

COMPARING 2-HANDLE ADDITIONS TO A GENUS 2 BOUNDARY COMPONENT

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ABSTRACT. We prove that knots obtained by attaching a band to a split link satisfy the cabling conjecture. We also give new proofs that unknotting number one knots are prime and that genus is superadditive under a band sum. Additionally, we prove a collection of results comparing two 2-handle additions to a genus 2 boundary component of a compact, orientable 3-manifold. These results give a near complete solution to a conjecture of Scharlemann and provide evidence for a conjecture of Scharlemann and Wu. The proofs make use of a new theorem concerning the effects of attaching a 2-handle to a suture in the boundary of a sutured manifold.

1. INTRODUCTION

Sutured manifold theory, since its creation, has been used for comparing the results of two Dehn surgeries on a knot in a 3-manifold. Probably the best known result along these lines is Gabai’s theorem [G2, Corollary 2.4] where at most one Dehn filling on a torus boundary component of a compact, orientable, atoroidal 3-manifold can decrease Thurston norm. In this paper we use a new theorem from [T3] concerning 2-handle addition to a sutured manifold to compare the results of two 2-handle additions to a genus 2 boundary component of a compact, orientable 3-manifold. We begin by stating our results, none of the statements of which involve sutured manifolds. In Section 2 we review the definition of “sutured manifold” and state the theorem (Theorem 2.1) from [T3] that is fundamental to the present work.

Throughout the paper we adopt the following conventions and notation. An **essential surface** in a 3-manifold is one that is incompressible, not a 2-sphere bounding a ball, and not boundary parallel. The regular neighborhood of a space X is denoted $\eta(X)$ and the interior of X is denoted $\overset{\circ}{X}$. N is a compact, orientable 3-manifold with a boundary component F containing an essential simple closed curve b . Let $N[b]$ denote the result of attaching a 2-handle to N along a regular neighborhood of b in F and let $\beta \subset N[b]$ denote the properly embedded arc that is the cocore of the 2-handle attached to b . Let $\partial_1 N = \partial_1 N[b]$ be the union of the components of $\partial N - F$ having genus at least 2. Let $\partial_0 N[b] = \partial N[b] - (\partial N - F)$. Note that $\partial_0 N[b]$ consists of one or two tori arising from the 2-handle attached to F .

1.1. Non-simple 2-handle additions. A 3-manifold is **simple** if it contains no essential, sphere, disc, annulus or torus.

If ∂N is compressible but $\partial N - b$ is incompressible, Jaco [J] showed that $\partial N[b]$ is incompressible. Eudave-Muñoz [EM2] gave upper bounds on the minimal number

Received by the editors November 3, 2011 and, in revised form, October 12, 2012.
2010 *Mathematics Subject Classification*. Primary 57M50.

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of times that an essential annulus or torus intersects β . Scharlemann [S2] showed that for any non-zero homology class $y \in H_2(N[b], \partial N[b])$ there is a Thurston-norm minimizing surface representing y in $N[b]$ disjoint from β . Since adding a 2-handle to ∂N along the boundary of an essential disc creates a reducible 3-manifold, each of these results can be viewed as a comparison of a reducible 2-handle addition and either a non-simple 2-handle addition or a Thurston-norm reducing 2-handle addition. Of course, a priori there may be reducible 2-handle additions that do not arise from attaching a 2-handle along the boundary of the disc. Also note that the aforementioned results concern 2-handle attachment to a non-simple 3-manifold.

Some work has been done comparing non-simple 2-handle attachments to the boundary of a simple 3-manifold. The curve along which such a 2-handle is attached is called a **degenerating** curve. Here is what was known, prior to this paper, concerning 2-handle attachments along degenerating curves a and b . None of these results use sutured manifold theory. We let $|a \cap b|$ denote the minimal intersection number between a and b .

Theorem. *Suppose that N is a simple 3-manifold and that $F \subset \partial N$ is a component of genus at least 2 containing essential simple closed curves a and b . Then*

- *If $N[b]$ is reducible and $N[a]$ is boundary-reducible, then either $|a \cap b| = 0$ or a and b can be isotoped to lie in a common once-punctured torus in ∂N [SW, Theorem 4.2].*
- *If $N[a]$ and $N[b]$ are both reducible, then $|a \cap b| \leq 4$ [ZQL].*
- *If $\text{genus}(F) = 2$, if a and b are both separating, and if $N[a]$ and $N[b]$ are both boundary-reducible, then $|a \cap b| = 0$ [LQZ].*

A degenerating curve $a \subset \partial N$ is called **basic** if a is separating or if there is no degenerating separating curve a^* bounding in ∂N a once-punctured torus containing a . Scharlemann and Wu conjecture:

Conjecture ([SW, Conjecture 2]). *If N is simple and if a and b are basic degenerating curves on ∂N , then $|a \cap b| \leq 5$.*

The next theorem gives some additional evidence for their conjecture. Its proof is logically independent from the prior results on 2-handle addition.

Corollary 5.4. *Suppose that N is a compact orientable simple 3-manifold and that $F \subset \partial N$ is a genus 2 boundary component. Let $a, b \subset F$ be essential simple closed curves with $|a \cap b| \geq 1$. Assume that b is separating and that $N[b]$ is reducible. Then $N[a]$ is irreducible and boundary-irreducible, and if a is a degenerating curve, then a is non-separating and $|a \cap b| = 2$. Furthermore, if $N[a]$ contains an essential annulus, then it contains one with boundary disjoint from $b \cap \partial N[a]$.*

1.2. Refilling meridians and boring. If N can be embedded in a 3-manifold M so that the genus 2 component $F \subset \partial N$ bounds a handlebody W in $M - N$ and if the curves a and b bound discs A and B in W , we say that the 3-manifolds $N[a]$ and $N[b]$ are obtained by **refilling** the meridians A and B respectively. We denote the core or cores of the solid torus or tori obtained from boundary reducing W using A by L_a and similarly define the knot or link L_b . If A and B cannot be isotoped to be disjoint, we say that L_a and L_b are related by **boring** and that L_a is obtained by boring L_b (and vice versa). The arcs α and β that are the cocores of $\eta(A)$ and $\eta(B)$ in $N[a]$ and $N[b]$ are called **boring arcs**.

The paper [T1] proves several theorems about knots and links obtained by boring a split link. In most ways, the results of this paper supercede that paper; however, the methods of that paper are of interest in their own right and are reasonably effective at studying homology classes $y \in H_2(N[b], \partial N[b])$ such that the projection of ∂y to $H_1(\partial_0 N[b])$ is zero. The techniques of this paper are not very useful for drawing conclusions about such classes when $b \subset F$ is separating.

Scharlemann [S5] conjectured that if M , N , and W satisfy certain rather mild hypotheses, then at least one of $N[a]$ or $N[b]$ is irreducible or boundary-irreducible. In [T1], considerable progress was made on a minor variation of the conjecture. The only significant case remaining was to prove that if M , N , and W satisfy certain mild conditions, then it is impossible for both $N[a]$ and $N[b]$ to be solid tori. The next result is a corollary of the more general Proposition 6.1:

Corollary 6.2. *Assume that W is a genus 2 handlebody embedded in a compact, orientable 3-manifold M such that M contains no lens space connected summands, any pair of curves in ∂M that compress in M are isotopic in M , and $N = M - \dot{W}$ is irreducible and boundary-irreducible. If a and b are essential closed curves on ∂W bounding discs in W that cannot be isotoped to be disjoint, then one of the following occurs:*

- (1) *One of $N[a]$ and $N[b]$ is irreducible and boundary-irreducible.*
- (2) *There exists an essential annulus $A \subset N$ such that one component of ∂A lies on a component of ∂M and the other lies on ∂W , is disjoint from a or b and bounds a disc in W .*

Thus, the only case of Scharlemann’s conjecture that remains unanswered is when there exists an essential annulus in N with one boundary component on ∂M and the other on either a or b .

1.3. Rational tangle replacement. A special type of boring is called rational tangle replacement. Consider a 2-tangle (D, τ) embedded in S^3 and consider rational tangles (D', r_a) and (D', r_b) such that $D' = S^3 - \dot{D}$ and $L_a = r_a \cup \tau$ and $L_b = r_b \cup \tau$ are each knots or 2-component links. We say that L_a and L_b are obtained from each other by **rational tangle replacement**. The genus 2 handlebody $W = D' \cup \eta(\tau)$ contains discs A and B such that reducing W along A creates a regular neighborhood of L_a and reducing W along B creates a regular neighborhood of L_b . (The discs A and B are the discs in D' separating the strands of r_a and r_b .) Assume that A and B have been properly isotoped in $D' - \partial\tau$ to minimize $d = |A \cap B|$. The number d is called the **distance** of the rational tangle replacement. It coincides with the usual notion of distance between the rational tangles (D', r_a) and (D', r_b) . Letting $a = \partial A$ and $b = \partial B$, we also have $d = |a \cap b|/2$. As usual, let α and β be the cocores of the 2-handles with cores A and B respectively.

Performing a crossing change on a knot or 2-component link is a rational tangle replacement of distance 2, and attaching a band to a knot or 2-component link is a rational tangle replacement of distance 1. If a crossing change on a non-trivial knot K converts it into the unknot, then K has **unknotting number one**, and if K is a knot formed by attaching a band to the components of a split link, then K is the **band sum** of the components of the split link.

For somewhat technical reasons, rational tangle replacement is particularly amenable to study by the techniques of this paper. We give three examples. The

first is a new proof of an old theorem [S1] of Scharlemann. A different sutured manifold theory proof of this result was given in [ST].

Theorem 7.2. *Unknotting number one knots are prime.*

The second example is a new proof of a result of Gabai and Scharlemann (proved in both cases using sutured manifold theory):

Theorem 7.3. *If a knot K is the band sum of knots K_1 and K_2 , then the genus of K is at least the sum of the genera of K_1 and K_2 .*

The advantage of our proofs is that the sutured manifold theory developed in [T3] and here has a number of other applications and can give more-or-less unified proofs of these results. The previous proofs were all idiosyncratic, particularly in the combinatorics, although each rested on showing that one of the arcs defining the rational tangle replacement could be made disjoint from a minimal genus Seifert surface. The technology of this paper allows a fairly streamlined approach to proving such a result.

Our final application to rational tangle replacement shows that band sums satisfy the cabling conjecture of González-Acuña and Short [GAS]. In fact, no surgery on a band sum produces a reducible 3-manifold. In addition to relying on the work of [T3], its proof also relies on Scharlemann's work in [S4].

Corollary 8.3. *If a knot K is formed by attaching a band to both components of a split link, then K is not a cable knot and no Dehn surgery on K produces a reducible 3-manifold.*

The proof is given in Section 8. This result is significant because Hayashi [H] has shown that knots in S^3 with an essential tangle decomposition into tangles without parallel strands satisfy the cabling conjecture. A knot that is a band sum, but not a connected sum, has an essential tangle decomposition into tangles that have parallel strands. Thus, this result gives additional evidence for the cabling conjecture for knots with essential meridional planar surfaces in their exteriors. Furthermore, a result stronger than Corollary 8.3 is actually proved: We show that the cabling conjecture is satisfied by any knot obtained by attaching a “complicated” band to a (not necessarily split) 2-component link.

2. SUTURED MANIFOLDS

Throughout we use Scharlemann's [S3] combinatorial version of Gabai's sutured manifold theory [G1, G2, G3]. A sutured manifold (N, γ) consists of a compact, orientable 3-manifold N and a collection of oriented simple closed curves (called **sutures**) $\gamma \subset \partial N$, such that:

- $\partial N - \mathring{\eta}(\gamma)$ consists of two (possibly disconnected) surfaces $R_- = R_-(\gamma)$ and $R_+ = R_+(\gamma)$ such that each component of γ is adjacent to both R_- and R_+ . Let $R(\gamma) = R_-(\gamma) \cup R_+(\gamma)$.
- $R_-(\gamma)$ has an inward normal orientation and $R_+(\gamma)$ has an outward normal orientation.
- Each component of $\partial R(\gamma)$ has orientation induced from that of $R(\gamma)$ that coincides with the orientation of the adjacent component of γ .

The **Thurston norm** of a connected surface S is

$$x(S) = \max(-\chi(S), 0).$$

The Thurston norm of a disconnected surface is the sum of the Thurston norms of its components. An oriented surface S is **taut** if S is incompressible and if out of all surfaces with the same boundary as S and homologous to S in $H_2(M, \partial S)$, the Thurston norm of S is minimal.

A sutured manifold (N, γ) is **taut** if N is irreducible and if R_- and R_+ are taut.

If $b \subset \partial N$ is a simple closed curve and if $Q \subset N$ is a properly embedded surface, then a **b -boundary compressing disc** is a disc D in N with interior disjoint from Q such that ∂D is the endpoint union of an arc embedded in b and an arc embedded in Q . Also, if $\partial_0 N[b]$ consists of two tori, we say that a non-zero homology class $y \in H_2(N[b], \partial N[b])$ is **Seifert-like** for $\partial_0 N[b]$ if the projection of y to the first homology of each component of $\partial_0 N[b]$ is non-zero. By the ‘‘half-lives/half-dies’’ theorem from algebraic topology, if $\partial_0 N[b]$ consists of two tori, there exists a Seifert-like homology class.

This paper gives a number of varied applications of the following theorem¹ that was proven in [T3].

Theorem 2.1. *Suppose that (N, γ) is a taut sutured manifold with a genus 2 boundary component F . Assume that $\gamma \cap F$ consists of either one separating or three non-separating curves and that b is a component of $\gamma \cap F$. Let $Q \subset N$ be a surface having no component a sphere or disc disjoint from γ . Assume that ∂Q intersects γ minimally up to isotopy and that $|\partial Q \cap b| \geq 1$. Then one of the following is true:*

- (1) Q has a compressing or b -boundary compressing disc.
- (2) $(N[b], \beta) = (M'_0, \beta'_0) \# (M'_1, \beta'_1)$, where M'_1 is a lens space and β'_1 is a core of a genus one Heegaard splitting of M'_1 .
- (3) The sutured manifold $(N[b], \gamma - b)$ is taut. The arc β can be properly isotoped to be embedded on a branched surface $B(\mathcal{H})$ associated to a taut sutured manifold hierarchy \mathcal{H} for $N[b]$. There is also a proper isotopy of β in $N[b]$ to an arc disjoint from the first decomposing surface in \mathcal{H} . If $|\gamma \cap F| = 3$, that first decomposing surface can be taken to represent $\pm y$ for any given non-zero $y \in H_2(N[b], \partial N[b])$. If $|\gamma \cap F| = 1$, the first decomposing surface can be taken to represent any Seifert-like homology class for $\partial_0 N[b]$.

(4)

$$-2\chi(Q) + |\partial Q \cap \gamma| \geq 2|\partial Q \cap b|.$$

Remark. The taut sutured manifold hierarchy \mathcal{H} constructed in conclusion (3) is constructed so that the first surface (which represents y) is ‘‘conditioned’’. A compact, orientable surface S in a sutured manifold (N, γ) is **conditioned** if all components of $S \cap \eta(\gamma)$ in each component of $\eta(\gamma)$ are coherently oriented arcs or circles intersecting γ minimally and if no collection of components of $\partial S \cap R(\gamma)$ is trivial in $H_1(R(\gamma), \partial R(\gamma))$. Thus, if $N[b] \subset M$ is formed by refilling a meridian of a genus 2 handlebody W with $\partial W = F$, and if L_b is null-homologous in M , then y can be chosen so that the first surface of \mathcal{H} is a minimal genus Seifert surface for L_b . This is discussed in greater detail in Section 10 of [T3].

To apply Theorem 2.1 effectively, we need to find surfaces Q having no compressing or b -boundary compressing discs in N . That is the topic of the next section.

¹In the statement below we have modified the theorem by specializing to the case when F has genus 2 and $|\gamma| \in \{1, 3\}$.

3. SUITABLY EMBEDDED SURFACES

Let N be a compact, orientable 3-manifold with a genus 2 boundary component F . Suppose that $a, b \subset F$ are essential simple closed curves. In this section, we show how to use a given essential surface \overline{R} in $N[a]$ to produce an incompressible and b -boundary incompressible surface $Q \subset N$.

First, we analyze ways in which an essential surface $\overline{R} \subset N[a]$ intersects N . If a is non-separating, there are multiple ways to obtain a manifold homeomorphic to $N[a]$. Certainly attaching a 2-handle to a is one such way, but there are other ways. If $a^* \subset F$ is a simple closed curve disjoint from a and bounding a once-punctured torus containing a , then attaching 2-handles to both a^* and a creates a manifold with a spherical boundary component. Filling in that sphere with a 3-ball creates a manifold homeomorphic to $N[a]$. We will often think of $N[a]$ as having been obtained in this fashion. If a is separating, define $a^* = \emptyset$.

Let $\overline{Q} \subset N[a]$ be an embedded surface and let $Q = \overline{Q} \cap N$. The surface \overline{Q} is **suitably embedded** if each component of $\partial Q - \partial \overline{Q}$ is a curve on F parallel to a or to some a^* . We denote the number of components of $\partial Q - \partial \overline{Q}$ parallel to a by $q = q(\overline{Q})$ and the number parallel to a^* by $q^* = q^*(\overline{Q})$. Let $\hat{q} = q + q^*$.

The next theorem is a particular case of [T2, Theorem 5.1]; however, for completeness and clarity we sketch the proof.

Theorem 3.1. *Suppose that N is a 3-manifold with a genus 2 boundary component F containing essential simple closed curves a and b that cannot be isotoped to be disjoint. Let $\overline{R} \subset N[a]$ be a suitably embedded, orientable, essential surface and suppose that $R = \overline{R} \cap N$ has a compressing disc or b -boundary compressing disc D and that $\hat{q}(\overline{R}) > 0$. Then there exists an essential surface $\overline{Q} \subset N[a]$ such that the following hold:*

- (1) *If D is a b -boundary compressing disc, Q is obtained from R by a b -boundary compression and possibly capping off inessential boundary components.*
- (2) *If D is a compressing disc and if $N[a]$ is reducible, then \overline{Q} is obtained by a trivial 2-surgery of \overline{R} and the discarding of a sphere component. If D is a compressing disc and if $N[a]$ is irreducible, then \overline{Q} is obtained by a proper isotopy of \overline{R} .*
- (3) $-\chi(\overline{Q}) \leq -\chi(\overline{R})$.
- (4) *The sum of the genera of the components of \overline{Q} is no more than the sum of the genera of the components of \overline{R} .*
- (5) $\hat{q}(\overline{Q}) < \hat{q}(\overline{R})$.
- (6) *If D is a b -boundary compressing disc, unless $\partial D \cap \partial N$ is an arc joining a component of $\partial R - \partial \overline{R}$ to itself, \overline{Q} is properly isotopic to \overline{R} in $N[a]$.*

Proof. If D is a compressing disc for R , then, since \overline{R} is essential, there is a disc $D' \subset \overline{R}$ with $\partial D' = \partial D$. If $N[a]$ is irreducible, $D \cup D'$ bounds a 3-ball in $N[a]$ and isotoping \overline{R} across the 3-ball reduces \hat{q} . If $N[a]$ is reducible, use D to perform a trivial 2-surgery on \overline{Q} ; discard the 2-sphere component of the resulting surface that contains D' .

Let D be a b -boundary compressing disc for R and let $\delta = \partial D \cap \partial N$. We organize the proof into cases based on what components of ∂R are connected by δ .

Case 1. δ joins two, not necessarily distinct, components of $\partial\bar{R}$.

Since $\hat{q}(\bar{R}) > 0$, D is a boundary compressing disc for \bar{R} in $N[a]$. In this case, it is since δ would lie on $\partial_0N[a]$, and \bar{R} is either compressible or is a boundary-parallel annulus. Either possibility contradicts the assumption that \bar{R} is essential in $N[a]$.

Case 2. δ joins a component c of $\partial\bar{R}$ to a component d of ∂R that is parallel to a^* or a .

In this case, there is a proper isotopy of \bar{R} to a surface \bar{Q} that moves \bar{R} across D and moves c across a copy of d in $\partial_0N[a]$. This reduces \hat{q} . Since \bar{Q} is isotopic to \bar{R} , we have the theorem in this case.

Case 3. δ joins distinct components of $\partial R - \partial\bar{R}$.

If δ joins components both parallel to a and lies in an annulus component of $F - \partial R$, then there is an isotopy of α across D through \bar{R} reducing \hat{q} . Reversing the isotopy gives an isotopy of \bar{R} reducing \hat{q} . Similarly, if δ joins two components parallel to a^* , there is an isotopy of \bar{R} reducing \hat{q} . If δ joins two components of ∂R , both parallel to a , but does not lie in an annulus component of $F - \partial R$, then isotoping \bar{Q} across D removes those two components of ∂R and introduces a third parallel to some a^* . If δ joins a component parallel to a^* to a component parallel to a , then there is an isotopy of \bar{R} that removes those two components of ∂R and introduces a third parallel to either a or a^* . Thus, \hat{q} is reduced by an isotopy of \bar{R} in all cases where δ joins distinct components of $\partial R - \partial\bar{R}$.

Case 4. δ joins a component c of $\partial R - \partial\bar{R}$ to itself.

Let $C \subset \eta(\alpha)$ be the disc bounded by c . Boundary compress $\bar{R} - \mathring{C}$ using D . Let \bar{Q} be the resulting surface. Clearly, conclusions (1) - (5) hold. We need only show that \bar{Q} is essential. Any compressing disc for \bar{Q} could be isotoped to be disjoint from $\eta(D)$ and so would be a compressing disc for \bar{R} , contrary to the hypothesis. The surface $\bar{R} - \mathring{C}$ can be obtained from \bar{Q} by tubing \bar{Q} to itself using an arc ζ lying on $\partial N[a]$ with endpoints on $\partial\bar{Q}$. If \bar{Q} is boundary-parallel with region of parallelism P , then if ζ is outside P , \bar{R} would also be boundary parallel, and if ζ is interior to P , \bar{R} would be compressible. Both contradict the hypothesis that \bar{R} is essential in $N[a]$. Thus, \bar{Q} is essential. \square

Suppose that \bar{R} and \bar{Q} are suitably embedded essential surfaces in $N[a]$ and that \bar{Q} is a non-empty surface obtained from \bar{R} by applying Theorem 3.1 to \bar{R} and possibly discarding one or more components. Then we say that $\bar{Q} < \bar{R}$. If for a given \bar{R} either $\hat{q}(\bar{R}) = 0$ or there does not exist an essential surface in $N[a]$ such that $\bar{Q} < \bar{R}$, then we say that \bar{R} is **minimal** with respect to b -boundary compressions. Extend the relation $<$ so that it is transitive.

The next lemma records some easy facts.

Lemma 3.2. *The following are true:*

- (1) *If $\bar{R} \subset N[a]$ is an essential properly embedded surface, then there exists an essential suitably embedded surface $\bar{Q} \subset N[a]$, minimal with respect to b -boundary compressions, such that $\bar{Q} \leq \bar{R}$.*
- (2) *Suppose that $\bar{Q} \subset N[a]$ is minimal with respect to b -boundary compressions but is not b -boundary incompressible. Then $\hat{q}(\bar{Q}) = 0$ and either there is*

a proper isotopy of \overline{Q} in $N[a]$ that reduces $|\partial\overline{Q} \cap b|$ or every b -boundary compressing disc for \overline{Q} has boundary intersecting a .

Proof. To prove (1), simply notice that each application of Theorem 3.1 reduces \widehat{q} and that the theorem cannot be applied if $\widehat{q} = 0$.

To prove (2), suppose that $\overline{Q} \subset N[a]$ is minimal with respect to b -boundary compressions, but has a b -boundary compression. Since we cannot apply Theorem 3.1 we must have $\widehat{q}(\overline{Q}) = 0$. If there were a b -boundary compressing disc disjoint from a , then \overline{Q} would be b -boundary compressible in $N[a]$. A b -boundary compressing disc for \overline{Q} adjacent to $\partial_0 N[a]$ and disjoint from a either gives a method for reducing $|\partial\overline{Q} \cap b|$ or shows that \overline{Q} is inessential. The latter is impossible. \square

It will occasionally be useful to double N along boundary components other than F . We make the following definitions. Let $D_a N$ denote the result of gluing a copy of $N_2 = N[a]$ to $N_1 = N$ by using the identity map on $\partial_1 N_1$. Let $F_1, a_1,$ and b_1 denote the copies of $F, a,$ and b in N_2 . Suppose that $\overline{Q} \subset N[a]$ is a suitably embedded surface. Let \overline{Q}' denote the result of isotoping \overline{Q} in $N[a]$ to be properly embedded. Let \overline{Q}_1 be the copy of \overline{Q} in N_1 and let \overline{Q}_2 be the copy of \overline{Q}' in N_2 . Let $D_a \overline{Q} = \overline{Q}_1 \cup \overline{Q}_2 \subset D_a N$. Notice that $D_a \overline{Q}$ is a suitably embedded surface in $D_a N[a_1]$. Let $D_a Q = D_a \overline{Q} \cap D_a N$.

Lemma 3.3. *Suppose that $N[a]$ is irreducible and boundary-irreducible. Suppose that $\overline{Q} \subset N[a]$ is a suitably embedded surface such that Q has no compressing or b -boundary compressing disc in N . Assume also that there is no boundary compressing disc for Q adjacent to $\partial_1 N$. Then $D_a N$ is irreducible and there is no compressing or b_1 -boundary compressing disc for $D_a Q$ in $D_a N$.*

Proof. Since $N[a]$ and N are irreducible and since $\partial N[a]$ is incompressible, any reducing sphere for $D_a N$ could be isotoped to lie in N or $N[a]$, a contradiction.

Suppose that E is a b -boundary compressing disc for $D_a Q$, chosen so that out of all such discs, $|E \cap \partial_1 N|$ is minimal. Since Q is b -boundary incompressible, $E \cap \partial_1 N$ is non-empty. An innermost disc argument, using the incompressibility of $\partial_1 N$, shows that $E \cap \partial_1 N$ consists of arcs. There is an arc bounding a disc $E' \subset E$ with interior disjoint from $\partial_1 N$ and with boundary disjoint from F . Then E' is a boundary compressing disc for \overline{Q} in $D_a N$. \square

4. ADDING SUTURES TO ∂N

To use Theorem 2.1 to prove statements that don't use the language of sutured manifolds requires methods of placing a sutured manifold structure on 3-manifolds with boundary. We explain how to do that in this section. Let $F \subset \partial N$ be a genus 2 component. Let $b \subset F$ be an essential simple closed curve. We begin by discussing sutures $\widehat{\gamma} \cup b$ on F .

If b is separating, define $\widehat{\gamma} = \emptyset$. If b is non-separating, define $\widehat{\gamma}$ to be any pair of essential disjoint simple closed curves on $\partial_0 N[b]$ that are disjoint from and separate the components of $\partial\eta(b)$. Thus, $F - (\widehat{\gamma} \cup b)$ is either the union of two once-punctured tori or the union of two thrice-punctured spheres.

Lemma 4.1. *Suppose that $\partial N - F$ is empty or consists of tori. Let $\widehat{\gamma}$ and b be simple closed curves on F as above. If N is irreducible and if $F - (\widehat{\gamma} \cup b)$ is incompressible in N , then $(N, \widehat{\gamma} \cup b)$ is a taut sutured manifold.*

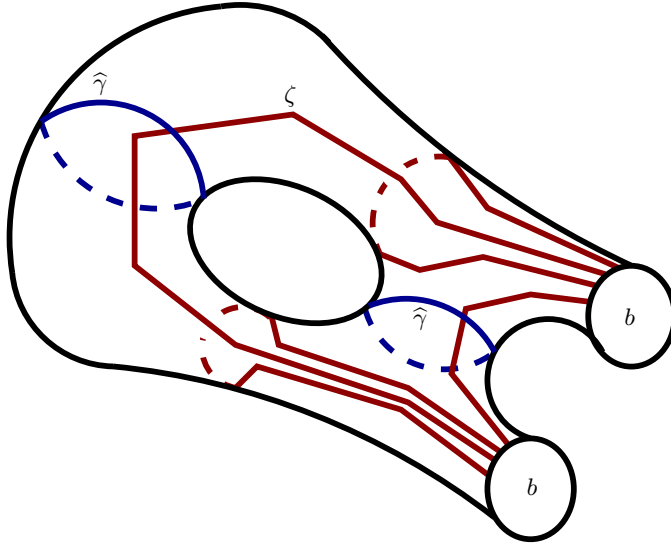


FIGURE 1. An example of sutures $\hat{\gamma} \subset F$ respecting a 1-manifold ζ on the twice-punctured torus $F - \mathring{\eta}(b)$.

Proof. Since N is irreducible, any torus component of ∂N is incompressible. Since $R_{\pm} \cap F$ is either a thrice-punctured sphere or a once-punctured torus, $x(R_{\pm}) = 1$. If S is a norm-minimizing, properly embedded surface in N with $\partial S = \partial R_{\pm}$ and homologous to R_{\pm} , then $x(S) \leq 1$. Without loss of generality, we may assume that S has no 2-sphere or inessential disc components. If $x(S) = 0$, then each component of S is either a disc or an annulus. Since $|\partial R_{\pm}|$ is odd, at least one component of S is a disc, contradicting the assumption that $F - (\hat{\gamma} \cup b)$ is incompressible. Thus, R_{\pm} and N are taut. \square

If b is non-separating, we will often want to be more precise about which curves $\hat{\gamma}$ we choose. Usually our choice will be determined by particular curves in F as follows.

Let $T = \partial_0 N[b] - \partial\beta$; note that T is a twice-punctured torus. Let $\zeta \subset F$ be the union of one or more simple closed curves on F such that no component of $\zeta \cap T$ is an inessential arc or circle. If an arc component of $\zeta \cap T$ has both endpoints at the same endpoint of β , then there are such arcs based at both endpoints of β and the essential circles formed by any two such arc components are isotopic on $\partial_0 N[b]$. If such arcs exist, choose $\hat{\gamma}$ so that each component is isotopic to these essential circles. If no such arc component of $\zeta \cap T$ exists, choose the components of $\hat{\gamma}$ so that each arc component of $\zeta \cap T$ intersects $\hat{\gamma}$ exactly once. (To see that this is possible, consider a homeomorphism of T that takes the arcs of $\zeta \cap T$ to “standard arcs” joining the punctures, choose curves intersecting the standard arcs exactly once each, and let $\hat{\gamma}$ be the inverse image of those curves.) Subject to these constraints, isotope $\hat{\gamma}$ so that it intersects ζ minimally. If $\hat{\gamma} = \emptyset$ or if $\hat{\gamma}$ is chosen as described, we say that $\hat{\gamma}$ **respects** ζ . If $a \subset F$ is an essential simple closed curve intersecting b minimally and not disjoint from b , and if $\overline{Q} \subset N[a]$ is a suitably embedded surface with $Q = \overline{Q} \cap N$, then $\hat{\gamma}$ **respects** \overline{Q} if it respects $\partial Q \cup a$. See Figure 1.

Let $\mathcal{M}_b(\zeta)$ denote the number of arcs of $\zeta \cap T$ that have both endpoints on the same component of $\partial\beta$.

Lemma 4.2. *Suppose that ζ is an embedded 1-manifold on F intersecting b minimally. Then for each component ζ_0 of ζ we have the following:*

- (1) *If b and ζ_0 are given orientations, then $\mathcal{M}_b(\zeta_0)$ is the number of changes of sign of $b \cap \zeta_0$ as ζ_0 is traversed.*
- (2) *$\mathcal{M}_b(\zeta)$ is even.*
- (3) *If $\mathcal{M}_b(\zeta_0) = 0$ and $|b \cap \zeta_0| > 0$, then both ζ_0 and b are non-separating and all points of intersection have the same sign.*
- (4) *If $\hat{\gamma}$ respects ζ , then*

$$\mathcal{M}_b(\zeta_0) = |\zeta_0 \cap b| - |\zeta_0 \cap \hat{\gamma}|.$$

Proof. Let b_1 and b_2 be the two endpoints of β on $\partial_0 N[b]$. Let $T = \partial_0 N[b] - \partial\beta$. If an arc of $\zeta_0 \cap T$ has both endpoints on either b_1 or b_2 , then the corresponding points of intersection of $\zeta_0 \cap b$ have opposite sign and are adjacent on ζ_0 . Conversely, any change of sign on ζ_0 must correspond to an arc of $\zeta_0 \cap T$ with both endpoints at either b_1 or b_2 , since each component of T is a once- or twice-punctured torus. Since ζ_0 is a circle, there must be an even number of changes of sign. Hence, (1). This is true for each component of ζ , so $\mathcal{M}_b(\zeta)$ is even. Hence, (2).

If $\mathcal{M}_b(\zeta_0) = 0$ and $|b \cap \zeta_0| > 0$, then b and ζ_0 intersect and all points of intersection have the same sign by (1). Consequently, both must be non-separating. Hence, (3).

If b is separating, (4) follows immediately, so assume b is non-separating. There are the same number of endpoints of $\zeta_0 \cap T$ at b_1 as at b_2 . Each arc either has both endpoints at one of b_1 or b_2 or has one endpoint at b_1 and the other b_2 . Thus, there are the same number of arcs of $\zeta_0 \cap T$ with both endpoints at b_1 as there are arcs of $\zeta_0 \cap T$ with both endpoints at b_2 . This is true for each component of ζ . If $\mathcal{M}_b(\zeta) > 0$, there exists at least one arc ζ' of $\zeta \cap T$ with both endpoints at b_1 and at least one arc ζ'' of $\zeta \cap T$ with both endpoints at b_2 and all other arcs of $\zeta \cap T$ are disjoint from these arcs. Since ζ intersects b minimally, in $\partial_0 N[b]$, ζ' and ζ'' are disjoint essential loops and are, therefore, parallel. Assume that $\hat{\gamma}$ respects ζ . The curves $\hat{\gamma}$ are parallel to these loops and intersect the other arcs of $\zeta \cap T$ minimally. Since $\partial_0 N[b] - (\zeta' \cup \zeta'')$ is a pair of annuli and since there is one component of $\hat{\gamma}$ in each annulus, we have

$$\mathcal{M}_b(\zeta_0) = |\zeta_0 \cap b| - |\zeta_0 \cap \hat{\gamma}|,$$

if $\mathcal{M}_b(\zeta) > 0$.

If $\mathcal{M}_b(\zeta) = 0$, then by the construction of ζ ,

$$\mathcal{M}_b(\zeta_0) = |\zeta_0 \cap b| - |\zeta_0 \cap \hat{\gamma}| = 0.$$

□

The next lemma shows how to usefully define sutures $\tilde{\gamma}$ on non-torus components of $\partial N - F$. Let $\gamma = \hat{\gamma} \cup \tilde{\gamma} \cup b$.

Lemma 4.3 ([T1, Lemma 4.1]). *Assume² that $\partial N - (\hat{\gamma} \cup b)$ is incompressible in N . Suppose also that N is irreducible and that if $\partial N - F$ contains a non-torus component, then there is no essential annulus in N with boundary on $\hat{\gamma} \cup b$. Then*

²This hypothesis was left out of the statement of the lemma in [T1]. It is used in the proof when [S4, Lemma 5.6] is applied.

$\tilde{\gamma}$ can be chosen so that (N, γ) is taut. Furthermore, if $c \subset \partial N - F$ is a collection of disjoint, non-parallel curves such that:

- $|c| \leq 2$,
- all components of c are on the same component of $\partial N - F$,
- no component of c cobounds an essential annulus in N with a curve of $\hat{\gamma} \cup b$,
- if $|c| = 2$, then there is no essential annulus in N with boundary c ,
- if $|c| = 2$ and b is separating, there is no essential thrice-punctured sphere in N with boundary $c \cup b$,

then $\tilde{\gamma}$ can be chosen to be disjoint from c .

If $\hat{\gamma}$ respects a suitably embedded surface $\bar{Q} \subset N[a]$, then we also say that γ respects \bar{Q} .

The following lemma records a handy bit of accounting that we will use repeatedly.

Lemma 4.4. *Suppose that γ respects a suitably embedded surface $\bar{Q} \subset N[a]$ and that $\mathcal{M}_b(a) > 0$. Then the inequality*

$$-2\chi(Q) + |\partial Q \cap \gamma| \geq 2|\partial Q \cap b|$$

implies the inequality

$$-2\chi(\bar{Q}) + |\partial \bar{Q} \cap \tilde{\gamma}| \geq q(\mathcal{M}_b(a) - 2) + q^*(\mathcal{M}_b(a^*) - 2) + \mathcal{M}_b(\partial \bar{Q}).$$

Proof. This is easily proven by piecing together Lemma 4.2 and the following two facts:

- $\chi(Q) = \chi(\bar{Q}) - \hat{q}(\bar{Q})$.
- For any simple closed curve ζ on ∂N ,

$$|\zeta \cap \gamma| = |\zeta \cap \hat{\gamma}| + |\zeta \cap \tilde{\gamma}| + |\zeta \cap b|.$$

□

5. DEGENERATING HANDLE ADDITIONS

In this section, we assume that N is simple and that attaching a 2-handle to N along b creates some sort of degeneration (usually either an essential disc or a sphere). In many instances we use this to deduce an inequality relating the intersection number of a and b to the euler characteristic of an essential surface in $N[a]$.

Proposition 5.1. *Suppose that N is a compact orientable simple 3-manifold with $F \subset \partial N$ a genus 2 component. Let a, b be two essential simple closed curves on ∂N that cannot be isotoped to be disjoint.*

Let $\bar{Q} \subset N[a]$ be a suitably embedded essential surface and assume that $Q = \bar{Q} \cap N$ is incompressible and b -boundary incompressible. Assume also that $\partial \bar{Q} \cap \partial_1 N$, if non-empty, consists of curves all parallel to a simple closed curve $c_1 \subset \partial_1 N$.

Assume that one of the following holds:

- (a) $N[b]$ is reducible.
- (b) $N[b]$ is a solid torus.
- (c) A simple closed curve c_b disjoint from c_1 and on the same component of $\partial_1 N$ compresses in $N[b]$.

- (d) *There is a non-zero class $y \in H_2(N[b], \partial N[b])$ such that no norm-minimizing conditioned surface representing $\pm y$ is disjoint from β . If b is separating, we also require that y is Seifert-like for $\partial_0 N[b]$.*

Then, one of the following occurs:

- (1) $\overline{Q} \subset N$ and \overline{Q} is disjoint from b .
- (2)

$$-2\chi(\overline{Q}) \geq q(\mathcal{M}_b(a) - 2) + q^*(\mathcal{M}_b(a^*) - 2) + \mathcal{M}_b(\partial\overline{Q}).$$
- (3) *a and b are both non-separating, and all intersection points of a and b have the same sign.*

Proof. Since N is simple, there is no essential disc in N . Hence, if $N[b]$ is a solid torus, an essential disc in $N[b]$ is norm-minimizing and not disjoint from β . Thus, hypothesis (b) implies hypothesis (d). If hypothesis (a), (b), or (d) is satisfied, let $c_2 = \emptyset$. If none of these is satisfied, but the curve c_1 compresses in $N[b]$, let $c_2 = c_1$. If none of (a), (b), or (d) is satisfied and if c_1 does not compress in $N[b]$, let $c_2 = c_b$. Define $c = c_1 \cup c_2$ and notice that c has two components if and only if c_1 does not compress in $N[b]$ and c_1 and c_2 are non-empty, disjoint, non-parallel loops on the same component of $\partial_1 N$.

We claim that c satisfies the hypotheses of Lemma 4.3. Since N contains no essential disc or annulus, we need only check that if $|c| = 2$ and b is separating, then there is no essential thrice-punctured sphere in N with boundary $c \cup b$. Since c_b compresses in $N[b]$, the existence of a thrice-punctured sphere in N with boundary on $c \cup b$ would imply that c_1 compresses in $N[b]$. But, by our choice of c_2 , this contradicts the hypothesis that $|c| = 2$. Thus, the hypotheses of Lemma 4.3 are satisfied. Let $\gamma = \tilde{\gamma} \cup \hat{\gamma} \cup b$ be sutures respecting \overline{Q} and disjoint from c . The sutured manifold (N, γ) is taut.

If $|\partial Q \cap b| = 0$, then $\hat{q}(\overline{Q}) = 0$ and $|\partial\overline{Q} \cap b| = 0$. This implies conclusion (1). Assume that conclusion (1) does not hold. Since γ respects Q , and since N is simple, no component of Q is a sphere or disc disjoint from γ . Thus, the hypotheses of Theorem 2.1 are satisfied. By hypothesis, Q has no compressing or b -boundary compressing disc. Since N is simple, β cannot intersect a reducing sphere in $N[b]$ exactly twice. By hypothesis and the construction of $\tilde{\gamma}$ to be disjoint from c_b , either $(N[b], \gamma - b)$ is not \emptyset -taut or no first surface in a taut sutured manifold hierarchy for $(N[b], \gamma - b)$ representing y is disjoint from β . Thus, by Theorem 2.1,

$$-2\chi(Q) + |Q \cap \gamma| \geq 2|Q \cap b|.$$

By Lemma 4.4, either (3) holds or this implies that

$$-2\chi(\overline{Q}) \geq q(\mathcal{M}_b(a) - 2) + q^*(\mathcal{M}_b(a^*) - 2) + \mathcal{M}_b(\partial\overline{Q}).$$

□

Corollary 5.2. *Suppose that N is a compact, orientable simple 3-manifold with a genus 2 boundary component F . Let $a, b \subset F$ be two essential curves that cannot be isotoped to be disjoint. Assume that $N[a]$ contains an essential sphere or disc \overline{R} .*

Assume also that one of the following holds:

- $N[b]$ is reducible.
- $N[b]$ is a solid torus.
- A simple closed curve c_b disjoint from, and on the same component of $\partial_1 N[b]$ as $\partial\overline{R}$, compresses in $N[b]$.

Then a and b are both non-separating, and all points of intersection between them have the same sign.

Proof. If $\mathcal{M}_b(a) = 0$, then by Lemma 4.2, a and b are both non-separating and all points of intersection between them have the same sign. Since $\mathcal{M}_b(a)$ is even, we may assume, therefore, that $\mathcal{M}_b(a) \geq 2$.

If $N[a]$ is reducible, let \overline{R} be an essential sphere in $N[a]$, and if $N[a]$ is irreducible, let \overline{R} be an essential disc in $N[a]$. Choose $\overline{Q} \leq \overline{R}$ to be minimal with respect to b -boundary compressions. Notice that if \overline{Q} is a disc, then \overline{R} is as well, and if $\partial\overline{Q} \subset \partial_1 N[b]$, then $\partial\overline{Q} = \partial\overline{R}$.

Since N is irreducible and boundary-irreducible, $\widehat{q}(\overline{Q}) \neq 0$. If a^* is non-empty and is disjoint from b , then a and b lie in a common once-punctured torus, in which case a and b are both non-separating and always intersect with the same sign. This contradicts our assumption. Hence, if $q^* \neq 0$, then $\mathcal{M}_b(a^*) \geq 2$ since a^* is separating.

If $q > 0$, by Proposition 5.1, we have

$$-2\chi(\overline{Q}) \geq q(\mathcal{M}_b(a) - 2) \geq 0.$$

But \overline{Q} is a sphere or a disc, so this is a contradiction.

If $q^* > 0$, then by Proposition 5.1, we have

$$-2\chi(\overline{Q}) \geq q^*(\mathcal{M}_b(a^*) - 2) \geq 0.$$

This is also a contradiction. Thus, a and b are non-separating, and all points of intersection between them have the same sign. □

Here is a version of Proposition 5.1 allowing for $\partial\overline{Q}$ to have non-isotopic components on $\partial_1 N$.

Proposition 5.3. *Suppose that N is a compact, orientable, simple 3-manifold and that $F \subset \partial N$ has genus 2. Let $a, b \subset F$ be two essential curves that cannot be isotoped to be disjoint. Suppose that $\overline{Q} \subset N[a]$ is a suitably embedded essential surface and that $Q = \overline{Q} \cap N$ is incompressible, b -boundary-incompressible, and has no boundary compressing disc adjacent to $\partial_1 N$. If $N[b]$ is reducible, then one of the following holds:*

- (1) a and b are non-separating and all points of $a \cap b$ have the same sign.
- (2) $\overline{Q} \subset N$ and $\partial\overline{Q}$ is disjoint from b .
- (3)

$$-4\chi(\overline{Q}) \geq q(\mathcal{M}_b(a) - 2) + q^*(\mathcal{M}_b(a^*) - 2) + \mathcal{M}_b(\partial\overline{Q}).$$

Proof. Assume neither (1) nor (2) holds. By Corollary 5.2, we may assume that $N[a]$ is irreducible and boundary-irreducible. Form the doubled manifold $D_a N$. Notice that $\partial D_a N$ consists of a single genus 2 surface and some tori. By Lemma 3.3, $D_a Q$ has no compressing or b_1 -boundary compressing disc. Choose $\widehat{\gamma} \subset F$ to respect $D_a Q$. By Lemma 4.1 $(D_a N, \widehat{\gamma} \cup b_1)$ is taut. Apply Theorem 2.1 to $D_a N$. One of the following must hold:

- (i) $(D_a N[b_1], \beta)$ has a (lens space, core) connected summand.
- (ii)

$$-2\chi(D_a \overline{Q}) \geq q(\mathcal{M}_b(a) - 2) + q^*(\mathcal{M}_b(a^*) - 2) + \mathcal{M}_b(\partial\overline{Q}).$$

The second conclusion together with the fact that $\chi(D_a\overline{Q}) = 2\chi(\overline{Q})$ gives our result immediately.

Assume, therefore, that there is a reducing sphere P for $D_aN[b]$ intersected twice by β . Out of all such spheres choose P to intersect ∂_1N minimally. Since N is simple, β is disjoint from ∂_1N , and $N[a]$ is boundary-irreducible, an innermost disc argument shows that any circle of $P \cap \partial_1N$ innermost on P bounds a disc intersecting β . Hence, $P \cap \partial_1N$ has exactly two circles innermost on P . Let D be one of them. The disc D is punctured exactly once by β . Thus $D \cap N_1$ is an essential annulus in N , contrary to the hypothesis that N is simple. \square

We can apply this to prove a special case of a conjecture of Scharlemann and Wu (as described in the introduction).

Corollary 5.4. *Suppose that N is a compact orientable simple 3-manifold and that $F \subset \partial N$ is a genus 2 boundary component. Let $a, b \subset F$ be essential simple closed curves that cannot be isotoped to be disjoint. Assume that b is separating and that $N[b]$ is reducible or a solid torus. Then $N[a]$ is irreducible and boundary-irreducible, and if a is a degenerating curve, then a is non-separating and $|a \cap b| = 2$. Furthermore, if $N[a]$ contains an essential annulus, then it contains one with boundary disjoint from $b \cap \partial N[a]$.*

Proof. By Corollary 5.2, $N[a]$ is irreducible and boundary-irreducible.

Let \overline{R} be an annulus or torus in $N[a]$. Let $\overline{Q} \leq \overline{R}$ be minimal. If \overline{Q} has a boundary-compressing disc adjacent to ∂_1N , it must be an annulus. Boundary-compressing it creates an essential disc, contrary to our initial observation. Thus, we may assume that \overline{Q} does not have a boundary compressing disc adjacent to $\partial_1N[b]$.

Since b is separating and since N is simple, by Propositions 5.1 and 5.3 and Lemma 4.2, we have

$$0 = -\epsilon\chi(\overline{Q}) \geq q(|a \cap b| - 2) + q^*(|a^* \cap b| - 2) + |\partial\overline{Q} \cap b| \geq 0,$$

for some $\epsilon \in \{2, 4\}$. This implies that $|\partial\overline{Q} \cap b| = 0$. Since if \overline{R} were an annulus \overline{Q} would be as well, we deduce that if $N[a]$ contains an essential annulus, then it contains one with boundary disjoint from $b \cap \partial N[a]$.

Recall that any two minimally intersecting, non-disjoint simple closed separating curves on a genus 2 surface must intersect at least four times. Since a^* is empty or separating, we must have $q^* = 0$. Since N is simple, we must have $q > 0$. Consequently, $|a \cap b| = 2$. Since b is separating, this implies that a is non-separating. \square

6. REFILLING MERIDIANS

We recall (from the introduction) the notion of “refilling meridians”. Let M be a compact orientable 3-manifold, and let $W \subset M$ be an embedded genus 2 handlebody. Let A and B be essential discs in W that cannot be isotoped to be disjoint. Assume that A and B intersect minimally. Define $a = \partial A$, $b = \partial B$, $F = \partial W$, and $N = M - \mathring{W}$. We say that the 3-manifolds $N[a]$ and $N[b]$ are obtained by **refilling** the meridians A and B respectively of W in M . Notice that an outermost arc of intersection between A and B on A cuts off a subdisc of A that is a meridian of one of the solid tori obtained by boundary-reducing W using B . In particular, this means that $\mathcal{M}_b(a) \geq 2$. Recall that L_a and L_b denote the

core or cores of the solid torus or tori $W - \mathring{\eta}(A)$ and $W - \mathring{\eta}(B)$ respectively. If a is non-separating and if $a^* \subset F$ bounds a once-punctured torus in F containing a , let A^* be a disc in W bounded by a^* .

In [S5], Scharlemann considered refilling meridians of a genus 2 handlebody W in a wide variety of compact, orientable 3-manifolds M . He showed in a number of situations that if both $N[a]$ and $N[b]$ are reducible or boundary-irreducible (with minor assumptions on the embedding of W in M), then either $M = S^3$ and W is unknotted or α and β are “aligned” in W . He conjectured that this is always the case (given his hypotheses on the pair (M, W)). In [T1], under slightly different hypotheses, significant progress was made on the conjecture. One case not covered by [T1] is the case when both $N[a]$ and $N[b]$ are solid tori. Corollary 6.2, a corollary to the next proposition, shows that if $|a \cap b| > 0$, it is impossible for both $N[a]$ and $N[b]$ to be solid tori. In fact, Corollary 6.2 gives a stronger result with weaker hypotheses than [T1, Theorem 6.1].

Proposition 6.1. *Suppose that W is a genus 2 handlebody embedded in a compact, orientable manifold M . Let A and B be essential discs in W which cannot be isotoped to be disjoint. Suppose that \overline{Q} is a suitably embedded essential surface in $N[a]$ such that $Q = \overline{Q} \cap N$ is incompressible and b -boundary-incompressible and not disjoint from b . Furthermore, assume that either $\partial\overline{Q} \cap \partial_1 N[b]$ is empty or that all of its components are parallel to a single simple closed curve c_a .*

Assume the following:

- (H1) *If $c_a \neq \emptyset$ and if a curve $c_b \subset \partial_1 N[b]$ compresses in $N[b]$, then c_a and c_b are on the same component of ∂M .*
- (H2) *If $P \subset M$ is a sphere such that either P is non-separating, or P bounds a lens space summand of M , or P is embedded and essential in $N[b]$ but inessential in M , then $|P \cap (L_b \cup \beta)| \geq 3$.*
- (H3) *If P is an essential disc in M , then $|P \cap (L_a \cup \alpha)| \geq 2$ and $|P \cap (L_b \cup \beta)| \geq 2$.*
- (H4) *$N = M - \mathring{W}$ is irreducible.*

Suppose one of the following:

- (1) *$N[b]$ is reducible.*
- (2) *$N[b]$ is boundary-reducible by a disc with boundary on $\partial_1 N[b]$.*
- (3) *There is a non-zero class $y \in H_2(N[b], \partial N[b])$ such that no conditioned norm-minimizing representative is disjoint from β . If b is separating, also assume that y is Seifert-like for $\partial_0 N[b]$.*

Then either c_a and c_b are both non-empty and they cannot be isotoped to be disjoint or

$$-2\chi(\overline{Q}) \geq q(\mathcal{M}_b(a) - 2) + q^*(\mathcal{M}_b(a^*) - 2) + \mathcal{M}_b(\partial\overline{Q}).$$

The proof is similar to that of Corollary 5.2.

Proof. Choose sutures $\widehat{\gamma}$ on ∂W so that $\widehat{\gamma} \cup b$ respects Q . Since \overline{Q} is not disjoint from b , any outermost disc of $A - B$ or $A^* - B$ is a meridian of $\eta(L_b)$. Thus, if $\widehat{\gamma} \neq \emptyset$, both components are meridians of L_b . If a component of $F - (\widehat{\gamma} \cup b)$ were compressible, it would violate (H2), so $F - \widehat{\gamma} \cup b$ is incompressible in N .

We desire to define sutures on N that both respect \overline{Q} and are disjoint from any boundary-compressing discs for $N[b]$ and $N[a]$, if such exist. To do so we will apply Lemma 4.3, and so we begin by defining the curves c used in that lemma. If $\partial\overline{Q} \cap \partial_1 N[b]$ is empty, let $c_a = \emptyset$. Whether or not c_a is empty, let $c_1 = c_a$. If

c_1 compresses in $N[b]$, or if $N[b]$ is reducible, let $c_2 = c_1$. Otherwise, let $c_2 = c_b$, if $c_b \neq \emptyset$. Thus, $|c| = 2$ if and only if c_1 and c_2 are non-empty and c_2 compresses in $N[b]$ but c_1 does not. Also note that the definition of c is independent of which hypothesis (1), (2), or (3) we are assuming.

Hypothesis (H3) implies that $\partial_1 N$ is incompressible in N . If there were an essential annulus in N with boundary on $\widehat{\gamma} \cup b$, it could be capped off in M to be a non-separating sphere intersecting $L_b \cup \beta$ twice. This contradicts hypothesis (H2).

If a curve of c bounded an essential annulus with a curve of $\widehat{\gamma} \cup b$, it could be capped off in M to be an essential disc intersecting $L_b \cup \beta$ exactly once. This contradicts hypothesis (H3).

If $|c| = 2$ and there were an essential annulus in N with boundary c , then c_1 would compress in $N[b]$, contradicting our construction of c .

If $|c| = 2$ and there were a thrice-punctured sphere with boundary $c \cup b$, then, once again, c_1 would compress in $N[b]$. This contradicts our construction of c .

We can, therefore, apply Lemma 4.3 to obtain sutures $\gamma \subset \partial N$ respecting \overline{Q} and disjoint from c . Notice that either $(N[b], \gamma - b)$ is not taut or that there is no conditioned taut representative of y disjoint from β .

Since \overline{Q} is essential in $N[a]$, since Q is incompressible and b -boundary incompressible, and since (N, γ) is taut, no component of Q is a sphere or disc disjoint from γ .

Thus, by Theorem 2.1 and Lemma 4.4 one of the following occurs:

- (i) $(N[b], \beta)$ has a (lens space, core) summand.
- (ii) $(N[b], \gamma - b)$ is taut and there is a conditioned norm-minimizing surface disjoint from β and representing $\pm y$.
- (iii)

$$-2\chi(\overline{Q}) \geq q(\mathcal{M}_b(a) - 2) + q^*(\mathcal{M}_b(a^*) - 2) + \mathcal{M}_b(\partial\overline{Q}).$$

The first of these is ruled out by hypothesis (H2). If $N[b]$ is reducible or if $c_b \neq \emptyset$, then the second is also ruled out. If $N[b]$ is a solid torus, then an unknotting disc for L_b represents some non-zero class $y \in H_2(N[b], \partial N[b])$. Such a disc is clearly norm-minimizing. We assume that no such norm-minimizing surface disjoint from β exists, and so (ii) cannot hold with any of our hypotheses. Consequently (iii) holds, as desired. □

The following corollary is phrased to make the connection to [S5, Conjecture 2] explicit.

Corollary 6.2. *Assume that W is a genus 2 handlebody embedded in a compact, orientable 3-manifold M such that every S^2 in M separates, M contains no lens space connected summands, any pair of curves in ∂M that compress in M are isotopic in ∂M , and $N = M - \overline{W}$ is irreducible and boundary-irreducible. If a and b are essential closed curves on ∂W bounding discs in W that cannot be isotoped to be disjoint, then one of the following occurs:*

- (1) *One of $N[a]$ and $N[b]$ is irreducible and boundary-irreducible.*
- (2) *There exists an essential annulus $A \subset N$ such that one component of ∂A lies on a component of ∂M and the other lies on ∂W , is disjoint from a or b and bounds a disc in W .*

Proof. Assume that $N[b]$ is reducible or boundary-reducible. If $N[b]$ is boundary-reducible but not reducible, then either $N[b]$ is a solid torus or there is a unique

simple closed curve $c \subset \partial_1 N$ that compresses in M (and in $N[b]$). Letting $c_a = c_b = c$, we see that (H1) holds.

Since M contains no lens space summands and no non-separating 2-spheres and since N is boundary-irreducible, (H2) is satisfied. Any essential disc in M must have boundary parallel to c . Suppose that P is such a disc with either $|P \cap (L_a \cup \alpha)| = 1$ or $|P \cap (L_b \cup \beta)| = 1$. Then (possibly after a small isotopy) the intersection of P with N is an annulus in N with one boundary component on ∂W disjoint from a or b and bounding a disc in W . This is conclusion (2). Hence, we may assume that (H3) is satisfied. (H4) is satisfied by hypothesis.

If $N[a]$ is reducible, let \overline{R} be an essential sphere. If $N[b]$ is irreducible, but boundary-reducible, let \overline{R} be an essential disc. Let $\overline{Q} \leq \overline{R}$ be minimal with respect to b -boundary compressions. Since N is irreducible and boundary-irreducible, $\widehat{q}(\overline{Q}) > 0$. Thus, Q is b -boundary incompressible and not disjoint from b . Since \overline{Q} is either a disc or a sphere, either \overline{Q} is a sphere or $\partial\overline{Q}$ is parallel to c . Thus, by Proposition 6.1,

$$-2 \geq q(\mathcal{M}_b(a) - 2) + q^*(\mathcal{M}_b(a^*) - 2) + \mathcal{M}_b(\partial\overline{Q}).$$

Each term of the right hand side of the inequality is non-negative. We have, therefore, encountered a contradiction. \square

If $M = S^3$ we define a boring arc β to be **complicated** if one of the following holds:

- L_b is a split link and β does not intersect a splitting sphere exactly once.
- L_b is not a split link and β is not disjoint from any minimal genus Seifert surface for L_b .

It turns out that the exterior of a knot or link and complicated boring arc is boundary-irreducible. (See also [J], [S2], and [T3, Theorem 10.1].)

Lemma 6.3. *Suppose that $L_b \subset S^3$ is a knot or 2-component link and that β is a complicated boring arc. Let N be the exterior of $L_b \cup \beta$. Then N is boundary-irreducible.*

Proof. Let $W = \eta(L_b \cup \beta)$ and let $F = \partial W = \partial N$. Suppose that D is a compressing disc for F in N , chosen so that out of all such discs, $|\partial D \cap b|$ is minimal.

Case 1. $\partial D \cap b = \emptyset$.

If ∂D bounds a disc in W , then either ∂D is parallel to b or b is non-separating and ∂D bounds a once-punctured torus in F containing b . In the former case, L_b is a split link and β intersects a splitting sphere exactly once. In the latter case, it is not difficult to see that there is a minimal genus Seifert surface for L_b disjoint from β . Thus, ∂D does not bound a disc in W . This implies that one component of L_b is an unknot and that D is an unknotting disc for that component disjoint from β . If b is non-separating this obviously contradicts the definition of a complicated boring arc. If b is separating, boundary compressing a component of $\partial\eta(L_b)$ using D produces a splitting sphere for L_b intersecting β exactly once. This also contradicts the definition of complicated boring arc. Consequently, this case cannot occur.

Case 2. $\partial D \cap b \neq \emptyset$.

Let $Q = \partial D$. Notice that by minimality of $|D \cap b|$, Q is b -boundary incompressible. It is obviously incompressible. Let $\widehat{\gamma} \subset F$ be sutures disjoint from b and

respecting Q . Let $\gamma = \hat{\gamma} \cup b$. It is argued in the proof of [T3, Theorem 10.1] that $|\partial Q \cap \hat{\gamma}| \leq |\partial Q \cap b|$. Thus, since Q is a disc,

$$-2\chi(Q) + |Q \cap \gamma| \leq 2|Q \cap b|.$$

Thus, by Theorem 2.1, $(N[b], \gamma - b)$ is \emptyset -taut, and a proper isotopy of β makes it disjoint from a minimal genus Seifert surface for L_b . This contradicts the definition of a complicated boring arc, and so this case cannot occur either. \square

The next corollary shows that many knots obtained by boring using a complicated boring arc are simple (i.e. have simple exteriors).

Corollary 6.4. *Suppose that L_a is a non-simple knot or 2-component link in S^3 obtained by boring a knot or 2-component link $L_b \subset S^3$ using a complicated boring arc. Then L_a is not a split link or unknot and one of the following occurs:*

- (1) *The exterior of L_a has an essential torus or annulus disjoint from α .*
- (2) *L_a or L_b is a knot and $\mathcal{M}_b(a) = 2$. If L_b is a 2-component link, this implies $|a \cap b| = 2$.*
- (3) *L_a and L_b are both knots, there exists a 2-component link L_{a^*} containing L_a as a component, L_{a^*} is non-simple, and a^* is an essential separating meridional curve on $\partial\eta(L_a \cup \alpha)$ disjoint from a such that $\mathcal{M}_b(a^*) = 2$.*

Furthermore, if L_a is a knot and L_b is a link, then if there is an essential annulus in the exterior of L_a , there is one with meridional boundary.

Remark. It may be worth remarking that the conclusion that $\mathcal{M}_b(a) = 2$ implies that if A and B are the discs in $\eta(L_a \cup \alpha)$ bounded by a and b , then all arcs of intersection of $A \cap B$ are parallel in A . This follows from the first conclusion of Lemma 4.2. If b is separating, then also by that lemma we have $\mathcal{M}_b(a) = |a \cap b|$.

Proof. Let W be the genus 2 handlebody that is a regular neighborhood of $L_b \cup \beta$. Notice that $N = S^3 - \mathring{W}$ is irreducible. By Lemma 6.3, ∂N is incompressible. By Corollary 6.2, L_a is not a split link or an unknot. Let \overline{R} be an essential annulus or torus in $N[a]$ and let $\overline{Q} \leq \overline{R}$ be minimal with respect to b -boundary compressions. Notice that if \overline{R} was an annulus, then \overline{Q} is as well.

Suppose that L_a is a knot and that \overline{Q} is an annulus. Then, if $\partial\overline{Q} \cap b = \emptyset$, $\partial\overline{Q}$ must be meridional on L_a since it is disjoint from the boundaries of outermost discs of $B - A$ lying in a neighborhood of L_a .

If $Q \cap b = \emptyset$, then $\overline{Q} \subset N$ and $\partial\overline{Q} \cap b = \emptyset$. This implies conclusion (1). Assume, therefore, that $Q \cap b \neq \emptyset$. If Q is compressible or b -boundary compressible, then $Q = \overline{Q}$ and it is b -boundary compressible since \overline{Q} is minimal with respect to b -boundary compressions. Performing the b -boundary compression creates an essential disc in N , contradicting Lemma 6.3. Thus, Q is incompressible and b -boundary incompressible and $Q \cap b \neq \emptyset$.

By Proposition 6.1, we have

$$0 = -2\chi(\overline{Q}) \geq q(\mathcal{M}_b(a) - 2) + q^*(\mathcal{M}_b(a^*) - 2) + \mathcal{M}_b(\partial\overline{Q}).$$

Each term of the right hand side is non-negative. Hence, each term is 0. If b is separating, this implies that

$$\mathcal{M}_b(\partial\overline{Q}) = |b \cap \partial\overline{Q}| = 0,$$

and so \overline{Q} is a torus or meridional annulus.

If $q > 0$, then $\mathcal{M}_b(a) = 2$ and we are done, since this is conclusion (2).

If $q = 0$ we must have $q^* > 0$, and so $\mathcal{M}_b(a^*) = 2$. If b were separating, we would have $|b \cap a^*| = 2$. Two essential separating curves on a genus 2 surface that intersect twice are actually parallel; hence $|a \cap b| = 0$, a contradiction. Thus, L_b is a knot. Since all components of $\partial Q - \partial \overline{Q}$ are parallel to a^* , the 2-component link L_{a^*} is not simple. This is conclusion (3). \square

Knots and links obtained by boring split links and unknots using complicated boring arcs are even more special; the dual boring is not complicated.

Theorem 6.5. *Suppose that $L_b \subset S^3$ is a knot or 2-component link obtained from a split link or unknot L_a using a complicated boring arc α . Then L_b is not a split link or unknot and L_b has a minimal genus Seifert surface disjoint from the dual boring arc β .*

Proof. By Corollary 6.2, L_b is not a split link or an unknot. Let N be the exterior of $L_a \cup \alpha$. By Lemma 6.3, N is boundary-irreducible. Let \overline{R} be a splitting sphere or unknotting disc for L_a and let $\overline{Q} \leq \overline{R}$ be minimal with respect to b -boundary compressions and assume \overline{Q} is connected. Since \overline{Q} is a sphere or disc,

$$-2\chi(\overline{Q}) < q(\mathcal{M}_b(a) - 2) + q^*(\mathcal{M}_b(a^*) - 2) + \mathcal{M}_b(\partial \overline{Q}).$$

Thus, by Proposition 6.1, for all classes $y \in H_2(N[b], \partial N[b])$, with the property that if b is separating, then the projection of ∂y to the first homology of each component of $\partial N[b]$ is non-zero, there is a conditioned norm-minimizing representative of $\pm y$ disjoint from β . Choosing y to be a class representable by Seifert surfaces of L_b , we see that L_b has a minimal genus Seifert surface disjoint from β . (Note that reversing the orientation of a Seifert surface of minimal genus in its homology class still gives a Seifert surface of minimal genus in its homology class, so the distinction between $\pm y$ does not matter if, as we do, consider unoriented knots and links.) \square

Remark. The proof of the theorem (stemming from the application of Theorem 2.1) actually gives the stronger result that β can be isotoped to lie on the branched surface associated to a certain taut sutured manifold hierarchy for the exterior of L_b .

7. RATIONAL TANGLE REPLACEMENT

Rational tangle replacement is a particularly tractable boring operation. If L_a and L_b are related by rational tangle replacement, then $\mathcal{M}_b(a) = |a \cap b| = \mathcal{M}_a(b)$. Furthermore, all the arcs of $b - a$ join a component of $\partial \eta(a)$ to itself and, together with an arc in $\eta(\partial \alpha) \subset \partial_0 N[a]$, form a meridian of L_a . Thus, if $\overline{Q} \leq \overline{R}$ and if \overline{R} has empty or meridional boundary, then \overline{Q} does as well. In fact, a closer examination of Theorem 3.1 gives the following result whose proof we leave as an exercise.

Lemma 7.1. *Suppose that \overline{R} is an essential surface in $N[a]$ with \overline{R} isotoped to intersect b minimally. Suppose that $\overline{Q} \leq \overline{R}$. Then one of the following is true:*

- (1) $\overline{Q} = \overline{R}$ and Q is incompressible and b -boundary incompressible.
- (2) $Q = \overline{Q} = \overline{R}$ and α is isotopic into Q .
- (3) $\overline{Q} < \overline{R}$ and \overline{Q} has meridional boundary on some component of L_a .

We can now give the promised new proofs that unknotting number one knots are prime and that genus is superadditive under band connected sum. The proof of the first of these relies on Corollary 6.4, and the proof of the second follows almost

immediately from Theorem 6.5. Both of those are consequences of Proposition 6.1, so we do have a unified sutured manifold theory framework for answering questions about rational tangle replacement.

Theorem 7.2 ([S1]). *Unknotting number one knots are prime.*

Proof. Suppose that L_a is a composite unknotting number one knot. Let L_b be the unknot obtained by changing a crossing of L_a .

Case 1. α is disjoint from an essential meridional annulus for L_a .

In this case, the crossing change occurs on a single summand of L_a . Such a crossing change can never unknot L_a .

Case 2. L_b has an unknotting disc with interior disjoint from β .

Let D be the 3-ball complementary to the site where the rational tangle replacement occurs so that $(D, L_a \cap B) = (D, L_b \cap B)$ is a 2-tangle. Isotope the unknotting disc so that it intersects ∂D minimally. It is not difficult to see that this implies that the strands of $L_a \cap D$ are parallel. It is then easy to construct a genus 1 Seifert surface for L_a . Since L_a is non-trivial, this is a minimal genus Seifert surface. Since L_a is composite, it must have genus at least 2. Thus, every unknotting disc for L_b has its interior intersected by β .

Case 3. The interior of every unknotting disc for L_b is intersected by β .

In this case β is a complicated boring arc. By Case 1, we may assume that α is not disjoint from any essential meridional annulus. By Corollary 6.4, $\mathcal{M}_b(a) = 2$. Since b is separating, we have $|a \cap b| = 2$. This contradicts the fact that $d = |a \cap b|/2 = 2$. □

Theorem 7.3 ([S3, Theorem 8.4] and [G4]). *If K is the band sum of knots K_1 and K_2 using a band β , then the genus of K is at least the sum of the genera of K_1 and K_2 . Equality holds if and only if the interior of β is disjoint from minimal genus Seifert surfaces for K_1 and K_2 .*

Proof. If the band sum is actually a connect sum, the result follows from the additivity of genus under a connect sum. Otherwise, the band β is complicated. Since (by the definition of “band sum”) $K_1 \cup K_2$ is a split link, by Theorem 6.5, K has a minimal genus Seifert surface disjoint from the cocore of the band. Easy cut and paste arguments, as in [S3, Theorem 8.4], finish the proof of the theorem. □

8. BAND SUMS SATISFY THE CABLING CONJECTURE

If $K \subset S^3$ is a knot such that the manifold $M_K(m/n)$ obtained by m/n Dehn surgery on K has an essential surface \widehat{Q} not disjoint from the core of the surgery solid torus, then K can be isotoped so that the surface $\widehat{Q} - \mathring{\eta}(K)$ is an essential surface with boundary having slope m/n on K . We use this observation to prove that band sums satisfy the cabling conjecture. The heart of the matter is contained in the following theorem:

Theorem 8.1. *Suppose that a knot L_a is obtained from a 2-component link L_b by attaching a band with complicated core. Then L_a is not a cable knot and no non-trivial surgery on L_a produces a reducible 3-manifold.*

We define $\alpha, \beta, W,$ and N as usual. Let L be the result of pushing b slightly into N . Before embarking on the proof we need a preliminary result.

Lemma 8.2. *Suppose that $\overline{Q} \subset N[a]$ is a properly embedded incompressible surface and that $Q' \subset N$ is obtained by maximally b -boundary compressing Q . If Q' is disjoint from b , then \overline{Q} can be isotoped to be disjoint from L .*

Proof. Each b -boundary compression can be achieved by an isotopy of \overline{Q} that makes it “dip into” W without making it intersect $L_a \cup \alpha$. Thus, Q' is obtained from Q in $S^3 - (L_a \cup \alpha)$ by an isotopy of \overline{Q} . Hence, if $Q' \cap b = \emptyset$, then \overline{Q} can be isotoped to be disjoint from L . □

Proof of Theorem 8.1. Suppose that a surgery on L_a of slope m/n with $(m, n) = 1$ produces a reducible manifold. By [GL], we know that $n = 1$. Then there is an essential connected planar surface \overline{R} in the exterior of L_a with boundary of slope $m/1$ on L_a . Isotope \overline{R} to intersect b minimally.

By Lemma 7.1, one of the following holds, for any surface $\overline{Q} \leq \overline{R}$:

- (1) $\overline{Q} = \overline{R}$ and $Q = \overline{Q} \cap N$ is incompressible and b -incompressible.
- (2) $Q = \overline{Q} = \overline{R}$ and there is a b -boundary compressing disc for \overline{Q} giving rise to an isotopy of α into \overline{Q} .
- (3) $\overline{Q} < \overline{R}$ and \overline{Q} has meridional boundary on some component of L_a .

In fact, possibility (3) cannot occur. Suppose that D is a b -boundary compressing disc for $R = \overline{R} \cap N$ with $\delta = \partial D \cap \partial N$ and that $R \neq \overline{R}$. Each component of $b - a$ is an arc joining a to itself with both endpoints on the same side of a . Furthermore, in $\partial N[a]$ each arc of $b - a$ is a meridian of L_a . Such an arc must, therefore, intersect $\partial \overline{R}$, which is integer-sloped. As in the proof of Theorem 3.1, since \overline{R} intersects b minimally, the arc δ must join $\partial \overline{R}$ to itself. As in Theorem 3.1, this implies that \overline{R} is boundary-compressible, a contradiction.

We now consider a special case; we then also rule out possibility (1).

Case 1. \overline{Q} is an annulus and α is isotopic into \overline{Q} .

Let D be the b -boundary compression giving rise to the isotopy of α into \overline{Q} . Performing the b -boundary compression creates an essential disc in N , contradicting the fact that β is complicated. (Lemma 6.3). □(Case 1).

Let $\gamma = b$ and recall that (N, γ) is a taut sutured manifold and that γ respects Q . If Q is incompressible and b -boundary incompressible, Proposition 6.1 implies that

$$-2\chi(\overline{Q}) \geq 2|\partial\overline{Q}|$$

since each component of $\partial\overline{Q}$ intersects b twice and $\hat{\gamma} = \emptyset$. But \overline{Q} is a planar surface and so we would have

$$-2 + |\partial\overline{Q}| \geq |\partial\overline{Q}|,$$

an obvious contradiction. Hence, Q is not incompressible and b -boundary incompressible. In fact, it must be incompressible but b -boundary compressible. Consequently, $Q = \overline{Q}$ and α is isotopic into \overline{Q} . Thus, in light of Case 1, we will be done if we can prove that it is possible to take \overline{Q} to be an annulus.

Let $Q_0 = \overline{Q}$. Each component of $\partial\overline{Q}$ intersects b exactly twice. Since \overline{Q} is incompressible in $N[a]$, any b -boundary compressing disc must join a component of $\partial\overline{Q}$ to itself (and cross $a \subset \partial N$). Assume that we have defined Q_i and let Q_{i+1} be obtained from Q_i by a b -boundary compression. We obtain a sequence

Q_0, Q_1, \dots, Q_p so that Q_p is b -boundary incompressible. (Possibly Q_p is disjoint from b .) Each b -boundary compression must join a component of $\partial\overline{Q}$ to itself and must cross a , since otherwise we would have a b -boundary compression for \overline{Q} in $N[a]$. Thus, each b -boundary compression removes a component of $\partial\overline{Q}$ and replaces it with two components disjoint from b . In particular, the boundary of each Q_i intersects b minimally up to isotopy.

Case 2. Q_p is not disjoint from b .

Since Q_p is incompressible and b -boundary incompressible in N and intersects b minimally, we may apply Theorem 2.1 to deduce that

$$-2\chi(Q_p) \geq |\partial Q_p \cap b|.$$

Since Q_p was obtained from $\overline{Q} = Q$ by b -boundary compressing p times, we have

$$-2\chi(Q_p) = -2(\chi(\overline{Q}) + p) = -2\chi(\overline{Q}) - 2p.$$

Each boundary compression converted a component of $\partial\overline{Q}$ that intersected b twice into two components of ∂Q_p that are disjoint from b . Thus, $|\partial Q_p \cap b| = 2(|\partial\overline{Q}| - p)$. Consequently,

$$-2\chi(\overline{Q}) - 2p \geq 2|\partial\overline{Q}| - 2p.$$

Hence,

$$-\chi(\overline{Q}) \geq |\partial\overline{Q}|,$$

which, as before, is a contradiction to the fact that \overline{Q} is a planar surface. Hence, this case cannot occur.

Case 3. Q_p is disjoint from b .

By Lemma 8.2, $\overline{Q} = Q$ can be isotoped (relative to its boundary) to be disjoint from L . Since L is the unknot in S^3 , \overline{Q} lies in a solid torus V . Performing m/n surgery on L_a in V produces a reducible manifold $V(m/n)$. Since β is a complicated boring arc, L_a cannot lie in a 3-ball in V . (I.e. the band sum is not a connected sum.) By [S4, Corollary 4.4], L_a is cabled and m/n is the slope of the cabling annulus. Hence, it is possible to take \overline{Q} to be an annulus, and we are done by Case 1. □

We conclude with a simple corollary that also draws on work of Scharlemann.

Corollary 8.3. *Suppose that K is the band sum of two knots K_1 and K_2 . Then K satisfies the cabling conjecture and is not a cable knot.*

Proof. If the core of the band is not complicated, then it intersects a splitting sphere for $K_1 \cup K_2$ exactly once. If it intersects a splitting sphere exactly once, the band sum is a connected sum. By Theorem 8.1, we may assume that the band sum is a connect sum. By [S4, Corollary 4.5], K satisfies the cabling conjecture. It is well known that cable knots are prime, and so we are done. □

ACKNOWLEDGEMENTS

Versions of the results of Sections 5, 6, and 7 appear in [T1]. The author is grateful to Marty Scharlemann for helpful conversations. Portions of the work in this paper were supported by a grant from the National Science Foundation.

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