A WEAK* SEPARABLE C(K)* SPACE WHOSE UNIT BALL IS NOT WEAK* SEPARABLE

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Abstract. We provide a ZFC example of a compact space K such that C(K)* is w*-separable but its closed unit ball B_{C(K)*} is not w*-separable. All previous examples of such kind had been constructed under CH. We also discuss the measurability of the supremum norm on that C(K) equipped with its weak Baire σ-algebra.

1. Introduction

Let K be a compact space (all our topological spaces are assumed to be Hausdorff) and let C(K) be the Banach space of all continuous real-valued functions on K (equipped with the supremum norm). One can consider the following list of properties related to the separability in K and C(K)*:

(a) K is separable;
(b) K carries a strictly positive measure of countable type;
(c) P(K) (the set of all Radon probability measures on K) is w*-separable;
(d) B_{C(K)*} (the closed unit ball of C(K)*) is w*-separable;
(e) C(K)* is w*-separable.

It is known that the following implications hold:

(a) ⇒ (b) ⇒ (c) ⇐⇒ (d) ⇒ (e)

and cannot be reversed in general. Indeed, the Stone space of the measure algebra of the Lebesgue measure on [0,1] satisfies (b) but not (a), while Talagrand [17] constructed under CH two examples showing that (c)⇒(b) and (e)⇒(c) do not hold. Recently, Džamonja and Plebanek [2] gave a ZFC counterexample to (c)⇒(b).

In this paper we provide a ZFC example of a compact space K witnessing that the implication (e)⇒(d) does not hold, i.e. C(K)* is w*-separable but B_{C(K)*} is not. The construction is given in Section 2 (see Theorem 2.1) and uses techniques which are similar to those of [2].

Section 3 is devoted to a discussion of further properties of that example which are relevant within the topic of measurability in Banach spaces. In every Banach space X one can consider the Baire σ-algebra of the weak topology, denoted by...
Ba\((X, w)\), which coincides with the \(\sigma\)-algebra on \(X\) generated by \(X^*\) (see [ordered reference]). While \(\text{Ba}(X, w)\) coincides with the Borel \(\sigma\)-algebra of the norm topology if \(X\) is separable, the two \(\sigma\)-algebras may be different if \(X\) is nonseparable. When \(X\) has a \(w^*\)-separable dual, then we can represent it as a subset of \(\mathbb{R}^N\). Namely, if \((x_n^*)\) is a \(w^*\)-dense sequence in \(X^*\), then the function \(X \to \mathbb{R}^N\) given by \(x \mapsto (x_n^*(x))\) is one-to-one. In particular, given any equivalent norm \(\| \cdot \|\) on \(X\), the function \(\| \cdot \| : X \to \mathbb{R}\) can be identified with a partial function on \(\mathbb{R}^N\). If \((x_n^*)\) was dense in the ball \(B_{X^*}\), then this norm function would be just the sup function, \(\|x\| = \sup_n |x_n^*(x)|\). But if we have a space \(X\) with \(w^*\)-separable dual and \(B_{X^*}\) is not \(w^*\)-separable, how good can the norm function be? Could it be a \(\text{Ba}(X, w)\)-measurable function? Note that \(\| \cdot \| \) is \(\text{Ba}(X, w)\)-measurable if and only if the closed unit ball \(B_X\) belongs to \(\text{Ba}(X, w)\). It is easy to check that:

\[
B_{X^*}\text{ is }w^*\text{-separable} \iff \| \cdot \| \text{ is }\text{Ba}(X, w)\text{-measurable} \implies X^* \text{ is }w^*\text{-separable}.
\]

None of the reverse implications holds in general [13].

It seems to be an open problem whether the weak Baire measurability of the supremum norm of a \(C(K)\) space is equivalent to either \(C(K)^*\) being \(w^*\)-separable or to \(B_{C(K)}^*\) being \(w^*\)-separable. Our compact space \(K\) of Section 2 would settle one of the two questions in the negative. In Section 3 we shall discuss whether the supremum norm of our space \(C(K)\) can be represented as a measurable function on a separating family of functionals of \(C(K)^*\), so that the norm would become \(\text{Ba}(C(K), w)\)-measurable.

Although we have been unable to determine the degree of measurability of the supremum norm on that \(C(K)\), we shall show, however, that at least there exists a norm dense set \(E \subseteq C(K)\) where the restriction of the supremum norm is relatively \(\text{Ba}(C(K), w)\)-measurable (Theorem 3.10). This set \(E\) can be taken to be the linear span of the characteristic functions of clopen subsets of \(K\) under the set-theoretic assumption that \(\kappa\) is a Kunen cardinal (Theorem 3.22).

**Terminology.** We write \(\mathcal{P}(S)\) to denote the power set of any set \(S\). The cardinality of \(S\) is denoted by \(|S|\). The letter \(\kappa\) stands for the cardinality of the continuum. A probability measure \(\nu\) is said to be of countable type if the space \(L^1(\nu)\) is separable. For a compact space \(K\), we usually identify the dual space \(C(K)^*\) with the space of all Radon measures on \(K\). Given a Boolean algebra \(\mathfrak{A}\), we write \(\mathfrak{A}^+\) to denote the set of all nonzero elements of \(\mathfrak{A}\). For the Boolean operations we use the usual symbols \(\cup, \cap, \text{ etc.}\) and we write \(0\) and \(1\) for the least and the greatest element.

The Stone space of all ultrafilters on \(\mathfrak{A}\) is denoted by \(\text{ULT}(\mathfrak{A})\). Recall that the Stone isomorphism between \(\mathfrak{A}\) and the algebra \(\text{Clop}(\text{ULT}(\mathfrak{A}))\) of clopen subsets of \(\text{ULT}(\mathfrak{A})\) is given by

\[
\mathfrak{A} \to \text{Clop}(\text{ULT}(\mathfrak{A})), \quad a \mapsto \hat{a} = \{ F \in \text{ULT}(\mathfrak{A}) : a \in F \}.
\]

Every measure \(\mu\) on \(\mathfrak{A}\) induces a measure \(\hat{\mu} \mapsto \mu(\hat{a})\) on \(\text{Clop}(\text{ULT}(\mathfrak{A}))\) which can be uniquely extended to a Radon measure on \(\text{ULT}(\mathfrak{A})\) (see e.g. [15 Chapter 5]); such a Radon measure is still denoted by the same letter \(\mu\).

Given a set \(S\) and a family \(\mathcal{D}\) of subsets of \(S\), the \(\sigma\)-algebra on \(S\) generated by \(\mathcal{D}\) is denoted by \(\sigma(\mathcal{D})\). If \((\Omega, \Sigma)\) is any measurable space and \(n \in \mathbb{N}\), we write \(\bigotimes_n \Sigma\) to denote the usual product \(\sigma\)-algebra on \(\Omega^n\), that is,

\[
\bigotimes_n \Sigma = \sigma\left(\{ A_1 \times \cdots \times A_n : A_i \in \Sigma \text{ for all } i = 1, \ldots, n\}\right).
\]
A cardinal \( \kappa \) is called a Kunen cardinal if the equality
\[
\mathcal{P}(\kappa \times \kappa) = \mathcal{P}(\kappa) \otimes \mathcal{P}(\kappa)
\]
holds true. Of course, in this case we have \( \mathcal{P}(\kappa^n) = \bigotimes_n \mathcal{P}(\kappa) \) for every \( n \in \mathbb{N} \). This notion was investigated by Kunen in his doctoral dissertation [9]. It is known that:

(i) any Kunen cardinal is less than or equal to \( \mathfrak{c} \);
(ii) \( \omega_1 \) is a Kunen cardinal;
(iii) \( \mathfrak{c} \) is a Kunen cardinal under Martin’s axiom, while \( \mathfrak{c} \) is not a Kunen cardinal in the Cohen model after adding \( \aleph_2 \) reals to a model of GCH or when \( \mathfrak{c} \) is real-valued measurable.

Kunen cardinals have been considered by Talagrand [16], Fremlin [6] and the authors [1] in connection with measurability properties of Banach spaces, and also by Todorcevic [18] in connection with universality properties of \( \ell_\infty/c_0 \).

2. The example

Fix any cardinal \( \kappa \) such that \( \omega_1 \leq \kappa \leq \mathfrak{c} \). Let \( \lambda \) be the usual product probability measure on the Baire \( \sigma \)-algebra of \( 2^\kappa \) and let \( \mathcal{B} \) be its measure algebra. Note that \( \mathcal{B} \) has cardinality \( \mathfrak{c} \) since every Baire subset of \( 2^\kappa \) is determined by countably many coordinates (see e.g. [3, 254M]). The letter \( \lambda \) will also stand for the corresponding probability measure on \( \mathcal{B} \). We shall work in the countable simple product \( \mathcal{C} := \mathcal{B}^\mathbb{N} \) of \( \mathcal{B} \) so that every \( c \in \mathcal{C} \) is a sequence \( c = (c(n))_n \) where \( c(n) \in \mathcal{B} \) for all \( n \in \mathbb{N} \). On the Boolean algebra \( \mathcal{C} \) we consider the sequence of probability measures \( \{\mu_n : n \in \mathbb{N}\} \) defined by

\[
\mu_n(c) := \lambda(c(n)) \quad \text{for all } c \in \mathcal{C}.
\]

Let \( \{N_b : b \in \mathcal{B}^+\} \) be a fixed independent family of subsets of \( \mathbb{N} \), i.e.

\[
\bigcap_{b \in s} N_b \setminus \bigcup_{b' \notin t} N_{b'} \neq \emptyset
\]

whenever \( s, t \subseteq \mathcal{B}^+ \) are finite and disjoint (see e.g. [4, p. 180, Exercise 3.6.F]); note that \( \bigcap_{b \in s} N_b = \mathbb{N} \) if \( s = \emptyset \) and that \( \bigcup_{b \notin t} N_{b'} = \emptyset \) if \( t = \emptyset \). For each \( b \in \mathcal{B}^+ \), define an element \( G_b \in \mathcal{C} \) by declaring

\[
G_b(n) := \begin{cases} b & \text{if } n \in N_b, \\ 0 & \text{otherwise}. \end{cases}
\]

Let \( \mathfrak{A} \) be the subalgebra of \( \mathcal{C} \) generated by \( \{G_b : b \in \mathcal{B}^+\} \) and write \( K := \text{ULT}(\mathfrak{A}) \).

This section is devoted to proving the following:

Theorem 2.1. \( B_{C(K)^*} \) is not \( w^* \)-separable, while \( \{\mu_n : n \in \mathbb{N}\} \) separates the points of \( C(K) \), so \( C(K)^* \) is \( w^* \)-separable.

In order to prove Theorem 2.1 we need some preliminary work. For any finite sets \( s, t \subseteq \mathcal{B}^+ \) we consider the following elements of \( \mathfrak{A} \):

\[
J(s) := \bigcap_{b \in s} G_b, \quad U(t) := \bigcup_{b \notin t} G_b \quad \text{and} \quad W(s, t) := J(s) \setminus U(t).
\]

The Boolean algebra \( \mathfrak{A} \) is completely determined up to isomorphism when given its set of generators \( \{G_b : b \in \mathcal{B}^+\} \) and which elements \( W(s, t) \) are zero. In this sense, Lemma 2.2 below is interpreted as the fact that \( \mathfrak{A} \) is isomorphic to the free Boolean algebra generated by \( \{G_b : b \in \mathcal{B}^+\} \) modulo the relations that \( W(s, t) = 0 \) if and only if \( s \cap t \neq \emptyset \) or \( \bigcap_{b \in s} b = 0 \).
Lemma 2.2. For two finite sets $s, t \subseteq \mathcal{B}^+$, we have $W(s, t) = 0$ if and only if $s \cap t \neq \emptyset$ or $\bigcap_{b \in s} b = 0$. In particular, for a finite $s \subseteq \mathcal{B}^+$ the following are equivalent:

1. $\bigcap_{b \in s} b = 0$;
2. $J(s) = 0$;
3. $W(s, t) = 0$ for all finite $t \subseteq \mathcal{B}^+ \setminus s$;
4. $W(s, t) = 0$ for some finite $t \subseteq \mathcal{B}^+ \setminus s$.

Proof. Let us observe that for every $n \in \mathbb{N}$ we have

$$J(s)(n) = \bigcap_{b \in s} G_b(n) = \begin{cases} \bigcap_{b \in s} b & \text{if } n \in \bigcap_{b \in s} N_b, \\ 0 & \text{otherwise.} \end{cases}$$

It follows from (2.1) that if $\bigcap_{b \in s} b = 0$, then $W(s, t) \subseteq J(s) = 0$ as well. The same immediately follows when $s \cap t \neq \emptyset$.

For the converse, suppose that $s \cap t = \emptyset$ and $\bigcap_{b \in s} b \neq 0$, and pick

$$n_0 \in \bigcap_{b \in s} N_b \setminus \bigcup_{b' \in t} N_{b'}.$$ 

Then

$$W(s, t)(n_0) = \bigcap_{b \in s} G_b(n_0) \setminus \bigcup_{b' \in t} G_{b'}(n_0) = \bigcap_{b \in s} b \neq 0,$$

and so $W(s, t) \neq 0$. The second part of the lemma, with the list of equivalences, follows from the first statement and the arguments above. \hfill \square

We next describe $K = \text{ULT}(\mathfrak{a})$. Let us consider the family of all subsets of $\mathfrak{a}$ with the finite intersection property, that is,

$$\text{FIP}(\mathfrak{a}) = \left\{ X \subseteq \mathfrak{a}^+ : b_1 \cap \ldots \cap b_n \neq 0 \text{ for every } b_1, \ldots, b_n \in X \right\}.$$ 

Given $X \in \text{FIP}(\mathfrak{a})$, let $\mathcal{F}_X$ be the filter on $\mathfrak{a}$ generated by

$$\{W(s, t) : s \subseteq X \text{ finite}, t \subseteq \mathfrak{a}^+ \setminus X \text{ finite}\}$$

(notice that this set has the finite intersection property by Lemma 2.2). In particular, when $X$ is empty, then $\mathcal{F}_X$ is simply generated by the complements of all $G_b$’s.

Lemma 2.3. $K = \{\mathcal{F}_X : X \in \text{FIP}(\mathfrak{a})\}$.

Proof. Every filter of the form $\mathcal{F}_X$ is an ultrafilter on $\mathfrak{a}$, because $\{G_b : b \in \mathfrak{a}^+\}$ is a set of generators of $\mathfrak{a}$, and, for each $b \in \mathfrak{a}^+$, we have either $G_b = W(\{b\}, 0) \in \mathcal{F}_X$ (if $b \in X$) or $1 \setminus G_b = W(\emptyset, \{b\}) \in \mathcal{F}_X$ (if $b \not\in X$). Conversely, let $\mathcal{F}$ be any ultrafilter on $\mathfrak{a}$ and consider $X := \{b \in \mathfrak{a}^+ : G_b \in \mathcal{F}\}$. Notice that $X \in \text{FIP}(\mathfrak{a})$ and that, for each $b \in \mathfrak{a}^+$, we have

$$G_b \in \mathcal{F}_X \iff G_b \in \mathcal{F}.$$ 

Since $\{G_b : b \in \mathfrak{a}^+\}$ is a set of generators of $\mathfrak{a}$, it follows that $\mathcal{F} = \mathcal{F}_X$. \hfill \square

Let $\text{FIP}_0(\mathfrak{a})$ be the family of all $s \in \text{FIP}(\mathfrak{a})$ which are finite. The next lemma says that the clopens of the form $\overline{W(s, t)}$ are a basis of the topology of $K$ and also that $\{\mathcal{F}_s : s \in \text{FIP}_0(\mathfrak{a})\}$ is dense in $K$. 

Lemma 2.4. Let \( a \in \mathfrak{B}^+ \) and \( \mathcal{F} \in \mathfrak{A} \subseteq K \).

(i) There exist \( s \in \text{FIP}_0(\mathfrak{B}) \) and a finite set \( t \subseteq \mathfrak{B}^+ \) such that \( \mathcal{F} \in \mathcal{W}(s, t) \subseteq \mathfrak{A} \).

(ii) If \( \mathcal{F} = \mathcal{F}_{s_0} \) for some \( s_0 \in \text{FIP}_0(\mathfrak{B}) \), then we can choose \( s = s_0 \) in (i).

Proof. (i) Take \( I \subseteq \mathfrak{B}^+ \) finite such that \( a \) belongs to the subalgebra of \( \mathfrak{A} \) generated by \( \{ G_b : b \in I \} \). The family \( \mathfrak{A}_0 \) of all elements of \( \mathfrak{A} \) which can be written as the union of finitely many elements of the form \( \mathcal{W}(s, t) \), where \( s \cup t = I \) and \( s \cap t = \emptyset \), is a subalgebra of \( \mathfrak{A} \) containing \( \{ G_b : b \in I \} \). Therefore \( a \in \mathfrak{A}_0 \). Since \( a \in \mathcal{F} \), there exist \( s \) and \( t \) as before such that \( \mathcal{W}(s, t) \subseteq \mathcal{F} \); hence \( s \in \text{FIP}_0(\mathfrak{B}) \) (by Lemma 2.2) and \( \mathcal{F} \subseteq \mathcal{W}(s, t) \subseteq \mathfrak{A} \).

(ii) Since \( \mathcal{W}(s, t) \subseteq \mathcal{F}_{s_0} \), we have \( G_b \in \mathcal{F}_{s_0} \) for all \( b \in s \) and \( G_b \not\in \mathcal{F}_{s_0} \) for all \( b \in t \). Hence \( s \subseteq s_0 \) and \( s_0 \cap t = \emptyset \). Therefore, \( \mathcal{F}_{s_0} \subseteq \mathcal{W}(s_0, t) \subseteq \mathcal{W}(s, t) \subseteq \mathfrak{A} \). □

Another ingredient to prove Theorem 2.1 is the result isolated in Lemma 2.6 below, which is a consequence of the following characterization of \( w^* \)-separability in spaces of measures due to Mägerl and Namioka [11].

Fact 2.5. Let \( L \) be a compact space. Then the space \( P(L) \) is \( w^* \)-separable if and only if there is a sequence \( \{ \nu_n : n \in \mathbb{N} \} \) in \( P(L) \) such that for every nonempty open set \( V \subseteq L \) there is \( n \in \mathbb{N} \) with \( \nu_n(V) > 1/2 \).

Lemma 2.6. Let \( \mathfrak{A}_1 \) and \( \mathfrak{A}_2 \) be Boolean algebras such that there is another Boolean algebra \( \mathfrak{A}_3 \) containing \( \mathfrak{A}_1 \) and \( \mathfrak{A}_2 \) as subalgebras and the following holds:

\[ (*) \quad \text{for every } b \in \mathfrak{A}_3^+ \text{ there is } a \in \mathfrak{A}_1^+ \text{ such that } a \leq b. \]

If \( P(\text{ULT}(\mathfrak{A}_2)) \) is not \( w^* \)-separable, then \( P(\text{ULT}(\mathfrak{A}_1)) \) is not \( w^* \)-separable either.

Proof. Let \( \{ \nu_n : n \in \mathbb{N} \} \) be a sequence of probability measures on \( \mathfrak{A}_1 \). For each \( n \in \mathbb{N} \), we can extend \( \nu_n \) to a probability measure \( \nu'_n \) on \( \mathfrak{A}_3 \) (see [10] or [12]) and we denote by \( \nu_n \) the restriction of \( \nu'_n \) to \( \mathfrak{A}_2 \). Since \( P(\text{ULT}(\mathfrak{A}_2)) \) is not \( w^* \)-separable, by Fact 2.5 there is \( b \in \mathfrak{A}_3^+ \) such that \( \nu_n(b) \leq 1/2 \) for every \( n \in \mathbb{N} \). Property \( (*) \) allows us to take \( a \in \mathfrak{A}_1^+ \) such that \( a \leq b \). Then

\[ \nu_n(a) = \nu'_n(a) \leq \nu'_n(b) = \nu_n(b) \leq 1/2 \quad \text{for every } n \in \mathbb{N}. \]

Another appeal to Fact 2.5 ensures that \( P(\text{ULT}(\mathfrak{A}_1)) \) is not \( w^* \)-separable. □

According to a result of Rosenthal (see [14] Theorem 3.1), the space \( L^\infty(\nu)^* \) is not \( w^* \)-separable whenever \( \nu \) is a probability measure of uncountable type. Since \( C(\text{ULT}(\mathfrak{B})) \) is isomorphic to \( L^\infty(\lambda) \), it follows that \( P(\text{ULT}(\mathfrak{B})) \) is not \( w^* \)-separable.

We are now ready to deal with Theorem 2.1.

Proof of Theorem 2.1

Let \( \mathfrak{B}^* \) be the subalgebra of \( \mathfrak{C} \) consisting of all constant sequences. Since \( \mathfrak{B}^* \) is isomorphic to \( \mathfrak{B} \), \( P(\text{ULT}(\mathfrak{B}^*)) \) is not \( w^* \)-separable. On the other hand, property \( (*) \) of Lemma 2.6 holds for \( \mathfrak{A}_1 = \mathfrak{A} \) and \( \mathfrak{A}_2 = \mathfrak{B}^* \), hence \( P(K) \) is not \( w^* \)-separable and so \( B_{C(K)^*} \) is not \( w^* \)-separable either.

We now prove that \( \{ \mu_n : n \in \mathbb{N} \} \) separates the points of \( C(K) \). Fix \( h \in C(K) \) \( \setminus \{ 0 \} \).

Step 1. Since \( \{ \mathcal{F}_s : s \in \text{FIP}_0(\mathfrak{B}) \} \) is dense in \( K \) (by Lemma 2.4(i)), there is \( s \in \text{FIP}_0(\mathfrak{B}) \) such that \( h(\mathcal{F}_s) \neq 0 \). Moreover, we can assume that

\[ (2.2) \quad h(\mathcal{F}_{s'}) = 0 \quad \text{whenever } s' \in \text{FIP}_0(\mathfrak{B}) \text{ satisfies } |s'| < |s|. \]
and that \( C := h(F_s) > 0 \). By the continuity of \( h \) and Lemma 2.4(ii), there is a finite set \( t \subseteq \mathfrak{B}^+ \setminus s \) such that, writing \( a := W(s, t) \), we have
\[
(2.3) \quad h(F) \geq \frac{C}{2} \quad \text{for all } F \in \hat{a}.
\]
Set \( \delta := \frac{C}{4} \lambda(\bigcap_{b \in s} b) > 0 \) and define \( \mathcal{R} := \{r \subseteq s : r \neq s\} \subseteq \text{FIP}_0(\mathfrak{B}) \).

**Step 2.** Fix \( r \in \mathcal{R} \). Since \( h \) is continuous and \( h(F_r) = 0 \) (by (2.2)), we can apply Lemma 2.4(ii) to find a finite set \( t_r' \subseteq \mathfrak{B}^+ \setminus r \) such that \( |h(F)| \leq \delta \) for all \( F \in \hat{W}(r, t_r') \). Writing \( t_r := t_r' \setminus s \), we have
\[
 a_r := W(r, t_r \cup (s \setminus r)) \subseteq W(r, t_r') \quad \text{in } \mathfrak{A},
\]
and therefore
\[
(2.4) \quad |h(F)| \leq \delta \quad \text{for all } F \in \hat{a}_r.
\]

**Step 3.** Define
\[
t^* := t \cup \bigcup_{r \in \mathcal{R}} t_r \subseteq \mathfrak{B}^+ \setminus s
\]
and choose \( n \in \bigcap_{b \in s} N_b \setminus \bigcup_{b' \in t^*} N_{b'} \) (which is nonempty by independence). Hence
\[
a(n) = \bigcap_{b \in s} b, \quad a_r(n) = \bigcap_{b \in r} b \setminus \bigcup_{b' \in s \setminus r} b'
\]
for every \( r \in \mathcal{R} \), and therefore
\[
(2.5) \quad 1 \setminus a(n) \subseteq \bigcup_{r \in \mathcal{R}} a_r(n) \quad \text{in } \mathfrak{B}.
\]

**Step 4.** Observe that
\[
\mu_n \left( (K \setminus \hat{a}) \setminus \bigcup_{r \in \mathcal{R}} \hat{a}_r \right) = \mu_n \left( (1 \setminus a) \setminus \bigcup_{r \in \mathcal{R}} a_r \right) = \lambda \left( (1 \setminus a(n)) \setminus \bigcup_{r \in \mathcal{R}} a_r(n) \right) = 0.
\]
Therefore
\[
\left| \int_{K \setminus \hat{a}} h \, d\mu_n \right| \leq \int_{\bigcup_{r \in \mathcal{R}} \hat{a}_r} |h| \, d\mu_n \leq \delta.
\]
It follows that
\[
\mu_n(h) = \int_{\hat{a}} h \, d\mu_n + \int_{K \setminus \hat{a}} h \, d\mu_n
\]
\[
\geq \int_{\hat{a}} h \, d\mu_n - \delta \geq \frac{C}{2} \mu_n(\hat{a}) - \delta = \frac{C}{2} \lambda(\bigcap_{b \in s} b) - \delta = \frac{C}{4} \lambda \left( \bigcap_{b \in s} b \right) > 0.
\]
Thus \( \mu_n(h) \neq 0 \). This finishes the proof. \( \square \)

3. **Weak Baire measurability of the norm**

In this section we address the question of whether the supremum norm is weakly Baire measurable on \( C(K) \), where \( K \) is the compact space constructed in Section 2.

As we have pointed out in the introduction, measurability of the norm is naturally related to \( w^* \)-separability. Throughout this section we follow the notation introduced in Section 2. The supremum norm on \( C(K) \) is denoted by \( \| \cdot \| \).

We first show that the family \( \{\mu_n : n \in \mathbb{N}\} \subseteq P(K) \), though separating the elements of \( C(K) \), is not rich enough to “measure” \( B_{C(K)} \).
Proposition 3.1. The supremum norm on $C(\mathbb{K})$ does not have to be measurable with respect to the $\sigma$-algebra generated by $\{\mu_n : n \in \mathbb{N}\}$.

Proof. We identify $\mathcal{P}(\mathbb{N})$ and $\{0,1\}^\mathbb{N}$ in the usual way. Let $\Omega \subseteq 2^\mathbb{N}$ be an independent family of subsets of $\mathbb{N}$ with $|\Omega| = \mathfrak{c}$ and let $\Sigma$ denote the trace of Borel($2^\mathbb{N}$) on $\Omega$. Then $|\Sigma| = \mathfrak{c}$, and so we can find $A \subseteq \Omega$ such that $A \notin \Sigma$ and $|A| = |\Omega \setminus A| = \mathfrak{c}$. Now we can choose an enumeration $\Omega = \{N_b : b \in \mathcal{B}^+\}$ such that

$$\lambda(b) = 1/2 \iff N_b \in A.$$  

Define a function $f : \Omega \rightarrow C(\mathbb{K})$ by

$$f(N_b) := \frac{1}{2\lambda(b)} \widetilde{G_b}.$$  

We claim that $f$ is measurable with respect to $\Sigma$ and the $\sigma$-algebra $\mathcal{C}$ on $C(\mathbb{K})$ generated by $\{\mu_n : n \in \mathbb{N}\}$. Indeed, for fixed $n \in \mathbb{N}$, we have

$$(\mu_n \circ f)(N_b) = \frac{1}{2\lambda(b)} \mu_n(G_b) = \frac{1}{2\lambda(b)} \lambda(G_b(n)) = \begin{cases} 1/2 & \text{if } n \in N_b, \\ 0 & \text{otherwise}. \end{cases}$$

It follows that $\mu_n \circ f = (1/2)\pi_n$, where $\pi_n : \Omega \subseteq 2^\mathbb{N} \rightarrow \{0,1\}$ denotes the $n$-th coordinate projection, hence $\mu_n \circ f$ is $\Sigma$-measurable.

On the other hand, the composition $\|f(\cdot)\| : \Omega \rightarrow \mathbb{R}$ is not $\Sigma$-measurable because

$$\|f(N_b)\| = 1 \iff N_b \in A$$

and $A \notin \Sigma$. This fact and the $\Sigma$-$\mathcal{C}$-measurability of $f$ imply that $\| \cdot \|$ is not $\mathcal{C}$-measurable. \hfill $\square$

3.1. Measurability on a norm dense set. Let $S(\mathbb{K})$ denote the dense subspace of $C(\mathbb{K})$ consisting of all linear combinations of characteristic functions of clopen subsets of $\mathbb{K}$. The next simple lemma provides a useful representation of the elements of $S(\mathbb{K})$.

Lemma 3.2. Let $g \in S(\mathbb{K})$. Then there exist $s \subseteq \mathcal{B}^+$ finite and a collection of real numbers $\{y_r : r \subseteq s\}$ such that

$$g = \sum_{r \subseteq s} y_r 1_{f(r)}.$$  

Proof. We can write $g$ as

$$g = \sum_{r' \subseteq s} z_{r'} 1_{W(r', s \setminus r')}$$

for some $s \subseteq \mathcal{B}^+$ finite and a certain collection of real numbers $\{z_{r'} : r' \subseteq s\}$. Note that there is a (unique) collection of real numbers $\{y_r : r \subseteq s\}$ such that

$$z_{r'} = \sum_{r \subseteq r'} y_r \quad \text{for every } r' \subseteq s.$$  

Indeed, the $y_r$’s can be obtained recurrently as follows:

$$y_0 := z_\emptyset,$$

$$y_{\{b\}} := z_{\{b\}} - y_0 \quad \text{for every } b \in s,$$

$$y_{\{b,b'\}} := z_{\{b,b'\}} - y_0 - y_{\{b\}} - y_{\{b'\}} \quad \text{for every } b,b' \in s \text{ with } b \neq b'.$$
and so on. Since
\[ J(r) = \bigcup_{r \subseteq r' \subseteq s} W(r', s \setminus r') \] for every \( r \subseteq s \)
we conclude that
\[ g = \sum_{r \subseteq s} y_{r'} W(r', s \setminus r') = \sum_{r \subseteq s} \sum_{r' \subseteq s} y_{r'} W(r', s \setminus r') = \sum_{r \subseteq s} y_{r'} \sum_{r' \subseteq s} 1_{W(r', s \setminus r')} \]
and the proof is over. \( \square \)

We denote by \( S'(K) \) the set of all \( g \in S(K) \) which can be written as
\[ g = \sum_{r \subseteq s} y_{r'} j_{(r)} \]
for some finite \( s \subseteq B^+ \) and some collection of nonzero real numbers \( \{y_r : r \subseteq s\} \).

It is easy to check (via Lemma 3.2) that \( S'(K) \) is norm dense in \( S(K) \), and so \( S'(K) \)
is norm dense in \( C(K) \). In Theorem 3.10 we shall prove that the restriction of the supremum norm to \( S'(K) \) is measurable with respect to the trace of \( B(\mathcal{C}(K), W) \) on \( S'(K) \). The proof requires some work, and the first step is to find a substantially larger family of measures to deal with.

**Lemma 3.3.** For each \( T \subseteq B^+ \) and each \( k \in \mathbb{N} \) there is a probability measure \( \mu^k_T \) on \( \mathfrak{A} \) such that, for every finite set \( r \subseteq B^+ \), we have
\[ \mu^k_T(J(r)) = \begin{cases} \lambda(\bigcap_{b \in r} b)^k & \text{if } r \subseteq T, \\ 0 & \text{otherwise.} \end{cases} \]
Moreover, for every finite disjoint sets \( r, s \subseteq B^+ \) we have
\[ \mu^1_T(W(r, s)) = \begin{cases} \lambda(\bigcap_{b \in r} b \setminus \bigcup_{b' \in s \cap T} b') & \text{if } r \subseteq T, \\ 0 & \text{otherwise.} \end{cases} \]
Sometimes we shall write \( \mu_T := \mu^1_T \).

**Proof.** Let \( \mathfrak{B}_k := \mathfrak{B} \otimes \cdots \otimes \mathfrak{B} \) denote the free product of \( k \) many copies of \( \mathfrak{B} \) and let \( \lambda_k \) denote the product measure on \( \mathfrak{B}_k \) (see e.g. [7, 2.25]). Consider the function
\[ \varphi^k_T : B^+ \to \mathfrak{B}_k \]
defined by
\[ \varphi^k_T(b) := \begin{cases} b \otimes \cdots \otimes b & \text{if } b \in T, \\ 0 & \text{otherwise.} \end{cases} \]
Then \( \varphi^k_T \) preserves disjointness, so there is a Boolean homomorphism \( \tilde{\varphi}^k_T : \mathfrak{A} \to \mathfrak{B}_k \)
such that \( \tilde{\varphi}^k_T(G_b) = \varphi^k_T(b) \) for all \( b \in B^+ \) (bear in mind that \( \mathfrak{A} \) is isomorphic to the Boolean algebra freely generated by the \( G_b \)’s modulo the relations that \( W(s, t) = 0 \) if and only if \( s \cap t \neq 0 \) or \( \bigcap_{b \in s} b = 0 \); see Lemma 2.2 and the preceding comments). Now, it is not difficult to check that the formula
\[ \mu^k_T(a) := \lambda_k(\tilde{\varphi}^k_T(a)) \]
defines a probability measure \( \mu^k_T \) on \( \mathfrak{A} \) satisfying the required property. \( \square \)

From the technical point of view, the following subsets of \( S(K) \) will play a relevant role in our proof of Theorem 3.10.
Definition 3.4. Let $D$ be a finite partition of $\mathfrak{B}^+$. We denote by $S_D(K)$ (resp. $S'_D(K)$) the set of all $g \in S(K)$ which can be written as

$$g = \sum_{r \subseteq s} y_r 1_{J(r)}$$

for some finite set $s \subseteq \mathfrak{B}^+$ such that $|T \cap s| \leq 1$ for every $T \in D$ and some collection of real numbers (resp. nonzero real numbers) $\{y_r : r \subseteq s\}$.

Definition 3.5. Let $D$ be a finite partition of $\mathfrak{B}^+$ and $C \subseteq D$. We define:

(i) $T_C := \bigcup_{T \in C} T \subseteq \mathfrak{B}^+$;

(ii) a signed measure $\nu^k_C$ on $\mathfrak{A}$ (for $k = 1, 2$) by

$$\nu^k_C := \sum_{B \subseteq C} (-1)^{|C \setminus B|} \mu^k_{T_B};$$

(iii) a function $\theta_C : C(K) \to \mathbb{R}$ by

$$\theta_C(g) := \frac{(\nu^1_C(g))^2}{\nu^2_C(g)}$$

if $\nu^2_C(g) \neq 0$ and $\theta_C(g) := 0$ otherwise;

(iv) a function $\eta_C : C(K) \to \mathbb{R}$ by

$$\eta_C(g) := \frac{\nu^1_C(g)}{\theta_C(g)}$$

if $\theta_C(g) \neq 0$ and $\eta_C(g) := 0$ otherwise.

Clearly, the functions $\theta_C$ and $\eta_C$ defined above are $\text{Ba}(C(K), w)$-measurable. The following lemma collects several useful properties of such functions.

Lemma 3.6. Let $D$ be a finite partition of $\mathfrak{B}^+$ and let $g \in S_D(K)$ be as in Definition 3.4. Fix $r \subseteq s$ and $k \in \{1, 2\}$. Writing

$$C_r := \{T \in D : T \cap r \neq 0\},$$

the following statements hold:

(i) If $C \subseteq C_r$, then $\nu^k_C(J(r)) = \mu^k_{T_C}(J(r))$.

(ii) If $C_r \subseteq C \subseteq D$, then $\nu^k_C(J(r)) = 0$.

(iii) $\nu^k_{C_r}(g) = y_r(\lambda(\bigcap_{b \in r} b))^k$.

(iv) If $J(r) \neq 0$, then $\theta_{C_r}(g) = y_r$.

(v) If $J(r) \neq 0$ and $y_r \neq 0$, then $\eta_{C_r}(g) = \lambda(\bigcap_{b \in r} b)$.

Proof. (i) For each $B \subseteq C$ we have $r \not\subseteq T_B$ (because $C \subseteq C_r$), and so $\mu^k_{T_B}(J(r)) = 0$. Hence

$$\nu^k_C(J(r)) = \sum_{B \subseteq C} (-1)^{|C \setminus B|} \mu^k_{T_B}(J(r)) = \mu^k_{T_C}(J(r)).$$
(ii) For each \( B \subseteq C \) with \( C_r \not\subseteq B \) we have \( r \not\subseteq T_B \), and so \( \mu^k_{T_B}(J(r)) = 0 \). On the other hand, given any \( C_r \subseteq B \subseteq C \) we have \( r \subseteq T_{C_r} \subseteq T_B \), and therefore
\[
\mu^k_{T_B}(J(r)) = (\lambda(\bigcap_{b \in r} b))^k.
\]
Hence
\[
\nu^k_C(J(r)) = \sum_{B \subseteq C} (-1)^{|C \setminus B|} \mu^k_{T_B}(J(r)) = \sum_{C_r \subseteq B \subseteq C} (-1)^{|C \setminus B|} \mu^k_{T_B}(J(r))
= \left( \lambda \left( \bigcap_{b \in r} b \right) \right)^k \cdot \sum_{C_r \subseteq B \subseteq C} (-1)^{|C \setminus B|}.
\]

The assumption \( C \neq C_r \) implies
\[
\sum_{C_r \subseteq B \subseteq C} (-1)^{|C \setminus B|} = \sum_{A \subseteq C \setminus C_r} (-1)^{|A|} = 0,
\]
as can be easily checked by induction on \(|C \setminus C_r|\). Therefore, \( \nu^k_C(J(r)) = 0 \).

(iii) Take any \( r' \subseteq s \). Note first that if \( r' \not\subseteq r \), then \( r' \not\subseteq T_B \) for every \( B \subseteq C_r \), hence
\[
\nu^k_{C_r}(J(r')) = \sum_{B \subseteq C_r} (-1)^{|C_r \setminus B|} \mu^k_{T_B}(J(r')) = 0.
\]
On the other hand, if \( r' \subseteq r \), then \( C_r \subseteq C_r \) and (ii) implies \( \nu^k_{C_r}(J(r')) = 0 \). Observe also that \( \nu^k_{C_r}(J(r)) = (\lambda(\bigcap_{b \in r} b))^k \) (by (i)). It follows that
\[
\nu^k_{C_r}(g) = \sum_{r' \subseteq s} y_{r'} \nu^k_{C_r}(J(r')) = y_r \nu^k_{C_r}(J(r)) = y_r \left( \lambda \left( \bigcap_{b \in r} b \right) \right)^k.
\]

(iv) Bearing in mind (iii) and the fact that \( J(r) \neq 0 \), we have \( y_r = 0 \) if and only if \( \nu^2_{C_r}(g) = 0 \). Thus, by the very definition of \( \theta_{C_r} \), the equality \( \theta_{C_r}(g) = y_r \) holds whenever \( y_r = 0 \). On the other hand, if \( y_r \neq 0 \), then
\[
\theta_{C_r}(g) = \frac{\nu^1_{C_r}(g)^2}{\nu^2_{C_r}(g)} = y_r
\]
and
\[
\eta_{C_r}(g) = \frac{\nu^1_{C_r}(g)}{\theta_{C_r}(g)} = \lambda \left( \bigcap_{b \in r} b \right),
\]
which proves (v).

Our next step is to prove that, for any \( Z \subseteq \mathfrak{B}^+ \) and \( p \in \mathbb{N} \), the mapping \( g \mapsto \mu_Z(g^p) \) is measurable with respect to the trace of \( \mathrm{Ba}(C(K), w) \) on \( S'(K) \) (see Lemma 3.9 below). We begin by checking the measurability on subsets of the form \( S'_{D}(K) \).

Lemma 3.7. Let \( Z \subseteq \mathfrak{B}^+ \) be a set, \( D \) a finite partition of \( \mathfrak{B}^+ \) finer than \( \{Z, \mathfrak{B}^+ \setminus Z\} \) and \( p \in \mathbb{N} \). Then the mapping
\[
\phi_{Z,p} : S'_{D}(K) \to \mathbb{R}, \quad \phi_{Z,p}(g) := \mu_Z(g^p),
\]
is measurable with respect to the trace of \( \mathrm{Ba}(C(K), w) \) on \( S'_{D}(K) \).
Proof. Write \( D(Z) := \{ T \in D : T \subseteq Z \} \) and let \( \Lambda \) be the set of all \( \beta = (\beta_C)_{C \subseteq D(Z)} \) such that \( \beta_C \in \mathbb{N} \cup \{0\} \) for all \( C \subseteq D(Z) \) and \( \sum_{C \subseteq D(Z)} \beta_C = p \). We write
\[
\binom{p}{\beta} := \frac{p!}{\prod_{C \subseteq D(Z)} \beta_C!} \quad \text{and} \quad C(\beta) := \bigcup_{C \subseteq D(Z)} C.
\]
To prove the measurability of \( \phi_{Z,p} \) with respect to the trace of \( \text{Ba}(C(K), w) \) on \( S'_D(K) \), it is sufficient to check that, for each \( g \in S'_D(K) \), the following equality holds:
\[
(3.3) \quad \phi_{Z,p}(g) = \sum_{\beta \in \Lambda} \binom{p}{\beta} \left( \prod_{C \subseteq D(Z)} \theta_{\beta_C}^{g}(g) \right) \eta_{C(\beta)}(g).
\]
Step 1. Write
\[
g = \sum_{r \subseteq s} y_r 1_{J(r)},
\]
where \( s \subseteq \mathfrak{B}^+ \) is finite, \( |T \cap s| \leq 1 \) for every \( T \in D \) and \( y_r \in \mathbb{R} \setminus \{0\} \) for all \( r \subseteq s \). Let \( \Delta \) be the set of all \( \delta = (\delta_r)_{r \subseteq s} \) such that \( \delta_r \in \mathbb{N} \cup \{0\} \) for all \( r \subseteq s \) and \( \sum_{r \subseteq s} \delta_r = p \). Writing
\[
\binom{p}{\delta} := \frac{p!}{\prod_{r \subseteq s} \delta_r!} \quad \text{and} \quad r(\delta) := \bigcup_{r \subseteq s} r,
\]
we have
\[
g^p = \sum_{\delta \in \Delta} \binom{p}{\delta} \left( \prod_{r \subseteq s} y_r^{\delta_r} \right) 1_{J(r(\delta))},
\]
and so
\[
(3.4) \quad \phi_{Z,p}(g) = \mu_Z(g^p) = \sum_{\delta \in \Delta} \binom{p}{\delta} \left( \prod_{r \subseteq s} y_r^{\delta_r} \right) \mu_Z(J(r(\delta))).
\]
Step 2. Let \( \delta \in \Delta \) such that \( \mu_Z(J(r(\delta))) \neq 0 \). For any \( r \subseteq s \) with \( \delta_r > 0 \) we have \( J(r) \supseteq J(r(\delta)) \). Hence \( \mu_Z(J(r)) \neq 0 \) (in particular, \( J(r) \neq 0 \)), and so \( r \subseteq Z \), which implies that \( C_r = \{ T \in D : T \cap r \neq \emptyset \} \subseteq D(Z) \). Set \( \beta = (\beta_C)_{C \subseteq D(Z)} \) by declaring
\[
(3.5) \quad \beta_C := \begin{cases} 
\delta_r & \text{if } C = C_r \text{ for some } r \subseteq s \text{ with } \delta_r > 0, \\
0 & \text{otherwise}.
\end{cases}
\]
Then \( \sum_{C \subseteq D(Z)} \beta_C = \sum_{r \subseteq s} \delta_r = p \), so that \( \beta \in \Lambda \). Moreover, we have
\[
C(\beta) = \bigcup_{C \subseteq D(Z)} C = \bigcup_{r \subseteq s} C_r = C_r(\delta).
\]
Since \( \mu_Z(J(r(\delta))) \neq 0 \), we have \( J(r(\delta)) \neq 0 \) and \( \mu_Z(J(r(\delta))) = \lambda(\bigcap_{r \subseteq s} \delta_r)^b \). Thus, from Lemma 3.6(v) it follows that \( \eta_{C(\beta)}(g) = \eta_{C_0(\delta)}(g) = \mu_Z(J(\delta))) \). On the
other hand, for each $r \subseteq s$ with $\delta_r > 0$ we have $y_r = \theta_{C_r}(g)$ by Lemma 3.6(iv), hence

$$\prod_{C \subseteq D(Z)} \theta_{\beta C}^{\beta}(g) = \prod_{r \subseteq s} \theta_{\delta_r}^{\delta_r}(g) = \prod_{r \subseteq s} y_{r}^{\delta_r}.$$  

Therefore

$$\left(\frac{p}{\beta}\right) \left(\prod_{C \subseteq D(Z)} \theta_{\beta C}^{\beta}(g)\right) \eta_{C(\beta)}(g) = \left(\frac{p}{\delta}\right) \left(\prod_{r \subseteq s} y_r^{\delta_r}\right) \mu_Z(J(r(\delta))).$$

This shows that each nonzero summand of (3.3) can be written as a summand of (3.4). Note also that if $\delta' \in \Delta$ satisfies $\mu_Z(J(r(\delta'))) \neq 0$ and we define $\beta' = (\beta_C')_{C \subseteq D(Z)} \in \Lambda$ as in (3.3) (with $\delta$ replaced by $\delta'$), then $\beta \neq \beta'$ whenever $\delta \neq \delta'$.

Step 3. Let $\beta \in \Lambda$ such that

$$\eta_{C(\beta)}(g) \neq 0.$$  

Fix $C \subseteq D(Z)$ with $\beta_C > 0$. We claim that $r_C := T_C \cap s$ satisfies $C_{r_C} = C$. Indeed, the inclusion $C_{r_C} \subseteq C$ is clear. To prove the reverse inclusion, we argue by contradiction. Suppose $C_{r_C} \not\subseteq C$. By Lemma 3.6(ii), we have $\nu_{C}^{1}(J(r')) = 0$ whenever $r' \subset T_C$. Since we also have $\nu_{C}^{1}(J(r')) = 0$ for every $r' \subseteq s$ with $r' \not\subseteq T_C$ (by the very definition of $\nu_{C}^{1}$), it follows that $\nu_{C}^{1}(g) = \sum_{r' \subseteq s} y_{r'} \nu_{C}^{1}(J(r')) = 0$, hence $\theta_{C}(g) = 0$, which contradicts (3.6). This shows that $C_{r_C} = C$, as claimed. Now Lemma 3.6(iii) ensures that

$$\nu_{C}^{1}(g) = \nu_{C_{r_c}}^{1}(g) = y_{r_C} \lambda\left(\bigcap_{b \in r_C} b\right).$$

Since $\nu_{C}^{1}(g) \neq 0$, the previous equality implies that $J(r_C) \neq 0$. From Lemma 3.6(iv) it follows that $y_{r_C} = \theta_{C_{r_C}}(g) = \theta_{C}(g)$.

Set $\delta = (\delta_r)_{r \subseteq s}$ by declaring

$$\delta_r := \begin{cases} \beta_C & \text{if } r = r_C \text{ for some } C \subseteq D(Z) \text{ with } \beta_C > 0, \\ 0 & \text{otherwise}. \end{cases}$$  

Then $\sum_{r \subseteq s} \delta_r = \sum_{C \subseteq D(Z)} \beta_C = p$, and hence $\delta \in \Delta$. From our previous considerations we deduce that

$$\prod_{C \subseteq D(Z)} \theta_{\beta C}^{\beta}(g) = \prod_{C \subseteq D(Z)} \theta_{\delta}^{\delta}(g) = \prod_{r \subseteq s} y_{r}^{\delta_r}.$$  

Moreover, we claim that $\eta_{C(\beta)}(g) = \mu_Z(J(r(\delta)))$. Indeed, since

$$C(\beta) = \bigcup_{C \subseteq D(Z)} C = \bigcup_{C \subseteq D(Z)} C_{r_C} = \bigcup_{r \subseteq s} C_{r} = C(r(\delta),$$

we have $\eta_{C(\beta)}(g) = \eta_{C_{r} (\delta)}(g)$, and so (3.6) implies that $\nu_{C_{r} (\delta)}^{1}(g) \neq 0$. Bearing in mind Lemma 3.6(iii) and the fact that $y_{r(\delta)} \neq 0$, we infer that $J(r(\delta)) \neq 0$. An
appeal to Lemma \ref{lem:partition} now yields $\eta_C(\beta)(g) = \lambda(\cap_{b \in r(\delta)} b)$. On the other hand, the fact that

$$r(\delta) = \bigcup_{r \subseteq s, \delta_c > 0} r = \bigcup_{\delta_c > 0} r_C = \bigcup_{\delta_c > 0} T_C \cap s \subseteq Z$$

ensures that $\mu_Z(J(r(\delta))) = \lambda(\cap_{b \in r(\delta)} b)$. It follows that $\eta_C(\beta)(g) = \mu_Z(J(r(\delta)))$.

Therefore

$$\left( \begin{array}{c} p \\ \beta \end{array} \right) \left( \prod_{r \subseteq s, \delta_c > 0} \theta^{\beta_C}(g) \right) \eta_C(\beta)(g) = \left( \begin{array}{c} p \\ \delta \end{array} \right) \left( \prod_{r \subseteq s, \delta_r > 0} y^{\delta_r} \right) \mu_Z(J(r(\delta))).$$

This shows that each nonzero summand of \ref{eq:summand} can be written as a summand of \ref{eq:summand}. Note that if $\beta' \in \Lambda$ satisfies \ref{eq:beta} (with $\beta$ replaced by $\beta'$) and we define $\delta' = (\delta'_r)_{r \subseteq s} \in \Delta$ as in \ref{eq:delta}, (with $\beta$ replaced by $\beta'$), then $\delta \neq \delta'$ whenever $\beta \neq \beta'$.

Thus, equality \ref{eq:summand} holds true and the proof is over. \hfill $\square$

The following folklore fact will allow us to prove Lemma \ref{lem:partition} as an easy consequence of Lemma \ref{lem:trace} above.

**Remark 3.8.** Let $(\Omega, \Sigma)$ be a measurable space. Write $\Omega = \bigcup_{j \in \mathbb{N}} \Omega_j$, where $\Omega_j \subseteq \Omega_{j+1}$ for all $j \in \mathbb{N}$. Let $A \subseteq \Omega$ be a set such that, for each $j \in \mathbb{N}$, the intersection $A \cap \Omega_j$ belongs to the trace of $\Sigma$ on $\Omega_j$. Then $A \in \Sigma$.

**Proof.** For each $j \in \mathbb{N}$ we have $A \cap \Omega_j = E_j \cap \Omega_j$ for some $E_j \in \Sigma$. We claim that $A = \bigcup_{k \in \mathbb{N}} \bigcap_{j \geq k} E_j$. Indeed, we have

$$A = \bigcup_{k \in \mathbb{N}} \bigcap_{j \geq k} A \cap \Omega_k = \bigcup_{k \in \mathbb{N}} \bigcap_{j \geq k} A \cap \Omega_j = \bigcup_{k \in \mathbb{N}} \bigcap_{j \geq k} E_j \cap \Omega_j \subseteq \bigcup_{k \in \mathbb{N}} \bigcap_{j \geq k} E_j.$$

On the other hand, for each $k \in \mathbb{N}$ we have

$$\bigcap_{j \geq k} E_j = \bigcap_{n \geq k} \bigcap_{j \geq k} E_j \cap \Omega_n \subseteq \bigcup_{n \geq k} E_n \cap \Omega_n = \bigcup_{n \geq k} A \cap \Omega_n = A.$$ 

It follows that $A = \bigcup_{k \in \mathbb{N}} \bigcap_{j \geq k} E_j \in \Sigma$. \hfill $\square$

**Lemma 3.9.** Let $Z \subseteq \mathfrak{B}^+$ be a set and $p \in \mathbb{N}$. Then the mapping

$$\phi_{Z,p} : S'(K) \to \mathbb{R}, \quad \phi_{Z,p}(g) := \mu_Z(g^p)$$

is measurable with respect to the trace of $\text{Ba}(C(K), w)$ on $S'(K)$.

**Proof.** Since $|\mathfrak{B}^+| = \mathfrak{c}$, there is a sequence $D(1), D(2), \ldots$ of finite partitions of $\mathfrak{B}^+$, each one being finer than $\{Z, \mathfrak{B}^+ \setminus Z\}$, such that:

- $D(j + 1)$ is finer than $D(j)$ for all $j \in \mathbb{N}$;
- for every $s \subseteq \mathfrak{B}^+$ finite there is $j \in \mathbb{N}$ such that $|T \cap s| \leq 1$ for all $T \in D(j)$.

Indeed, let $\xi : \{0,1\}^\mathbb{N} \to \mathfrak{B}^+$ be any bijection and, for each $j \in \mathbb{N}$ and $\sigma \in \{0,1\}^j$, set

$$E^\sigma_j := \{x \in \{0,1\}^\mathbb{N} : x(i) = \sigma(i) \text{ for every } i = 1, \ldots, j\}.$$

Then the partitions

$$D(j) := \{\xi(E^\sigma_j) \cap Z : \sigma \in \{0,1\}^j\} \cup \{\xi(E^\sigma_j) \setminus Z : \sigma \in \{0,1\}^j\}, \quad j \in \mathbb{N},$$

fulfill the required properties.
Clearly, \( S'(K) = \bigcup_{j \in \mathbb{N}} S'_{D(j)}(K) \) and \( S'_{D(j)}(K) \subseteq S'_{D(j+1)}(K) \) for all \( j \in \mathbb{N} \). The measurability of \( \phi_Z \), with respect to the trace of \( \mathrm{Ba}(C(K), w) \) on \( S'(K) \) now follows from Lemma 3.7 and Remark 3.8.

We have already gathered all the tools needed to prove the main result of this subsection:

**Theorem 3.10.** The restriction of the supremum norm to \( S'(K) \) is measurable with respect to the trace of \( \mathrm{Ba}(C(K), w) \) on \( S'(K) \).

**Proof.** We fix a countable algebra \( Z \) on \( \mathcal{B}^+ \) which separates the points of \( \mathcal{B}^+ \) (the algebra of clopen subsets of \( \{0,1\}^\mathbb{N} \) can be transferred to \( \mathcal{B}^+ \) via any bijection between \( \{0,1\}^\mathbb{N} \) and \( \mathcal{B}^+ \)). We claim that

\[
(3.8) \quad \|g\| = \sup_{Z \in \mathcal{Z}} \limsup_{p \to \infty} \left( \mu_Z(g^{2p}) \right)^{1/2p} \quad \text{for every } g \in C(K).
\]

Indeed, the inequality “\( \geq \)” is obvious (each \( \mu_Z \) is a probability measure). To verify the reverse inequality, fix \( g \in C(K) \) and take \( \varepsilon > 0 \). By Lemma 2.4(i) there exist finite disjoint sets \( r, s \subseteq \mathcal{B}^+ \) such that \( |g(\mathcal{F})| \geq \|g\| - \varepsilon \) for every \( \mathcal{F} \in \hat{W}(r,s) \neq \emptyset \). Since \( Z \) separates the points of \( \mathcal{B}^+ \), we can find \( Z \in \mathcal{Z} \) such that \( r \subseteq Z \) and \( s \cap Z = \emptyset \), hence \( \mu_Z(W(r,s)) = \lambda(\bigcap_{b \in r} b) > 0 \). Since

\[
\left( \mu_Z(g^{2p}) \right)^{1/2p} \geq \left( \|g\| - \varepsilon \right) \left( \mu_Z(W(r,s)) \right)^{1/2p} \quad \text{for every } p \in \mathbb{N},
\]

we have

\[
\limsup_{p \to \infty} \left( \mu_Z(g^{2p}) \right)^{1/2p} \geq \left( \|g\| - \varepsilon \right) \lim_{p \to \infty} \left( \mu_Z(W(r,s)) \right)^{1/2p} = \|g\| - \varepsilon.
\]

As \( \varepsilon > 0 \) is arbitrary, equality (3.8) holds true.

Once we know that \( \|g\| \) is expressed by formula (3.8), the assertion follows from Lemma 3.9.

\[ \square \]

### 3.2. Measurability on the set of simple functions

Any element of \( S(K) \) admits a representation which cannot be simplified in a sense, as the following lemma shows.

**Lemma 3.11.** Let \( g \in S(K) \). Then there exist a finite set \( s \subseteq \mathcal{B}^+ \) and a collection of real numbers \( \{z_r : r \subseteq s\} \) such that:

(i) \( g = \sum_{r \subseteq s} z_r 1_{W(r,s)} \);

(ii) there is no \( s' \subset s \) such that

\[
z_r = z_r' \quad \text{whenever} \quad r \cap s' = r' \cap s', \quad \bigcap_{b \in r} b \neq 0 \quad \text{and} \quad \bigcap_{b \in r'} b \neq 0.
\]

**Proof.** Of course, we can write \( g \) as in (i). To get a representation satisfying (ii), we proceed by induction on \( |s| \). The case \( s = \emptyset \) being obvious, we assume that \( s \neq \emptyset \) and that the induction hypothesis holds. Assume that (ii) fails and fix \( s' \subset s \) such that

\[
z_r = z_r' \quad \text{whenever} \quad r \cap s' = r' \cap s', \quad \bigcap_{b \in r} b \neq 0 \quad \text{and} \quad \bigcap_{b \in r'} b \neq 0.
\]
For any \( t \subseteq s' \) with \( \cap_{b \in L_t} b \neq 0 \), let \( A_t \) be the collection of all \( r \subseteq s \) such that \( r \cap s' = t \) and \( \cap_{b \in L_t} b \neq 0 \). Then \( z_r = z_t \) for every \( r \in A_t \) and \( W(t, s' \setminus t) = \bigcup_{r \in A_t} W(r, s \setminus r) \), as can be easily checked. Hence

\[
g = \sum_{r \subseteq s} z_r 1_{W(r, s \setminus r)} = \sum_{t \subseteq s'} \sum_{r \subseteq s, r \cap s' = t} z_r 1_{W(r, s \setminus r)} = \sum_{t \subseteq s'} \sum_{r \in A_t} \sum_{b \in L_t, b \neq 0} z_r 1_{W(r, s \setminus r)} = \sum_{t \subseteq s'} \sum_{r \in A_t} z_r 1_{W(r, s \setminus r)} = \sum_{t \subseteq s'} z_r 1_{W(t, s' \setminus t)}.
\]

Since \( |s'| < |s| \), the induction hypothesis now ensures that \( g \) admits a representation satisfying both (i) and (ii). \( \square \)

**Definition 3.12.** Let \( D \) be a finite partition of \( \mathcal{B}^+ \). We denote by \( A_D(K) \) the set of all \( g \in S(K) \) which can be written as

\[
g = \sum_{r \subseteq s} z_r 1_{W(r, s \setminus r)}
\]

for some finite set \( s \subseteq \mathcal{B}^+ \) and some collection of real numbers \( \{z_r : r \subseteq s\} \) such that:

- (i) \( |T \cap s| = 1 \) for every \( T \in D \);
- (ii) there is no \( s' \subseteq s \) such that \( z_r = z_{r'} \) whenever \( r \cap s' = r' \cap s' \), \( \cap_{b \in r} b \neq 0 \) and \( \cap_{b \in r'} b \neq 0 \).

Our next step is to prove that the sets \( A_D(K) \) defined above belong to the trace of \( \text{Ba}(C(K), w) \) on \( S(K) \) (see Corollary 3.16 below). From now on we fix a countable algebra \( Z \) on \( \mathcal{B}^+ \) which separates the points of \( \mathcal{B}^+ \) (as in the proof of Theorem 3.10).

**Lemma 3.13.** Let \( D \) be a finite partition of \( \mathcal{B}^+ \) with \( D \subseteq Z \) and let \( g \in A_D(K) \). Then for each \( T_0 \in D \) there is \( T \in Z \) such that \( \mu_T \setminus T_0(g) \neq \mu_T \cup T_0(g) \).

**Proof.** Our proof is by contradiction. Suppose there is \( T_0 \in D \) such that

\[
\mu_T \setminus T_0(g) = \mu_T \cup T_0(g) \quad \text{for every } T \in Z.
\]

Write \( g = \sum_{r \subseteq s} z_r 1_{W(r, s \setminus r)} \) as in Definition 3.12. Let \( s' := s \setminus T_0 \subseteq s \). In order to reach a contradiction, we claim that

\[
\text{Indeed, assume that } r \neq r' \text{ and proceed by induction on } |r \cap s'| = |r' \cap s'|.
\]

Suppose first that \( r \cap s' = r' \cap s' = \emptyset \). Then we have \( r = \emptyset \) and \( r' = T_0 \cap s \) (or vice versa). By (3.9) we have \( \mu_0(g) = \mu_{T_0}(g) \). Note that \( \mu_0(g) = z_0 \) and, writing \( T_0 \cap s = \{b_0\} \), we have \( \mu_{T_0}(g) = z_0(1 - \lambda(b_0)) + z_{T_0 \cap s} \lambda(b_0) \). It follows that

\[
z_0 = z_0(1 - \lambda(b_0)) + z_{T_0 \cap s} \lambda(b_0),
\]

hence \( z_0 = z_{T_0 \cap s} \), as required.
Suppose now that \( r \cap s' = r' \cap s' \neq \emptyset \), together with \( \bigcap_{b \in r} b \neq \emptyset \neq \bigcap_{b \in r'} b \) and the inductive hypothesis. Since \( r \neq r' \), we have either \( b_0 \in r \) and \( b_0 \notin r' \) or vice versa. We assume for instance that \( b_0 \in r \) and \( b_0 \notin r' \). Then
\[
\begin{align*}
r &= \{b_0\} \cup (r \cap s') \quad \text{and} \quad r' = r \cap s'.
\end{align*}
\]
By the inductive hypothesis,
\[
(3.10) \quad z_{r_0} = z_{r_0 \cup \{b_0\}} \quad \text{for every} \quad r_0 \subsetneq r'.
\]
Set
\[
T_1 := T_0 \cup \bigcup \{T \in D : T \cap s \subseteq r'\} \in \mathcal{Z}
\]
and observe that \( T_1 \cap s = r \). Writing
\[
w(t, t') := \bigcap_{b \in t} b \setminus \bigcup_{b' \in t'} b' \in \mathcal{B}
\]
for any pair of finite sets \( t, t' \subseteq \mathcal{B}^+ \), we have
\[
(3.11) \quad \mu_{T_1}(g) = \sum_{r_0 \subseteq r} z_{r_0} \lambda(w(r_0, r \setminus r_0)) = \sum_{r_0 \subseteq r'} z_{r_0} \lambda(w(r_0, r' \setminus r_0))
\]
\[
+ \sum_{r_0 \subseteq r'} z_{r_0} \lambda(w(r_0, r' \setminus r)),
\]
For each \( r_0 \subseteq r' \), the elements \( w(r_0, r \setminus r_0) \) and \( w(r_0 \cup \{b_0\}, r' \setminus r_0) \) are disjoint and their union is \( w(r_0, r' \setminus r_0) \), hence \((3.10)\) yields
\[
(3.12) \quad \mu_{T_1}(g) = z_{r'} \lambda(w(r', \{b_0\})) + z_r \lambda(w(r, \emptyset)) + \sum_{r_0 \subseteq r'} z_{r_0} \lambda(w(r_0, r' \setminus r_0)).
\]
Bearing in mind that \((T_1 \setminus T_0) \cap s = r'\), we also have
\[
(3.13) \quad \mu_{T_1 \setminus T_0}(g) = \sum_{r_0 \subseteq r'} z_{r_0} \lambda(w(r_0, r' \setminus r_0)) = z_{r'} \lambda(w(r', \emptyset)) + \sum_{r_0 \subseteq r'} z_{r_0} \lambda(w(r_0, r' \setminus r_0)),
\]
Since \( \mu_{T_1 \setminus T_0}(g) = \mu_{T_1}(g) \) (by \((3.9)\)) and \((3.12)\) and \((3.13)\) yield
\[
(3.14) \quad z_{r'} \lambda(w(r', \{b_0\})) + z_r \lambda(w(r, \emptyset)) = z_{r'} \lambda(w(r', \emptyset)),
\]
therefore \( z_r \lambda(\bigcap_{b \in r} b) = z_{r'} \lambda(\bigcap_{b \in r'} b) \), and so \( z_r = z_{r'} \). This finishes the proof. \( \square \)

**Remark 3.14.** Let \( g \in S(K) \) be written as \( g = \sum_{s \subseteq s} z_r 1_{\{W_s(r, s)\}} \) for some finite set \( s \subseteq \mathcal{B}^+ \) and \( z_r \in \mathbb{R} \). If \( T, T' \subseteq \mathcal{B}^+ \) satisfy \( T \cap s = T' \cap s \), then \( \mu_T(g) = \mu_{T'}(g) \).

**Proof.** For every \( r \subseteq s \) we have \( r \subseteq T \) if and only if \( r \subseteq T' \). In this case,
\[
\mu_T(W(r, s)) = \lambda \left( \bigcap_{b \in r} b \setminus \bigcap_{b \in s \cap T} b \right) = \mu_{T'}(W(r, s)).
\]
Hence
\[
\mu_T(g) = \sum_{r \subseteq T} z_r \mu_T(W(r, s)) = \sum_{r \subseteq T'} z_r \mu_{T'}(W(r, s)) = \mu_{T'}(g). \quad \square
\]
Lemma 3.15. Let $D$ be a finite partition of $\mathfrak{B}^+$ with $D \subseteq \mathcal{Z}$ and let $g \in S(K)$. Then $g \in A_D(K)$ if and only if the following two statements hold:

(⋆) for each $T_0 \in D$ there is $T \in \mathcal{Z}$ such that $\mu_{T \setminus T_0}(g) \neq \mu_{T \cup T_0}(g)$;

(⋆⋆) for each $T_0 \in D$ and each finite partition $D_0$ of $T_0$ with $D_0 \subseteq \mathcal{Z}$, there is $Z_0 \in D_0$ such that $\mu_{T \setminus Z}(g) = \mu_{T \cup Z}(g)$ for every $Z \in D_0 \setminus \{Z_0\}$ and $T \in \mathcal{Z}$.

Proof. “Only if” part. Suppose $g \in A_D(K)$ and write $g = \sum_{r \subseteq s} z_r 1_{W_{(r,s \setminus r)}}$ as in Definition 3.12. Statement (⋆) holds by Lemma 3.13. To check (⋆⋆), take $T_0 \in D$ and fix a finite partition $D_0$ of $T_0$ with $D_0 \subseteq \mathcal{Z}$. Since $T_0 \cap s = \emptyset$ for every $Z \in D_0 \setminus \{Z_0\}$, and so for any $T \in \mathcal{Z}$ we have $(T \setminus Z) \cap s = (T \cup Z) \cap s$, hence $\mu_{T \setminus Z}(g) = \mu_{T \cup Z}(g)$ (Remark 3.14).

“If” part. Write $g = \sum_{r \subseteq s} z_r 1_{W_{(r,s \setminus r)}}$ for some finite set $s \subseteq \mathfrak{B}^+$ and some collection of real numbers $\{z_r : r \subseteq s\}$. By Lemma 3.11 this representation can be chosen in such a way that there is no $s' \subseteq s$ such that $z_r = z_{r'}$ whenever $r \cap s' = r' \cap s'$, $\bigcap_{b \in r} b \neq 0$ and $\bigcap_{b \in r'} b \neq 0$.

In order to prove that $g \in A_D(K)$ we only have to check that $|T_0 \cap s| = 1$ for every $T_0 \in D$. Observe first that, for each $T_0 \in D$, condition (⋆) tells us that there is $T \in \mathcal{Z}$ such that $\mu_{T \setminus T_0}(g) \neq \mu_{T \cup T_0}(g)$, hence $(T \setminus T_0) \cap s \neq (T \cup T_0) \cap s$ (Remark 3.14) and so $T_0 \cap s = \emptyset$. Thus, we can find a finite partition $D' \subseteq \mathfrak{B}^+$ finer than $D$ such that $|T' \cap s| = 1$ for every $T' \in D'$. Therefore, $g \in A_D(K)$.

Fix $T_0 \in D$ and set $D_0 := \{T' \in D' : T' \subseteq T_0\}$. By Lemma 3.13 applied to $g$ and $D'$, for each $T' \in D_0$ there is $T'' \in \mathcal{Z}$ such that $\mu_{T'' \setminus T'}(g) \neq \mu_{T'' \cup T'}(g)$. This fact and condition (⋆⋆) yield $|D_0| = 1$, that is, $D_0 = \{T_0\}$, and so $|T_0 \cap s| = 1$. As $T_0 \in D$ is arbitrary, $g \in A_D(K)$ and the proof is over.

Corollary 3.16. Let $D$ be a finite partition of $\mathfrak{B}^+$. Then $A_D(K)$ belongs to the trace of $\text{Ba}(C(K), w)$ on $S(K)$.

Proof. We can assume without loss of generality (by enlarging $\mathcal{Z}$ if necessary) that $D \subseteq \mathcal{Z}$. Since $\mathcal{Z}$ is countable, Lemma 3.15 gives the result. □

Our next task is to prove that, under the assumption that $c$ is a Kunen cardinal, the restriction of the supremum norm to any set of the form $A_D(K)$ is relatively $\text{Ba}(C(K), w)$-measurable (Lemma 3.21).

Lemma 3.17. Let $\Phi : \mathfrak{B}^+ \to 2^\mathcal{Z}$ be the mapping defined by $\Phi(b) := (1_T(b))_{T \in \mathcal{Z}}$. Set $\Omega := \Phi(\mathfrak{B}^+)$ and let $\Sigma$ be the trace of $\text{Borel}(2^\mathcal{Z})$ on $\Omega$. Then for each $n \in \mathbb{N}$ the mapping

$$
\Phi^n : ((\mathfrak{B}^+)^n, \otimes_n \sigma(\mathcal{Z})) \to (\Omega^n, \otimes_n \Sigma), \quad \Phi^n(b_1, \ldots, b_n) := (\Phi(b_1), \ldots, \Phi(b_n))
$$

is an isomorphism of measurable spaces.

Proof. It suffices to prove the case $n = 1$. Clearly, $\Phi$ is one-to-one (because $\mathcal{Z}$ separates the points of $\mathfrak{B}^+$) and $\sigma(\mathcal{Z})$-$\Sigma$-measurable. On the other hand, for each $T_0 \in \mathcal{Z}$ we have

$$
\Phi(T_0) = \{(1_T(b))_{T \in \mathcal{Z}} : b \in T_0\} = \Omega \cap \{(x_T)_{T \in \mathcal{Z}} \in 2^\mathcal{Z} : x_{T_0} = 1\} \in \Sigma,
$$

hence $\Phi^{-1}$ is $\Sigma$-$\sigma(\mathcal{Z})$-measurable. □
Lemma 3.18. Let $D$ be a finite partition of $\mathcal{B}^+$ with $D \subseteq \mathcal{Z}$, and let $T_0 \in D$ and $T \in \mathcal{Z}$. Let $g \in A_D(K)$. Write $g = \sum_{r \leq s} z_r 1_{W(r,s \setminus r)}$ as in Definition 3.12 and $T_0 \cap s = \{b_0\}$.

1. The following statements are equivalent:
   (i) $b_0 \in T$;
   (ii) $\mu_T(g) = \mu_{T \cup T_0}(g)$ for every $T \in \mathcal{Z}$ such that $T \cap T_0 = T \cap T_0$.
2. The following statements are equivalent:
   (i') $b_0 \not\in T$;
   (ii') $\mu_T(g) = \mu_{T \setminus T_0}(g)$ for every $T \in \mathcal{Z}$ such that $T \cap T_0 = T \cap T_0$.

Proof. (i)⇒(ii) Let $T \in \mathcal{Z}$ be such that $T \cap T_0 = T \cap T_0$. Since $b_0 \in T$ by assumption, we have $b_0 \in T \cap T_0 \subseteq T$, and so $(T \cup T_0) \cap s = T \cap s$. Bearing in mind Remark 3.14, we get $\mu_T(g) = \mu_{T \cup T_0}(g)$. A similar argument yields (i')⇒(ii').

Now, in order to prove (ii)⇒(i) and (ii')⇒(i'), it is enough to check that statements (ii) and (ii') cannot hold simultaneously. To this end, pick $T^* \in \mathcal{Z}$ such that $\mu_{T^* \cup T_0}(g) \neq \mu_{T \cup T_0}(g)$ (we apply Lemma 3.13) and set

$$
\bar{T} := (T^* \setminus T_0) \cup (T \cap T_0) \in \mathcal{Z}.
$$

Clearly, $T \cup T_0 = T^* \cup T_0$ and $T \cap T_0 = T^* \cap T_0$, hence we have either $\mu_{T \cup T_0}(g) \neq \mu_T(g)$ or $\mu_{T^* \cup T_0}(g) \neq \mu_T(g)$. Since $T \cap T_0 = T \cap T_0$, this shows that either (ii) or (ii') fails.

Remark 3.19. Let $D = \{T_1, \ldots, T_n\}$ be a finite partition of $\mathcal{B}^+$ with $D \subseteq \mathcal{Z}$.

1. Let $i \in \{1, \ldots, n\}$. Given $T \in \mathcal{Z}$, since statements (ii) and (ii') in Lemma 3.18 (applied to $T_i$) are independent of the representation of $g \in A_D(K)$, there is a mapping $\psi_{D,T,T_i} : A_D(K) \to \{0, 1\}$ such that, for any $g = \sum_{r \leq s} z_r 1_{W(r,s \setminus r)}$ as in Definition 3.12 and writing $T_i \cap s = \{b_i\}$, we have

$$
\psi_{D,T,T_i}(g) := \begin{cases} 1 & \text{if } b_i \in T, \\ 0 & \text{if } b_i \not\in T. \end{cases}
$$

$\psi_{D,T,T_i}$ is measurable with respect to the trace of $\text{Ba}(C(K), w)$ on $A_D(K)$, thanks to Lemma 3.18 (applied to $T_i$). Define $\psi_{D,T_i} : A_D(K) \to 2^\mathcal{Z}$ by $\psi_{D,T_i}(g) := (\psi_{D,T,T_i}(g))_{T \in \mathcal{Z}}$. Observe that $\psi_{D,T_i}(A_D(K)) \subseteq \Omega$,

because for any $g \in A_D(K)$ as above we have $\psi_{D,T_i}(g) = \Phi(b_i)$.

2. Thus, we can consider the mapping

$$
\psi_D : A_D(K) \to \Omega^n, \quad \psi_D(g) := (\psi_{D,T_1}(g), \ldots, \psi_{D,T_n}(g)).
$$

Clearly, $\psi_D$ is measurable with respect to $\otimes_n \mathcal{S}$ and the trace of $\text{Ba}(C(K), w)$ on $A_D(K)$.

3. Let $P \subseteq \{1, \ldots, n\}$. Define $\zeta_{n,P} : (\mathcal{B}^+)^n \to \mathbb{R}$ by

$$
\zeta_{n,P}(b_1', \ldots, b_n') := \lambda \left( \bigcap_{i \in P} b_i' \right).
$$

Then the mapping $L_{D,P} : A_D(K) \to \mathbb{R}$ given by $L_{D,P} := \zeta_{n,P} \circ (\Phi^n)^{-1} \circ \psi_D$ satisfies $L_{D,P}(g) = \lambda(\bigcap_{i \in P} b_i)$ for every $g \in A_D(K)$ as above.

From now on we deal with the additional assumption that $\kappa$ is a Kunen cardinal.
Lemma 3.20. Suppose \( c \) is a Kunen cardinal. Then there is a countable algebra \( Z_0 \) on \( B^+ \) separating the points of \( B^+ \) such that, for each \( n \in \mathbb{N} \) and \( P \subseteq \{1, \ldots, n\} \), the mapping \( \zeta_{n,P} \) is \( \otimes_n \sigma(Z_0) \)-measurable.

Proof. Since \( |B^+| = c \) is a Kunen cardinal, each \( \zeta_{n,P} \) is \( \otimes_n \mathcal{P}(B^+) \)-measurable. Thus, we can find a countable family \( \mathcal{C} \) of subsets of \( B^+ \) such that \( \zeta_{n,P} \) is \( \otimes_n \sigma(\mathcal{C}) \)-measurable for every \( n \in \mathbb{N} \) and every \( P \subseteq \{1, \ldots, n\} \). Now, it is enough to choose any countable algebra \( Z_0 \) on \( B^+ \) which separates the points of \( B^+ \) and contains \( \mathcal{C} \).

\[ \square \]

Lemma 3.21. Suppose \( c \) is a Kunen cardinal. Let \( D \) be a finite partition of \( B^+ \). Then the restriction of the supremum norm to \( A_D(K) \) is measurable with respect to the trace of \( \text{Ba}(C(K),w) \) on \( A_D(K) \).

Proof. Write \( D = \{T_1, \ldots, T_n\} \). We can suppose without loss of generality (by enlarging \( Z \) if necessary) that \( D \subseteq Z \) and that all functions \( \zeta_{n,P} \) are \( \otimes_n \sigma(Z) \)-measurable (see Lemma 3.20). For each \( P \subseteq \{1, \ldots, n\} \), define

\[ N_{D,P} : A_D(K) \to \{0, 1\}, \quad N_{D,P}(g) := \begin{cases} 1 & \text{if } L_{D,P}(g) \neq 0, \\ 0 & \text{if } L_{D,P}(g) = 0. \end{cases} \]

Since \( L_{D,P} \) is measurable with respect to the trace of \( \text{Ba}(C(K),w) \) on \( A_D(K) \) (combine Lemma 3.17 and Remark 3.19), the same holds for \( N_{D,P} \).

Fix \( g \in A_D(K) \) and write

\[ g = \sum_{r \subseteq s} y_r 1_{\overline{J(r)}} \]

for some \( s \subseteq B^+ \) finite with \( |T_i \cap s| = 1 \) for every \( i \in \{1, \ldots, n\} \) and some collection of real numbers \( \{y_r : r \subseteq s\} \) (see the proof of Lemma 3.2). Lemma 3.6 iv) ensures that

\[ y_r = \theta_{C_r}(g) \quad \text{for every } r \subseteq s \text{ with } J(r) \neq 0. \]

Hence \( g = \sum_{r \subseteq s} \theta_{C_r}(g) 1_{\overline{J(r)}} \), and therefore

\[ g = \sum_{r' \subseteq s} \left( \sum_{r \subseteq r'} \theta_{C_r}(g) \right) 1_{\overline{W(r',s \setminus r')}} \]

(see again the proof of Lemma 3.2). Write \( T_i \cap s = \{b_i\} \) for every \( i \in \{1, \ldots, n\} \) and observe that for each \( P \subseteq \{1, \ldots, n\} \) we have

\[ N_{D,P}(g) = \begin{cases} 1 & \text{if } \bigcap_{i \in P} b_i \neq 0, \\ 0 & \text{if } \bigcap_{i \in P} b_i = 0. \end{cases} \]

Define \( C(Q) := \{T_i : i \in Q\} \) for every \( Q \subseteq \{1, \ldots, n\} \). From (3.14) it follows that

\[ \|g\| = \sup_{r' \subseteq s, J(r') \neq 0} \left| \sum_{r \subseteq r'} \theta_{C_r}(g) \right| = \sup_{P \subseteq \{1, \ldots, n\}} \left| \sum_{Q \subseteq P} \theta_{C(Q)}(g) \right| \cdot N_{D,P}(g). \]

As \( g \in A_D(K) \) is arbitrary, the norm function \( \| \cdot \| \) coincides with

\[ \sup_{P \subseteq \{1, \ldots, n\}} \left| \sum_{Q \subseteq P} \theta_{C(Q)}(\cdot) \right| \cdot N_{D,P}(\cdot) \]
on $A_D(K)$, and so $\| \cdot \|$ is measurable with respect to the trace of $\text{Ba}(C(K), w)$ on $A_D(K)$. The proof is over. \hfill \Box

Finally, we arrive at the main result of this subsection:

**Theorem 3.22.** Suppose $c$ is a Kunen cardinal. Then the restriction of the supremum norm to $S(K)$ is measurable with respect to the trace of $\text{Ba}(C(K), w)$ on $S(K)$.

**Proof.** Let $\Pi$ be the collection of all partitions of $\mathcal{B}^+$ into finitely many elements of $\mathcal{Z}$. By Lemma 3.11, we can write $S(K) = \bigcup_{D \in \Pi} A_D(K)$. Since $\Pi$ is countable and each $A_D(K)$ belongs to the trace of $\text{Ba}(C(K), w)$ on $S(K)$ (see Corollary 3.16), the result follows from Lemma 3.21. \hfill \Box

### 3.3. Some open problems.

(A) Let $L$ be a compact space. Is the $\text{Ba}(C(L), w)$-measurability of the supremum norm on $C(L)$ equivalent to the $w^*$-separability of $B_{C(L)}^*$ or $C(L)^*$? What about the compact space $K$ considered in this paper?

(B) Let $(X, \| \cdot \|)$ be a Banach space and suppose $A \subseteq X$ is a norm dense set (or linear subspace) such that $\| \cdot \|$ is relatively $\text{Ba}(X, w)$-measurable on $A$. Does this imply that $\| \cdot \|$ is $\text{Ba}(X, w)$-measurable on $X$?

Note that the analogous question for the property “$B_X$ is $w^*$-separable” has a positive answer (just apply the Hahn-Banach theorem), while for the property “$X$ is $w^*$-separable” it has a negative answer; see [5, Example 1.1].

(C) Let $L$ be a compact space and consider the ‘square’ mapping

$$S: C(L) \to C(L), \quad S(g) := g^2.$$ 

Which conditions on $L$ ensure that $S$ is $\text{Ba}(C(L), w)$-measurable?

Note that if $L$ carries a strictly positive measure, say $\mu$, then the supremum norm on $C(L)$ can be computed as

$$\|g\| = \lim_{n \to \infty} \left( \int_L f^{2n} \, d\mu \right)^{\frac{1}{2n}},$$

hence $\| \cdot \|$ is $\text{Ba}(C(L), w)$-measurable whenever $S$ is $\text{Ba}(C(L), w)$-measurable. What about the compact space $K$ considered in this paper?

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