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GREEN FUNCTION ESTIMATES FOR SUBORDINATE BROWNIAN MOTIONS: STABLE AND BEYOND

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Abstract. A subordinate Brownian motion X is a Lévy process which can be obtained by replacing the time of the Brownian motion by an independent subordinator. In this paper, when the Laplace exponent ϕ of the corresponding subordinator satisfies some mild conditions, we first prove the scale invariant boundary Harnack inequality for X on arbitrary open sets. Then we give an explicit form of sharp two-sided estimates of the Green functions of these subordinate Brownian motions in any bounded $C^{1,1}$ open set. As a consequence, we prove the boundary Harnack inequality for X on any $C^{1,1}$ open set with explicit decay rate. Unlike previous work of Kim, Song and Vondraček, our results cover geometric stable processes and relativistic geometric stable process, i.e. the cases when the subordinator has the Laplace exponent

$$\phi(\lambda) = \log(1 + \lambda^{\alpha/2}) \quad (0 < \alpha \le 2, d > \alpha)$$

and

$$\phi(\lambda) = \log(1 + (\lambda + m^{2/\alpha})^{\alpha/2} - m) \quad (0 < \alpha < 2,\, m > 0, d > 2)\,.$$

1. Introduction

Let d be a positive integer, let $W = (W_t, \mathbb{P}_x)$ be a Brownian motion in $\mathbb{R}^d$ starting at x and let $S = (S_t : t \ge 0)$ be a subordinator independent of W, i.e. a Lévy process taking values in $[0, \infty)$ and starting at 0.

The Laplace exponent of a subordinator is a Bernstein function and hence has the representation

$$\phi(\lambda) = b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t})\, \mu(dt)\,, \tag{1.1}$$

where $b \ge 0$ and μ is a measure on $(0, \infty)$ satisfying $\int_{(0,\infty)} (1 \wedge t)\, \mu(dt) < \infty$, usually called the Lévy measure of ϕ. If the measure μ has a completely monotone density, the Laplace exponent ϕ is called a complete Bernstein function.

We define the subordinate Brownian motion $X = (X_t, \mathbb{P}_x)$ by $X_t = W_{S_t}$.

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The aim of this paper is to obtain the following two-sided estimates of the Green function $G_D(x,y)$ of X in a bounded $C^{1,1}$ open set $D \subset \mathbb{R}^d$ in terms of the Laplace exponent ϕ of the subordinator:

$$G_D(x,y) \asymp \left(1 \wedge \frac{\phi(|x-y|^{-2})}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})}}\right) \frac{\phi'(|x-y|^{-2})}{|x-y|^{d+2}\phi(|x-y|^{-2})^2},$$

where $\delta_D(x)$ denotes the distance of the point x to D^c and $a \wedge b := \min\{a,b\}$. Here and in the sequel, $f \asymp g$ means that the quotient $\frac{f}{g}$ stays bounded between two positive numbers on their common domain of definition.

The process X is, in particular, a rotationally symmetric Lévy process. Recently there has been huge interest in studying the potential theory of such processes. See, for example, [KMR, KSV12a, KSV12b, KSV12c, RSV06] and the references therein. The purpose of this paper is to extend recent results in [KSV12b, KSV12c] by covering geometric stable processes and much more.

Estimates of the Green function for discontinuous Markov processes were first studied for rotationally symmetric α-stable processes in [CS98] and in [Kul97] independently. These results were later extended to relativistic α-stable processes and to sums of two independent stable processes in [Ryz02] and [CKS10] respectively. Recently, the first named author with R. Song and Z. Vondraček succeeded to obtain such estimates for a large class of subordinate Brownian motions in [KSV12b].

Still, the class considered in [KSV12b] does not include some interesting cases like geometric stable processes or, more generally, the class of subordinate Brownian motions with a Laplace exponent that varies slowly at infinity. Our approach covers a large class of such processes.

Another feature of our approach is that it is unifying in the following sense: the sharp estimates of the Green function are given only in terms of the Laplace exponent ϕ and its derivative.

Let us give a few examples of transient processes that are covered by our approach.

Example 1 (Geometric stable processes)**.**

$$\phi(\lambda) = \log(1+\lambda^{\beta/2}) \quad (0 < \beta \le 2,\ d > \beta).$$

Example 2 (Iterated geometric stable processes)**.**

$$\begin{aligned} \phi_1(\lambda) &= \log(1+\lambda^{\beta/2}) \quad (0<\beta\le 2), \\ \phi_{n+1} &= \phi_1 \circ \phi_n \quad n \in \mathbb{N}, \end{aligned}$$

with an additional condition $d > 2^{1-n}\beta^n$.

Example 3 (Relativistic geometric stable processes)**.**

$$\phi(\lambda) = \log\left(1 + \left(\lambda + m^{\beta/2}\right)^{2/\beta} - m\right) \quad (m>0,\ 0<\beta<2,\ d>2).$$

In order to obtain the sharp Green function estimates we first obtain the uniform boundary Harnack principle, with its constant not depending on the open set itself. Such a uniform boundary Harnack principle was first proved in [BKK08] and very recently generalized to a larger class of rotationally symmetric Lévy processes in [KSV12c]. We adapt the approach in the latter paper in order to cover the class of subordinate Brownian motions with slowly varying Laplace exponents. Unlike the approach in [KSV12c], instead of the use of the Harnack inequality, we use estimates of the Green function of balls near boundary obtained in [KM12].

Further, our uniform boundary Harnack principle can be used to prove sharp Green function estimates for bounded $C^{1,1}$ open sets by adapting the method in [KSV12b]. Even though we follow the roadmap in [KSV12b], we needed to make significant changes due to the fact that now we do not have necessarily regularly varying Laplace exponents.

To overcome such difficulties we use new types of estimates (not only in terms of the Laplace exponent itself, but also in terms of its derivative) of the jumping kernel and the potential kernel of the subordinate Brownian motions, which were obtained for the first time in [KM12]. This type of estimate is essential in our approach.

Let us be more precise now. In this paper we will always assume the following three conditions on the Laplace exponent ϕ of the subordinator S:

(A-1) ϕ is a complete Bernstein function;

(A-2) the Lévy density μ of ϕ is infinite, i.e. $\mu(0,\infty)=\infty$;

(A-3) there exist constants $\sigma>0$, $\lambda_0>0$ and $\delta\in(0,1]$ such that

$$\frac{\phi'(\lambda x)}{\phi'(\lambda)} \le \sigma\, x^{-\delta} \quad \text{for all } \ x\ge 1 \ \text{ and } \ \lambda\ge\lambda_0\,. \tag{1.2}$$

Either in the case $d\le 2$ and $\delta>1-d/2$ or in the case $d\ge 2$ and $0<\delta\le\frac{1}{2}$ we will sometimes further assume two technical conditions below. Note that **(A-4)**, related to transience of the corresponding subordinate Brownian motion, is used in [KM12] to obtain the asymptotic of the jumping kernel and the Green function of the subordinate Brownian motion. Unlike [KM12] we state **(A-4)** for $d=2$ and $d=1$ separately to make it clear.

(A-4) If $d=2$, we assume that there are $\sigma_0>0$ and $\delta_0\in(0,2)$ such that

$$\frac{\phi'(\lambda x)}{\phi'(\lambda)} \ge \sigma_0\, x^{-\delta_0} \quad \text{for all } \ x\ge 1 \ \text{ and } \ \lambda\ge\lambda_0. \tag{1.3}$$

If $d=1$, we assume that the constant δ in **(A-3)** satisfies $\delta>\frac{1}{2}$ and that there are $\sigma_0>0$ and $\delta_0\in(\frac{1}{2},2\delta-\frac{1}{2})$ such that (1.3) holds.

(A-5) If $d\ge 2$ and the constant δ in **(A-3)** satisfies $0<\delta\le\frac{1}{2}$, then we assume that there exist constants $\sigma_1>0$ and $\delta_1\in[\delta,1)$ such that

$$\frac{\phi(\lambda x)}{\phi(\lambda)} \ge \sigma_1\, x^{1-\delta_1} \quad \text{for all } \ x\ge 1 \ \text{ and } \ \lambda\ge\lambda_0\,. \tag{1.4}$$

Remark 1.1. (a) Note that **(A-3)** implies $b=0$ in (1.1), by letting $\lambda\to\infty$.

(b) The condition **(A-3)** is implied by the following stronger condition:

$$\forall\, x>0 \qquad \lim_{\lambda\to\infty}\frac{\phi'(\lambda x)}{\phi'(\lambda)} = x^{\frac{\alpha}{2}-1} \quad (0\le\alpha<2)\,. \tag{1.5}$$

In other words, (1.5) says that ϕ' varies regularly at infinity with index $\frac{\alpha}{2}-1$. A novelty here is the case $\alpha=0$.

(c) The condition **(A-4)** is used only to obtain Green function estimates.

Now we state the main result of this paper. By $\mathrm{diam}(D)$ we denote the diameter of D.

Theorem 1.2. *Suppose that $X = (X_t, \mathbb{P}_x : t \geq 0, x \in \mathbb{R}^d)$ is a transient subordinate Brownian motion whose characteristic exponent is given by $\Phi(\theta) = \phi(|\theta|^2)$, $\theta \in \mathbb{R}^d$, with ϕ satisfying* **(A-1)**–**(A-5)**.

Then for every bounded $C^{1,1}$ open set D (see Definition 3.4) in $\mathbb{R}^d$ with characteristics (R, Λ), there exists $c = c(diam(D), R, \Lambda, \phi, d) > 1$ such that the Green function $G_D(x, y)$ of X in D satisfies

$$c^{-1} g_D(x, y) \leq G_D(x, y) \leq c g_D(x, y) \tag{1.6}$$

with

$$g_D(x, y) = \left(1 \wedge \frac{\phi(|x-y|^{-2})}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})}}\right) \frac{\phi'(|x-y|^{-2})}{|x-y|^{d+2}\phi(|x-y|^{-2})^2}\,. \tag{1.7}$$

Before we discuss a corollary of Theorem 1.2, we record a simple fact.

Lemma 1.3. *If $\delta_* \in (0, 1)$ and ψ is a Bernstein function satisfying*

$$\frac{\psi(\lambda x)}{\psi(\lambda)} \geq \sigma_* \, x^{1-\delta_*} \quad \textit{for all} \quad x \geq 1 \quad \textit{and} \quad \lambda \geq \lambda_*\,, \tag{1.8}$$

for some $\sigma_*, \lambda_* > 0$, then there exists a constant $c > 0$ such that $\psi(\lambda) \leq c\lambda\psi'(\lambda)$ for all $\lambda \geq \lambda_*$.

Proof. Let $a_1 = 2 \vee (\frac{2}{\sigma_*})^{\frac{1}{1-\delta_*}}$. Since ψ' is decreasing,

$$(a_1 - 1)\lambda\psi'(\lambda) \geq \int_{\lambda}^{a_1\lambda} \psi'(t)dt = \psi(a_1\lambda) - \psi(\lambda). \tag{1.9}$$

Let $\lambda \geq \lambda_*$. Since $\psi(a_1\lambda) \geq \sigma_* \, a_1^{1-\delta_*}\psi(\lambda)$ by (1.8), we get from (1.9)

$$(a_1 - 1)\lambda\psi'(\lambda) \geq (\sigma_* a_1^{1-\delta_*} - 1)\psi(\lambda) \geq \psi(\lambda).$$

□

Now we consider the following upper and lower scaling conditions on the Laplace exponent ϕ with exponents in the range $(0, 1)$: there exist constants $c_1, c_2, \lambda_1 > 0$, $\alpha, \beta \in (0, 2)$ and $\alpha \leq \beta$ such that

$$c_1 \, x^{\alpha/2} \leq \frac{\phi(\lambda x)}{\phi(\lambda)} \leq c_2 \, x^{\beta/2} \quad \text{for all} \quad x \geq 1 \quad \text{and} \quad \lambda \geq \lambda_1\,. \tag{1.10}$$

Define

$$\widehat{g}_D(x, y) = \left(1 \wedge \frac{\phi(|x-y|^{-2})}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})}}\right) \frac{1}{|x-y|^d \, \phi(|x-y|^{-2})}. \tag{1.11}$$

If ϕ is a complete Bernstein function such that (1.10) holds, then

$$\liminf_{x\to\infty} \phi(x) \geq c_1\lambda_1^{-\alpha/2}\phi(\lambda_1) \liminf_{x\to\infty} x^{\alpha/2} = \infty.$$

Thus **(A-1)**–**(A-2)** hold. Moreover, applying Lemma 1.3 and (2.3) below, (1.10) implies that $\lambda\phi'(\lambda) \leq \phi(\lambda) \leq c\lambda\phi'(\lambda)$ for all $\lambda \geq \lambda_0$, and so **(A-3)** and **(A-5)** hold and (1.6) is equivalent to (1.12). Furthermore, **(A-4)** holds when $d = 2$. Therefore Theorem 1.2 gives the following extension of the main result in [KSV12b].

Corollary 1.4. *Suppose that $X = (X_t : t \geq 0)$ is a transient subordinate Brownian motion whose characteristic exponent is given by $\Phi(\theta) = \phi(|\theta|^2)$, $\theta \in \mathbb{R}^d$, where $\phi : (0, \infty) \to [0, \infty)$ is a complete Bernstein function such that* (1.10) *holds. We further assume that* **(A-4)** *holds with $\delta_0 = 1 - \beta/2$ when $d = 1$.*

Then for every bounded $C^{1,1}$ open set D in $\mathbb{R}^d$ with characteristics (R, Λ), there exists $c = c(diam(D), R, \Lambda, \phi, d) > 1$ such that the Green function $G_D(x, y)$ of X in D satisfies the following estimates:

$$c^{-1}\widehat{g}_D(x, y) \leq G_D(x, y) \leq c\widehat{g}_D(x, y)\,, \tag{1.12}$$

where $\widehat{g}_D(x, y)$ is defined in (1.11).

In [KSV12b], the above result is proved when, instead of (1.10), ϕ satisfies

$$\phi(\lambda) \asymp \lambda^{\alpha/2}\ell(\lambda),\ \lambda \to \infty \quad (0 < \alpha < 2)\,, \tag{1.13}$$

where ℓ varies slowly at infinity, i.e.

$$\forall\, x > 0 \qquad \lim_{\lambda\to\infty} \frac{\ell(\lambda x)}{\ell(\lambda)} = 1\,.$$

By Potter's theorem (see [BGT87, Theorem 1.5.6(i)]), (1.13) clearly implies (1.10).

Using Green function estimates in Theorem 1.2 for $d \geq 3$ and a dimension reduction argument (see the proof of Theorem 5.6), we prove the boundary Harnack principle for subordinate Brownian motions satisfying **(A-1)**, **(A-2)**, **(A-3)** and **(A-5)** in a $C^{1,1}$ open set. We emphasize that in the next theorem we do not assume either the transience or **(A-4)**.

Theorem 1.5. *Suppose that $X = (X_t, \mathbb{P}_x : t \geq 0, x \in \mathbb{R}^d)$ is a (not necessarily transient) subordinate Brownian motion satisfying* **(A-1)**, **(A-2)**, **(A-3)** *and* **(A-5)** *and that D is a (possibly unbounded) $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristics (R, Λ). Then there exists $c = c(R, \Lambda, \phi) > 0$ such that for every $r \in (0, \frac{R\wedge 1}{4}]$, $z \in \partial D$ and any nonnegative function u in $\mathbb{R}^d$ that is harmonic in $D \cap B(z, r)$ with respect to X and vanishes continuously on $D^c \cap B(z, r)$, we have*

$$\frac{u(x)}{u(y)} \leq c\sqrt{\frac{\phi(\delta_D(y)^{-2})}{\phi(\delta_D(x)^{-2})}} \qquad \text{for every } x, y \in D \cap B(z, \tfrac{r}{2}). \tag{1.14}$$

We remark that Theorem 1.5 covers the processes in Examples 1-3 without the assumptions on transience.

By the same argument used to obtain Corollary 1.4 from Theorem 1.2, Theorem 1.5 gives the following extension of the boundary Harnack principle in [KSV12b].

Corollary 1.6. *Suppose that $X = (X_t, \mathbb{P}_x : t \geq 0, x \in \mathbb{R}^d)$ is a subordinate Brownian motion whose characteristic exponent is given by $\Phi(\theta) = \phi(|\theta|^2)$, $\theta \in \mathbb{R}^d$, where $\phi : (0, \infty) \to [0, \infty)$ is a complete Bernstein function such that* (1.10) *holds, and that D is a $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristics (R, Λ). Then there exists $c = c(R, \Lambda, \phi) > 0$ such that for every $r \in (0, \frac{R\wedge 1}{4}]$, $z \in \partial D$ and any nonnegative function u in $\mathbb{R}^d$ that is harmonic in $D \cap B(z, r)$ with respect to X and vanishes continuously on $D^c \cap B(z, r)$, we have* (1.14).

Our paper is organized as follows. In Section 2 we record some preliminary results concerning subordinate Brownian motions obtained in [KM12]. We start Section 3 by analyzing special harmonic functions in half-space and use these results to obtain key probabilistic estimates on $C^{1,1}$ open sets. Section 4 contains estimates of a Poisson kernel on balls which are used in Section 5 to obtain the uniform boundary Harnack principle on arbitrary open sets. After proving sharp Green function estimates in Lipschitz domains in Section 6, we finally obtain in Section 7 the boundary Harnack principle and sharp Green function estimates in $C^{1,1}$ open sets.

Notation. Throughout the paper we use the notation $f(r) \asymp g(r),\ r \to a$ to denote that $\frac{f(r)}{g(r)}$ stays between two positive constants as $r \to a$. We say that $f\colon \mathbb{R} \to \mathbb{R}$ is increasing if $s \le t$ implies $f(s) \le f(t)$, and analogously for a decreasing function. For $a, b \in \mathbb{R}$, we set $a \wedge b := \min\{a, b\}$, $a \vee b := \max\{a, b\}$. For a Borel set $A \subset \mathbb{R}^d$, we also use $|A|$ to denote its Lebesgue measure. We will use ":=" to denote a definition, which is read as "is defined to be".

We will use the following conventions in this paper. The values of the constants C_1, C_2, C_3, C_4 and ε_1 will remain the same throughout this paper, while $c, c_1, c_2, \dots$ stand for constants whose values are unimportant and which may change from one appearance to another. All constants are positive finite numbers. The labeling of the constants $c_1, c_2, \dots$ starts anew in the proof of each result. The dependence of the constants $c, c_1, c_2, \dots$ on the dimension d will not be mentioned explicitly.

2. Preliminaries

By concavity, we see that every Bernstein function ψ satisfies

$$\psi(t\lambda) \le \lambda\psi(t) \qquad \text{for all } \lambda \ge 1, t > 0. \tag{2.1}$$

Thus

$$\lambda \mapsto \frac{\psi(\lambda)}{\lambda} \ \text{ is decreasing,} \tag{2.2}$$

which implies

$$\lambda\psi'(\lambda) \le \psi(\lambda) \qquad \text{for all } \lambda > 0. \tag{2.3}$$

We first recall the following results from [KM12].

Lemma 2.1 ([KM12, Lemma 4.1])**.** *Suppose that ψ is a special Bernstein function, i.e., $\lambda \mapsto \frac{\lambda}{\psi(\lambda)}$ is also a Bernstein function. Then the functions $\eta_1, \eta_2\colon (0,\infty) \to (0,\infty)$ given by*

$$\eta_1(\lambda) = \lambda^2\psi'(\lambda) \quad \text{and} \quad \eta_2(\lambda) = \lambda^2\frac{\psi'(\lambda)}{\psi(\lambda)^2}$$

are increasing.

The next result is a simple consequence of Lemma 2.1 and we will use it several times in this paper.

Corollary 2.2. *Suppose that ψ is a special Bernstein function. For every $d \ge 1$, $a > 1$, $\lambda > 0$ and $b \in (0,1)$ we have*

$$ba^{-d-3}\lambda^{-d-2}\tfrac{\psi'(\lambda^{-2})}{\psi(\lambda^{-2})^2} \le t^{-d-2}\tfrac{\psi'(t^{-2})}{\psi(t^{-2})^2} \le ab^{-d-3}\lambda^{-d-2}\tfrac{\psi'(\lambda^{-2})}{\psi(\lambda^{-2})^2} \quad \forall t \in [b\lambda, a\lambda].$$

Proof. We use the fact that $t \to t^{-4}\frac{\psi'(t^{-2})}{\psi(t^{-2})^2}$ is decreasing (by Lemma 2.1) and $t \to \frac{\psi'(t^{-2})}{\psi(t^{-2})^2}$ is increasing. When $d \ge 2$ for all $0 < b\lambda \le t \le a\lambda$,

$$a^{-d-2}\lambda^{-d-2}\tfrac{\psi'(\lambda^{-2})}{\psi(\lambda^{-2})^2} \le t^{-d-2}\tfrac{\psi'(t^{-2})}{\psi(t^{-2})^2} \le b^{-d-2}\lambda^{-d-2}\tfrac{\psi'(\lambda^{-2})}{\psi(\lambda^{-2})^2}.$$

If $d = 1$, then for every $0 < b\lambda \le t \le a\lambda$,

$$t^{-3}\tfrac{\psi'(t^{-2})}{\psi(t^{-2})^2} = t\left(t^{-4}\tfrac{\psi'(t^{-2})}{\psi(t^{-2})^2}\right) \le a\lambda\left(b^{-4}\lambda^{-4}\tfrac{\psi'((b\lambda)^{-2})}{\psi((b\lambda)^{-2})^2}\right)$$
$$\le ab^{-4}\lambda^{-3}\tfrac{\psi'((b\lambda)^{-2})}{\psi((b\lambda)^{-2})^2} \le ab^{-4}\lambda^{-3}\tfrac{\psi'(\lambda^{-2})}{\psi(\lambda^{-2})^2},$$

and similarly

$$t^{-3}\tfrac{\psi'(t^{-2})}{\psi(t^{-2})^2} \ge ba^{-4}\lambda^{-3}\tfrac{\psi'(\lambda^{-2})}{\psi(\lambda^{-2})^2}.$$

□

Recall that we will always assume that the Laplace exponent ϕ of S satisfies **(A-1)**–**(A-3)**. We also recall the following elementary fact from [KM12] which says that **(A-3)** controls the growth of ϕ.

Lemma 2.3 ([KM12, Lemma 3.2 (ii)])**.** *For every $\varepsilon > 0$ there exists $c(\varepsilon, \sigma) > 1$ such that*

$$\frac{\phi(\lambda x)}{\phi(\lambda)} \le c\, x^{1-\delta+\varepsilon} \quad \textit{for all} \quad x \ge 1 \quad \textit{and} \quad \lambda \ge \lambda_0\,. \tag{2.4}$$

The analysis of one-dimensional subordinate Brownian motions will be crucial in our approach in this paper. Therefore we now consider a one-dimensional subordinate Brownian motion $(Z_t, \mathbb{P}_x)$ with the characteristic exponent $\phi(\theta^2)$, $\theta \in \mathbb{R}$.

Let

$$\overline{Z}_t := \sup\{0 \vee Z_s : 0 \le s \le t\}$$

be the supremum process of Z and let $L = (L_t : t \ge 0)$ be a local time of $\overline{Z} - Z$ at 0. The right continuous inverse L_t^{-1} of L is a subordinator and it is called the ladder time process of Z. The process $\overline{Z}_{L_t^{-1}}$ is also a subordinator, called the ladder height process of Z. (For the basic properties of the ladder time and ladder height processes, we refer the reader to [Ber96, Chapter 6].)

Let κ be the Laplace exponent of the ladder height process of Z. It follows from [Fri74, Corollary 9.7] that

$$\kappa(\lambda) = \exp\left\{\frac{1}{\pi}\int_0^\infty \frac{\log(\phi(\lambda^2\theta^2))}{1+\theta^2}d\theta\right\}, \quad \forall\lambda > 0. \tag{2.5}$$

By our assumptions and [KSV12a, Proposition 13.3.7] or [KMR, Proposition 2.1] we see that the ladder height process of Z has no drift and is not compound Poisson, and so the process Z does not creep upwards. Since Z is symmetric, we know that Z also does not creep downwards.

Denote by V the potential measure of the ladder height process of Z. We will slightly abuse notation and use the same letter V to denote the renewal function of the ladder height process of Z, that is, $V(t) = V((0,t))$. V is a smooth function by [KSV12a, Corollary 13.3.8].

Combining [KSV12a, Proposition 13.3.7] and [Ber96, Proposition III.1] the following result holds.

Proposition 2.4. *There exists a constant $c > 1$ such that for all $r > 0$,*

$$\frac{c^{-1}}{\sqrt{\phi(r^{-2})}} \le V(r) \le \frac{c}{\sqrt{\phi(r^{-2})}}.$$

We next consider multi-dimensional subordinate Brownian motions. Let $W = (W_t = (W_t^1, \ldots, W_t^d) : t \geq 0)$ be a Brownian motion in $\mathbb{R}^d$ with

$$\mathbb{E}\left[e^{i\theta\cdot(W_t - W_0)}\right] = e^{-t|\theta|^2}, \qquad \forall\, \theta \in \mathbb{R}^d, t > 0\,,$$

and let S be a subordinator independent of W with Laplace exponent ϕ. In the remainder of this paper, we always assume that $X = (X_t, \mathbb{P}_x)$ is a subordinate process defined by $X_t = W_{S_t}$. This process is a pure-jump symmetric Lévy process with the characteristic exponent $\Phi(\xi) = \phi(|\xi|^2)$, i.e.

$$\mathbb{E}_0\left[e^{i\xi\cdot X_t}\right] = e^{-t\Phi(\xi)} = e^{-t\phi(|\xi|^2)}\,.$$

Moreover, Φ has the representation

$$\Phi(\xi) = \int_{\mathbb{R}^d} (1 - \cos(\xi\cdot y))j(|x|)\,dx$$

with the Lévy measure of the form $\Pi(dx) = j(|x|)\,dx$, where

$$j(r) = \int_{(0,\infty)} (4\pi t)^{-d/2} \exp\left(-\tfrac{r^2}{4t}\right)\mu(dt),\ r > 0.$$

For any open set D, let us denote by τ_D the first exit time of D, i.e.

$$\tau_D = \inf\{t > 0 :\ X_t \notin D\}\,.$$

Using Proposition 2.4, the proof of the next result is the same as the one of [KSV12c, Proposition 3.2]. So we skip the proof.

Lemma 2.5. *There exists $c > 0$ such that for any $r \in (0,\infty)$ and $x_0 \in \mathbb{R}^d$,* $\mathbb{E}_x[\tau_{B(x_0,r)}] \leq c\,V(r)V(r - |x - x_0|) \asymp \frac{1}{\sqrt{\phi(r^{-2})\phi((r-|x-x_0|)^{-2})}}$ *for* $x \in B(x_0, r)$.

The process X has a transition density $p(t,x,y)$ given by

$$p(t,x,y) = \int_0^\infty (4\pi t)^{-d/2} \exp\left(-\tfrac{|x-y|^2}{4t}\right)\mathbb{P}(S_t \in ds)\,. \tag{2.6}$$

When X is transient, we can define the Green function (potential) by

$$G(x,y) = g(|y-x|) = \int_0^\infty p(t,x,y)\,dt\,.$$

Note that g and j are decreasing.

The following result is proved in [KM12]. Note that there is an error in the statement in [KM12, Proposition 4.2]. It is clear from the proof of [KM12, Proposition 4.2] that [KM12, Proposition 4.2] holds under the conditions (A-1), (A-3) and (B) in [KM12].

Proposition 2.6. *Suppose ϕ satisfies* **(A-1)**–**(A-4)**. *Then we have*

$$j(r) \asymp r^{-d-2}\phi'(r^{-2}), \quad r \to 0+\ . \tag{2.7}$$

If X is transient, then

$$g(r) \asymp r^{-d-2}\frac{\phi'(r^{-2})}{\phi(r^{-2})^2},\ r \to 0+\ . \tag{2.8}$$

□

As a consequence of (2.7) it follows that if ϕ satisfies **(A-1)**–**(A-4)**, then for any $K > 0$, there exists $c = c(K) > 1$ such that

$$j(r) \leq c\, j(2r), \qquad \forall r \in (0, K). \tag{2.9}$$

Since ϕ is a complete Bernstein function, there exists a constant $c > 0$ such that $\mu(t) \leq c\, \mu(t+1)$ for all $t \geq 1$ (see [KSV12b, Lemma 2.1]). Thus, using this and [KM12, Proposition 3.3], by the proof of [KSV12a, Proposition 13.3.5] we see that the function j also enjoys the following property: if ϕ satisfies **(A-1)**–**(A-4)**, then there is a constant $c > 0$ such that

$$j(r+1) \leq j(r) \leq cj(r+1) \quad \text{for all} \quad r \geq 1\,. \tag{2.10}$$

Let $D \subset \mathbb{R}^d$ be an open subset. The killed process X^D is defined by

$$X_t^D = X_t \quad \text{if} \quad t < \tau_D \qquad \text{and} \qquad X_t^D = \Delta \quad \text{otherwise,}$$

where Δ is an extra point adjoined to D (usually called a cemetery).

The transition density of X^D is given by

$$p_D(t, x, y) = p(t, x, y) - \mathbb{E}_x \left[p(t - \tau_D, X_{\tau_D}, y); \tau_D < t \right].$$

A subset D of $\mathbb{R}^d$ is said to be Greenian (for X) if X^D is transient. When $d \geq 3$, any nonempty open set $D \subset \mathbb{R}^d$ is Greenian. An open set $D \subset \mathbb{R}^d$ is Greenian if and only if D^c is nonpolar for X (or equivalently, has positive capacity with respect to X). For any Greenian open set D in $\mathbb{R}^d$ let $G_D(x, y) = \int\limits_0^\infty p_D(t, x, y)\, dt$ be the Green function of X^D. $G_D(x, y)$ is symmetric and, for fixed $y \in D$, $G_D(\cdot, y)$ is harmonic (with respect to X) in $D \setminus \{y\}$.

The next two results are the key estimates in [KM12].

Proposition 2.7. *Suppose X is transient and ϕ satisfies* **(A-1)**–**(A-4)**. *There exist constants $c_1, c_2 > 0$ and $b_1, b_2 \in (0, \frac{1}{2})$, $2b_1 < b_2$ such that for all $x_0 \in \mathbb{R}^d$ and $r \in (0, 1)$ we have*

$$c_1 \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})}\, \mathbb{E}_y \tau_{B(x_0,r)} \leq G_{B(x_0,r)}(x, y) \leq c_2 \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})}\, \mathbb{E}_y \tau_{B(x_0,r)} \tag{2.11}$$

for all $x \in B(x_0, b_1 r)$ and $y \in B(x_0, r) \setminus B(x_0, b_2 r)$.

Proposition 2.8. *Suppose X is transient and ϕ satisfies* **(A-1)**–**(A-4)**. *There exist constants $c_1 > 0$ and $a \in (0, \frac{1}{3})$ so that for $x_0 \in \mathbb{R}^d$ and $r \in (0, 1)$ we have*

$$\mathbb{E}_x[\tau_{B(x_0,r)}] \geq \frac{c_1}{\phi(r^{-2})} \qquad \text{for any} \quad x \in B(x_0, ar)\,.$$

Before we state the Harnack inequality, we recall the definition of harmonic functions.

Definition 2.9. Let D be an open subset of $\mathbb{R}^d$. A function u defined on $\mathbb{R}^d$ is said to be

(i) harmonic in D with respect to X if

$$\mathbb{E}_x \left[|u(X_{\tau_B})| \right] < \infty \quad \text{and} \quad u(x) = \mathbb{E}_x \left[u(X_{\tau_B}) \right],\ x \in B\,,$$

for every open set B whose closure is a compact subset of D;

(ii) regular harmonic in D with respect to X if it is harmonic in D with respect to X and

$$u(x) = \mathbb{E}_x \left[u(X_{\tau_D}) \right] \quad \text{for any} \quad x \in D\,.$$

The following Harnack inequality is the main result of [KM12].

Theorem 2.10 (Harnack inequality). *Suppose that ϕ satisfies* **(A-1)**–**(A-3)**. *There exists a constant $c > 0$ such that for all $x_0 \in \mathbb{R}^d$ and $r \in (0,1)$ we have*

$$h(x_1) \le c\, h(x_2) \quad \text{for all} \quad x_1, x_2 \in B(x_0, r/2)$$

and for every nonnegative function $h\colon \mathbb{R}^d \to [0,\infty)$ which is harmonic in $B(x_0, r)$.

Using Theorem 2.10 and the standard chain argument to (2.11), we have

Corollary 2.11. *Under the assumptions of Proposition 2.7 there exist constants $c_1, c_2 > 0$ so that for any $r \in (0,1)$ and $x_0 \in \mathbb{R}^d$,*

$$c_1 \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})}\, \mathbb{E}_y \tau_{B(x_0,r)} \le G_{B(x_0,r)}(x,y) \le c_2 \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})}\, \mathbb{E}_y \tau_{B(x_0,r)}$$

for all $x \in B(x_0, r/2)$ and $y \in B(x_0, r) \setminus B(x_0, 3r/4)$.

By the result of Ikeda and Watanabe (see [IW62, Theorem 1]) the following formula is true:

$$\mathbb{P}_x(X_{\tau_D} \in F) = \int_F \int_D G_D(x,y) j(|z-y|)\, dy\, dz \tag{2.12}$$

for any $F \subset \overline{D}^c$. We define the Poisson kernel of the set D by

$$K_D(x,z) = \int_D G_D(x,y) j(|z-y|)\, dy, \tag{2.13}$$

so that $\mathbb{P}_x(X_{\tau_D} \in F) = \int_F K_D(x,z)\, dz$ for any $F \subset \overline{D}^c$.

Proposition 2.12. *Suppose X is transient and ϕ satisfies* **(A-1)**–**(A-4)**. *There exists $c_1 = c_1(\phi) > 0$ and $c_2 = c_2(\phi) > 0$ such that for every $r \in (0,1)$ and $x_0 \in \mathbb{R}^d$,*

$$K_{B(x_0,r)}(x,y) \le c_1 \frac{j(|y-x_0|-r)}{\sqrt{\phi(r^{-2})\phi((r-|x-x_0|)^{-2})}} \tag{2.14}$$

$$\le c_1 \frac{j(|y-x_0|-r)}{\phi(r^{-2})} \tag{2.15}$$

for all $(x,y) \in B(x_0,r) \times \overline{B(x_0,r)}^c$ and

$$K_{B(x_0,r)}(x_0,y) \ge c_2 \frac{j(|y-x_0|)}{\phi(r^{-2})} \qquad \text{for all } y \in \overline{B(x_0,r)}^c. \tag{2.16}$$

Proof. First using (2.9) and (2.10) to (2.13), then applying Lemma 2.5 and Proposition 2.8, (2.14) and (2.16) follow easily (see the proof of [KSV12a, Proposition 13.4.10] for the details). (2.15) follows from (2.14) and the fact that ϕ is increasing. □

3. Analysis on half-space and $C^{1,1}$ open sets

In this section we establish key estimates which will be used in sections later in this paper.

Recall that $X = (X_t : t \ge 0)$ is the d-dimensional subordinate Brownian motion defined by $X_t = W_{S_t}$ where $W = (W^1, \dots, W^d)$ is a (not necessarily transient) d-dimensional Brownian motion and $S = (S_t : t \ge 0)$ an independent subordinator with the Laplace exponent ϕ satisfying **(A-1)-(A-3)**. In this section, we further assume that **(A-4)** holds.

Let $Z = (Z_t : t \ge 0)$ be the one-dimensional subordinate Brownian motion defined by $Z_t := W^d_{S_t}$.

Recall that V denotes the renewal function of the ladder height process of Z. We use the notation

$$\mathbb{R}^d_+ := \{x = (x_1, \dots, x_{d-1}, x_d) := (\tilde{x}, x_d) \in \mathbb{R}^d : x_d > 0\}$$

for the half-space.

Set $w(x) := V((x_d)^+)$. Since $Z_t = W^d_{S_t}$ has a transition density, by using [Sil80, Theorem 2], the proof of the next result is the same as the one of [KSV12b, Theorem 4.1]. We omit the proof.

Theorem 3.1. *The function w is harmonic in $\mathbb{R}^d_+$ with respect to X and, for any $r > 0$, regular harmonic in $\mathbb{R}^{d-1} \times (0, r)$ for X.*

Using Theorem 3.1, (2.9) and (2.10), the proof of the next result is the same as the one of [KSV, Proposition 3.3].

Proposition 3.2. *For all positive constants r_0 and L, we have*

$$\sup_{x \in \mathbb{R}^d :\, 0 < x_d < L} \int_{B(x, r_0)^c \cap \mathbb{R}^d_+} w(y) j(|x - y|)\, dy < \infty\,.$$

Define an operator $(\mathcal{A}, \mathfrak{D}(\mathcal{A}))$ by

$$\begin{aligned}\mathcal{A}f(x) &:= \text{p.v.} \int_{\mathbb{R}^d} (f(y) - f(x))\, j(|y - x|)\, dy \\ &:= \lim_{\varepsilon \downarrow 0} \int_{\{y \in \mathbb{R}^d : |x-y| > \varepsilon\}} (f(y) - f(x))\, j(|y - x|)\, dy\end{aligned}$$

$$\mathfrak{D}(\mathcal{A}) := \left\{ f : \mathbb{R}^d \to \mathbb{R} : \lim_{\varepsilon \downarrow 0} \int_{\{y \in \mathbb{R}^d : |x-y| > \varepsilon\}} (f(y) - f(x))\, j(|y - x|)\, dy \right.$$

$$\left. \text{exists and it is finite} \right\}. \tag{3.1}$$

Let C^2_0 be the collection of C^2 functions in $\mathbb{R}^d$ vanishing at infinity. It is well known that $C^2_0 \subset \mathfrak{D}(\mathcal{A})$ and that by the rotational symmetry of X, $\mathcal{A}$ restricted to C^2_0 coincides with the infinitesimal generator $\mathcal{L}$ of the process X (see e.g. [Sat99, Theorem 31.5]).

Since V is smooth by [KSV12a, Corollary 13.3.8], using our Theorem 3.1, (2.9) and (2.10), the proof of the next result is the same as [KSV, Proposition 3.3] or [KSV12b, Proposition 4.2], so we skip the proof.

Theorem 3.3. *$\mathcal{A}w(x)$ is well defined and $\mathcal{A}w(x) = 0$ for all $x \in \mathbb{R}^d_+$.*

In the rest of this section we aim to prove two key estimates of the exit probability and the exit time for $C^{1,1}$ open sets. Let us recall the definition of a $C^{1,1}$ open set.

Definition 3.4. An open set D in $\mathbb{R}^d$ ($d \geq 2$) is said to be a $C^{1,1}$ open set if there exist a localization radius $R > 0$ and a constant $\Lambda > 0$ such that for every $z \in \partial D$, there exist a $C^{1,1}$-function $\psi = \psi_z : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\psi(0) = 0$, $\nabla\psi(0) = (0, \dots, 0)$,

$$\|\nabla\psi\|_\infty \leq \Lambda, \quad |\nabla\psi(x) - \nabla\psi(w)| \leq \Lambda|x - w|, \quad x, w \in \mathbb{R}^{d-1}$$

and an orthonormal coordinate system CS_z: $y = (y_1, \dots, y_{d-1}, y_d) := (\widetilde{y},\, y_d)$ with origin at z such that

$$B(z,R) \cap D = \{y = (\widetilde{y},\, y_d) \in B(0,R) \text{ in } CS_z : y_d > \psi(\widetilde{y})\}.$$

The pair (R,Λ) is called the characteristic of the $C^{1,1}$ open set D. By a $C^{1,1}$ open set in $\mathbb{R}$ we mean an open set which can be expressed as the union of disjoint intervals so that the minimum of the lengths of all these intervals is positive and the minimum of the distances between these intervals is positive.

Remark 3.5. In some literature, the $C^{1,1}$ open set defined above is called a *uniform* $C^{1,1}$ open set since (R,Λ) is universal for all $z \in \partial D$.

For $x \in \mathbb{R}^d$, let $\delta_{\partial D}(x)$ denote the Euclidean distance between x and ∂D. Recall that for any $x \in \mathbb{R}^d$, $\delta_D(x)$ is the Euclidean distance between x and D^c. It is well known that any $C^{1,1}$ open set D with characteristics (R,Λ) there exists $r_1 > 0$ so that the following holds true:

(i) *uniform interior ball condition*, i.e. for every $x \in D$ with $\delta_D(x) < r_1$ there exists $z_x \in \partial D$ so that
$$|x - z_x| = \delta_{\partial D}(x) \quad \text{and} \quad B(x_0, r_1) \subset D,$$
for $x_0 = z_x + r_1 \frac{x - z_x}{|x - z_x|}$;

(ii) *uniform exterior ball condition*, i.e. for every $y \in \mathbb{R}^d \setminus D$ with $\delta_{\partial D}(y) < r_1$ there exists $z_y \in \partial D$ so that
$$|y - z_y| = \delta_{\partial D}(y) \quad \text{and} \quad B(y_0, r_1) \subset \mathbb{R}^d \setminus D,$$
for $y_0 = z_y + r_1 \frac{y - z_y}{|y - z_y|}$.

Assume for the rest of this section that D is a $C^{1,1}$ open set with characteristics (R,Λ) satisfying the uniform interior ball condition and the uniform exterior ball condition with the radius $R \le 1$ (by choosing R smaller if necessary).

Before we prove our technical Lemma 3.7 below, we need some preparation.

Lemma 3.6. *Suppose that* $d \ge 2$ *and the constant* δ *in* **(A-3)** *satisfies* $0 < \delta \le \frac{1}{2}$. *If* **(A-5)** *holds, then for every* $M > 0$,

$$\sup_{x \in [0, M/4]} \int_0^M v(s/6) \left(\phi(|s-x|^{-2})|s-x| + \int_{|s-x|}^M \phi(r^{-2}) dr \right) ds = c(M,\phi) < \infty.$$

Proof. For $x \in [0, M/4]$, let

$$I := \int_0^{x/2} v(s/6) \left(\phi(|s-x|^{-2})|s-x| + \int_{|s-x|}^M \phi(r^{-2}) dr \right) ds,$$

$$II := \int_{x/2}^{2x} v(s/6) \left(\phi(|s-x|^{-2})|s-x| + \int_{|s-x|}^M \phi(r^{-2}) dr \right) ds,$$

and

$$III := \int_{2x}^{M} v(s/6) \left(\phi(|s-x|^{-2})|s-x| + \int_{|s-x|}^{M} \phi(r^{-2})dr \right) ds.$$

We consider these three parts separately.

First, for $s \in (0, x/2)$, we have $x \geq x - s = |x-s| \geq x/2$. Thus using (2.1) and Proposition 2.4,

$$\begin{aligned} I \leq & x\phi(4x^{-2}) \int_0^{x/2} v(s/6)ds + \int_0^{x/2} v(s/6)ds \int_{x/2}^{M} \phi(r^{-2})dr \\ \leq & 6\left(4x\phi(x^{-2}) + \int_{x/2}^{M} \phi(r^{-2})dr\right) V(x/12) \leq c_1 x\phi(x^{-2})^{1/2} + c_1 \int_{x/2}^{M} \tfrac{\phi(r^{-2})}{\phi(x^{-2})^{1/2}} dr. \end{aligned}$$

By (2.2), the first term is uniformly bounded by $c_1(M/4)\phi((M/4)^{-2})^{1/2}$ for $x \in [0, M/4]$. On the other hand, by Lemma 2.3 with $\varepsilon = \delta/2$,

$$\int_{x/2}^{M} \tfrac{\phi(r^{-2})}{\phi(x^{-2})^{1/2}} dr \leq c_2 \int_{x/2}^{M} \phi(r^{-2})^{1/2} \leq c_3 \int_{x/2}^{M} r^{-(1-\delta+\varepsilon)} dr \leq c_4 M^{\delta-\varepsilon} < \infty.$$

Applying Proposition 2.4, we deduce

$$v(s/6) \leq \frac{6}{s}\int_0^{s/6} v(t)dt = \frac{6}{s}V(s/6) \leq c_5 \frac{1}{s}\phi(s^{-2})^{-1/2}, \quad \text{for all } \ s > 0. \tag{3.2}$$

By (2.1) and (3.2),

$$\begin{aligned} II \leq & c_5 x^{-1}\phi(x^{-2})^{-1/2} \int_{x/2}^{2x} \phi(|s-x|^{-2})|s-x| ds \\ & + c_5 x^{-1}\phi(x^{-2})^{-1/2} \int_{x/2}^{2x} \int_{|s-x|}^{M} \phi(r^{-2}) dr ds \\ \leq & c_6 x^{-1}\phi(x^{-2})^{1/2} \int_0^{x} t\tfrac{\phi(t^{-2})}{\phi(x^{-2})} dt \\ & + c_5 x^{-1} \int_{x/2}^{2x} \tfrac{\phi(|s-x|^{-2})^{1/2}}{\phi(x^{-2})^{1/2}} \int_{|s-x|}^{M} \tfrac{\phi(r^{-2})^{1/2}}{\phi(|s-x|^{-2})^{1/2}} \phi(r^{-2})^{1/2} dr ds. \end{aligned}$$

Applying Lemma 2.3 twice with $\varepsilon = \delta/2$ to $\frac{\phi(t^{-2})}{\phi(x^{-2})}$ and $\phi(x^{-2})^{1/2}$, we get

$$x^{-1}\phi(x^{-2})^{1/2}\int_0^x t\frac{\phi(t^{-2})}{\phi(x^{-2})}dt \le c_7 x^{-1}x^{\delta-\varepsilon-1}\int_0^x t\left(\tfrac{t}{x}\right)^{-2+2(\delta-\varepsilon)}dt$$
$$= c_7 x^{-(\delta-\varepsilon)}\int_0^x t^{-1+2(\delta-\varepsilon)}dt \le c_8 x^{\delta-\varepsilon} \le c_8 M^{\delta-\varepsilon} < \infty.$$

On the other hand, since $|s-x|\le 3x$ for $s\le 2x$, (2.1), Lemma 2.3 with $\varepsilon=\delta/2$ and **(A-5)** imply

$$x^{-1}\int_{x/2}^{2x}\frac{\phi(|s-x|^{-2})^{1/2}}{\phi(x^{-2})^{1/2}}\left(\int_{|s-x|}^{M}\frac{\phi(r^{-2})^{1/2}}{\phi(|s-x|^{-2})^{1/2}}\phi(r^{-2})^{1/2}dr\right)ds$$
$$\le c_9 x^{-1}\int_{x/2}^{2x}\frac{x^{1-\delta+\varepsilon}}{|s-x|^{1-\delta+\varepsilon}}\left(\int_{|s-x|}^{M}\left(\tfrac{|s-x|}{r}\right)^{1-\delta_1}r^{-(1-\delta+\varepsilon)}dr\right)ds$$
$$= c_9 x^{-\delta+\varepsilon}\int_{x/2}^{2x}|s-x|^{-\delta_1+\delta-\varepsilon}\left(\int_{|s-x|}^{M}r^{-2+\delta_1+\delta-\varepsilon}dr\right)ds$$
$$\le c_{10}x^{-\delta+\varepsilon}\int_0^x t^{-\delta_1+\delta-\varepsilon}\left(\int_t^M r^{-2+\delta_1+\delta-\varepsilon}dr\right)dt =: A.$$

If $2-\delta_1-\delta+\varepsilon>1$,

$$A \le c_{11}x^{-\delta+\varepsilon}\int_0^x t^{-1+2(\delta-\varepsilon)}ds \le c_{12}x^{\delta-\varepsilon} \le c_{12}M^{\delta-\varepsilon} < \infty.$$

If $2-\delta_1-\delta+\varepsilon=1$, integration by parts yields

$$A \le c_{13}x^{-\delta+\varepsilon}\int_0^x t^{-\delta_1+\delta-\varepsilon}\ln(M/t)dt \le c_{14}x^{-\delta+\varepsilon}x^{1-\delta_1+\delta-\varepsilon}\ln(M/x)$$
$$\le c_{14}\sup_{x\in[0,M/4]}x^{1-\delta_1}\ln(M/x) < \infty.$$

If $2-\delta_1-\delta+\varepsilon<1$,

$$A \le c_{10}x^{-\delta+\varepsilon}\int_0^x t^{-\delta_1+\delta-\varepsilon}\left(\int_0^M r^{-2+\delta_1+\delta-\varepsilon}dr\right)dt \le c_{15}x^{1-\delta_1} \le c_{15}M^{1-\delta_1} < \infty.$$

Thus $II<\infty$.

For III, we note that $s \geq s - x = |s - x| \geq s/2$ for $s \geq 2x$. Using this, (2.1), (3.2), Lemma 2.3 with $\varepsilon = \delta/2$ and **(A-5)**, we get

$$
\begin{aligned}
III &\leq \int_{2x}^{M} v(s/6) s\phi(4s^{-2}) ds + \int_{2x}^{M} v(s/6) \int_{s/2}^{M} \phi(r^{-2}) dr ds \\
&\leq c_{16} \int_{2x}^{M} \phi(s^{-2})^{1/2} ds + c_{16} \int_{2x}^{M} s^{-1} \int_{s/2}^{M} \tfrac{\phi(r^{-2})^{1/2}}{\phi(s^{-2})^{1/2}} \phi(r^{-2})^{1/2} dr ds \\
&\leq c_{17} \int_{0}^{M} s^{-1+(\delta-\varepsilon)} ds + c_{17} \int_{2x}^{M} s^{-1} \int_{s/2}^{M} (s/r)^{1-\delta_1} \phi(r^{-2})^{1/2} dr ds.
\end{aligned}
$$

Clearly the first term is finite. Using Lemma 2.3 with $\varepsilon = \delta/2$, the second term is bounded by

$$
B := c_{18} \int_{2x}^{M} s^{-\delta_1} \int_{s/2}^{M} r^{-2+\delta_1+\delta-\varepsilon} dr ds.
$$

Thus if $2 - \delta_1 - \delta + \varepsilon > 1$,

$$
B \leq c_{19} \int_{2x}^{M} s^{-(1-\delta+\varepsilon)} ds \leq c_{19} \int_{0}^{M} s^{-(1-\delta+\varepsilon)} ds < \infty.
$$

If $2 - \delta_1 - \delta + \varepsilon = 1$, using integration by parts we obtain

$$
B \leq c_{20} \int_{2x}^{M} s^{-\delta_1} \ln(M/s) ds \leq c_{21} x^{1-\delta_1} \ln(M/x) \leq c_{21} \sup_{x \in [0, M/4]} x^{1-\delta_1} \ln(M/x) < \infty.
$$

Finally, If $2 - \delta_1 - \delta + \varepsilon < 1$,

$$
B \leq \; c_{18} \int_{2x}^{M} s^{-\delta_1} \int_{0}^{M} r^{-(2-\delta_1-\delta+\varepsilon)} dr ds \leq c_{22} \int_{2x}^{M} s^{-\delta_1} ds \leq c_{22} \int_{0}^{M} s^{-\delta_1} ds < \infty.
$$

Thus $III < \infty$, and so we have proved the lemma. $\square$

Lemma 3.7. *Assume additionally that* **(A-5)** *holds. Fix $Q \in \partial D$ and let*

$$
h(y) = \begin{cases} V(\delta_D(y)), & y \in B(Q, R) \cap D, \\ 0, & \textit{otherwise}. \end{cases}
$$

There exists $C_1 = C_1(\Lambda, R, \phi) > 0$ independent of the point $Q \in \partial D$ such that $\mathcal{A}h$ is well defined in $D \cap B(Q, \frac{R}{4})$ and

$$
|\mathcal{A}h(x)| \leq C_1 \quad \textit{for all } x \in D \cap B(Q, \tfrac{R}{4}). \tag{3.3}
$$

Proof. We first note that when $d = 1$, the lemma follows from Proposition 3.2 and Theorem 3.3 by following the same proof as the one in [KSV12b, Lemma 4.4].

Assume now that $d \geq 2$. Fix $x \in D \cap B(Q, \frac{R}{4})$ and let $x_0 \in \partial D$ such that $\delta_D(x) = |x - x_0|$.

Denote by ψ a $C^{1,1}$ function and by $CS = CS_{x_0}$ an orthonormal coordinate system with x_0 chosen so that $x = (\widetilde{0}, x_d)$ and

$$B(x_0, R) \cap D = \{y = (\widetilde{y}, y_d) \text{ in } CS \colon y \in B(0,R),\ y_d > \psi(\widetilde{y})\}.$$

We fix such ψ and the coordinate system CS.

Define two auxiliary functions $\psi_1, \psi_2 \colon B(\widetilde{0}, R) \to \mathbb{R}$ by

$$\psi_1(\widetilde{y}) = R - \sqrt{R^2 - |\widetilde{y}|^2} \quad \text{and} \quad \psi_2(\widetilde{y}) = -\left(R - \sqrt{R^2 - |\widetilde{y}|^2}\right).$$

By the interior/exterior uniform ball conditions (with radius R) it follows that

$$\psi_2(\widetilde{y}) \le \psi(\widetilde{y}) \le \psi_1(\widetilde{y}) \quad \text{for any} \quad y \in D \cap B(x, \tfrac{R}{4}). \tag{3.4}$$

Now we define a function $h_x(y) = V(\delta_{H^+}(y))$, where

$$H^+ = \{y = (\widetilde{y}, y_d) \text{ in } CS \colon y_d > 0\}$$

denotes the half-space in CS.

Since $\delta_{H^+}(y) = (y_d)^+$ in CS, we can use Theorem 3.3 to deduce that

$$\mathcal{A}h_x(y) = 0, \quad \forall y \in H^+.$$

Now the idea is to show that $\mathcal{A}(h - h_x)(x)$ is well defined and that there exists a constant $C_1 = C_1(\Lambda, R, \phi) > 0$ so that

$$\int\limits_{\{y \in D \cup H^+ \colon |y-x| > \varepsilon\}} |h(y) - h_x(y)| j(|y-x|)\, dy \le C_1 \quad \text{for any} \quad \varepsilon > 0. \tag{3.5}$$

To do this we estimate the integral in (3.5) by the sum of the following three integrals:

$$I_1 = \int\limits_{B(x, \frac{R}{4})^c} (h(y) + h_x(y)) j(|y-x|)\, dy,$$

$$I_2 = \int\limits_A (h(y) + h_x(y)) j(|y-x|)\, dy, \quad \text{where}$$

$$A := \{y \in (D \cup H^+) \cap B(x, \tfrac{R}{4}) \colon \psi_2(\widetilde{y}) \le y_d \le \psi_1(\widetilde{y})\},$$

$$I_3 = \int\limits_E |h(y) - h_x(y)| j(|y-x|)\, dy, \quad \text{where } E := \{y \in B(x, \tfrac{R}{4}) \colon y_d > \psi_1(\widetilde{y})\},$$

and prove that $I_1 + I_2 + I_3 \le C_1$.

To estimate I_1 note that, by definition of h, $h = 0$ on $B(Q,R)^c$, which gives

$$I_1 \le \sup_{\substack{z \in \mathbb{R}^d \\ 0 < z_d < R}} \int\limits_{B(z, \frac{R}{4})^c \cap H^+} V(y_d) j(|z-y|)\, dy + c_1 \int\limits_{B(0, \frac{R}{4})^c} j(|y|) dy < \infty.$$

Here we have used Proposition 3.2 and the fact that the Lévy measure is a finite measure away from the origin.

Now we estimate I_2. Denoting by $m_{d-1}(dy)$ the surface measure, we obtain

$$I_2 \le \int_0^{\frac{R}{4}} \int\limits_{|\widetilde{y}| = r} \mathbf{1}_A(y)(h_x(y) + h(y)) j\left(\sqrt{r^2 + |y_d - x_d|^2}\right) m_{d-1}(dy)\, dr.$$

Since V is increasing and

$$R-\sqrt{R^2-|\widetilde{y}|^2}\le \tfrac{|\widetilde{y}|^2}{R}\le |\widetilde{y}|,$$

we can use (3.4) to deduce

$$h_x(y)+h(y)\le 2V\left(\psi_1(\widetilde{y})-\psi_2(\widetilde{y})\right)\le 2V(2|\widetilde{y}|)\,.$$

Then, by the fact that j decreases, Proposition 2.4 and (2.7), we get

$$\begin{aligned}I_2 &\le 2\int_0^{\frac{R}{4}}\int_{|\widetilde{y}|=r}\mathbf{1}_A(y)V(2|\widetilde{y}|)j(r)\,m_{d-1}(dy)\,dr\\ &\le c_2\int_0^{\frac{R}{4}} r^{-d-2}\frac{\phi'(r^{-2})}{\sqrt{\phi(r^{-2})}}\,m_{d-1}(\{y\in A\colon |\widetilde{y}|=r\})\,dr\,.\end{aligned}$$

Noting that $|\psi_2(\widetilde{y})-\psi_1(\widetilde{y})|\le \frac{2|\widetilde{y}|^2}{R}=\frac{2r^2}{R}$ for $|\widetilde{y}|=r$, we obtain

$$m_{d-1}(\{y\colon |\widetilde{y}|=r,\ \psi_2(\widetilde{y})\le y_d\le \psi_1(\widetilde{y})\})\le c_3r^d\quad\text{for}\quad r\le \tfrac{R}{4}\,.$$

Thus, by the previous observation and integration by parts we get

$$\begin{aligned}I_2 &\le c_4\int_0^{\frac{R}{4}} r^{-2}\frac{\phi'(r^{-2})}{\sqrt{\phi(r^{-2})}}\,dr\;=\;c_4\int_0^{\frac{R}{4}} r\left(-\sqrt{\phi(r^{-2})}\right)'\,dr\\ &\le c_4\left[\lim_{r\downarrow 0} r\sqrt{\phi(r^{-2})}+\int_0^{\frac{R}{4}}\sqrt{\phi(r^{-2})}\,dr\right]\,.\end{aligned}$$

By Lemma 2.3 applied to a fixed $\varepsilon<\delta$ we see that there is a constant $c_5=c_5(\varepsilon)>0$ so that

$$\phi(r^{-2})\le c_5r^{-2(1-\delta+\varepsilon)},$$

which gives

$$I_2\le c_4\int_0^{\frac{R}{4}}\sqrt{\phi(r^{-2})}\,dr\le c_4\sqrt{c_5}\int_0^{\frac{R}{4}} r^{-1+\delta-\varepsilon}\,dr<\infty\,.$$

In order to estimate I_3, we consider two cases. First, if $0<y_d=\delta_{_{H+}}(y)\le\delta_D(y)$,

(3.6)

$$h(y)-h_x(y)\le V(y_d+R^{-1}|\widetilde{y}|^2)-V(y_d)=\int_{y_d}^{y_d+R^{-1}|\widetilde{y}|^2} v(z)dz\le R^{-1}|\widetilde{y}|^2v(y_d),$$

since v is decreasing.

If $y_d=\delta_{_{H+}}(y)>\delta_D(y)$ and $y\in E$, using the fact that $\delta_D(y)$ is greater than or equal to the distance between y and the graph of ψ_1 and

$$y_d-R+\sqrt{|\widetilde{y}|^2+(R-y_d)^2}=\tfrac{|\widetilde{y}|^2}{\sqrt{|\widetilde{y}|^2+(R-y_d)^2}+(R-y_d)}\;\le\;\tfrac{|\widetilde{y}|^2}{2(R-y_d)}\le\tfrac{|\widetilde{y}|^2}{R},$$

we obtain

(3.7)
$$h_x(y) - h(y) \le \int_{R-\sqrt{|\widetilde{y}|^2+(R-y_d)^2}}^{y_d} v(z)dz \le R^{-1}|\widetilde{y}|^2\, v\left(R-\sqrt{|\widetilde{y}|^2+(R-y_d)^2}\right).$$

By (3.6) and (3.7),

$$\begin{aligned} I_3 \le& R^{-1}\int_{E\cap\{y:y_d\le\delta_D(y)\}} |\widetilde{y}|^2 v(y_d) j(|x-y|)dy \\ &+R^{-1}\int_{E\cap\{y:y_d>\delta_D(y)\}} |\widetilde{y}|^2 v\left(R-\sqrt{|\widetilde{y}|^2+(R-y_d)^2}\right) j(|x-y|)dy \\ =:&R^{-1}(L_1+L_2). \end{aligned}$$

Since

$$E\subset\{z=(\widetilde{z},z_d)\in\mathbb{R}^d:\ |\widetilde{z}|<\frac{R}{4}\wedge\sqrt{2Rz_d-z_d^2}\text{ and } 0<z_d\le\frac{R}{2}\},$$

changing to polar coordinates for $\widetilde{y}$ and using (2.1), (2.3), (2.7) and Proposition 2.4, yields

$$\begin{aligned} L_1 &\le c_6\int_0^{\frac{R}{2}} v(y_d)\left(\int_0^{\frac{R}{4}\wedge\sqrt{2Ry_d-y_d^2}} \frac{r^d\phi'((r^2+|y_d-x_d|^2)^{-1})}{(r^2+|y_d-x_d|^2)^{(d+2)/2}}dr\right)dy_d \\ &\le c_7\int_0^{\frac{R}{2}} v(y_d/6)\left(\int_0^R \frac{r^d\phi'((r+|y_d-x_d|)^{-2})}{(r+|y_d-x_d|)^{d+2}}dr\right)dy_d =: c_7\widehat{L}_1. \end{aligned}$$

If $\delta\neq\frac{1}{2}$, by **(A-3)**

$$\begin{aligned} &\int_0^R \frac{\phi'((r+|y_d-x_d|)^{-2})}{(r+|y_d-x_d|)^2}dr \\ =&\phi'((R+|y_d-x_d|)^{-2})\int_0^R \frac{\phi'((r+|y_d-x_d|)^{-2})}{\phi'((R+|y_d-x_d|)^{-2})}\frac{dr}{(r+|y_d-x_d|)^2} \\ \le&c_8\phi'((R+|y_d-x_d|)^{-2})\int_0^R \left(\frac{(r+|y_d-x_d|)^{-2}}{(R+|y_d-x_d|)^{-2}}\right)^{-\delta}\frac{dr}{(r+|y_d-x_d|)^2} \\ =&c_8\phi'((R+|y_d-x_d|)^{-2})(R+|y_d-x_d|)^{-2\delta}\int_0^R (r+|y_d-x_d|)^{-2+2\delta}dr \end{aligned}$$

$$\le c_9\phi'((R+|y_d-x_d|)^{-2})(R+|y_d-x_d|)^{-2\delta}|y_d-x_d|^{-(1-2\delta)_+}. \tag{3.8}$$

Thus, in the case $\delta > \frac{1}{2}$, (3.8) implies

$$\widehat{L}_1 \quad \leq \quad c_{11} \int_0^{\frac{R}{2}} v(y_d/6)dy_d \leq c_{12} V(\tfrac{R}{12}) < \infty. \tag{3.9}$$

For the case $\delta \leq \frac{1}{2}$, we first note that by using (2.3) we obtain

$$\begin{aligned}
&\int_0^R \frac{r^d \phi'((r+|y_d-x_d|)^{-2})}{(r+|y_d-x_d|)^{d+2}} dr \\
\leq\ & \int_0^{|s-x_d|} \frac{r^d \phi((r+|s-x_d|)^{-2})}{(r+|s-x_d|)^d} dr + \int_{|s-x_d|}^R \frac{r^d \phi((r+|s-x_d|)^{-2})}{(r+|s-x_d|)^d} dr \\
\leq\ & \frac{\phi(|s-x_d|^{-2})}{|s-x_d|^d} \int_0^{|s-x_d|} r^d dr + \int_{|s-x_d|}^R \phi(r^{-2}) dr \\
=\ & (d+1)^{-1}\phi(|s-x_d|^{-2})|s-x_d| + \int_{|s-x_d|}^R \phi(r^{-2}) dr.
\end{aligned}$$

Thus, by Lemma 3.6,

$$\widehat{L}_1 \leq c_{13} \int_0^{\frac{R}{2}} v(s/6) \left(\phi(|s-x_d|^{-2})|s-x_d| + \int_{|s-x_d|}^R \phi(r^{-2}) dr \right) ds < \infty.$$

Let us estimate L_2. Switching to polar coordinates for $\widetilde{y}$, and by the use of (2.7), we get

$$\begin{aligned}
L_2 &\leq c_{20} \int_0^{x_d+\frac{R}{4}} \left(\int_0^{\sqrt{2Ry_d-y_d^2}} v(R-\sqrt{r^2+(R-y_d)^2}) r^d j((r^2+|y_d-x_d|^2)^{1/2}) dr \right) dy_d \\
&\leq c_{21} \int_0^{x_d+\frac{R}{4}} \left(\int_0^{\sqrt{2Ry_d-y_d^2}} \frac{v(R-\sqrt{r^2+(R-y_d)^2})\phi'((r^2+|y_d-x_d|^2)^{-1})}{(r^2+|y_d-x_d|^2)^{(d+2)/2}} r^d dr \right) dy_d \\
&\leq c_{22} \int_0^{x_d+\frac{R}{4}} \left(\int_0^{\sqrt{2Ry_d-y_d^2}} \frac{v(R-\sqrt{r^2+(R-y_d)^2})\phi'((r+|y_d-x_d|)^{-2})}{(r+|y_d-x_d|)^{2+d}} r^d dr \right) dy_d.
\end{aligned}$$

Since, for $0 < r < R$,

$$R - \sqrt{r^2+(R-y_d)^2} = \frac{\left(\sqrt{2Ry_d-y_d^2}+r\right)\left(\sqrt{2Ry_d-y_d^2}-r\right)}{R+\sqrt{r^2+(R-y_d)^2}} \geq \frac{\sqrt{y_d}}{3\sqrt{R}} \left(\sqrt{2Ry_d-y_d^2} - r \right)$$

and $\sqrt{2Ry_d - y_d^2} < \sqrt{R/2}\sqrt{2R-y_d} < R$ for $0 < y_d < x_d + \frac{R}{4}$, we have

$$L_2 \le c_{22} \int_0^{x_d+\frac{R}{4}} \int_0^{\sqrt{2Ry_d-y_d^2}} \cdot \frac{v\left(\sqrt{y_d}(\sqrt{2Ry_d-y_d^2}-r)/(3\sqrt{R})\right)\phi'((r+|y_d-x_d|)^{-2})}{(r+|y_d-x_d|)^{2+d}} r^d dr dy_d\,.$$

Using (2.3), we see that with $a := \sqrt{2Ry_d - y_d^2}$ and $b := |y_d - x_d|$,

$$\begin{aligned}
&\int_0^a \frac{v(\sqrt{y_d}(a-r)/(3\sqrt{R}))\phi'((r+b)^{-2})}{(r+b)^{2+d}} r^d dr \\
&\le \int_0^{a/2} \frac{v(\sqrt{y_d}(a-r)/(3\sqrt{R}))\phi'((r+b)^{-2})}{(r+b)^{2+d}} r^d dr \\
&\quad + \int_{a/2}^a \frac{v(\sqrt{y_d}(a-r)/(3\sqrt{R}))\phi((r+b)^{-2})}{(r+b)^{d}} r^d dr \\
&\le v(\sqrt{y_d}a/(6\sqrt{R})) \int_0^{a/2} \frac{\phi'((r+b)^{-2})}{(r+b)^{2+d}} r^d dr + \phi((b+a/2)^{-2}) \int_{a/2}^a v(\sqrt{y_d}(a-r)/(3\sqrt{R}))dr \\
&\le v(\sqrt{y_d}a/(6\sqrt{R})) \int_0^{R} \frac{\phi'((r+b)^{-2})}{(r+b)^{2+d}} r^d dr + c_{23}\phi((b+a/2)^{-2}) \frac{1}{\sqrt{y_d}} V(\sqrt{y_d}a/(6\sqrt{R})) \\
&:= B_1(y_d) + B_2(y_d).
\end{aligned}$$

First, note that $\sqrt{y_d R} < \sqrt{2Ry_d - y_d^2} = a \le \sqrt{y_d}\sqrt{2R}$. Thus

$$\int_0^{\frac{R}{2}} B_1(y_d)dy_d \le c_{24} \int_0^{\frac{R}{2}} v(y_d/6) \int_0^R \frac{\phi'((r+|y_d-x_d|)^{-2})}{(r+|y_d-x_d|)^{2+d}} r^d dr dy_d = c_{24}\widehat{L}_1 < \infty.$$

Since $\sqrt{y_d R} < a \le \sqrt{y_d}\sqrt{2R}$, by Proposition 2.4 and (2.1),

$$B_2(y_d) \le c_{25} y_d^{-1/2} \phi((|y_d - x_d| + \sqrt{y_d})^{-2}) \phi({y_d}^{-2})^{-1/2}.$$

Using the inequality $y_d/\sqrt{R} \le \sqrt{y_d} \le |y_d - x_d| + \sqrt{y_d}$, we have

$$\phi((|y_d - x_d| + \sqrt{y_d})^{-2}) \le \phi((y_d/\sqrt{R})^{-2})^{1/2} \phi(y_d^{-1})^{1/2}.$$

Thus, by (2.1) and Lemma 2.3 with $\varepsilon = \delta/2$ we have

$$B_2(y_d) \le c_{26} y_d^{-1/2} \phi(y_d^{-1})^{1/2} \le c_{27} y_d^{-1/2} y_d^{(-1+\delta-\varepsilon)/2} = c_{27} y_d^{-1+(\delta-\varepsilon)/2}.$$

Thus

$$\int_0^{\frac{R}{2}} B_2(y_d)dy_d \le c_{27} \int_0^{\frac{R}{2}} y_d^{-1+(\delta-\varepsilon)/2} dy_d < \infty.$$

Therefore $L_2 < \infty$.

Now we see that $\mathcal{A}(h-h_x)(x)$ is well defined. Indeed, since $h_x(x)=h(x)$ and

$$\mathbf{1}_{\{y\in D\cup H^+ :\, |y-x|>\varepsilon\}}|h(y)-h_x(y)|j(|y-x|)$$
$$\le \mathbf{1}_{A\cup B(x,\frac{R}{4})^c}(h(y)+h_x(y))j(|y-x|)+\mathbf{1}_E|h(y)-h_x(y)|j(|y-x|)\in L^1(\mathbb{R}^d),$$

we can use the dominated convergence theorem to deduce that the limit

$$\lim_{\varepsilon\downarrow 0}\int\limits_{\{y\in D\cup H^+ :\ |y-x|>\varepsilon\}}(h(y)-h_x(y))j(|y-x|)\,dy$$

exists. Moreover, $\mathcal{A}h(x)$ is then also well defined and satisfies $|\mathcal{A}h(x)|\le C_1$. $\square$

For $a,b>0$, we define $D_Q(a,b):=\{y\in D: a>\rho_Q(y)>0,\, |\widetilde{y}|<b\}$.

Lemma 3.8. *Assume additionally that* **(A-5)** *holds. There are constants* $R_1 = R_1(R,\Lambda,\phi)\in(0,\frac{R}{16\sqrt{1+(1+\Lambda)^2}})$ *and* $c_i=c_i(R,\Lambda,\phi)>0$*,* $i=1,2$*, such that for every* $r\le R_1$*,* $Q\in\partial D$ *and* $x\in D_Q(r,r)$*,*

$$\mathbb{P}_x\left(X_{\tau_{D_Q(r,r)}}\in D\right)\ge c_1 V(\delta_D(x)) \tag{3.10}$$

and

$$\mathbb{E}_x\left[\tau_{D_Q(r,r)}\right]\ \le\ c_2 V(\delta_D(x)). \tag{3.11}$$

Proof. Without loss of generality we may assume that $Q=0$ and that $\psi\colon\mathbb{R}^{d-1}\to\mathbb{R}$ is a $C^{1,1}$ function such that in the coordinate system CS_0,

$$B(0,R)\cap D=\{(\widetilde{y},y_d)\in B(0,R) \text{ in } \ CS_0\colon y_d>\psi(\widetilde{y})\}\,.$$

The function ρ defined by $\rho(y)=y_d-\psi(\widetilde{y})$ satisfies

$$\frac{\rho(y)}{\sqrt{1+\Lambda^2}}\le\delta_D(y)\le\rho(y)\quad\text{for all}\quad y\in B(0,R)\cap D. \tag{3.12}$$

Define for $a>0$,

$$D_a=\{y\in D\colon 0<\rho(y)<a, |\widetilde{y}|<a\}$$

and the function

$$h(y)=\begin{cases}V(\delta_D(y)), & y\in B(0,R)\cap D,\\ 0, & \text{otherwise}\,.\end{cases}$$

Using the Dynkin formula and the same approximation argument as in the proof of Lemma 4.5 in [KSV12b], from our Lemma 3.7 we have the following estimate for any open set $U\subset B(0,\frac{R}{4})\cap D$:

$$h(x)-C_1\mathbb{E}_x\tau_U\le\mathbb{E}_x h(X_{\tau_U})\le h(x)+C_1\mathbb{E}_x\tau_U, \tag{3.13}$$

where $C_1>0$ is the constant from Lemma 3.7.

By choosing $A:=\frac{R}{4\sqrt{1+(1+\Lambda)^2}}$ we obtain

$$D_r\subset D_A\subset D(0,\tfrac{R}{4})\cap D\quad\text{for all}\ \ r\le A\,.$$

Indeed, for $y\in D_r$ and $r>0$ the following is true:

$$|y|^2=|\widetilde{y}|^2+|y_d|^2\le r^2+(|y_d-\psi(\widetilde{y})|+|\psi(\widetilde{y})|)^2\le(1+(1+\Lambda)^2)r^2\,. \tag{3.14}$$

In particular, for $r\le A$,

$$|y|\le\sqrt{1+(1+\Lambda)^2}A=\tfrac{R}{4}\,.$$

The idea is to choose $\lambda_2 \geq 1$ large enough so that (3.10) and (3.11) hold for $r \leq \lambda_2^{-1} A$ and $x \in D_r$.

We are going to show that there are constants $c_1, c_2 > 0$ such that for any $\lambda \geq 4$ and $x \in D_{\lambda^{-1}A}$ the following two inequalities hold:

$$\mathbb{E}_x[h(X_{\tau_{D_{\lambda^{-1}A}}})] \geq c_1 \left(\sqrt{\phi(16\lambda^2 R^{-2})} - \sqrt{\phi(R^{-2})}\right) \mathbb{E}_x \tau_{D_{\lambda^{-1}A}}, \tag{3.15}$$

$$\mathbb{P}_x\left(X_{\tau_{D_{\lambda^{-1}A}}} \in D\right) \geq c_2 \left(\phi(16\lambda^2 R^{-2}) - \phi(R^{-2})\right) \mathbb{E}_x \tau_{D_{\lambda^{-1}A}}. \tag{3.16}$$

Once we prove this, we can choose $\lambda_2 > 4$ so that

$$\sqrt{\phi(16\lambda_2^2 R^{-2})} > \sqrt{\phi(R^{-2})} + \tfrac{2C_1}{c_1}\,.$$

Then, for any $\lambda \geq \lambda_2$ and $x \in D_{\lambda^{-1}A}$ we can use

$$c_1\left(\sqrt{\phi(16\lambda^2 R^{-2})} - \sqrt{\phi(R^{-2})}\right) - C_1 > C_1$$

on (3.15) and (3.13) to get

$$V(\delta_D(x)) = h(x) \geq \mathbb{E}_x[h(X_{\tau_{D_{\lambda^{-1}A}}})] - C_1 \mathbb{E}_x \tau_{D_{\lambda^{-1}A}} \geq C_1 \mathbb{E}_x \tau_{D_{\lambda^{-1}A}},$$

which proves (3.11) with $R_1 = \lambda_2^{-1} A$.

Similarly, by (3.13) and (3.16), for any $\lambda \geq \lambda_2$ and $x \in D_{\lambda^{-1}A}$ we have

$$\begin{aligned} V(\delta_D(x)) \ &= \ h(x) \ \leq \ \mathbb{E}_x[h(X_{\tau_{D_{\lambda^{-1}A}}})] + C_1 \mathbb{E}_x \tau_{D_{\lambda^{-1}A}} \\ &\leq V(R)\mathbb{P}_x\left(X_{\tau_{D_{\lambda^{-1}A}}} \in D\right) + C_1 c_2^{-1}\left(\phi(16\lambda_2^2 R^{-2}) - \phi(R^{-2})\right)^{-1} \mathbb{P}_x\left(X_{\tau_{D_{\lambda^{-1}A}}} \in D\right), \end{aligned}$$

where the first term is obtained by estimating h by $V(R)$ and noting that $h(x) = 0$ unless $x \in D$. This yields

$$\mathbb{P}_x\left(X_{\tau_{D_{\lambda^{-1}A}}} \in D\right) \geq \frac{V(\delta_D(x))}{V(R) + C_1 c_2^{-1}\left(\phi(16\lambda_2^2 R^{-2}) - \phi(R^{-2})\right)^{-1}}\,.$$

This proves (3.10) with $R_1 = \lambda_2^{-1} A$.

Now we prove (3.15). Note that for $z \in D_{\lambda^{-1}A}$ and $y \notin B(0, \lambda^{-1}\frac{R}{r})$,

$$|z| \leq \sqrt{1 + (1+\lambda^2)}\lambda^{-1} A = \lambda^{-1}\tfrac{R}{4} \leq |y| \tag{3.17}$$

implies

$$j(|z-y|) \geq j(2|y|) \geq c_3 j(|y|)\,.$$

Then the Ikeda-Watanabe formula implies

$$\begin{aligned} \mathbb{E}_x[h(X_{\tau_{D_{\lambda^{-1}A}}})] &\geq \int_{B(0,r)\cap D\setminus D_{\lambda^{-1}A}} \int_{D_{\lambda^{-1}A}} G_{D_{\lambda^{-1}A}}(x,z) j(|z-y|) V(\delta_D(y))\, dz\, dy \\ &\geq c_3 \left(\int_{D_{\lambda^{-1}A}} G_{D_{\lambda^{-1}A}}(x,z)\, dz\right) \int_{B(0,R)\cap D\setminus D_{\lambda^{-1}A}} V(\delta_D(y)) j(|y|)\, dy \\ &\geq c_3 \mathbb{E}_x \tau_{D_{\lambda^{-1}A}} \int_{B(0,R)\cap D\setminus D_{\lambda^{-1}A}} j(|y|) V\left(\tfrac{y_d - \psi(\widetilde{y})}{\sqrt{1+\Lambda^2}}\right) dy, \end{aligned}$$

since $\frac{y_d - \psi(\widetilde{y})}{\sqrt{1+\Lambda^2}} \leq \delta_D(y)$ by (3.12) .

On the set $E := \{(\widetilde{y}, y_d)\colon 2\Lambda|\widetilde{y}| < y_d,\ \lambda^{-1}\frac{R}{4} < |y| < R\}$ we have

$$|y| \le \sqrt{1+4\Lambda^2}\, y_d \quad \text{and} \quad y_d - \psi(\widetilde{y}) \ge y_d - \Lambda|\widetilde{y}| \ge \frac{|y|}{2\sqrt{1+4\Lambda^2}}\,.$$

Since $E \subset B(0,R) \setminus D_{\lambda^{-1}A}$ because of the first inequality in (3.17), changing to polar coordinates gives

$$\mathbb{E}_x[h(X_{\tau_{D_{\lambda^{-1}A}}})] \ge c_4 \mathbb{E}_x[\tau_{D_{\lambda^{-1}A}}] \int_{\lambda^{-1}\frac{R}{4}}^{R} j(r) V\left(\frac{r}{2\sqrt{1+4\Lambda^2}\sqrt{1+\Lambda^2}}\right) r^{d-1}\, dr$$

with constant $c_4 > 0$ depending on Λ and d.

Then (2.7) and Proposition 2.4 imply

$$\begin{aligned}
\mathbb{E}_x[h(X_{\tau_{D_{\lambda^{-1}A}}})] &\ge c_5 \mathbb{E}_x[\tau_{D_{\lambda^{-1}A}}] \int_{\lambda^{-1}\frac{R}{4}}^{R} r^{-3} \frac{\phi'(r^{-2})}{\sqrt{\phi(r^{-2})}}\, dr \\
&= c_5 \mathbb{E}_x[\tau_{D_{\lambda^{-1}A}}] \left(\sqrt{\phi(16\lambda^2 R^{-2})} - \sqrt{\phi(R^{-2})}\right)\,.
\end{aligned}$$

We prove (3.16) similarly by the same computation as above without V:

$$\begin{aligned}
\mathbb{P}_x\left(X_{\tau_{D_{\lambda^{-1}A}}} \in D\right) &\ge \mathbb{P}_x\left(X_{\tau_{D_{\lambda^{-1}A}}} \in B(0,R) \cap D \setminus B(0, \lambda^{-1}\tfrac{R}{4})\right) \\
&\ge c_6 \mathbb{E}_x[\tau_{D_{\lambda^{-1}A}}] \int_{\lambda^{-1}\frac{R}{4}}^{R} j(r) r^{d-1}\, dr \\
&\ge c_7 \mathbb{E}_x[\tau_{D_{\lambda^{-1}A}}] \int_{\lambda^{-1}\frac{R}{4}}^{R} r^{-3}\phi'(r^{-2})\, dr \\
&= 2^{-1} c_7 \mathbb{E}_x[\tau_{D_{\lambda^{-1}A}}] \left(\phi(16\lambda^2 R^{-2}) - \phi(R^{-2})\right)\,.
\end{aligned}$$

□

4. Analysis of the Poisson kernel

In this section we always assume that the Laplace exponent ϕ of the subordinator $S = (S_t\colon t \ge 0)$ satisfies **(A-1)**–**(A-4)** and the corresponding subordinate Brownian motion $X = (X_t, \mathbb{P}_x)$ is transient.

First we record an inequality.

Lemma 4.1. *For every $R_0 > 0$, there exists a constant $c(R_0, \phi) > 0$ such that*

$$\lambda^2 \int_0^{\lambda^{-1}} r^{-1}\phi'(r^{-2})dr + \int_{\lambda^{-1}}^{R_0} r^{-3}\phi'(r^{-2})dr \le c\,\phi(\lambda^2), \quad \forall \lambda \ge \tfrac{1}{R_0}. \tag{4.1}$$

Proof. Assume $\lambda \geq \lambda_0 \vee \frac{1}{R_0}$. By (1.2), $\phi'(r^{-2}) \leq c_1 r^{2\delta}\lambda^{2\delta}\phi'(\lambda^2)$ for $r \leq \lambda^{-1}$. Thus

$$\begin{aligned}
&\lambda^2 \int_0^{\lambda^{-1}} r^{-1}\phi'(r^{-2})dr + \int_{\lambda^{-1}}^{R_0} r^{-3}\phi'(r^{-2})dr \\
&= \lambda^2\phi'(\lambda^2) \int_0^{\lambda^{-1}} r^{-1}\frac{\phi'(r^{-2})}{\phi'(\lambda^2)}dr - \frac{1}{2}\int_{\lambda^{-1}}^{R_0} (\phi(r^{-2}))'dr \\
&\leq c_2\phi'(\lambda^2)\lambda^{2+2\delta} \int_0^{\lambda^{-1}} r^{-1+2\delta}dr + c_2\phi(\lambda^2) \;\leq\; c_3(\phi'(\lambda^2)\lambda^2 + \phi(\lambda^2)) \leq 2c_3\phi(\lambda^2),
\end{aligned}$$

where we have used (2.3) in the last inequality.

If $\frac{1}{R_0} > \lambda_0$ and $\frac{1}{R_0} \leq \lambda \leq \lambda_0$, then clearly the left hand side of (4.1) is bounded above by

$$\lambda_0^2 \int_0^{R_0} r^{-1}\phi'(r^{-2})dr + \int_{\lambda_0^{-1}}^{R_0} r^{-3}\phi'(r^{-2})dr = c_4 \leq c_5\phi(\lambda^2).$$

□

Recall that the infinitesimal generator $\mathcal{L}$ of X is given by

$$\mathcal{L}f(x) = \int_{\mathbb{R}^d} \left(f(x+y) - f(x) - y\cdot\nabla f(x)\mathbf{1}_{\{|y|\leq\varepsilon\}}\right) j(|y|)dy \tag{4.2}$$

for every $\varepsilon > 0$ and $f \in C_b^2(\mathbb{R}^d)$, where $C_b^2(\mathbb{R}^d)$ is the collection of bounded C^2 functions in $\mathbb{R}^d$.

Using Lemma 4.1, we now prove [KSV12c, Lemma 4.2] under a weaker assumption.

Lemma 4.2. *There exists a constant $c = c(\phi) > 0$ such that for every $f \in C_b^2(\mathbb{R}^d)$ with $0 \leq f \leq 1$,*

$$|\mathcal{L}f_r(x)| \leq c\phi(r^{-2})\left(1+\sup_y \sum_{j,k}\left|\frac{\partial^2 f}{\partial y_j\partial y_k}(y)\right|\right)+b_0, \quad \textit{for every } x\in\mathbb{R}^d \textit{ and } r\in(0,1],$$

where $f_r(y) := f(\frac{y}{r})$ and $b_0 := 2\int_{|z|>1} j(|z|)dz < \infty$.

Proof. Set $L_1 = \sup_{y\in\mathbb{R}^d}\sum_{j,k}|\frac{\partial^2 f(y)}{\partial y_j\partial y_k}|$. Then

$$|f(z+y) - f(z) - y\cdot\nabla f(z)| \leq \tfrac{1}{2}L_1|y|^2,$$

which implies the following estimate:

$$|f_r(z+y) - f_r(z) - y\cdot\nabla f_r(z)\mathbf{1}_{\{|y|\leq r\}}| \leq \tfrac{L_1}{2}\tfrac{|y|^2}{r^2}\mathbf{1}_{\{|y|\leq r\}} + 2\cdot\mathbf{1}_{\{|y|\geq r\}}\ .$$

Now, (2.7) and (4.1) yield

$$\begin{aligned}
&|\mathcal{L}f_r(z)| \\
\leq\ & \int_{\mathbb{R}^d} |f_r(z+y)-f_r(z)-y\cdot\nabla f_r(z)\mathbf{1}_{\{|y|\leq r\}}|\, j(|y|)dy \\
\leq\ & \tfrac{L_1}{2}\int_{\mathbb{R}^d} \mathbf{1}_{\{|y|\leq r\}}\tfrac{|y|^2}{r^2} j(|y|)dy + 2\int_{\mathbb{R}^d} \mathbf{1}_{\{r\leq|y|\leq 1\}} j(|y|)dy + 2\int_{\mathbb{R}^d} \mathbf{1}_{\{|y|\geq 1\}} j(|y|)dy \\
\leq\ & c\phi(r^{-2})\left(2+\tfrac{L_1}{2}\right) + 2\int_{\{|y|\geq 1\}} j(|y|)dy\,,
\end{aligned}$$

where the constant c is independent of $r\in(0,1]$. □

Lemma 4.3. *For every $a\in(0,1)$, there exists a positive constant $c=c(a,\phi)>0$ such that for any $r\in(0,1)$ and any open set D with $D\subset B(0,r)$,*

$$\mathbb{P}_x\left(X_{\tau_D}\in B(0,r)^c\right)\ \leq\ c\,\phi(r^{-2})\,\mathbb{E}_x[\tau_D] \quad \textit{for all} \quad x\in D\cap B(0,ar)\,.$$

Proof. Using Lemma 4.2, the proof of the lemma is similar to that of [KSV12a, Lemma 13.4.15]. We omit the details. □

Let $A(x,a,b):=\{y\in\mathbb{R}^d\colon a\leq|y-x|<b\}$ and recall that the Poisson kernel $K_D(x,z)$ of X in D is defined in (2.13).

Unlike [KSV12c], instead of the Harnack inequality we use Corollary 2.11 (which is a combination of Proposition 2.7 and the Harnack inequality) in the next proposition.

Proposition 4.4. *Let $p\in(0,1)$. Then there exists a constant $c(\phi,p)>0$ such that for any $r\in(0,1)$ we have*

$$\int_{\frac{1+p}{2}r}^{|z|} K_{B(0,s)}(x,z)\,ds \leq c\tfrac{r}{\phi(r^{-2})}\,j(|z|)$$

for all $x\in B(0,pr)$ and $z\in A(0,\frac{1+p}{2}r,r)$.

Proof. We split the Poisson kernel into two parts:

$$K_{B(0,s)}(x,z)=\int_{B(0,s)} G_{B(0,s)}(x,y)j(|z-y|)\,dy = I_1(s)+I_2(s),$$

where

$$I_1(s)=\int_{B(0,3s/4)} G_{B(0,s)}(x,y)j(|z-y|)\,dy,$$

$$I_2(s)=\int_{A(0,3s/4,s)} G_{B(0,s)}(x,y)j(|z-y|)\,dy.$$

First we consider $I_1(s)$. Since $|z-y| \geq \frac{1}{4}|z|$, we conclude from (2.1) and (2.8) that

$$I_1(s) \leq j\left(\tfrac{|z|}{4}\right) \int_{B(0,3s/4)} G(x,y)\,dy \leq j\left(\tfrac{|z|}{4}\right) \int_{B(x,2s)} G(x,y)\,dy$$
$$\leq c_1 j\left(|z|\right) \int_0^{2s} t^{-3} \tfrac{\phi'(t^{-2})}{\phi(t^{-2})^2} dt = \frac{c_1}{2} j\left(|z|\right) \int_0^{2s} \left(\tfrac{1}{\phi(t^{-2})}\right)' dt \leq c_2 \tfrac{j(|z|)}{\phi(s^{-2})}\,.$$

Then, since $|z| \leq r$,

$$\int_{\frac{1+p}{2}r}^{|z|} I_1(s)\,ds \leq c_2 j(|z|) \int_{\frac{1+p}{2}r}^{|z|} \tfrac{ds}{\phi(s^{-2})}$$
$$\leq c_2 j(|z|) \tfrac{|z|-\frac{1+p}{2}r}{\phi(r^{-2})} \leq c_2 j(|z|) \tfrac{r}{\phi(r^{-2})}\,.$$

On the other hand, by Corollary 2.11 and Lemma 2.5,

$$I_2(s) \leq c_3 s^{-d-2} \tfrac{\phi'(s^{-2})}{\phi(s^{-2})} \int_{A(0,3s/4,s)} \mathbb{E}_y[\tau_{B(0,s)}]\, j(|z-y|)\,dy$$
$$\leq c_4 s^{-d-2} \tfrac{\phi'(s^{-2})}{\phi(s^{-2})} \int_{A(0,3s/4,s)} \tfrac{j(|z-y|)}{\sqrt{\phi(s^{-2})\phi((s-|y|)^{-2})}}\,dy$$
$$\leq c_4 s^{-d-2} \tfrac{\phi'(s^{-2})}{\phi(s^{-2})^{3/2}} \int_{A(0,3s/4,s)} \tfrac{j(|z-y|)}{\sqrt{\phi(|z-y|^{-2})}}\,dy,$$

since $s-|y| \leq |z-y|$.

Observing that $A(z,3s/4,s) \subset B(z,s) \subset A(0,|z|-s,2r)$ we arrive at

$$I_2(s) \leq c_4 s^{-d-2} \tfrac{\phi'(s^{-2})}{\phi(s^{-2})^{3/2}} \int_{A(0,|z|-s,2r)} \tfrac{j(|v|)}{\sqrt{\phi(|v|^{-2})}}\,dv$$
$$= c_5 s^{-d-2} \tfrac{\phi'(s^{-2})}{\phi(s^{-2})^{3/2}} \int_{|z|-s}^{2r} t^{-3} \tfrac{\phi'(t^{-2})}{\sqrt{\phi(t^{-2})}}\,dv$$
$$\leq c_6 s^{-d-2} \tfrac{\phi'(s^{-2})}{\phi(s^{-2})^{3/2}} \sqrt{\phi((|z|-s))^{-2}}.$$

Then using the fact that $s \mapsto \phi'(s^{-2})$ and $s \mapsto \phi(s^{-2})^{-1}$ are increasing, we obtain

$$\int_{\frac{1+p}{2}r}^{|z|} I_2(s)\,ds \leq c_6 \int_{\frac{1+p}{2}r}^{|z|} \tfrac{s^{-d-2}\phi'(s^{-2})}{\phi(s^{-2})^{3/2}} \sqrt{\phi((|z|-s)^{-2})}\,ds$$
$$\leq c_6 \tfrac{\left(\frac{1+p}{2}r\right)^{-d-2}\phi'(|z|^{-2})}{\phi(|z|^{-2})^{3/2}} \int_0^{|z|-\frac{1+p}{2}r} \sqrt{\phi(t^{-2})}\,dt\,. \tag{4.3}$$

By Lemma 2.3 with $\varepsilon = \frac{\delta}{2} > 0$ for any $a \in (0,1)$ we have

$$\int_0^a \sqrt{\phi(s^{-2})}\, ds = \int_0^a \frac{\sqrt{\phi(s^{-2})}}{\sqrt{\phi(a^{-2})}}\, ds \sqrt{\phi(a^{-2})}$$

$$\leq c_7 a^{1-\delta/2}\sqrt{\phi(a^{-2})} \int_0^a s^{-1+\delta/2} ds \leq c_8 a \sqrt{\phi(a^{-2})}. \tag{4.4}$$

Since $\frac{1+p}{2}r \leq |z| \leq r$, (4.3)–(4.4) together with (2.2) and (2.7) give

$$\int_{\frac{1+p}{2}r}^{|z|} I_2(s)\, ds \leq c_9 \frac{\left(\frac{1+p}{2}r\right)^{-d-2}\phi'(|z|^{-2})}{\phi(|z|^{-2})^{3/2}} \left(|z| - \tfrac{1+p}{2}r\right) \phi\left(\left(|z| - \tfrac{1+p}{2}r\right)^{-2}\right)^{1/2}$$

$$\leq c_9 \frac{\left(\frac{1+p}{2}r\right)^{-d-2}\phi'(|z|^{-2})}{\phi(|z|^{-2})^{3/2}} |z| \sqrt{\phi\left(|z|^{-2}\right)} \leq c_9 |z|^{-d-2}\phi'(|z|^{-2}) \frac{r}{\phi(r^{-2})}$$
$$\leq c_{10} j(|z|) \frac{r}{\phi(r^{-2})}.$$

□

5. Uniform boundary Harnack principle

In this section we give a proof of the uniform boundary Harnack principle for X in an arbitrary open set with the constant not depending on the open set itself. This type of boundary Harnack principle was first obtained in [BKK08] for rotationally symmetric stable processes. Since, using results of the previous section, the proofs in this section are almost identical to the one in [KSV12c, Section 5], we give details only on parts that require extra explanation.

Recall that $X = (X_t, \mathbb{P}_x)$ is a subordinate process defined by $X_t = W_{S_t}$ where $W = (W_t, \mathbb{P}_x)$ is a Brownian motion in $\mathbb{R}^d$ independent of the subordinator S and the Laplace exponent ϕ of the subordinator S satisfies **(A-1)**–**(A-3)**.

Using (2.9), (2.10), Proposition 2.12, Proposition 4.4 and the fact that for $U \subset D$

$$K_D(x,z) = K_U(x,z) + \mathbb{E}_x\left[K_D(X_{\tau_U}, z)\right], \qquad (x,z) \in U \times D^c, \tag{5.1}$$

the proof of the next result is the same as the one of [KSV12c, Lemma 5.2].

Lemma 5.1. *Assume that X is transient and satisfies* **(A-1)**–**(A-4)**. *For every $p \in (0,1)$, there exists $c = c(\phi, p) > 0$ such that for every $r \in (0,1)$, $z_0 \in \mathbb{R}^d$, $U \subset B(z_0, r)$ and for any $(x,y) \in (U \cap B(z_0, pr)) \times B(z_0, r)^c$,*

$$K_U(x,y) \leq c\, \frac{1}{\phi(r^{-2})} \left(\int_{U \setminus B\left(z_0, \frac{(1+p)r}{2}\right)} j(|z-z_0|) K_U(z,y) dz + j(|y - z_0|) \right).$$

The process X satisfies the hypothesis **H** in [Szt00]. Therefore, by [Szt00, Theorem 1], for a Lipschitz open set $V \subset \mathbb{R}^d$ and an open subset $U \subset V$,

$$\mathbb{P}_x(X_{\tau_U} \in \partial V) = 0 \qquad \text{and} \qquad \mathbb{P}_x(X_{\tau_U} \in dz) = K_U(x,z)dz \quad \text{on } V^c. \tag{5.2}$$

Using (5.2) and Lemma 5.1, the proof of the next result is the same as the one of [KSV12c, Lemma 5.3].

Lemma 5.2. *Assume that X is transient and satisfies* **(A-1)**–**(A-4)**. *For every $p \in (0,1)$, there exists $c = c(\phi, p) > 0$ such that for every $r \in (0,1)$, for every $z_0 \in \mathbb{R}^d$, $U \subset B(z_0, r)$ and any nonnegative function u in $\mathbb{R}^d$ which is regular harmonic in U with respect to X and vanishes in $U^c \cap B(z_0, r)$, we have*

$$u(x) \le c\,\tfrac{1}{\phi(r^{-2})} \int\limits_{\left(U\setminus B(z_0,\frac{(1+p)r}{2})\right)\cup B(z_0,r)^c} j(|y-z_0|)u(y)dy, \quad x \in U \cap B(z_0, pr).$$

We give a detailed proof of the next result.

Lemma 5.3. *Assume that X is transient and satisfies* **(A-1)**–**(A-4)**. *There exists $C_2 = C_2(d, \phi) > 1$ such that for every $r \in (0,1)$, for every $z_0 \in \mathbb{R}^d$, $U \subset B(z_0, r)$ and for any $(x, y) \in U \cap B(z_0, \frac{r}{2}) \times B(z_0, r)^c$,*

$$\begin{aligned} & C_2^{-1}\,\mathbb{E}_x[\tau_U]\left(\int\limits_{U\setminus B(z_0,\frac{r}{2})} j(|z-z_0|)K_U(z,y)dz + j(|y-z_0|)\right) \\ \le \quad & K_U(x,y) \le C_2\,\mathbb{E}_x[\tau_U]\left(\int\limits_{U\setminus B(z_0,\frac{r}{2})} j(|z-z_0|)K_U(z,y)dz + j(|y-z_0|)\right). \end{aligned}$$

Proof. Without loss of generality, we assume $z_0 = 0$. Fix $r \in (0,1)$ and let $U_1 := U \cap B(0, \frac{1}{2}r)$, $U_2 := U \cap B(0, \frac{2}{3}r)$ and $U_3 := U \cap B(0, \frac{3}{4}r)$. Let $x \in U \cap B(0, \frac{r}{2})$, $y \in B(0, r)^c$. By (5.1),

$$\begin{aligned} K_U(x,y) &= \mathbb{E}_x[K_U(X_{\tau_{U_2}}, y)] + K_{U_2}(x,y) \\ &= \int\limits_{U\setminus U_2} K_U(z,y)\mathbb{P}_x(X_{\tau_{U_2}} \in dz) + K_{U_2}(x,y) \\ &= \int\limits_{U_3\setminus U_2} K_U(z,y)\mathbb{P}_x(X_{\tau_{U_2}} \in dz) \\ &\qquad + \int\limits_{U\setminus U_3} K_U(z,y)K_{U_2}(x,z)dz + K_{U_2}(x,y) \\ &= \int\limits_{U_3\setminus U_2} K_U(z,y)\mathbb{P}_x(X_{\tau_{U_2}} \in dz) \\ &\qquad + \int\limits_{U\setminus U_3} K_U(z,y)\int\limits_{U_2} G_{U_2}(x,w)j(|z-w|)dwdz \\ &\qquad + \int\limits_{U_2} G_{U_2}(x,w)j(|y-w|)dw =: I_1 + I_2 + I_3. \end{aligned}$$

From Lemma 4.3 and Lemma 5.1, we see that there exist c_1 and c_2 such that

$$\begin{aligned} I_1 &\leq c_1 \left(\sup_{z\in U_3} K_U(z,y) \right) \phi(r^{-2})\mathbb{E}_x[\tau_{U_2}] \\ &\leq c_2 \mathbb{E}_x[\tau_{U_2}] \left(\int\limits_{U\setminus U_3} j(|z|)K_U(z,y)dz + j(|y|) \right). \end{aligned} \tag{5.3}$$

Now using (2.9) and (2.10) one can check as in [KSV12c] that there exists $c_5 = c_5(d,\phi) > 1$ such that

$$c_5^{-1}\mathbb{E}_x[\tau_{U_2}] \int\limits_{U\setminus U_3} j(|z|)K_U(z,y)dz \leq I_2 \leq c_5\mathbb{E}_x[\tau_{U_2}] \int\limits_{U\setminus U_3} j(|z|)K_U(z,y)dz \tag{5.4}$$

and

$$c_5^{-1}\mathbb{E}_x[\tau_{U_2}]j(|y|) \leq I_3 \leq c_5\mathbb{E}_x[\tau_{U_2}]j(|y|)\,. \tag{5.5}$$

The upper bound follows from (5.3)–(5.5).

Using the strong Markov property, we get

$$\begin{aligned} \mathbb{E}_x[\tau_U] &= \mathbb{E}_x[\tau_{U_2}] + \mathbb{E}_x\left[\mathbb{E}_{X_{\tau_{U_2}}}[\tau_U]\right] \\ &\leq \mathbb{E}_x[\tau_{U_2}] + \left(\sup_{z\in U}\mathbb{E}_z[\tau_U]\right)\mathbb{P}_x\big(X_{\tau_{U_2}} \in B(0,\tfrac{2r}{3})^c\big) \\ &\leq \mathbb{E}_x[\tau_{U_2}] + c_6\phi(r^{-2})^{-1}\,\phi((\tfrac{2r}{3})^{-2})\mathbb{E}_x[\tau_{U_2}] \;\leq\; c_7\mathbb{E}_x[\tau_{U_2}], \end{aligned}$$

where in the second inequality we have used Lemma 2.5 and Lemma 4.3 and in last inequality we have used (2.1).

Since

$$\begin{aligned} \int\limits_{U\setminus U_1} j(|z|)K_U(z,y)dz &= \int\limits_{U\setminus U_3} j(|z|)K_U(z,y)dz + \int\limits_{U_3\setminus U_1} j(|z|)K_U(z,y)dz \\ &\leq \int\limits_{U\setminus U_3} j(|z|)K_U(z,y)dz \\ &\quad + \left(\sup_{z\in U_3} K_U(z,y)\right) \int\limits_{A(0,r/2,3r/4)} j(|y|)dy, \end{aligned}$$

by (2.7) and Lemma 5.1,

$$\begin{aligned}
&\int\limits_{U\setminus U_1} j(|z|)K_U(z,y)dz \\
\le\ & \left(1+\frac{c_8}{\phi(r^{-2})}\int_{\frac{r}{2}}^{\frac{3r}{4}} s^{-3}\phi'(s^{-2})ds\right)\left(\int\limits_{U\setminus U_3} j(|z|)K_U(z,y)dz + j(|y|)\right) \\
=\ & \left(1-2\frac{c_8}{\phi(r^{-2})}\int_{\frac{r}{2}}^{\frac{3r}{4}} (\phi(s^{-2}))'ds\right)\left(\int\limits_{U\setminus U_3} j(|z|)K_U(z,y)dz + j(|y|)\right) \\
\le\ & \left(1+c_9\frac{\phi(4r^{-2})}{\phi(r^{-2})}\right)\left(\int\limits_{U\setminus U_3} j(|z|)K_U(z,y)dz + j(|y|)\right). \qquad (5.6)
\end{aligned}$$

Combining (2.1) and (5.4)–(5.6), we finish the proof of the lower bound. □

Using Lemmas 5.2 and 5.3, the proof of the next result is the same as the one of [KSV12c, Lemma 5.5].

Lemma 5.4. *Assume that X is transient and satisfies* **(A-1)**–**(A-4)**. *For every $z_0 \in \mathbb{R}^d$, every open set $U \subset B(z_0,r)$ and for any nonnegative function u in $\mathbb{R}^d$ which is regular harmonic in U with respect to X and vanishes a.e. on $U^c \cap B(z_0,r)$,*

$$C_2^{-1}\mathbb{E}_x[\tau_U]\int\limits_{B(z_0,\frac{r}{2})^c} j(|y-z_0|)u(y)dy \le u(x) \le C_2\mathbb{E}_x[\tau_U]\int\limits_{B(z_0,\frac{r}{2})^c} j(|y-z_0|)u(y)dy,$$

for every $x \in U\cap B(z_0,\frac{r}{2})$ (where C_2 is the constant from Lemma 5.3).

As in [KSV12c, Corollary 5.6], the last two lemmas immediately imply the following approximate factorization of the Poisson kernel.

Corollary 5.5. *Assume that X is transient and satisfies* **(A-1)**–**(A-4)**. *Let $z_0 \in \mathbb{R}^d$ and $D \subset \mathbb{R}^d$ be open. Then for every $r\in(0,1)$ and all $(x,y) \in (D\cap B(z_0,\frac{r}{2}))\times(D^c\cap B(z_0,r)^c)$ it holds that*

$$C_2^{-1}\mathbb{E}_x[\tau_{D\cap B(z_0,r)}]A_D(z_0,r,y) \le K_D(x,y) \le C_2\,\mathbb{E}_x[\tau_{D\cap B(z_0,r)}]A_D(z_0,r,y)\,, \qquad (5.7)$$

where

$$\begin{aligned}
A_D(z_0,r,y) \ :=\ & \int\limits_{(D\cap B(z_0,r))\setminus B(z_0,\frac{r}{2})} j(|z-z_0|)K_{D\cap B(z_0,r)}(z,y)\,dz \\
& + j(|y-z_0|) + \int\limits_{B(z_0,\frac{r}{2})^c} j(|z-z_0|)\mathbb{E}_z\left[K_D(X_{\tau_{D\cap B(z_0,r)}},y)\right]dz\,.
\end{aligned}$$

Lemma 5.4 and (5.7) imply the following uniform boundary Harnack principle. Note that the constants in the following theorem do not depend on the open set itself. That is why this type of result is called the uniform boundary Harnack principle.

Theorem 5.6. *Suppose that ϕ satisfies* **(A-1)**–**(A-3)**. *There exists a constant $c = c(\phi) > 0$ such that*

(i) *For every $z_0 \in \mathbb{R}^d$, every open set $D \subset \mathbb{R}^d$, every $r \in (0,1)$ and for any nonnegative functions u, v in $\mathbb{R}^d$ which are regular harmonic in $D \cap B(z_0, r)$ with respect to X and vanish a.e. on $D^c \cap B(z_0, r)$, we have*

$$\frac{u(x)}{v(x)} \le c\,\frac{u(y)}{v(y)}$$

for all $x, y \in D \cap B(z_0, \frac{r}{2})$.

(ii) *If X is, additionally, transient and satisfies* **(A-4)**, *then for every $z_0 \in \mathbb{R}^d$, every Greenian open set $D \subset \mathbb{R}^d$, and every $r \in (0,1)$, we have*

$$K_D(x_1, y_1)K_D(x_2, y_2) \le cK_D(x_1, y_2)K_D(x_2, y_1)$$

for all $x_1, x_2 \in D \cap B(z_0, \frac{r}{2})$ and all $y_1, y_2 \in \overline{D}^c \cap B(z_0, r)^c$.

Proof. Under the assumption of transience and **(A-1)**–**(A-4)** the result follows from Lemma 5.4 and Corollary 5.5 (see the proof of [KSV12c, Theorem 1.1]).

If the process X is not transient, we can use an argument similar to the proof of [KM12, Theorem 1.2, p. 17], where it is shown how to deduce Harnack inequality in dimensions $d = 1, 2$ from Harnack inequality in dimension $d \ge 3$ (since in the latter case the process is always transient). Since we will use the argument in the proof of Theorem 1.5 again, here we provide the detail for the readers' convenience.

We use the notation $\tilde{x} = (x^1, \dots, x^{d-1})$ for $x = (x^1, \dots, x^{d-1}, x^d) \in \mathbb{R}^d$ and $X = ((\widetilde{X}_t, X_t^d), \mathbb{P}_{(\widetilde{x}, x^d)})$. As in the proof of [KM12, Theorem 1.2, p. 17], we have that for every $x^d \in \mathbb{R}$, $\widetilde{X} = (\widetilde{X}_t, \mathbb{P}_{\widetilde{x}})$ is a $(d-1)$-dimensional subordinate Brownian motion with characteristic exponent $\widetilde{\Phi}(\tilde{\xi}) = \phi(|\tilde{\xi}|^2)$ for $\tilde{\xi} \in \mathbb{R}^{d-1}$.

Suppose (i) is true for for some $d \ge 2$ and let D be an open subset of $\mathbb{R}^{d-1}$ and $u, v \colon \mathbb{R}^{d-1} \to [0, \infty)$ be functions that are regular harmonic in $D \cap B(\widetilde{x}_0, r)$ with respect to $\widetilde{X}$ and vanish on $D^c \cap B(\widetilde{x}_0, r)$ a.e. with respect to the $(d-1)$-dimensional Lebesgue measure.

Let f and $g : \mathbb{R}^d \to [0, \infty)$ be defined by

$$f(\widetilde{x}, x^d) = u(\widetilde{x}) \quad \text{and} \quad g(\widetilde{x}, x^d) = v(\widetilde{x}).$$

Since

$$\tau_{(B(\widetilde{x}_0, s) \cap D) \times \mathbb{R}} = \inf\{t > 0 : \widetilde{X}_t \notin B(\widetilde{x}_0, s) \cap D\},$$

by the strong Markov property, f and g are regular harmonic in $B(\widetilde{x}_0, r) \times \mathbb{R}$ with respect to X. Clearly f and g vanish on $(B(\widetilde{x}_0, r) \times \mathbb{R}) \cap (D \times \mathbb{R})^c$ a.e. with respect to the d-dimensional Lebesgue measure. Thus, by applying the result to f and g, we see that there exists a constant $c > 0$ such that for all $\widetilde{x}_0 \in \mathbb{R}^{d-1}$, open set $D \subset \mathbb{R}^{d-1}$ and $r \in (0,1)$,

$$\frac{u(\widetilde{x}_1)}{v(\widetilde{x}_1)} = \frac{f((\widetilde{x}_1, 0))}{g((\widetilde{x}_1, 0))} \le c\,\frac{f((\widetilde{x}_2, 0))}{g((\widetilde{x}_2, 0))} = c\,\frac{u(\widetilde{x}_2)}{v(\widetilde{x}_2)} \quad \text{for all } \ \widetilde{x}_1, \widetilde{x}_2 \in D \cap B(\widetilde{x}_0, \tfrac{r}{2}).$$

Applying this argument first to $d = 3$ and then to $d = 2$, we finish the proof of the theorem. ∎

6. Green function estimates on bounded Lipschitz domain

The purpose of this section is to establish sharp two-sided Green function estimates for X in any bounded Lipschitz domain D of $\mathbb{R}^d$.

Recall that we have assumed that $X = (X_t, \mathbb{P}_x)$ is the subordinate process defined by $X_t = W_{S_t}$ where $W = (W_t, \mathbb{P}_x)$ is a Brownian motion in $\mathbb{R}^d$ independent of the subordinator S and the Laplace exponent ϕ of the subordinator S satisfies **(A-1)**–**(A-3)**. In this section we further assume that X is transient and that **(A-4)** also holds.

We will first establish the interior estimates using Proposition 2.6 and Theorem 2.10. As in [KSV12b], once we have the interior estimates, we can apply Theorem 2.10 and the boundary Harnack principle (Theorem 5.6), and use the arguments of [Bog00, Han05] to get the full estimates for bounded Lipschitz domain D.

Lemma 6.1. *For every bounded domain $D \subset \mathbb{R}^d$, there exists a constant $C_3 = C_3(d, \phi,\ diam(D)) > 0$ such that*

$$G_D(x,y) \le C_3 \frac{|x-y|^{-d-2}\phi'(|x-y|^{-2})}{\phi(|x-y|^{-2})^2} \qquad \text{for all } x, y \in D\,, \tag{6.1}$$

and for all $x, y \in D$ with $b_2^{-1}|x-y| \le \delta_D(x) \wedge \delta_D(y)$,

$$G_D(x,y) \ \ge\ C_3^{-1}\, \frac{|x-y|^{-d-2}\phi'(|x-y|^{-2})}{\phi(|x-y|^{-2})^2}, \tag{6.2}$$

where $b_2 \in (0, \frac{1}{2})$ is the constant from Proposition 2.7.

Proof. Since $G_D(x,y) \le g(|x-y|)$ and D is bounded, (6.1) is an immediate consequence of Proposition 2.6.

Now we show (6.2). We have two cases:

Case 1: $|x-y| \le b_2$. Since $B(x, b_2^{-1}|x-y|) \subset D$ and $y \in A(x, |x-y|, b_2^{-1}|x-y|)$, we can use Proposition 2.7 to get

$$\begin{aligned} G_D(x,y) &\ge G_{B(x,b_2^{-1}|x-y|)}(x,y) \ge c_1 \tfrac{b_2^{d+2}|x-y|^{-d-2}\phi'(b_2^2|x-y|^{-2})}{\phi(b_2^2|x-y|^{-2})} \mathbb{E}_x[\tau_{B(x,b_2^{-1}|x-y|)}] \\ &\ge c_2 \tfrac{|x-y|^{-d-2}\phi'(|x-y|^{-2})}{\phi(|x-y|^{-2})^2}, \end{aligned}$$

where in the last inequality we have used Proposition 2.8, **(A-3)** and the facts that $b_2 \in (0, \frac{1}{2})$ and that the function $r \mapsto \frac{1}{\phi(r)}$ is decreasing.

Case 2: $|x-y| > b_2$. In this case it follows that $\delta_D(x) \wedge \delta_D(y) > 1$. Let $x_0 \in \partial B(y, b_2)$. Then

$$b_2^{-1}|x_0 - y| = 1 < \delta_D(x) \wedge \delta_D(y),$$

and so, by the Case 1, we obtain

$$G_D(x_0, y) \ge c_2 \tfrac{b_2^{-d-2}\phi'(b_2^{-2})}{\phi(b_2^{-2})^2}. \tag{6.3}$$

Since $G_D(\cdot, y)$ is harmonic in $B(x_0, \frac{b_2}{2}) \cup B(x, \frac{b_2}{2})$ (with respect to X), we can use Theorem 2.10 to deduce

$$\begin{aligned} G_D(x,y) &= \mathbb{E}_x[G_D(X_{\tau_{B(x,b_2/4)}}, y)] \ge \mathbb{E}_x[G_D(X_{\tau_{B(x,b_2/4)}}, y); X_{\tau_{B(x,b_2/4)}} \in B(x_0, \tfrac{b_2}{4})] \\ &\ge c_3 G_D(x_0,y)\mathbb{P}_x(X_{\tau_{B(x,b_2/4)}} \in B(x_0, \tfrac{b_2}{4}))\,. \end{aligned} \tag{6.4}$$

By Proposition 2.12, (2.12) and (2.13) we get

$$\begin{aligned}\mathbb{P}_x(X_{\tau_{B(x,b_2/4)}} \in B(x_0, \tfrac{b_2}{4})) &= \int_{B(x_0,\frac{b_2}{4})} K_{B(x,\frac{b_2}{4})}(x,z)\,dz \\ &\geq \tfrac{c_4}{\phi(16b_2^{-2})} \int_{B(x_0,\frac{b_2}{4})} j(|z-x|)\,dz. \qquad (6.5)\end{aligned}$$

Since $|z-x| \leq \mathrm{diam}(D)$, by the monotonicity of j we deduce

$$\mathbb{P}_x(X_{\tau_{B(x,b_2/4)}} \in B(x_0, \tfrac{b_2}{4})) \geq c_5 \tfrac{b_2^d j(\mathrm{diam}(D))}{\phi(16b_2^{-2})}\,.$$

Therefore, using (6.3)–(6.5) we conclude that

$$G_D(x,y) \geq c_6 \geq c_7 \tfrac{|x-y|^{-d-2}\phi'(|x-y|^{-2})}{\phi(|x-y|^{-2})^2}.$$

In the last inequality we use the fact that $b_2 < |x-y| \leq \mathrm{diam}(D)$ and Corollary 2.2. □

An open set D is said to be a Lipschitz domain if there is a localization radius $R_1 > 0$ and a constant $\Lambda > 0$ such that for every $z \in \partial D$, there is a Lipschitz function $\phi_z : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying

$$|\phi_z(x) - \phi_z(w)| \leq \Lambda|x-w|,$$

and an orthonormal coordinate system CS_z with origin at z such that

$$B(z,R_1) \cap D = B(z,R_1) \cap \{y = (\tilde{y}, y_d) \text{ in } CS_z : y_d > \phi_z(\tilde{y})\}.$$

The pair (R_1, Λ) is called the characteristic of the Lipschitz domain D.

Unlike [KSV12b] we will assume that D is a bounded Lipschitz domain instead of a κ-fat open set. The main reason we assume that D is a bounded Lipschitz domain is Theorem 2.10 and the Harnack chain argument. Note that in [KSV12b], [KSV12b, Theorem 2.14] is used instead of Theorem 2.10 and the Harnack chain argument. Unfortunately, it seems that, under our assumptions, such a result is not true for certain harmonic functions like $u(x) := \mathbb{P}_x(X_{\tau_{B(x_1,r)}} \in B(x_0,r))$ when the distance between x_0 and x_1 is large and r is small.

Lemma 6.2. *For every $L > 0$ and bounded Lipschitz domain D with the characteristics (R_1, Λ), there exists $c = c(L, d, \phi, R_1, \Lambda, diam(D)) > 0$ such that for every $x, y \in D$ with $|x-y| \leq L(\delta_D(x) \wedge \delta_D(y))$,*

$$G_D(x,y) \geq c\frac{|x-y|^{-d-2}\phi'(|x-y|^{-2})}{\phi(|x-y|^{-2})^2}. \qquad (6.6)$$

Proof. By symmetry of G_D we may assume $\delta_D(x) \leq \delta_D(y)$. Moreover, by Lemma 6.1 we can assume that $L > b_2$ and so we only need to show (6.6) for $b_2\delta_D(x) \leq |x-y| \leq L\delta_D(x)$.

Choose a point $w \in \partial B(x, b_2\delta_D(x))$. Then Lemma 6.1 gives

$$G_D(x,w) \;\geq\; c_1 \frac{(b_2\delta_D(x))^{-d-2}\phi'((b_2\delta_D(x))^{-2})}{\phi((b_2\delta_D(x))^{-2})^2}.$$

Since $|y-w| \leq |x-y|+|x-w| \leq (L+1)\delta_D(x)$ and $G_D(x,\,\cdot\,) = G_D(\,\cdot\,,x)$ is harmonic with respect to X in $B(y, b_2\delta_D(x)) \cup B(w, b_2\delta_D(x))$, using the assumption that D

is a bounded Lipschitz domain, Theorem 2.10 and the Harnack chain argument we obtain

$$G_D(x,y) \ge c_2 G_D(x,w) \ge c_3 \frac{(b_2\delta_D(x))^{-d-2}\phi'((b_2\delta_D(x))^{-2})}{\phi((b_2\delta_D(x))^{-2})^2}.$$

By Corollary 2.2

$$\begin{aligned} G_D(x,y) &\ge c_2 G_D(x,w) \ge c_3 \frac{(b_2\delta_D(x))^{-d-2}\phi'((b_2\delta_D(x))^{-2})}{\phi((b_2\delta_D(x))^{-2})^2} \\ &\ge c_4 \frac{|x-y|^{-d-2}\phi'(|x-y|^{-2})}{\phi(|x-y|^{-2})^2}. \end{aligned}$$

□

For the remainder of this section, we assume that D is a bounded Lipschitz domain with characteristics (R_1, Λ).

Without loss of generality we may assume that $R_1 \le \frac{1}{4}$. Since D is Lipschitz, there exists $\kappa = \kappa(\Lambda) \in (0, \frac{1}{2})$ such that for each $Q \in \partial D$ and $r \in (0, R_1)$, there exists a point

$$A_r(Q) \in D \cap B(Q,r) \ \text{ satisfying } \ B(A_r(Q), \kappa r) \subset D \cap B(Q,r).$$

Recall that $G_D(\cdot, y)$ is regular harmonic in $D \setminus \overline{B(y,\varepsilon)}$ for every $\varepsilon > 0$ and vanishes outside D.

Fix $z_0 \in D$ with $\kappa R_1 < \delta_D(z_0) < R_1$ and set $\varepsilon_1 := \frac{\kappa R_1}{24}$. Define

$$r(x,y) := \delta_D(x) \vee \delta_D(y) \vee |x-y|, \ x, y \in D,$$

and

(6.7)
$$\mathcal{B}(x,y) := \begin{cases} \{A \in D : \delta_D(A) > \frac{\kappa}{2} r(x,y), \ |x-A| \vee |y-A| < 5r(x,y)\} & \text{if } r(x,y) < \varepsilon_1, \\ \{z_0\} & \text{if } r(x,y) \ge \varepsilon_1. \end{cases}$$

Note that for every $(x,y) \in D \times D$ with $r(x,y) < \varepsilon_1$,

$$\tfrac{1}{6}\delta_D(A) \le r(x,y) \le 2\kappa^{-1}\delta_D(A), \qquad A \in \mathcal{B}(x,y). \tag{6.8}$$

Set

$$C_4 := C_3 \mathrm{diam}(D) (\tfrac{\delta_D(z_0)}{2})^{-d-3} \frac{\phi'((\frac{\delta_D(z_0)}{2})^{-2})}{\phi((\frac{\delta_D(z_0)}{2})^{-2})^2}.$$

By (6.1) and Corollary 2.2 (with $a = 2\mathrm{diam}(D)/\delta_D(z_0)$ and $b = 1$) we see that

$$G_D(x, z_0) \le C_4 \ \text{ for } \ x \in D \setminus B(z_0, \tfrac{\delta_D(z_0)}{2}).$$

Now we define

$$g_D(x) := G_D(x, z_0) \wedge C_4. \tag{6.9}$$

We note that for $\delta_D(z) \le 6\varepsilon_1$,

$$g_D(z) = G_D(z, z_0),$$

since $6\varepsilon_1 < \frac{\delta_D(z_0)}{4}$, and thus $|z - z_0| \ge \delta_D(z_0) - 6\varepsilon_1 \ge \frac{\delta_D(z_0)}{2}$.

The following lemma follows from Theorem 2.10 and the standard Harnack chain argument:

Lemma 6.3. *There exists $c > 1$ such that for every $x \in D$ satisfying $\delta_D(x) \geq \frac{\kappa^3 \varepsilon_1}{64}$ we have*

$$c^{-1} \leq g_D(x) \leq c\,.$$

Theorem 6.4. *Suppose X is transient and ϕ satisfies* **(A-1)**–**(A-4)**. *If D is a bounded Lipschitz domain with characteristics (R_1, Λ), then there exists $c = c(diam(D), R_1, \Lambda, \phi) > 1$ such that for every $x, y \in D$ and $A \in \mathcal{B}(x, y)$,*

$$c^{-1} \frac{g_D(x)g_D(y)\phi'(|x-y|^{-2})}{g_D(A)^2|x-y|^{d+2}\phi(|x-y|^{-2})^2} \leq G_D(x,y) \leq c\, \frac{g_D(x)g_D(y)\phi'(|x-y|^{-2})}{g_D(A)^2|x-y|^{d+2}\phi(|x-y|^{-2})^2}, \tag{6.10}$$

where g_D and $\mathcal{B}(x,y)$ are defined by (6.9) *and* (6.7) *respectively.*

Proof. Since the proof is an adaptation of the proofs of [Bog00, Proposition 6] and [Han05, Theorem 2.4], we only give the proof when $\delta_D(x) \leq \delta_D(y) \leq \frac{\kappa}{4}|x-y|$. In this case, we have $r(x, y) = |x - y|$.

By Theorem 2.10, we see that for all $x, y \in D$ and $A_1, A_2 \in \mathcal{B}(x, y)$,

$$g_D(A_1) \text{ is comparable to } g_D(A_2)\,.$$

Set $r = \frac{|x-y| \wedge \varepsilon_1}{2}$ and choose

$$Q_x, Q_y \in \partial D \quad \text{with} \quad |Q_x - x| = \delta_D(x) \quad \text{and} \quad |Q_y - y| = \delta_D(y)\,.$$

Pick points $x_1 = A_{\kappa r/2}(Q_x)$ and $y_1 = A_{\kappa r/2}(Q_y)$ so that

$$x, x_1 \in B(Q_x, \kappa r/2) \quad \text{and} \quad y, y_1 \in B(Q_y, \kappa r/2)\,.$$

Then one can easily check that $|z_0 - Q_x| \geq \kappa r$ and $|y - Q_x| \geq r$.

Then Theorem 5.6 implies

$$c_1^{-1} \frac{G_D(x_1, y)}{g_D(x_1)} \leq \frac{G_D(x,y)}{g_D(x)} \leq c_1 \frac{G_D(x_1, y)}{g_D(x_1)}$$

for some $c_1 > 1$.

Also, since $|z_0 - Q_y| \geq r$ and $|x_1 - Q_y| \geq r$, by Theorem 5.6 again,

$$c_1^{-1} \frac{G_D(x_1, y_1)}{g_D(y_1)} \leq \frac{G_D(x_1,y)}{g_D(y)} \leq c_1 \frac{G_D(x_1, y_1)}{g_D(y_1)}.$$

Therefore

$$c_1^{-2} \frac{G_D(x_1, y_1)}{g_D(x_1)g_D(y_1)} \leq \frac{G_D(x,y)}{g_D(x)g_D(y)} \leq c_1^2 \frac{G_D(x_1, y_1)}{g_D(x_1)g_D(y_1)}.$$

Now we can use Lemma 6.2 for the lower and Lemma 6.1 for the upper bound to get

$$\begin{aligned} \frac{c_2^{-1}c_1^{-2}}{g_D(x_1)g_D(y_1)} \frac{|x_1-y_1|^{-d-2}\phi'(|x_1-y_1|^{-2})}{\phi(|x_1-y_1|^{-2})^2} &\leq \frac{G_D(x,y)}{g_D(x)g_D(y)} \\ &\leq \frac{c_2c_1^2}{g_D(x_1)g_D(y_1)} \frac{|x_1-y_1|^{-d-2}\phi'(|x_1-y_1|^{-2})}{\phi(|x_1-y_1|^{-2})^2} \end{aligned} \tag{6.11}$$

for some $c_2 > 1$.

Since $\frac{|x-y|}{3} < |x_1 - y_1| < 2|x-y|$, Corollary 2.2 yields

$$\frac{|x_1-y_1|^{-d-2}\phi'(|x_1-y_1|^{-2})}{\phi(|x_1-y_1|^{-2})^2} \leq 2 \cdot 3^{d+3} \frac{|x-y|^{-d-2}\phi'(9|x-y|^{-2})}{\phi(9|x-y|^{-2})^2} \leq 2 \cdot 3^{d+3} \frac{|x-y|^{-d-2}\phi'(|x-y|^{-2})}{\phi(|x-y|^{-2})^2}$$

and

$$\frac{|x_1-y_1|^{-d-2}\phi'(|x_1-y_1|^{-2})}{\phi(|x_1-y_1|^{-2})^2} \geq 3^{-1}\cdot 2^{-d-3}\frac{|x-y|^{-d-2}\phi'(4^{-1}|x-y|^{-2})}{\phi(4^{-1}|x-y|^{-2})^2}$$
$$\geq 3^{-1}\cdot 2^{-d-3}\frac{|x-y|^{-d-2}\phi'(|x-y|^{-2})}{\phi(|x-y|^{-2})^2}.$$

Therefore,

(6.12)
$$\frac{2^{-d-3}c_2^{-1}c_1^{-2}}{3g_D(x_1)g_D(y_1)}\frac{|x-y|^{-d-2}\phi'(|x-y|^{-2})}{\phi(|x-y|^{-2})^2} \leq \frac{G_D(x,y)}{g_D(x)g_D(y)} \leq \frac{2\cdot 3^{d+3}c_2c_1^2}{g_D(x_1)g_D(y_1)}\frac{|x-y|^{-d-2}\phi'(|x-y|^{-2})}{\phi(|x-y|^{-2})^2}.$$

If $r=\frac{\varepsilon_1}{2}$, then $r(x,y)=|x-y|\geq\varepsilon_1$, and so

$$g_D(A)=g_D(z_0)=C_4 \quad\text{and}\quad \delta_D(x_1)\wedge\delta_D(y_1)\geq\tfrac{\kappa^2 r}{2}=\tfrac{\kappa^2\varepsilon_1}{4}.$$

Thus, in this case, Lemma 6.3 yields

$$c_3^{-1}\leq\frac{g_D(A)^2}{g_D(x_1)g_D(y_1)}\leq c_3 \tag{6.13}$$

for some $c_3>1$.

In the case $r<\frac{\varepsilon_1}{2}$ we have $r(x,y)=|x-y|<\varepsilon_1$ and $r=\frac12 r(x,y)$. Hence

$$\delta_D(x_1)\wedge\delta_D(y_1)\geq\tfrac{\kappa^2 r}{2}=\tfrac{\kappa^2 r(x,y)}{4}.$$

Since $|x_1-A|\vee|y_1-A|\leq 5r(x,y)+|x_1-x|+|y_1-y|\leq 5r(x,y)+2\kappa r\leq 6r(x,y)$, Theorem 2.10 applied to g_D gives

$$c_4^{-1}\leq\frac{g_D(A)}{g_D(x_1)}\leq c_4 \quad\text{and}\quad c_4^{-1}\leq\frac{g_D(A)}{g_D(y_1)}\leq c_4 \tag{6.14}$$

for some constant $c_4>0$. Combining (6.12)-(6.14), we get

$$c_5^{-1}\frac{g_D(x)g_D(y)}{g_D(A)^2}\frac{|x-y|^{-d-2}\phi'(|x-y|^{-2})}{\phi(|x-y|^{-2})^2}\leq G_D(x,y)\leq c_5\frac{g_D(x)g_D(y)}{g_D(A)^2}\frac{|x-y|^{-d-2}\phi'(|x-y|^{-2})}{\phi(|x-y|^{-2})^2}$$

for all $A\in\mathcal{B}(x,y)$. □

7. Explicit Green function estimates on bounded $C^{1,1}$ open sets

The purpose of this section is to establish the explicit Green function estimates from Theorem 6.4 in the case of bounded $C^{1,1}$ open sets.

Theorem 7.1. *Suppose that $X=(X_t:t\geq 0)$ is a transient d-dimensional subordinate Brownian motion where the corresponding subordinator S has the Laplace exponent ϕ satisfying* **(A-1)**–**(A-5)**. *If D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ with $C^{1,1}$ characteristics (R,Λ), then there exists $c=c(R,\Lambda,\phi,diam(D))>0$ such that*

$$c^{-1}\,(V(\delta_D(x))\wedge 1)\leq g_D(x)\leq c\,(V(\delta_D(x))\wedge 1)\quad \textit{for all}\quad x\in D. \tag{7.1}$$

Proof. The proof follows the proof of [KSV12b, Theorem 4.6] by using our Proposition 2.6, Lemma 3.8 and Theorem 5.6. □

Proof of Theorem 1.2. Only using the fact that V is increasing and subadditive, the following is proved in [KSV12b, (4.38)]:

$$\frac{(V(\delta_D(x))\wedge 1)(V(\delta_D(y))\wedge 1)}{(V(\delta_D(x)\vee\delta_D(y)\vee|x-y|)\wedge 1)^2}\asymp\frac{V(\delta_D(x))V(\delta_D(y))}{V^2(\delta_D(x)\vee\delta_D(y)\vee|x-y|)}: \tag{7.2}$$

Thus, when D is connected, Theorem 1.2 follows from (7.2) and our Theorems 6.4 and 7.1.

Next we assume that D is not connected. The proof below is similar to the one in [CKSV10, Theorem 3.4].

Let (R,Λ) be the $C^{1,1}$ characteristics of D. Note that D has only finitely many components and the distance between any two distinct components of D is at least $R>0$.

Assume first that x and y are in two distinct components of D. Let $D(x)$ be the component of D that contains x. Then by the strong Markov property and (2.12) we obtain

$$G_D(x,y)=\mathbb{E}_x\left[G_D\big(X_{\tau_{D(x)}},y\big)\right]=\mathbb{E}_x\left[\int_0^{\tau_{D(x)}}\left(\int_{D\setminus D(x)} j(|X_s-z|)G_D(z,y)dz\right)ds\right].$$

Consequently,

$$\begin{aligned} j(\mathrm{diam}(D))\,\mathbb{E}_x[\tau_{D(x)}]\int_{D\setminus D(x)} G_D(y,z)dz &\le G_D(x,y)\\ &\le j(R)\,\mathbb{E}_x[\tau_{D(x)}]\int_{D\setminus D(x)} G_D(y,z)dz. \end{aligned} \tag{7.3}$$

Applying the two-sided estimates (1.6) established in the first part of this proof to $D(x)$, after integrating out the second variable we get

$$\frac{c_1^{-1}}{\sqrt{\phi(\delta_D(x)^{-2})}}=\frac{c_1^{-1}}{\sqrt{\phi(\delta_{D(x)}(x)^{-2})}}\le\mathbb{E}_x\left[\tau_{D(x)}\right]\le\frac{c_1^{-1}}{\sqrt{\phi(\delta_{D(x)}(x)^{-2})}}=\frac{c_1^{-1}}{\sqrt{\phi(\delta_D(x)^{-2})}}. \tag{7.4}$$

By (7.4) we get

$$\int_{D\setminus D(x)} G_D(y,z)dz\ge\int_{D(y)} G_{D(y)}(y,z)dz=\mathbb{E}_y[\tau_{D(y)}]\ge\frac{c_2}{\sqrt{\phi(\delta_D(y)^{-2})}}.$$

On the other hand, (2.12) and (7.4) imply

$$\begin{aligned} \int_{D\setminus D(x)} G_D(y,z)dz &\le \mathbb{E}_y\big[\tau_D\big]=\mathbb{E}_y\Big[\tau_{D(y)}\Big]+\mathbb{E}_y\Big[\mathbb{E}_{X_{\tau_{D(y)}}}[\tau_D]\Big]\\ &\le \frac{c_3}{\sqrt{\phi(\delta_D(y)^{-2})}}+\mathbb{E}_y\left[\int_0^{\tau_{D(y)}}\int_{D\setminus D(y)} j(|X_s-z|)\,\mathbb{E}_z[\tau_D]dzds\right]\\ &\le \frac{c_3}{\sqrt{\phi(\delta_D(y)^{-2})}}+j(R)\mathbb{E}_y\big[\tau_{D(y)}\big]|D|\,\mathbb{E}_0[\tau_{B(0,\mathrm{diam(D)})}]\\ &\le \frac{c_3}{\sqrt{\phi(\delta_D(y)^{-2})}}+c_4\mathbb{E}_y\big[\tau_{D(y)}\big]\le\frac{c_5}{\sqrt{\phi(\delta_D(y)^{-2})}}. \end{aligned}$$

We conclude from the last three displays and (7.3) that there is a constant $c_6\ge 1$ such that

$$\frac{c_6^{-1}}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})}}\le G_D(x,y)\le\frac{c_6}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})}}. \tag{7.5}$$

Noting that

$$R\le|x-y|\le\mathrm{diam}(D)$$

when x and y are in different components of D, by Corollary 2.2 we obtain (1.6).

Now we assume that x, y are in the same component U of D. Applying (1.6) to U we get

$$\begin{aligned} G_D(x,y) \geq G_U(x,y) &\geq c_7\left(1\wedge \frac{\phi(|x-y|^{-2})}{\sqrt{\phi(\delta_U(x)^{-2})\phi(\delta_U(y)^{-2})}}\right)\frac{|x-y|^{-d-2}\phi'(|x-y|^{-2})}{\phi(|x-y|^{-2})^2} \\ &= c_7\left(1\wedge \frac{\phi(|x-y|^{-2})}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})}}\right)\frac{|x-y|^{-d-2}\phi'(|x-y|^{-2})}{\phi(|x-y|^{-2})^2}. \end{aligned}$$

For the upper bound, we use the strong Markov property, (2.12) and (7.4)–(7.5) to get

$$\begin{aligned} &G_D(x,y) \\ = \;& G_U(x,y) + \mathbb{E}_x\left[G_D(X_{\tau_U},y)\right] \\ \leq \;& c_8\left(1\wedge \frac{\phi(|x-y|^{-2})}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})}}\right)\frac{|x-y|^{-d-2}\phi'(|x-y|^{-2})}{\phi(|x-y|^{-2})^2} \\ & +\mathbb{E}_x\left[\int_0^{\tau_U}\int_{D\setminus U} j(|X_s-z|)G_D(z,y)dzds\right] \\ \leq \;& c_8\left(1\wedge \frac{\phi(|x-y|^{-2})}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})}}\right)\frac{|x-y|^{-d-2}\phi'(|x-y|^{-2})}{\phi(|x-y|^{-2})^2} \\ & + j(R)\,\mathbb{E}_x[\tau_U]\int_{D\setminus U} G_D(y,z)dz \\ \leq \;& c_8\left(1\wedge \frac{\phi(|x-y|^{-2})}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})}}\right)\frac{|x-y|^{-d-2}\phi'(|x-y|^{-2})}{\phi(|x-y|^{-2})^2} \\ & + \frac{c_9\int_{D\setminus U}\frac{dz}{\sqrt{\phi(\delta_D(z)^{-2})}}dz}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})}}. \end{aligned} \tag{7.6}$$

Since D is bounded, we get

$$\begin{aligned} \frac{1}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})}}\int_{D\setminus U}\frac{dz}{\sqrt{\phi(\delta_D(z)^{-2})}} &\leq \frac{|D|}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})\phi(\mathrm{diam}(D)^{-2})}} \\ &\leq c_{10}\left(1\wedge \frac{\phi(|x-y|^{-2})}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})}}\right), \end{aligned}$$

which together with (7.6) and Corollary 2.2 gives

$$\begin{aligned} G_D(x,y) &\leq c_{11}\left(1\wedge \frac{\phi(|x-y|^{-2})}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})}}\right) \\ &\leq c_{12}\left(1\wedge \frac{\phi(|x-y|^{-2})}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})}}\right)\frac{\phi'(|x-y|^{-2})}{|x-y|^{d+2}\phi(|x-y|^{-2})^2}. \qquad \square \end{aligned}$$

Proof of Theorem 1.5. When $d = 1$, the theorem follows from Proposition 2.4, Theorem 3.1 and Theorem 5.6 (i).

Note that the result in [CKSV12, Lemma 4.2] is true in our case too. By this result, Theorem 2.10, Theorem 5.6 (i) and Theorem 1.2, the proof of Theorem 1.5 is the same as the proof of [KSV12b, Theorem 1.3] when $d \geq 3$.

Note that if D is a $C^{1,1}$ open set in $\mathbb{R}^{d-1}$ with characteristics (R, Λ), then $D \times \mathbb{R}$ is clearly a $C^{1,1}$ open set in $\mathbb{R}^d$ with the same characteristics (R, Λ). Thus the case $d = 2$ can be handled in the same way as in the proof of Theorem 5.6 (i). $\square$

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