subgroup. Replacing g_i by some element in $g_iN(G)$, we may assume that $g_i \in H$.

Suppose $H \mid H^{1,1}(X)$ is non-abelian. Then the commutator subgroup $[H, H] \mid H^{1,1}(X)$ is non-trivial and is contained in $[G, G] \mid H^{1,1}(X)$, while the latter is contained in the group $N(G) \mid H^{1,1}(X)$ because $G/N(G)$ is abelian. Since $H \mid H^{1,1}(X)$ is nilpotent, its centre $(\leq [H, H] \mid H^{1,1}(X) \leq N(G) \mid H^{1,1}(X))$ contains a non-trivial element $z \mid H^{1,1}(X)$ (of course commutative with all $g_i \mid H^{1,1}(X)$). The proof of Lemma 2.11 shows that $z \mid H^{1,1}(X) \in G \mid H^{1,1}(X)$ is of finite order, contradicting the torsion freeness assumption of $G \mid H^{1,1}(X)$.

Thus, we may assume that $H \mid H^{1,1}(X)$ is abelian (and free because $G \mid H^{1,1}(X)$ is as well). Replacing H by $\langle g_1, \dots, g_{n-1} \rangle$, we may assume that $H \mid H^{1,1}(X)$ is free abelian and of rank $n - 1$ because it is generated by $n - 1$ elements and dominates $\mathbb{Z}^{\oplus n-1}$ via the *surjective* composite below (cf. the display (*) above):

$$\begin{aligned} H \mid H^{1,1}(X) &\rightarrow (H \mid H^{1,1}(X)) / ((H \mid H^{1,1}(X)) \cap (N(G) \mid H^{1,1}(X))) \\ &\rightarrow (G \mid H^{1,1}(X)) / (N(G) \mid H^{1,1}(X)) \cong \langle \overline{g_1}, \dots, \overline{g_{n-1}} \rangle \cong \mathbb{Z}^{\oplus n-1}. \end{aligned}$$

Now the same dominance (between free abelian groups of the same rank) implies that $(H \mid H^{1,1}(X)) \cap (N(G) \mid H^{1,1}(X)) = \{\text{id}\}$ (or else $\mathbb{Z}^{\oplus n-1}$ is dominated by a free abelian group, a quotient of H , of rank $\leq n - 2$, absurd!). Thus the display of and hence the entire proposition follow. □

Lemma 2.13. $L_{g_s^{-1}}$ ($1 \leq s \leq n - 1$) are all parallel to $L_{g_1^{-1}}$ in $H^{1,1}(X)$.

Proof. Since $L_{g_i^{-1}}$ is unique up to a multiple (cf. Lemma 2.10) and is parallel to $L_{g_i^{-t}}$ for any $t > 0$, we may assume that $G = N(G)H$ as in Proposition 2.12. Since $g_i \mid H^{1,1}(X) \in H \mid H^{1,1}(X)$ ($1 \leq i \leq n - 1$) are commutative to each other, we may assume that both L_{g_i} and $L_{g_i^{-1}}$ are common nef eigenvectors of H . Set $L := L_{g_1^{-1}}$ and write $g_s^*L = \mu_s L$ with $\mu_1 = 1/d_1(g_1^{-1}) = 1/d_1(g_1) = 1/\lambda_1$ (cf. Lemma 2.9). Applying g_s^* to the non-zero scalar $L_{g_1} \dots L_{g_{n-1}} \cdot L$ in $N^n(X) \cong \mathbb{R}$ in Lemma 2.9, we get

$$L_{g_1} \dots L_{g_{n-1}} \cdot L = (d_1(g_s) \mu_s)(L_{g_1} \dots L_{g_{n-1}} \cdot L).$$

Hence $1 = d_1(g_s) \mu_s$ and $\mu_s = 1/d_1(g_s) = 1/d_1(g_s^{-1})$. Thus $(g_s^{-1})^*L = d_1(g_s^{-1})L$ and the lemma follows (cf. Lemma 2.10). □

2.14. Proof of Proposition 2.3. Since $G \mid H^{1,1}(X)$ is solvable and Z -connected and hence upper-triangularizable by the Lie-Kolchin theorem and by Proposition 2.12, we may assume that $G = H = \langle g_1, \dots, g_{n-1} \rangle$. By Lemma 2.13, $L'_1 := L_{g_1^{-1}}$ is a nef common eigenvector of G . As in [19, Proof of Theorem 1.1; §2.2], we may construct a quasi-nef sequence $L'_1 \dots L'_k$ ($1 \leq k \leq n - 1$) so that $g^*(L'_1 \dots L'_k) = (\chi'_1(g) \dots \chi'_k(g))(L'_1 \dots L'_k)$ and the homomorphism

$$\begin{aligned} \varphi' : G &\rightarrow (\mathbb{R}^{\oplus n-1}, +), \\ g &\mapsto (\log \chi'_1(g), \dots, \log \chi'_{n-1}(g)) \end{aligned}$$

satisfies

$$\text{Ker}(\varphi') = N(G), \quad \text{Im}(\varphi') \cong \mathbb{Z}^{\oplus n-1}.$$

As in the proof of Lemma 2.8, the projection $\text{Im}(\log \chi'_1) \cong \mathbb{Z}$ and hence is generated by some $\log \mu := \log \chi'_1(g_b)$. As in the proof of Lemma 2.13, $d_1(g_s)L'_1 = (g_s^{-1})^*L'_1 = \chi'_1(g_s^{-1})L'_1 = \mu^t L'_1$ with $t \in \mathbb{Z}_{>0}$ depending on g_s . Hence $d_1(g_s) = d_1(g_b)^t$. Replacing g_i 's by their powers, we may assume that $\lambda := d_1(g_i)$ is independent of $i \in \{1, \dots, n-1\}$.

For every $g \in G \setminus N(G)$, since $G \mid H^{1,1}(X)$ is now commutative, we may assume that L_g is a nef common eigenvector of G . If $L_{g_1} \dots L_{g_{n-1}}.L_g \neq 0$, then applying g^* we get

$$\chi_1(g) \dots \chi_{n-1}(g).d_1(g) = 1.$$

If $L_{g_1} \dots L_{g_{n-1}}.L_g = 0$, then as in Lemma 2.9, [6, Lemma 4.4] implies that $d_1(g) = \chi_i(g)$ for some i . Thus, $d_1(g)$ is either $\chi_i(g)$ or $\prod_{j=1}^{n-1} \chi_j(g^{-1})$, which is an integer power of λ . If we express g as a product of powers of g_i , use Lemma 2.9(7) and note that $\chi_i(g_j) = 1$ ($i \neq j$). This proves Proposition 2.3.

2.15. With the notation and assumption of Theorem 1.2, suppose that $c_2(X) \neq 0$ as an element in $N^{n-2}(X)$, i.e., as a linear form on $N^1(X) \times \dots \times N^1(X)$, $(n-2)$ of them (cf. [16, pages 265-267], [22, Definition 2.4]). Since X is minimal, Miyaoka's pseudo-effectivity of $c_2(X)$ implies that $c_2(X) \geq 0$ on $\text{Nef}(X) \times \dots \times \text{Nef}(X)$, $(n-2)$ of them (cf. [16, Theorem 4.1] and the references therein). Hence $c_2(X) > 0$ on the self-product of the ample cone $\text{Amp}(X) \times \dots \times \text{Amp}(X)$, since $N^1(X)$ is spanned by $\text{Amp}(X)$, which is an open cone.

Since $c_2(X).\text{Amp}(X)$ is a non-zero cone of $N^3(X)$ and is stable under the natural action of G , and since $G \mid N^1(X)$ is solvable and Z -connected, there are nef (and indeed ample) divisors $P(t)$ such that

$$c_2(X).M_1 := \lim_{t \rightarrow \infty} c_2(X).P(t)$$

is non-zero in $N^3(X)$ and a common G -eigenvector (cf. [9, Theorem 1.1]); note that $M_1 \in N^1(X)$ may *not* be nef since the cone $c_2(X).\text{Nef}(X)$ may not be closed in $N^3(X)$. By the same reasoning, we can construct $c_2(X).M_1.M_2$. Continuing the process, we get:

Lemma 2.16. *With the notation and assumption of Theorem 1.2, suppose that $c_2(X) \neq 0$ in $N^2(X)$. Then we have:*

- (1) *There is a sequence $0 \neq c_2(X).M_1 \dots M_k \in N^{k+2}(X)$ ($1 \leq k \leq n-3$) (called a pseudo-effective sequence) which is positive on the self-product of the ample cone $\text{Amp}(X) \times \dots \times \text{Amp}(X)$ ($n-k-2$ of them), such that*

$$g^*(c_2(X).M_1 \dots M_k) = \chi'_1(g) \dots \chi'_k(g)(c_2(X).M_1 \dots M_k)$$

for all k with characters $\chi'_i : G \rightarrow (\mathbb{R}_{>0}, \times)$.

- (2) *In particular, $C := c_2(X).M_1 \dots M_{n-3}$ is a non-zero element in the closed cone $\overline{\text{NE}}(X)$ of effective 1-cycles (which is dual to the nef cone $\text{Nef}(X)$).*

2.17. **Proof of Theorem 1.2.** The proof will almost fill up the rest of the paper. We may and will freely replace G by its finite-index subgroups. We may assume that $G = N(G)H$ with $H = \langle g_1, \dots, g_{n-1} \rangle$ so that $H \mid \text{NS}_{\mathbb{C}}(X) \cong \mathbb{Z}^{\oplus n-1}$ as in Theorem 2.2 and satisfies the four assertions there, but with $H^{1,1}(X)$ replaced by $\text{NS}_{\mathbb{C}}(X)$ (cf. Remark 2.4). We use the notation in the proof of Theorem 2.2 and let

$$A_H := L_{g_1} + \dots + L_{g_{n-1}} + L_{g_1^{-1}}$$

be the nef and big divisor, where $L_{g_j^\pm} \in \text{Nef}(X)$ can be chosen to be common eigenvectors of H since $H \mid \text{NS}_{\mathbb{R}}(X)$ is commutative.

- Lemma 2.18.** (1) *Let $D \in N^s(X)$ ($0 < s < n$) such that $h^*D = D$ for all $h \in H$. Then $D \cdot A_H^{n-s} = 0$.*
 (2) *In particular, for the Chern classes $c_i(X)$ ($i = 1, 2$), we have $c_i(X) \cdot A_H^{n-i} = 0$.*
 (3) *Hence $K_X \sim_{\mathbb{Q}} 0$, i.e., a positive multiple of K_X is linearly equivalent to zero.*

Proof. Take $M := L_1^{i_1} \cdots L_n^{i_n}$ with $\sum_{k=1}^n i_k = n - s$. For $h \in H$, we have $h^*M = e(h)M$ with $e(h) = \chi_1(h)^{i_1} \cdots \chi_n(h)^{i_n}$. Since $\mathbb{R} \ni M \cdot D = h^*M \cdot h^*D = e(h)M \cdot D$ and since A_H^{n-s} is a combination of such M , it suffices to show that $e(h) \neq 1$ for some $h \in H$ (so that $M \cdot D = 0$). Suppose to the contrary that $e(h) = 1$ for all $h \in H$. Taking log and using $\chi_1 \cdots \chi_n = 1$, we have $(i_1 - i_n) \log \chi_1 + \cdots + (i_{n-1} - i_n) \log \chi_{n-1} = 0$ on H . Since the image of the homomorphism $\varphi = (\log \chi_1, \dots, \log \chi_{n-1})$ is a spanning lattice in $\mathbb{R}^{\oplus n-1}$ of rank $n - 1$ (cf. Section 2.7), this happens only when $i_1 - i_n = \cdots = i_{n-1} - i_n = 0$. Thus $n - 1 \geq n - s = \sum_{k=1}^n i_k = ni_1$, so $i_1 = 0$ and hence $s = n$. This is absurd.

Since X is minimal and hence K_X is nef, the vanishing of $K_X \cdot A_H^{n-1}$ and [13, Lemma 2.2] imply that $K_X \equiv 0$ (numerically). Hence $K_X \sim_{\mathbb{Q}} 0$ (cf. [8, Theorem 8.2]). This proves Lemma 2.18. □

Lemma 2.19. *Theorem 1.2 is true when $c_2(X) = 0$ in $N^2(X)$.*

We prove the lemma. Set $A := A_H$. The vanishing of $c_2(X)$ in $N^2(X)$ implies the vanishing of the orbifold second Chern class of X (cf. [16, Proposition 1.1]). This and the vanishing of $c_1(X)$ in Lemma 2.18 imply the existence of a finite surjective morphism $T' \rightarrow X$ from an abelian variety T' , based on a deep result of S. T. Yau (cf. [1], [5, Theorem 7.6]). This is the place we need the singularities of X to be of quotient type (cf. also Remark 1.3).

Since $K_{T'} \sim 0 \sim_{\mathbb{Q}} K_X$, the map $T' \rightarrow X$ is étale in codimension one. Let $T \rightarrow X$ be the Galois cover corresponding to the unique maximal lattice L in $\pi_1(X \setminus \text{Sing } X)$ so that T is an abelian variety. Then $X = T/F$ with

$$F = \pi_1(X \setminus \text{Sing } X)/L = \text{Gal}(T/X),$$

and there is an exact sequence

$$1 \rightarrow F \rightarrow \tilde{G} \xrightarrow{\gamma} G \rightarrow 1$$

where \tilde{G} (the lifting of the original G) acts faithfully on T (cf. [1, §3, especially the Proof of Prop. 3] applied to étale-in-codimension-one covers, and also [13, Prop. 3.5]).

By [20, Lemma 2.6] or [12, Lemma A.8], $N(\tilde{G}) = \gamma^{-1}(N(G))$. Hence

$$\tilde{G}/N(\tilde{G}) \cong G/N(G) \cong \mathbb{Z}^{\oplus n-1}, \quad r(\tilde{G}) = r(G) = n - 1.$$

Since $N(\tilde{G}) \mid \text{NS}_{\mathbb{C}}(X)$ is virtually unipotent (cf. Proposition 2.12), we may assume that $\tilde{G} \mid \text{NS}_{\mathbb{C}}(X)$ is solvable and Z -connected after G is replaced by its finite-index subgroup.

If Theorem 1.2(1) is not true, then $N(\tilde{G}) \mid \text{NS}_{\mathbb{R}}(T)$ is also an infinite group. Thus [22, Proof of Theorem 1.1(3), page 2338] shows the existence of a \tilde{G} -equivariant

fibration $T \rightarrow T/B$ with $0 < B < T$ a subtorus of T fixed by the unipotent elements of $\tilde{G} \mid H^{1,1}(T)$. This leads to $\text{rank } r(G) = r(\tilde{G}) \leq n - 2$ (cf. [19, Proof of Lemma 2.10]), contradicting the assumption $r(G) = n - 1$. Hence assertion (1) of Theorem 1.2 is true. (2) follows from (1) and Lemma 2.6.

For assertion (3), take an ample divisor (or a Kähler class for the purpose of Theorem 2.1) M' on X . Then $M := \sum h_t^* M'$, where h_t runs in the finite group $N(G) \mid \text{NS}_{\mathbb{R}}(X)$, is an ample divisor (or a Kähler class) and is stable under the action of $N(G)$. Hence $N(G) \leq \text{Aut}_M(X) := \{f \in \text{Aut}(X) \mid f^* M = M\}$, where $|\text{Aut}_M(X) : \text{Aut}_0(X)| < \infty$ by [11, Proposition 2.2] or [7, Theorem 4.8]. Thus for

$$N_0 := N(G) \cap \text{Aut}_0(X)$$

we have

$$N(G)/N_0 \cong (\text{Aut}_0(X).N(G))/\text{Aut}_0(X) \leq \text{Aut}_M(X)/\text{Aut}_0(X),$$

where the latter is a finite group. If N_0 is finite, then so is $N(G)$, and the first case of assertion (3) is true (cf. Lemma 2.6).

Suppose that N_0 is infinite. We shall show that the second case of assertion (3) occurs. First, $\text{Aut}_0(X) \neq 1$. If the linear part of $\text{Aut}_0(X)$ is non-trivial, then X is ruled (and hence uniruled) (cf. [7, Proposition 5.10]), contradicting the assumption that X is a minimal variety with only canonical singularities and hence non-uniruled by the well-known Miyaoka-Mori uniruledness criterion.

Therefore, the linear part $\text{Aut}_0(X)$ is trivial. Then $\text{Aut}_0(X)$ is a complex torus and (cf. [11, Theorem 3.12] or [7, Theorem 5.5])

$$1 \leq \dim \text{Aut}_0(X) \leq \dim \text{Alb}(X) = q(X),$$

where $\text{Alb}(X)$ is the Albanese variety of X and $q(X) = h^1(X, \mathcal{O}_X)$ is the irregularity of X . By [19, Lemma 2.13] applied to a resolution of X and using [8, Lemma 8.1], the albanese map $\text{alb}_X : X \rightarrow \text{Alb}(X)$ is a birational and surjective morphism.

Claim 2.20. X has no positive-dimensional subvariety Y' which is G -periodic (i.e., Y' is stabilized by a finite-index subgroup of G).

Proof. We prove the claim. We have already proved the finiteness of $N(\tilde{G}) \mid \text{NS}_{\mathbb{C}}(T)$. Replacing \tilde{G} by its finite-index subgroup and by Lemma 2.6, we may assume that $\tilde{G} \mid \text{NS}_{\mathbb{C}}(T)$ and even $\tilde{G}/(\tilde{G} \cap \text{Aut}_0(T))$ are free abelian groups of rank $n - 1$.

Suppose a positive-dimensional subvariety $Y' \subset X$ is G -periodic. Then a subvariety $Y \subset T$ (dominating Y') is \tilde{G} -periodic and is even stabilized by \tilde{G} after this group is replaced by its finite-index subgroup. By the proof of [21, Lemma 2.11], there is a \tilde{G} -equivariant homomorphism $T \rightarrow T/B$ with $\dim(T/B) \in \{1, \dots, n - 1\}$. Thus the rank $r(\tilde{G}) \leq \dim T - 2 = n - 2$ (cf. [19, Lemma 2.10]). This contradicts the fact that $r(\tilde{G}) = r(G) = n - 1$. This proves Claim 2.20. \square

We return to the proof of Lemma 2.19.

The action of (the original group) G on X induces an action of G on the Albanese variety $\text{Alb}(X)$ so that alb_X is G -equivariant, by the universal property of the albanese variety. If alb_X is not an isomorphism, then its exceptional locus (where the map is not isomorphic) is G -periodic and positive-dimensional by Zariski's main theorem, contradicting Claim 2.20. Thus alb_X is an isomorphism, and hence X is an abelian variety. Since $r(G) = n - 1$, [19, Lemma 2.14] implies that the Zariski-closure $\overline{N_0}$ of N_0 in the translation group $\text{Aut}_0(X) \cong X$ acts (as translations) on

the torus X with a Zariski-dense open orbit and is hence equal to $\text{Aut}_0(X)$. So the second case of assertion (3) occurs.

For Theorem 1.2(4), since $F \triangleleft \tilde{G}$, the fixed locus

$$T^F = \{t \in T \mid f(t) = t \text{ for some } \text{id} \neq f \in F\}$$

is \tilde{G} -stable. If this locus is positive-dimensional, then its image in X is G -periodic, contradicting Claim 2.20. If X is not yet an abelian variety, then by the proved assertion (3), $N(\tilde{G}) = \gamma^{-1}(N(G))$ is finite. Applying Lemma 2.6 to

$$(\tilde{G}/F)/(N(\tilde{G})/F) \cong G/N(G) \cong \mathbb{Z}^{\oplus n-1}$$

there is a rank $n - 1$, free abelian, finite-index subgroup \tilde{G}_1/F of $\tilde{G}/F = G$. Applying Lemma 2.6 to \tilde{G}_1/F , our $\gamma : \tilde{G} \rightarrow G$ maps a rank $n - 1$, free abelian, finite-index subgroup G_1 of \tilde{G} isomorphically onto a subgroup of $\tilde{G}_1/F \leq G$. This proves Theorem 1.2(4), Lemma 2.19 and the entire Theorem 1.2, provided that $c_2(X) = 0$ in $N^2(X)$.

Now Theorem 1.2 follows from the following:

Lemma 2.21. $c_2(X) = 0$ in $N^2(X)$.

We prove the lemma. Suppose to the contrary that $c_2(X) \neq 0$ in $N^2(X)$. We shall show that this contradicts the minimality assumption on the pair (X, G) . There is an extremal birational contraction $\tau = \tau_H : X \rightarrow X_H$ corresponding to the rational polyhedral face

$$F_H := \{\alpha \in \overline{\text{NE}}(X) \mid A_H \cdot \alpha = 0\}$$

of the closed cone of effective curves $\overline{\text{NE}}(X)$ (cf. [2, Proof of Theorem 3.9.1]), so that $\tau(C)$ is a point for a curve C on X if and only if the class $[C] \in F_H$, and $A_H = \tau^*A'_H$ (resp. $L_{g_i} = \tau^*L'_{g_i}$) for an ample \mathbb{R} -divisor A'_H (resp. a nef \mathbb{R} -divisor L'_{g_i}). By [22, Proof of Claim 2.11], X_H has only canonical singularities and $K_{X_H} \sim 0$, because the same assertions are true for X . Since A_H is the sum of nef H -eigenvectors, F_H is H -stable. So the extremal contraction τ is H -equivariant.

Claim 2.22. X_H has no H -periodic subvariety D of dimension s in $\{1, \dots, n - 1\}$. Hence every H -periodic subvariety of X (especially the exceptional locus of τ_H) is contracted to a point(s) by τ_H .

Proof. We prove the claim. Since $\tau : X \rightarrow X_H$ is H -equivariant and $L_{g_i} = \tau^*L'_{g_i}$, our L'_{g_i} and L_{g_i} give rise to the same character χ_i on H . Now the proof of Lemma 2.18 shows that $(A'_H)^s \cdot D = 0$. Since A'_H is ample, this contradicts Nakai's ampleness criterion (generalized to \mathbb{R} -divisors by Campana and Peternell). This proves Claim 2.22. □

We return back to the proof of Lemma 2.21.

Take $u \in N(G)$ and set $H_u := u^{-1}Hu$. Then $G = N(G)H_u$ and it satisfies the four assertions of Theorem 2.2 (with H replaced by H_u). Set $g'_i := u^{-1}g_iu$ so that $H_u = \langle g'_1, \dots, g'_{n-1} \rangle$. Since $(g'_i)^*(u^*L_{g_i}) = u^*g_i^*(u^{-1})^*(u^*L_{g_i}) = d_1(g_i)u^*L_{g_i}$, we may take $L_{g'_i} = u^*L_{g_i}$ and similarly $L_{g'_1} = u^*L_{g_1}$ (cf. Lemma 2.13), which are all nef common H_u -eigenvectors. Set

$$A_{H_u} = L_{g'_1} + \dots + L_{g'_{n-1}} + L_{g'_1}^{-1}, \quad u^*F_H := \{u^*(\alpha) \mid \alpha \in F_H\}.$$

Then $A_{H_u} = u^*A_H$ and

$$u^*F_H = F_{H_u} := \{\alpha \in \overline{\text{NE}}(X) \mid A_{H_u}.\alpha = 0\}.$$

Since each u^*F_H is spanned by finitely many extremal rays in the cone $\overline{\text{NE}}(X)$, there are finitely many $u_i \in N(G)$ ($i = 1, \dots, t$) such that

$$F_G := \bigcap_{g \in G} g^*F_H = \bigcap_{u \in N(G)} u^*F_H = \bigcap_{i=1}^t u_i^*F_H$$

$$= \{\alpha \in \overline{\text{NE}}(X) \mid A_{H_{u_i}}.\alpha = 0, 1 \leq i \leq t\} = \{\alpha \in \overline{\text{NE}}(X) \mid \sum_{i=1}^t A_{H_{u_i}}.\alpha = 0\},$$

where the second equality is true because $G = N(G)H$ and A_H is the sum of nef H -eigenvectors.

Claim 2.23. $F_G = 0$.

Proof. We prove the claim. Suppose to the contrary that $F_G \neq 0$. Note that F_G is G -stable. As argued before Claim 2.22, there is a non-isomorphic G -equivariant birational extremal contraction $\tau_G : X \rightarrow X_G$ so that $\tau_G(C)$ is a point for a curve C on X if and only if the class $[C] \in F_G$. Also,

$$A_G := \sum_{i=1}^t A_{H_{u_i}} = \tau_G^*A'_G, \quad L_{g'_i} = \tau_G^*L'_{g'_i}$$

for some ample divisor A'_G and nef divisor $L'_{g'_i}$ on X_G ($L_{g'_i}$, w.r.t. every given u , having been defined preceding this claim) and X_G has at worst canonical singularities. The same proof of Claim 2.22 (applied to all H_u) and the ampleness of A'_G show that X_G has no positive-dimensional G -periodic subvariety and hence $\dim \text{Sing } X_G \leq 0$. This contradicts the minimality of the pair (X, G) . Therefore, $F_G = 0$ and Claim 2.23 is true. \square

We return to the proof of Lemma 2.21. Let $\sigma : Z \rightarrow X$ be some Hironaka's blowup with centre in the exceptional locus of $\tau = \tau_H : X \rightarrow X_H =: Y$, such that $-E$ is $\tau\sigma$ -relatively ample for some effective $\tau\sigma$ -exceptional divisor E on Z . Since $A = \tau^*A'$ for some ample divisor A' on Y (with $A := A_H$ and $A' := A'_H$), $\sigma^*A - E$ is an ample divisor on Z if A is replaced by a positive multiple (cf. [10, Proposition 1.45]). By Claim 2.22 and the construction of the blowup σ , we have $\dim \tau\sigma(E) \leq 0$, so $\sigma^*A.E = (\tau\sigma)^*A'.E = 0$. The non-vanishing of $c_2(X)$ in $N^2(X)$ and Lemma 2.16 imply the existence of a non-zero effective 1-cycle $C := c_2(X).M_1 \dots M_{n-3}$. Since ampleness is an open condition, replacing M_i by a small positive multiple we may write

$$\sigma^*A - E = \sigma^*M_i + P_i$$

for some ample \mathbb{R} -divisor P_i . Since the difference $\sigma_*c_2(Z) - c_2(X)$ (as $(n - 2)$ -cycles) lies in the centre of the blowup of $\sigma : Z \rightarrow X$ (i.e., in the exceptional locus of $\tau : X \rightarrow Y$ by the construction of σ) and hence is contracted by τ to a finite subset of Y , and since $A = \tau^*A'$, we have $\sigma_*c_2(Z).A = c_2(X).A$ as $(n - 3)$ -cycles on X .

Consider the 1-cycle $\ell(i_1, \dots, i_s) := \sigma_*(c_2(Z).P_{i_1} \dots P_{i_s})A^{n-s-3}$ on X . Since $\sigma : Z \rightarrow X$ is isomorphic outside the exceptional locus of $\tau : X \rightarrow Y$ by the construction and the intersection of $c_2(X)$ with $\text{Nef}(X) \times \dots \times \text{Nef}(X)$ ($n - 3$ of them)

lies in $\overline{NE}(X)$ (Miyaoka’s pseudo-effectivity), our $\ell(i_1, \dots, i_s) = \ell_1(i_1, \dots, i_s) + \ell_2(i_1, \dots, i_s)$, with ℓ_1 effective and ℓ_2 supported on the exceptional locus of τ , and hence $\ell_2(i_1, \dots, i_s).A = \ell_2(i_1, \dots, i_s).\tau^*A' = 0$ (cf. Claim 2.22).

Now Miyaoka’s pseudo-effectivity of c_2 for minimal variety, the construction of the pseudo-effective sequence in Lemma 2.16, $\sigma_*c_2(Z).A = c_2(X).A$, the projection formula, $\sigma^*A.E = 0$ and Lemma 2.18 imply:

$$\begin{aligned} 0 &\leq c_2(X).M_1 \dots M_{n-3}.A = \sigma_*c_2(Z).M_1 \dots M_{n-3}.A \\ &= c_2(Z).\sigma^*M_1 \dots \sigma^*M_{n-3}.\sigma^*A \\ &= c_2(Z).(\sigma^*A - P_1 - E) \dots (\sigma^*A - P_{n-3} - E).\sigma^*A \\ &= c_2(Z).(\sigma^*A - P_1) \dots (\sigma^*A - P_{n-3}).\sigma^*A \\ &= c_2(Z).(\sigma^*A)^{n-2} - \sum_{i_1, \dots, i_s} (c_2(Z).P_{i_1} \dots P_{i_s})(\sigma^*A)^{n-s-2} \\ &= \sigma_*c_2(Z).A^{n-2} - \sum_{i_1, \dots, i_s} \sigma_*(c_2(Z).P_{i_1} \dots P_{i_s})A^{n-s-2} \\ &= c_2(X).A^{n-2} - \sum_{i_1, \dots, i_s} (\ell_1(i_1, \dots, i_s) + \ell_2(i_1, \dots, i_s))A \\ &= c_2(X).A^{n-2} - \sum_{i_1, \dots, i_s} \ell_1(i_1, \dots, i_s).A \leq c_2(X).A^{n-2} = 0. \end{aligned}$$

Thus the effective 1-cycle $C = c_2(X).M_1 \dots M_{n-3}$ satisfies $C.A_H = 0$ (with $A = A_H$). Since C is a G -eigenvector (cf. Lemma 2.16), we then have $C.u_i^*A_H = 0$ (with $u_i^*A_H = A_{H_{u_i}}$) for all i . Hence $C \in F_G = 0$. This contradicts $C \neq 0$ (cf. Lemma 2.16). We have completed the proof of Lemma 2.21 and also Theorem 1.2.

2.24. Proof of Theorem 2.1. We will use the argument prior to Section 2.14. We may and will freely replace G by its finite-index subgroups. We may assume that $G = N(G)H$ with $H = \langle g_1, \dots, g_{n-1} \rangle$ so that $H|H^{1,1}(X) \cong \mathbb{Z}^{\oplus n-1}$ as in Theorem 2.2 and satisfies the four assertions there. We use the notation in the proof of Theorem 2.2 and let

$$A_H := L_{g_1} + \dots + L_{g_{n-1}} + L_{g_1^{-1}}$$

be the nef and big class, where $L_{g_j^\pm} \in \overline{K}(X)$ can be chosen to be common eigenvectors of H since $H|H^{1,1}(X)$ is commutative.

As in Lemma 2.18, utilizing [13, Lemma 2.2] (writing the nef and big class A_H as the sum of a Kähler class and a positive real current), K_X equals zero (cohomologically). Thus there is an étale finite Galois covering $\tilde{X} = T \times S \times Y \rightarrow X$ such that T is a complex torus, S is a product of hyperkähler manifolds S_i , Y is a product of (projective) Calabi-Yau manifolds Y_j , and G , replaced by its finite-index subgroup, lifts to some group

$$\tilde{G} \leq \text{Aut}(T) \times \prod \text{Aut}(S_i) \times \prod \text{Aut}(Y_j)$$

with $\tilde{G}/\text{Gal}(\tilde{X}/X) = G$ (cf. [1, §3]). As in Theorem 1.2, $N(\tilde{G})$ is the preimage of $N(G)$, via the quotient map $\tilde{G} \rightarrow G$, so that $\tilde{G}/N(\tilde{G}) = \mathbb{Z}^{\oplus n-1}$, and hence the rank $r(\tilde{G}) = n - 1$. Since the projections of \tilde{X} to its factors T , S_i and Y_j are \tilde{G} -equivariant, the maximality of $r(\tilde{G})$ implies that \tilde{X} equals T , $S = S_1$ or $Y = Y_1$,

i.e., is a complex torus, a hyperkähler manifold, or a projective Calabi-Yau manifold (cf. [19, Lemma 2.10]).

Since X is non-algebraic, so is \tilde{X} , and hence \tilde{X} equals T or S . If \tilde{X} is hyperkähler, then we reach a contradiction: $2 \leq n-1 = r(\tilde{G}) \leq 1$ (cf. [9, Theorem 4.6]). Thus \tilde{X} is a complex torus. Now the argument of Lemma 2.19 (easier now) implies Theorem 2.1.

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REFERENCES

- [1] Arnaud Beauville, *Some remarks on Kähler manifolds with $c_1 = 0$* , Classification of algebraic and analytic manifolds (Katata, 1982), Progr. Math., vol. 39, Birkhäuser Boston, Boston, MA, 1983, pp. 1–26. MR728605 (86c:32031)
- [2] Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James McKernan, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc. **23** (2010), no. 2, 405–468, DOI 10.1090/S0894-0347-09-00649-3. MR2601039 (2011f:14023)
- [3] Garrett Birkhoff, *Linear transformations with invariant cones*, Amer. Math. Monthly **74** (1967), 274–276. MR0214605 (35 #5454)
- [4] F. Campana, F. Wang and D.-Q. Zhang, *Automorphism groups of positive entropy on projective threefolds*, Trans. Amer. Math. Soc. **366** (2014), no. 3, 1621–1638. MR3145744
- [5] S. Cantat and A. Zeghib, *Holomorphic actions of higher rank lattices in dimension three*, Preprint 2009.
- [6] Tien-Cuong Dinh and Nessim Sibony, *Groupes commutatifs d’automorphismes d’une variété kählérienne compacte* (French, with English and French summaries), Duke Math. J. **123** (2004), no. 2, 311–328, DOI 10.1215/S0012-7094-04-12323-1. MR2066940 (2005g:32020)
- [7] Akira Fujiki, *On automorphism groups of compact Kähler manifolds*, Invent. Math. **44** (1978), no. 3, 225–258. MR0481142 (58 #1285)
- [8] Yujiro Kawamata, *Minimal models and the Kodaira dimension of algebraic fiber spaces*, J. Reine Angew. Math. **363** (1985), 1–46, DOI 10.1515/crll.1985.363.1. MR814013 (87a:14013)
- [9] JongHae Keum, Keiji Oguiso, and De-Qi Zhang, *Conjecture of Tits type for complex varieties and theorem of Lie-Kolchin type for a cone*, Math. Res. Lett. **16** (2009), no. 1, 133–148. MR2480567 (2010e:14039)
- [10] János Kollár and Shigefumi Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti. Translated from the 1998 Japanese original. MR1658959 (2000b:14018)
- [11] David I. Lieberman, *Compactness of the Chow scheme: applications to automorphisms and deformations of Kähler manifolds*, Fonctions de plusieurs variables complexes, III (Sém. François Norguet, 1975–1977), Lecture Notes in Math., vol. 670, Springer, Berlin, 1978, pp. 140–186. MR521918 (80h:32056)
- [12] Noboru Nakayama and De-Qi Zhang, *Building blocks of étale endomorphisms of complex projective manifolds*, Proc. Lond. Math. Soc. (3) **99** (2009), no. 3, 725–756, DOI 10.1112/plms/pdp015. MR2551469 (2011c:14040)
- [13] Noboru Nakayama and De-Qi Zhang, *Polarized endomorphisms of complex normal varieties*, Math. Ann. **346** (2010), no. 4, 991–1018, DOI 10.1007/s00208-009-0420-y. MR2587100 (2011c:14103)
- [14] Keiji Oguiso, *Automorphisms of hyperkähler manifolds in the view of topological entropy*, Algebraic geometry, Contemp. Math., vol. 422, Amer. Math. Soc., Providence, RI, 2007, pp. 173–185, DOI 10.1090/conm/422/08060. MR2296437 (2008a:14055)
- [15] Daniel Segal, *Polycyclic groups*, Cambridge Tracts in Mathematics, vol. 82, Cambridge University Press, Cambridge, 1983. MR713786 (85h:20003)

- [16] N. I. Shepherd-Barron and P. M. H. Wilson, *Singular threefolds with numerically trivial first and second Chern classes*, J. Algebraic Geom. **3** (1994), no. 2, 265–281. MR1257323 (95h:14033)
- [17] Joseph A. Wolf, *Growth of finitely generated solvable groups and curvature of Riemannian manifolds*, J. Differential Geometry **2** (1968), 421–446. MR0248688 (40 #1939)
- [18] De-Qi Zhang, *Automorphism groups and anti-pluricanonical curves*, Math. Res. Lett. **15** (2008), no. 1, 163–183. MR2367182 (2009f:14072)
- [19] De-Qi Zhang, *A theorem of Tits type for compact Kähler manifolds*, Invent. Math. **176** (2009), no. 3, 449–459, DOI 10.1007/s00222-008-0166-2. MR2501294 (2010d:32013)
- [20] De-Qi Zhang, *Dynamics of automorphisms on projective complex manifolds*, J. Differential Geom. **82** (2009), no. 3, 691–722. MR2534992 (2010k:14078)
- [21] De-Qi Zhang, *The g -periodic subvarieties for an automorphism g of positive entropy on a compact Kähler manifold*, Adv. Math. **223** (2010), no. 2, 405–415, DOI 10.1016/j.aim.2009.08.010. MR2565534 (2011a:32036)
- [22] De-Qi Zhang, *Automorphism groups of positive entropy on minimal projective varieties*, Adv. Math. **225** (2010), no. 5, 2332–2340, DOI 10.1016/j.aim.2010.04.022. MR2680167 (2011m:14070)
- [23] De-Qi Zhang, *Algebraic varieties with automorphism groups of maximal rank*, Math. Ann. **355** (2013), no. 1, 131–146, DOI 10.1007/s00208-012-0783-3. MR3004578

DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF SINGAPORE, 10 LOWER KENT RIDGE ROAD, SINGAPORE 119076

E-mail address: matzdzq@nus.edu.sg