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FIBERS OF CHARACTERS IN GELFAND-TSETLIN CATEGORIES

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Abstract. For a class of noncommutative rings, called Galois orders, we study the problem of an extension of characters from a commutative subalgebra. We show that for Galois orders this problem is always solvable in the sense that all characters can be extended, moreover, in finitely many ways, up to isomorphism. These results can be viewed as a noncommutative analogue of liftings of prime ideals in the case of integral extensions of commutative rings. The proposed approach can be applied to the representation theory of many infinite dimensional algebras including universal enveloping algebras of reductive Lie algebras (in particular gl_n), Yangians and finite W-algebras. As an example we recover the theory of Gelfand-Tsetlin modules for gl_n.

Contents

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1. Introduction

The functors of restriction to subalgebras and induction from subalgebras are important tools in representation theory. The effectiveness of these tools depends on the choice of a subalgebra. For a commutative algebra A denote by $\operatorname{Specm} A$ ($\operatorname{Spec} A$) the space of maximal (prime) ideals in A, endowed with the Zarisky topology. In the classical commutative algebra setup, an integral extension $A \subset B$ of two commutative rings (e.g. $A = B^G$, where G is a finite subgroup of the automorphism group of B) induces a map $\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$, whose fibers are nonempty for every point of $\operatorname{Spec} A$. In particular, every character of A can be extended to a character of its integral extension B. Moreover, if B is finite over A, then all fibers $\varphi^{-1}(I)$, $I \in \operatorname{Spec} A$, are finite, and hence the number of extensions of a character of A is finite. The Hilbert-Noether theorem provides an example of such a situation with B being the symmetric algebra on a finite dimensional vector space V and $A = B^G$, where G is a finite subgroup of $GL(V)$.

The primary goal of this paper is to generalize these results to "semicommutative" pairs $\Gamma \subset U$, where U is an associative (noncommutative) algebra over a base field $\Bbbk$ and Γ is an integral domain. The canonical embedding $\Gamma \subset U$ induces a functor from the category of U-modules which are direct sums of finite dimensional Γ-modules (*Gelfand-Tsetlin modules*[1] *with respect to* Γ) to the category of torsion Γ-modules. This functor induces a "multivalued function" from $\operatorname{Specm}\Gamma$ associating to an ideal $\mathsf{m} \in \operatorname{Specm}\Gamma$ the *fiber* $\Phi(\mathsf{m})$ of left maximal ideals of U that contain m. Our goal is to find natural sufficient conditions for the fibers to be nonempty and finite for any point in $\operatorname{Specm}\Gamma$. On the other hand, for a maximal left ideal $I \subset U$ such that U/I is a Gelfand-Tsetlin module, it is interesting to investigate its support in $\operatorname{Specm}\Gamma$ (i.e. the set of $\mathsf{m} \in \operatorname{Specm}\Gamma$, such that Γ/m is a subquotient of U/I as a Γ-module) and find the multiplicity of Γ/m in U/I.

A motivation for the study of such pairs (U, Γ) comes from representation theory. The classical framework of *Harish-Chandra modules* ([D], Ch. 9) is related to a pair of a reductive Lie algebra $\mathcal{F}$ and its reductive subalgebra $\mathcal{F}'$, where U and Γ are their universal enveloping algebras respectively. A more general concept of Harish-Chandra modules (related to a pair (U, Γ)) was introduced in [DFO2].

The case when U is the universal enveloping algebra of a reductive finite dimensional Lie algebra and Γ is the universal enveloping algebra of a Cartan subalgebra leads to the theory of Harish-Chandra modules with respect to this Cartan subalgebra, commonly known as generalized weight modules. Classification of such simple modules is well known for gl_2 and for any simple finite dimensional Lie algebra for modules with finite dimensional weight spaces, due to Fernando [Fe] and Mathieu [Ma]. It remains an open problem in general. To approach this classification problem, the full subcategory of weight Gelfand-Tsetlin $U(\mathrm{gl}_n)$-modules with respect to the *Gelfand-Tsetlin subalgebra* was introduced in [DFO1]. This class is based on

[1] In the literature sometimes the spelling is "Zetlin" instead of "Tsetlin".

natural properties of a Gelfand-Tsetlin basis for finite dimensional representations of simple classical Lie algebras [GTs], [Zh], [M], [MazO], [Maz1], [Maz2], [MazC].

Gelfand-Tsetlin subalgebras were considered in [FM] in connection with the solutions of the Euler equation, in [Vi] in connection with subalgebras of maximal Gelfand-Kirillov dimension in the universal enveloping algebra of a simple Lie algebra, in [KW1], [KW2] in connection with classical mechanics, and also in [Gr1], [Gr2] in connection with general hypergeometric functions on the Lie group $GL(n,\mathbb{C})$. A similar approach was used by Okunkov and Vershik in their study of the representations of the symmetric group S_n [OV], with U being the group algebra of S_n and Γ being the maximal commutative subalgebra generated by the Jucys-Murphy elements

$$(1i)+\ldots+(i-1i)\,, \quad i=1,\ldots,n.$$

In this case the elements of $\operatorname{Specm}\Gamma$ parametrize basis elements in all irreducible representations of U. Recent advances in the representation theory of Yangians ([FMO]) and finite W-algebras ([FMO1]) are also based on similar techniques.

What is the intrinsic reason for the existence of Gelfand-Tsetlin formulae and for the successful study of Gelfand-Tsetlin representations of various classes of algebras? This question led us to the introduction in [FO1] of the concepts of *Galois rings* and *Galois orders* in invariant skew monoidal rings.

For the rest of the paper we assume that Γ *is a commutative domain,* K *the field of fractions of* Γ*,* $K\subset L$ *a finite Galois extension, and* $\mathcal{M}\subset \operatorname{Aut} L$ *a submonoid closed under conjugation by the elements of the Galois group* $G=G(L/K)$*. We will always assume that for any* $\varphi_1,\varphi_2\in\mathcal{M}$ *the equality*

$$\varphi_1|_K=\varphi_2|_K$$

implies $\varphi_1=\varphi_2$ *(we call such* $\mathcal{M}$ separating *with respect to* K*).*

The group G acts on the skew monoidal ring $L*\mathcal{M}$ via the action $(l\varphi)^g=l^g\varphi^g$, where $\varphi^g=g^{-1}\varphi g$. We denote by $\mathcal{K}$ the subring $(L*\mathcal{M})^G$ of G-invariants in $L*\mathcal{M}$.

Definition 1.1. A *Galois ring* U over Γ is a finitely generated over Γ subring in $\mathcal{K}$ such that $KU=UK=\mathcal{K}$. A Galois ring U over Γ is called a *right (respectively left) order* if for any finite dimensional right (respectively left) K-subspace $W\subset\mathcal{K}$ (respectively $W\subset\mathcal{K}$), $W\cap U$ is a finitely generated right (respectively left) Γ-module. A Galois ring is an *order* if it is both a right and a left order ([FO1]).

Galois orders are natural versions of "noncommutative orders" in skew monoidal rings of invariants. In comparison with the classical notion of an order we note that $\Gamma\subset U$ is not central.

The class of Galois orders includes, in particular, the following subrings in the corresponding skew group rings ([FO1]): Generalized Weyl algebras over integral domains with infinite order automorphisms, e.g. the n-th Weyl algebra A_n, the quantum plane, the q-deformed Heisenberg algebra, quantized Weyl algebras, the Witten-Woronowicz algebra ([Ba], [BO]), the universal enveloping algebra of gl_n over the Gelfand-Tsetlin subalgebra ([DFO1], [DFO2]), associated shifted Yangians and finite W-algebras ([FMO], [FMO1]), and certain rings of invariant differential operators on a torus. In Section 2 we present some necessary facts about Galois orders.

In this paper we develop a representation theory of Galois orders. The main tool for our investigation of categories of representation is a technique from [DFO2]. In

Section 3 we give a detailed exposition of the results from [DFO2] adapted for the case of a commutative subalgebra considered in this paper.

The last two sections are devoted to the representation theory of Galois orders. We emphasize that the theory of Galois orders unifies the representation theories of universal enveloping algebras and generalized Weyl algebras. Our main result establishes sufficient conditions for the fiber $\Phi(\mathsf{m})$ to be nontrivial and finite. Let ℓ_{m} be any lifting of m to the integral closure of Γ in L, and $\mathcal{M}_{\mathsf{m}}$ be the stabilizer of ℓ_{m} in $\mathcal{M}$. Note that the group $\mathcal{M}_{\mathsf{m}}$ is defined uniquely up to G-conjugation, hence its cardinality is well defined.

Our main result is the following

Main Theorem. *Let Γ be a commutative domain which is finitely generated as a $\Bbbk$-algebra, U a right Galois order over Γ, and $\mathsf{m} \in \operatorname{Specm}\Gamma$. Suppose that $\mathcal{M}_{\mathsf{m}}$ is finite.*

- *The fiber $\Phi(\mathsf{m})$ is nonempty.*
- *If U is a Galois order over Γ, then the fiber $\Phi(\mathsf{m})$ is finite.*

For any $\mathsf{m} \in \operatorname{Specm}\Gamma$ with finite $\mathcal{M}_{\mathsf{m}}$ we obtain an effective estimate for the number of isomorphism classes of simple Gelfand-Tsetlin modules M whose support contains m and for the dimension of generalized weight spaces $M(\mathsf{m})$ (Theorem 4.12). In particular, for $U = U(\mathrm{gl}_n)$ these numbers are bounded by $1!\,2!\ldots(n-1)!$.

We note here an important connection which arose in the case when $U = U(\mathrm{gl}_n)$ and $\Gamma \subset U$ is the Gelfand-Tsetlin subalgebra. In this case an important role is played by the variety of the so-called strongly nilpotent matrices ([Ov2]). It was shown in [Ov2] that this variety is a complete intersection. In particular, this implies that U is free both as a right and as a left Γ-module ([FO2]). Kostant and Wallach ([KW1], [KW2]) introduced a generalization of the variety of strongly nilpotent matrices and revealed a deep relation between this variety and the hamiltonian mechanics. A connection between Gelfand-Tsetlin representations of U and the structure of the Kostant-Wallach variety is evidently important and should be a topic of further study.

In this paper we apply the theory only to Lie algebras of type A, but we believe it can be extended to other types. This technique was used in [FMO1] to address the classification problem of irreducible Gelfand-Tsetlin modules for finite *W-algebras* and *shifted Yangians* associated with gl_n and to prove an analogue of the Gelfand-Kirillov conjecture for these algebras.

2. Preliminaries

All fields in the paper contain the base field $\Bbbk$, which is algebraically closed of characteristic 0. All rings in the paper are $\Bbbk$-algebras. If A is an associative ring, then we let $A-\mathrm{mod}$ denote the category of finitely generated left A-modules. Let $\mathcal{C}$ be a $\Bbbk$-category, i.e. all $\mathrm{Hom}_{\mathcal{C}}$-sets are endowed with the structure of a $\Bbbk$-vector space and all the composition maps are $\Bbbk$-bilinear. The category $\mathcal{C}-\mathrm{Mod}$ of *$\mathcal{C}$-modules* is defined as the category of $\Bbbk$-linear functors $M : \mathcal{C} \longrightarrow \Bbbk - \mathrm{Mod}$, where $\Bbbk - \mathrm{Mod}$ is the category of $\Bbbk$-vector spaces. Submodules and quotients are defined naturally. A $\mathcal{C}$-module M is called *locally finitely generated* if there exists a family of finite dimensional subspaces $\{M'(i) \mid i \in \operatorname{Ob}\mathcal{C}\}$, such that the submodule $N \subset M$ generated by all $M'(i)$, i.e. the minimal submodule with the property

$M'(i) \subset N(i)$, coincides with M. The module M is called *finitely generated* if the $M'(i)$'s above can be chosen such that only finitely many of them are nonzero.

We denote by $\mathcal{C}-\mathrm{mod}$ the category of locally finitely generated $\mathcal{C}$-modules. If G is a group and X a G-set, then by X/G we denote the corresponding set of orbits and by X^G the set of G-invariants. For a set X by $|X|$ we denote the cardinality of X.

2.1. **Integral extensions.** Let A be an integral commutative domain, K its field of fractions and $\tilde{A}$ the integral closure of A in K. The ring A is called *normal* if $A = \tilde{A}$. The following is standard (e.g. [AM]).

Proposition 2.1. *Let A be a normal noetherian ring, $K \subset L$ a finite Galois extension and let $\bar{A}$ be the integral closure of A in L. Then $\bar{A}$ is a finitely generated A-module.*

Let $\imath : A \hookrightarrow B$ be an integral extension of $\Bbbk$-algebras. Then it induces a surjective map $\operatorname{Specm} B \to \operatorname{Specm} A$ ($\operatorname{Spec} B \longrightarrow \operatorname{Spec} A$). In particular, for any character $\chi : A \to \Bbbk$ there exists a character $\tilde{\chi} : B \to \Bbbk$ such that $\tilde{\chi}|_A = \chi$. If, in addition, B is finitely generated as an A-module, then the number of different characters of B which correspond to the same character of A is finite.

Corollary 2.2 ([S], Ch. III, Prop. 11, Prop. 16). *If A is a finitely generated $\Bbbk$-algebra, then for any character $\chi : A \to \Bbbk$ there exist finitely many characters $\tilde{\chi} : \bar{A} \to \Bbbk$ such that $\tilde{\chi}|_A = \chi$.*

The following statement is probably well known, but in [FO1] we included the proof for the convenience of the reader.

Proposition 2.3 ([FO1]). *Let $i : A \hookrightarrow B$ be an embedding of integral domains with a regular A. Assume that the induced morphism of varieties $i^* : \operatorname{Specm} B \to \operatorname{Specm} A$ is surjective (e.g. $A \subset B$ is an integral extension). If $b \in B$ and $ab \in A$ for some nonzero $a \in A$, then $b \in A$.*

We consider Proposition 2.3 as a motivation for introducing the notion of a Galois order.

2.2. **Skew monoidal rings.** Below we present the necessary facts from [FO1]. Let $\mathcal{M}$ be a monoid acting on a set S. The result of the left action of $\varphi \in \mathcal{M}$ on $s \in S$ will be denoted as $\varphi \cdot s$ (or s^{φ} when convenient), so we have

$$\varphi_2 \cdot (\varphi_1 \cdot s) = (\varphi_2 \varphi_1) \cdot s,\ (s^{\varphi_1})^{\varphi_2} = s^{\varphi_1 \varphi_2},\ \varphi_1, \varphi_2 \in \mathcal{M}.$$

By $\varphi \cdot S'$ or S'^{φ} we denote the induced actions of $\mathcal{M}$ on the subset $S' \subset S$ and by $S^{\mathcal{M}} \subset S$ the set of all $\mathcal{M}$ invariants.

We use the following notation from [FO1]. Let H be a group acting on a set X, X/H the set of orbits, and let $F(x)$ be an expression depending on $x \in X$ such that $F(x)$ is constant on the orbit. Then the notation $\sum\limits_{x \in X/H} F(x)$ means that the sum is taken over some set of representatives of the orbits and the sum does not depend on this choice. In particular, we use this notation in the case where H is a subgroup of a finite group G and the sum $\sum\limits_{x \in G/H} F(x)$ is taken over the set of left cosets (e.g., see (2)). Similarly for $\bigoplus\limits_{x \in X/H}$.

Let R be a $\Bbbk$-algebra, $\operatorname{Aut} R$ the group of all $\Bbbk$-algebra automorphisms of R, $\mathcal{M} \subset \operatorname{Aut} R$ a submonoid and G a finite subgroup of $\operatorname{Aut} R$, normalizing $\mathcal{M}$, i.e. $\mathcal{M} = g\mathcal{M}g^{-1}$ for any $g \in G$. We will use the notation φ^g for $g\varphi g^{-1}$, $g \in G$, $\varphi \in \mathcal{M}$. The *skew monoidal ring* $R * \mathcal{M}$ is a free left R-module with a basis $\mathcal{M}$ and with the multiplication

$$(r_1\varphi_1) \cdot (r_2\varphi_2) = (r_1 r_2^{\varphi_1})(\varphi_1\varphi_2), \quad \varphi_1, \varphi_2 \in \mathcal{M},\ r_1, r_2 \in R.$$

Under the assumptions above, G acts on $R * \mathcal{M}$ by automorphisms as follows:

$$(r\varphi)^g = r^g \varphi^g, r \in R,\ \varphi \in \mathcal{M},\ g \in G.$$

Denote by $\mathcal{K} = (R * \mathcal{M})^G$ the algebra of invariants of this action. If $x \in R * \mathcal{M}$ and $\varphi \in \mathcal{M}$, then denote by x_φ the element in R such that $x = \sum\limits_{\varphi \in \mathcal{M}} x_\varphi \varphi$. Set

$$\operatorname{supp} x = \{\varphi \in \mathcal{M} | x_\varphi \neq 0\}.$$

For $\varphi \in \mathcal{M}$ denote its G-stabilizer and G-orbit by

$$H_\varphi = \{h \in G | \varphi^h = \varphi\},\ \mathcal{O}_\varphi = \{\varphi^g \mid g \in G\}, \tag{1}$$

respectively. Also set

$$\mathcal{K}_\varphi = \big\{[a\varphi] \mid a \in R^{H_\varphi}\big\}.$$

The following lemma describes a set of additive generators of $\mathcal{K}$.

Lemma 2.4 ([FO1], Lemma 2.1)**.** *Under the assumption above, the following holds:*

(a) *$x \in R * \mathcal{M}$ is G-invariant if and only if $x_{\varphi^g} = x_\varphi^g$ for all $\varphi \in \mathcal{M}, g \in G$, and in particular x_φ is constant on the classes $\mathcal{M}/H_\varphi$. In this case* $\operatorname{supp} x \subset \mathcal{M}$ *is a finite G-invariant set.*

(b) *Let $\varphi \in \mathcal{M}, a \in R^{H_\varphi}$. Then*

$$[a\varphi] = \sum_{g \in G/H_\varphi} a^g \varphi^g \tag{2}$$

is G-invariant.

(c) *Let $\varphi \in \mathcal{M}$. Then $\mathcal{K}_\varphi$ is an R^{H_φ}-bimodule (hence R^G-bimodule), where R^{H_φ} acts on $\mathcal{K}_\varphi$ by left and right multiplication:*

$$r \cdot [a\varphi] = [(ra)\varphi],\ [a\varphi] \cdot r = [(ar^\varphi)\varphi],\ r \in R^{H_\varphi}.$$

(d) *As an R^G-bimodule*

$$\mathcal{K} = \bigoplus_{\varphi \in \mathcal{M}/G} \mathcal{K}_\varphi.$$

2.3. **Galois rings.** We will use the notation and results from Subsection 2.2 in the case when $R = L$ is a field, $K \subset L$ is a finite Galois extension of fields, and $G = G(L/K)$ its Galois group. We will denote by $\imath$ the canonical embedding $K \hookrightarrow L$. Denote by i_Γ the canonical embedding $\Gamma \hookrightarrow U$,

Recall that the monoid $\mathcal{M} \subset \operatorname{Aut} L$ from the definition of a Galois ring is assumed to be *separating* (with respect to K) if for any $\varphi_1, \varphi_2 \in \mathcal{M}$ the equality $\varphi_1|_K = \varphi_2|_K$ implies $\varphi_1 = \varphi_2$. An automorphism $\varphi : L \longrightarrow L$ is called *separating* (with respect to K) if the monoid generated by $\{\varphi^g \mid g \in G\}$ in $\operatorname{Aut} L$ is separating.

Lemma 2.5 ([FO1], Lemma 2.2). *Let $\mathcal{M}$ be a separating monoid with respect to K. Then*

(a) $\mathcal{M} \cap G = \{e\}$.
(b) *For any $\varphi \in \mathcal{M}, \varphi \neq e$, there exists $\gamma \in K$ such that $\varphi(\gamma) \neq \gamma$.*
(c) *If $G\varphi_1 G = G\varphi_2 G$ for some $\varphi_1, \varphi_2 \in \mathcal{M}$, then there exists $g \in G$ such that $\varphi_1 = \varphi_2^g$.*
(d) *If $\mathcal{M}$ is a group, then statements* (a), (b), (c) *are equivalent and each of them implies that $\mathcal{M}$ is separating.*

Let $S \subset \mathcal{M}$ be a finite G-invariant subset and $B \subset \mathcal{K}$ a Γ-subbimodule. Then we introduce the following Γ-subbimodule in B:

$$B(S) = \{x \in B \mid \operatorname{supp} x \subset S\}. \tag{3}$$

Under the assumptions above both the right and the left dimensions of $\mathcal{K}(S)$ over K coincide with $|S/G|$. In particular, both dimensions of $\mathcal{K}_\varphi$ over K are equal to

$$[L^{H_\varphi} : K] = [G : H_\varphi] = |G\varphi| \tag{4}$$

for any $\varphi \in \mathcal{M}$. It was shown in [FO1] that $\mathcal{K}_\varphi$ is irreducible as a K-bimodule (there it was denoted by $V(\varphi)$).

A finitely generated K-bimodule V is called *balanced (over L)* provided that the L-bimodule $L \otimes_K V \otimes_K L$ is a direct sum of L-subbimodules, which are one dimensional over L both as left and right spaces ([FO1]). It was shown in [FO1] that any simple balanced K-bimodule has the form $V(\varphi) = L^{H_\varphi}$ for some $\varphi \in \operatorname{Aut}_{\Bbbk} L$ where $x \cdot 1 = x$, $1 \cdot x = x^\varphi$, $1 \in L^{H_\varphi}$, $x \in K$. Moreover, for $\varphi \in \mathcal{M}$ and $a \in L^{H_\varphi}$ we have $K\varphi K \simeq K[a\varphi]K \simeq V(\varphi)$. Finally, $\mathcal{K}$ is a balanced K-bimodule (in particular, semisimple) and

$$\mathcal{K} = \sum_{\varphi \in \mathcal{M}/G} K\varphi K \simeq \bigoplus_{\varphi \in \mathcal{M}/G} V(\varphi).$$

The details can be found in [FO1].

Remark 2.6. Below we will always identify the structure of a K-bimodule with the structure of a left $K \otimes_{\Bbbk} K$-module, where the first factor acts from the left and the second factor acts from the right. Similarly, we identify a Γ-bimodule structure with a $\Gamma \otimes_{\Bbbk} \Gamma$-module structure.

Lemma 2.7 ([FO1], Lemma 4.1). *Let $u \in U$ be a nonzero element, and*

$$T = \operatorname{supp} u,\ u = \sum_{\varphi \in T/G} [a_\varphi \varphi].$$

Then

$$K(\Gamma u \Gamma) = (\Gamma u \Gamma)K = KuK = \bigoplus_{m \in T/G} V(a_m m),$$

where $V(a_\varphi \varphi) = K[a_\varphi \varphi]K$ is an irreducible K-bimodule.

In particular, this shows that for every $\varphi \in \mathcal{M}$ the algebra U contains some elements $[b_1\varphi]$, ..., $[b_k\varphi]$ such that $b_1, \ldots, b_k$ is a K-basis of L^{H_φ}.

Note also that $\mathcal{K}$ is a left and right torsion free Γ-module and we have the canonical isomorphisms

$$U \otimes_\Gamma K \longrightarrow \mathcal{K},\ u \otimes x \mapsto ux, u \in U, x \in K,$$
$$K \otimes_\Gamma U \longrightarrow \mathcal{K},\ x \otimes u \mapsto xu, u \in U, x \in K \tag{5}$$

of $\Gamma - K$ and $K - \Gamma$-bimodules respectively.

Let $e \in \mathcal{M}$ be the unit element and $U_e = U \cap Le$.

Theorem 2.8 ([FO1], Theorem 4.1)**.** *Let U be a Galois ring over Γ. Then*

(a) $U_e \subset K$.
(b) *$U \cap K$ is a maximal commutative $\Bbbk$-subalgebra in U.*
(c) *The center $Z(U)$ of U equals $U \cap K^{\mathcal{M}}$.*

2.4. **Galois orders and Harish-Chandra subalgebras.** In this section we recall basic properties of Galois orders following [FO1]. For simplicity we only consider right Galois orders. Let M be a right Γ-submodule of a Galois order U over Γ. Set

$$\mathbb{D}_r(M) = \{u \in U \mid \text{there exists } \gamma \in \Gamma, \gamma \neq 0 \text{ such that } u \cdot \gamma \in M\}.$$

We have the following characterization of Galois orders.

Proposition 2.9 ([FO1], Corollary 5.1)**.** *A Galois ring U over a noetherian Γ is a* right order *if and only if for every finitely generated right Γ-module $M \subset U$, the right Γ-module $\mathbb{D}_r(M)$ is finitely generated.*

In particular, if U is right integral, then $\Gamma \subset U_e$ is an integral extension and U_e is a normal ring.

Recall that Γ is called a *Harish-Chandra subalgebra* in U if $\Gamma u \Gamma$ is finitely generated both as a left and as a right Γ-module for any $u \in U$. This is a particular (commutative) case of a general notion of a Harish-Chandra subalgebra introduced in [DFO2]. We will also say that Γ is a *right (left) Harish-Chandra subalgebra* if $\Gamma u \Gamma$ is finitely generated as a right (left) Γ-module for any $u \in U$. Note that this property is enough to check for some set of generators of the ring U over Γ. If U is a Galois order over Γ and Γ is a noetherian $\Bbbk$-algebra, then Γ is a Harish-Chandra subalgebra in U ([FO1], Corollary 5.4).

The next lemma is the main technical tool in our study of representations of Galois orders. Following Remark 2.6, the K-bimodule structure on $\mathcal{K}$ will be considered as the left $K \otimes_\Bbbk K$-module structure: $\nu_1 v \nu_2 = (\nu_1 \otimes \nu_2)v, \nu_1, \nu_2 \in K, v \in \mathcal{K}$.

Denote by $\bar{\Gamma}$ the integral closure of Γ in L. Let $S \subset \mathcal{M}$ be a finite G-invariant subset. As in (3) consider a Γ-subbimodule $U(S) \subset U$. For every $f \in \Gamma$ consider $f_S^r \subset \Gamma \otimes_\Bbbk K$ as follows:

$$f_S^r = \prod_{s \in S} (f \otimes 1 - 1 \otimes f^{s^{-1}}) = \sum_{i=0}^{|S|} f^{|S|-i} \otimes T_i \ (T_0 = 1). \tag{6}$$

Similarly we define $f_S^l = \prod\limits_{s \in S} (f^s \otimes 1 - 1 \otimes f) \in K \otimes_\Bbbk \Gamma$. If $\varphi^{-1}(\Gamma) \subset \bar{\Gamma}$ ($\varphi(\Gamma) \subset \bar{\Gamma}$ respectively) for all $\varphi \in \mathcal{M}$, then for any G-invariant subset $S \subset \mathcal{M}$ and $f_S = f_S^r$ ($f_S = f_S^l$ respectively) there holds $f_S \in \Gamma \otimes_\Bbbk \Gamma$. We formulate the following lemma for f_S^r; the case f_S^l is analogous.

Lemma 2.10 ([FO1], Lemma 5.2)**.** *Let $\varphi^{-1}(\Gamma) \subset \bar{\Gamma}$, $S \subset \mathcal{M}$ a G-invariant subset, and $f_S = f_S^r \in \Gamma \otimes_\Bbbk \Gamma$.*

(a) *An element* $u \in U$ *belongs to* $U(S)$ *if and only if* $f_S \cdot u = 0$ *for every* $f \in \Gamma$.

(b) *If* $T = \operatorname{supp} u \setminus S$, *then* $f_T \cdot u \in U(S)$ *for every* $f \in \Gamma$.

(c) *If* $f_S = \sum_{i=1}^{n} f_i \otimes g_i$, $[a\varphi] \in L * \mathcal{M}$, *then*

$$f_S \cdot [a\varphi] = [(\sum_{i=1}^{n} f_i g_i^{\varphi} a)\varphi] = [\prod_{s\in S}(f - f^{\varphi s^{-1}})a\varphi].$$

(d) *Let* S *be a* G*-orbit and* T *a* G*-invariant subset in* $\mathcal{M}$. *Then either the* Γ*-bimodule homomorphism* $P_S^T (= P_S^T(f)) : U(T) \longrightarrow U(S)$, $u \mapsto f_{T\setminus S} \cdot u, f \in \Gamma$, *is zero or* $\operatorname{Ker} P_S^T = U(T \setminus S)$.

(e) *Let* $S = S_1 \sqcup \cdots \sqcup S_n$ *be a decomposition of* S *into* G*-orbits and* $P_{S_i}^S : U(S) \longrightarrow U(S_i)$, $i = 1, \dots, n$, *be the nonzero homomorphisms defined in* (d). *Then the homomorphism*

$$P^S : U(S) \longrightarrow \bigoplus_{i=1}^{n} U(S_i),\ P^S = (P_{S_1}^S, \dots, P_{S_n}^S), \tag{7}$$

is a monomorphism.

We have the following equivalent conditions for a Galois ring to be a Galois order.

Theorem 2.11 ([FO1], Theorem 5.1). *Let* U *be a Galois ring over a noetherian* Γ *and assume that* Γ *is a right (left) Harish-Chandra* $\Bbbk$*-subalgebra of* U. *Then the following statements are equivalent:*

(a) U *is a right (respectively left) Galois order over* Γ.

(b) $U(S)$ *is a finitely generated right (respectively left)* Γ*-module for any finite* G*-invariant* $S \subset \mathcal{M}$.

(c) $U(G \cdot \varphi)$ *is a finitely generated right (respectively left)* Γ*-module for any* $\varphi \in \mathcal{M}$.

Theorem 2.12 ([FO1], Theorem 5.2). *Let* U *be a Galois ring over a noetherian* Γ *and* $\mathcal{M}$ *a subgroup of* $\operatorname{Aut} L$.

(a) *If* U_e *is an integral extension of* Γ *and* $\varphi^{-1}(\Gamma) \subset \bar{\Gamma}$ *(respectively* $\varphi(\Gamma) \subset \bar{\Gamma}$*) for any* $\varphi \in \mathcal{M}$, *then* U *is a right (respectively left) Galois order.*

(b) *If* U_e *is an integral extension of* Γ *and* Γ *is a Harish-Chandra* $\Bbbk$*-subalgebra in* U, *then* U *is a Galois order over* Γ.

3. Gelfand-Tsetlin categories

3.1. **Motivation.** The constructions of this section are the main tools we will use to investigate the class of Gelfand-Tsetlin U-modules. The first such constructions appeared in [DFO2] in the general setting, but for our purposes here we consider a special case of a commutative subalgebra Γ and present it in detail. In this section we assume that Γ is a commutative Harish-Chandra subalgebra of a finitely generated associative algebra U.

Before going into detail we give some motivation for the constructions below (see [GR], Section 3.4). Let U be a finite dimensional associative algebra over $\Bbbk$, $R \subset U$ its Levi subalgebra, and $\Gamma \subset U$ the center of R. Then $\Gamma = \bigoplus_{i=1}^{n} \Bbbk e_i$, where $\{e_1, \dots, e_n\}$ is the complete (i.e. $e_1 + \cdots + e_n = 1$) family of mutually orthogonal idempotents. Obviously, $\Gamma \subset U$ is a Harish-Chandra subalgebra.

The algebra U is isomorphic to the algebra of $n \times n$ matrices of the form $(e_j U e_i)_{i,j=1,\dots,n}$ via the map

$$u \longmapsto \begin{pmatrix} e_1 u e_1 & \dots & e_1 u e_n \\ \vdots & \ddots & \vdots \\ e_n u e_1 & \dots & e_n u e_n \end{pmatrix}. \tag{8}$$

This presentation is called the *two-sided Pierce decomposition of* U. In addition, we can associate to U a $\Bbbk$-linear category

$$\mathcal{A} = \mathcal{A}(U;\Gamma), \text{ where } \operatorname{Ob}\mathcal{A} = \{1,\dots,n\},\ \mathcal{A}(i,j) = e_j U e_i,$$

and the composition of morphisms is defined by the multiplication in U. One simple but important observation is the existence of an equivalence

$$U - \operatorname{Mod} \simeq \mathcal{A} - \operatorname{Mod}. \tag{9}$$

If $\Gamma = R$ or, equivalently, U is a basic (or Morita reduced) algebra, then the category $\mathcal{A}$ is usually presented as a quiver with relations. This presentation is the key feature in the study of finite dimensional representations of U (see e.g. [DK], [GR] for details).

In the last case the definition of $\mathcal{A}$ can be rewritten as follows. Define the category $\mathcal{A}'$ with objects $\operatorname{Specm}\Gamma = \{\mathsf{m}_1,\dots,\mathsf{m}_n\}$, where m_i is the kernel of the projection of Γ onto $\Bbbk e_i$, and morphisms $\mathcal{A}'(\mathsf{m}_i,\mathsf{m}_j) = \Gamma/\mathsf{m}_j \otimes_\Gamma U \otimes_\Gamma \Gamma/\mathsf{m}_i$ for all i and j. Then there exist obvious canonical isomorphisms $\Gamma/\mathsf{m}_j \otimes_\Gamma U \otimes_\Gamma \Gamma/\mathsf{m}_i \simeq e_j U e_i$ and, hence, $\mathcal{A}$ is isomorphic to $\mathcal{A}'$. It allows us to endow $\mathcal{A}$ with the composition of morphisms induced by the multiplication in U.

The construction of the category $\mathcal{A}$ in [DFO2] can be considered as a generalization of the two-sided Pierce decomposition. The construction below (see Definition 3.5) is a special case of [DFO2] in the case of a commutative Harish-Chandra subalgebra $\Gamma \subset U$, where U is not necessarily finite dimensional. As above, we associate with the pair $\Gamma \subset U$ a category $\mathcal{A}$ with $\operatorname{Ob}\mathcal{A} = \operatorname{Specm}\Gamma$. Unfortunately, there is no equivalence between the categories of U-modules and $\mathcal{A}$-modules. Instead we have a weaker result for the full subcategory of Gelfand-Tsetlin U-modules (see Theorem 3.14).

3.2. **Gelfand-Tsetlin modules.** We assume U is an algebra over $\Bbbk$ and $\Gamma \subset U$ is a commutative finitely generated subalgebra. We will always assume that Γ is a Harish-Chandra subalgebra. The following is the key notion of this paper.

Definition 3.1. A finitely generated U-module M is called a *Gelfand-Tsetlin module (with respect to* Γ*)* provided that the restriction $M|_\Gamma$ is a direct sum of Γ-modules

$$M|_\Gamma = \bigoplus_{\mathsf{m} \in \operatorname{Specm}\Gamma} M(\mathsf{m}), \tag{10}$$

where

$$M(\mathsf{m}) = \{v \in M \mid \mathsf{m}^k v = 0 \text{ for some } k \geq 0\}.$$

Since Γ is commutative, the Γ-submodules $M(\mathsf{m})$ are canonically defined. For a Gelfand-Tsetlin U-module M and $\mathsf{m} \in \operatorname{Specm}\Gamma$ denote by $p_{\mathsf{m}} : M \longrightarrow M(\mathsf{m})$ the canonical projection.

A Gelfand-Tsetlin module M is called a *weight module (with respect to* Γ*)* provided that $\mathsf{m}x = 0$ for all $\mathsf{m} \in \operatorname{Specm}\Gamma$ and all $x \in M(\mathsf{m})$. We denote by M_{m} the subspace of m-*weight* vectors with respect to $\mathsf{m} \in \operatorname{Specm}\Gamma$, i.e.

$$M_{\mathsf{m}} = \{x \in M(\mathsf{m}) \mid \mathsf{m}x = 0\}. \tag{11}$$

More generally, for a left (right) Γ-module X and $\mathsf{m} \in \operatorname{Specm}\Gamma$ we call an element $x \in X$ left (right) m-*nilpotent*, provided that $\mathsf{m}^k x = 0$ ($x\mathsf{m}^k = 0$) for some $k \geq 1$.

All Gelfand-Tsetlin modules form a full, abelian and extension closed subcategory $\mathbb{H}(U, \Gamma)$ of $U - \mathrm{mod}$. We denote by $\mathbb{H}W(U, \Gamma)$ the full subcategory of $\mathbb{H}(U, \Gamma)$ consisting of all weight modules.

The *support* of a Gelfand-Tsetlin module M is the set

$$\operatorname{supp} M = \{\mathsf{m} \in \operatorname{Specm}\Gamma \mid M(\mathsf{m}) \neq 0\}.$$

For $D \subset \operatorname{Specm}\Gamma$ denote by $\mathbb{H}(U, \Gamma, D)$ the full subcategory in $\mathbb{H}(U, \Gamma)$ formed by M such that $\operatorname{supp} M \subset D$.

Since $\Bbbk$ is algebraically closed and Γ is finitely generated as an algebra, for a given $\mathsf{m} \in \operatorname{Specm}\Gamma$ we have the canonical isomorphism $\Gamma/\mathsf{m} \simeq \Bbbk$. Then we denote by $\chi_{\mathsf{m}} : \Gamma \longrightarrow \Gamma/\mathsf{m} \simeq \Bbbk$ the corresponding character of Γ. Conversely, for a character $\chi : \Gamma \longrightarrow \Bbbk$ denote $\mathsf{m}_\chi = \operatorname{Ker}\chi$, so we will identify the set of all characters of Γ with $\operatorname{Specm}\Gamma$. If there exists a Gelfand-Tsetlin module M with $M(\mathsf{m}) \neq 0$, then we say that m (and the character χ_{m}) *lifts*, or *extends* to M.

For a Γ-bimodule V, any pair $(\mathsf{m}, \mathsf{n}) \in \operatorname{Specm}\Gamma \times \operatorname{Specm}\Gamma$ and $m, n \geq 0$ we will use the following notation:

$$_{\mathsf{n}^n}V = V/\mathsf{n}^n V, \quad V_{\mathsf{m}^m} = V/V\mathsf{m}^m, \quad {}_{\mathsf{n}^n}V_{\mathsf{m}^m} = V/(\mathsf{n}^n V + V\mathsf{m}^m).$$

Lemma 3.2. *Let $a \in U$, $V = \Gamma a \Gamma$ and $\mathsf{m}, \mathsf{n} \in \operatorname{Specm}\Gamma$. Then under the assumptions above the following holds:*

(a) *All the modules $_{\mathsf{n}^n}V$ and V_{m^m} are finite dimensional.*
(b) *The following conditions are equivalent:*
   1. Γ/n *is a subquotient of V_{m^m} as a left* Γ*-module;*
   2. Γ/m *is a subquotient of $_{\mathsf{n}^n}V$ as a right* Γ*-module;*
   3. $\Gamma/\mathsf{n} \otimes_\Gamma \Gamma a \Gamma \otimes_\Gamma \Gamma/\mathsf{m} \neq 0$.

   We denote by $X(a)$ the subset of $\operatorname{Specm}\Gamma \times \operatorname{Specm}\Gamma$ *consisting of* (m, n) *satisfying these equivalent conditions.*
(c) *For* $\mathsf{m}, \mathsf{n} \in \operatorname{Specm}\Gamma$ *define the set*

$$X_a(\mathsf{m}) = \{\mathsf{n} \in \operatorname{Specm}\Gamma \mid (\mathsf{m}, \mathsf{n}) \in X(a)\}$$

   and the set

$$X^a(\mathsf{n}) = \{\mathsf{m} \in \operatorname{Specm}\Gamma \mid (\mathsf{m}, \mathsf{n}) \in X(a)\}.$$

   Then both $X_a(\mathsf{m})$ and $X^a(\mathsf{n})$ are finite.
(d) *Let M be a Gelfand-Tsetlin U-module,* $\mathsf{m} \in \operatorname{Specm}\Gamma$*. Then*

$$aM(\mathsf{m}) \subset \sum_{\mathsf{n} \in X_a(\mathsf{m})} M(\mathsf{n}). \tag{12}$$

(e) *Let M be a Gelfand-Tsetlin U-module. Then for every $\mathsf{m} \notin X^a(\mathsf{n})$ we have*

$$p_{\mathsf{n}}(aM(\mathsf{m})) = 0. \tag{13}$$

(f) *If X is a finite dimensional $\Gamma-$module, then $U \otimes_\Gamma X$ is a Gelfand-Tsetlin module.*

Proof. We will prove the statements for V_{m^m}; the case of ${}_{\mathsf{n}^n}V$ is analogous.

Since Γ is finitely generated, we have $\dim_k \Gamma/\mathsf{m}^m < \infty$. Then $V_{\mathsf{m}^m} \simeq \Gamma a \Gamma \otimes_\Gamma \Gamma/\mathsf{m}^m$ is finite dimensional, since $\Gamma a \Gamma$ is finitely generated as a right Γ-module. This proves (a). If $V_{\mathsf{m}^m} = \bigoplus\limits_{k=1}^{t} L_k$ is a decomposition into a direct sum of indecomposable left Γ-modules, then for every $k = 1, \dots, t$ there exists $\mathsf{n}_k \in \operatorname{Specm}\Gamma$ and $n_k \geq 1$ such that $\mathsf{n}_k^{n_k} L_k = 0$. In particular, all subquotients of L_k are isomorphic to Γ/n_k. On the other hand, $\Gamma/\mathsf{n} \otimes_\Gamma L_k \simeq L_k/\mathsf{n}L_k$ is nonzero if and only if $\mathsf{n} = \mathsf{n}_k$, which, together with (a), implies (b) and (c). To prove (d) consider any $x \in M(\mathsf{m})$. Then there exists $m > 1$, such that $\mathsf{m}^m x = 0$. It follows that the left Γ-submodule $\Gamma a \Gamma x \subset M$ is a subquotient of V_{m^m}. Then (d) follows from (c). Statement (e) is proved analogously. To show (f) it is enough to consider the case $\dim_{\Bbbk} V = 1$. But then the statement follows from (a). $\square$

Denote by Δ the minimal equivalence on $\operatorname{Specm}\Gamma$ containing all $X(a)$, $a \in U$, and by $\Delta(U, \Gamma)$ the set of the $\Delta-$equivalence classes on $\operatorname{Specm}\Gamma$.

Lemma 3.3. *Let M, M' be Gelfand-Tsetlin modules,* $\operatorname{supp} M \subset D$, *and* $\operatorname{supp} M' \subset D'$, *where D and D' are different classes of Δ-equivalence. Then*

$$\operatorname{Hom}_U(M, M') = 0, \ \operatorname{Ext}^1_U(M, M') = 0.$$

Proof. Obviously, $\operatorname{Hom}_\Gamma(M, M') = 0$. As $\operatorname{Hom}_U(M, M') \subset \operatorname{Hom}_\Gamma(M, M')$, it follows that $\operatorname{Hom}_U(M, M') = 0$. It is enough to prove that every exact sequence

$$0 \longrightarrow M' \longrightarrow N \longrightarrow M \longrightarrow 0$$

splits in $U - \mathrm{mod}$.

Since $D \cap D' = \varnothing$, for every $\mathsf{m} \in \operatorname{supp} N$ we have

$$N(\mathsf{m}) = M'(\mathsf{m}) \oplus M(\mathsf{m}),$$

where either $M'(\mathsf{m}) = 0$ or $M(\mathsf{m}) = 0$. For $a \in U$ we have $aM'(\mathsf{m}) \subset M'$ and $aM(\mathsf{m}) \subset M$, by Lemma 3.2, (d). Hence M' and M are U-submodules in N. $\square$

Immediately from Lemma 3.2, (d), (e) and Lemma 3.3 we obtain the following.

Corollary 3.4.

$$\mathbb{H}(U, \Gamma) = \bigoplus_{D \in \Delta(U, \Gamma)} \mathbb{H}(U, \Gamma, D).$$

For $D \in \Delta(U, \Gamma)$ the subcategory $\mathbb{H}(U, \Gamma, D)$ will be called a *block* of $\mathbb{H}(U, \Gamma)$.

3.3. **The category $\mathcal{A}$.** We assume Γ to be noetherian. For $\mathsf{m} \in \operatorname{Specm}\Gamma$ denote by $\widehat{\Gamma}_{\mathsf{m}}$ the completion $\varprojlim\limits_{m} \Gamma/\mathsf{m}^m$ of Γ by the maximal ideal m.

For a Γ-bimodule V denote by ${}_{\mathsf{n}}\widehat{V}_{\mathsf{m}}$ the I-adic completion of the $\Gamma \otimes_{\Bbbk} \Gamma$-module V, where $I \subset \Gamma \otimes \Gamma$ is the maximal ideal $I = \mathsf{n} \otimes \Gamma + \Gamma \otimes \mathsf{m}$; in other words,

$${}_{\mathsf{n}}\widehat{V}_{\mathsf{m}} \ = \ \varprojlim_{n,m} {}_{\mathsf{n}^n}V_{\mathsf{m}^m}.$$

It has the structure of a completed Γ-bimodule or a completed $\Gamma - \Gamma$-module, that is, ${}_{\mathsf{n}}\widehat{V}_{\mathsf{m}}$ is a topological $(\widehat{\Gamma \otimes \Gamma})_I$-module, where $(\widehat{\Gamma \otimes \Gamma})_I$ is the completion of $\Gamma \otimes_{\Bbbk} \Gamma$ by the ideal $I = \mathsf{n} \otimes \Gamma + \Gamma \otimes \mathsf{m}$. We will denote $(\widehat{\Gamma \otimes \Gamma})_I$ by $\widehat{\Gamma}_{\mathsf{n}} \widehat{\otimes} \widehat{\Gamma}_{\mathsf{m}}$. Moreover, let

V and W be completed $\widehat{\Gamma}_{\mathsf{p}} - \widehat{\Gamma}_{\mathsf{n}}$ and $\widehat{\Gamma}_{\mathsf{n}} - \widehat{\Gamma}_{\mathsf{m}}$ bimodules correspondingly. Then the *completed tensor product* $V \widehat{\otimes}_{\widehat{\Gamma}_{\mathsf{n}}} W$ is defined as

$$V \widehat{\otimes}_{\widehat{\Gamma}_{\mathsf{n}}} W = \varprojlim_{k,l} V \otimes_{\widehat{\Gamma}_{\mathsf{n}}} W / (V \mathsf{m}^k \otimes W + V \otimes \mathsf{m}^l W).$$

There exists the canonical $\widehat{\Gamma}_{\mathsf{p}} - \widehat{\Gamma}_{\mathsf{m}}$ bimodule homomorphism $\varphi : V \otimes_{\widehat{\Gamma}_{\mathsf{n}}} W \to V \widehat{\otimes}_{\widehat{\Gamma}_{\mathsf{n}}} W$ as the morphism to the inverse limit.

Let B be a Γ-bimodule satisfying the following *Harish-Chandra condition: every finitely generated bimodule in B is finitely generated both as a left and as a right module.* If Γ is a Harish-Chandra subalgebra in U, then every finitely generated Γ-subbimodule has this property. Denote by $F(B)$ the set of all finitely generated Γ-subbimodules in B. For $V \in F(B)$ we have ${}_{\mathsf{n}}\widehat{V}_{\mathsf{m}} = (\widehat{\Gamma \otimes \Gamma})_I \otimes_{\Gamma \otimes_{\Bbbk} \Gamma} V$, hence if $V, W \in F(B)$ and $V \subset W$, then the canonical embedding induces a *monomorphism* ${}_{\mathsf{n}}\widehat{V}_{\mathsf{m}} \hookrightarrow {}_{\mathsf{n}}\widehat{W}_{\mathsf{m}}$ ([Mat], Theorems 8.7 and 8.8).

The *finitary completion* ${}_{\mathsf{n}}\widetilde{B}_{\mathsf{m}}$ of the bimodule B by the ideal $I = \mathsf{n} \otimes \Gamma + \Gamma \otimes \mathsf{m}$ is defined as

$$(14) \qquad {}_{\mathsf{n}}\widetilde{B}_{\mathsf{m}} = \varinjlim_{V \in F(B)} {}_{\mathsf{n}}\widehat{V}_{\mathsf{m}}.$$

For any $V \in F(B)$, $\mathsf{m}, \mathsf{n} \in \operatorname{Specm} \Gamma$, $m, n > 0$, we have the following Γ-bimodule morphisms:

$${}_{\mathsf{n}}\widehat{V}_{\mathsf{m}} \longrightarrow {}_{\mathsf{n}^n} V_{\mathsf{m}^m} \longrightarrow {}_{\mathsf{n}^n} B_{\mathsf{m}^m},$$

where the first one is the canonical map from the projective limit and the second one is induced by the embedding $V \subset B$. This family defines a homomorphism

$$\Psi : {}_{\mathsf{n}}\widetilde{B}_{\mathsf{m}} = \varinjlim_{V \in F(B)} {}_{\mathsf{n}}\widehat{V}_{\mathsf{m}} \longrightarrow \varprojlim_{n,m} {}_{\mathsf{n}^n} B_{\mathsf{m}^m} = {}_{\mathsf{n}}\widehat{B}_{\mathsf{m}}.$$

If B is finitely generated as a Γ-bimodule, then Ψ is an isomorphism.

Definition 3.5. The category $\mathcal{A} = \mathcal{A}_{U,\Gamma}$ is defined to have objects $\operatorname{Ob} \mathcal{A} = \operatorname{Specm} \Gamma$. The space of morphisms from m to n is defined as follows:

$$\mathcal{A}(\mathsf{m}, \mathsf{n}) \;=\; {}_{\mathsf{n}}\widetilde{U}_{\mathsf{m}} \;\Big(= \varinjlim_{V \in F(U)} {}_{\mathsf{n}}\widehat{V}_{\mathsf{m}} = \varinjlim_{V \in F(U)} \varprojlim_{n,m} {}_{\mathsf{n}^n} V_{\mathsf{m}^m}\Big).$$

The rest of this subsection is devoted to the definition of the composition of morphisms in $\mathcal{A}$. The spaces $\mathcal{A}(\mathsf{m}, \mathsf{n})$ are endowed with the standard topology defined by the limits. Further, any $\mathcal{A}(\mathsf{m}, \mathsf{n})$ is endowed with the canonical structure of a completed $\widehat{\Gamma}_{\mathsf{n}} - \widehat{\Gamma}_{\mathsf{m}}$-bimodule.

For a (not necessarily finitely generated) Γ-bimodule $V \subset U$ and $\mathsf{p} \in \operatorname{Specm} \Gamma$ set

$$\begin{aligned} {}^{\mathsf{p}}V_{\mathsf{m}^m} &= \{\bar{a} \in V_{\mathsf{m}^m} \mid \mathsf{p}^p \bar{a} = 0 \text{ for some } p \geq 1\}, \\ {}_{\mathsf{n}^n} V^{\mathsf{p}} &= \{\bar{b} \in {}_{\mathsf{n}^n} V \mid \bar{b} \mathsf{p}^p = 0 \text{ for some } p \geq 1\}. \end{aligned}$$

By Lemma 3.2, (c), for every $a \in V$ there exists a finite set $X_a(\mathsf{m}) = \{\mathsf{n}_1, \cdots, \mathsf{n}_k\}$ and $N = N(a, \mathsf{m}) \geq 1$ such that $\mathsf{n}_1^N \dots \mathsf{n}_k^N$ annihilates the class $\bar{a}$ of a in $V/V\mathsf{m}^m$.

Hence, by the Chinese remainder theorem we have

$$V_{\mathsf{m}^m} = \bigoplus_{\mathsf{p}\in X_{\mathsf{m}}(V)} {}^{\mathsf{p}}V_{\mathsf{m}^m}, \text{ where } X_{\mathsf{m}}(V) = \bigcup_{a\in V} X_a(\mathsf{m})$$

(15) and

$$_{\mathsf{n}^n}V = \bigoplus_{\mathsf{p}\in X^{\mathsf{n}}(V)} {}_{\mathsf{n}^n}V^{\mathsf{p}}, \text{ where } X^{\mathsf{n}}(V) = \bigcup_{a\in V} X^a(\mathsf{n}).$$

Lemma 3.6. (a) *Let $D \neq D'$ be Δ-equivalence classes with $\mathsf{m} \in D$, $\mathsf{n} \in D'$. Then $\mathcal{A}(\mathsf{m},\mathsf{n}) = 0$.*

(b) *We have the following decomposition of $\mathcal{A}$ into a direct sum of its full subcategories:*

$$\mathcal{A} = \bigoplus_{D\in\Delta(U,\Gamma)} \mathcal{A}_D,$$

where $\mathcal{A}_D$ is the full subcategory of $\mathcal{A}$ such that $\operatorname{Ob}\mathcal{A}_D = D$.

(c) *If $\mathsf{n} \notin X_a(\mathsf{m})$ for $a \in U$, then the class of a in $\mathcal{A}(\mathsf{m},\mathsf{n})$ equals 0.*

(d) *If $\mathsf{m} \notin X^a(\mathsf{n})$ for $a \in U$, then the class of a in $\mathcal{A}(\mathsf{m},\mathsf{n})$ equals 0.*

Proof. Let $\mathsf{m} \in \operatorname{Specm}\Gamma$ and $m \geq 1$. Then for every Γ-subbimodule $V \subset U$,

$$_{\mathsf{n}^n}V_{\mathsf{m}^m} = \Gamma/\mathsf{n}^n \otimes_\Gamma V_{\mathsf{m}^m} = \Gamma/\mathsf{n}^n \otimes_\Gamma \bigoplus_{\mathsf{p}\in X_m(V)} {}^{\mathsf{p}}V_{\mathsf{m}^m} = \bigoplus_{\mathsf{p}\in X_m(V)} \Gamma/\mathsf{n}^n \otimes_\Gamma {}^{\mathsf{p}}V_{\mathsf{m}^m}.$$

If m and n belong to different classes of Δ-equivalence, then all summands in the last sum equal 0, since $\mathsf{n} \neq \mathsf{p}$. This proves (a), and (b) follows. Statements (c) and (d) are proved analogously to (a). □

Let V and W be finitely generated Γ-subbimodules in U. Since Γ is a Harish-Chandra subalgebra in U, the bimodule $T \subset U$, spanned by all products vw, $v \in V$, $w \in W$, is finitely generated. Denote by $\mu : V \otimes_\Gamma W \longrightarrow T$ the map $\mu(v\otimes w) = vw$, $v \in V, w \in W$.

For a Γ-bimodule B denote by $_{\mathsf{n}^n}\mathbb{1}_{\mathsf{m}^m}$ $(= {}_{\mathsf{n}^n}\mathbb{1}_{\mathsf{m}^m}(B))$ the canonical epimorphism

$$_{\mathsf{n}^n}\mathbb{1}_{\mathsf{m}^m} : B \longrightarrow B/(\mathsf{n}^n B + B\mathsf{m}^m).$$

Lemma 3.7. *Let $V, W \subset U$ be finitely generated Γ-bimodules, $T = VW$, $S(V,W) = X^{\mathsf{n}}(V) \cap X_{\mathsf{m}}(W)$, $\mathsf{m},\mathsf{n} \in \operatorname{Specm}\Gamma$, $m, n \geq 0$.*

(a) *For $\mathsf{p},\mathsf{p}' \in \operatorname{Specm}\Gamma$, $\mathsf{p} \neq \mathsf{p}'$ we have $_{\mathsf{n}^n}V^{\mathsf{p}} \otimes_\Gamma {}^{\mathsf{p}'}W_{\mathsf{m}^n} = 0$.*

(b) *The induced by the decomposition (15) homomorphism*

$$\Phi : \bigoplus_{\mathsf{p}\in S(V,W)} \left({}_{\mathsf{n}^n}V^{\mathsf{p}} \otimes_\Gamma {}^{\mathsf{p}}W_{\mathsf{m}^n}\right) \longrightarrow {}_{\mathsf{n}^n}V \otimes_\Gamma W_{\mathsf{m}^m}$$

is an isomorphism of Γ-bimodules.

(c) *There exists $P = P(\mathsf{m},\mathsf{n},m,n,V,W) > 0$ such that for any $p \geq P$ the canonical projections*

$$_{\mathsf{n}^n}\pi_{\mathsf{p}^p} : {}_{\mathsf{n}^n}V \longrightarrow {}_{\mathsf{n}^n}V_{\mathsf{p}^p}, \ {}_{\mathsf{p}^p}\pi_{\mathsf{m}^m} : W_{\mathsf{m}^m} \longrightarrow {}_{\mathsf{p}^p}W_{\mathsf{m}^m}$$

induce the canonical isomorphisms

$$_{\mathsf{n}^n}\pi^{\mathsf{p}^p} : {}_{\mathsf{n}^n}V^{\mathsf{p}} \longrightarrow {}_{\mathsf{n}^n}V_{\mathsf{p}^p}, \ {}^{\mathsf{p}^p}\pi_{\mathsf{m}^m} : {}^{\mathsf{p}}W_{\mathsf{m}^m} \longrightarrow {}_{\mathsf{p}^p}W_{\mathsf{m}^m}.$$

(d) *There exists P such that for all $p \geq P$ there exists a unique homomorphism*

$$\widehat{\mu}(\mathsf{m}^m, \mathsf{n}^n, p) : \bigoplus_{\mathsf{p}\in S(V,W)} \left({}_{\mathsf{n}^n}V_{\mathsf{p}^p} \otimes_{\widehat{\Gamma}_{\mathsf{p}}} {}_{\mathsf{p}^p}W_{\mathsf{m}^n}\right) \longrightarrow {}_{\mathsf{n}^n}T_{\mathsf{m}^m},$$

which makes the diagram below commutative:

$$\begin{tikzcd}
V \otimes_\Gamma W \arrow[r, "{}_{\mathsf{n}^n}\mathbb{1}_{\mathsf{m}^m}"] \arrow[d, "\mu"'] & {}_{\mathsf{n}^n}V \otimes_\Gamma W_{\mathsf{m}^m} \arrow[d, "{}_{\mathsf{n}^n}\mu_{\mathsf{m}^m}"'] & \Phi\bigoplus\limits_{\mathsf{p}\in S(V,W)} \left({}_{\mathsf{n}^n}V^{\mathsf{p}} \otimes_\Gamma {}^{\mathsf{p}}W_{\mathsf{m}^n}\right) \arrow[l] \arrow[d, "\oplus_{\mathsf{p}} c_{\mathsf{p}}(p)"'] \\
T \arrow[r, "{}_{\mathsf{n}^n}\mathbb{1}_{\mathsf{m}^m}"] & {}_{\mathsf{n}^n}T_{\mathsf{m}^m} & \bigoplus\limits_{\mathsf{p}\in S(V,W)} \left({}_{\mathsf{n}^n}V_{\mathsf{p}^p} \otimes_{\widehat{\Gamma}_{\mathsf{p}}} {}_{\mathsf{p}^p}W_{\mathsf{m}^n}\right) \arrow[l, Rightarrow, "{\widehat{\mu}(\mathsf{m}^m,\mathsf{n}^n,p)}"']
\end{tikzcd}$$

Here ${}_{\mathsf{n}^n}\mu_{\mathsf{m}^m}$ is induced by μ, Φ is given by claim (b) *and*

$$c_{\mathsf{p}}(p) : {}_{\mathsf{n}^n}V^{\mathsf{p}} \otimes_\Gamma {}^{\mathsf{p}}W_{\mathsf{m}^n} \xrightarrow{{}_{\mathsf{n}^n}\pi^{\mathsf{p}^p} \otimes {}^{\mathsf{p}^p}\pi_{\mathsf{m}^m}} {}_{\mathsf{n}^n}V_{\mathsf{p}^p} \otimes_\Gamma {}_{\mathsf{p}^p}W_{\mathsf{m}^n} \longrightarrow {}_{\mathsf{n}^n}V_{\mathsf{p}^p} \otimes_{\widehat{\Gamma}_{\mathsf{p}}} {}_{\mathsf{p}^p}W_{\mathsf{m}^n}.$$

(e) *Assume that $p > P$, where P satisfies* (c) *and* (d). *Then the diagram*

$$\begin{tikzcd}
\bigoplus\limits_{\mathsf{p}\in S(V,W)} \left({}_{\mathsf{n}^n}V_{\mathsf{p}^{p+1}} \otimes_{\widehat{\Gamma}_{\mathsf{p}}} {}_{\mathsf{p}^{p+1}}W_{\mathsf{m}^n}\right) \arrow[r, "{\widehat{\mu}(\mathsf{m}^m,\mathsf{n}^n,p+1)}"] \arrow[d, "c_{\mathsf{p}}(p)(c_{\mathsf{p}}(p+1))^{-1}"] & {}_{\mathsf{n}^n}T_{\mathsf{m}^m} \arrow[d, equal] \\
\bigoplus\limits_{\mathsf{p}\in S(V,W)} \left({}_{\mathsf{n}^n}V_{\mathsf{p}^p} \otimes_{\widehat{\Gamma}_{\mathsf{p}}} {}_{\mathsf{p}^p}W_{\mathsf{m}^n}\right) \arrow[r, "{\widehat{\mu}(\mathsf{m}^m,\mathsf{n}^n,p)}"] & {}_{\mathsf{n}^n}T_{\mathsf{m}^m}
\end{tikzcd}$$

is commutative. After identification of the bimodules ${}_{\mathsf{n}^n}V_{\mathsf{p}^p} \otimes_{\widehat{\Gamma}_{\mathsf{p}}} {}_{\mathsf{p}^p}W_{\mathsf{m}^n}$ for $p > P$ through the isomorphisms $c_{\mathsf{p}}(p)(c_{\mathsf{p}}(p+1))^{-1}$, we use the notation

$$\widehat{\mu}(\mathsf{m}^m, \mathsf{n}^n) : \bigoplus_{\mathsf{p}\in S(V,W)} {}_{\mathsf{n}^n}V_{\mathsf{p}^p} \otimes_{\widehat{\Gamma}_{\mathsf{p}}} {}_{\mathsf{p}^p}W_{\mathsf{m}^n} \to {}_{\mathsf{n}^n}T_{\mathsf{m}^m}$$

instead of $\widehat{\mu}(\mathsf{m}^m, \mathsf{n}^n, p)$.

(f) *Denote*

$$\widehat{\mu}_{\mathsf{p}}(\mathsf{m}^m, \mathsf{n}^n) : {}_{\mathsf{n}^n}V_{\mathsf{p}^p} \otimes_{\widehat{\Gamma}_{\mathsf{p}}} {}_{\mathsf{p}^p}W_{\mathsf{m}^n} \longrightarrow {}_{\mathsf{n}^n}T_{\mathsf{m}^m}$$

as the restriction of $\widehat{\mu}(\mathsf{m}^m, \mathsf{n}^n)$, i.e.

$$\widehat{\mu}(\mathsf{m}^m, \mathsf{n}^n) = \sum_{\mathsf{p}\in S(V,W)} \widehat{\mu}_{\mathsf{p}}(\mathsf{m}^m, \mathsf{n}^n).$$

For all $p > P$, where P satisfies (c) *and* (d)*, there exists $S \geq 1$ such that for every $\mathsf{p} \in S(V,W)$, $v \in V$, $w \in W$, the corresponding classes $\bar{v} \in {}_{\mathsf{n}^n}V_{\mathsf{p}^p}$, $\bar{w} \in {}_{\mathsf{p}^p}W_{\mathsf{m}^n}$, $s \geq S$, and any $\gamma \in \Gamma$ satisfying*

$$\gamma \equiv 1 \bmod \mathsf{p}^s, \gamma \equiv 0 \bmod \mathsf{q}^s, \mathsf{q} \in S(V,W), \mathsf{q} \neq \mathsf{p} \tag{16}$$

(such γ exists by the Chinese remainder theorem), the following holds:

$$\widehat{\mu}_{\mathsf{p}}(\mathsf{m}^m, \mathsf{n}^n)(\bar{v} \otimes \bar{w}) = \overline{v\gamma w},$$

where $\overline{v\gamma w}$ is the class of $v\gamma w$ in ${}_{\mathsf{n}^n}T_{\mathsf{m}^m}$.

(g) *Let $m' \leq m$, $n' \leq n$ be integers. Then for all $p > P$, where P is given by* (c) *and* (d), *we have the following commutative diagram of $\widehat{\Gamma}_{\mathsf{n}} - \widehat{\Gamma}_{\mathsf{m}}$ homomorphisms:*

$$\begin{tikzcd}
{}_{\mathsf{n}^n}V_{\mathsf{p}^p} \otimes_{\widehat{\Gamma}_{\mathsf{p}}} {}_{\mathsf{p}^p}W_{\mathsf{m}^n} \arrow[r] \arrow[d, "\widehat{\mu}_{\mathsf{p}}(\mathsf{m}^m, \mathsf{n}^n)"] & {}_{\mathsf{n}^{n'}}V_{\mathsf{p}^p} \otimes_{\widehat{\Gamma}_{\mathsf{p}}} {}_{\mathsf{p}^p}W_{\mathsf{m}^{m'}} \arrow[d, "\widehat{\mu}_{\mathsf{p}}(\mathsf{m}^{m'}, \mathsf{n}^{n'})"] \\
{}_{\mathsf{n}^n}T_{\mathsf{m}^m} \arrow[r] & {}_{\mathsf{n}^{n'}}T_{\mathsf{m}^{m'}},
\end{tikzcd}$$

where the horizontal arrows are induced by the canonical projections.

(h) *Let $V \subset V'$, $W \subset W'$ be finitely generated Γ-subbimodules in U, $T' = V'W'$. Then for $p > P$, where P is given by* (d), *we have the following commutative diagram of $\widehat{\Gamma}_{\mathsf{n}} - \widehat{\Gamma}_{\mathsf{m}}$ homomorphisms:*

$$\begin{tikzcd}
{}_{\mathsf{n}^n}V_{\mathsf{p}^p} \otimes_{\widehat{\Gamma}_{\mathsf{p}}} {}_{\mathsf{p}^p}W_{\mathsf{m}^n} \arrow[r] \arrow[d, "\widehat{\mu}_{\mathsf{p}}(\mathsf{m}^m, \mathsf{n}^n)"] & {}_{\mathsf{n}^n}V'_{\mathsf{p}^p} \otimes_{\widehat{\Gamma}_{\mathsf{p}}} {}_{\mathsf{p}^p}W'_{\mathsf{m}^m} \arrow[d, "\widehat{\mu}_{\mathsf{p}}(\mathsf{m}^m, \mathsf{n}^n)"] \\
{}_{\mathsf{n}^n}T_{\mathsf{p}^p} \arrow[r] & {}_{\mathsf{n}^n}T'_{\mathsf{p}^p},
\end{tikzcd}$$

where the horizontal arrows are induced by the canonical embeddings.

Proof. Statement (a) easily follows from the Chinese remainder theorem. Statement (b) follows from decomposition (15) and from statement (a). To prove statement (c) note first that the sequence of finite dimensional modules ${}_{\mathsf{n}^n}V_{\mathsf{p}^p}$ stabilizes when p exceeds some P. Hence ${}_{\mathsf{n}^n}V^{\mathsf{p}}$ is a quotient of ${}_{\mathsf{n}^n}V_{\mathsf{p}^p}$, $p \geq P$. On the other hand every ${}_{\mathsf{n}^n}\pi_{\mathsf{p}^p}$ factorizes through ${}_{\mathsf{n}^n}V^{\mathsf{p}}$, which proves (c) for V. The case of W is considered analogously.

The left square in the diagram from (d) is obviously commutative. From (c) it follows that for sufficiently large p all c_{p} are isomorphisms: the first map in the definition of c_{p} is an isomorphism due to (c), and the second map is an isomorphism, since both factors of the tensor product are finite dimensional p-torsion modules. Hence, the third vertical arrow in the diagram is an isomorphism. In addition, Φ is an isomorphism by (b) and

$$\widehat{\mu}_{\mathsf{p}}(\mathsf{m}^m, \mathsf{n}^n, p) = {}_{\mathsf{n}^n}\mu_{\mathsf{m}^m} \circ \Phi \circ (\bigoplus_{\mathsf{p}} c_{\mathsf{p}}(p))^{-1}, \tag{17}$$

which implies (d). The commutativity of the diagram from (e) follows from (17).

Let $\bar{v} = \sum\limits_{\mathsf{p} \in X^{\mathsf{n}}(V)} \bar{v}_{\mathsf{p}}$, $\bar{w} = \sum\limits_{\mathsf{p} \in X_{\mathsf{m}}(W)} {}_{\mathsf{p}}\bar{w}$ be the decompositions from (15). Then for a fixed $\mathsf{p} \in S(V, W)$, for $\gamma', \gamma'' \in \Gamma$ satisfying (16) and for s big enough we have $\bar{v}\gamma' = \bar{v}_{\mathsf{p}}$, $\gamma''\bar{w} = {}_{\mathsf{p}}\bar{w}$, where $\bar{v}$ and $\bar{w}$ are the classes of v and w in ${}_{\mathsf{n}^n}V_{\mathsf{p}^p}$ and ${}_{\mathsf{p}^p}W_{\mathsf{m}^n}$, respectively.

From the commutativity of the diagram in (d) we obtain

$${}_{\mathsf{n}^n}\mathbb{1}_{\mathsf{m}^m} \circ \mu = ({}_{\mathsf{n}^n}\mu_{\mathsf{m}^m}) \circ ({}_{\mathsf{n}^n}\mathbb{1}_{\mathsf{m}^m}) = \widehat{\mu}(\mathsf{m}^m, \mathsf{n}^n, p) \circ (\bigoplus_{\mathsf{p}} c_{\mathsf{p}}) \circ \Phi^{-1} \circ {}_{\mathsf{n}^n}\mathbb{1}_{\mathsf{m}^m}.$$

Applying the first morphism to $v\gamma' \otimes \gamma'' w$ we obtain $\overline{v\gamma'\gamma'' w}$, and applying the third morphism we obtain $\widehat{\mu}(\mathsf{m}^m, \mathsf{n}^n, p)(v_{\mathsf{p}} \otimes {}_{\mathsf{p}}w)$, which, by definition, equals

$$\widehat{\mu}_{\mathsf{p}}(\mathsf{m}^m, \mathsf{n}^n)(v_{\mathsf{p}} \otimes {}_{\mathsf{p}}w).$$

Setting $\gamma = \gamma'\gamma''$ implies (f).

Statements (g) and (h) follow from (f) in the following way. Consider an element $v\otimes w\in V\otimes_\Gamma W$. The upper horizontal arrows send a class of $v\otimes w$ to its class in respective tensor products. Assume that p satisfies the conditions of (c) and (d) for all tensor products in (g) and (h). Following (f), for a sufficiently large s we can choose $\gamma\in\Gamma$ such that all four vertical arrows send a class of $v\otimes w$ into a class of $v\gamma w$. So in both statements the restriction on the class of $v\otimes w$ gives the commutative diagram

$$\begin{tikzcd} \overline{v\otimes w} \arrow[r, mapsto] \arrow[d, mapsto] & \overline{v\otimes w} \arrow[d, mapsto] \\ \overline{v\gamma w} \arrow[r, mapsto] & \overline{v\gamma w}, \end{tikzcd}$$

since in both cases the lower horizontal arrows send the class to the class of the same element from T. $\square$

Now the composition in the category $\mathcal{A}$ is defined as follows. Since direct limits commute with the tensor product, for $\mathsf{p},\mathsf{m},\mathsf{n}\in\operatorname{Spec}\Gamma$ we can write

$$\mathcal{A}(\mathsf{p},\mathsf{n})\otimes_{\widehat{\Gamma}_{\mathsf{p}}}\mathcal{A}(\mathsf{m},\mathsf{p})\simeq\varinjlim_{V\in F(U)}\varinjlim_{W\in F(U)}{}_{\mathsf{n}}\widehat{V}_{\mathsf{p}}\otimes_{\widehat{\Gamma}_{\mathsf{p}}}{}_{\mathsf{p}}\widehat{W}_{\mathsf{m}}.$$

Then for sufficiently large m,n and p we have the following composition:

$$(18)\qquad {}_{\mathsf{n}^n}V_{\mathsf{p}^p}\otimes_{\widehat{\Gamma}_{\mathsf{p}}}{}_{\mathsf{p}^p}W_{\mathsf{m}^m}\xrightarrow{\widehat{\mu}_{\mathsf{p}}(\mathsf{m}^m,\mathsf{n}^n)}{}_{\mathsf{n}^n}T_{\mathsf{m}^m}\xrightarrow{i_{\mathsf{n}^n,\mathsf{m}^m}}\mathcal{A}(\mathsf{m},\mathsf{n}),$$

where the first homomorphism is constructed above and the second is the canonical map in the direct limit. Taking the inverse limit by m,n,p from the commutative diagram in Lemma 3.7, (g), we have the well-defined maps

$${}_{\mathsf{n}}V_{\mathsf{p}}\,\widehat{\otimes_{\widehat{\Gamma}_{\mathsf{p}}}}\,{}_{\mathsf{p}}W_{\mathsf{m}}\xrightarrow{\widehat{\mu}_{\mathsf{p}}(\mathsf{m},\mathsf{n})}{}_{\mathsf{n}}\widehat{T}_{\mathsf{m}}\xrightarrow{\imath_{\mathsf{n},\mathsf{m}}}\mathcal{A}(\mathsf{m},\mathsf{n}).$$

Since the first member in the sequence above is just ${}_{\mathsf{n}}\widehat{V}_{\mathsf{p}}\widehat{\otimes}_{\widehat{\Gamma}_{\mathsf{p}}}{}_{\mathsf{p}}\widehat{W}_{\mathsf{m}}$, we obtain well-defined bimodule maps

$$(19)\qquad {}_{\mathsf{n}}\widehat{V}_{\mathsf{p}}\otimes_{\widehat{\Gamma}_{\mathsf{p}}}{}_{\mathsf{p}}\widehat{W}_{\mathsf{m}}\xrightarrow{\varphi}{}_{\mathsf{n}}\widehat{V}_{\mathsf{p}}\widehat{\otimes}_{\widehat{\Gamma}_{\mathsf{p}}}{}_{\mathsf{p}}\widehat{W}_{\mathsf{m}}\xrightarrow{\widehat{\mu}_{\mathsf{p}}(\mathsf{m},\mathsf{n})}{}_{\mathsf{n}}\widehat{T}_{\mathsf{m}}\xrightarrow{\imath_{\mathsf{n},\mathsf{m}}}\mathcal{A}(\mathsf{m},\mathsf{n}).$$

Taking the direct limit by V,W and T we obtain the required composition in $\mathcal{A}$,

$$\mathcal{A}(\mathsf{p},\mathsf{n})\times\mathcal{A}(\mathsf{m},\mathsf{p})\longrightarrow\mathcal{A}(\mathsf{p},\mathsf{n})\otimes_{\widehat{\Gamma}_{\mathsf{p}}}\mathcal{A}(\mathsf{m},\mathsf{p})\longrightarrow\mathcal{A}(\mathsf{m},\mathsf{n}),$$

where the first mapping is the standard bilinear mapping and the second one is the limit bimodule mapping.

Remark 3.8. Let $v,w\in U$. To calculate the product of their classes in $\mathcal{A}(\mathsf{m},\mathsf{p})$ and $\mathcal{A}(\mathsf{p},\mathsf{n})$, respectively, we could use the following procedure. Consider two finitely generated subbimodules V,W in U such that $v\in V$, $w\in W$ (e.g. $V=\Gamma a\Gamma$, $W=\Gamma w\Gamma$). Then for $m,n\geq 0$ consider $\gamma=\gamma_{m,n}$ satisfying the conditions of Lemma 3.7, (f) and the class $\overline{v\gamma_{m,n}w}$ of $v\gamma_{m,n}w$ in ${}_{\mathsf{n}^n}(VW)_{\mathsf{m}^m}$. Then the element $\varprojlim_{m,n}\overline{v\gamma_{m,n}w}\in{}_{\mathsf{n}}(VW)_{\mathsf{m}}$ is the necessary product since all mappings in direct limits are injective.

Lemma 3.9. *The above-defined composition in $\mathcal{A}$ is associative and the image of $1\in U$ in $\mathcal{A}(\mathsf{m},\mathsf{m})$, $\mathsf{m}\in\operatorname{Specm}\Gamma$, is the identity morphism.*

Proof. For $i=1,2,3,4$ let $\mathsf{n}_i \in \operatorname{Specm}\Gamma$, $V_i \subset U$ be finitely generated Γ-bimodules and $v_i \in V_i$. For $i=1,2,3$ denote by $\overline{v}_i$ the class of v_i in $\mathcal{A}(\mathsf{n}_i, \mathsf{n}_{i+1})$. We need to prove that $(\overline{v}_3\overline{v}_2)\overline{v}_1 = \overline{v}_3(\overline{v}_2\overline{v}_1) \in \mathcal{A}(\mathsf{n}_1, \mathsf{n}_4)$. Denote $T = V_3V_2V_1$ and fix $n_1, n_4 > 1$. Using Lemma 3.7, (f) we prove that the classes of both products above in ${}_{\mathsf{n}_4^{n_4}}T_{\mathsf{n}_1^{n_1}}$ coincide, which implies the lemma. First compute a representative of $(\overline{v}_3\overline{v}_2)\overline{v}_1$ in ${}_{\mathsf{n}_4^{n_4}}T_{\mathsf{n}_1^{n_1}}$. Using Lemma 3.7, (c) choose $n_3, n_2, n_2' > 0$ such that

$$(20) \qquad {}_{\mathsf{n}_4^{n_4}}(V_3)^{\mathsf{n}_3} = {}_{\mathsf{n}_4^{n_4}}(V_3)_{\mathsf{n}_3^{n_3}}, \; {}_{\mathsf{n}_3^{n_3}}(V_2)^{\mathsf{n}_2} = {}_{\mathsf{n}_3^{n_3}}(V_2)_{\mathsf{n}_2^{n_2}}, \; {}^{\mathsf{n}_2}(V_1)_{\mathsf{n}_1^{n_1}} = {}_{\mathsf{n}_2^{n_2'}}(V_1)_{\mathsf{n}_1^{n_1}}.$$

We may assume $n_2 \geq n_2'$. With such a choice of n_i's, by Lemma 3.7, (d) we may compute the class $(\overline{v}_3\overline{v}_2)\overline{v}_1$ in ${}_{\mathsf{n}_4^{n_4}}T_{\mathsf{n}_1^{n_1}}$ as

$$(21) \qquad \widehat{\mu}(\mathsf{n}_1^{n_1}, \mathsf{n}_4^{n_4})\Big(\widehat{\mu}(\mathsf{n}_2^{n_2}, \mathsf{n}_4^{n_4})\big(v_{3\,\mathsf{n}_3} \otimes {}_{\mathsf{n}_3}v_2\big) \otimes {}_{\mathsf{n}_2}v_1\Big).$$

By Lemma 3.7, (f) this equals the class of $(v_3\gamma_3v_2)\gamma_2v_1$, $\gamma_3, \gamma_2 \in \Gamma$, where $\gamma_3 - 1$ (respectively, $\gamma_2 - 1$) belongs to a sufficiently large power of n_3 (respectively, n_2) and γ_3 (respectively, γ_2) belongs to sufficiently large powers of some finite set of maximal ideals of Γ.

Computing analogously the class of $\overline{v}_3(\overline{v}_2\overline{v}_1)$, we see that it coincides with the class of $v_3\gamma_3'(v_2\gamma_2'v_1)$, $\gamma_3', \gamma_2' \in \Gamma$, where $\gamma_3' - 1$ and $\gamma_2' - 1$ belong to sufficiently large powers of n_3 and n_2 respectively, while γ_3' (respectively, γ_2') belongs to sufficiently large powers of some finite set of maximal ideals of Γ. Then by the Chinese remainder theorem we can choose $\gamma_3 = \gamma_3', \gamma_2 = \gamma_2'$, and the associativity follows. The statement about units is analogous. □

Corollary 3.10. *The canonical map* $i_{\mathsf{m}} : \widehat{\Gamma}_{\mathsf{m}} \longrightarrow \mathcal{A}(\mathsf{m}, \mathsf{m})$ *is a monomorphism of algebras.*

Proof. Let $V = \Gamma \cdot 1 \cdot \Gamma$. Then ${}_{\mathsf{m}}\widehat{V}_{\mathsf{m}} = \widehat{\Gamma}_{\mathsf{m}}$, which proves injectivity. Remark 3.8 shows that i_{m} is a homomorphism. □

3.4. **Generalized Pierce decomposition.** We start from the following categorical statement. Assume that $\mathcal{C}$ is a $\Bbbk$-category with sums and products and $\{C_i \mid i \in \mathcal{I}\}$ is a family of objects from $\mathcal{C}$. Denote by (*) the following properties of this family:

(*) for every $j \in \mathcal{I}$ there exist finitely many $i \in \mathcal{I}$, such that $\mathcal{C}(C_i, C_j)$ and $\mathcal{C}(C_j, C_i)$ are nonzero.

Consider the vector space

$$\Pi_{\mathcal{I}} = \prod_{(i,j) \in \mathcal{I} \times \mathcal{I}} \mathcal{C}(C_j, C_i),$$

written as $\mathcal{I} \times \mathcal{I}$ matrices, where the j's correspond to columns and the i's correspond to rows. In general, the standard "column-by-row" product of such matrices is not well defined. Denote by $\mathrm{M}_{\mathcal{I}}$ the subspaces of $\Pi_{\mathcal{I}}$, formed by the matrices with finitely many nonzero elements in any column and in any row. Then the "column-by-row" product turns it into a $\Bbbk$-algebra.

Let $\mathcal{C}_{\mathcal{I}}$ be the full subcategory of $\mathcal{C}$ whose objects are $C_i, i \in \mathcal{I}$, and $\mathrm{M}_{\mathcal{I}}-\mathrm{Mod}_r$ be the full subcategory in $\mathrm{M}_{\mathcal{I}}-\mathrm{Mod}$ consisting of modules M, such that $M = \bigoplus_{i \in \mathcal{I}} e_{ii}M$, where $e_{ii} = Id_{C_i}, i \in \mathcal{I}$.

Lemma 3.11. *Assume that the family $\{C_i \mid i \in \mathcal{I}\}$ of objects from $\mathcal{C}$ satisfies (*). Then the following holds:*

(a) *There exists a canonical isomorphism of $\Bbbk$-algebras,*

$$\mathrm{M}_{\mathcal{I}} \simeq \mathrm{End}_{\mathcal{C}}(\bigoplus_{i\in\mathcal{I}} C_i),$$

where $\mathrm{End}_{\mathcal{C}}$ denotes the endomorphism ring in the category $\mathcal{C}$.

(b) *There exist a canonical equivalence provided by the following functors:*

$$F_{\mathcal{I}} : \mathcal{C}_{\mathcal{I}} - \mathrm{Mod} \simeq \mathrm{M}_{\mathcal{I}} - \mathrm{Mod}_r,\ G_{\mathcal{I}} : \mathrm{M}_{\mathcal{I}} - \mathrm{Mod}_r \simeq \mathcal{C}_{\mathcal{I}} - \mathrm{Mod},$$

where $F_{\mathcal{I}}(N) = \bigoplus_{i\in\mathcal{I}} N(C_i)$ for $N \in \mathcal{C}_{\mathcal{I}} - \mathrm{Mod}$, $F_{\mathcal{I}}(f) = \bigoplus_{i\in\mathcal{I}} f(i)$ for $f : N \to N'$, $G_{\mathcal{I}}(M)(C_i) = e_{ii}M$ with a natural action of $G_{\mathcal{I}}(M)$ on the morphisms of $\mathcal{C}_{\mathcal{I}}$, for $M \in \mathrm{M}_{\mathcal{I}} - \mathrm{Mod}_r$. For $f : M \to M'$ we have $G_{\mathcal{I}}(f)(C_i, C_j) = f_{ij} = M'(e_{jj})fM(e_{ii})$, $i, j \in \mathcal{I}$.

Proof. Every element $(f_{ij} \mid i, j \in \mathcal{I}) \in \Pi_{\mathcal{I}}$ defines canonically a homomorphism $f : \bigoplus_{i\in\mathcal{I}} C_i \longrightarrow \prod_{i\in\mathcal{I}} C_i$. From condition (*) it follows that the image of f belongs to $\bigoplus_{i\in\mathcal{I}} C_i \subset \prod_{i\in\mathcal{I}} C_i$. The second statement is standard (see for example [GR], Section 2, Example 3). □

The following statement is an analogue of the two-sided Pierce decomposition for the pair $\Gamma \subset U$. Denote by $\mathrm{M}_{\mathcal{A}}$ the algebra $\mathrm{M}_{\mathrm{Ob}\,\mathcal{A}}$.

Theorem 3.12. *For $u \in U$ denote by $[u]$ the matrix from $\mathrm{M}_{\mathcal{A}}$ such that $[u]_{\mathsf{m},\mathsf{n}} = u_{\mathsf{n},\mathsf{m}}$, $\mathsf{m}, \mathsf{n} \in \operatorname{Specm}\Gamma$, where $u_{\mathsf{n},\mathsf{m}}$ is the image of u in $\mathcal{A}(\mathsf{n}, \mathsf{m})$.*

(a) *The mapping $\Omega : U \longrightarrow \mathrm{M}_{\mathcal{A}}$, which sends $u \in U$ to $[u]$, is a homomorphism of $\Bbbk$-algebras.*

(b) *Let $D \subset \operatorname{Specm}\Gamma$ be a class of Δ-equivalence and $p_D : \mathrm{M}_{\mathcal{A}} \to \mathrm{M}_D$ the canonical projection. Then the image of Ω_D,*

$$\Omega_D : U \xrightarrow{\ \Omega\ } \mathrm{M}_{\mathcal{A}} \xrightarrow{\ p_D\ } \mathrm{M}_D,$$

is dense in M_D, where the topology on M_D is induced from Π_D, endowed with the topology of a direct product.

Proof. Following Lemma 3.6, (c), (d), the matrix $[u]$ has finitely many nonzero elements in any row and any column, hence $[u] \in \mathrm{M}_{\mathcal{A}}$. Obviously, $[1]$ is the unit matrix. Fix $v, w \in U$, $\mathsf{m}, \mathsf{n} \in \operatorname{Specm}\Gamma$, $m, n \geq 1$. Also fix V, W, T satisfying the conditions of Lemma 3.7 and $P > 0$ satisfying Lemma 3.7, (c) and (d). As in the proof of Lemma 3.7, (f) it follows that

$$\overline{vw} = \sum_{\mathsf{p}\in S(V,W)} \widehat{\mu}_{\mathsf{p}}(\mathsf{m}^m, \mathsf{n}^n)(\bar{v}_{\mathsf{p}} \otimes {}_{\mathsf{p}}\bar{w}),$$

where $\overline{vw}$ is the class of vw in ${}_{\mathsf{n}^n}T_{\mathsf{m}^m}$. Following (18) and (19) we obtain that the product $\widehat{\mu}(\mathsf{m}^m, \mathsf{n}^n)(\bar{v}_{\mathsf{p}} \otimes {}_{\mathsf{p}}\bar{w})$ converges to $v_{\mathsf{p},\mathsf{n}}w_{\mathsf{m},\mathsf{p}}$ when $m, n \longrightarrow \infty$. In the limit we obtain

$$[vw]_{\mathsf{m},\mathsf{n}} = \sum_{\mathsf{p}\in S(V,W)} [v]_{\mathsf{n},\mathsf{p}}[w]_{\mathsf{p},\mathsf{m}},$$

which proves the first statement.

To prove the second statement note that, by definition, any D is at most countable. Let $\mathsf{m}, \mathsf{n} \in D$. Consider $\mathcal{A}(\mathsf{m}, \mathsf{n})$ as a subset in M_D, formed by the matrices from M_D whose entries are all zero except the one in position (n, m). It is enough to show that $\mathcal{A}(\mathsf{m}, \mathsf{n})$ belongs to the closure of $[U]$. First note that the image of U is dense in the image of $\mathcal{A}(\mathsf{m}, \mathsf{n}) \subset \mathrm{M}_D$. Let $f \in \mathcal{A}(\mathsf{m}, \mathsf{n})$ be the class of $u \in U$. Consider an increasing sequence of finite subsets $S_i \subset D$, $i = 1, 2, \ldots$, such that $\bigcup_{i=1}^{\infty} S_i = D$ and some elements $\mu_i, \nu_i \in \Gamma$ such that

$$\begin{cases} \mu_i & \equiv & 1 & \mathrm{mod} & \mathsf{m}^i, \\ \mu_i & \equiv & 0 & \mathrm{mod} & \mathsf{m}'^i, & \mathsf{m}' \in S_i, \mathsf{m}' \neq \mathsf{m} \end{cases}$$

and

$$\begin{cases} \nu_i & \equiv & 1 & \mathrm{mod} & \mathsf{n}^i, \\ \nu_i & \equiv & 0 & \mathrm{mod} & \mathsf{n}'^i, & \mathsf{n}' \in S_i, \mathsf{n}' \neq \mathsf{n}. \end{cases}$$

Then $\Omega_D(\mu_i)$ (respectively $\Omega_D(\nu_i)$) converges in M_D to the diagonal matrix unit in position m (respectively n). Hence the sequence $\Omega_D(\nu_i u_i \mu_i) = \Omega_D(\nu_i)\ \Omega_D(u_i)\ \Omega_D(\mu_i)$ converges to f since it converges to 0 in all positions except (n, m) and to f in position (n, m). □

3.5. **Gelfand-Tsetlin modules as $\mathcal{A}$-modules.** We consider the category $\Bbbk - \mathrm{Mod}$ endowed with the discrete topology and the category $\mathcal{A} - \mathrm{Mod}_d$ of continuous functors $M : \mathcal{A} \longrightarrow \Bbbk - \mathrm{Mod}$ (called *discrete modules* in [DFO2]). That is, for any $M \in \mathcal{A} - \mathrm{Mod}_d$ and every $\mathsf{m} \in \operatorname{Specm} \Gamma$ there exists $m = m(\mathsf{m}) \geq 0$ such that $\mathsf{m}^m M(\mathsf{m}) = 0$. The corresponding subcategory of $\mathrm{M}_{\mathcal{A}} - \mathrm{Mod}$ will be denoted by $\mathrm{M}_{\mathcal{A}} - \mathrm{Mod}_d$.

Corollary 3.13. (a) *Let* $\mathsf{m}, \mathsf{n} \in \operatorname{Specm} \Gamma$ *and* P *be a Gelfand-Tsetlin module. Then the action of* U *on* P *induces the mapping* $\mathcal{A}(\mathsf{m}, \mathsf{n}) \times P(\mathsf{m}) \to P(\mathsf{n})$, *such that for every* $u \in U$ *and* $x \in P(\mathsf{m})$ *we have* ${}_{\mathsf{m}}u_{\mathsf{n}} x = p_{\mathsf{n}}(ux)$.

(b) *The action above defines an action* $\mathcal{M}_{\mathcal{A}} \times P \to P$, *and for every* $u \in U$ *and* $x \in P(\mathsf{m})$ *we have* $ux = [u]x$. *This endows* P *with the structure of an* $\mathrm{M}_{\mathcal{A}}$*-module.*

(c) *The* $\mathrm{M}_{\mathcal{A}}$*-module structure introduced in* (b) *defines a faithful functor* $\Omega_* : \mathbb{H}(U, \Gamma) \to \mathrm{M}_{\mathcal{A}} - \mathrm{Mod}_d$, *which is the identity both on objects and morphisms.*

Proof. Consider a sequence $\nu_n u \mu_n \in \Gamma u \Gamma$, $n \geq 1$, such that $\nu_n \to 1$ in $\widehat{\Gamma}_{\mathsf{n}}$ and $\nu_n \to 0$ in $\widehat{\Gamma}_{\mathsf{n}'}$ for any $\mathsf{n}' \in X_u(\mathsf{m})$, $\mathsf{n}' \neq \mathsf{n}$ and such that $\mu_n \to 1$ in $\widehat{\Gamma}_{\mathsf{m}}$ and $\mu_n \to 0$ in $\widehat{\Gamma}_{\mathsf{m}'}$ for any $\mathsf{m}' \in X^u(\mathsf{n})$, $\mathsf{m}' \neq \mathsf{m}$, while n tends to ∞. The image of $\nu_n u \mu_n$ in $\mathcal{A}(\mathsf{m}, \mathsf{n})$ tends to ${}_{\mathsf{m}}u_{\mathsf{n}}$ and $\nu_n u \mu_n x \in P$ stabilizes for large enough n and equals $p_{\mathsf{n}}(ux)$.

To prove that this action defines on P the structure of an $\mathrm{M}_{\mathcal{A}}$-module, it is enough to show that $[uv] \cdot x = [u] \cdot ([v \cdot x])$, where $u, v \in U$, $x \in P(\mathsf{m})$ and the action "$\cdot$" is extended from the action defined in (a) by the "row to column" rule. Denote $S(u, v) = X^u \cap X_v$. We have

$$\begin{aligned} {}_{\mathsf{m}}(uv)_{\mathsf{n}} \cdot x &= p_{\mathsf{n}}((uv)x) = p_{\mathsf{n}}(u(vx)) \\ &= \sum_{\mathsf{p} \in S(u,v)} p_{\mathsf{n}}(u p_{\mathsf{p}}(vx)) = \sum_{\mathsf{p} \in S(u,v)} {}_{\mathsf{p}}u_{\mathsf{n}} \cdot (p_{\mathsf{p}}(vx)) = \sum_{\mathsf{p} \in S(u,v)} {}_{\mathsf{p}}u_{\mathsf{n}} \cdot ({}_{\mathsf{m}}v_{\mathsf{p}} \cdot x), \end{aligned}$$

which proves (b). Statement (c) is obvious. □

We define the functor $\mathbb{G} : \mathbb{H}(U,\Gamma) \longrightarrow \mathcal{A} - \mathrm{Mod}_d$ as the composition

$$\mathbb{G} : \mathbb{H}(U,\Gamma) \xrightarrow{[\ ]_*} \mathrm{M}_{\mathcal{A}} - \mathrm{Mod}_d \xrightarrow{G_{\mathcal{A}}} \mathcal{A} - \mathrm{Mod}_d\,. \tag{22}$$

For a discrete $\mathcal{A}-$module M define a Gelfand-Tsetlin $U-$module $\mathbb{F}(M)$ by setting

$$\mathbb{F}(M) = \bigoplus_{\mathsf{m}\in \operatorname{Specm}\Gamma} M(\mathsf{m}). \tag{23}$$

For $x \in M(\mathsf{m})$ and $u \in U$ set

$$ux = \sum_{\mathsf{n}\in \operatorname{Specm}\Gamma} u_{\mathsf{m},\mathsf{n}}x.$$

If $f : M \longrightarrow N$ is a morphism in $\mathcal{A} - \mathrm{mod}_d$, then define $\mathbb{F}(f) = \bigoplus_{\mathsf{m}\in \operatorname{Specm}\Gamma} f(\mathsf{m})$. Consider the composition

$$\mathcal{A} - \mathrm{Mod}_d \xrightarrow{F_{\mathcal{A}}} \mathrm{M}_{\mathcal{A}} - \mathrm{Mod}_d \xrightarrow{[\ ]^*} U - \mathrm{Mod}$$

of the canonical equivalence $F_{\mathcal{A}}$ (Lemma 3.11, (b)) and the functor of restriction Ω^*, induced by the homomorphism $[\ ]$. Since the image of this functor belongs to $\mathbb{H}(U,\Gamma)$, it induces the functor $\Omega^*_d : \mathrm{M}_{\mathcal{A}} - \mathrm{Mod}_d \longrightarrow \mathbb{H}(U,\Gamma)$. Then the functors $\Omega^*_d \circ F_{\mathcal{A}}$ and $\mathbb{F}$ are isomorphic.

Theorem 3.14 ([DFO2], Theorem 17)**.** *The functors $\mathbb{F}$ and $\mathbb{G}$ are mutually quasi-inverse and hence define an equivalence of categories $\mathcal{A}-\mathrm{mod}_d$ and $\mathbb{H}(U,\Gamma)$. Moreover, $\mathbb{F}$ induces a functorial isomorphism*

$$\mathrm{Ext}^1_{\mathcal{A}}(\mathbb{F}(X), \mathbb{F}(Y)) \simeq \mathrm{Ext}^1_U(X,Y),$$

for every $X, Y \in \mathbb{H}(U,\Gamma)$.

Proof. It is enough to prove that the functors $[\ ]_* : \mathbb{H}(U,\Gamma) \longrightarrow \mathrm{M}_{\mathcal{A}} - \mathrm{Mod}_d$ and $[\ ]^*_d : \mathrm{M}_{\mathcal{A}} - \mathrm{Mod}_d \longrightarrow \mathbb{H}(U,\Gamma)$ are quasi-inverse. By definition both functors are identical on objects and on morphisms, which implies the first statement. The statement about Ext^1 follows from the fact that both $\mathbb{F}$ and $\mathbb{G}$ preserve exact sequences and from the easy observation that $\mathcal{A} - \mathrm{mod}_d$ is a Serre subcategory of $\mathcal{A} - \mathrm{mod}$. $\square$

We will identify a discrete $\mathcal{A}$-module N with the corresponding Gelfand-Tsetlin module $\mathbb{F}(N)$. For $\mathsf{m} \in \operatorname{Specm}\Gamma$ denote by $\widehat{\mathsf{m}}$ the completion of m. Consider the two-sided ideal $I \subset \mathcal{A}$ generated by $\widehat{\mathsf{m}}$ for all $\mathsf{m} \in \operatorname{Specm}\Gamma$ and set $\mathcal{A}(W) = \mathcal{A}/I$. Then Theorem 3.14 implies, by restriction, the following corollary.

Corollary 3.15. *The categories $\mathbb{H}W(U,\Gamma)$ and $\mathcal{A}(W) - \mathrm{mod}_d$ are equivalent.*

The following statement is standard. It can be shown using the standard restriction and induction technique. We leave the details to the reader.

Lemma 3.16. *Let M be a simple $\mathcal{A}$-module, $\mathsf{m} \in \operatorname{Specm}\Gamma$, $M(\mathsf{m}) \neq 0$, and N a simple $\mathcal{A}(\mathsf{m},\mathsf{m})$-module. Then the correspondences*

$$M \longmapsto M|_{\mathcal{A}(\mathsf{m},\mathsf{m})} \ \text{ and } \ N \longmapsto (\mathcal{A} \otimes_{\mathcal{A}(\mathsf{m},\mathsf{m})} N)/J,$$

where J is the unique maximal $\mathcal{A}$-submodule of $\mathcal{A}\otimes_{\mathcal{A}(\mathsf{m},\mathsf{m})} N$, realizes a bijection between the sets of isomorphism classes of simple $\mathcal{A}(\mathsf{m},\mathsf{m})$-modules and isomorphism classes of simple $\mathcal{A}$-modules M such that $M(\mathsf{m}) \neq 0$.

Lemma 3.16 shows usefulness of the category $\mathcal{A}$ for the study of simple Gelfand-Tsetlin modules over U.

The subalgebra Γ is called *big* at $\mathsf{m} \in \operatorname{Specm}\Gamma$ if $\mathcal{A}(\mathsf{m},\mathsf{m})$ is finitely generated as a right $\widehat{\Gamma}_{\mathsf{m}}-$module. The importance of this concept is described in the following statement.

Lemma 3.17 ([DFO2], Corollary 19)**.** *If Γ is big at $\mathsf{m} \in \operatorname{Specm}\Gamma$, then there exist only finitely many nonisomorphic irreducible Gelfand-Tsetlin $U-$modules M such that $M(\mathsf{m}) \neq 0$. For any such module M holds $\dim_{\Bbbk} M(\mathsf{m}) < \infty$.*

3.6. **Examples of computation of $\mathcal{A}$.** The first example was given at the beginning of Section 3, namely, the presentation of a basic associative algebra as a quiver with relations.

Now we illustrate our techniques by considering representations of skew group algebras and by obtaining well-known results on the irreducible representations of a finite group $G = N \rtimes H$ with abelian N.

The case of a skew group algebra is summarized in the following statement.

Proposition 3.18. *Let Λ be a commutative noetherian algebra, $\mathcal{M}$ a group acting on Λ, and $U = \Lambda * \mathcal{M}$ the corresponding skew-group algebra (see Subsection 2.2). For $\mathsf{m}, \mathsf{n} \in \operatorname{Specm}\Lambda$ set*

$$\mathcal{M}(\mathsf{m},\mathsf{n}) = \{\varphi \in \mathcal{M} \mid \varphi \cdot \mathsf{m} = \mathsf{n}\}.$$

Then $\Lambda \subset U$ is a Harish-Chandra subalgebra and the following hold:

(a) *Blocks of the category $\mathcal{A} = \mathcal{A}_{U,\Lambda}$ correspond to elements from $\operatorname{Specm}\Lambda/\mathcal{M}$ (i.e. to orbits of $\mathcal{M}$ on $\operatorname{Specm}\Lambda$). For $D \in \operatorname{Spec}\Lambda/\mathcal{M}$ the set of objects of $\mathcal{A}_D$ coincides with D, all objects in D are isomorphic and for $\mathsf{m} \in D$ we have $\mathcal{A}(\mathsf{m},\mathsf{m}) \simeq \widehat{\Lambda}_{\mathsf{m}} * \mathcal{M}(\mathsf{m},\mathsf{m})$.*

(b) *Every block contains a simple Gelfand-Tsetlin module. Moreover, if $\mathsf{m} \in \operatorname{Specm}\Lambda$ belongs to a block, then simple modules in this block are in a natural bijection with simple $\widehat{\Lambda}_{\mathsf{m}} * \mathcal{M}(\mathsf{m},\mathsf{m})$-modules.*

(c) *If the action of $\mathcal{M}$ on D is free, then $\mathcal{A}_D-\operatorname{mod}_d$ is equivalent to the category of finite dimensional modules over $\widehat{\Lambda}_{\mathsf{m}}$ for any $\mathsf{m} \in D$.*

Proof. We compute $\mathcal{A}(\mathsf{m},\mathsf{n})$ using Definition 3.5. Note that every finitely generated Λ-bimodule in U is contained in a bimodule $V = \sum\limits_{\varphi\in S} \Lambda\varphi\Lambda$ for some finite $S \subset \mathcal{M}$. By Lemma 2.4, (c) we have $V = \bigoplus\limits_{\varphi\in S} \Lambda\varphi$ and

$$V/(\mathsf{n}^n V + V\mathsf{m}^m) \simeq \bigoplus_{\varphi\in S\cap\mathcal{M}(\mathsf{m},\mathsf{n})} (\Lambda/\mathsf{n}^k)\varphi, \text{ where } k = \min\{m,n\}.$$

In the definition of $\mathcal{A}(\mathsf{m},\mathsf{n})$ we can consider only V containing $\mathcal{M}(\mathsf{m},\mathsf{n})$, hence taking the inverse limit we obtain ${}_{\mathsf{n}}\widehat{V}_{\mathsf{m}} \simeq \widehat{\Lambda}_{\mathsf{n}}\varphi \simeq \varphi\widehat{\Lambda}_{\mathsf{m}}$. The last isomorphism is defined as the inverse limit of the isomorphisms of Λ-bimodules $(\Lambda/\mathsf{n}^k)\varphi \simeq \varphi(\Lambda/\mathsf{m}^k), k \geq 1$. The direct limit gives

$$\mathcal{A}(\mathsf{m},\mathsf{n}) \simeq \bigoplus_{\varphi\in\mathcal{M}(\mathsf{m},\mathsf{n})} \widehat{\Lambda}_{\mathsf{n}}\varphi \simeq \bigoplus_{\varphi\in\mathcal{M}(\mathsf{m},\mathsf{n})} \varphi\widehat{\Lambda}_{\mathsf{m}}.$$

The equality $(\Lambda\varphi)(\Lambda\psi) = \Lambda\varphi\psi$ together with the isomorphism above defines the composition in $\mathcal{A}$. In particular, if $\mathcal{M}(\mathsf{m}, \mathsf{n}) \neq \varnothing$, then every $\varphi \in \mathcal{M}(\mathsf{m}, \mathsf{n})$ defines an isomorphism between m and n, and $\mathcal{A}(\mathsf{m}, \mathsf{m})$ is isomorphic to $\bigoplus_{\varphi \in \mathcal{M}(\mathsf{m},\mathsf{m})} \widehat{\Lambda}_{\mathsf{m}}\varphi$ as a Λ-bimodule, which is in turn isomorphic to $\widehat{\Lambda}_{\mathsf{m}} * \mathcal{M}(\mathsf{m}, \mathsf{m})$ as an algebra. This proves (a).

By Lemma 3.16, statement (a) reduces the problem of classification of simple Gelfand-Tsetlin U-modules to the problem of classification of simple $\widehat{\Lambda}_{\mathsf{m}} * \mathcal{M}(\mathsf{m}, \mathsf{m})$-modules, $\mathsf{m} \in \operatorname{Specm} \Lambda$. But the Λ-bimodule $\widehat{\Lambda}_{\mathsf{m}}$ is a direct summand of $\widehat{\Lambda}_{\mathsf{m}} * \mathcal{M}(\mathsf{m}, \mathsf{m})$ as a $\widehat{\Lambda}_{\mathsf{m}}$-bimodule, hence $U/U\mathsf{m} \neq 0$. Statement (b) follows. Statement (c) is obvious, since in this case $\mathcal{M}(\mathsf{m}, \mathsf{m}) = \{e\}$. □

Next we will show how the theory of Gelfand-Tsetlin modules can be applied to study group representations induced from subgroups (cf. [FO3]). Let G be a finite group and N an abelian normal subgroup in G. We allow the field $\Bbbk$ to have a positive characteristic which is coprime with $|G|$. We set $U = \Bbbk[G]$ and $\Gamma = \Bbbk[N]$. Obviously, Γ is a Harish-Chandra subalgebra in U. Denote by $\check{N}$ the set of characters of N. The group G acts on N by conjugation which induces an action of G on $\check{N}$. Denote by $\operatorname{St}_G \chi$ the stabilizer of $\chi \in \check{N}$.

Proposition 3.19 ([FO3]). *Let $\mathcal{Y} = \mathcal{Y}(G, N)$ be the groupoid such that*

$$\operatorname{Ob} \mathcal{Y} = \check{N}, \quad \mathcal{Y}(\chi_1, \chi_2) = \{g \in G \mid \chi_1 = \chi_2^g\}, \quad \chi_1, \chi_2 \in \check{N},$$

and $\mathcal{N}$ be a subgroupoid of $\mathcal{Y}$, such that $\mathcal{N}(\chi, \chi) = N$ for any $\chi \in \check{N}$ and empty otherwise, $\mathcal{X} = \mathcal{Y}/\mathcal{N}$ and $\Bbbk\mathcal{X}$ be its $\Bbbk$-linear envelope. Then there exists a canonical isomorphism of categories $\Phi : \Bbbk\mathcal{X} \longrightarrow \mathcal{A}$ which sends a character from $\check{N}$ to its kernel. In addition

1. $\mathcal{A}(\chi, \chi) \simeq \Bbbk[\operatorname{St}_G \chi]$.
2. *$\chi_i, \chi_j \in \operatorname{Ob} \mathcal{A}$ are isomorphic if and only if $\chi_i^g = \chi_j$ for some $g \in G$.*

4. Representations of Galois orders

4.1. **Extension of characters for Galois orders.** If $\ell \in \operatorname{Specm} \bar{\Gamma}$ projects onto $\mathsf{m} \in \operatorname{Specm} \Gamma$, then we will write $\ell = \ell_{\mathsf{m}}$, $\pi(\ell) = \mathsf{m}$ and say that ℓ_{m} is lying over m. Note that given $\mathsf{m} \in \operatorname{Specm} \Gamma$ the number of different ℓ_{m} is finite due to Corollary 2.2. Given $\mathsf{m} \in \operatorname{Specm} \Gamma$ fix $\ell_{\mathsf{m}} \in \operatorname{Specm} \bar{\Gamma}$ and denote by $\mathcal{M}_{\ell_{\mathsf{m}}} \subset \mathcal{M}$ ($G_{\ell_{\mathsf{m}}} \subset G$) the stabilizer of ℓ_{m} in $\mathcal{M}$ (in G). The ideal m defines $\mathcal{M}_{\ell_{\mathsf{m}}}$ and $G_{\ell_{\mathsf{m}}}$ uniquely up to G-conjugation. This allows us to use the notation $\mathcal{M}_{\mathsf{m}}$ instead of $\mathcal{M}_{\ell_{\mathsf{m}}}$ and G_{m} instead of $G_{\ell_{\mathsf{m}}}$.

Let $\mathsf{m} \in \operatorname{Specm} \Gamma$ and $f \in \Gamma$. Then $f(\mathsf{m})$ will denote the "evaluation" of f in m, i.e. the image of f under the natural projection $\Gamma \to \Gamma/\mathsf{m}$. If $\pi(\ell) = \mathsf{m}$, then set $f(\ell) = f(\mathsf{m})$.

Let ℓ_{m} and ℓ_{n} be some maximal ideals of $\bar{\Gamma}$ lying over m and n, respectively. Denote by $S(\mathsf{m}, \mathsf{n})$ the following G-invariant subset in $\mathcal{M}$:

$$S(\mathsf{m}, \mathsf{n}) = \{\mu \in \mathcal{M} \mid \ell_{\mathsf{n}} \in GmG \cdot \ell_{\mathsf{m}}\} \tag{24}$$

$$= \{\mu \in \mathcal{M} \mid g_2\ell_{\mathsf{n}} = \mu g_1 \ell_{\mathsf{m}} \text{ for some } g_1, g_2 \in G\}.$$

Obviously, this definition does not depend on the choice of ℓ_{m} and ℓ_{n}.

Lemma 4.1. *Let* $\mathsf{m} \in \operatorname{Specm}\Gamma$. *Assume that* $\mathcal{M}$ *is a group. Then*

(a) $|\mathcal{M}_{\mathsf{m}}| < \infty$ *if and only if for some* $\mathsf{n} \in \operatorname{Specm}\Gamma$ *at least one from the sets* $S(\mathsf{m},\mathsf{n})$ *or* $S(\mathsf{n},\mathsf{m})$ *is nonempty and finite. If* $|\mathcal{M}_{\mathsf{m}}| < \infty$, $S(\mathsf{m},\mathsf{n}) \neq \varnothing$ *and* $S(\mathsf{n},\mathsf{m}) \neq \varnothing$, *then* $|\mathcal{M}_{\mathsf{n}}| < \infty$ *and* $|S(\mathsf{m},\mathsf{n})| = |S(\mathsf{n},\mathsf{m})|$.

(b)
$$|S(\mathsf{m},\mathsf{n})| \leq \frac{|G|^2|\mathcal{M}_{\mathsf{m}}|}{|G_{\mathsf{m}}||G_{\mathsf{n}}|}.$$

(c)
$$|S(\mathsf{m},\mathsf{n})/G| \leq |\{\varphi \in \mathcal{M} \mid \pi(\varphi\ell_{\mathsf{m}}) = \mathsf{n}\}|.$$

Proof. The set $S(\mathsf{m},\mathsf{n})$ is infinite if and only if for some $g \in G$, ℓ_{m} and ℓ_{n} the set
$$\mathcal{M}(\ell_{\mathsf{m}},\ell_{\mathsf{n}},g) = \{\mu \in \mathcal{M} \mid \mu g\ell_{\mathsf{m}} = \ell_{\mathsf{n}}\}$$
is infinite. In this case for any $\tau \in \mathcal{M}(\ell_{\mathsf{m}},\ell_{\mathsf{n}},g)$ we have
$$g^{-1}\tau^{-1}\mathcal{M}(\ell_{\mathsf{m}},\ell_{\mathsf{n}},g)g \subset \mathcal{M}_{\mathsf{m}},\ \mathcal{M}(\ell_{\mathsf{m}},\ell_{\mathsf{n}},g)\tau^{-1} \subset \mathcal{M}_{\mathsf{n}}.$$

Observe that $\mu \in S(\mathsf{m},\mathsf{n})$ if and only if $\mu^{-1} \in S(\mathsf{n},\mathsf{m})$. This proves (a). If $|\mathcal{M}_{\mathsf{m}}| < \infty$, then there exist at most $|\mathcal{M}_{\mathsf{m}}|$ elements $\varphi \in \mathcal{M}$ such that $\ell_{\mathsf{n}} = \varphi\ell_{\mathsf{m}}$. Considering the G-orbits of ℓ_{m} and ℓ_{n}, which have lengths $\dfrac{|G|}{|G_{\mathsf{m}}|}$ and $\dfrac{|G|}{|G_{\mathsf{n}}|}$, respectively, we obtain inequality (b). Assume $\varphi g_1\ell_{\mathsf{m}} = g_2\ell_{\mathsf{n}}$, $g_1, g_2 \in G$, $\varphi \in \mathcal{M}$. Then $(g_1^{-1}\varphi g_1)\ell_{\mathsf{m}} = g_1^{-1}g_2\ell_{\mathsf{n}}$, which proves (c). $\square$

The property of the Galois ring U to be a Galois order has the following immediate implication on the representation theory of U. We will consider right Galois orders. The case of left orders is analogous. In addition we are using the identification stated in Remark 2.6.

Lemma 4.2. *Let* U *be a Galois ring over* Γ, Γ *a noetherian algebra which is a right Harish-Chandra subalgebra of* U, $\mathsf{m} \in \operatorname{Specm}\Gamma$ *such that* $\mathcal{M}_{\mathsf{m}}$ *is finite, and* $S = S(\mathsf{m},\mathsf{m})$. *If* $U = U\mathsf{m}$, *then for every* $k \geq 1$ *there exist* $\gamma_k \in \Gamma \setminus \mathsf{m}$, $v_j \in U$, $\nu_j \in \mathsf{m}^k$, $j = 1,\dots,N$, *such that*

$$\gamma_k = \sum_{j=1}^{N} v_j\nu_j \tag{25}$$

and $\operatorname{supp} v_j \subset S$, $j = 1,\dots,n$.

Proof. The condition $U = U\mathsf{m}$ is equivalent to the fact that $1 \in U\mathsf{m}$, i.e. we have

$$1 = \sum_{i=1}^{n} u_i\mu_i \text{ for some } u_i \in U, \mu_i \in \mathsf{m}. \tag{26}$$

We use induction on k to prove the statement of the lemma without the condition $\operatorname{supp} v_i \subset S$, $i = 1,\dots,n$. The base $k = 1$ is formula (26). The induction step is obtained by plugging the right hand side of (26) into (25): if

$$1 = \sum_{j=1}^{N} w_j\nu_j,\ \text{where } w_j \in U, \nu_j \in \mathsf{m}^k, j = 1,\dots,N, \tag{27}$$

then

$$1 = \sum_{j=1}^{N} w_j \nu_j = \sum_{j=1}^{N} w_j \cdot 1 \cdot \nu_j = \sum_{j=1}^{N} w_j \big(\sum_{i=1}^{n} u_i \mu_i \big) \nu_j = \sum_{j=1}^{N} \sum_{i=1}^{n} w_j u_i (\mu_i \nu_j).$$

This proves the induction step, since all $\mu_i \nu_j \in \mathsf{m}^{k+1}$.

Denote $T = \bigcup_{i=1}^{N} \operatorname{supp} w_i \setminus S$. Since $T \bigcap S(\mathsf{m}, \mathsf{m}) = \varnothing$, the ideals ℓ_{m}^t and ℓ_{m} belong to different G-orbits for every $t \in T$. Then there exists $f \in \Gamma$ such that $f(\ell_{\mathsf{m}}) \neq f(\ell_{\mathsf{m}}^t)$ for every $t \in T$. Consider the element $f_T = f_T^r$ as in Lemma 2.10. Without loss of generality we can assume that (see Lemma 2.10, (c))

$$\prod_{t \in T} (f(\ell_{\mathsf{m}}) - f^{t^{-1}}(\ell_{\mathsf{m}})) = \prod_{t \in T} (f(\ell_{\mathsf{m}}) - f(\ell_{\mathsf{m}}^t)) = 1.$$

In particular, this implies that $f_T \in 1 + \mathsf{m} \otimes \Gamma + \Gamma \otimes \mathsf{m}$. Then by Lemma 2.10, (c), $\gamma_k = f_T \cdot 1 \in 1 + \mathsf{m}$. In addition, by Lemma 2.10, (b), $v_j = f_T \cdot w_j$ belongs to $U(S)$. Moreover, $f_T \cdot (w_j \nu_j) = (f_T \cdot w_j)\nu_j$ due to the fact that $f_T \in 1 + \mathsf{m} \otimes \Gamma + \Gamma \otimes \mathsf{m}$ and the commutativity of Γ. Applying f_T to equality (27) we obtain

$$\gamma_k = \sum_{j=1}^{N} v_j \nu_j, \text{ where } \gamma_k \in \Gamma \setminus \mathsf{m},\ v_j \in U(S), \nu_j \in \mathsf{m}^k,$$

which completes the proof. □

Corollary 4.3. *Let Γ be a noetherian algebra, U a right Galois order over Γ, and $\mathsf{m} \in \operatorname{Specm} \Gamma$, such that $|\mathcal{M}_{\mathsf{m}}| < \infty$. Then $U\mathsf{m} \neq U$. In particular, there exists a simple left Gelfand-Tsetlin U-module M generated by $x \in M$ such that $\mathsf{m} \cdot x = 0$.*

Proof. By Lemma 4.1, (a) the set $S = S(\mathsf{m}, \mathsf{m})$ is finite. Since U is a Galois order, by Theorem 2.11, (b), the right Γ-module $U(S)$ is finitely generated. Applying the Artin-Rees Lemma (Theorem 8.5 of [Mat]) to the right Γ- module $U(S)$ and its submodule Γ, we conclude that there exists $c \geq 0$ such that for every $k \geq c$,

$$U(S)\mathsf{m}^k \cap \Gamma = (U(S)\mathsf{m}^c \cap \Gamma)\mathsf{m}^{k-c}.$$

But by Lemma 4.2, for every $k > c$ there exists $\gamma_k \in U(S)\mathsf{m}^k \cap \Gamma$ such that $\gamma_k \notin \mathsf{m}$. Hence $\gamma_k \notin (U(S)\mathsf{m}^c \cap \Gamma)\mathsf{m}^{k-c}$, provided that $k - c > 0$. The obtained contradiction shows that $U \neq U\mathsf{m}$.

If $U\mathsf{m} \neq U$, then $U/U\mathsf{m}$ has a nonzero simple quotient M which is a Gelfand-Tsetlin module by Lemma 3.2, (f). Moreover, M is generated by the image v of 1 and the latter satisfies $\mathsf{m}v = 0$. □

Lemma 4.4. *Let M be a finitely generated right Γ-module. Then the set of $\mathsf{m} \in \operatorname{Specm} \Gamma$ such that $\operatorname{Tor}_1^\Gamma(M, \Gamma/\mathsf{m}) = 0$ contains an open dense subset $X \subset \operatorname{Specm} \Gamma$.*

Proof. Let $R^\bullet \ : \ \ldots \xrightarrow{d^2} \Gamma^{n_2} \xrightarrow{d^1} \Gamma^{n_1} \xrightarrow{d^0} \Gamma^{n_0} \longrightarrow 0 \ldots$ be a free resolution of M, where all d^i are understood as matrices over Γ. Since K is a flat Γ-module, we construct the resolution $R^\bullet \otimes_\Gamma K$ of $M \otimes_\Gamma K$ consisting of K-vector spaces. Set $l = \dim_K M \otimes_\Gamma K$. Then the conditions $\operatorname{Tor}_0^\Gamma(M, K) = M \otimes_\Gamma K$ and $\operatorname{Tor}_1^\Gamma(M, K) = 0$ imply that the matrices of $d^0 \otimes \mathrm{id}_K$ and $d^1 \otimes \mathrm{id}_K$ have the maximal possible rank, $n_0 - l$ and $n_1 - n_0 - l$, respectively. Therefore there exist maximal minors $\gamma_0, \gamma_1 \in \Gamma$ in these matrices which are nonzero. Then for $\mathsf{m} \in \operatorname{Specm} \Gamma$ the matrices of differentials in $R^\bullet \otimes_\Gamma \Gamma/\mathsf{m}$ are obtained from those in $R^\bullet$ by specialization of the coefficients at m. Hence, for any m from the open set defined by the conditions $\gamma_1(\mathsf{m}) \neq 0$ and $\gamma_2(\mathsf{m}) \neq 0$, we have $\operatorname{Tor}_1^\Gamma(M, \Gamma/\mathsf{m}) = 0$. □

We have the following version of the Harish-Chandra theorem.

Proposition 4.5. *Let U be a Galois order. Then for a nonzero $u \in U$ there exists a massive set $\Omega_u \subset \operatorname{Specm}\Gamma$ such that for every $\mathsf{m} \in \Omega_u$ the image $\bar{u}$ of u in $U/U\mathsf{m}$ is nonzero.*

Proof. Let $\mathsf{m} \in \operatorname{Specm}\Gamma$, $N = u\Gamma$. Then $\bar{u} = 0$ if and only if $u \cdot (1 + U\mathsf{m}) = 0$, or, equivalently, if $u = \sum\limits_{i=1}^{n} u_i \mu_i$ for some $u_i \in U, \mu_i \in \mathsf{m}$, $i = 1, \dots, n$. Denote by $S = \bigcup_{i=1}^n \operatorname{supp} u_i$ and $M = U(S)$ ((3)), in particular $u_1, \dots, u_n \in U(S)$. Consider the exact sequence of right Γ-modules,

$$0 \longrightarrow u\Gamma \longrightarrow U(S) \longrightarrow U(S)/u\Gamma \longrightarrow 0.$$

It becomes nonexact after tensoring with Γ/m. Indeed, $u\Gamma \otimes_\Gamma \Gamma/\mathsf{m}$ is a one dimensional space, but the image of $u \otimes \Gamma/\mathsf{m}$ in $U(S) \otimes \Gamma/\mathsf{m}$ is zero:

$$u \otimes \bar{1} = (\sum_{i=1}^{n} u_i \mu_i) \otimes \bar{1} = \sum_{i=1}^{n} u_i \otimes \mu_i \bar{1} = 0,$$

so $\operatorname{Tor}_1^\Gamma(U(S)/u\Gamma, \Gamma/\mathsf{m}) \neq 0$. Then for a fixed G-invariant finite $S \subset \operatorname{Specm}\Gamma$ set

$$Z(u, S) = \{\mathsf{m} \in \operatorname{Specm}\Gamma \mid \operatorname{Tor}_1^\Gamma(U(S)/u\Gamma, \Gamma/\mathsf{m}) \neq 0\}.$$

Following Lemma 4.4 this set is closed and $Z(u, S) \neq \operatorname{Specm}\Gamma$. Then we can set Ω_u to be equal to the complement of $\bigcup_{S \subset \mathcal{M}} Z(u, S)$. □

4.2. **Finiteness of extensions of characters for Galois orders.** In this subsection we assume that U is a Galois order over Γ, where Γ is normal noetherian over $\Bbbk$. In particular, $\Gamma = \tilde{\Gamma} = U_e$ and $\bar{\Gamma}$ is finite over Γ by Proposition 2.1. Also Γ is a Harish-Chandra subalgebra by Proposition 2.9.

Lemma 4.6. *Let $\mathsf{m}, \mathsf{n} \in \operatorname{Specm}\Gamma$, $S = S(\mathsf{m}, \mathsf{n})$, $m, n \geq 0$. Then $U = U(S) + \mathsf{n}^n U + U\mathsf{m}^m$. Moreover, for every $u \in U$ and any $k \geq 0$ there exists $u_k \in U(S)$ such that $u \in u_k + \mathsf{n}^{[k/2]} u\Gamma + \Gamma u \mathsf{m}^{[k/2]}$.*

Proof. Fix $u \in U$ and denote $T = \operatorname{supp} u \setminus S$. If $T = \varnothing$, then $u \in U(S)$. Let $T \neq \varnothing$. We show by induction on k that there exists $u_k \in U(S)$ such that

$$u \in u_k + \sum_{i=0}^{k} \mathsf{n}^{k-i} u \mathsf{m}^i \text{ (that is, } u_{k+1} - u_k \in \sum_{i=0}^{k} \mathsf{n}^{k-i} u \mathsf{m}^i). \tag{28}$$

Since ℓ_m^t and ℓ_n belong to different G-orbits if $t \notin S$, there exists $f \in \Gamma$ such that $f(\ell_\mathsf{n}) \neq f(\ell_\mathsf{m}^t)$ for every $t \in T$. Without loss of generality we can assume that $\prod\limits_{t \in T} (f(\ell_\mathsf{n}) - f^{t^{-1}}(\ell_\mathsf{m})) = 1$, which implies $f_T \in 1 + \mathsf{n} \otimes \Gamma + \Gamma \otimes \mathsf{m}$. Set $u_1 = f_T \cdot u$. Then u_1 belongs to $u + \mathsf{n}u\Gamma + \Gamma u \mathsf{m}$ and, hence, $u \in u_1 + \mathsf{n}u\Gamma + \Gamma u \mathsf{m}$. Moreover, $u_1 \in U(S)$ by Lemma 2.10, (b). Now we prove the induction step $k \to k+1$.

Applying (28) to itself recursively, we have

$$\begin{aligned} u \in u_k + \sum_{i=0}^{k} \mathsf{n}^{k-i}(u_k + \sum_{j=0}^{k} \mathsf{n}^{k-j} u \mathsf{m}^j)\mathsf{m}^i \\ \subset u_k + \sum_{i=0}^{k} \mathsf{n}^{k-i} u_k \mathsf{m}^i + \sum_{i=0}^{k+1} \mathsf{n}^{k+1-i} u \mathsf{m}^i, \end{aligned}$$

which proves the induction step, since $u_k + \sum_{i=0}^{k} \mathsf{n}^{k-i} u_k \mathsf{m}^i \subset U(S)$. □

Theorem 4.7. *For any* $\mathsf{m}, \mathsf{n} \in \operatorname{Specm}\Gamma$ *such that* $S = S(\mathsf{m}, \mathsf{n})$ *is finite, the completed* $\widehat{\Gamma}_{\mathsf{n}} - \widehat{\Gamma}_{\mathsf{m}}$*-bimodule* $\mathcal{A}(\mathsf{m}, \mathsf{n})$ *(see Subsection* 3.5*) is finitely generated. Moreover,* $\mathcal{A}(\mathsf{m}, \mathsf{n})$ *coincides with the image of* $U(S)$ *in* $\mathcal{A}$ *(identifying the elements of* U *with the corresponding morphisms in* $\mathcal{A}$*). Furthermore,* $\mathcal{A}(\mathsf{m}, \mathsf{n})$ *is finitely generated both as a right* $\widehat{\Gamma}_{\mathsf{m}}$*-module and as a left* $\widehat{\Gamma}_{\mathsf{n}}$*-module.*

Proof. In view of Lemma 4.6 and formula (3.5) we have an embedding,

$$\mathcal{A}(\mathsf{m}, \mathsf{n}) \subset \varprojlim_{n,m} U(S)/\big((\mathsf{n}^n U + U\mathsf{m}^m) \cap U(S)\big). \tag{29}$$

Since $U(S)$ is a noetherian Γ-bimodule by Theorem 2.11, the generators of $U(S)$ as a Γ-bimodule generate any $U(S)/((\mathsf{n}^n U + U\mathsf{m}^m) \cap U(S))$ as a Γ-bimodule, and hence generate $\mathcal{A}(\mathsf{m}, \mathsf{n})$ as a complete $\widehat{\Gamma}_{\mathsf{n}} - \widehat{\Gamma}_{\mathsf{m}}$-bimodule ([Mat], Theorem 8.7). The statement that $\mathcal{A}(\mathsf{m}, \mathsf{n})$ is finitely generated both from the left and from the right follows from Theorem 2.11, (b) and from Theorem 8.7 of [Mat]. This completes the proof. □

Note that Theorem 4.7 and Definition 3.5 imply that the embedding (29) is an isomorphism. From Lemma 3.17 we have.

Corollary 4.8. *Under the assuptions of Theorem* 4.7*,* Γ *is big at* m*. In particular, if* $\mathsf{m} \in \operatorname{Specm}\Gamma$*, then there exist only finitely many nonisomorphic extensions of* m *to simple* U*-modules.*

4.3. **Proof of the Main Theorem.** Corollary 4.3 implies the first statement of the Main Theorem. The condition $|\mathcal{M}_{\mathsf{m}}| < \infty$ implies the finiteness of $S(\mathsf{m}, \mathsf{m})$ (Lemma 4.1, (a)). Consider $\chi : \Gamma \longrightarrow \Bbbk$ such that $\mathsf{m} = \operatorname{Ker}\chi$. If Γ is not normal, then $\tilde{\Gamma}$ is a finite Γ-module and χ admits finitely many extensions to $\tilde{\Gamma}$ by Corollary 2.2. Hence, it is enough to prove the statement for normal Γ. But in this case the statement follows from Corollary 4.8, which completes the proof of the Main Theorem.

4.4. **Module theoretic characterization of Galois orders.** Combining the Main Theorem and Corollary 4.10 we obtain the following.

Corollary 4.9. *Let* U *be a Galois ring over a noetherian algebra* Γ*. Assume that* $\mathcal{M}$ *is a group and* $\varphi^{-1}(\Gamma) \subset \bar{\Gamma}$ $(\varphi(\Gamma) \subset \bar{\Gamma})$ *for any* $\varphi \in \mathcal{M}$*. Then every character* $\chi : \Gamma \to \Bbbk$ *extends to a simple left (right)* U*-module if and only if* U *is a right (left) Galois order.*

We also have the following converse statement to the Main Theorem.

Corollary 4.10. *Let $U \subset L * \mathcal{M}$ be a Galois ring over a noetherian Γ. If every character $\chi : \Gamma \longrightarrow \Bbbk$ extends to a representation of U, then $U_e \subset \bar{\Gamma} \cap K$. If, in addition, $\mathcal{M}$ is a group and Γ is a Harish-Chandra subalgebra, then U is a Galois order.*

Proof. If χ extends to a representation of U, then it extends, in particular, to a representation of $U_e \subset K$. Proposition 2.3 implies that U_e is contained in the integral closure of Γ in K. The second statement follows immediately from Theorem 2.12. $\square$

4.5. **Bounds for dimensions and blocks of the category of Gelfand-Tsetlin modules.** Denote by $r(\mathsf{m}, \mathsf{n})$ the minimal number of generators of $U(S(\mathsf{m}, \mathsf{n}))$ as a right Γ-module. Since Γ is a Harish-Chandra subalgebra, from $|S(\mathsf{m}, \mathsf{m})| < \infty$ it follows that $r(\mathsf{m}, \mathsf{n}) < \infty$ and, by Theorem 4.7, $\mathcal{A}(\mathsf{m}, \mathsf{m})$ is finitely generated as a right $\widehat{\Gamma}_{\mathsf{m}}$-module (that is, Γ is *big* at m). In particular, there exist only finitely many nonisomorphic simple $\mathcal{A}$-modules M such that $M(\mathsf{m}) \neq 0$. Moreover, in any such module $M(\mathsf{m})$ is finite dimensional (Lemma 3.17).

Lemma 4.11. *Let $\mathsf{m}, \mathsf{n} \in \operatorname{Specm}\Gamma$, $S = S(\mathsf{m}, \mathsf{n})$, $M = U \otimes_\Gamma \Gamma/\mathsf{m}$, and $x = 1 \otimes (1 + \mathsf{m}) \in M(\mathsf{m})$. Let $p_{\mathsf{n}} : M \to M(\mathsf{n})$ be the canonical projection with respect to decomposition* (10)*. Then*

$$\mathcal{A}(\mathsf{m}, \mathsf{n}) \cdot x = p_{\mathsf{n}}(Ux) = p_{\mathsf{n}}(U(S)x)$$

and

$$\dim_{\Bbbk} M(\mathsf{n}) \leq \dim_{\Bbbk}(U(S)x),\ \dim_{\Bbbk} M(\mathsf{n}) \leq r(\mathsf{m}, \mathsf{n}).$$

Analogous statements hold for any U-module N generated by a nonzero $y \in N(\mathsf{m})$ such that $\mathsf{m}y = 0$.

Proof. The first equality follows from Lemma 3.2, (e). To prove the second equality consider some $u \in U$. Then by the Chinese remainder theorem there exists $\gamma \in \Gamma$ such that $\gamma ux = p_{\mathsf{n}}(ux)$ and we replace u by γu. Then in the notation of Lemma 4.6 set $k = 2t$, where $\mathsf{n}^t ux = 0$. We have $p_{\mathsf{n}}(ux) = p_{\mathsf{n}}(u_k x)$, where $u_k \in U(S)$. The second statement follows from the first one and the fact that any such N is a quotient of M. $\square$

Theorem 4.12. *Let U be a Galois order over a normal noetherian Γ.*

(a) *If $\mathcal{M}_{\mathsf{m}}$ is finite for some $\mathsf{m} \in D$ and $M \in \mathbb{H}(U, \Gamma, D)$ is simple, then $M(\mathsf{m})$ is finite dimensional. Both $\dim_{\Bbbk} M(\mathsf{m})$ and the number of isomorphism classes of simple modules N satisfying $N(\mathsf{m}) \neq 0$ are bounded by $r(\mathsf{m}, \mathsf{m})$.*

(b) *If, in addition, $\mathcal{M}$ is a group, then for any $\mathsf{n} \in D$,*

$$\dim_{\Bbbk} M(\mathsf{n}) \leq r(\mathsf{m}, \mathsf{n}) < \infty.$$

(c) *Assume that U is free as a right Γ-module. If M is a U-module, generated by some $x \in M$ such that $\mathsf{m}x = 0$ for some $\mathsf{m} \in \operatorname{Specm}\Gamma$, then*

$$\dim_{\Bbbk} M(\mathsf{n}) \leq |S(\mathsf{m}, \mathsf{n})/G|.$$

Proof. The statements (a) and (b) follow from Lemma 4.11 and Lemma 4.1, respectively. Let $S = S(\mathsf{m}, \mathsf{n})$. To prove (c) let $p : F \longrightarrow U(S)$ be a free right $\Gamma-$cover of $U(S)$ and $q : F \longrightarrow U$ be the composition of p with the canonical embedding. Consider the mapping of right K-vector spaces:

$$q \otimes_\Gamma \mathrm{id}_K : F \otimes_\Gamma K \longrightarrow U \otimes_\Gamma K.$$

Following (5) we have $U \otimes_\Gamma K \simeq \mathcal{K}$, and the image of $q \otimes_\Gamma \mathrm{id}_K$ in $\mathcal{K}$ coincides with $KU(S) = \mathcal{K}(S)$. Recall that $\dim_K \mathcal{K}(S) = |S/G|$. From the semicontinuity of the dimension of the image of a mapping between free modules we obtain that for

$$q \otimes_\Gamma \mathrm{id}_{\Gamma/\mathsf{m}} : F \otimes_\Gamma \Gamma/\mathsf{m} \longrightarrow U \otimes_\Gamma \Gamma/\mathsf{m} \simeq U/U\mathsf{m} = U_\mathsf{m}$$

we have the inequality $\dim_\Bbbk \mathrm{Im}(q \otimes_\Gamma \mathrm{id}_{\Gamma/\mathsf{m}}) \leq \dim_K \mathcal{K}(S) = |S/G|$. However, $\mathrm{Im}(q \otimes_\Gamma \mathrm{id}_{\Gamma/\mathsf{m}}) = U_\mathsf{m}(n)$, which completes the proof. □

Let D be an equivalence class of Δ, $\mathsf{m}, \mathsf{n} \in D$ and $\varphi \in S(\mathsf{m}, \mathsf{n})$. We say that m and n are *connected* by φ if there exist $[a_-\varphi^{-1}]$, $[a_+\varphi] \in U$, such that

(a) a_+ is defined in n and a_- is defined in m (as rational functions).

(b) $a_- a_+^{\varphi^{-1}} \notin g \cdot \ell_\mathsf{m}$ for any $g \in G$ and $a_-^{\varphi} a_+ \notin g \cdot \ell_\mathsf{n}$ for any $g \in G$.

(c)

$$\{\varphi^{-1}\varphi^g \mid g \in G\} \cap S(\mathsf{m}, \mathsf{m})\} = \{e\},$$
$$\{\varphi(\varphi^{-1})^g \mid g \in G\} \cap S(\mathsf{n}, \mathsf{n})\} = \{e\}.$$

Endow D with the structure of a nonoriented graph as follows. The vertices are elements of D. An edge between m and n exists if and only if there exists some $\varphi \in \mathcal{M}$ that connects m and n. The following statement is a generalization of Theorem 32 from [DFO1].

Proposition 4.13. *If $\mathsf{m}, \mathsf{n} \in D$ are connected by $\varphi \in \mathcal{M}$, then $\mathsf{m} \simeq \mathsf{n}$ in $\mathcal{A}_D$. Moreover, if m and n belong to the same connected component of the graph D, then they are isomorphic in $\mathcal{A}_D$.*

Proof. We show that the images $f \in \mathcal{A}(\mathsf{m}, \mathsf{n})$ of $[a_+\varphi]$ and $g \in \mathcal{A}(\mathsf{n}, \mathsf{m})$ of $[a_-\varphi^{-1}]$ are isomorphisms. By condition (c) and the Chinese remainder theorem, for every $k \geq 0$ there exist $\gamma_k, \nu_k \in \Gamma$, such that

$$\gamma_k \equiv 1 \bmod \ell_\mathsf{m}^k, \gamma_k \equiv 0 \bmod (\varphi^x \cdot \ell_\mathsf{m})^k \text{ for } x \in G : \pi(\ell_\mathsf{m}) \neq \pi(\varphi^x \cdot \ell_\mathsf{m}),$$
$$\nu_k \equiv 1 \bmod \ell_\mathsf{n}^k, \nu_k \equiv 0 \bmod (\varphi^x \cdot \ell_\mathsf{n})^k \text{ for } x \in G : \pi(\ell_\mathsf{n}) \neq \pi(\varphi^x \cdot \ell_\mathsf{n}),$$

where $\pi : \mathrm{Specm}\,\bar{\Gamma} \to \mathrm{Specm}\,\Gamma$ is the canonical projection. Consider the elements $f_k = \nu_k[a_+\varphi]\gamma_k$, $g_k = \gamma_k[a_-\varphi^{-1}]\nu_k$. The images of f_k in $\mathcal{A}(\mathsf{m}, \mathsf{n})$ and g_k in $\mathcal{A}(\mathsf{n}, \mathsf{m})$ converge to f and g, respectively, when k tends to infinity. We prove that gf is an isomorphism, and the case of fg is analogous. We find the image of gf in $\mathcal{A}(\mathsf{m}, \mathsf{m})$ as the limit of the images of $g_k f_k$. The images of $g_k f_k$ and $(g_k f_k)_e$ in $\mathcal{A}(\mathsf{m}, \mathsf{m})$ coincide since $g_k f_k$ is a sum of elements of the form $[a_{x_1,x_2}(\varphi^{-1})^{x_1}\varphi^{x_2}] = [a_{x_1,x_2}^{x_1^{-1}}\varphi\varphi^{x_1^{-1}x_2}]$, $x_1, x_2 \in G$. But following condition (c), if $\varphi\varphi^{x_1^{-1}x_2} \neq e$, then the class of corresponding elements in $\mathcal{A}(\mathsf{m}, \mathsf{m})$ equals 0. Immediate calculation gives us

$$(g_k f_k)_e = \sum_{x \in G/H_\varphi} a_-^{x^{-1}} a_+^{(\varphi^{-1})^x} \big(\gamma_k^{(\varphi^{-1})^x}\big)^2 \nu_k^2.$$

Due to (a), all summands here are defined in some localization of $\Lambda = \bar{\Gamma}$. The canonical embedding $\widehat{\Lambda}_\mathsf{m} \longrightarrow \prod_{\ell \in \pi^{-1}(\mathsf{m})} \Lambda_{\ell_\mathsf{m}}$, condition (b) and the choice of γ_k show that the class of $g_k f_k$ converges to the class of $a_- a_+^{\varphi^{-1}}$ in $\widehat{\Gamma}_\mathsf{m}$, which is invertible.

Let $gf = \lambda$, $fg = \mu$. Then $f(g\mu^{-1}) = (fg)\mu^{-1} = \mathbb{1}_\mathsf{n}$. Moreover, we also have $fgf = \mu f = f\lambda$, and hence $\mu^{-1} f = f\lambda^{-1}$. This implies $(g\mu^{-1})f = g\mu^{-1}f = gf\lambda^{-1} = \mathbb{1}_\mathsf{n}$. Hence f is invertible, completing the proof. □

Immediately from Proposition 4.13 follows

Theorem 4.14. *Let D be a class of Δ-equivalence. Suppose $\mathcal{M}$ is a group, D has a finite number of connected components and for some $\mathsf{m} \in D$ the group $\mathcal{M}_{\mathsf{m}}$ is finite. Then the module $U/U\mathsf{m}$ is of finite length.*

Proof. By Proposition 4.13 the skeleton $\mathcal{B}_D$ of the category $\mathcal{A}_D$ ([AHS]) contains a finite number of objects, say $\operatorname{Ob}\mathcal{B}_D = \{\mathsf{n}_1, \dots, \mathsf{n}_k\}$. Consider $U/U\mathsf{m}$ as an element in $\mathcal{A} - \operatorname{Mod}_d$. Denote by M the $\mathcal{B}_D$-module $(U/U\mathsf{m})|_{\mathcal{B}_D}$. Then, by Theorem 4.12, (b), $\dim_k M(\mathsf{n}_i) \leq r(\mathsf{m}, \mathsf{n}_i)$ for any $i = 1, \dots, k$. Since the categories $\mathcal{A} - \operatorname{Mod}_d$ and $\mathcal{B}_D - \operatorname{Mod}_d$ are equivalent, the claim follows. □

From the proof above we obtain

Corollary 4.15. *Under the assumptions of Theorem 4.14 the length of $U/U\mathsf{m}$ and the number of simple objects in the block $\mathbb{H}(U, \Gamma, D)$ do not exceed $\sum_{i=1}^{k} r(\mathsf{m}, \mathsf{n}_i)$, where n_i runs through a set of representatives of connected components of D.*

4.6. **Further properties of Gelfand-Tsetlin modules.** In this subsection we assume that U is a Galois order over a noetherian Γ. Under the assumptions of Theorem 4.7 we are able to prove the following generalization of Corollary 4.3.

Theorem 4.16. *Let U be a Galois order over a noetherian algebra. Assume that $\mathsf{m}, \mathsf{n} \in \operatorname{Specm}\Gamma$ such that $S(\mathsf{m}, \mathsf{n})$ and $|\mathcal{M}_{\mathsf{m}}|$, $|\mathcal{M}_{\mathsf{n}}|$ are finite and $S(\mathsf{m}, \mathsf{n}) \neq \varnothing$. Then the image of $U(S(\mathsf{m}, \mathsf{n}))$ in $\mathcal{A}(\mathsf{m}, \mathsf{n})$ (under the map $u \mapsto u_{\mathsf{m},\mathsf{n}}$ for any $u \in U$) is nonzero.*

Proof. Note that for a nonempty G-invariant $S \subset \mathcal{M}$ the Γ-bimodule $U(S)$ is nonzero since $KU(S) \subset \mathcal{K}$ is nonzero. Following Remark 2.6 we consider U as a $\Gamma \otimes \Gamma$-module and denote by I the ideal $\mathsf{n} \otimes \Gamma + \Gamma \otimes \mathsf{n}$ in $\Gamma \otimes \Gamma$. Then the class of $u \in U$ in $\mathcal{A}(\mathsf{m}, \mathsf{n})$ is 0 if $u \in I^N U = \mathsf{n}^N U + U\mathsf{m}^N$ for any $N \geq 0$ (see Definition 3.5).

We prove the following statement: If $U(S) \subset I^N U$ for some $N \geq 0$, then $U(S) \subset I^N U(S)$. Assume $U(S) \subset I^N U$, that is, if $v_1, \dots, v_k$ are generators of $U(S)$ as a $\Gamma \otimes \Gamma$-module, then

$$v_i = \sum_{j=1}^{l} \nu_{ij} u_{ij}, \text{ for some } \nu_{ij} \in I^N \subset \Gamma \otimes \Gamma, u_{ij} \in U, i = 1, \dots, k. \tag{30}$$

Set $T = \bigcup_{i,j} \operatorname{supp} u_{ij} \setminus S$. As in the proof of Lemma 4.6 construct the element

$$f_T = 1 - F \in 1 + \mathsf{n} \otimes \Gamma + \Gamma \otimes \mathsf{m}$$

such that for all $v_{ij} = f_T \cdot u_{ij}$ we have $\operatorname{supp} v_{ij} \subset S$. Applying f_T to both sides of equality (30) we obtain

$$v_i = F \cdot v_i + \sum_{j=1}^{k} \nu_{ij} v_{ij}, \ \nu_{ij} \in I^N, v_{ij} \in U(S), i = 1, \dots, k. \tag{31}$$

Substituting recursively v_i on the right hand side of (31), we obtain

$$v_i = F^2 \cdot v_i + (F + 1) \sum_{j=1}^{k} \nu_{ij} v_{ij}.$$

Iterating this procedure $N-1$ times we obtain

$$v_i = F^N \cdot v_i + (F^{N-1} + \cdots + F + 1)\sum_{j=1}^{l} \nu_{ij} v_{ij},\ i = 1, \ldots, k.$$

This shows that $v_i \in I^N U(S)$, since v_i and all v_{ij} belong to $U(S)$ and $F^N \in I^N$. In particular, this means that $U(S) = \bigcap_{n=1}^{\infty} I^N U(S)$. Then, by Krull's Theorem (Theorem 8.9 of [Mat]), there exists $a \in 1 + I$, such that $a \cdot U(S) = 0$. Since $\Gamma \otimes \Gamma$ acts on $\mathcal{K}$, the element a acts on $V = U(S)K \subset \mathcal{K}$ and $a \cdot V = 0$ by the above. Then $a \in 1 + I \subset \Gamma \otimes \Gamma$ can be written in the form

$$a = 1 + \sum_{i=1}^{m} \nu_i \otimes \alpha_i + \sum_{j=1}^{n} \beta_j \otimes \mu_j, \alpha_i, \beta_j \in \Gamma, \mu_i \in \mathsf{m}, \nu_j \in \mathsf{n},$$

using the noetherianity of Γ.

Following Lemma 2.7, all irreducible summands of V as a K-bimodule are of the form $V(\varphi)$ for some $\varphi \in \mathcal{M}$. Since $\operatorname{supp} V = S(\mathsf{m}, \mathsf{n})$, there exist ℓ_{m} and ℓ_{n} which project onto m and n, respectively, such that $\ell_{\mathsf{n}} = \ell_{\mathsf{m}}^{\varphi}$. Note that the K-bimodule $V(\varphi)$ is isomorphic to L^{H_φ}, endowed with the natural structure of a K-bimodule.

The action of a on $1 \in L^{H_\varphi}$ is zero. But, on the other hand, we have

$$a \cdot 1 = (1 + \sum_{i=1}^{m} \nu_i \otimes \alpha_i + \sum_{j=1}^{n} \beta_j \otimes \mu_j) \cdot 1 = 1 + \sum_{i=1}^{m} \nu_i \alpha_i^{\varphi} + \sum_{j=1}^{n} \beta_j \mu_j^{\varphi} \in 1 + \ell_{\mathsf{n}},$$

because all the elements in the formulas above belong to $\overline{\Gamma}$, $\nu_i \in \mathsf{n}$ and $\mu_j^{\varphi} \in \mathsf{n}$, since $\ell_{\mathsf{m}}^{\varphi} = \ell_{\mathsf{n}}$ lies over m^{φ}. But $0 \notin 1 + \ell_{\mathsf{n}}$, and this contradiction completes the proof. $\square$

Note that this theorem (in the case $\mathsf{m} = \mathsf{n}$), together with Theorem 3.14, gives another proof of Corollary 4.3.

Let $\mathsf{m} \in \operatorname{Specm}\Gamma$ and $\ell_1, \ldots, \ell_k$ be all possible extensions of m in $\operatorname{Specm}\bar{\Gamma}$, that is, $\pi(\ell_i) = \mathsf{m}$ for all i and $\pi^{-1}(\mathsf{m}) = \{\ell_1, \ldots, \ell_k\}$. The following lemma describes the set $X_u(\mathsf{m})$ for $u \in U$ (see Lemma 3.2). The set $X^u(\mathsf{m})$ is described analogously.

Lemma 4.17. *Let $\varphi \in \mathcal{M}$, $a \in L^{H_\varphi}$ and $V = \Gamma[a\varphi]\Gamma$. Then the set of simple quotients of the left Γ-module $V \otimes_\Gamma \Gamma/\mathsf{m}$ coincides with the set of simples of the form Γ/n, $\mathsf{n} \in \pi(\varphi(\pi^{-1}(\mathsf{m}))) = \{\pi(\ell_1^{\varphi}), \ldots, \pi(\ell_k^{\varphi})\}$. In addition, for $u \in U$ and $\mathsf{m} \in \operatorname{Specm}\Gamma$ one has $X_u(\mathsf{m}) = \pi(\operatorname{supp} u \cdot \pi^{-1}(\mathsf{m}))$.*

Proof. Denote $\chi : \Gamma \to \Gamma/\mathsf{m} = \Bbbk$. Then m^{φ} is an ideal of Γ^{φ}. Let $\chi^{\varphi} : \Gamma^{\varphi} \to \Gamma^{\varphi}/\mathsf{m}^{\varphi} = \Bbbk$ be the homomorphism induced by χ and φ. By Lemma 2.4, (c), we have $V/V\mathsf{m} \simeq \Gamma\Gamma^{\varphi}/\Gamma\mathsf{m}^{\varphi}$. Since $\Gamma^{\varphi} \subset \Gamma\Gamma^{\varphi}$ is a finite integral extension, we have $(\Gamma\Gamma^{\varphi})\mathsf{m}^{\varphi} = \Gamma\mathsf{m}^{\varphi} \neq \Gamma\Gamma^{\varphi} \subset \bar{\Gamma}$. The kernels of homomorphisms $\widehat{\chi^{\varphi}} : \bar{\Gamma} \to \Bbbk$ extending $\chi^{\varphi} : \Gamma^{\varphi} \to \Gamma^{\varphi}/\mathsf{m}^{\varphi}$ form the G^{φ}-orbit $\{\pi(\ell_1^{\varphi}), \ldots, \pi(\ell_k^{\varphi})\}$, whose restrictions to Γ uniquely define all characters of $\Gamma\Gamma^{\varphi}$ extending χ^{φ}. This proves the first statement.

Let $W = \Gamma u \Gamma$. Then Lemma 2.10, (e) reduces the second statement to the case of a Γ-bimodule W generated by elements of the form $[a_1\varphi], \ldots, [a_k\varphi]$. Hence the second statement follows from the first one. $\square$

Theorem 4.18. *Let D be an equivalence class of Δ (see Subsection 3.2).*

(a) *If $S(\mathsf{m},\mathsf{n})=\varnothing$ for some $\mathsf{m},\mathsf{n}\in\operatorname{Specm}\Gamma$, then $\mathcal{A}(\mathsf{m},\mathsf{n})=0$.*

(b) *Let Δ' be the minimal equivalence containing all $(\mathsf{m},\mathsf{n})\in\operatorname{Specm}\Gamma\times\operatorname{Specm}\Gamma$ such that $S(\mathsf{m},\mathsf{n})\neq\varnothing$. Then $\Delta=\Delta'$. If the assumptions of Theorem 4.16 are satisfied, then the category $\mathcal{A}_D$ does not split into a nontrivial direct sum and acts faithfully on $\mathbb{H}(U,\Gamma,D)$.*

Proof. Statement (a) follows from Lemma 4.6. By Lemma 4.17, we have that $\varphi\in S(\mathsf{m},\mathsf{n})$ if and only if Γ/n is a right subquotient of $\Gamma[a\varphi]\Gamma/\Gamma[a\varphi]\mathsf{m}$. This proves the first statement from (b). To prove the second statement note that $U(\mathsf{m},\mathsf{n})\neq 0$ if and only if $S(\mathsf{m},\mathsf{n})\neq\varnothing$ and, following Theorem 4.16 and (a), if and only if $\mathcal{A}(\mathsf{m},\mathsf{n})\neq 0$. On the other hand, if $a\in\mathcal{A}(\mathsf{m},\mathsf{n})$, $a\neq 0$, then there exists $N\geq 1$ such that $a\notin\mathcal{A}(\mathsf{m},\mathsf{n})\mathsf{m}^N$, hence a acts nontrivially on $U/U\mathsf{m}^N$. Statement (b) follows. □

5. Gelfand-Tsetlin modules for gl_n

Consider the general linear Lie algebra gl_n with the standard basis of matrix units $e_{ij}, i,j=1,\dots,n$. Set $U=U(\mathrm{gl}_n)$, $U_m=U(\mathrm{gl}_m)$, $1\leq m\leq n$. Let Z_m be the center of U_m. Then Z_m is a polynomial algebra in m variables $\{c_{mk}\mid k=1,\dots,m\}$,

$$c_{mk} \;=\; \sum_{(i_1,\dots,i_k)\in\{1,\dots,m\}^k} e_{i_1i_2}e_{i_2i_3}\dots e_{i_ki_1}. \tag{32}$$

We identify gl_m for $m\leqslant n$ with the Lie subalgebra of gl_n spanned by $\{e_{ij}\mid i,j=1,\dots,m\}$, so that we have the following chain of inclusions:

$$\mathrm{gl}_1\subset\mathrm{gl}_2\subset\dots\subset\mathrm{gl}_n\,.$$

It induces inclusions $U_1\subset U_2\subset\dots\subset U_n$ of the universal enveloping algebras. The *Gelfand-Tsetlin* subalgebra Γ in U ([DFO1]) is generated by $\{Z_m\mid m=1,\dots,n\}$. The algebra Γ is a polynomial algebra in $\dfrac{n(n+1)}{2}$ variables $\{c_{ij}\mid 1\leqslant j\leqslant i\leqslant n\}$ ([Zh]). Denote by K the field of fractions of Γ.

Let $\mathcal{L}\simeq\Bbbk^{\frac{n(n+1)}{2}}$ be a subspace of $\Bbbk^{n^2}$ generated by sequences (ℓ_{ij}), $1\leqslant j\leqslant i\leqslant n$. Let $\mathcal{M}\subset\mathcal{L}$ be the free abelian group generated by δ^{ij}, $1\leqslant j\leqslant i\leqslant n-1$, $(\delta^{ij})_{kl}=1$ if $i=k$, $j=l$ and 0 otherwise, we have $\mathcal{M}\simeq\mathbb{Z}^{\frac{n(n-1)}{2}}$. For $i=1,\dots,n$ denote by S_i the i-th symmetric group and set $G=S_1\times\dots\times S_n$. The group G acts on $\mathcal{L}$ as follows: $(s\cdot\ell)_{ij}=\ell_{i\,s_i(j)}$ for $\ell=(\ell_{ij})\in\mathcal{L}$ and $s=(s_i)\in G$. Also the group $\mathcal{M}$ acts on $\mathcal{L}$ as follows: $\delta^{ij}\cdot\ell=\ell+\delta^{ij}$, $\delta^{ij}\in\mathcal{M}$.

Let Λ be the polynomial algebra in variables $\{\lambda_{ij}\mid 1\leqslant j\leqslant i\leqslant n\}$ and L be the fraction field of Λ (note that $\operatorname{Specm}\Lambda$ can be identified with $\mathcal{L}$).

Let $\imath:\Gamma\longrightarrow\Lambda$ be the embedding given by

$$\imath(c_{mk}) \;=\; \sum_{i=1}^{m}(\lambda_{mi}+m)^k\prod_{j\neq i}\Big(1-\frac{1}{\lambda_{mi}-\lambda_{mj}}\Big).$$

The image of $\imath$ coincides with the subalgebra of $G-$invariant polynomials in Λ (see [Zh]) which we identify with Γ. The homomorphism $\imath$ can be extended to the embedding $K\subset L$ of fields, where $K=L^G$. Then $G=G(L/K)$. The action of the group G by conjugation on $\mathcal{M}$ induces an action of G on $L*\mathcal{M}$.

Let e be the identity element of the group $\mathcal{M}$. Consider the linear map $\mathrm{t} : U \longmapsto \mathcal{K}$ given by

$$(33)\quad \mathrm{t}(e_{mm}) = \imath(e_{mm})e,\ \mathrm{t}(e_{m\,m+1}) = \sum_{i=1}^{m} A^{+}_{mi}\delta^{mi},\ \mathrm{t}(e_{m+1\,m}) = \sum_{i=1}^{m} A^{-}_{mi}(\delta^{mi})^{-1},$$

where

$$A^{\pm}_{mi} = \mp \frac{\prod_j (\lambda_{m\pm 1,j} - \lambda_{mi})}{\prod_{j\neq i}(\lambda_{mj} - \lambda_{mi})}.$$

Lemma 5.1. *The map* t *is an algebra homomorphism.*

Proof. A straightforward computation shows that the Serre relations are satisfied by the images $\mathrm{t}(e_{ij})$. On the other hand, one can argue that these relations are given by some rational functions and that these rational functions agree on a dense set formed by certain vectors in finite dimensional modules (due to Gelfand and Tsetlin [GTs] and Zhelobenko [Zh]). Hence, all Serre relations are satisfied. □

Proposition 5.2 ([FO1], Proposition 7.2)**.** *We have:*

- t *is an embedding;*
- $UK = KU \simeq (L * \mathbb{Z}^m)^G$, $m = n(n-1)/2$;
- U *is a Galois order over* Γ.

Note that the first statement of Proposition 5.2 also appears in [Maz3], [Ov1] and in [Kh].

To estimate the number of isomorphism classes of simple Gelfand-Tsetlin modules for $U(\mathrm{gl}_n)$ we recall that $U(\mathrm{gl}_n)$ is free both as a left and as a right Γ-module ([Ov1], [FO2]).

Set

$$Q_n = \prod_{i=1}^{n-1} i!.$$

The following is a refinement of the results from [Ov1].

Corollary 5.3. *Let* $U = U(\mathrm{gl}_n)$, $\Gamma \subset U$ *be the Gelfand-Tsetlin subalgebra,* D *a* Δ*-class, and* $\mathsf{m} \in D$. *Then we have*

(a) *For a* U*-module* M*, such that* $M(\mathsf{m}) \neq 0$ *and* M *is generated by some* $x \in M(\mathsf{m})$ *(in particular for a simple module), one has*

$$\dim_{\Bbbk} M(\mathsf{m}) \leq Q_n.$$

(b) *The number of isomorphism classes of simple* U*-modules* N *such that* $N(\mathsf{m}) \neq 0$ *is always nonzero and does not exceed* Q_n.

Proof. Note that a simple M such that $M(\mathsf{m}) \neq 0$ is generated by any nonzero vector from $M(\mathsf{m})$. Since U is a free Γ-module, we can apply Theorem 4.12, (c) and obtain $\dim_{\Bbbk} M(\mathsf{m}) \leq |S(\mathsf{m},\mathsf{m})/G|$. On the other hand, by Lemma 4.1, (c) the right hand side here is bounded by the cardinality of the set

$$B = \{m \in \mathbb{Z}^{n(n-1)/2}) \mid \pi(m + \ell_{\mathsf{m}}) = \ell_{\mathsf{m}}\}.$$

Equivalently, $m \in B$ if and only if the i-th rows of ℓ_{m} and $\ell_{\mathsf{m}} + m$ differ by a permutation from S_i for every $i = 1, \ldots, n-1$. This gives us at most $|S_1| \cdot |S_2| \cdot \ldots \cdot |S_{n-1}|$ possibilities for $\mu \in \mathcal{M}$ and implies (a). By Lemma 3.16, the number of isomorphism classes of simple U-modules N such that $N(\mathsf{m}) \neq 0$ equals the number of isomorphism classes of simple $\mathcal{A}(\mathsf{m},\mathsf{m})$-modules, and the correspondence is given by $M \leftrightarrow M(\mathsf{m})$. Therefore, if $X = U/U\mathsf{m}$, then the $\mathcal{A}(\mathsf{m},\mathsf{m})$-module $X(\mathsf{m})$ covers any simple $\mathcal{A}(\mathsf{m},\mathsf{m})$-module. Together with (a) this implies (b). □

Remark 5.4. We believe that the bound Q_n in (a) cannot be improved. It is known to be exact for $n = 2$ and $n = 3$ [DFO1].

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