TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 366, Number 8, August 2014, Pages 4127–4150
S 0002-9947(2014)06113-8
Article electronically published on March 26, 2014

WEIGHTED BERGMAN SPACES AND THE $\bar{\partial}$–EQUATION

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Dedicated to Professor Jinhao Zhang on the occasion of his seventieth birthday

ABSTRACT. We give a Hörmander type L^2–estimate for the $\bar{\partial}$–equation with respect to the measure $\delta_\Omega^{-\alpha} dV$, $\alpha < 1$, on any bounded pseudoconvex domain with C^2–boundary. Several applications to the function theory of weighted Bergman spaces $A^2_\alpha(\Omega)$ are given, including a corona type theorem, a Gleason type theorem, together with a density theorem. We investigate in particular the boundary behavior of functions in $A^2_\alpha(\Omega)$ by proving an analogue of the Levi problem for $A^2_\alpha(\Omega)$ and giving an optimal Gehring type estimate for functions in $A^2_\alpha(\Omega)$. A vanishing theorem for $A^2_1(\Omega)$ is established for arbitrary bounded domains. Relations between the weighted Bergman kernel and the Szegő kernel are also discussed.

1. Introduction

Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex domain and let φ be a C^2 plurisubharmonic (psh) function on Ω. A fundamental theorem of Hörmander (cf. [23, 26]; see also [1, 13]) states that for any $\bar{\partial}$–closed $(0,1)$–form v, there exists a solution u to the equation $\bar{\partial} u = v$ such that

$$\int_\Omega |u|^2 e^{-\varphi} dV \le \int_\Omega |v|^2_{i\partial\bar{\partial}\varphi} e^{-\varphi} dV \tag{1}$$

provided the right-hand side is finite.

In 1983, Donnelly-Fefferman [14] made a striking discovery that under certain conditions, the $\bar{\partial}$–equation may have solutions of finite L^2–norm with some *non-psh* weight. Such a discovery was extended and simplified substantially by a number of mathematicians (see e.g. [4, 6, 9, 17, 33]), and may now be formulated as follows: if ψ is another C^2 psh function on Ω satisfying $i\alpha\partial\bar{\partial}\psi \ge i\partial\psi \wedge \bar{\partial}\psi$ for some $0 < \alpha < 1$, then the $L^2(\Omega, \varphi)$–minimal solution of the $\bar{\partial}$–equation enjoys the estimate

$$\int_\Omega |u|^2 e^{\psi-\varphi} dV \le \text{const}_\alpha \int_\Omega |v|^2_{i\partial\bar{\partial}(\varphi+\psi)} e^{\psi-\varphi} dV \tag{2}$$

provided the right-hand side is finite. In particular, if we take $\psi = -\frac{\alpha}{\alpha_0}\log(-\rho)$, where ρ is a negative C^2 psh function verifying $-\rho \asymp \delta_\Omega^{\alpha_0}$, $\alpha_0 > \alpha > 0$ and δ_Ω is the boundary distance function, then (2) implies

$$\int_\Omega |u|^2 e^{-\varphi} \delta_\Omega^{-\alpha} dV \le \text{const}_{\alpha,\Omega} \int_\Omega |v|^2_{i\partial\bar{\partial}\varphi} e^{-\varphi} \delta_\Omega^{-\alpha} dV, \tag{3}$$

Received by the editors July 29, 2012.
2010 *Mathematics Subject Classification.* Primary 32A25, 32A36, 32A40, 32W05.
Key words and phrases. $\bar{\partial}$–equation, L^2–estimate, Bergman space, weighted Bergman kernel, Szegő kernel.
This work was supported by the Key Program of NSFC No. 11031008.

which has significant applications in the study of regularities of the Bergman projection (cf. [6]; see also [34]). In case Ω has a C^2-boundary, Diederich-Fornæss [15] proved the existence of such a ρ, where α_0 is called a Diederich-Fornæss exponent. On the other side, there are pseudoconvex domains (so-called worm domains) whose Diederich-Fornæss exponents are arbitrarily small (cf. [16]).

In this paper, we shall prove the following

Theorem 1.1. *Let $\Omega \Subset \mathbb{C}^n$ be a pseudoconvex domain with C^2-boundary and φ a C^2 psh function on Ω. Then for each $\alpha < 1$ and each $\bar\partial-$closed $(0,1)-$form v with $\int_\Omega |v|^2_{i\partial\bar\partial\varphi} e^{-\varphi}\delta_\Omega^{-\alpha} dV < \infty$, there is a solution u to the equation $\bar\partial u = v$ such that* (3) *holds.*

We shall give various applications of this result to the function theory of the weighted Bergman space $A^2_\alpha(\Omega)$, that is, the Hilbert space of holomorphic functions f on Ω with

$$\|f\|^2_\alpha := \int_\Omega |f|^2 \delta_\Omega^{-\alpha} dV < \infty.$$

The spaces $A^2_\alpha(\Omega)$ coincide with the usual Sobolev spaces of holomorphic functions for $\alpha < 1$, i.e.,

$$A^2_\alpha(\Omega) = \mathscr{O}(\Omega) \cap W^\alpha(\Omega)$$

(see Ligocka [32]). Despite the deep results achieved for strongly pseudoconvex domains (see e.g., [2, 18]), little progress has been made in the case of weakly pseudoconvex domains.

Corona type theorem. *Let $\Omega \Subset \mathbb{C}^n$ be a pseudoconvex domain with C^2-boundary. Let $f_1, f_2 \in \mathscr{O}(\Omega)$ and $\delta > 0$ be such that*

$$\delta^2 \le |f_1|^2 + |f_2|^2 \le 1.$$

Then for each $h \in A^2_\alpha(\Omega)$, $\alpha < 1$, there are functions $g_1, g_2 \in A^2_\alpha(\Omega)$ satisfying

$$f_1 g_1 + f_2 g_2 = h.$$

Gleason type theorem. *Let $\Omega \Subset \mathbb{C}^2$ be a pseudoconvex domain with C^2-boundary. If $w \in \Omega$ and $h \in A^2_\alpha(\Omega)$, $\alpha < 1$, then there are functions $g_1, g_2 \in A^2_\alpha(\Omega)$ satisfying*

$$h(z) - h(w) = (z_1 - w_1)g_1(z) + (z_2 - w_2)g_2(z), \quad \forall\, z \in \Omega.$$

Density theorem. *Let $\Omega \Subset \mathbb{C}^n$ be a pseudoconvex domain with C^2-boundary.*

(a) *For each $\alpha < 1$, $A^2_\alpha(\Omega)$ is dense in the space $\mathscr{O}(\Omega)$, equipped with the topology of uniform convergence on compact subsets.*

(b) *For any $\alpha_1 < \alpha_2 < 1$, $A^2_{\alpha_2}(\Omega)$ is dense in $A^2_{\alpha_1}(\Omega)$.*

The following result is an analogue of the Levi problem for $A^2_\alpha(\Omega)$, which also generalizes an old result of Pflug (cf. [38]):

Theorem 1.2. *Let $\Omega \Subset \mathbb{C}^n$ be a pseudoconvex domain with C^2-boundary. Then for each $\alpha < 1$, there are $\beta > 0$ and $f \in A^2_\alpha(\Omega)$ such that for all $\zeta \in \partial\Omega$,*

$$\limsup_{z\to\zeta} |f(z)|\delta_\Omega(z)^{1-\frac{\alpha}{2}} |\log \delta_\Omega(z)|^\beta = \infty.$$

It should be noted that each bounded pseudoconvex domain with $C^\infty-$*boundary* is the domain of existence of a function in $A^\infty(\overline{\Omega}) := \mathscr{O}(\Omega) \cap C^\infty(\overline{\Omega})$ (cf. [10]; see also [22]).

On the other side, we have the following Gehring type estimate:

Theorem 1.3. *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with C^2−boundary and let $f \in A^2_\alpha(\Omega)$, $\alpha < 1$. Then for almost all $\zeta \in \partial\Omega$,*

$$|f(z)| = \mathrm{o}(\delta_\zeta(z)^{-\frac{1-\alpha}{2}}) \qquad \textit{uniformly},$$

as z approaches ζ admissibly. Here $\delta_\zeta(z)$ = minimum of $\delta_\Omega(z)$ and the distance from z to the tangent space at ζ, and $A = \mathrm{o}(B)$ means $\lim A/B = 0$.

The concept of admissible approach was introduced by Stein [41] in his far-reaching generalization of Fatou's theorem for holomorphic functions in a bounded domain with C^2−boundary.

It turns out that the above bound is optimal for the case of the unit ball:

Theorem 1.4. *Let $\mathbb{B}^n$ be the unit ball in $\mathbb{C}^n$ and $\mathbb{S}^n$ the unit sphere. For each $\alpha < 1$, there is a number $t_\alpha > 1$ such that for each $\varepsilon > 0$, there exists a function $f \in A^2_\alpha(\mathbb{B}^n)$ so that for each $\zeta \in \mathbb{S}^n$,*

$$\limsup |f(z)|(1-|z|)^{\frac{1-\alpha}{2}} |\log(1-|z|)|^{\frac{1+\varepsilon}{2}} > 0$$

as $z \to \zeta$ from the inside of the Koranyi region $\mathcal{A}_{t_\alpha}(\zeta)$ defined by

$$\mathcal{A}_{t_\alpha}(\zeta) = \left\{ z \in \mathbb{B}^n : |1 - z \cdot \bar{\zeta}| < t_\alpha(1-|z|) \right\}.$$

Stein [41] suggested studying the relation between the Bergman and Szegő kernels. In [12], Chen-Fu obtained a comparison of the Szegő and Bergman kernels for so-called δ−regular domains including domains of finite type and domains with psh defining functions. Here we shall prove the following natural connection between the weighted Bergman kernels K_α and the Szegő kernel S, which seems not to have been noticed in the literature:

Theorem 1.5. *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with C^2−boundary. Then*

$$(1-\alpha)^{-1} K_\alpha(z,w) \to S(z,w)$$

locally uniformly in z, w as $\alpha \to 1-$. In particular,

$$\left.\frac{\partial K_\alpha(z,w)}{\partial \alpha}\right|_{1-} := \lim_{\alpha \to 1-} \frac{K_\alpha(z,w) - K_1(z,w)}{\alpha - 1} = -S(z,w).$$

For general bounded domains, a fundamental question immediately arises:

When is $A^2_\alpha(\Omega)$ trivial or non-trivial?

Clearly, $A^2_\alpha(\Omega)$ is always non-trivial for $\alpha \le 0$. On the other side, we have the following vanishing theorem:

Theorem 1.6. *Let Ω be a bounded domain in $\mathbb{C}^n$.*

(a) *For each $f \in \mathcal{O}(\Omega)$ with $\int_\Omega |f|^2 \delta_\Omega^{-1} (1 + |\log \delta_\Omega|)^{-1} dV < \infty$, we have $f = 0$. In particular, $A^2_\alpha(\Omega) = \{0\}$ for each $\alpha \ge 1$.*

(b) *Let $\Omega_\varepsilon = \{z \in \Omega : \delta_\Omega(z) > \varepsilon\}$ and let $c(\varepsilon) := \mathrm{cap}\left(\overline{\Omega}_\varepsilon, \Omega\right)$ denote the capacity of $\overline{\Omega}_\varepsilon$ in Ω. Suppose there is a sequence $\varepsilon_j \to 0+$ so that $c(\varepsilon_j) = O(\varepsilon_j^{-\alpha})$; then $A^2_\alpha(\Omega) = \{0\}$.*

As a consequence of Theorem 1.6, we have

Theorem 1.7. *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain. For each $\varepsilon > 0$, there does not exist a continuous psh function $\rho < 0$ on Ω such that*

$$-\rho \le \mathrm{const}_\varepsilon \delta_\Omega \left(1 + |\log \delta_\Omega|\right)^{-\varepsilon}.$$

In particular, the order of hyperconvexity of Ω is no larger than 1. In case $\partial\Omega$ is of class C^2, this result is a direct consequence of the Hopf lemma.

2. Proof of Theorem 1.1

Let $\Omega \Subset \mathbb{C}^n$ be a pseudoconvex domain with C^2-boundary. Let φ be a real-valued C^2-smooth function on Ω. Let $L^{p,q}_{(2)}(\Omega,\varphi)$ denote the space of $(p,q)-$forms u on Ω satisfying

$$\|u\|^2_\varphi := \int_\Omega |u|^2 e^{-\varphi} dV < \infty.$$

Let $\bar\partial^*_\varphi$ denote the adjoint of the operator $\bar\partial$ with respect to the corresponding inner product $(\cdot,\cdot)_\varphi$. We recall the following twisted Morrey-Kohn-Hörmander formula, which goes back to Ohsawa-Takegoshi (cf. [4, 9, 33, 36, 37, 40]):

Proposition 2.1. *Let ρ be a C^2-defining function of Ω. Let u be a $(0,1)-$form that is continuously differentiable on $\overline{\Omega}$ and satisfies the $\bar\partial-$Neumann boundary conditions on $\partial\Omega$, $\partial\rho\cdot u = 0$, and let η and φ be real-valued functions that are twice continuously differentiable on $\overline{\Omega}$ with $\eta \geq 0$. Then*

$$\begin{aligned}\|\sqrt{\eta}\bar\partial u\|^2_\varphi + \|\sqrt{\eta}\bar\partial^*_\varphi u\|^2_\varphi =& \sum_{j,k=1}^n \int_{\partial\Omega} \eta \frac{\partial^2\rho}{\partial z_j \partial\bar z_k} u_j \bar u_k e^{-\varphi} \frac{d\sigma}{|\nabla\rho|} + \sum_{j=1}^n \int_\Omega \eta \left|\frac{\partial u_j}{\partial \bar z_j}\right|^2 e^{-\varphi} dV \\ &+ \sum_{j,k=1}^n \int_\Omega \left(\eta \frac{\partial^2\varphi}{\partial z_j\partial\bar z_k} - \frac{\partial^2\eta}{\partial z_j \partial \bar z_k}\right) u_j \bar u_k e^{-\varphi} dV \\ &+ 2\mathrm{Re} \int_\Omega (\partial\eta\cdot u)\overline{\bar\partial^*_\varphi u} e^{-\varphi} dV.\end{aligned}$$

Now we prove Theorem 1.1. It is well known that *locally* the Diederich-Fornæss exponents can be arbitrarily close to 1 (cf. [15], Remark b), p. 133). Thus for any given $\alpha < 1$, there exists a cover $\{U_j\}_{1\le j\le m_\alpha}$ of $\partial\Omega$ and C^2 psh functions $\rho_j < 0$ on $\Omega\cap U_j$ such that

$$C^{-1}\delta_\Omega(z)^{\frac{\alpha+1}{2}} \leq -\rho_j(z) \leq C\delta_\Omega(z)^{\frac{\alpha+1}{2}}, \quad z\in\Omega\cap U_j,\ 1\le j\le m_\alpha.$$

(Throughout this section, C denotes a generic positive constant depending only on α and Ω.) Take an open subset $U_0 \Subset \Omega$ such that $\{U_j\}_{0\le j\le m_\alpha}$ forms a cover of $\overline{\Omega}$. Clearly, we can take a negative C^2 psh function ρ_0 on U_0 such that

$$C^{-1}\delta_\Omega(z)^{\frac{\alpha+1}{2}} \leq -\rho_0(z) \leq C\delta_\Omega(z)^{\frac{\alpha+1}{2}}, \quad z\in U_0$$

(for example, $\rho_0(z) = |z|^2 - \sup_\Omega |z|^2 - 1$).

Put $\varphi_\tau(z) = \varphi(z) + \tau|z|^2$, $\tau > 0$, and $\Omega_\varepsilon := \{z\in\Omega : \delta_\Omega(z) > \varepsilon\}$, $\varepsilon \ll 1$. By Proposition 2.1, we have

$$\begin{aligned}&\int_{\Omega_\varepsilon} (\eta + c(\eta)^{-1})|\bar\partial^*_{\varphi_\tau} w|^2 e^{-\varphi_\tau} dV + \int_{\Omega_\varepsilon} \eta|\bar\partial w|^2 e^{-\varphi_\tau} dV \\ &\geq \sum_{k,l} \int_{\Omega_\varepsilon} \left(\eta\frac{\partial^2\varphi_\tau}{\partial z_k\partial\bar z_l} - \frac{\partial^2\eta}{\partial z_k\partial\bar z_l}\right) w_k \bar w_l\, e^{-\varphi_\tau} dV - \int_{\Omega_\varepsilon} c(\eta)\left|\sum_k \frac{\partial\eta}{\partial z_k} w_k\right|^2 e^{-\varphi_\tau} dV, \end{aligned} \tag{4}$$

where $w = \sum_k w_k d\bar z_k$ lies in Dom $\bar\partial^*_{\varphi_\tau}$ and is continuously differentiable on $\overline{\Omega}_\varepsilon$ (i.e., it satisfies the $\bar\partial-$Neumann boundary condition on $\partial\Omega_\varepsilon$), $\eta \geq 0$, $\eta \in C^2(\Omega)$ and c is a positive continuous function on $\mathbb{R}^+$.

Let $\{\chi_j\}_{0\le j\le m_\alpha}$ be a partition of unity subordinate to the cover $\{U_j\}_{0\le j\le m_\alpha}$ of $\overline{\Omega}$. The point is that $w^j=\chi_j w$ still lies in $\mathrm{Dom}\,\bar\partial^*_{\varphi_\tau}$. Now we choose a real-valued function $\tilde\chi_j\in C_0^\infty(U_j)$ so that $\tilde\chi_j=1$ on $\mathrm{supp}\,\chi_j$. Put $\psi_j=-\frac{2\alpha}{\alpha+1}\log(-\rho_j)$. Applying (4) to each w^j with $\eta=e^{-\tilde\chi_j\psi_j}$ and $c(\eta)=\frac{1-\alpha}{2\alpha}e^{\tilde\chi_j\psi_j}$, we get

$$\begin{aligned}&\sum_{k,l}\int_{\Omega_\varepsilon\cap U_j}\frac{\partial^2\varphi_\tau}{\partial z_k\partial\bar z_l}|\chi_j|^2 w_k\bar w_l\, e^{-\varphi_\tau-\psi_j}dV\\ \le\ &\int_{\Omega_\varepsilon\cap U_j}|\bar\partial(\chi_j w)|^2 e^{-\varphi_\tau-\psi_j}dV+\frac{1+\alpha}{1-\alpha}\int_{\Omega_\varepsilon\cap U_j}|\bar\partial^*_{\varphi_\tau}(\chi_j w)|^2 e^{-\varphi_\tau-\psi_j}dV\end{aligned}$$

because

$$-i(\partial\bar\partial\eta+c(\eta)\partial\eta\wedge\bar\partial\eta)=ie^{-\psi_j}\left(\partial\bar\partial\psi_j-\frac{\alpha+1}{2\alpha}\partial\psi_j\wedge\bar\partial\psi_j\right)\ge 0$$

holds on $\Omega\cap\mathrm{supp}\,\chi_j$. Since $e^{-\psi_j}\asymp\delta_\Omega^\alpha$ on $\Omega\cap U_j$, we get

$$\begin{aligned}&\sum_{k,l}\int_{\Omega_\varepsilon\cap U_j}\frac{\partial^2\varphi_\tau}{\partial z_k\partial\bar z_l}|\chi_j|^2 w_k\bar w_l\, e^{-\varphi_\tau}\delta_\Omega^\alpha dV\\ &\quad\le C\int_{\Omega_\varepsilon\cap U_j}(|\bar\partial(\chi_j w)|^2+|\bar\partial^*_{\varphi_\tau}(\chi_j w)|^2)e^{-\varphi_\tau}\delta_\Omega^\alpha dV.\end{aligned}\tag{5}$$

Thus

$$\begin{aligned}&\sum_{k,l}\int_{\Omega_\varepsilon}\frac{\partial^2\varphi_\tau}{\partial z_k\partial\bar z_l}w_k\bar w_l\, e^{-\varphi_\tau}\delta_\Omega^\alpha dV\\ =\ &\sum_{k,l}\int_{\Omega_\varepsilon}\frac{\partial^2\varphi_\tau}{\partial z_k\partial\bar z_l}\left(\sum_{j=0}^{m_\alpha}\chi_j\right)^2 w_k\bar w_l e^{-\varphi_\tau}\delta_\Omega^\alpha dV\\ \le\ &(m_\alpha+1)\sum_{j=0}^{m_\alpha}\sum_{k,l}\int_{\Omega_\varepsilon\cap U_j}\frac{\partial^2\varphi_\tau}{\partial z_k\partial\bar z_l}|\chi_j|^2 w_k\bar w_l\, e^{-\varphi_\tau}\delta_\Omega^\alpha dV\\ \le\ &(m_\alpha+1)C\sum_{j=0}^{m_\alpha}\int_{\Omega_\varepsilon\cap U_j}(|\bar\partial(\chi_j w)|^2+|\bar\partial^*_{\varphi_\tau}(\chi_j w)|^2)e^{-\varphi_\tau}\delta_\Omega^\alpha dV\end{aligned}$$

by (5). Since

$$\bar\partial(\chi_j w)=\chi_j\bar\partial w+\bar\partial\chi_j\wedge w,\quad \bar\partial^*_{\varphi_\tau}(\chi_j w)=\chi_j\bar\partial^*_{\varphi_\tau}w-\bar\partial\chi_j\lrcorner w,$$

thus by Schwarz's inequality,

$$\begin{aligned}&\sum_{k,l}\int_{\Omega_\varepsilon}\frac{\partial^2\varphi_\tau}{\partial z_k\partial\bar z_l}w_k\bar w_l\, e^{-\varphi_\tau}\delta_\Omega^\alpha dV\\ \le\ &2(m_\alpha+1)C\sum_{j=0}^{m_\alpha}\int_{\Omega_\varepsilon\cap U_j}(|\bar\partial w|^2+|\bar\partial^*_{\varphi_\tau}w|^2+2|w|^2|\bar\partial\chi_j|^2)e^{-\varphi_\tau}\delta_\Omega^\alpha dV\\ \le\ &2(m_\alpha+1)^2C\int_{\Omega_\varepsilon}(|\bar\partial w|^2+|\bar\partial^*_{\varphi_\tau}w|^2)e^{-\varphi_\tau}\delta_\Omega^\alpha dV\\ &+4(m_\alpha+1)C\int_{\Omega_\varepsilon}|w|^2\sum_j|\bar\partial\chi_j|^2 e^{-\varphi_\tau}\delta_\Omega^\alpha dV.\end{aligned}\tag{6}$$

Since $\partial\bar\partial\varphi_\tau = \partial\bar\partial\varphi + \tau\partial\bar\partial|z|^2$, then when $\tau = \tau(\alpha,\Omega)$ is sufficiently large, the second term on the right-hand side of (6) may be absorbed by the left-hand side and we get the following basic inequality:

$$(7)\qquad \sum_{k,l}\int_{\Omega_\varepsilon}\frac{\partial^2\varphi}{\partial z_k\partial\bar z_l}w_k\bar w_l\, e^{-\varphi_\tau}\delta_\Omega^\alpha dV \le C\int_{\Omega_\varepsilon}(|\bar\partial w|^2+|\bar\partial^*_{\varphi_\tau}w|^2)e^{-\varphi_\tau}\delta_\Omega^\alpha dV.$$

The remaining argument is standard. By Hörmander [23], Proposition 2.1.1, the same inequality holds for any $w \in L^{0,1}_{(2)}(\Omega_\varepsilon,\varphi_\tau)\cap \mathrm{Dom}\,\bar\partial \cap \mathrm{Dom}\,\bar\partial^*_{\varphi_\tau}$ (Note that $C_\varepsilon^{-1}\le \delta_\Omega^\alpha \le C_\varepsilon$ on Ω_ε.) In particular, if $\bar\partial w = 0$, then

$$\sum_{k,l}\int_{\Omega_\varepsilon}\frac{\partial^2\varphi}{\partial z_k\partial\bar z_l}w_k\bar w_l\, e^{-\varphi_\tau}\delta_\Omega^\alpha dV \le C\int_{\Omega_\varepsilon}|\bar\partial^*_{\varphi_\tau}w|^2e^{-\varphi_\tau}\delta_\Omega^\alpha dV.$$

By Schwarz's inequality,

$$\begin{aligned}\left|\int_{\Omega_\varepsilon}\langle v,w\rangle e^{-\varphi_\tau}dV\right|^2 &\le \int_{\Omega_\varepsilon}|v|^2_{i\partial\bar\partial\varphi}e^{-\varphi_\tau}\delta_\Omega^{-\alpha}dV\sum_{k,l}\int_{\Omega_\varepsilon}\frac{\partial^2\varphi}{\partial z_k\partial\bar z_l}w_k\bar w_l\, e^{-\varphi_\tau}\delta_\Omega^\alpha dV\\ &\le C\int_{\Omega}|v|^2_{i\partial\bar\partial\varphi}e^{-\varphi_\tau}\delta_\Omega^{-\alpha}dV\int_{\Omega_\varepsilon}|\bar\partial^*_{\varphi_\tau}w|^2e^{-\varphi_\tau}\delta_\Omega^\alpha dV.\end{aligned}$$

For general $w \in \mathrm{Dom}\,\bar\partial^*_{\varphi_\tau}$, one has the orthogonal decomposition $w = w_1 + w_2$ where $w_1 \in \mathrm{Ker}\,\bar\partial$ and $w_2 \in (\mathrm{Ker}\,\bar\partial)^\perp \subset \mathrm{Ker}\,\bar\partial^*_{\varphi_\tau}$. Thus

$$\begin{aligned}\left|\int_{\Omega_\varepsilon}\langle v,w\rangle e^{-\varphi_\tau}dV\right|^2 &= \left|\int_{\Omega_\varepsilon}\langle v,w_1\rangle e^{-\varphi_\tau}dV\right|^2\\ \le\ & C\int_{\Omega}|v|^2_{i\partial\bar\partial\varphi}e^{-\varphi_\tau}\delta_\Omega^{-\alpha}dV\int_{\Omega_\varepsilon}|\bar\partial^*_{\varphi_\tau}w_1|^2e^{-\varphi_\tau}\delta_\Omega^\alpha dV\\ =\ & C\int_{\Omega}|v|^2_{i\partial\bar\partial\varphi}e^{-\varphi_\tau}\delta_\Omega^{-\alpha}dV\int_{\Omega_\varepsilon}|\bar\partial^*_{\varphi_\tau}w|^2e^{-\varphi_\tau}\delta_\Omega^\alpha dV.\end{aligned}$$

Applying the Hahn-Banach theorem to the anti-linear map

$$\delta_\Omega^{\frac{\alpha}{2}}\bar\partial^*_{\varphi_\tau}w \mapsto \int_{\Omega_\varepsilon}\langle v,w\rangle e^{-\varphi_\tau}dV,$$

together with the Riesz representation theorem, we get a solution u_ε of the equation $\bar\partial(\delta_\Omega^{\frac{\alpha}{2}}u_\varepsilon) = v$ on Ω_ε with the estimate

$$\int_{\Omega_\varepsilon}|u_\varepsilon|^2e^{-\varphi_\tau}dV \le C\int_{\Omega}|v|^2_{i\partial\bar\partial\varphi}e^{-\varphi_\tau}\delta_\Omega^{-\alpha}dV.$$

Taking a weak limit of $\delta_\Omega^{\frac{\alpha}{2}}u_\varepsilon$ as $\varepsilon \to 0+$, we immediately obtain the desired solution. □

Remark 2.2. (a) The additional weight $t|z|^2$ is somewhat inspired by Kohn [30].

(b) The following variation of Theorem 1.1 is more convenient for applications, which may be proved similarly, together with an additional approximation argument.

Theorem 1.1′. *Let $\Omega \Subset \mathbb{C}^n$ be a pseudoconvex domain with C^2-boundary and let $\hat\Omega \subset \Omega$ be a pseudoconvex domain. Let φ be a psh function on $\hat\Omega$ such that $i\partial\bar\partial\varphi \ge i\partial\bar\partial\psi$ in the sense of distribution, where ψ is a C^2 psh function on $\hat\Omega$. Then* *for each* $\alpha < 1$ *and each* $\bar\partial$−*closed* $(0,1)$−*form* v *with* $\int_{\hat\Omega} |v|^2_{i\partial\bar\partial\psi} e^{-\varphi}\delta_\Omega^{-\alpha} dV < \infty$, *there is a solution* u *to the equation* $\bar\partial u = v$ *on* $\hat\Omega$ *such that*

$$\int_{\hat\Omega} |u|^2 e^{-\varphi}\delta_\Omega^{-\alpha} dV \le \mathrm{const}_{\alpha,\Omega} \int_{\hat\Omega} |v|^2_{i\partial\bar\partial\psi} e^{-\varphi}\delta_\Omega^{-\alpha} dV.$$

3. Some consequences of Theorem 1.1

3.1. We first prove the Corona type theorem. Following Wolff's approach to Carleson's theorem (cf. [19], p. 315), we put

$$g_1 = h\frac{\bar f_1}{|f|^2} - uf_2, \quad g_2 = h\frac{\bar f_2}{|f|^2} + uf_1,$$

where $|f|^2 = |f_1|^2 + |f_2|^2$. Clearly, $f_1g_1 + f_2g_2 = h$, so the problem is reduced to choose $u \in L^2_\alpha(\Omega)$, i.e., $\int_\Omega |u|^2\delta_\Omega^{-\alpha} dV < \infty$, so that g_1, g_2 are holomorphic. Thus it suffices to solve

$$\bar\partial u = h\frac{\overline{f_2\partial f_1} - \overline{f_1\partial f_2}}{|f|^4} =: v$$

such that $u \in L^2_\alpha(\Omega)$. Applying Theorem 1.1 with $\varphi = \log|f|^2$, we get a solution u satisfying

$$\int_\Omega |u|^2|f|^{-2}\delta_\Omega^{-\alpha} dV \le \mathrm{const}_{\alpha,\Omega} \int_\Omega |v|^2_{i\partial\bar\partial\varphi}|f|^{-2}\delta_\Omega^{-\alpha} dV.$$

A straightforward calculation shows

$$\partial\bar\partial\varphi = \frac{(f_1\partial f_2 - f_2\partial f_1)\wedge\overline{(f_1\partial f_2 - f_2\partial f_1)}}{|f|^4}$$

so that $|v|^2_{i\partial\bar\partial\varphi} \le |h|^2/|f|^4 \le |h|^2/\delta^4$. Thus

$$\int_\Omega |u|^2\delta_\Omega^{-\alpha} dV \le \mathrm{const}_{\alpha,\Omega}\, \delta^{-6} \int_\Omega |h|^2\delta_\Omega^{-\alpha} dV.$$

□

3.2. Next we prove the Gleason type theorem. The argument is a slight modification of Section 3.1. Without loss of generality, we assume $w = 0$, $h(0) = 0$, $|z|^2 < e^{-1}$ on Ω. Put $f_k = z_k$, $k = 1, 2$, and $\varphi = -\log(-\log|f|^2)$. Then we have

$$\partial\bar\partial\varphi \ge \frac{(f_1\partial f_2 - f_2\partial f_1)\wedge\overline{(f_1\partial f_2 - f_2\partial f_1)}}{|f|^4(-\log|f|^2)}.$$

Let g_k, v be defined as above and put $\hat\Omega = \Omega\backslash\{f_1 = 0\}$. By Theorem 1.1', we may solve the equation $\bar\partial u = v$ on $\hat\Omega$ such that

$$\int_{\hat\Omega} |u|^2\delta_\Omega^{-\alpha} dV \le \int_{\hat\Omega} |u|^2 e^{-\varphi}\delta_\Omega^{-\alpha} dV \le \mathrm{const}_{\alpha,\Omega} \int_{\hat\Omega} |v|^2_{i\partial\bar\partial\varphi} e^{-\varphi}\delta_\Omega^{-\alpha} dV$$

since the last term is bounded by

$$\begin{aligned}
& \mathrm{const}_{\alpha,\Omega} \int_{\hat\Omega} |h|^2 |f|^{-4} (\log |f|^2)^2 \delta_\Omega^{-\alpha} dV \\
= \ & \mathrm{const}_{\alpha,\Omega} \int_{\hat\Omega \cap \{|z|<\varepsilon\}} |h|^2 |f|^{-4} (\log |f|^2)^2 \delta_\Omega^{-\alpha} dV \\
& + \mathrm{const}_{\alpha,\Omega} \int_{\hat\Omega \setminus \{|z|<\varepsilon\}} |h|^2 |f|^{-4} (\log |f|^2)^2 \delta_\Omega^{-\alpha} dV \\
\le \ & \mathrm{const}_{\alpha,\Omega} \int_{\{|z|<\varepsilon\}} |z|^{-2} (\log |z|)^2 dV + \mathrm{const}_{\alpha,\Omega} \int_\Omega |h|^2 \delta_\Omega^{-\alpha} dV < \infty,
\end{aligned}$$

where $\varepsilon > 0$ is so small that $\{|z| \le \varepsilon\} \subset \Omega$. Thus g_1, g_2 are holomorphic on $\hat\Omega$ such that

$$\int_{\hat\Omega} |g_k|^2 \delta_\Omega^{-\alpha} dV < \infty, \quad k = 1, 2.$$

The assertion follows immediately from Riemann's removable singularities theorem. □

Remark 3.1. It is possible to extend both the Corona and Gleason type theorems to general cases by using the Koszul complex technique introduced by Hörmander [24]. But the argument will be substantially longer and not very enlightening, so we shall not treat it here.

3.3. Finally, we prove the density theorem. (a) Let K be a compact subset of Ω and $f \in \mathscr{O}(\Omega)$. We take a strictly psh exhaustion function $\psi \in C^\infty(\Omega)$ such that $K \subset \{\psi < 0\}$. Let κ be a C^∞ convex increasing function such that $\kappa = 0$ on $(-\infty, 0]$ and $\kappa' > 0$, $\kappa'' > 0$ on $(0, +\infty)$. Let $\rho < 0$ be a bounded strictly psh exhaustion function on Ω. Choose $\varepsilon > 0$ so small that $\{\psi \le 0\} \subset \{\rho < -\varepsilon\}$. Let $\chi \in C_0^\infty(\Omega)$ be a real-valued function satisfying $\chi = 1$ in a neighborhood of $\{\rho \le -\varepsilon\}$. We construct a $2-$parameter family of weight functions as follows:

$$\varphi_{t,s}(z) = |z|^2 + t\chi(z)\kappa(\psi(z)) + s\kappa(\rho(z) + \varepsilon), \quad t, s > 0.$$

It is easy to see that for any $t > 0$ there is a sufficiently large number $s = s(t) > 0$ such that $\partial\bar\partial\varphi_{t,s} \ge \partial\bar\partial|z|^2$. Let $\hat\chi \in C_0^\infty(\Omega)$ such that $\hat\chi = 1$ in a neighborhood of $\{\psi \le 0\}$ and $\hat\chi(z) = 0$ if $\rho(z) \ge -\varepsilon$. By Theorem 1.1, we may solve the equation

$$\bar\partial u_t = f\bar\partial\hat\chi$$

such that

$$\begin{aligned}
\int_\Omega |u_t|^2 e^{-\varphi_{t,s}} \delta_\Omega^{-\alpha} dV \le \ & \mathrm{const}_{\alpha,\Omega} \int_\Omega |f|^2 |\bar\partial\hat\chi|^2 e^{-\varphi_{t,s}} \delta_\Omega^{-\alpha} dV \\
\le \ & \mathrm{const}_{\alpha,\Omega} \int_{\mathrm{supp}\, \bar\partial\hat\chi} |f|^2 e^{-t\kappa\circ\psi} \delta_\Omega^{-\alpha} dV \to 0
\end{aligned}$$

as $t \to +\infty$. Since $\varphi_{t,s}(z) = |z|^2$ whenever $\psi(z) \le 0$, we conclude that

$$\int_{\{\psi \le 0\}} |u_t|^2 dV \to 0$$

as $t \to +\infty$, and so is the function $f_t - f$ where $f_t := \hat\chi f - u_t$. On the other hand, $f_t \in A_\alpha^2(\Omega)$ because $\varphi_{t,s}$ is a bounded function. Since $f_t - f$ is holomorphic on

$\{\psi < 0\}$, a standard compactness argument yields

$$\sup_K |f_t - f| \to 0$$

as $t \to +\infty$.

(b) We take a C^2 psh function $\rho < 0$ on Ω such that $-\rho \asymp \delta_\Omega^a$ for some $a > 0$. Let $0 \le \tilde{\chi} \le 1$ be a cut-off function on $\mathbb{R}$ such that $\tilde{\chi}|_{(-\infty,-\log 2)} = 1$ and $\tilde{\chi}|_{(0,\infty)} = 0$. Let $f \in A^2_{\alpha_1}(\Omega)$ be given. For each $\varepsilon > 0$, we define

$$v_\varepsilon = f\bar{\partial}\tilde{\chi}(-\log(-\rho+\varepsilon)+\log 2\varepsilon), \quad \varphi_\varepsilon = -\frac{\alpha_2-\alpha_1}{a}\log(-\rho+\varepsilon).$$

By Theorem 1.1, we have a solution of $\bar{\partial}u_\varepsilon = v_\varepsilon$ so that

$$\begin{aligned}\int_\Omega |u_\varepsilon|^2 e^{-\varphi_\varepsilon}\delta_\Omega^{-\alpha_2}dV &\le \text{const.}\int_\Omega |v_\varepsilon|^2_{i\partial\bar{\partial}\varphi_\varepsilon} e^{-\varphi_\varepsilon}\delta_\Omega^{-\alpha_2}dV\\ &\le \text{const.}\int_{\varepsilon\le-\rho\le 3\varepsilon}|f|^2\delta_\Omega^{-\alpha_1}dV\end{aligned}$$

for $i\partial\bar{\partial}\varphi_\varepsilon \ge \frac{\alpha_2-\alpha_1}{a} i\partial\log(-\rho+\varepsilon)\wedge\bar{\partial}\log(-\rho+\varepsilon)$. Put

$$f_\varepsilon = f\tilde{\chi}(-\log(-\rho+\varepsilon)+\log 2\varepsilon) - u_\varepsilon.$$

Since φ_ε is bounded and

$$e^{-\varphi_\varepsilon} \ge e^{\frac{\alpha_2-\alpha_1}{a}\log(-\rho)} \asymp \delta_\Omega^{\alpha_2-\alpha_1},$$

we conclude that $f_\varepsilon \in A^2_{\alpha_2}(\Omega)$ and

$$\begin{aligned}\int_\Omega |f_\varepsilon - f|^2\delta_\Omega^{-\alpha_1}dV &\le 2\int_{-\rho\le 3\varepsilon}|f|^2\delta_\Omega^{-\alpha_1}dV + 2\int_\Omega |u_\varepsilon|^2\delta_\Omega^{-\alpha_1}dV\\ &\le 2\int_{-\rho\le 3\varepsilon}|f|^2\delta_\Omega^{-\alpha_1}dV + \text{const.}\int_\Omega |u_\varepsilon|^2 e^{-\varphi_\varepsilon}\delta_\Omega^{-\alpha_2}dV\\ &\le 2\int_{-\rho\le 3\varepsilon}|f|^2\delta_\Omega^{-\alpha_1}dV + \text{const.}\int_{\varepsilon\le-\rho\le 3\varepsilon}|f|^2\delta_\Omega^{-\alpha_1}dV\\ &\to 0\end{aligned}$$

as $\varepsilon \to 0+$. □

Question 3.2. Is the Hardy space $H^2(\Omega)$ dense in $A^2_\alpha(\Omega)$ for each $\alpha < 1$?

Remark 3.3. The referee of this paper pointed out the following:

a) Bell and Boas have proved a theorem related to the above Density theorem (cf. [3], Theorem 1).

b) There is a standard argument as follows, which is perhaps more straightforward than the author's proof. Choose a cover $\{U_j\}_{j=1}^m$ of the boundary and vectors n_j such that $z - \varepsilon n_j \in \Omega$ for $1 \le j \le m$, $z \in U_j$, $\varepsilon \le \varepsilon_0$. Choose $\phi_0 \in C_0^\infty(\Omega)$ and $\phi_j \in C_0^\infty(U_j)$, $1 \le j \le m$, with $\sum \phi_j = 1$ in a neighborhood of $\overline{\Omega}$. Set

$$f_\varepsilon(z) = \phi_0(z)f(z) + \sum_{j=1}^m \phi_j(z)f(z-\varepsilon n_j).$$

Then $f_\varepsilon \to f$ in the norm with weight $\delta_\Omega^{-\alpha}$. The theorem now follows by correcting f_ε via

$$\bar{\partial}f_\varepsilon = f\bar{\partial}\phi_0 + \sum_{j=1}^m f(z-\varepsilon n_j)\bar{\partial}\phi_j = \sum_{j=1}^m [f(z-\varepsilon n_j) - f(z)]\bar{\partial}\phi_j$$

(because $\sum_{j=0}^m \bar\partial\phi_j = 0$ on Ω). The norm of the right-hand side tends to zero; so if we solve the $\bar\partial-$equation with the estimate that was shown, the corrections we make to the f_ε tend to zero as well in norm, and we are done.

4. Proof of Theorem 1.2

4.1. Let $\Omega \subset \mathbb{C}^n$ be a bounded domain. We define the pluricomplex Green function $g_\Omega(\cdot, w)$ with pole at $w \in \Omega$ as

$$g_\Omega(z,w) = \sup\left\{u(z) : u \in PSH(\Omega), u < 0, \limsup_{z\to w}(u(z) - \log|z-w|) < \infty\right\}.$$

It is well known that $g_\Omega(\cdot, w) \in PSH(\Omega)$ for each fixed w and $g_\Omega \in C(\overline{\Omega} \times \Omega\backslash\{z = w\})$ when Ω is hyperconvex (cf. [29]). We need the following estimate of g_Ω:

Theorem 4.1 (Blocki [7]). *Let $\Omega \Subset \mathbb{C}^n$ be a pseudoconvex domain. Suppose there is a negative psh function ρ on Ω satisfying*

$$C_1\delta_\Omega^a(z) \le -\rho(z) \le C_2\delta_\Omega^b(z), \quad z \in \Omega,$$

where $C_1, C_2 > 0$ and $a \ge b \ge 0$ are constants. Then there are positive numbers δ_0, C such that

$$\{g_\Omega(\cdot,w) \le -1\} \subset \{C^{-1}\delta_\Omega(w)^{\frac{a}{b}}|\log\delta_\Omega(w)|^{-\frac{1}{b}} \le \delta_\Omega \le C\delta_\Omega(w)^{\frac{b}{a}}|\log\delta_\Omega(w)|^{\frac{n}{a}}\}$$

holds for any $w \in \Omega$ with $\delta_\Omega(w) \le \delta_0$.

4.2. Let K_α be the Bergman kernel of $A^2_\alpha(\Omega)$.

Proposition 4.2. *Suppose* $\lim_{z\to\partial\Omega} K_\alpha(z)\eta(z) = \infty$ *where η is a positive continuous function on Ω. Then there exists a function $f \in A^2_\alpha(\Omega)$ such that*

$$\limsup_{z\to\zeta} |f(z)|\sqrt{\eta(z)} = \infty, \quad \forall\,\zeta \in \partial\Omega.$$

Proof. The argument is standard (see e.g. [27], pp. 416–417). We claim that the following assertion holds:

For each $\zeta \in \partial\Omega$ and each sequence of points in Ω with $z_j \to \zeta$, there exists a function $f \in A^2_\alpha(\Omega)$ such that $\sup_j |f(z_j)|\sqrt{\eta(z_j)} = \infty$.

Suppose there is a point $\zeta \in \partial\Omega$ and a sequence of points in Ω such that $z_j \to \zeta$ such that $\sup_j |f(z_j)|\sqrt{\eta(z_j)} < \infty$, $\forall\, f \in A^2_\alpha(\Omega)$. Applying the Banach-Steinhaus theorem to the linear functional $f \to f(z_j)\sqrt{\eta(z_j)}$, we get

$$\sup_j |f(z_j)|\sqrt{\eta(z_j)} \le \text{const.}\|f\|$$

for all $f \in A^2_\alpha(\Omega)$. Thus $K_\alpha(z_j)\sqrt{\eta(z_j)} \le$ const., contradictory.

Now we construct the desired function f. Pick a non-decreasing sequence of compact subsets $\{K_j\}$ of Ω such that $D = \bigcup K_j$. Fix a dense sequence $\{z_j\} \subset \Omega$. We reorder the points of the sequence as follows:

$$z_1, z_1, z_2, z_1, z_2, z_3, z_1, \cdots$$

and denote the new sequence by $\{w_j\}$. Put $B_j = B(w_j, \delta_\Omega(w_j))$, where $B(z,r)$ is the Euclidean ball with center z and radius r. By the above claim, we may construct inductively the sequences

$$\{j_\nu\} \subset \mathbb{Z}^+, \quad \{\zeta_\nu\} \subset \Omega, \quad \{\theta_\nu\} \subset \mathbb{R}, \quad \{f_\nu\} \subset A^2_\alpha(\Omega)$$

such that

$$\zeta_\nu \in (B_\nu \backslash K_{j_\nu}) \cap K_{j_{\nu+1}}, \quad \|f_\nu\| = 1, \quad \left|\sum_{\mu=1}^{\nu} \frac{f_\mu(\zeta_\nu)e^{i\theta_\nu}}{\mu^3(1+\|f_\mu\|_{K_{j_\mu}})}\right| \geq \frac{\nu}{\sqrt{\eta(\zeta_\nu)}},$$

where $\|f_\mu\|_{K_{j_\mu}} = \sup_{K_{j_\mu}} |f_\mu|$. It suffices to take $f(z) = \sum_{\nu=1}^\infty \frac{f_\nu(z)e^{i\theta_\nu}}{\nu^3(1+\|f_\nu\|_{K_{j_\nu}})}$. □

4.3. Now we prove Theorem 1.2. The argument is essentially the same as [12]. First fix an arbitrary point w sufficiently close to $\partial\Omega$. Put $g_j = \max\{g_\Omega(\cdot, w), -j\}$, $j = 1, 2, \cdots$. Since Ω is hyperconvex, g_j is continuous on Ω and $g_j \downarrow g_\Omega(\cdot, w)$ as $j \to \infty$. By Richberg's theorem (cf. [39]), there is a C^∞ strictly psh function $\psi_j < 0$ on Ω such that $|\psi_j(z) - g_j(z)| < 1/j$, $z \in \Omega$. Put

$$\varphi = 2ng_\Omega(\cdot, w) - \log(-g_\Omega(\cdot, w) + 1), \quad \varphi_j = 2n\psi_j - \log(-\psi_j + 1).$$

Let $\chi : \mathbb{R} \to [0, 1]$ be a C^∞ cut-off function satisfying $\chi|_{(-\infty,-1)} = 1$ and $\chi|_{(-\log 2,\infty)} = 0$. Put

$$v_j = \bar\partial\chi(-\log(-\psi_j))\frac{K_\Omega(\cdot, w)}{\sqrt{K_\Omega(w)}},$$

where K_Ω denotes the unweighted Bergman kernel of Ω. By Theorem 1.1, there is a solution of the equation $\bar\partial u_j = v_j$ such that

$$\begin{aligned}\int_\Omega |u_j|^2 e^{-\varphi_j}\delta_\Omega^{-\alpha} dV &\leq \text{const}_{\alpha,\Omega} \int_\Omega |v_j|^2_{i\partial\bar\partial\varphi_j} e^{-\varphi_j}\delta_\Omega^{-\alpha} dV \\ &\leq \text{const}_{\alpha,\Omega} \int_{\text{supp}\,\bar\partial\chi(\cdot)} \frac{|K_\Omega(\cdot, w)|^2}{K_\Omega(w)}\delta_\Omega^{-\alpha} dV,\end{aligned}$$

where the second inequality follows from

$$i\partial\bar\partial\varphi_j \geq \frac{i\partial\psi_j \wedge \bar\partial\psi_j}{(-\psi_j + 1)^2}.$$

By Blocki's theorem, we have

$$\begin{aligned}\text{supp}\,\bar\partial\chi(\cdot) \subset\ &\{\psi_j \leq -2\} \subset \{g_\Omega(\cdot, w) \leq -1\} \\ \subset\ &\{C^{-1}\delta_\Omega(w)|\log\delta_\Omega(w)|^{-\frac{1}{a}} \leq \delta_\Omega\}, \quad j \gg 1,\end{aligned}$$

where a is a Diederich-Fornaess exponent for Ω. Thus

$$\int_\Omega |u_j|^2 e^{-\varphi_j}\delta_\Omega^{-\alpha} dV \leq \text{const}_{\alpha,\Omega} \frac{|\log\delta_\Omega(w)|^{\frac{\alpha}{a}}}{\delta_\Omega(w)^\alpha}.$$

Let u be a weak limit of a subsequence of $\{u_j\}$. Thus

$$f := \chi(-\log(-g_\Omega(\cdot, w)))K_\Omega(\cdot, w)/\sqrt{K_\Omega(w)} - u$$

is holomorphic on Ω. Since u is holomorphic in a neighborhood of w and

$$\int_\Omega |u|^2 e^{-\varphi}\delta_\Omega^{-\alpha} dV \leq \text{const}_{\alpha,\Omega} \frac{|\log\delta_\Omega(w)|^{\frac{\alpha}{a}}}{\delta_\Omega(w)^\alpha},$$

we conclude that $u(w) = 0$. Thus $f(w) = \sqrt{K_\Omega(w)}$ and

$$\int_\Omega |f|^2\delta_\Omega^{-\alpha} dV \leq \text{const}_{\alpha,\Omega} \frac{|\log\delta_\Omega(w)|^{\frac{\alpha}{a}}}{\delta_\Omega(w)^\alpha}.$$

Thus

$$K_\alpha(w) \geq \frac{|f(w)|^2}{\int_\Omega |f|^2 \delta_\Omega^{-\alpha} dV} \geq \text{const}_{\alpha,\Omega}\, K_\Omega(w) \frac{\delta_\Omega(w)^\alpha}{|\log \delta_\Omega(w)|^{\frac{\alpha}{a}}} \geq \frac{\text{const}_{\alpha,\Omega}}{\delta_\Omega(w)^{2-\alpha} |\log \delta_\Omega(w)|^{\frac{\alpha}{a}}}$$

as $w \to \partial\Omega$, where the last inequality follows from the Ohsawa-Takegoshi extension theorem (cf. [36]). Applying Proposition 4.2 with $\eta(z) = \delta_\Omega(z)^{2-\alpha} |\log \delta_\Omega(z)|^{\frac{2\alpha}{a}}$, we conclude the proof. ☐

5. Proof of Theorem 1.3

We closely follow Stein's book [41]. For each $\zeta \in \partial\Omega$, let ν_ζ denote the unit outward normal at ζ and T_ζ the tangent plane at ζ. For each $t > 0$, we define an approach region $\mathcal{A}_t(\zeta)$ with vertex ζ by

$$\mathcal{A}_t(\zeta) = \left\{ z \in \Omega : |(z-\zeta)\cdot \bar{\nu}_\zeta| < (1+t)\delta_\zeta(z),\ |z-\zeta|^2 < t\delta_\zeta(z) \right\},$$

where $\delta_\zeta(z) = \min\{\delta_\Omega(z), d(z, T_\zeta)\}$. We shall say that $|f(z)| = o(\delta_\Omega(z)^{-\beta})$ uniformly as $z \to \zeta$ admissibly for some $\beta \geq 0$ if for each $t > 0$,

$$\limsup \delta_\Omega(z)^\beta |f(z)| = 0,$$

as $z \to \zeta$ from the inside of $\mathcal{A}_t(\zeta)$. For each $\zeta_0 \in \partial\Omega$ and $r > 0$, we put

$$\begin{aligned} B_1(\zeta_0, r) &= \{\zeta \in \partial\Omega : |\zeta - \zeta_0| < r\}, \\ B_2(\zeta_0, r) &= \left\{\zeta \in \partial\Omega : |(\zeta-\zeta_0)\cdot \bar{\nu}_{\zeta_0}| < r,\ |\zeta-\zeta_0|^2 < r\right\} \end{aligned}$$

and

$$f_j^*(\zeta_0) = \sup_{r>0} \frac{1}{\sigma(B_j(\zeta_0, r))} \int_{B_j(\zeta_0,r)} |f(\zeta)| d\sigma(\zeta), \quad j = 1, 2,$$

where $f \in L^p(\partial\Omega)$ and $d\sigma$ is the surface measure for $\partial\Omega$. The maximal function is defined by

$$(Mf)(\zeta) = (f_1^*)_2^*(\zeta).$$

Theorem 5.1 (cf. [41]; see also [25]). 1) $\|Mf\|_p \leq \text{const}_p \|f\|_p$, $\forall f \in L^p(\partial\Omega)$, $1 < p \leq \infty$.

2) *Let u be a psh function on Ω which is continuous on $\overline{\Omega}$ and let $f = u|_{\partial\Omega}$. Then*

$$\sup_{z \in \mathcal{A}_t(\zeta)} |u(z)| \leq \text{const}_p\, (Mf)(\zeta).$$

Now choose a cover of Ω by finitely many subdomains $\Omega_0, \Omega_1, \cdots, \Omega_m \subset \Omega$ with the following properties:

(a) $\partial\Omega_j$ is C^2.

(b) $\partial\Omega_j - (\partial\Omega_j \cap \partial\Omega) \subset \Omega$.

(c) There exists a domain $W_j \subset \partial\Omega_j \cap \partial\Omega$ such that $\{W_j\}_{j=0}^m$ forms a cover of $\partial\Omega$.

(d) There exists an outward unit normal ν_j at a point in $\partial\Omega_j \cap \partial\Omega$ such that

$$\overline{\Omega}_j - \varepsilon v_j \subset \Omega, \qquad \forall\, 0 \leq \varepsilon \ll 1.$$

It suffices to work on a single subdomain, say Ω_0. Let ε_0 be a sufficiently small number. In order to apply Gehring's method (cf. [20]), we define for each $t > 0$,

$0 < \varepsilon < \varepsilon_0/2$, $\zeta \in W_0$,

$$\begin{aligned} U_\varepsilon^{(t)}(\zeta) &= \{z \in \mathcal{A}_t(\zeta) : 2\varepsilon < \delta_\zeta(z) < \varepsilon_0\}, \\ V_\varepsilon^{(t)}(\zeta) &= \left\{z \in \mathcal{A}_t(\zeta) - \varepsilon\nu_0 : \delta_\zeta(z) < \frac{3}{2}\varepsilon_0\right\}. \end{aligned}$$

Lemma 5.2. *For each $t > 0$, we may choose $\varepsilon_0 > 0$ so that*

$$U_\varepsilon^{(t)}(\zeta) \subset V_\varepsilon^{(s)}(\zeta) \subset \Omega_0, \quad s := 2 + 4t,$$

for all $\varepsilon < \varepsilon_0/2$ and $\zeta \in W_0$.

Proof. For each $z \in U_\varepsilon^{(t)}(\zeta)$, we have $\delta_\zeta(z) > 2\varepsilon$. Thus

$$\begin{aligned} \delta_\zeta(z + \varepsilon\nu_0) &\geq \delta_\zeta(z) - \varepsilon > \varepsilon, \\ \delta_\zeta(z + \varepsilon\nu_0) &\leq \delta_\zeta(z) + \varepsilon < \frac{3}{2}\varepsilon_0 \end{aligned}$$

for all $\varepsilon < \varepsilon_0/2$. Since

$$|(z - \zeta) \cdot \bar{\nu}_\zeta| < (1 + t)\delta_\zeta(z), \quad |z - \zeta| < (t\delta_\zeta(z))^{1/2},$$

we get

$$\begin{aligned} |(z + \varepsilon\nu_0 - \zeta) \cdot \bar{\nu}_\zeta| &\leq |(z - \zeta) \cdot \bar{\nu}_\zeta| + \varepsilon < (1 + t)\delta_\zeta(z) + \varepsilon \leq (3 + 2t)\delta_\zeta(z + \varepsilon\nu_0) \\ |z + \varepsilon\nu_0 - \zeta|^2 &\leq 2|z - \zeta|^2 + 2\varepsilon^2 < 2t\delta_\zeta(z) + 2\varepsilon \leq (2 + 4t)\delta_\zeta(z + \varepsilon\nu_0). \end{aligned}$$

Thus $z+\varepsilon\nu_0 \in V_\varepsilon^{(s)}(\zeta)$ where $s = 2+4t$, and we get the first inclusion in the lemma.

On the other hand, for each $z \in V_\varepsilon^{(s)}(\zeta)$, we have $|z-\zeta|^2 < s\delta_\zeta(z) \leq \frac{3}{2}s\varepsilon_0$; hence $V_\varepsilon^{(s)}(\zeta) \subset \Omega_0$ for all $\varepsilon < \varepsilon_0/2$, provided ε_0 small enough. □

For each $f \in A_\alpha^2(\Omega)$, we define

$$u_\varepsilon^{(t)}(\zeta) = \sup_{z \in U_\varepsilon^{(t)}(\zeta)} |f(z)| \quad \text{and} \quad v_\varepsilon^{(s)}(\zeta) = \sup_{z \in V_\varepsilon^{(s)}(\zeta)} |f(z)|.$$

Put $f_\varepsilon(z) = f(z - \varepsilon\nu_0)$, $z \in \Omega_0$. Clearly, $|f_\varepsilon|$ is psh in Ω_0 and continuous on $\overline{\Omega}_0$. Let $M_0 f_\varepsilon$ be the corresponding maximal function on $\partial\Omega_0$. Take $0 < c < 1$ so that

$$\Omega_0 - \varepsilon\nu_0 =: \Omega_0^\varepsilon \subset \Omega_{c\varepsilon} := \{z \in \Omega : \delta_\Omega(z) > c\varepsilon\}.$$

Let $d\sigma_0$ and $d\sigma_{c\varepsilon}$ denote the surface measures on $\partial\Omega_0$ and $\partial\Omega_{c\varepsilon}$ respectively, and let C denote a generic constant which is independent of ε but probably depends on α, t, s. By Theorem 5.1 and Lemma 5.2, we have

$$u_\varepsilon^{(t)}(\zeta) \leq v_\varepsilon^{(s)}(\zeta) \leq C(M_0 f_\varepsilon)(\zeta), \quad \forall\, \zeta \in W_0,$$

so that

$$\begin{aligned} \int_{W_0} |u_\varepsilon^{(t)}(\zeta)|^2 d\sigma_0(\zeta) &\leq C \int_{\partial\Omega_0} |M_0 f_\varepsilon|^2 d\sigma_0 \leq C \int_{\partial\Omega_0} |f_\varepsilon|^2 d\sigma_0 \\ &= C \int_{\partial\Omega_0^\varepsilon} |f|^2 d\sigma_0 \leq C \int_{\partial\Omega_{c\varepsilon}} |f|^2 d\sigma_{c\varepsilon} \end{aligned}$$

because of the following.

Lemma 5.3. *There is a constant $C > 0$ independent of ε and f such that*

$$\int_{\partial\Omega_0^\varepsilon} |f|^2 d\sigma_0 \leq C \int_{\partial\Omega_{c\varepsilon}} |f|^2 d\sigma_{c\varepsilon}$$

for all sufficiently small $\varepsilon > 0$.

Thus for suitable small number $c_0 > 0$ we have

$$\int_0^{c_0} \varepsilon^{-\alpha} \int_{W_0} |u_\varepsilon^{(t)}(\zeta)|^2 d\sigma_0(\zeta) d\varepsilon \leq C \int_0^{c_0} \int_{\partial\Omega_{c\varepsilon}} |f|^2 \varepsilon^{-\alpha} d\sigma_{c\varepsilon} d\varepsilon$$
$$\leq C \int_\Omega |f|^2 \delta_\Omega^{-\alpha} dV < \infty,$$

so that for σ_0-almost every $\zeta \in W_0$,

$$\int_0^{c_0} \varepsilon^{-\alpha} |u_\varepsilon^{(t)}(\zeta)|^2 d\varepsilon < \infty.$$

Hence

$$\int_0^{\varepsilon'} \varepsilon^{-\alpha} |u_\varepsilon^{(t)}(\zeta)|^2 d\varepsilon = \mathrm{o}(1)$$

as $\varepsilon' \to 0$. Given $z \in \mathcal{A}_t(\zeta)$, we let $\varepsilon' = \delta_\zeta(z)/2$. Since $z \in U_\varepsilon^{(t)}(\zeta)$ for each $\varepsilon < \varepsilon'$, we have $u_\varepsilon^{(t)}(\zeta) \geq |f(z)|$; thus

$$|f(z)| = \mathrm{o}(\delta_\zeta(z)^{-\frac{1-\alpha}{2}}) \qquad \text{uniformly}$$

as $z \to \zeta$ from the inside of $\mathcal{A}_t(\zeta)$. □

Proof of Lemma 5.3. The idea is essentially implicit in [12]. Let $P(z,w)$, $P_\varepsilon(z,w)$, $P_0(z,w)$ and $P_{0,\varepsilon}(z,w)$ denote the Poisson kernels of Ω, $\Omega_{c\varepsilon}$, Ω_0 and Ω_0^ε respectively. Put

$$g(z) = \int_{\partial\Omega_{c\varepsilon}} P_\varepsilon(z,w)|f(w)|^2 d\sigma_\varepsilon(w).$$

Then g is a harmonic majorant of $|f|^2$ on $\Omega_{c\varepsilon}$. Fix a point z_0 in Ω_0. Since $P_\varepsilon(z_0, \pi_\varepsilon^{-1}(\zeta))$ converges uniformly on $\partial\Omega$ to $P(z_0,\zeta)$ where π_ε is the normal projection from $\partial\Omega_{c\varepsilon}$ to $\partial\Omega$,

$$g(z_0) \leq 2C_1 \int_{\partial\Omega_{c\varepsilon}} |f(w)|^2 d\sigma_\varepsilon(w)$$

for all sufficiently small $\varepsilon > 0$ where $C_1 = \sup_{\zeta\in\partial\Omega} P(z_0,\zeta)$. On the other hand,

$$\begin{aligned} g(z_0) &= \int_{\partial\Omega_0^\varepsilon} P_{0,\varepsilon}(z_0,w) g(w) d\sigma_0 \\ &\geq \frac{C_2}{2} \int_{\partial\Omega_0^\varepsilon} g(w) d\sigma_0 \geq \frac{C_2}{2} \int_{\partial\Omega_0^\varepsilon} |f(w)|^2 d\sigma_0 \end{aligned}$$

for all sufficiently small $\varepsilon > 0$ where $C_2 = \inf_{\zeta\in\partial\Omega_0} P_0(z_0,\zeta)$. The proof is complete. □

Remark 5.4. In various studies of boundary behavior of functions in Hardy spaces, the approach region defined as above is only best possible for strongly pseudoconvex domains (see e.g., [31,35]). It is probably the same in the case of weighted Bergman spaces.

6. Proof of Theorem 1.5

Let $\|\cdot\|_\alpha$ and $\|\cdot\|_{\partial\Omega}$ denote the corresponding norms of the weighted Bergman space $A^2_\alpha(\Omega)$ and the Hardy space $H^2(\Omega)$ respectively. Note first that for each $f \in H^2(\Omega)$, and any sufficiently small $\varepsilon_0 > 0$,

$$\begin{aligned}(1-\alpha)\int_\Omega |f|^2\delta_\Omega^{-\alpha}dV &= (1-\alpha)\int_{\Omega_{\varepsilon_0}} |f|^2\delta_\Omega^{-\alpha}dV + (1-\alpha)\int_{\Omega\backslash\Omega_{\varepsilon_0}} |f|^2\delta_\Omega^{-\alpha}dV \\ &\leq (1-\alpha)\int_{\Omega_{\varepsilon_0}} |f|^2\delta_\Omega^{-\alpha}dV + \varepsilon_0^{1-\alpha}\sup_{0<\varepsilon<\varepsilon_0}\|f\|^2_{\partial\Omega_\varepsilon}.\end{aligned}$$

Applying this inequality with $f(z) = S(z,w)$ for fixed $w \in \Omega$, we get

$$\liminf_{\alpha\to 1-}(1-\alpha)^{-1}K_\alpha(w) \geq \liminf_{\alpha\to 1-}(1-\alpha)^{-1}\frac{|f(w)|^2}{\|f\|^2_\alpha} = \frac{S(w)^2}{\sup_{0<\varepsilon<\varepsilon_0}\|S(\cdot,w)\|^2_{\partial\Omega_\varepsilon}}$$

locally uniformly in w and uniformly in ε_0. Let S_ε denote the Szegő kernel of Ω_ε. It was proved by Boas [8] that $S_\varepsilon(z,w) \to S(z,w)$ locally uniformly in z, w and

$$\|S_\varepsilon(\cdot,w) - S(\cdot,w)\|_{\partial\Omega_\varepsilon} \to 0$$

locally uniformly in w as $\varepsilon \to 0+$. Thus

$$\begin{aligned}\liminf_{\alpha\to 1-}(1-\alpha)^{-1}K_\alpha(w) &\geq \lim_{\varepsilon_0\to 0+}\frac{S(w)^2}{\sup_{0<\varepsilon<\varepsilon_0}\|S_\varepsilon(\cdot,w)\|^2_{\partial\Omega_\varepsilon}} \\ &= \lim_{\varepsilon_0\to 0+}\frac{S(w)^2}{\sup_{0<\varepsilon<\varepsilon_0}S_\varepsilon(w)} = S(w)\end{aligned}$$

locally uniformly in w. On the other side, for any sufficiently small $\varepsilon > 0$,

$$\begin{aligned}&\int_{\partial\Omega_\varepsilon}\left|(1-\alpha)^{-1}K_\alpha(z,w) - S_\varepsilon(z,w)\right|^2 d\sigma_\varepsilon(z) \\ = \ & (1-\alpha)^{-2}\|K_\alpha(\cdot,w)\|^2_{\partial\Omega_\varepsilon} + \|S_\varepsilon(\cdot,w)\|^2_{\partial\Omega_\varepsilon} \\ & -2(1-\alpha)^{-1}\mathrm{Re}\int_{\partial\Omega_\varepsilon}K_\alpha(z,w)\overline{S_\varepsilon(z,w)}d\sigma_\varepsilon(z) \\ = \ & (1-\alpha)^{-2}\|K_\alpha(\cdot,w)\|^2_{\partial\Omega_\varepsilon} + S_\varepsilon(w) - 2(1-\alpha)^{-1}K_\alpha(w).\end{aligned}$$

Put $f_\alpha(z) := (1-\alpha)^{-1/2}K_\alpha(z,w)/\sqrt{K_\alpha(w)}$. Following [12], we introduce

$$\lambda_\alpha(\varepsilon) := \|f_\alpha\|^2_{\partial\Omega_\varepsilon} = \int_{\partial\Omega_\varepsilon}|f_\alpha|^2 d\sigma_\varepsilon.$$

Clearly, λ_α is continuous on $(0,a]$ for some sufficiently small $a > 0$ (independent of α). For any sufficiently small $0 < \varepsilon_1 < \varepsilon_2 < a$, λ_α assumes the minimum at some point $\varepsilon^* = \varepsilon^*(\varepsilon_1,\varepsilon_2,\alpha)$ in $[\varepsilon_1,\varepsilon_2]$. Thus

$$1 = (1-\alpha)\|f_\alpha\|^2_\alpha \geq (1-\alpha)\int_{\varepsilon_1\leq\delta_\Omega\leq\varepsilon_2}|f_\alpha|^2\delta_\Omega^{-\alpha}dV \geq \left(\varepsilon_2^{1-\alpha} - \varepsilon_1^{1-\alpha}\right)\lambda_\alpha(\varepsilon^*),$$

so that

$$\|K_\alpha(\cdot,w)\|^2_{\partial\Omega_{\varepsilon^*}} \leq (1-\alpha)\left(\varepsilon_2^{1-\alpha} - \varepsilon_1^{1-\alpha}\right)^{-1}K_\alpha(w).$$

Thus

$$\begin{aligned}
&\int_{\partial\Omega_{\varepsilon^*}} \left|(1-\alpha)^{-1}K_\alpha(z,w) - S_{\varepsilon^*}(z,w)\right|^2 d\sigma_{\varepsilon^*}(z) \\
\leq\ & S_{\varepsilon^*}(w) - (1-\alpha)^{-1}\left(2 - \left(\varepsilon_2^{1-\alpha} - \varepsilon_1^{1-\alpha}\right)^{-1}\right) K_\alpha(w) \\
=\ & \left(2 - \left(\varepsilon_2^{1-\alpha} - \varepsilon_1^{1-\alpha}\right)^{-1}\right)\left(S(w) - (1-\alpha)^{-1}K_\alpha(w)\right) \\
& + \left(\left(\varepsilon_2^{1-\alpha} - \varepsilon_1^{1-\alpha}\right)^{-1} - 1\right) S(w) + S_{\varepsilon^*}(w) - S(w).
\end{aligned}$$

It follow that

$$\limsup_{\varepsilon_2\to 0+}\ \limsup_{\alpha\to 1-}\ \limsup_{\varepsilon_1\to 0+} \int_{\partial\Omega_{\varepsilon^*}} \left|(1-\alpha)^{-1}K_\alpha(z,w) - S_{\varepsilon^*}(z,w)\right|^2 d\sigma_{\varepsilon^*}(z) = 0$$

locally uniformly in w. Let $P_\varepsilon(z,\zeta)$ denote the Poisson kernel of Ω_ε. For each compact set M in Ω and $z,w\in M$, we have

$$\begin{aligned}
& \left|(1-\alpha)^{-1}K_\alpha(z,w) - S_{\varepsilon^*}(z,w)\right|^2 \\
\leq\ & \int_{\partial\Omega_{\varepsilon^*}} P_{\varepsilon^*}(z,\zeta)\left|(1-\alpha)^{-1}K_\alpha(\zeta,w) - S_{\varepsilon^*}(\zeta,w)\right|^2 d\sigma_{\varepsilon^*}(\zeta) \\
\leq\ & \mathrm{const}_M \int_{\partial\Omega_{\varepsilon^*}} \left|(1-\alpha)^{-1}K_\alpha(\zeta,w) - S_{\varepsilon^*}(\zeta,w)\right|^2 d\sigma_{\varepsilon^*}(\zeta)
\end{aligned}$$

provided ε^* is sufficiently small. Thus $(1-\alpha)^{-1}K_\alpha(z,w)\to S(z,w)$ uniformly in $z,w\in M$ as $\alpha\to 1-$. The second assertion follows immediately from this fact and Theorem 1.6. □

Question 6.1. Does $(1-\alpha)^{-1}K_\alpha(z,w)$ admit an asymptotic expansion in powers of $1-\alpha$ as $\alpha\to 1-$?

7. Proof of Theorem 1.4

Let $ds^2_{\mathbb{B}^n} = \partial\bar\partial(-\log(1-|z|^2))$ be the Bergman metric of $\mathbb{B}^n$ and $d(z,w)$ the Bergman distance between two points z,w. Here we omit the factor $n+1$ in the classical definition of the Bergman metric for the sake of convenience. For each $w\in\mathbb{B}^n$, $\tau>0$ and $0<r<1$, we put

$$B_\tau(w) = \{z\in\mathbb{B}^n : d(z,w)<\tau\}, \qquad \mathbb{B}_r(w) = \{z\in\mathbb{B}^n : |z-w|<r\}.$$

Note that

$$B_\tau(0) = \mathbb{B}_r(0) \iff \tau = \frac{1}{2}\log\frac{1+r}{1-r}.$$

Let vol_B and vol_E denote the Bergman and Euclidean volumes respectively.

Proposition 7.1. (i) *For each $\tau>0$, there is a constant $C_\tau>1$ such that for each $w\in\mathbb{B}^n$,*

$$B_\tau(w) \subset \left\{z\in\mathbb{B}^n : C_\tau^{-1}(1-|w|) < 1-|z| < C_\tau(1-|w|)\right\},$$

$$C_\tau^{-1}(1-|w|)^{n+1} \leq \mathrm{vol}_E\left(B_\tau(w)\right) \leq C_\tau(1-|w|)^{n+1}.$$

(ii) *For each $r<1$,*

$$\mathrm{vol}_B\left(\mathbb{B}_r(0)\right) \leq \mathrm{const}_n(1-r)^{-n}.$$

(iii) For each $\tau > 0$, there is a constant $t > 1$ such that for each $\zeta \in \mathbb{S}^n$ and each $w \in L_\zeta$, where L_ζ is the segment determined by $0, \zeta$, we have

$$B_\tau(w) \subset \mathcal{A}_t(\zeta).$$

Proof. (i) See [43], Lemma 2.20 and Lemma 1.23.

(ii) The Bergman volume form is

$$\text{const}_n(1 - |z|^2)^{-n-1}dV.$$

Thus

$$\text{vol}_B\left(\mathbb{B}_r(0)\right) = \text{const}_n \int_0^r (1 - s^2)^{-n-1} s^{2n-1} ds,$$

from which the assertion immediately follows.

(iii) By [43], Lemma 2.20, there is a constant $C_\tau > 0$ such that

$$|1 - z \cdot \bar{w}| < C_\tau(1 - |w|), \qquad \forall\, z \in B_\tau(w).$$

Thus

$$|1 - z \cdot \bar{\zeta}| \leq |1 - z \cdot \bar{w}| + \left| z \cdot \overline{(w - \zeta)} \right| \leq (C_\tau + 1)(1 - |w|) \leq t(1 - |z|)$$

for suitable $t \gg 1$ by (i). □

Definition 7.2 (see e.g., [28])**.** A subset $\Gamma = \{w_j\}_{j=1}^\infty$ of $\mathbb{B}^n$ is said to be τ*-separated* for $\tau > 0$ if $d(w_j, w_k) \geq \tau$ for all $j \neq k$, and a $\tau-$separated subset is called maximal if no more points can be added to Γ without breaking the condition.

A basic observation is the following.

Lemma 7.3. *Let $\Gamma = \{w_j\}_{j=1}^\infty$ be a $\tau-$separated sequence such that $0 \notin \Gamma$. For any $\varepsilon > 0$,*

$$\sum_{j=1}^\infty \frac{(1 - |w_j|)^n}{\left(\log \frac{1}{1-|w_j|}\right)^{1+\varepsilon}} < \infty.$$

Proof. The argument is standard (compare [42], Theorem XI. 7 and Theorem XI. 8). For each $0 < r < 1$, let n_r denote the number of points w_j which are contained in the ball $\mathbb{B}_r(0) = B_{\frac{1}{2}\log\frac{1+r}{1-r}}(0)$. Since $\{B_{\tau/2}(w_j)\}_{j=1}^\infty$ do not overlap, we have

$$\begin{aligned} n_r \text{vol}_B\left(B_{\tau/2}(0)\right) \leq\ & \text{vol}_B\left(B_{\frac{1}{2}\log\frac{1+r}{1-r}+\frac{\tau}{2}}(0)\right) = \text{vol}_B\left(\mathbb{B}_{\frac{e^\tau(1+r)-(1-r)}{e^\tau(1+r)+(1-r)}}(0)\right) \\ \leq\ & \text{const}_{n,\tau}(1 - r)^{-n} \end{aligned}$$

by Proposition 7.1(ii). Take $r_0 > 0$ such that $|w_j| \geq r_0$ for each j. Thus

$$\begin{aligned} \sum_{|w_j|<r<1} \frac{(1 - |w_j|)^n}{\left(\log \frac{1}{1-|w_j|}\right)^{1+\varepsilon}} &= \int_{r_0}^r \frac{(1 - s)^n}{\left(\log \frac{1}{1-s}\right)^{1+\varepsilon}} dn_s \\ &\leq \frac{(1 - r)^n}{\left(\log \frac{1}{1-r}\right)^{1+\varepsilon}} n_r + \int_{r_0}^r \left(\frac{(1 - s)^n}{\left(\log \frac{1}{1-s}\right)^{1+\varepsilon}} \right)' n_s ds \\ &\leq \frac{\text{const}_{n,\tau}}{\left(\log \frac{1}{1-r}\right)^{1+\varepsilon}} + \text{const}_{n,\tau,\varepsilon} \int_{r_0}^r \frac{1}{(1 - s)\left(\log \frac{1}{1-s}\right)^{1+\varepsilon}} ds = \mathrm{O}(1) \end{aligned}$$

as $r \to 1-$. □

Lemma 7.4. *There is a constant $C_n > 0$ such that for each $\alpha < 1$, $\varepsilon > 0$ and each $2\tau-$separated sequence $\Gamma = \{w_j\}_{j=1}^\infty$ with $0 \notin \Gamma$ and $\tau \geq \frac{C_n}{\sqrt{1-\alpha}}$, there exists a function $f \in A^2_\alpha(\mathbb{B}^n)$ such that*

$$f(w_j) = (1-|w_j|)^{-\frac{1-\alpha}{2}} \left(\log \frac{1}{1-|w_j|}\right)^{-\frac{1+\varepsilon}{2}}, \qquad \forall\, j.$$

Proof. Take a C^∞ cut-off function $\chi : \mathbb{R} \to [0,1]$ such that $\chi|_{(-\infty,1/4)} = 1$, $\chi|_{(1/2,+\infty)} = 0$ and $\chi' \leq 0$. Put $d_j(z) = d(z, w_j)$ and

$$\begin{aligned} \psi(z) &= \sum_j \chi(d_j(z)/\tau) \log d_j(z)/\tau, \\ \varphi(z) &= -\frac{1-\alpha}{2} \log(1-|z|^2) + 2n\psi(z). \end{aligned}$$

A straightforward calculation shows

$$\begin{aligned} \partial\bar\partial\psi &= \sum_j \chi''(\cdot) \frac{\partial d_j \wedge \bar\partial d_j}{\tau^2} \log d_j/\tau + 2\chi'(\cdot) \frac{\partial d_j \wedge \bar\partial d_j}{\tau d_j} \\ &\quad + \chi'(\cdot) \frac{\partial\bar\partial d_j}{\tau} \log d_j/\tau + \chi(\cdot) \partial\bar\partial \log d_j. \end{aligned} \tag{8}$$

Since $ds^2_{\mathbb{B}^n}$ has negative Riemannian sectional curvature, it follows from [21] that $\log d_j$ is psh (so is d_j) on $\mathbb{B}^n$. Neglecting the last two semipositive terms in (8), we get

$$\partial\bar\partial\psi \geq -\frac{C_n^2}{8n\tau^2} ds^2_{\mathbb{B}^n}$$

for a suitable constant $C_n > 0$. If $\tau \geq C_n/\sqrt{1-\alpha}$, then

$$\partial\bar\partial\varphi \geq \frac{1-\alpha}{4} ds^2_{\mathbb{B}^n}.$$

By Theorem 1.1, we may solve the equation

$$\bar\partial u = \sum_j (1-|w_j|)^{-\frac{1-\alpha}{2}} \left(\log \frac{1}{1-|w_j|}\right)^{-\frac{1+\varepsilon}{2}} \bar\partial\chi(d_j/\tau) =: v$$

such that

$$\begin{aligned} \int_{\mathbb{B}^n} |u|^2 e^{-\varphi} (1-|z|)^{-\frac{1+\alpha}{2}} dV &\leq \text{const}_{n,\alpha} \int_{\mathbb{B}^n} |v|^2_{i\partial\bar\partial\varphi} e^{-\varphi} (1-|z|)^{-\frac{1+\alpha}{2}} dV \\ \leq\ \text{const}_{n,\alpha,\tau} \sum_j (1-|w_j|)^{-1+\alpha} &\left(\log \frac{1}{1-|w_j|}\right)^{-1-\varepsilon} \int_{B_\tau(w_j)} (1-|z|)^{-\alpha} dV \\ \leq\ \text{const}_{n,\alpha,\tau} \sum_{j=1}^\infty &\frac{(1-|w_j|)^n}{\left(\log \frac{1}{1-|w_j|}\right)^{1+\varepsilon}} < \infty, \end{aligned}$$

where the last inequality follows from Proposition 7.1(i). To get the desired function, we only need to take

$$f := \sum_j \chi(d_j/\tau)(1-|w_j|)^{-\frac{1-\alpha}{2}} \left(\log \frac{1}{1-|w_j|}\right)^{-\frac{1+\varepsilon}{2}} - u.$$

□

Proof of Theorem 1.4. Take $\tau = C_n/\sqrt{1-\alpha}$ as in Lemma 7.4. Pick a maximal 2τ−separated sequence $\Gamma = \{w_j\}_{j=1}^{\infty}$ with $0 \notin \Gamma$. It is easy to see that the geodesic balls $B_\tau(w_j)$ are disjoint and $\{B_{3\tau}(w_j)\}_{j=1}^{\infty}$ forms a cover of $\mathbb{B}^n$. In particular,

$$B_{4\tau}(w) \cap \Gamma \neq \emptyset, \qquad \forall\, w \in \mathbb{B}^n.$$

By Proposition 7.1(*iii*) and the completeness of $ds^2_{\mathbb{B}^n}$, we conclude that there is a constant $t > 1$ such that for each $\zeta \in \mathbb{S}^n$, the set $\mathcal{A}_t(\zeta)$ contains a sequence of disjoint geodesic balls of radius 4τ whose centers approach ζ. Consequently, this set contains a subsequence of Γ. On the other hand, there is a function $f \in A^2_\alpha(\mathbb{B}^n)$ such that

$$f(w_j) = (1-|w_j|)^{-\frac{1-\alpha}{2}} \left(\log \frac{1}{1-|w_j|}\right)^{-\frac{1+\varepsilon}{2}}, \quad \forall\, j$$

by virtue of Lemma 7.4. Thus the proof is complete. □

8. Proof of Theorems 1.6 and 1.7

Let $dz = dz_1 \wedge \cdots \wedge dz_n$ and $\widehat{d\bar{z}_j} = d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_{j-1} \wedge d\bar{z}_{j+1} \wedge \cdots \wedge d\bar{z}_n$. The Bochner-Martinelli kernel is defined to be

$$K_{BM}(\zeta - z) = \frac{(n-1)!}{(2\pi i)^n} \sum_{j=1}^{n} \frac{(-1)^{j-1}(\bar{\zeta}_j - \bar{z}_j)}{|\zeta - z|^{2n}} \widehat{d\bar{\zeta}_j} \wedge d\zeta.$$

Bochner-Martinelli formula. *Let $D \subset \mathbb{C}^n$ be a bounded domain with C^1−boundary. Let $f \in C^1(\overline{D})$. Then for each $z \in D$,*

$$f(z) = \int_{\partial D} f(\zeta) K_{BM}(\zeta - z) - \frac{(n-1)!}{(2\pi i)^n} \int_D \sum_{j=1}^{n} (\bar{\zeta}_j - \bar{z}_j) \frac{\partial f}{\partial \bar{\zeta}_j} \frac{d\bar{\zeta} \wedge d\zeta}{|\zeta - z|^{2n}}.$$

Proof of Theorem 1.6. Without loss of generality, we assume that the diameter $d(\Omega)$ of Ω is less than 1/2.

(a) Put $\delta(z) := d(z, \partial\Omega)$, $z \in \mathbb{C}^n$. Clearly, $|\delta(z) - \delta(w)| \leq |z - w|$ for all $z, w \in \mathbb{C}^n$. To apply the B-M formula, we need to approximate $\delta(z)$ first by C^1−smooth functions with uniformly bounded gradients by a standard argument as follows. Let $\kappa \geq 0$ be a C^∞ function in $\mathbb{C}^n$ satisfying the following properties: κ depends only on $|z|$, $\operatorname{supp} \kappa \subset \mathbb{B}^n$ and $\int_{\mathbb{C}^n} \kappa(z) dV = 1$. For each $\varepsilon > 0$, we put $\kappa_\varepsilon(z) = \varepsilon^{-2n} \kappa(z/\varepsilon)$ and $\delta_\varepsilon = \delta * \kappa_\varepsilon$. Clearly, δ_ε converges uniformly on $\overline{\Omega}$ to δ, and the gradient $\bigtriangledown \delta_\varepsilon$ of δ_ε verifies

$$\bigtriangledown \delta_\varepsilon(z) = \int_{\mathbb{C}^n} \delta(\zeta) \bigtriangledown_z \kappa_\varepsilon(\zeta - z) dV_\zeta = \int_{\mathbb{C}^n} (\delta(\zeta) - \delta(z)) \bigtriangledown_z \kappa_\varepsilon(\zeta - z) dV_\zeta$$

because $\int_{\mathbb{C}^n} \kappa_\varepsilon(\zeta - z) dV_\zeta = 1$. Thus

$$|\bigtriangledown \delta_\varepsilon(z)| \leq \int_{\mathbb{C}^n} |\delta(\zeta) - \delta(z)| \cdot |\bigtriangledown_z \kappa_\varepsilon(\zeta - z)|\, dV_\zeta \leq \text{const}_n.$$

Let $f \in \mathscr{O}(\Omega)$ and $z_0 \in \Omega$ be arbitrarily fixed. For any sufficiently small $\varepsilon > 0$, there is a positive number ε_1 such that

$$\left\{z \in \Omega : \varepsilon \leq \delta_{\varepsilon_1}(z) \leq \sqrt{\varepsilon}\right\} \subset \Omega_{\frac{\varepsilon}{2}} \backslash \Omega_{2\sqrt{\varepsilon}}$$

and $\delta_{\varepsilon_1} \asymp \delta_\Omega$ holds on $\Omega_{\frac{\varepsilon}{2}} \backslash \Omega_{2\sqrt{\varepsilon}}$ (with implicit constants independent of $\varepsilon, \varepsilon_1$). Now take a cut-off function χ on $\mathbb{R}$ such that $\chi|_{(-\infty,-\log 2)} = 1$ and $\chi|_{(0,\infty)} = 0$. Applying the B-M formula to the function

$$\chi(\log\log 1/\delta_{\varepsilon_1} - \log\log 1/\varepsilon) f^2$$

with ε sufficiently small, we obtain

$$f^2(z_0) = -\frac{(n-1)!}{(2\pi i)^n} \int_\Omega \frac{f^2(\zeta)\chi'(\,\cdot\,)}{\delta_{\varepsilon_1}(\zeta)\log\delta_{\varepsilon_1}(\zeta)} \sum_{j=1}^n (\bar{\zeta}_j - \bar{z}_{0,j}) \frac{\partial \delta_{\varepsilon_1}}{\partial \bar{\zeta}_j}(\zeta) \frac{d\bar{\zeta} \wedge d\zeta}{|\zeta - z_0|^{2n}}.$$

Thus

$$|f(z_0)|^2 \le \mathrm{const}_{n,z_0} \int_{\Omega_{\frac{\varepsilon}{2}} \backslash \Omega_{2\sqrt{\varepsilon}}} |f|^2 \delta_\Omega^{-1} |\log \delta_\Omega|^{-1} dV \to 0 \quad (\varepsilon \to 0+),$$

provided

$$\int_\Omega |f|^2 \delta_\Omega^{-1} |\log \delta_\Omega|^{-1} dV < \infty.$$

(b) Recall first that for each compact set $M \subset \Omega$, the capacity of M in Ω is defined by

$$\mathrm{cap}\,(M, \Omega) = \inf \int_\Omega |\nabla \phi|^2 \, dV$$

where the infimum is taken over all $\phi \in C_0^\infty(\Omega)$ such that $0 \le \phi \le 1$ and $\phi = 1$ in a neighborhood of M. For each j, we may choose a function $\phi_j \in C_0^\infty(\Omega)$ with $0 \le \phi_j \le 1$, $\phi_j = 1$, in a neighborhood of $\overline{\Omega}_{\varepsilon_j}$, so that

$$\int_\Omega |\nabla \phi_j|^2 \, dV \le 2c(\varepsilon_j).$$

Let $f \in A_\alpha^2(\Omega)$ and $z_0 \in \Omega$ be arbitrarily fixed. Applying the B-M formula to the function $\phi_j f$ with j sufficiently large, we get

$$f(z_0) = -\frac{(n-1)!}{(2\pi i)^n} \int_\Omega f(\zeta) \sum_{k=1}^n (\bar{\zeta}_k - \bar{z}_{0,k}) \frac{\partial \phi_j}{\partial \bar{\zeta}_k}(\zeta) \frac{d\bar{\zeta} \wedge d\zeta}{|\zeta - z_0|^{2n}}$$

so that

$$\begin{aligned}
|f(z_0)| &\le \mathrm{const}_{n,z_0} \int_\Omega |\nabla \phi_j| \, |f| \, dV \\
&\le \mathrm{const}_{n,z_0} \left(\int_{\Omega \backslash \Omega_{\varepsilon_j}} |\nabla \phi_j|^2 \, \delta_\Omega^\alpha \, dV \right)^{1/2} \left(\int_{\Omega \backslash \Omega_{\varepsilon_j}} |f|^2 \delta_\Omega^{-\alpha} dV \right)^{1/2} \\
&\le \mathrm{const}_{n,z_0} \, c(\varepsilon_j)^{1/2} \varepsilon_j^{\alpha/2} \left(\int_{\Omega \backslash \Omega_{\varepsilon_j}} |f|^2 \delta_\Omega^{-\alpha} dV \right)^{1/2} \to 0
\end{aligned}$$

as $j \to \infty$. □

On the other side, we have

Proposition 8.1. *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain and put $V(\varepsilon) = \mathrm{vol}_E\,(\Omega \backslash \Omega_\varepsilon)$. If*

$$\alpha < \liminf_{\varepsilon \to 0+} \frac{\log V(\varepsilon)}{\log \varepsilon},$$

then $H^\infty(\Omega) \subset A_\alpha^2(\Omega)$.

Proof. It suffices to show that $1\in A^2_\alpha(\Omega)$. Fix β such that $\alpha<\beta<\liminf_{\varepsilon\to 0+}\frac{\log V(\varepsilon)}{\log\varepsilon}$. Note that

$$\mathrm{vol}_E(\Omega\backslash\Omega_\varepsilon) < \mathrm{const}_\beta\, \varepsilon^\beta$$

for all $\varepsilon > 0$. Without loss of generality, we assume $\delta_\Omega < 1$ on Ω and $\alpha \ge 0$. Then we have

$$\begin{aligned}\int_\Omega \delta_\Omega^{-\alpha} dV &\le \sum_{j=0}^\infty \int_{\Omega_{2^{-j-1}}\backslash\Omega_{2^{-j}}} 2^{\alpha(j+1)} dV \le \sum_{j=0}^\infty 2^{\alpha(j+1)}\mathrm{vol}_E(\Omega\backslash\Omega_{2^{-j}})\\ &\le \mathrm{const}_{\alpha,\beta}\sum_{j=0}^\infty 2^{-(\beta-\alpha)j} < \infty.\end{aligned}$$

□

It is reasonable to introduce the following.

Definition 8.2. Let Ω be a bounded domain in $\mathbb{C}^n$. The critical exponent $\alpha(\Omega)$ of Ω for weighted Bergman spaces $A^2_\alpha(\Omega)$ is defined to be

$$\alpha(\Omega) := \sup\left\{\alpha : A^2_\alpha(\Omega) \neq \{0\}\right\} = \inf\left\{\alpha : A^2_\alpha(\Omega) = \{0\}\right\}.$$

From Proposition 8.1 and Theorem 1.6, we know that

$$\beta(\Omega) := \liminf_{\varepsilon\to 0+}\frac{\log V(\varepsilon)}{\log\varepsilon} \le \alpha(\Omega) \le \min\left\{1, \liminf_{\varepsilon\to 0+}\frac{\log c(\varepsilon)}{\log 1/\varepsilon}\right\} =: \gamma(\Omega).$$

Note that $2n-\beta(\Omega)$ is nothing but the classical Minkowski dimension of $\partial\Omega$. Thus $\alpha(\Omega)=1$ in case $\partial\Omega$ is non-fractal, i.e., $\beta(\Omega)=1$. This is the case, for instance, when Ω is a bounded domain in $\mathbb{C}^n$ with Lipschitz boundary or a domain in $\mathbb{C}$ whose boundary is a rectifiable Jordan curve. Unfortunately, the author is unable to find an example with $\alpha(\Omega)<1$.

Proof of Theorem 1.7. Without loss of generality, we may assume that $\rho > -e^{-1}$ and $d(\Omega) \le 1/2$. Suppose to the contrary that there is a continuous psh function $\rho < 0$ on Ω such that

$$-\rho \le \mathrm{const}_\varepsilon \delta_\Omega \left|\log\delta_\Omega\right|^{-\varepsilon}.$$

Then we have

$$(-\rho)(-\log(-\rho))^{1+\varepsilon/2} \le \mathrm{const}_\varepsilon\delta_\Omega|\log\delta_\Omega|. \tag{9}$$

By Richberg's theorem, we may also assume that ρ is C^∞ and strictly psh on Ω. Fix $z_0\in\Omega$. Put $\phi = -\log(-\rho)$ and

$$\varphi(z) = 2n\log|z-z_0|, \quad \psi = \phi - \frac{\varepsilon}{2}\log\phi.$$

Note that $\bar\partial\psi = \bar\partial\phi - \frac{\varepsilon}{2}\frac{\bar\partial\phi}{\phi}$ and

$$i\partial\bar\partial\psi = \left(1-\frac{\varepsilon}{2\phi}\right)i\partial\bar\partial\phi + \frac{\varepsilon}{2}\frac{i\partial\phi\wedge\bar\partial\phi}{\phi^2} \ge \left(1-\frac{\varepsilon}{2\phi}+\frac{\varepsilon}{2\phi^2}\right)i\partial\phi\wedge\bar\partial\phi,$$

so that

$$|\bar\partial\psi|^2_{i\partial\bar\partial\psi} \le \frac{1-\frac{\varepsilon}{\phi}+\frac{\varepsilon^2}{4\phi^2}}{1-\frac{\varepsilon}{2\phi}+\frac{\varepsilon}{2\phi^2}}. \tag{10}$$

Let χ be as in the proof of Theorem 1.6 and put $v=\bar\partial\chi(2|z-z_0|/\delta_\Omega(z_0)-1)$. We need to solve the equation $\bar\partial u = v$ on Ω together with a Donnelly-Fefferman type estimate by using a trick from Berndtsson-Charpentier [6] essentially as in [11]. Let $m > 0$ be sufficiently large and u_m the minimal solution of $\bar\partial u = v$ in $L^2(\Omega_{1/m}, \varphi)$. Then we have $u_m e^{\psi} \perp \operatorname{Ker} \bar\partial$ in $L^2(\Omega_{1/m}, \varphi+\psi)$. Thus by Hörmander's estimate (1),

$$\begin{aligned}
\int_{\Omega_{1/m}} |u_m|^2 e^{-\varphi+\psi} dV &\leq \int_{\Omega_{1/m}} |\bar\partial(u_m e^{\psi})|^2_{i\partial\bar\partial(\varphi+\psi)} e^{-\varphi-\psi} dV \\
&\leq \int_{\Omega_{1/m}} |v + \bar\partial\psi \wedge u_m|^2_{i\partial\bar\partial\psi} e^{-\varphi+\psi} dV \\
&\leq \int_{\Omega_{1/m}} \left(1 + \frac{4\phi}{\varepsilon}\right) |v|^2_{i\partial\bar\partial\psi} e^{-\varphi+\psi} dV \\
&\quad + \int_{\Omega_{1/m}} \left(1 + \frac{\varepsilon}{4\phi}\right) |\bar\partial\psi|^2_{i\partial\bar\partial\psi} |u_m|^2 e^{-\varphi+\psi} dV.
\end{aligned}$$

Together with (10), we get

$$\int_{\Omega_{1/m}} |u_m|^2 \phi^{-1} e^{-\varphi+\psi} dV \leq \text{const}_\varepsilon \int_{\Omega} \left(1 + \frac{4\phi}{\varepsilon}\right) |v|^2_{i\partial\bar\partial\psi} e^{-\varphi+\psi} dV < \infty, \tag{11}$$

for we can make ϕ sufficiently large if ρ is replaced by ρ/C with $C \gg 1$.

Now put $f_m(z) := \chi(2|z - z_0|/\delta_\Omega(z_0) - 1) - u_m(z)$. Let f be a weak limit of $\{f_m\}_{m=1}^\infty$. Clearly, $f \in \mathscr{O}(\Omega)$, $f(z_0) = 1$ and by (9) and (11),

$$\int_\Omega |f|^2 \delta_\Omega^{-1} |\log \delta_\Omega|^{-1} dV \leq \text{const}_\varepsilon \int_\Omega |f|^2 \phi^{-1} e^{\psi} dV < \infty.$$

This contradicts Theorem 1.6. □

Acknowledgement

The author thanks Dr. Xu Wang for pointing out several inaccuracies in a draft of this paper. He also thanks the referee and Professor Peter Pflug for valuable comments.

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