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CONFIGURATIONS OF LINES IN DEL PEZZO SURFACES WITH GOSSET POLYTOPES

JAE-HYOUK LEE

Abstract. In this article, we classify and describe the configuration of the divisor classes of del Pezzo surfaces, which are written as the sum of distinct lines with fixed intersection according to combinatorial data in Gosset polytopes.

We introduce the k-Steiner system and cornered simplexes, and characterize the configurations of positive degree $m(\leq 3)$-simplexes with them via monoidal transforms.

Higher dimensional m $(4 \leq m \leq 7)$-simplexes of 1-degree exist in 4_{21} in the Picard group of del Pezzo surface of degree 1, and their configurations are nontrivial. The configurations of 4- and 7-simplexes are related to rulings in S_8, and the configurations of 5- and 6-simplexes correspond to the skew 3-lines and skew 7-lines in S_8. In particular, the seven lines in a 6-simplex produce a Fano plane.

1. Introduction

In this article, we study the configuration of lines in del Pezzo surfaces with fixed intersection along the combinatorial data of the Gosset polytopes whose vertices corresponded to the lines.

A *del Pezzo surface* is a smooth irreducible surface S_r whose anticanonical class $-K_{S_r}$ is ample. Each del Pezzo surface can be constructed by blowing up $r \leq 8$-points from $\mathbb{P}^2$ unless it is $\mathbb{P}^1 \times \mathbb{P}^1$. A *line* in *Pic* S_r is a divisor class l with $l^2 = l \cdot K_{S_r} = -1$, which contains a rational smooth curve in S_r by the adjunction formula. This rational curve is embedded to a line in $\mathbb{P}^{9-r}$ along the embedding given by $|-K_{S_r}|$, when S_r is very ample. The set of lines L_r in *Pic* S_r is finite, and its symmetry group is the Weyl group E_r. In particular, 27 lines on a cubic surface S_6 are well known, and the configuration of these lines was studied along the action of the Weyl group action E_6 [8], [9], [11]. In fact, the set of 27-lines in S_6 are bijective to the set of vertices of a Gosset polytope 2_{21}, and the bijection was applied to study the geometry of 2_{21} by Coxeter [6]. Here, the *Gosset polytopes* $(r-4)_{21}$, $3 \leq r \leq 8$, are the r-dimensional semiregular polytopes discovered by Gosset, and their symmetry groups are the Coxeter group E_r. The vertex figure of $(r-4)_{21}$ is $(r-5)_{21}$ and the facets of $(r-4)_{21}$ are regular $(r-1)$-dimensional simplexes α_{r-1} and $(r-1)$-dimensional crosspolytopes β_{r-1}. But all the lower-dimensional faces are regular simplexes.

The bijection between the set of vertices in $(r-4)_{21}$ and the set of lines L_r is well known and applied in many different research fields [6], [15]. In particular, the classical approach to the configurations of lines in del Pezzo surfaces can be found

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in the study of Du Val [10], and recently another application of the configurations of lines via representation theory was studied by Manivel [16]. In fact, we can derive deeper relationships than the bijection between vertices in $(r-4)_{21}$ and lines in $Pic\ S_r$. In [12] the author showed that the convex hull of L_r in $Pic\ S_r$ is the Gosset polytope $(r-4)_{21}$ and extended the bijection to the correspondences between special divisors (resp. skew m-lines $1 \leq m \leq r$, rulings, and exceptional divisors) in $Pic\ S_r$ and faces (resp. $(m-1)$-simplexes $1 \leq m \leq r$, crosspolytope facets, and $(r-1)$-simplex facets) in $(r-4)_{21}$. As we consider the configurations of lines in $Pic\ S_r$ via the action of the symmetry group E_r, the configurations of lines are studied according to the combinatorial data of the polytope $(r-4)_{21}$, and the above correspondences between basic divisors and faces play important roles in this article.

Main question. In this article, we consider divisor classes of del Pezzo surfaces S_r written as a sum of lines and describe their properties by exploring the configuration of lines consisting of the divisors. As we work with Gosset polytopes and corresponding Weyl actions on $Pic\ S_r$, we also add a condition that lines of each divisor have a common intersection. We define the configuration set $\mathcal{A}^r_{m+1}(b)$ by

$$\mathcal{A}^r_m(b) := \{\{l_1, ..., l_{m+1}\} \mid l_i \in L_r,\ l_i \cdot l_j = b \text{ for } i \neq j\}$$

and its divisor set $\mathcal{D}^r_m(b)$ by

$$\mathcal{D}^r_m(b) := \{l_1 + ... + l_{m+1} \in Pic(S_r) \mid \{l_1, ..., l_{m+1}\} \in \mathcal{A}^r_m(b)\}.$$

In this article, we classify the divisor set $\mathcal{D}^r_m(b)$ and give a complete description of the configuration of $\mathcal{A}^r_m(b)$.

A special case of such divisors is called a ruling, which is a sum of two lines in del Pezzo surfaces with fixed intersections. Rulings are studied in many different directions. As the lines play the generators in Cox rings, the rulings determine the relations for the Cox rings as shown by Batyrev and Popov [1]. Rulings are also applied to the geometry of the line bundles over del Pezzo surfaces via the representation theory by Leung and Zhang [13], [14].

According to the correspondence between lines in $Pic\ S_r$ and vertices of Gosset polytopes $(r-4)_{21}$, each element in $\mathcal{A}^r_m(b)$ corresponds to a set of vertices consisting of an m-simplex. We call the element in $\mathcal{A}^r_m(b)$ an *$A^r_m(b)$-vertex,* and the corresponding m-simplex in $(r-4)_{21}$ is called a *b-degree m-simplex in* $(r-4)_{21}$ or an *$A^r_m(b)$-simplex.* Also, the sum of the line $l_1 + ... + l_{m+1}$ in $\mathcal{D}^r_m(b)$ is called *the center of the $A^r_m(b)$-simplex* or an *$A^r_m(b)$-divisor.* Here the b-degree is the common intersection between vertices. When b is zero, the $A^r_m(0)$-simplexes are honest faces in $(r-4)_{21}$, but for $b > 0$, $A^r_m(b)$-vertices do not produce faces in $(r-4)_{21}$.

$Pic\ S_r$	$(r-4)_{21}$
$\{l_1, ..., l_{m+1}\} \in \mathcal{A}^r_m(b)$, $A^r_m(b)$-vertex	m-simplex
$l_1 + ... + l_{m+1} \in \mathcal{A}^r_m(b)$, $A^r_m(b)$-divisor	center of m-simplex

From the intersection between lines in $Pic\ S_r$, we only consider $0 \leq b \leq 3$.

(1) Cases with $b = 0$ and rulings, i.e. $A^r_1(1)$-simplexes are honest subpolytopes in $(r-4)_{21}$ which are regular simplexes and $(r-1)$-crosspolytopes, respectively. These cases are studied in [12] (also see subsection 2.2) along with the Gosset polytopes.

(2) Cases with $1 \leq b \leq 3$ are the main issues of this article. In particular, the higher degree simplexes, i.e. $b = 2, 3$, exist only for $r = 7, 8$, and especially the

$b = 3$ case appears only for $r = 8$. When $r = 8$ and $b = 1$, $A^8_m(1)$-simplexes exist up to $m = 7$ because the root space of S_8 is 8-dimensional, and in fact 1-degree $A^r_m(1)$-simplexes of $4 \leq m \leq 7$ exist only for $r = 8$.

We characterize the configuration sets into two types, *deterministic* and *indeterministic*. (1) Deterministic cases are when the divisor sets $\mathcal{D}^r_m(b)$ are either sets of a single element or sets correspond to these single element sets by monoidal transforms of del Pezzo surfaces. For these sets with a single element, we introduce the Steiner system and study the deterministic cases with the Steiner system and monoidal transform. (2) Other cases are indeterministic. These cases only appear for 1-degree $A^8_m(1)$-vertices in $r = 8$ with $4 \leq m \leq 7$. These configuration sets are very nontrivial, and we need to provide new combinatorial approaches. Note that for $A^8_3(1)$-vertices in 4_{21}, both deterministic and indeterministic cases appear. These two cases are related to the two E_8-orbits of exceptional systems (divisor class D_t with $D_t^2 = 1$ and $K_{S_r} D_t = -3$) in *Pic* S_8 and are keys to studying indeterministic cases.

A. Deterministic cases: Steiner systems and monoidal transforms. For $A^r_m(1)$-vertices in 2_{21}, 3_{21} and 4_{21} for $m \leq 3$, the configurations of lines in the simplexes can be obtained by k-Steiner systems and monoidal transforms.

A *Steiner system* $S(x, y, z)$ is a type of block design system given by a family of y-element subsets, called blocks, in the z-element total set where each x-element in the total set is contained in exactly one subset of the family. For example, $S(2, 3, 7)$ represents the famous Fano projective plane. Whenever we choose the $(y - 1)$-element subset in a Steiner system $S(x, y, z)$, we can determine the y-th element if the $(y - 1)$-element subset is in a block. This deterministic nature of blocks of Steiner systems can be found in the Steiner triplet in cubic surfaces. Namely, when we consider two lines in a cubic S_6, if these two lines intersect by one, the third line is uniquely determined by considering the Steiner triplet (see also section 3). Even though the system of Steiner triplets may not be an honest Steiner system, we focus on blocks consisted of lines with the common intersection number determining the systems and define $\mathcal{S}(k, S_r)$, *k-Steiner system in S_r*, as a family of subsets of lines L_r where each set of $k - 1$ lines in L_r with constant intersections to each other determines exactly one subset in the family. A typical type of k-Steiner system is found when divisor set $\mathcal{D}^r_{k-1}(b)$ consists of one element. In other words, all blocks in the k-Steiner system correspond to the $A^r_{k-1}(b)$-vertices with a common center. In section 3, we consider k-Steiner systems $\mathcal{S}_A(2, S_7)$, $\mathcal{S}_A(2, S_8)$, $\mathcal{S}_B(3, S_6)$, $\mathcal{S}_B(3, S_8)$, and $\mathcal{S}_C(4, S_7)$, and relate them to $A^7_1(2)$-, $A^8_1(3)$-, $A^6_2(1)$-, $A^8_2(2)$- and $A^7_3(1)$-vertices respectively.

By applying the monoidal transform, we can extend the deterministic nature of the Steiner system to study the configuration of a so-called, *tied* simplex (or an $A^r_m(b)$-vertex). An $A^r_m(b)$-vertex in $(r - 4)_{21}$ is called *tied* if there is a line l in L_r whose vertex figure contains the $A^r_m(b)$-vertex; otherwise it is called *untied*. Here the *vertex figure of* l in $(r-4)_{21}$ is the subset of lines in L_r with 0-intersection with l. Thus a simplex tied by a line l is preserved by the blow down map $\pi^r_l : S_r \to S_{r-1}$ given by the line l. All the $A^8_m(1)$-vertices $(m \geq 4)$ in 4_{21} are untied since there are no $A^r_m(1)$-vertices in 2_{21}, 3_{21}. When $m = 3$, $A^8_3(1)$-vertices in 4_{21} can be either tied or untied. This is the major reason why the configuration of lines in *Pic* S_8 is nontrivial. Here, the centers of $A^8_3(1)$-vertices in 4_{21} are bijectively related to the exceptional system in *Pic* S_8 which consists of two Weyl orbits. The centers of tied (resp. untied) $A_3^8(1)$-vertices correspond to a Weyl orbit of roots (skew 8-lines) in S_8 (Proposition 8, Corollaries 9, 10).

Typical examples of deterministic cases with monoidal transforms are $A_2^r(1)$-vertices. The $A_2^r(1)$-vertices exist for $r = 6, 7, 8$. Since all the $A_2^6(1)$-vertices in 2_{21} share a center, the configuration of $A_2^6(1)$-vertices in 2_{21} is given by $\mathcal{S}_B(3, S_6)$. The set of $A_2^7(1)$ (resp. $A_2^8(1)$)-vertices is equivalently the set of lines in $Pic\ S_7$ (resp. skew 2-lines in $Pic\ S_8$), and the configuration of $A_2^7(1)$ (resp. $A_2^8(1)$)-vertices of a common center is also given by $\mathcal{S}_B(3, S_6)$ along the monoidal transform (Theorem 5).

Similarly, the configurations of $A_3^7(1)$ are determined by $\mathcal{S}_C(4, S_7)$ (Theorem 12), and the configuration of tied $A_3^8(1)$-vertices of a fixed center is also given by $\mathcal{S}_C(4, S_7)$ along the monoidal transform. On the other hand, the set of untied $A_3^8(1)$-vertices is equivalent to the set of a 7-simplex in 4_{21}, and the configuration of untied $A_3^8(1)$-vertices of a fixed center is obtained as the set of 4-skew edges in a 7-simplex in 4_{21} (Proposition 14).

The configurations of $A_m^r(b)$-vertices in $(r-4)_{21}$ in degree $b > 1$ are also obtained by the k-Steiner system and the monoidal transforms.

B. Indeterministic $A_m^8(1)$-vertices in 4_{21}. The $A_m^r(1)$-vertices for $4 \leq m \leq 7$ exist only in 4_{21}. As $A_m^8(1)$-vertices $(4 \leq m \leq 7)$ are untied, the configuration for these is far from being uniform. But from the fact that the untied $A_3^8(1)$-vertices are related to the 7-simplexes in 4_{21}, we figure out that the $A_5^8(1)$ (resp.$A_6^8(1)$)-vertices are related to skew 3-lines (resp. skew 7-lines), and the $A_4^8(1)-$ and $A_7^8(1)$-vertices are related to rulings in $Pic\ S_8$. These are the bottom lines of indeterministic cases.

(1) **(**$m = 4$**)** An $A_4^8(1)$-vertex can be obtained by adding the proper line l to an $A_3^8(1)$-vertex (Theorem 20). Here, if the $A_3^8(1)$-vertex is chosen to be tied, the $A_3^8(1)$-vertex corresponds to a root, equivalently a line l in $Pic\ S_8$. Furthermore, these two lines l and l' form a ruling in $Pic\ S_8$. In fact, it turns out that this is the general characterization of an $A_4^8(1)$-vertex. Thus for an $A_4^8(1)$-divisor D_5 of $A_4^8(1)$-vertex, there is a unique line l_{D_5} in L_8 and the $A_3^8(1)$-divisor A_{D_5} of a tied $A_3^8(1)$-vertex such that $D_5 = A_{D_5} + l_{D_5}^{S_8}$. Furthermore, the set $\mathcal{D}_4^8(1)$ of an $A_4^8(1)$-divisor in 4_{21} is bijective to the following set of ordered pairs of lines with 1-intersection defined as

$$\tilde{F}_8 := \{(l_1, l_2) \mid l_1, l_2 \in L_8 \text{ with } l_1 \cdot l_2 = 1\}.$$

For the configuration set $\mathcal{A}_4^8(1)$, all the $A_4^8(1)$-vertices with a common center D_5 in 4_{21} share a common line l_{D_5} and the common $A_3^8(1)$-divisor $D_5 - l_{D_5}$ which is the common center of the uniquely determined tied $A_3^8(1)$-vertex in each $A_4^8(1)$-vertex (Theorems 22, 21).

(2) **(**$m = 5$**)** The set of $A_5^8(1)$-divisors of $A_5^8(1)$-vertices is equivalent to the set of skew 3-lines in $Pic\ S_8$. As a skew 3-line is given by a unique triple of lines with 0-intersection, an $A_5^8(1)$-vertex induces a unique triple of tied $A_3^8(1)$-vertices in it (Lemma 24). This characterization gives the configuration of $A_5^8(1)$-vertices (Theorem 25).

(3) **(**$m = 6$**)** Just like $A_5^8(1)$-vertices, each center of an $A_6^8(1)$-vertex gives a skew 7-line in 4_{21} which is given by the unique choice of seven lines with 0-intersection, and this uniqueness is the key to studying the configuration of $A_6^8(1)$-vertices in 4_{21} (Theorem 27). Thus for each $A_6^8(1)$-vertex, there are seven tied $A_3^8(1)$-vertices in it (Lemma 26). Furthermore, for each tied $A_3^8(1)$-vertex in an $A_6^8(1)$-vertex, the remaining three lines form an $A^8_2(1)$-vertex which is not contained in any tied $A^8_3(1)$-vertex in the 6-simplex (Proposition 28). We call the triplet of lines a *Fano block* and show that each $A^8_6(1)$-vertex contains seven Fano blocks. Moreover, the seven lines in an $A^8_6(1)$-vertex and its Fano blocks produce a Steiner system $S(2,3,7)$ which is known as *Fano plane* (Theorem 29).

(4) ($m = 7$) The configuration of $A^8_7(1)$-vertices in 4_{21} is similar to that of $A^8_4(1)$-vertices. Here each center of $A^8_7(1)$-vertices gives a ruling in 4_{21}. Furthermore, each center of $A^8_7(1)$-vertices can be written as the sum of two centers of tied $A^8_3(1)$-vertices where they are in the vertex figures of the antipodal pair of lines in the corresponding ruling (Theorem 30).

The classification of $A^r_k(b)$-divisors and configuration of $A^r_k(b)$-vertices are summarized as follows.

TABLE 1

$A^r_k(b)$	Configuration
$A^r_k(0)$, $0 \le k \le r-1$	$(k-1)$-simplexes in$(r-4)_{21}$, skew k-lines
$A^r_1(1)$	crosspolytopes, rulings
$A^6_2(1), A^7_2(1), A^8_2(1)$	$\mathcal{S}_B(3, S_6)$, Theorem 5
$A^7_3(1), A^8_3(1)$(tied)	$\mathcal{S}_C(4, S_7)$, Proposition 14
$A^8_3(1)$(untied)	7-simplexes in 4_{21},Theorem 20
$A^8_4(1)$	rulings, Theorems 21,22
$A^8_5(1)$	skew 3-lines, Lemma 24, Theorem25
$A^8_6(1)$	skew 7-lines, Fano block, Theorem 27
$A^8_7(1)$	rulings, Theorem 30
$A^7_1(2)$	$\mathcal{S}_A(2, S_7)$
$A^8_2(2)$	$\mathcal{S}_B(3, S_8)$
$A^8_1(3)$	$\mathcal{S}_A(2, S_8)$

2. PRELIMINARIES

2.1. **Regular polytopes and Gosset polytopes.** In this subsection, we review the general theory on regular polytopes that we use in this article and a family of semiregular polytopes known as Gosset figures (k_{21} according to Coxeter). Here, we only present general facts, and further details on them can be found in [3], [4], [5] and [12].

Let P_n be a convex n-simplex in an n-dimensional Euclidean space. For each vertex V, the midpoints of all the edges emanating from a vertex V in P_n form an $(n-1)$-simplex if they lie in a hyperplane, and this $(n-1)$-simplex is called the *vertex figure* of P_n at V. In this article, the vertices on the other ends of the edges emanating from the vertex V also form an $(n-1)$-polytope, and we also call this $(n-1)$-polytope the *vertex figure* of V in P_n.

A *regular* polytope P_n ($n \ge 2$) is a polytope whose facets and vertex figure at each vertex are regular. In particular, a polygon P_2 is regular if it is equilateral and equiangular. Naturally, the facets of regular P_n are all congruent, and the vertex figures are all the same.

In this article, we consider two classes of regular polytopes.

(1) A **regular simplex** α_n is an n-dimensional simplex with equilateral edges. Note α_n is a pyramid based on α_{n-1}. Thus the facets of a regular simplex α_n is a regular simplex α_{n-1}, and the vertex figure of α_n is also α_{n-1}. For example, α_1 is a line-segment, α_2 is an equilateral triangle, and α_3 is a tetrahedron. For a regular simplex α_n, only the regular simplex α_k, $0 \leq k \leq n-1$, appears as a face.

(2) A **crosspolytope** β_n is an n-dimensional polytope whose $2n$-vertices are the intersections between an n-dimensional Cartesian coordinate frame and a sphere centered at the origin. Note β_n is a bipyramid based on β_{n-1}, and the n-vertices in β_n form α_{n-1} if the choice is made as one vertex from each Cartesian coordinate line. So the vertex figure of a crosspolytope β_n is also a crosspolytope β_{n-1}, and the facets of β_n is α_{n-1}. For instance, β_1 is a line-segment, β_2 is a square, and β_3 is an octahedron. For a crosspolytope β_n, only the regular simplex α_k, $0 \leq k \leq n-1$, appears as a face.

A polytope P_n is called *semiregular* if its facets are regular and its vertices are equivalent, namely, the symmetry group of P_n acts transitively on the vertices of P_n.

Here, we consider the semiregular k_{21} polytopes discovered by Gosset which are $(k+4)$-dimensional polytopes whose symmetry groups are the Coxeter group E_{k+4}. Note that the vertex figure of k_{21} is $(k-1)_{21}$ and the facets of k_{21} are regular simplexes α_{k+3} and crosspolytopes β_{k+3}.

For $k \neq -1$, the facets of k_{21}-polytopes are the regular simplex α_{k+3} and the crosspolytope β_{k+3}. But all the lower-dimensional faces are regular simplexes. When $k = -1$, the vertex figure in -1_{21} is an isosceles triangle instead of an equilateral triangle, and its facets are the regular triangle α_2 and the square β_2.

2.2. **Gosset polytopes in the Picard groups of del Pezzo surfaces.** The del Pezzo surfaces are smooth irreducible surfaces S_r whose anticanonical class $-K_{S_r}$ is ample. We can construct the del Pezzo surfaces by blowing up $r \leq 8$ points from $\mathbb{P}^2$ unless it is $\mathbb{P}^1 \times \mathbb{P}^1$. In particular, it is very well known that there are 27 lines on a cubic surface S_6 and the configuration of these lines is acted upon by the Weyl group E_6 [8], [9], [11]. The set of 27-lines in S_6 are bijective to the set of vertices of a Gosset 2_{21} polytope. A similar correspondence is found between the 28-bitangents in S_7 and 3_{21} polytopes, and between the tritangent planes for S_8 and 4_{21} polytopes. The correspondence between lines in S_6 and vertices in 2_{21} is applied to study the geometry of 2_{21} by Coxeter [6], and the correspondence is extended to each $3 \leq r \leq 8$ in [15].

We denote such a del Pezzo surface by S_r and the corresponding blow up by $\pi_r : S_r \to \mathbb{P}^2$. Also, $K_{S_r}^2 = 9 - r$ is called the degree of the del Pezzo surface. Each exceptional curve and the corresponding class given by blowing up is denoted by e_i, and both the class of $\pi_r^*(h)$ in S_r and the class of a line h in $\mathbb{P}^2$ are referred to as h. Then, we have

$$h^2 = 1,\ h \cdot e_i = 0,\ e_i \cdot e_j = -\delta_{ij} \quad \text{for } 1 \leq i, j \leq r,$$

and the Picard group of S_r is

$$Pic\ S_r \simeq \mathbb{Z}h \oplus \mathbb{Z}e_1 \oplus ... \oplus \mathbb{Z}e_r$$

with the signature $(1, -r)$. We also have that $K_{S_r} = -3h + \sum_{i=1}^{r} e_i$.

The inner product given by the intersection on $Pic\ S_r$ induces a negative definite metric on $(\mathbb{Z}K_{S_r})^{\perp}$ in $Pic\ S_r$ where we can also define natural reflections. To define reflections on $(\mathbb{Z}K_{S_r})^{\perp}$ in $Pic\ S_r$, we consider a root system

$$R_r := \{d \in Pic\ S_r \mid d^2 = -2,\ d \cdot K_{S_r} = 0\},$$

with simple roots $d_0 = h - e_1 - e_2 - e_3, d_i = e_i - e_{i+1},\ 1 \le i \le r-1$. Each element d in R_r defines a reflection on $(\mathbb{Z}K_{S_r})^{\perp}$ in $Pic\ S_r$,

$$\sigma_d(D) := D + (D \cdot d)\, d \ \text{ for } D \in (\mathbb{Z}K_{S_r})^{\perp},$$

and the corresponding Weyl group $W(S_r)$ is E_r, where $3 \le r \le 8$. Furthermore, the reflection σ_d on $(\mathbb{Z}K_{S_r})^{\perp}$ can be used to obtain a transformation both on $Pic\ S_r$ and on $Pic\ S_r \otimes \mathbb{Q} \simeq \mathbb{Q}h \oplus \mathbb{Q}e_1 \oplus ... \oplus \mathbb{Q}e_r$ via the linear extension of the intersections of divisors in $Pic\ S_r$. Here $Pic\ S_r \otimes \mathbb{Q}$ is a vector space with the signature $(1, -r)$.

In this article, we deal with divisor classes D satisfying $D \cdot K_{S_r} = \alpha, D^2 = \beta$ where α and β are integers, which are preserved by the extended action of $W(S_r)$. In particular, $W(S_r)$ acts as a reflection group on the set of divisor classes with $D \cdot K_{S_r} = \alpha$. Therefore, we define an *affine hyperplane section* in $Pic\ S_r \otimes \mathbb{Q}$ by

$$\tilde{H}_b := \{D \in Pic\ S_r \ \otimes \mathbb{Q} \mid -D \cdot K_{S_r} = b\},$$

where b is an arbitrary real number and an affine hyperplane section $H_b := \tilde{H}_b \cap Pic\ S_r$ in $Pic\ S_r$. By the ample condition of $-K_{S_r}$ and the Hodge index theorem, the inner product on $Pic\ S_r$ induces a negative definite metric on H_b. In fact, the induced metric is defined on $Pic\ S_r \otimes \mathbb{Q}$, and we can also consider the induced norm by fixing a center $\frac{b}{9-r}K_{S_r}$ in the affine hyperplane section $-D \cdot K_{S_r} = b$ in $Pic\ S_r \otimes \mathbb{Q}$. This norm is also negative definite. Furthermore, the reflection σ_d on $Pic\ S_r \otimes \mathbb{Q}$ induces a reflection on $\tilde{H}_b$. Generically, the hyperplanes in $Pic\ S_r \otimes \mathbb{Q}$ induce affine hyperplanes in $\tilde{H}_b$ and they may not share a common point. But the reflection hyperplane of each reflection σ_d in $Pic\ S_r \otimes \mathbb{Q}$ gives a hyperplane in $\tilde{H}_b$ containing the center because it is given by a condition $K_{S_r} \cdot d = 0$. Thus, the Weyl group $W(S_r)$ acts on $\tilde{H}_b$ and H_b as a reflection group.

Now, we want to construct Gosset polytopes $(r-4)_{21}$ in $Pic\ S_r \otimes \mathbb{Q}$ as the convex hull of the set of special classes in $Pic\ S_r$, which are known as lines. A *line* in $Pic\ S_r$ is equivalently a divisor class l with $l^2 = -1$ and $K_{S_r} \cdot l = -1$, and the set of lines is given as

$$L_r := \{D \in Pic(S_r) \mid D^2 = -1, K_{S_r} D = -1\}.$$

As the Weyl group $W(S_r)$ acts as an affine reflection group on the affine hyperplane given by $D \cdot K_{S_r} = -1$, $W(S_r)$ acts on the set of lines in $Pic\ S_r$. Therefore, we construct a semiregular polytope in $Pic\ S_r \otimes \mathbb{Q}$ whose vertices are exactly the lines in $Pic\ S_r$. Since the symmetry group of the polytope is $W(S_r)$, the polytope is actually a Gosset polytope $(r-4)_{21}$.

Remark. Each line l in L_r contains an exceptional curve which produces a blow down map from S_r to S_{r-1}. We denote the map as $\pi_l^r : S_r \to S_{r-1}$.

For a Gosset polytope $(r-4)_{21}$, faces are regular simplexes except for the facets which consist of $(r-1)$-simplexes and $(r-1)$-crosspolytopes. Since the faces in $(r-4)_{21}$ are basically configurations of vertices, we obtain the natural characterization of faces in $(r-4)_{21}$ as divisor classes in $Pic\ S_r$.

Remark. To identify each face in $(r-4)_{21}$, we want to use the barycenter of the face. Each vertex of the polytope $(r-4)_{21}$ represents a line in S_r, and the honest centers of simplexes (resp. crosspolytopes) are written as $(l_1 + ... + l_k)/k$ (resp.$(l_1' + l_2')/2$)

in $\tilde{H}_1$ which may not be elements in $Pic\ S_r$. Therefore, alternatively, we choose $(l_1 + ... + l_k)$ as the center of a face so that $(l_1 + ... + l_k)$ is in $Pic\ S_r$.

We use the algebraic geometry of del Pezzo surfaces to identify the divisor classes corresponding to the faces in $(r-4)_{21}$. For this purpose, we consider the following set of divisor classes which are called *skew a-lines, exceptional systems* and *rulings* in $Pic\ S_r$:

$$
\begin{aligned}
L_r^a &:= \{D \in Pic(S_r) \mid D = l_1 + ... + l_a,\ l_i \text{ disjoint lines in } S_r\},\\
\mathcal{E}_r &:= \{D \in Pic(S_r) \mid D^2 = 1,\ K_{S_r} \cdot D = -3\},\\
F_r &:= \{D \in Pic(S_r) \mid D^2 = 0,\ K_{S_r} \cdot D = -2\}.
\end{aligned}
$$

A *skew a-line* in L_r^a is an extension of the definition of lines in S_r. Each skew a-line represents an $(a-1)$-simplex in an $(r-4)_{21}$ polytope. Furthermore, for each skew a-line, there is only one set of disjoint lines in L_8 to present it. The skew a-lines also have $D^2 = -a$ and $D \cdot K_{S_r} = -a$, and the divisor classes with these conditions are equivalently skew a-lines for $a \leq 3$.

An *exceptional system* in $\mathcal{E}_r$ is a divisor class in $Pic\ S_r$ whose linear system gives a regular map from S_r to $\mathbb{P}^2$. As this regular map corresponds to a blowing up from $\mathbb{P}^2$ to S_r, naturally exceptional systems are related to $(r-1)$-simplexes in $(r-4)_{21}$ polytopes, which is one of two types of facets appearing in $(r-4)_{21}$ polytopes. In fact, by a transformation Φ from $\mathcal{E}_r$ to L_r^r by

$$\Phi(D_t) := K_{S_r} + 3D_t \text{ for } D_t \in \mathcal{E}_r,$$

the set $\mathcal{E}_r$ is bijective to the set of the $(r-1)$-simplexes in $(r-4)_{21}$ polytopes, for $3 \leq r \leq 7$. When $r = 8$, the set of exceptional systems has two orbits. One orbit with 17280 elements corresponds to the set of skew 8-lines in S_8, and the other orbit with 240 elements corresponds to the set of E_8-roots because $-3K_{S_8} + 2d$ is an exceptional system for each E_8-root d. Thus an exceptional system D_t in $\mathcal{E}_r$ that satisfies either $-3K_{S_8} + 2d = D_t$ for a root d or $3D_t + K_{S_8} = D_1$ for D_1 is a skew 8-line.

A *ruling* in F_r is a divisor class in $Pic\ S_r$ which gives a fibration of S_r over $\mathbb{P}^1$, and we show that the F_r is bijective to the set of $(r-1)$-crosspolytopes in the $(r-4)_{21}$ polytope. Furthermore, we explain the relationships between lines and rulings according to the incidence between the vertices and $(r-1)$-crosspolytopes. This leads us to the fact that a pair of proper crosspolytopes in the $(r-4)_{21}$ give the blowing down maps from S_r to $\mathbb{P}^1 \times \mathbb{P}^1$.

After proper comparison between divisor classes obtained from the geometry of the polytope $(r-4)_{21}$ and those given by the geometry of a del Pezzo surface, we come to the following correspondences:

del Pezzo surface S_r	Gosset polytopes $(r-4)_{21}$
lines	vertices
skew a-lines $1 \leq a \leq r$	$(a-1)$-simplexes $1 \leq a \leq r$
exceptional systems	$(r-1)$-simplexes $(r < 8)$
rulings	$(r-1)$-crosspolytopes

Lines in a ruling. Now, we know rulings in $Pic\ S_r$ correspond to $(r-1)$-crosspolytopes in the Gosset polytopes $(r-4)_{21}$. Since an $(r-1)$-crosspolytope has $2(r-1)$ vertices in it, we want to have a criterion that a line l in L_r is one of the vertices of an $(r-1)$-crosspolytope corresponding to a ruling f. Here, the following are equivalently stating the criterion:

(1) $f \cdot l = 0$.

(2) $f - l$ is a line.

(3) The vertex represented by l in $(r-4)_{21}$ is one of the vertices of the $(r-1)$-crosspolytope corresponding to f.

Gieser transform and Bertini transform on lines. In [12], we define the *Gieser transform on lines in* S_7 or simply the Gieser transform by

$$G(l) := -\left(K_{S_7} + l\right) \text{ for } l \in L_7.$$

This is an automorphism of L_7 which is closely related to the deck transformation of the covering from S_7 to $\mathbb{P}^2$ given by $|-K_{S_7}|$. Similarly a transformation B on lines in S_8 defined by

$$B(l) := -\left(2K_{S_8} + l\right) \text{ for } l \in L_8$$

is referred to as the *Bertini transform on lines* or simply Bertini transform.

The Gieser transform G and the Bertini transform B preserve the intersection between two lines. Therefore, these transforms can be extended to symmetries on the configurations of lines in S_7 and S_8, which are also symmetries on 3_{21} and 4_{21} respectively (see [12]).

2.3. **Monoidal transform of del Pezzo surfaces.** Since a del Pezzo surface S_r is obtained by the blowing up of one point on S_{r-1}, we can describe divisor classes in S_{r-1} producing lines in S_r after blowing up. Let l be the exceptional divisor in S_r given by a blow up of a point from S_{r-1}, namely $\pi_l^r : S_r \to S_{r-1}$. The proper transform of a divisor D in $Pic\ S_{r-1}$ producing a line in S_r satisfies

$$\left(\pi_l^{r*}(D) - ml\right)^2 = -1,\ \left(\pi_l^{r*}(D) - ml\right) \cdot \left(\pi_l^{r*}\left(K_{S_{r-1}}\right) + l\right) = -1$$

for a nonnegative integer m. Therefore, we consider a divisor D in $Pic(S_{r-1})$ with

$$D^2 = m^2 - 1,\ D \cdot K_{S_{r-1}} = -m - 1.$$

By the Hodge index theorem (see section 6 in [12]), the list of possible m is

$$m = \begin{cases} 0, 1, & 4 \le r \le 6, \\ 0, 1, 2, & r = 7, \\ 0, 1, 2, 3, & r = 8. \end{cases}$$

On the other hand, the integer m is the intersection between l and a line $\pi_l^{r*}(D) - ml$. Therefore, the above divisors in S_{r-1} characterize the subsets of L_r according to the intersection with l. Here we consider a set $N_k(l, S_r)$ defined as

$$N_k(l, S_r) := \{\ l' \ \in L_r \mid l' \cdot l = m\} \text{ for } k \ge -1,$$

and obtain the following bijections:

$$\begin{aligned}
&N_0(l, S_r) \approx L_{r-1} \text{ for } 4 \le r \le 8, \\
&N_1(l, S_r) \approx F_{r-1} \text{ for } 4 \le r \le 8, \\
&N_2(l, S_8) \approx N_0(-2K_{S_8} - l, S_8) \approx L_7,\ N_2(l, S_7) = \{-K_{S_7} - l\}, \\
&N_3(l, S_8) = \{-2K_{S_8} - l\}.
\end{aligned}$$

It will be useful to observe that the Gieser transform G on L_7 gives

$$G(N_{-1}(l,S_7)) = N_2(l,S_7) \text{ and } G(N_0(l,S_7)) = N_1(l,S_7),$$

and the Bertini transform B on L_8 induces

$$B(N_{-1}(l,S_8)) = N_3(l,S_8),\ B(N_0(l,S_8)) = N_2(l,S_8) \text{ and } B(N_1(l,S_8)) = N_1(l,S_8).$$

Lines of a vertex figure. In this article, the subset $N_0(l,S_r)$ in L_r plays a very important role. When we consider a rational map given by the blowing down of the exceptional curve in l, $N_0(l,S_r)$ in L_r is bijectively mapped to the lines in S_{r-1} which are also vertices of the corresponding Gosset polytope $(r-5)_{21}$. We call $N_0(l,S_r)$ in L_r *lines of a vertex figure* of l or simply a vertex figure of l. In fact, each line in $N_0(l,S_r)$ is joined to l with an edge, and the set of midpoints of their edges is the vertex figure of the vertex l in $(r-4)_{21}$.

3. Steiner systems of del Pezzo surfaces

A *Steiner system* $S(x,y,z)$ is a type of block design system which is a family of subsets S consisting of y-elements in a set T of z-elements satisfying that each x-element of T is contained in exactly one subset in the family. Here the subset S in the system is called a *block*.

For example, $S(2,3,7)$ represents the famous Fano projective plane which is a family of 3-point subsets (called lines) in a 7-point set where each two points in the set determines a line. This is one special case of Steiner triple systems $S(2,3,n)$ containing $n(n-1)/6$ blocks. An analog of the Steiner system can be considered to be the set of lines in the Picard group of S_6 whose block is given as a subset (called a triplet) consisting of three lines in S_6 with 1-intersection to each other. Because the sum of these three lines in a block equals $-K_{S_6}$, a block is defined by each two lines with 1-intersection. However, the total number of blocks in this system is much less than the expected number of blocks in the ordinary Steiner triple system $S(2,3,27)$. Therefore, it is natural to add a few combinatorial conditions to define an analogue of the Steiner system on the family of divisors of del Pezzo surfaces. In this section, we define Steiner systems on del Pezzo surfaces as analogous to the triplets in cubic surfaces and search for Steiner systems in the configurations of lines in del Pezzo surfaces.

In this article, we work on divisor classes D given as a sum of lines with fixed intersections. The divisor class D satisfies equations $D^2 = \alpha$ and $D \cdot K_{S_8} = \beta$. For certain integers α and β, the equations have a unique solution which gives a condition determining a block in the following k-Steiner system.

Definition 1. A family of subsets of lines in a del Pezzo surface S_r written $\mathcal{S}(k,S_r)$ is called a k-Steiner system on S_r if it is a family of subsets S of k-lines in $Pic\ S_r$ where each set of $k-1$ lines in $Pic\ S_r$ with constant intersection to each other exactly determines a subset S in $\mathcal{S}(k,S_r)$.

Note: Each block in $\mathcal{S}(k,S_r)$ has k-lines.

For example, we consider a system given by Steiner triplets in cubic surfaces S_6. For each pair of lines l_1 and l_2 with $l_1 \cdot l_2 = 1$ in S_6 there is a line l_3 with $l_1 \cdot l_3 = l_2 \cdot l_3 = 1$ determined by $l_1 + l_2 + l_3 = -K_{S_6}$. Thus there is a 3-Steiner system on S_6. This is an example of what we mean by a deterministic relationship.

Similarly, we can consider the following deterministic relationships between lines:

(1) $l_1 + l_2 = -K_{S_7}$ for $l_1, l_2 \in L_7$ with $l_1 \cdot l_2 = 2$,
(2) $l_1 + l_2 = -2K_{S_8}$ for $l_1, l_2 \in L_8$ with $l_1 \cdot l_2 = 3$,
(3) $l_1 + l_2 + l_3 = -K_{S_6}$ for $l_i, l_j \in L_6$ with $l_i \cdot l_j = 1$,
(4) $l_1 + l_2 + l_3 = -3K_{S_8}$ for $l_i, l_j \in L_8$ with $l_i \cdot l_j = 2$,
(5) $l_1 + l_2 + l_3 + l_4 = -2K_{S_7}$ for $l_i, l_j \in L_8$ with $l_i \cdot l_j = 1$.

Equivalently, we define the following k-Steiner systems:

(1) $\mathcal{S}_A(2, S_7) := \{\{l_1, l_2\} \mid l_1 \cdot l_2 = 2 \text{ for } l_1,\ l_2\ \in L_7\},\ |\mathcal{S}_A(2, S_7)| = 56$,

(2) $\mathcal{S}_A(2, S_8) := \{\{l_1, l_2\} \mid l_1 \cdot l_2 = 3 \text{ for } l_1,\ l_2\ \in L_8\},\ |\mathcal{S}_A(2, S_8)| = 240$,

(3) $\mathcal{S}_B(3, S_6) := \{\{l_1, l_2, l_3\} \mid l_i \cdot l_j = 1 \text{ for } l_i, l_j \in L_6\},\ |\mathcal{S}_B(3, S_6)| = 45$,

(4) $\mathcal{S}_B(3, S_8) := \{\{l_1, l_2, l_3\} \mid l_i \cdot l_j = 2 \text{ for } l_i, l_j \in L_8\},\ |\mathcal{S}_B(3, S_8)| = 2240$,

(5) $\mathcal{S}_C(4, S_7) := \{\{l_1, l_2, l_3, l_4\} \mid l_i \cdot l_j = 1 \text{ for } l_i, l_j \in L_7\},\ |\mathcal{S}_C(4, S_7)| = 720$.

Remark.

(1) A 2-Steiner block in $\mathcal{S}_A(2, S_7)$ can be written as $\{l, -(K_{S_7} + l)\} = \{l, G(l)\}$ where $G(l)$ is the Gieser transform of l. Thus $|\mathcal{S}_A(2, S_7)| = |L_7| = 56$. In fact, each 2-Steiner block in $\mathcal{S}_A(2, S_7)$ on S_7 represents a bitangent of degree 2 covering from S_7 to $\mathbb{P}^2$ given by $|-K_{S_7}|$ (see chapter 8 of [9]).

(2) A 2-Steiner block in $\mathcal{S}_A(2, S_8)$ is $\{l, -(2K_{S_8} + l)\} = \{l, B(l)\}$ for Bertini transform B, and $|\mathcal{S}_A(2, S_8)| = |L_8| = 240$. Again, this 2-Steiner block on S_8 is related to a tritangent plane of degree 2 covering from S_8 to $\mathbb{P}^2$ given by $|-2K_{S_8}|$ (see chapter 8 of [9]).

(3) For each line l in S_8, $l + K_{S_8}$ is a root in S_8, and vice versa. Thus, any three roots d_1, d_2 and d_3 in S_8 with $d_i \cdot d_j = 1$, $i \neq j$, correspond to three lines l_1, l_2 and l_3 with

$$l_i \cdot l_j = (d_i - K_{S_8}) \cdot (d_j - K_{S_8}) = d_i \cdot d_j + 1 = 2,$$

and, because $d_1 + d_2 + d_3 = l_1 + l_2 + l_3 + 3K_{S_8} = 0$, we can also define a 3-Steiner system of the roots on S_8.

(4) A 3-Steiner block in $\mathcal{S}_B(3, S_6)$ can be written as $\{l_1, l_2, -(K_{S_6} + l_1 + l_2)\}$ with $l_1 \cdot l_2 = 1$. Thus

$$3!\,|\mathcal{S}_B(3, S_6)| = |L_6|\,(|L_6| - |L_5| - 1) = 27(27 - 16 - 1)$$

and $|\mathcal{S}_B(3, S_6)| = 45$. Similarly, a 3-Steiner block in $\mathcal{S}_B(3, S_8)$ can be written as $\{l_1, l_2, -(3K_{S_8} + l_1 + l_2)\}$, where $l_1 \cdot l_2 = 2$. Thus $3!\,|\mathcal{S}_B(3, S_8)| = |L_8|\,|L_7| = 240 \cdot 56$ and $|\mathcal{S}_B(3, S_8)| = 2240$. Here we use $l_1 \cdot (-2K_{S_8} - l_2) = 0$ for a line $-2K_{S_8} - l_2$ corresponding to l_2.

(5) A 4-Steiner block in $\mathcal{S}_C(4, S_7)$ can be written as $\{l_{1,}, l_2, l_3, -(2K_{S_7} + l_1 + l_2 + l_3)\}$ with $l_i \cdot l_j = 1$. We observe that two divisors $\pi^8_{l_1 *}(l_2)$ and $\pi^8_{l_1 *}(l_3)$ given by the monoidal transform π are rulings in S_6 with $\pi^8_{l_1 *}(l_2) \cdot \pi^8_{l_1 *}(l_2) = 2$. Moreover, these divisors correspond to lines $-K_{S_6} - \pi^8_{l_1 *}(l_2)$ and $-K_{S_6} - \pi^8_{l_1 *}(l_3)$ with $\left(-K_{S_6} - \pi^8_{l_1 *}(l_2)\right) \cdot \left(-K_{S_6} - \pi^8_{l_1 *}(l_3)\right) = 1$. Therefore

$$4!\,|\mathcal{S}_C(4, S_7)| = |L_7|\,|L_6|\,(|L_6| - |L_5| - 1) = 56 \cdot 27(27 - 16 - 1),$$

and we have $|\mathcal{S}_C(4, S_7)| = 720$.

As an application of this 3-Steiner system on S_8, we have the following theorem. We also provide another proof given by the monoidal transform of lines. The argument in the second proof is very useful in this article.

Theorem 2. *For a del Pezzo surface S_8, the canonical class K_{S_8} can be written as*

$$K_{S_8} = l_1 - l_2 - l_3,$$

where l_1, l_2 and l_3 are lines with $l_1 \cdot l_2 = l_1 \cdot l_3 = 0$ and $l_2 \cdot l_3 = 2$.

Proof (1). Each line l_2 and l_3 with $l_2 \cdot l_3 = 2$ determines a line $-3K_{S_8} - l_2 - l_3$ as in a 3-Steiner block. Again, as in the 2-Steiner system, we get a line

$$l_1 = -\left(2K_{S_8} + (-3K_{S_8} - l_2 - l_3)\right) = K_{S_8} + (l_2 + l_3).$$

This gives the theorem.

Proof (2). Consider a line l_1 in L_8 and the corresponding blow down map $\pi^8_{l_1} : S_8 \to S_7$. Recall that $-K_{S_7}$ can be written as the sum of two lines l_2 and l_3 in L_7 with $l_2 \cdot l_3 = 2$. Since $K_{S_8} = \pi^{8*}_{l_1}(K_{S_7}) + l_1$ and $l_2 \cdot l_1 = l_3 \cdot l_1 = 0$, we have

$$K_{S_8} = -l_2 - l_3 + l_1 \ .$$

□

Remark. The first proof above implies that for a divisor class $l_2 + l_3$ with $l_2 \cdot l_3 = 2$, the divisor class $K_{S_8} + l_2 + l_3$ is a line with 0-intersection to l_2 and l_3.

4. Higher degree simplexes in Gosset polytopes

In this section, we study the convex regular polytopes whose vertices are the subsets of vertices of Gosset polytopes. As is the nature of the study on Gosset polytopes including the symmetry of high degree, the characterization of these polytopes is very complicated. Since we are dealing with the divisor classes written as a sum of lines with fixed intersection, we focus on the fundamental regular polytopes such as $A^r_m(b)$-simplexes in this article.

As in [12], we identify the barycentric centers of $A^r_m(b)$-simplexes in the Gosset polytopes and study the configuration of corresponding $A^r_m(b)$-vertices. Since we are working along the configuration of lines in the del Pezzo surface, the centers are chosen in $Pic\ S_r$ instead of $Pic\ S_r \otimes \mathbb{Q}$ after multiplying proper integers.

4.1. 1-degree simplexes in Gosset polytopes.

Configuration sets and divisor sets. We observe that for any two distinct lines l_1 and l_2 in $Pic\ S_r$, we have

$$(l_1 - l_2)^2 = -2 - 2l_1 \cdot l_2,\ (l_1 - l_2)\ \cdot K_{S_r} = 0.$$

Since the metric on $K^{\perp}_{S_r}$ is negative definite, $l_1 - l_2$ has length $\sqrt{2\,(1 + l_1 \cdot l_2)}$. The line segment joining lines l_1 and l_2 which are two vertices in a Gosset polytope in $Pic\ S_r$ are called a *b-degree edge* if $l_1 \cdot l_2 = b$. The 0-degree edges are honest edges appearing as faces in the Gosset polytopes. Since the Gosset polytopes are convex, any polytope whose vertices form a subset of the vertices of the Gosset polytopes is inscribed in them. Here we consider the following configuration sets and their divisor sets.

Definition 3. We define the *configuration set* $\mathcal{A}^r_{m+1}(b)$ by

$$\mathcal{A}^r_m(b) := \{\{l_1, ..., l_{m+1}\} \mid l_i \in L_r,\ l_i \cdot l_j = b \text{ for } i \neq j\}$$

and its *divisor set* $\mathcal{D}^r_m(b)$ by

$$\mathcal{D}^r_m(b) := \{l_1 + ... + l_{m+1} \in Pic(S_r) \mid \{l_1, ..., l_{m+1}\} \in \mathcal{A}^r_m(b)\}\,.$$

Here, each element in $\mathcal{A}^r_m(b)$ (resp. $\mathcal{D}^r_m(b)$) is called an $A^r_m(b)$-*vertex* (resp. $A^r_m(b)$-*divisor*). The m-simplex consisting of an $A^r_m(b)$-*vertex* is called a *b-degree m-simplex in* $(r-4)_{21}$ or an $A^r_m(b)$-*simplex*. Since an $A^r_m(b)$-*divisor* is the sum of lines, $l_1+...+l_{m+1}$, we also call it *the center of the* $A^r_m(b)$-*simplex*. Note that, when b is zero, the $A^r_m(0)$-simplexes are from the faces of $(r-4)_{21}$, but for $b>0$, $A^r_m(b)$-vertices do not produce faces in $(r-4)_{21}$.

As the monoidal transform is useful in this article, we want to characterize $A^r_m(b)$-simplexes in $(r-4)_{21}$ which are preserved by a blow down from S_r to S_{r-1}. Thus this type of $A^r_m(b)$-simplex can be treated not only in $(r-4)_{21}$ but also in $(r-5)_{21}$.

Definition 4. An $A^r_m(b)$-vertex (or simplex) in a Gosset polytope $(r-4)_{21}$ of a del Pezzo surface S_r is called *tied* if there is a line l in L_r where the $A^r_m(b)$-vertex of the simplex is the subset of the vertex figure of l. Otherwise, the simplex is called untied.

4.1.1. 1-*degree* 1-*simplexes and* 2-*simplexes.* In the following, the centers of $A^r_m(b)$-simplexes $m\leq 3$ in $(r-4)_{21}$ are either unique or correspond to the lines for the tied simplexes issues. Thus the configurations of $A^r_m(b)$-simplexes $m\leq 3$ in $(r-4)_{21}$ are naturally related to the k-Steiner systems along the monoidal transform.

A. 1-degree 1-simplexes for $3\leq r\leq 8$.

By definition, the $A^r_1(1)$-divisor set $\mathcal{D}^r_1(1)$ is a subset of the divisor classes D with $D\cdot K_{S_r}=-2$ and $D^2=0$, which is a ruling. Since each ruling can be written as the sum of two lines with 1-intersection (subsection 2.2), we have $\mathcal{D}^r_1(1)=F_r$. Furthermore, the configuration of the $A^r_1(1)$-simplexes with a fixed center D is the set of antipodal pairs of lines in a crosspolytope in $(r-4)_{21}$ whose center is D.

B. 1-degree 2-simplex for $6\leq r\leq 8$.

The 1-degree 2-simplexes ($A^r_2(1)$-vertices) exist when $6\leq r\leq 8$.

The center of each $A^r_2(1)$-simplex, namely, an $A^r_2(1)$-divisor $D:=l_1+l_2+l_3$ where l_1,l_2 and l_3 are lines with $l_i\cdot l_j=1$, satisfies $D^2=(l_1+l_2+l_3)^2=3$ and $D\cdot K_{S_r}=(l_1+l_2+l_3)\cdot K_{S_r}=-3$.

(a) 1-degree 2-simplex in 2_{21}.

The class $-K_{S_6}$ is the only divisor class satisfying the above equations for the center. Thus, all the $A^6_2(1)$-vertices in 2_{21} share one common $A^6_2(1)$-divisor, $-K_{S_6}$. By definition, the configuration set of $\mathcal{A}^6_2(1)$ is the 3-Steiner system $\mathcal{S}_B(3,S_6)$ in section 3.

(b) 1-degree 2-simplex in 3_{21}.

When $r=7$, for each $A^7_2(1)$-divisor D_1 satisfying the above center equations, $D_1^2=3$ and $D_1\cdot K_{S_7}=-3$. Here we observe that any divisor in $Pic\ S_7$ with $D^2=3$ and $D\cdot K_{S_7}=-3$ gives a new divisor defined by $l^{S_7}_D:=D+K_{S_7}$, which is in fact a line in L_7.

Claim. Each divisor D in $Pic\ S_7$ with $D^2=3$ and $D\cdot K_{S_7}=-3$ is an $A^7_2(1)$-divisor.

Proof. Because $l^{S_7}_D\cdot D=0$, we can deduce that such a divisor D in $Pic\ S_7$ corresponds to an $A^6_2(1)$-divisor $\pi_{l^{S_7}_D *}(D)$ in $Pic\ S_6$ by monoidal transform to S_6 given by $l^{S_7}_D$. Thus there is an $A^6_2(1)$-vertex where its $A^6_2(1)$-divisor is $\pi_{l^{S_7}_D *}(D)$, and by pulling back the $A^6_2(1)$-vertex, we obtain an $A^7_2(1)$-vertex with the $A^7_2(1)$-divisor D. $\square$

We note that the $A_2^7(1)$-vertex of the $A_2^7(1)$-divisor D in the Claim is contained in the vertex figure of $l_D^{S_7}$.

Thus we know the set of $A_2^7(1)$-divisors is bijective to the set $\{D \in Pic\ S_7 \mid D^2 = 3 = -D \cdot K_{S_7}\}$ which is also bijective to L_7. To apply a monoidal transform to the configuration set $\mathcal{A}_2^7(1)$, we need to know that each of the $A_2^7(1)$-vertices, l_1, l_2 and l_3, sharing a common $A_2^7(1)$-divisor D, is in the vertex figure of $l_D^{S_7}$. This is true because

$$l_D \cdot l_i = (l_1 + l_2 + l_3 + K_{S_7}) \cdot l_i = 0 \quad \text{for } i = 1, 2, 3.$$

Thus l_1, l_2 and l_3 are in $N_0(l_D, S_7)$, namely, the vertex figure of $l_D^{S_7}$, and the $A_2^7(1)$-simplex is tied. In conclusion, the configuration of $A_2^7(1)$-vertices with a fixed $A_2^7(1)$-divisor in $Pic\ S_7$ equals the 3-Steiner system $\mathcal{S}(3, S_6)$ in S_6 by monoidal transforms.

Remark. The choice of line $l_D^{S_7}$ is natural once we observe its role in the following configuration of lines in 3_{21}. Recall that each pair of l_1, l_2 and l_3 in an $A_2^7(1)$-simplex forms a ruling corresponding to a crosspolytope (see subsection 2.2 or [12]). As the crosspolytopes in 3_{21} have six pairs of antipodal vertices in it, a ruling can be obtained from the six pairs of lines with intersection 1. We choose $l_1 + l_2$ for a ruling, then $G(l_3)$, the Gieser transform of l_3, and $l_D^{S_7}$ are another antipodal pair of lines in the ruling because $G(l_3) + l_D^{S_7} = (-K_{S_7} - l_3) + (l_1 + l_2 + l_3 + K_{S_7}) = l_1 + l_2$, and we deduce $l_D^{S_7} \cdot l_1 = l_D^{S_7} \cdot l_2 = 0$. From the other two choices of rulings from l_1, l_2 and l_3 in the $A_2^7(1)$-simplex, we have similar results containing $l_D^{S_7}$ in common. The line $l_D^{S_7}$ is the common vertex of the crosspolytopes corresponding to the three rulings. Thus l_1, l_2 and l_3 are in the vertex figure of $l_D^{S_7}$.

(c) 1-degree 2-simplex in 4_{21}.

As in the case of 3_{21}, the monoidal transforms from S_8 to S_6 are the key to describing the configuration. But here, we need to find two disjoint lines to blow down to S_6.

As above, we consider a divisor D in $Pic\ S_8$ with $D^2 = 3$ and $D \cdot K_{S_8} = -3$, and a new divisor defined by $D + K_{S_8}$. Then $D + K_{S_8}$ is a skew 2-line in $Pic\ S_8$ because $(D + K_{S_8})^2 = -2$ and $(D + K_{S_8}) \cdot K_{S_8} = -2$. By subsection 2.2, $D + K_{S_8}$ is a skew 2-line in $Pic\ S_8$ which is represented by the sum of two disjoint lines in S_8. Since $D \cdot (D + K_{S_8}) = 0$, by applying the monoidal transform from S_8 to S_6 given by the two disjoint lines, we conclude that this divisor D is in fact an $A_2^8(1)$-divisor. Thus we have the bijection between the $A_2^8(1)$-divisor set $\mathcal{D}_2^8(1)$ and the set of the skew 2-lines in L_8^2.

Again, we need to see that all the $A_2^8(1)$-vertices with a common $A_2^8(1)$-divisor are properly tied by two disjoint lines so as to apply the monoidal transforms. Assume l_1, l_2 and l_3 form an $A_2^8(1)$-vertex in 4_{21} where its center is $D_1 = l_1 + l_2 + l_3$. Then $D_1 + K_{S_8}$ is a skew 2-line which can be written as a sum of two disjoint lines, namely, $D_1 + K_{S_8} = l_a^{123} + l_b^{123}$, where l_a^{123} and l_b^{123} in L_8 with $l_a^{123} \cdot l_b^{123} = 0$. Here the choice of l_a^{123} and l_b^{123} is unique since a skew 2-line is given by a unique pair of lines. Furthermore we have

$$\left(l_a^{123} + l_b^{123}\right) \cdot l_i = (l_1 + l_2 + l_3 + K_{S_8}) \cdot l_i = 0 \text{ for } i = 1, 2, 3.$$

Here, if $l_a^{123} \cdot l_1 = -1$ and $l_b^{123} \cdot l_1 = 1$, then $l_a^{123} = l_1$ and $1 = l_b^{123} \cdot l_1 = l_b^{123} \cdot l_a^{123} = 0$, which is a contradiction. Thus we obtain that $l_a^{123} \cdot l_i = l_b^{123} \cdot l_i = 0$ for $i = 1, 2, 3$, and we conclude the l_1, l_2 and l_3 are in the vertex figures of l_a^{123} and l_b^{123}. Here the $A_2^8(1)$-vertices are tied by disjoint two lines l_a^{123} and l_b^{123}.

Since the skew 2-line $l_a^{123}+l_b^{123}$ induces a blow down map from S_8 to S_6, it sends the $A_2^8(1)$-divisor in 4_{21} given by l_1, l_2 and l_3 to an $A_2^6(1)$-divisor in 2_{21}. Thus the configuration of the $A_2^8(1)$-simplexes in 4_{21} with fixed $A_2^8(1)$-divisor equals the 3-Steiner system $\mathcal{S}_B(3, S_6)$ in S_6.

Now, the configuration of 1-degree 2-simplexes are summarized as follows.

Theorem 5. *The* 3*-Steiner system* $\mathcal{S}_B(3, S_6)$ *in* S_6 *determines the configuration of* $A_2^r(1)$*-vertices in* $(r-4)_{21}$ $(r = 6, 7, 8)$ *with a common* $A_2^r(1)$*-divisor.*

Corollary 6. *Each divisor* D *in Pic* S_r*,* $r = 6, 7, 8$*, satisfying* $D^2 = 3$ *and* $D \cdot K_{S_r} = -3$ *is an* $A_2^r(1)$*-divisor, namely the sum of three lines with intersection* 1*.*

4.1.2. 1*-degree* 3*-simplexes.* The 1-degree 3-simplexes, $A_3^r(1)$-simplexes, exist when $r = 7, 8$. Each $A_3^r(1)$-divisor D given by l_1, l_2, l_3 and l_4 lines with $l_i \cdot l_j = 1$, $i \neq j$, satisfies $D^2 = (l_1 + l_2 + l_3 + l_4)^2 = 8$ and $D \cdot K_{S_r} = (l_1 + l_2 + l_3 + l_4) \cdot K_{S_r} = -4$.

A. 1-degree 3-simplex in 3_{21}.

By the Hodge index theorem, $-2K_{S_7}$ is the only divisor class satisfying $D^2 = 8$ and $D \cdot K_{S_7} = -4$ on S_7. Thus all $A_3^7(1)$-vertices share a common center $-2K_{S_7}$, namely $\mathcal{D}_3^7(1) = \{-2K_{S_7}\}$. By definition, the configuration set of $A_3^7(1)$-vertices, $\mathcal{A}_3^7(1)$, equals the 4-Steiner $\mathcal{S}_C(4, S_7)$ on S_7 in section 3.

B. 1-degree 3-simplex in 4_{21}.

As before, we want to apply the monoidal transform to describe the configuration of $A_3^8(1)$-vertices. But it turns out that some $A_3^8(1)$-vertices are not properly tied to survive after the blowing down.

First, we consider divisor class D in Pic S_8 satisfying $D^2 = 8$ and $D \cdot K_{S_8} = -4$. The divisor D in Pic S_8 can be transformed to an exceptional system $D + K_{S_8}$ because $(D + K_{S_8})^2 = 1$ and $(D + K_{S_8}) \cdot K_{S_8} = -3$. According to subsection 2.2, $D + K_{S_8}$ can be in one of two E_8 orbits which correspond to the set of roots and the set of skew 8-lines in Pic S_8.

Now, assume D is an $A_3^8(1)$-divisor class $D = l_1 + l_2 + l_3 + l_4$ given by four lines with intersection 1. From subsection 2.2, we know that if $(l_1 + l_2 + l_3 + l_4) + K_{S_8}$ corresponds to a root, there is a root d such that $(l_1 + l_2 + l_3 + l_4) + K_{S_8} = -3K_{S_8} + 2d$. Furthermore, by checking the following two examples, we conclude that $A_3^8(1)$-divisors can be related to both E_8 orbits in exceptional systems.

(1) We choose the four lines

$$\begin{aligned}(l_1 + l_2 + l_3 + l_4) &= ((h - e_1 - e_2) + (h - e_3 - e_4) + (h - e_5 - e_6) + (h - e_7 - e_8)) \\ &= h - K_{S_8}.\end{aligned}$$

Also, $(h - K_{S_8}) + 4K_{S_8} = h + 3K_{S_8}$ cannot be $2d$ for any root d.

(2) However, if we choose four lines

$$\begin{aligned}(l_1 + l_2 + l_3 + l_4) &= \left(\begin{array}{c} e_1 + (h - e_1 - e_2) + (2h - e_1 - e_3 - e_4 - e_5 - e_6) \\ +(3h - e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - 2e_7) \end{array}\right) \\ &= -2K_{S_8} + 2e_8,\end{aligned}$$

then we have $(-2K_{S_8} + 2e_8) + 4K_{S_8} = 2(K_{S_8} + e_8)$, where $(K_{S_8} + e_8)$ is a root.

In fact, the center $(l_1 + l_2 + l_3 + l_4)$ can correspond to an element in the set of skew 8-lines in S_8 or the set of roots in S_8. Furthermore, by applying $W(E_8)$ action on the set of $A_3^8(1)$-divisors, $\mathcal{D}_3^8(1)$ ($W(E_8)$ action on L_8 can be extended to $\mathcal{D}_3^8(1)$), one can show that $\mathcal{D}_3^8(1)$ is bijective to the set of exceptional systems on S_8 which is the union of the two sets which bijectively correspond to the set of skew 8-lines and the set of roots in S_8. Thus we conclude the following lemma.

Lemma 7. *Each divisor D in Pic S_8 with $D^2 = 8$ and $D \cdot K_{S_8} = -4$ is an $A_3^8(1)$-divisor, i.e. it can be written as a sum of four lines with 1-intersection.*

(a) Tied 1-degree 3-simplex in 4_{21}.

Before we describe the configurations of $A_3^8(1)$-vertices in 4_{21}, we need to find a better way to separate the $A_3^8(1)$-vertices in 4_{21} into the two different sets above. Here we consider the condition *tied*. If an $A_3^8(1)$-vertex in 4_{21} is tied, there is a line l where all the lines in the $A_3^8(1)$-vertex are in the vertex figure of l. Since lines of S_8 correspond to roots of S_8, the line l whose vertex figure containing the tied $A_3^8(1)$-vertex gives a root $l + K_{S_8}$. This roughly implies that the centers of tied $A_3^8(1)$-vertices in 4_{21} correspond to the roots of S_8. In fact, we can explain explicitly the correspondence in the following proposition.

Proposition 8. *Let D be an $A_3^8(1)$-divisor in 4_{21}. The divisor class D corresponds to a root d of S_8 by $D + 4K_S = 2d$ if and only if each $A_3^8(1)$-vertex with the center D is tied by the line $l = d - K_{S_8}$.*

Proof. Suppose D gives a root d by $D + 4K_S = 2d$. Then we can define a line $l = d - K_{S_8} = D/2 + K_{S_8}$. For each line l' in the $A_3^8(1)$-vertex in 4_{21} with center D, we have $l' \cdot l = l' \cdot (D/2 + K_{S_8}) = 0$. Therefore, all the lines consisting of the $A_3^8(1)$-vertex are in the vertex figure of l, and the $A_3^8(1)$-simplex is tied by l.

Assume the $A_3^8(1)$-simplex with the center D is tied; then there is a line l where each line in the $A_3^8(1)$-vertex is in the vertex figure of l. Furthermore, the divisor class D and the line l are related by $D = 2l - 2K_{S_8}$. This relation can be verified by checking $(2l - 2K_{S_8} - D)^2 = 0$ and $(2l - 2K_{S_8} - D) \cdot K_{S_8} = 0$ with $D \cdot l = 0$. Now the divisor d given by $l + K_{S_8}$ satisfies

$$2d = 2(l + K_{S_8}) = (2l - 2K_{S_8}) + 4K_{S_8} = D + 4K_S.$$

□

Remark. The tied condition in the proposition is determined by the $A_3^8(1)$-divisor of the $A_3^8(1)$-vertex rather than the lines in the $A_3^8(1)$-vertex. Therefore, all the $A_3^8(1)$-vertices in 4_{21} with a common $A_3^8(1)$-divisor D are tied by a common line if D corresponds to a root d as above.

Corollary 9. *For a tied $A_3^8(1)$-vertex in 4_{21} given by lines l_1, l_2, l_3 and l_4, the line l in the above proposition is written as*

$$l = K_{S_8} + \frac{(l_1 + l_2 + l_3 + l_4)}{2},$$

and l is the only line in L_8 where its vertex figure contains l_1, l_2, l_3 and l_4.

Proof. The expression of l is from Proposition 8, and we only need to prove the uniqueness of line l. Assume l_a is another line in L_8 where $\{l_1, l_2, l_3, l_4\} \subset N_0(l_a, S_8)$. By checking the possible intersection $l \cdot l_a$, we show that $l = l_a$.

The intersection $l_a \cdot l$ is not zero because two skew lines l_a and l produce a blow down from S_8 to S_6 inducing a 1-degree 3-simplex in 2_{21} which does not exist. If $l_a \cdot l = 1$, then $l_a + l$ is a ruling on S_8 where the lines l_1, l_2, l_3 and l_4 are the vertices of the 7-crosspolytope in 4_{21} corresponding to $l_a + l$ (see subsection 2.2 on lines in a ruling). But a line in a 7-crosspolytope must have only one other line in the 7-crosspolytope with 1-intersection, while l_1 has more than one other line in it. Thus $l_a \cdot l \neq 1$. If $l_a \cdot l = 2$, by Theorem 2 $l_a + l + K_{S_8}$ gives another line l_b such that $l_i \cdot l_b = l_i \cdot (l_a + l + K_{S_8}) = -1$ for $i = 1, 2, 3, 4$. Thus $l_i = l_b$ for $i = 1, 2, 3, 4$, and this is a contradiction. Therefore $l_a \cdot l \neq 2$. If $l_a \cdot l = 3$, then $l_a = B(l) = -2K_{S_8} - l$. But since $l_1 \cdot l = 0$, i.e. l_1 is in the vertex figure of l, we have a contradiction that $l_a \cdot l_1 = (-2K_{S_8} - l) \cdot l_1 = 2$. At last, $l_a \cdot l \neq 3$. By subsection 2.3, the intersection of two lines in L_8 is no larger than 3. Thus from above, $l_a \cdot l$ must be -1. Thus $l = l_a$. $\square$

Recall that the $W(E_8)$ action gives two orbits in the exceptional systems $\mathcal{E}_8$ in *Pic* S_8, which correspond to the set of roots in *Pic* S_8 and skew 7-lines in 4_{21}. In subsection 2.2, the set of the centers of the tied $A_3^8(1)$-vertices in 4_{21} lead us to the set of roots in S_8, and we expect the set of the centers of the untied $A_3^8(1)$-vertices in 4_{21} to correspond to the set of skew 8-lines in *Pic* S_8. The proof of the following corollary is similar to Proposition 8, and we leave it to the reader.

Corollary 10. *Let D be a center of an $A_3^8(1)$-simplex in 4_{21}. The divisor class D corresponds to a skew 8-line D_1 in L_1^8 as $3D + 4K_{S_8} = D_1$ if and only if the $A_3^8(1)$-simplex is untied.*

Remark. This corollary implies that all the $A_3^8(1)$-vertices in 4_{21} with center D corresponding to skew 8-lines, namely 7-simplexes, are untied.

From Corollary 9 above, we can obtain the following important proposition.

Proposition 11. *Let l_i, $1 \leq i \leq 5$, be lines in L_8 such that $l_i \cdot l_j = 1$ for $i \neq j$. If l_1, l_2, l_3 and l_4 form a tied $A_3^8(1)$-simplex in 4_{21}, another $A_3^8(1)$-simplex given by l_1, l_2, l_3 and l_5 is untied.*

Proof. Let l be the line where l_1, l_2, l_3 and l_4 are in the vertex figure of l. By Corollary 9, l is $K_{S_8} + \frac{l_1+l_2+l_3+l_4}{2}$. Now we assume the $A_3^7(1)$-simplex given by l_1, l_2, l_3 and l_5 is tied. Then there is a line l' given by $K_{S_8} + \frac{l_1+l_2+l_3+l_5}{2}$. We consider

$$l \cdot l' = \left(K_{S_8} + \frac{l_1 + l_2 + l_3 + l_4}{2}\right) \cdot \left(K_{S_8} + \frac{l_1 + l_2 + l_3 + l_5}{2}\right) = -\frac{1}{2}.$$

Since the intersection must be an integer, this is a contradiction to the assumption. This gives the proposition. $\square$

Now the configuration of tied $A_3^8(1)$-vertices in 4_{21} is given as follows.

Let D be a tied $A_3^8(1)$-divisor in 4_{21}. Then we get a line $l = D/2 + K_{S_8}$, where each $A_3^8(1)$-vertex of D is in the vertex figure of l. By a blow down map $\pi_l^8 : S_8 \to S_7$ given by l, the simplex in 4_{21} is mapped to $A_3^7(1)$-vertices in 3_{21}, which is a 4-Steiner block in $\mathcal{S}_C(4, S_7)$ on S_7. Therefore, the configuration of tied $A_3^8(1)$-vertices in 4_{21} with a common $A_3^8(1)$-divisor equals the 4-Steiner system in $\mathcal{S}_C(4, S_7)$ on S_7.

By combining this and the result from the case $r = 7$ of $A_3^7(1)$-simplexes, we have the following theorem.

Theorem 12. *Both the configuration of tied $A_3^8(1)$-vertices in 4_{21} with a common $A_3^8(1)$-divisor and the configuration of $A_3^7(1)$-vertices in 3_{21} equal the 4-Steiner system in $\mathcal{S}_C(4, S_7)$ on S_7.*

(b) Untied 1-degree 3-simplex in 4_{21}.

From the previous subsubsection, we know that each center of $A^8_2(1)$-vertices in 4_{21} gives a skew 2-line which corresponds to an edge in 4_{21}, namely a skew 2-line is given by a unique pair of lines. Thus the center also determines two disjoint lines. By using this fact, we have the following facts for the configuration of untied $A^8_3(1)$-simplexes in 4_{21}.

Definition 13. Two edges D_1 and D_2, namely, skew 2-lines, in $(r-4)_{21}$ of a del Pezzo surface S_r are called *skew edges* if they are two disjoint edges in a (0-degree) 3-simplex in $(r-4)_{21}$, equivalent to two disjoint edges with $D_1 \cdot D_2 = 0$.

Proposition 14. *Let D be a center of an $A^8_3(1)$-vertex in 4_{21}. If the vertex is untied, the four lines in the $A^8_3(1)$-vertex induce four pairwise skew edges in a 7-simplex in 4_{21} whose center is $3D + 4K_{S_8}$.*

Proof. Let l_1, l_2, l_3 and l_4 be four lines in the $A^8_3(1)$-vertex in 4_{21} with center $D = l_1 + l_2 + l_3 + l_4$. There are four $A^8_2(1)$-vertices in 4_{21}, each given by three choices from $\{l_1, l_2, l_3, l_4\}$. For example, we consider an $A^8_2(1)$-vertex in 4_{21} given by l_1, l_2 and l_3 with the center $l_1 + l_2 + l_3$. Then there is a skew 2-line $l^{123}_a + l^{123}_b$ given by $l_1 + l_2 + l_3 + K_{S_8} = l^{123}_a + l^{123}_b$, and l_1, l_2 and l_3 are in the vertex figures of l^{123}_a and l^{123}_a as in the previous subsubsection. But $l^{123}_i \cdot l_4 \geq 1$ for $i = a, b$ because the $A^8_3(1)$-vertex is untied. In fact, we have $l^{123}_i \cdot l_4 = 1$ for $i = a, b$ since

$$2 = (l_1 + l_2 + l_3 + K_{S_8}) \cdot l_4 = \left(l^{123}_a + l^{123}_b\right) \cdot l_4 = l^{123}_a \cdot l_4 + l^{123}_b \cdot l_4.$$

Thus from the lines l_1, l_2 and l_3, providing an $A^8_2(1)$-vertex in the untied $A^8_3(1)$-simplex in 4_{21}, we have two disjoint lines l^{123}_a and l^{123}_b such that

$$l_j \cdot l^{123}_i = 0 \text{ and } l_4 \cdot l^{123}_i = 1 \text{ for } j = 1, 2, 3 \text{ and } i = a, b.$$

By performing this process for the other $A^8_2(1)$-vertices, we have four pairs of edges $l^{lmn}_a + l^{lmn}_b$ for $lmn \in S =: \{123, 124, 134, 234\}$ in 4_{21}. Here all of the eight lines in l^{lmn}_i must be distinct because of their intersections to l_1, l_2, l_3 and l_4. Therefore, we have four pairs of edges $l^{lmn}_a + l^{lmn}_b$ for $lmn \in S$ that are pairwise disjoint. Furthermore, we observe

$$\left(l^{234}_a + l^{234}_b\right) \cdot \left(l^{134}_a + l^{134}_b\right) = (l_2 + l_3 + l_4 + K_{S_8}) \cdot (l_1 + l_3 + l_4 + K_{S_8}) = 0$$

and conclude that the four disjoint edges are skew edges to each other. This implies that the distinct eight lines of l^{lmn}_i have 0-intersection to each other. Therefore, they are the vertices of a 7-simplex in 4_{21}. Moreover, the center D_1 of the 7-simplex is

$$D_1 = \sum_{lmn\ \in S} (l^{lmn}_a + l^{lmn}_b) = 3\sum_{j=1}^{4} l_j + 4K_{S_8}.$$

This completes the proposition. □

Remark. This proposition shows how to identify the skew 8-lines in Corollary 10.

As the converse of the above proposition, we have the following theorem.

Theorem 15. *Let D_1 be a center of a 7-simplex in 4_{21}. For each set of four pairwise skew edges in a 7-simplex in 4_{21}, there is an untied $A^8_3(1)$-vertex in 4_{21}. Moreover, each edge in the set of the 4-skew edges gives a line in the $A^8_3(1)$-vertex.*

Proof. Let a_1, a_2, a_3 and a_4 be 4-skew edges in a 7-simplex in 4_{21} with $a_1+a_2+a_3+a_4=D_1$. We denote $a_i=l_i^1+l_i^2$ for $i=1,2,3,4$, where l_i^1, l_i^2 in L_8. Since we have skew edges, l_i^j's are distinct 8-lines which form the 7-simplex with center D_1. By Corollary 10, the center of a 7-simplex with center D_1 gives the center D of an $A_3^8(1)$-simplex in $4_{21}$by $3D+4K_{S_8}=D_1$. Thus $\left(D_1-4K_{S_8}\right)/3$ is an integral class in *Pic* S_8, and we can define a class l_{a_i} in *Pic* S_8 for the edge a_i as

$$l_{a_i}:=\frac{\left(D_1-K_{S_8}\right)}{3}-a_i \text{ for } i=1,2,3,4.$$

The classes of l_{a_i} are lines because $l_{a_i}\cdot K_{S_8}=-1$ and $l_{a_i}^2=-1$. Furthermore, we have $l_{a_i}\cdot l_{a_j}=1+a_i\cdot a_j=1$ for $i\neq j$ and

$$l_{a_1}+l_{a_2}+l_{a_3}+l_{a_4}=\frac{4\left(D_1-K_{S_8}\right)}{3}-(a_1+a_2+a_3+a_4)=D.$$

Therefore, the l_{a_i}'s are four vertices of an untied $A_3^8(1)$-simplex in 4_{21} with the center D. This gives the theorem. □

By the above proposition and theorem, we have the following configuration of untied $A_3^8(1)$-vertices in 4_{21}.

Theorem 16. *Let D be the center of an untied $A_3^8(1)$-simplex in 4_{21} and D_1 be the corresponding skew 8-lines in 4_{21}. The configuration of the $A_3^8(1)$-simplexes with a common center D in 4_{21} equals the family of subsets consisting of 4-skew edges in the 7-simplex with center D_1 in 4_{21}.*

(c) Construction of tied $A_3^8(1)$-vertices from $A_2^8(1)$-vertices in 4_{21}.

Recall that each $A_2^8(1)$-vertex in 4_{21} corresponds to skew 2-line, and an $A_2^8(1)$-vertex is in the vertex figure of each line in the skew 2-line. For example, we have three lines l_1, l_2 and l_3 making an $A_2^8(1)$-simplex, and $l_a^{123}+l_b^{123}$ is the corresponding skew 2-line with $l_1+l_2+l_3+K_{S_8}=l_a^{123}+l_b^{123}$. Now, we observe that the $A_2^8(1)$-vertex given by l_1, l_2 and l_3 can be extended to exactly one tied $A_3^8(1)$-simplex in the vertex figure of l_a^{123} by using monoidal transform given by l_a^{123} and the Steiner system $\mathcal{S}_C(4,S_7)$. Similarly, we can obtain a unique tied $A_3^8(1)$-simplex containing l_1, l_2 and l_3 in the vertex figure of l_b^{123}. Therefore, we study the relationship between the above $A_2^8(1)$-vertex and the corresponding skew 2-lines for the configuration of tied $A_3^8(1)$-simplexes containing the $A_2^8(1)$-vertex.

One of the important features of a tied $A_3^8(1)$-vertex is that there is a line whose vertex figure contains the $A_3^8(1)$-vertex. For a line in L_8 and the vertex figure, we consider the blow down map $\pi_l^8: S_8\to S_7$ and observe that the Gieser transform G on *Pic* S_7 can also be defined on each vertex figure in 4_{21}. This gives the following definition.

Definition 17. Let l be a line in L_8 and l' be a line in the vertex figure of l. The line $G_l(l')$, defined as

$$G_l(l'):=\pi_l^{8*}\left(G\left(\pi_{l*}^8(l')\right)\right)=-(K_{S_8}-l)-l',$$

is called the Gieser transform of l' for l in 4_{21}.

Remark.

(1) It is easy to see that $G_l(l')$ is in the vertex figure of l, and the definition implies $G_l(l')\cdot l'=2$.

(2) Because $l \cdot l' = 0$, we have two roots $\pm (l - l')$, and one of the roots $l - l'$ determines the line $G_l(l')$ as

$$G_l(l') = -K_{S_8} + (l - l')$$

and the other root determines a line $G_{l'}(l)$.

The Gieser transform of l' for l in 4_{21} and the Bertini transform B on 4_{21} are related as follows.

Lemma 18. *Let l and l' be lines in L_8 with $l \cdot l' = 0$. Then we have (1) $G_l(l') = B(G_{l'}(l))$ and (2) $B(l') = G_{G_l(l')}(l)$.*

Proof. (1) Observe $G_l(l') + G_{l'}(l) = -2K_{S_8}$. Thus $G_l(l') = -2K_{S_8} - G_{l'}(l) = B(G_{l'}(l))$.

(2) By the definition of G_l, we have $G_l(G_l(l')) = l'$, and from (1) we obtain

$$G_{G_l(l')}(l) = B(G_l(G_l(l'))) = B(l').$$

□

The role of the Gieser transform in 4_{21} on an $A_2^8(1)$-vertex in 4_{21} is explained in the following lemma.

Lemma 19. *Let l_1, l_2 and l_3 be three lines in L_8 giving an $A_2^8(1)$-simplex in 4_{21}, and $l_a^{123} + l_b^{123}$ be the corresponding skew 2-lines.*

(1) The line $G_{l_a^{123}}(l_b^{123})$ can be written as $G_{l_a^{123}}(l_b^{123}) = 2\left(l_a^{123} - K_{S_8}\right) - D$ where $D = (l_1 + l_2 + l_3)$.

(2) l_1, l_2, l_3 and $G_{l_a^{123}}(l_b^{123})$ (resp. $G_{l_b^{123}}(l_a^{123})$) form a tied $A_3^8(1)$-simplex in the vertex figure of l_a^{123} (resp. l_b^{123}).

(3) $G_{l_a^{123}}(l_b^{123})$ and $G_{l_b^{123}}(l_a^{123})$ are transformed to each other by the Bertini transform B.

Proof. (1) It is directly given by $(l_1 + l_2 + l_3) + K_{S_8} = l_a^{123} + l_b^{123}$.

(2)We just need to check $l_i \cdot G_{l_a^{123}}(l_b^{123}) = l_i \cdot \left(-K_{S_8} + l_a^{123} - l_b^{123}\right) = 1$ for $i = 1, 2, 3$.

(3) This is given by Lemma 18. □

4.2. **1-degree $m(4 \leq m \leq 7)$-simplexes in 4_{21}.** In 4_{21}, the 1-degree m-simplexes, $A_m^8(1)$-vertices exist for $m \leq 7$ for the following reason.

The center of each $A_m^8(1)$-vertex represents an $A_m^8(1)$-divisor $l_1 + ... + l_{m+1}$ where $l_1, ..., l_{m+1}$ lines with $l_i \cdot l_j = 1$, $i \neq j$, which satisfies $(l_1 + ... + l_{m+1})^2 = m^2 - 1$ and $(l_1 + ... + l_{m+1}) \cdot K_{S_r} = -(m+1)$. Since lines l in L_8 correspond to roots $l + K_{S_8}$, the divisor class $l_1 + ... + l_{m+1}$ is transformed to a divisor class

$$D_1 := (l_1 + K_{S_8}) + ... + (l_{m+1} + K_{S_8})$$

with $D_1^2 = -2(m+1)$ and $D_1 \cdot K_{S_8} = 0$. Since $(l_i + K_{S_8}) \cdot (l_j + K_{S_8}) = l_i \cdot l_j - 1$, the divisor D_1 is a sum of perpendicular roots. Because the space of roots in $Pic\ S_8$ is 8-dimensional, the biggest number of m is 7. Since cases with $m \leq 3$ are discussed as above, $m = 4, 5, 6$ and 7 are the remaining cases we study in this subsection.

According to chapter 4 of [2], the set of divisors D with $D^2 = -2(m+1)$ and $D \cdot K_{S_8} = 0$ correspond to the set of vectors with norm $2(m+1)$ in E_8 lattices. Here, each divisor set $\mathcal{D}_m^8(1)$ of $A_m^8(1)$-vertices $4 \leq m \leq 7$ in 4_{21} is a subset of the set of vectors with norm 10, 12, 14 and 16 in E_8-lattices, respectively. Here the total number of vectors with norm 10, 12, 14 and 16 in E_8-lattices are 30240,

$60480, 82560$ and 140400, respectively. Moreover, each set of vertices with norm 10 and 12 which contains divisors $l_1 + ... + l_{m+1} + (m+1)K_{S_8}$ for $m = 4, 5$ are transitively acted upon by the Weyl group $W(E_8)$ (chapter 4 of [2]). Since there is only one orbit of $W(E_8)$ action on each set of vectors with norm 10 and 12, the numbers of $A^8_4(1)$-divisors and $A^8_5(1)$-simplexes are 30240 and 60480, respectively. Here we note that the number of skew 3-lines in S_8 is also 60480.

Remark. As before, by $W(E_8)$ action, we conclude that each divisor D with $D^2 = 15$ and $D \cdot K_{S_8} = -5$ (resp. $D^2 = 24$ and $D \cdot K_{S_8} = -6$) is an $A^8_4(1)$-divisor (resp. an $A^8_5(1)$-divisor), namely a sum of 4-lines (resp. 5-lines) with 1-intersection.

4.2.1. *Rulings and* 1*-degree* 4*-simplexes.* Because the 1-degree 4-simplexes, $A^8_4(1)$-vertices exist only for $r = 8$, there is no line in $Pic\ S_8$ whose vertex figure contains an $A^8_4(1)$-vertex. Thus every $A^8_4(1)$-vertex is untied. For each tied $A^8_3(1)$-vertex, by adding a line which intersects with each line in the $A^8_3(1)$-vertex, we can obtain an $A^8_4(1)$-vertex. But the choice of a new line is too arbitrary to describe the configuration of $A^8_4(1)$-vertices. Thus we begin with $A^8_4(1)$-vertices constructed from untied $A^8_3(1)$-vertices in 4_{21}. This leads us to the configuration of $A^8_4(1)$-vertices in 4_{21}.

Let l_1, l_2, l_3 and l_4 be lines in L_8 which form an untied $A^8_3(1)$-vertex in 4_{21}. We consider the skew 2-lines $l^{123}_a + l^{123}_b$ corresponding to $l_1 + l_2 + l_3$, namely, $l_1 + l_2 + l_3 + K_{S_8} = l^{123}_a + l^{123}_b$. By Lemma 19, we have a tied $A^8_3(1)$-vertex in 4_{21} given by $\{l_1, l_2, l_3, G_{l^{123}_a}(l^{123}_b)\}$. In fact we obtain an $A^8_4(1)$-vertex in 4_{21} as follows.

Theorem 20. *Suppose l_1, l_2, l_3 and l_4, lines in L_8 give an untied $A^8_3(1)$-vertex in 4_{21}, and $l^{123}_a + l^{123}_b$ is the skew 2-line corresponding to $l_1 + l_2 + l_3$. The five lines l_1, l_2, $l_3, G_{l^{123}_a}(l^{123}_b)$ and l_4 form an $A^8_4(1)$-vertex in 4_{21}.*

Proof. We only need to check $G_{l^{123}_a}(l^{123}_b) \cdot l_4 = 1$, and we need to get $l^{123}_a \cdot l_4$ and $l^{123}_b \cdot l_4$ first.

We consider another set of skew 2-lines $l^{124}_a + l^{124}_b$ given by $l_1 + l_2 + l_4 + K_{S_8} = l^{124}_a + l^{124}_b$. Since the 3-simplex is untied, by Theorem 15 $l^{124}_a + l^{124}_b$ and $l^{124}_a + l^{124}_b$ are skew edges. Thus we have $l^{123}_a \cdot \left(l^{124}_b + l^{124}_b\right) = l^{123}_b \cdot \left(l^{124}_a + l^{124}_b\right) = 0$. Furthermore, $l^{123}_a \cdot l_4 = 1 = l^{123}_b \cdot l_4$.

Now, we have

$$G_{l^{123}_a}(l^{123}_b) \cdot l_4 = \left(-K_{S_8} + l^{123}_a - l^{123}_b\right) \cdot l_4 = 1,$$

and $\{l_1, l_2, l_3, G_{l^{123}_a}(l^{123}_b), l_4\}$ gives an $A^r_m(b)$-vertex 1-degree 4-simplex in 4_{21}. □

Remark. This theorem implies that any line l where $\{l_1, l_2, l_3, l\}$ gives an untied $A^8_3(1)$-vertex in 4_{21} satisfies $G_{l^{123}_a}(l^{123}_b) \cdot l = G_{l^{123}_b}(l^{123}_a) \cdot l = 1$.

In Theorem 20, the $A^8_4(1)$-vertex in 4_{21} given by $\{l_1, l_2, l_3, G_{l^{123}_a}(l^{123}_b), l_4\}$ contains a tied $A^8_3(1)$-vertex consisting of $\{l_1, l_2, l_3, G_{l^{123}_a}(l^{123}_b)\}$. In fact this is the only possible tied $A^8_3(1)$-vertex in the $A^8_4(1)$-vertex because of Proposition 11. It turns out that this is true for all the $A^8_4(1)$-vertices in 4_{21} as follows.

Theorem 21. *Let $\{l_i\ 1 \leq i \leq 5\}$ be the set of lines in L_8 giving an $A^8_4(1)$-vertex in 4_{21}. There is a unique subset in $\{l_i\ 1 \leq i \leq 5\}$ which gives a tied $A^8_3(1)$-vertex.*

Proof. **(Existence)** Suppose all the $A^8_3(1)$-vertices in $\{l_i\ 1 \leq i \leq 5\}$ are untied. Then we consider an $A^8_2(1)$-vertex given by $\{l_1, l_2, l_3\}$ and the corresponding skew 2-lines $l^{123}_a + l^{123}_b$. By Lemma 19 and the remark of Theorem 20, we have

$G_{l_a^{123}}(l_b^{123}) \cdot l_4 = G_{l_a^{123}}(l_b^{123}) \cdot l_5 = 1$ because $\{l_1, l_2, l_3, l_4\}$ and $\{l_1, l_2, l_3, l_5\}$ are untied $A_3^8(1)$-vertices. Thus an $A_5^8(1)$-vertex in 4_{21} is given by $\{l_1, l_2, l_3, l_4, l_5, G_{l_a^{123}}(l_b^{123})\}$. Similarly, for skew 2-lines $l_a^{124} + l_b^{124}$ corresponding to the $A_2^8(1)$-vertex of $\{l_1, l_2, l_4\}$, we obtain another $A_5^8(1)$-vertex $\{l_1, l_2, l_3, l_4, l_5, G_{l_a^{124}}(l_b^{124})\}$ in 4_{21}. Furthermore, we observe that $G_{l_a^{123}}(l_b^{123}) \cdot G_{l_a^{124}}(l_b^{124}) = 1$ is verified as

$$G_{l_a^{123}}(l_b^{123}) \cdot G_{l_a^{124}}(l_b^{124}) = \left(-K_{S_8} + l_a^{123} - l_b^{123}\right) \cdot \left(-K_{S_8} + l_a^{124} - l_b^{124}\right) = 1.$$

Here $l_a^{123} \cdot l_a^{124} = l_a^{123} \cdot l_b^{124} = l_b^{123} \cdot l_a^{124} = l_b^{123} \cdot l_b^{124} = 0$ because $l_a^{123} + l_b^{123}$ and $l_a^{124} + l_b^{124}$ are two skew edges given from $\{l_1, l_2, l_3, l_4\}$ by Proposition 14. Thus any three lines from $\{l_1, l_2, l_3, l_4\}$ give a line so that

$$\{l_1, l_2, l_3, l_4, l_5, G_{l_a^{123}}(l_b^{123}), G_{l_a^{124}}(l_b^{124}), G_{l_a^{134}}(l_b^{134}), G_{l_a^{234}}(l_b^{234})\}$$

makes an $A_8^8(1)$-vertex in 4_{21}. But this 1-degree 8-simplex may not exist in 4_{21}. Thus we conclude that there exists a subset in $\{l_i \ 1 \le i \le 5\}$ which gives a tied $A_3^8(1)$-vertex.

(Uniqueness) Assume $\{l_1, l_2, l_3, l_4\}$ is a tied $A_3^8(1)$-vertex from $\{l_i \ 1 \le i \le 5\}$. There are four other possible choices of $A_3^8(1)$-vertices in $\{l_i \ 1 \le i \le 5\}$ which are obtained by replacing one of the lines in $\{l_1, l_2, l_3, l_4\}$ by l_5. But by Proposition 11, none of these four $A_3^8(1)$-vertices can be tied. Thus there is only one tied $A_3^8(1)$-vertex from $\{l_i \ 1 \le i \le 5\}$. $\square$

Recall that the number of $A_4^8(1)$-divisors is 30240. We can observe that this number equals the number of ordered pairs of lines in L_8 with 1-intersection, namely, $|L_8|\,|N_1(l, S_8)| = 240 \times 126 = 240 \times (5^3 + 1^3) = 30240$. In the following theorem, we show that the coincidence of these two numbers leads us to the configuration of $A_4^8(1)$-simplexes in 4_{21}.

Theorem 22. *Let D be an $A_4^8(1)$-divisor. There is a unique line $l_D^{S_8}$ in L_8 and the center A_D of a tied $A_3^8(1)$-vertex such that $D = A_D + l_D^{S_8}$. Furthermore, the divisor set $\mathcal{D}_4^8(1)$ in 4_{21} is bijective to the ordered set $\tilde{F}_8$ defined as*

$$\tilde{F}_8 := \{(l_1, l_2) \mid l_1, l_2 \in L_8 \text{ with } l_1 \cdot l_2 = 1\}.$$

Proof. From Theorem 21 above, each $A_4^8(1)$-divisor D in 4_{21} can be written as the sum of a center B_1 of a tied $A_3^8(1)$-vertex and a line l_1 in L_8. Suppose B_2 and l_2 are another center of another tied $A_3^8(1)$-vertex and a line in L_8 with $D = B_1 + l_1 = B_2 + l_2$, $(B_1, l_1) \neq (B_2, l_2)$.

Since B_1 is the center of a tied $A_3^8(1)$-vertex, by Corollary 9 there is a uniquely determined line l as $l = K_{S_8} + B_1/2$ whose vertex figure contains the tied $A_3^8(1)$-vertex with a center B_1. Thus $B_1 \cdot l = 0$, and $D \cdot l = 1$ because $l_1 \cdot l = l_1 \cdot (K_{S_8} + B_1/2) = 1$. This implies that $l \cdot B_2 + l \cdot l_2 = 1$. Now we consider the following cases for $l \cdot l_2$. Since at most one line of B_2 coincides with l_1, we only consider $-1 \le l \cdot l_2 \le 2$.

(a) If $l \cdot l_2 = -1$, then $l \cdot B_2 = 2$. Since $l = l_2$, we have $l_2 \cdot B_2 = l \cdot B_2 = 2$. But by the definition of the $A_4^8(1)$-vertex, $l_2 \cdot B_2$ must be 4. Thus $l \cdot l_2 \neq -1$.

(b) If $l \cdot l_2 = 0$, then $l \cdot B_2 = 1$. Here l cannot be in any tied $A_3^8(1)$-vertex with a common center B_2, because in this case $l \cdot B_2$ must be 2 instead of 1 by the definition of the $A_3^8(1)$-vertex. Now since l is not in any tied $A_3^8(1)$-vertex with a common center B_2, for each tied $A_3^8(1)$-vertex C with its $A_3^8(1)$-divisor B_2, there is a unique line l_C with $l \cdot l_C = 1$ and $C \backslash \{l_C\}$ is in the vertex figure of l. Thus

$C\backslash\{l_C\}$ and l_2 form an $A_3^8(1)$-vertex tied by the line l. But by Theorem 21, this $A_3^8(1)$-vertex cannot be tied. Thus $l \cdot l_2 \neq 0$.

(c) If $l \cdot l_2 = 1$, then $l \cdot B_2 = 0$. Again, l is not in any tied $A_3^8(1)$-vertex with a common center B_2, and thus l has 0-intersection to each line in tied $A_3^8(1)$-vertices with a common center B_2. This implies that l and B_2 correspond by Corollary 9. Therefore, we have $B_1 = B_2$, which is a contradiction. Thus $l \cdot l_2 \neq 1$.

(d) If $l \cdot l_2 = 2$, then $l \cdot B_2 = -1$. Here l must be one of the lines in each tied $A_3^8(1)$-vertex with a common center B_2. But in that case $l \cdot B_2 = 2$ instead of -1. Thus $l \cdot l_2 \neq -1$.

From the above cases, we conclude that the choice of center B_1 and a line l_1 for $D = B_1 + l_1$ is unique.

By Corollary 9, we get a line l' from the center B_1, and l' has $l' \cdot l_1 = 1$ which is an ordered pair (l'_1, l_1) in $\tilde{F}_8$.

As we know $\left|\mathcal{D}_4^8(1)\right| = \left|\tilde{F}_8\right|$, obviously, each ordered pair of lines in $\tilde{F}_8$ gives an $A_4^8(1)$-divisor in 4_{21}.

Thus we have the theorem. □

Remark. Even though each ordered pair (l_1, l_2) in $\tilde{F}_8$ corresponds to the $A_4^8(1)$-divisor in 4_{21}, not all the $A_3^8(1)$-vertices given by a line l_1 produce $A_4^8(1)$-vertices after combining the line l_2.

By the above theorem, we have the following theorem for configuration of $A_4^8(1)$-vertices in 4_{21}. Here, the configuration of the set of tied $A_3^8(1)$-vertices with a common center is determined in Theorem 12.

Theorem 23. *Suppose D is an $A_4^8(1)$-divisor in 4_{21}. All the $A_4^8(1)$-vertices with center D in 4_{21} share a common line l, and moreover $D - l$ is the common center of the uniquely determined tied $A_3^8(1)$-vertices in each $A_4^8(1)$-vertex with center D.*

4.2.2. *Skew 3-lines and 1-degree 5-simplexes.* Recall that the number of $A_5^8(1)$-divisors in 4_{21} equals the number of skew 3-lines in 4_{21} where each skew 3-line can be written as the sum of unique three lines in L_8. The uniqueness for the skew 3-lines leads us to the following lemma, and therefore, we get the configuration of $A_5^8(1)$-vertices in 4_{21}.

Lemma 24. *For each of the six lines in L_8 consisting of an $A_5^8(1)$-vertex in 4_{21}, there exist three tied $A_3^8(1)$-vertices in it, and the choice is unique. Furthermore, the given six lines can be labelled as $l_i, 1 \leq i \leq 6$, where $\{l_1, l_2, l_3, l_4\}$,$\{l_1, l_2, l_5, l_6\}$ and $\{l_3, l_4, l_5, l_6\}$ are the tied $A_3^8(1)$-vertices.*

Proof. Recall that for each of the five lines from the given six lines, there exists a tied $A_3^8(1)$-simplex by Theorem 21. We choose four lines from the six lines which give a tied $A_3^8(1)$-simplex, and label them as l_1, l_2, l_3 and l_4. By Proposition 11, the lines from another tied $A_3^8(1)$-simplex and $\{l_1, l_2, l_3, l_4\}$ share 1 or 2 lines. Since there are only six lines in the $A_5^8(1)$-vertex, they share 2-lines, and we can relabel six lines so that $\{l_1, l_2, l_3, l_4\}$ and $\{l_1, l_2, l_5, l_6\}$ produce tied $A_3^8(1)$-simplexes. By applying Theorem 21 and Proposition 11 to each $\{l_1, l_2, l_3, l_4, l_5, l_6\}\backslash\{l_i\}, 1 \leq i \leq 6$, we deduce that there is one more tied $A_3^8(1)$-simplex given by $\{l_3, l_4, l_5, l_6\}$ which also shares two lines with $\{l_1, l_2, l_3, l_4\}$ and $\{l_1, l_2, l_5, l_6\}$. This gives the lemma. □

Remark. According to Lemma 24, for each $A_5^8(1)$-vertex A in 4_{21}, there are three unique disjoint subsets $A_i, i = 1, 2, 3$, of A consisting of two lines such that each $A - A_i$ produces a tied $A_3^8(1)$-vertex.

Theorem 25. *Let D be the center of an $A_5^8(1)$-vertex in 4_{21}. The center D corresponds to skew 3-lines $l_a + l_b + l_c$ by $D + 3K_{S_8} = l_a + l_b + l_c$. Furthermore, each $A_5^8(1)$-vertex in 4_{21} with center D has three uniquely determined tied $A_3^8(1)$-vertices where they are in the vertex figure of l_a,l_b and l_c respectively.*

Proof. Since the center D satisfies $D^2 = 24$, $D \cdot K_{S_8} = -6$, the new divisor class $D+3K_{S_8}$ also satisfies $(D + 3K_{S_8})^2 = -3$, $(D + 3K_{S_8}) \cdot K_{S_8} = -3$, and it is a skew 3-line. Since a skew 3-line is the sum of three unique disjoint lines (subsection 2.2), we have $D + 3K_{S_8} = l_a + l_b + l_c$, where l_a, l_b and l_c are disjoint lines in L_8.

On the other hand, by Lemma 24 above, we label the given six lines as $l_i, 1 \leq i \leq 6$, so that $\{l_1, l_2, l_3, l_4\}$,$\{l_1, l_2, l_5, l_6\}$ and $\{l_3, l_4, l_5, l_6\}$ are tied $A_3^8(1)$-simplexes. By Corollary 9, these tied $A_3^8(1)$-simplexes produce lines l^a, l^b and l^c in L_8. Here, the intersection $l^a \cdot l^b$ is

$$\left(\frac{l_1 + l_2 + l_3 + l_4}{2} + K_{S_8}\right) \cdot \left(\frac{l_1 + l_2 + l_5 + l_6}{2} + K_{S_8}\right) = 0,$$

and similarly l^a, l^b and l^c are disjoint from each other. Furthermore, we have

$$\begin{aligned} l^a + l^b + l^c &= (l_1 + l_2 + l_3 + l_4 + l_5 + l_6) + 3K_{S_8} \\ &= D + 3K_{S_8} = l_a + l_b + l_c. \end{aligned}$$

Thus l^a, l^b and l^c form the skew 3-line $l_a + l_b + l_c$, and by the uniqueness of the skew 3-lines, we have $\{l^a, l^b, l^c\} = \{l_a, l_b, l_c\}$. This gives the theorem. □

4.2.3. *Fano planes and $A_6^8(1)$-simplexes.* The set of $A_6^8(1)$-divisors in 4_{21} is a subset of the set of vectors with norm 14 in an E_8-lattice which has more than one E_8-orbit. But we figure out that the configuration of $A_6^8(1)$-vertices in 4_{21} is somewhat similar to that of $A_5^8(1)$-vertices. Here each $A_6^8(1)$-divisor gives skew 7-lines in 4_{21} where the choice of lines in it is unique, and the uniqueness is the key to studying the configuration of $A_6^8(1)$-simplexes in 4_{21}.

Lemma 26. *For each of the seven lines in L_8 consisting of an $A_6^8(1)$-vertex in 4_{21}, there exist seven tied $A_3^8(1)$-vertices in it, and the choice is unique. Furthermore, the given seven lines can be labelled as $l_i, 1 \leq i \leq 7$, where*

$$\begin{aligned} &\{l_1, l_2, l_3, l_4\}, \{l_1, l_2, l_5, l_6\}, \{l_3, l_4, l_5, l_6\}, \{l_1, l_3, l_5, l_7\}, \\ &\{l_2, l_4, l_5, l_7\}, \{l_1, l_4, l_6, l_7\} \textit{ and } \{l_2, l_3, l_6, l_7\} \end{aligned}$$

are the tied $A_3^8(1)$-vertices.

Proof. We choose six lines from the given seven lines. By applying Lemma 24 on the six lines, we can label them as $l_i, 1 \leq i \leq 6$, where $\{l_1, l_2, l_3, l_4\}, \{l_1, l_2, l_5, l_6\}$ and $\{l_3, l_4, l_5, l_6\}$ are the three tied $A_3^8(1)$-vertices in it. Thus the remaining one line is denoted by l_7.

Now, we consider another set of six lines $\{l_1, l_2, l_3, l_4, l_5, l_7\}$. By Lemma 24 and Proposition 11, $\{l_1, l_3, l_5, l_7\}$, $\{l_1, l_4, l_5, l_7\}$, $\{l_2, l_4, l_5, l_7\}$ and $\{l_2, l_3, l_5, l_7\}$ are possible subsets of tied $A_3^8(1)$-vertices in addition to one given by $\{l_1, l_2, l_3, l_4\}$. Here, we observe that even if we exchange l_1 (resp. l_3) with l_2 (resp. l_4), the tied 3-vertices from $\{l_1, l_2, l_3, l_4, l_5, l_6\}$ do not change. Thus, after relabelling if necessary, we can set $\{l_1, l_3, l_5, l_7\}$ and $\{l_2, l_4, l_5, l_7\}$ to give new tied $A_3^8(1)$-vertices. Next, we consider another set of six lines $\{l_1, l_2, l_3, l_4, l_6, l_7\}$. Here, by Lemma 24 and Proposition 11, $\{l_1, l_4, l_6, l_7\}$ and $\{l_2, l_3, l_6, l_7\}$ are the only possible subsets of tied $A_3^8(1)$-vertices from $\{l_1, l_2, l_3, l_4, l_6, l_7\}$. Also, one can check that there are no more tied $A_3^8(1)$-vertices from the other choices of six lines from $\{l_i \; 1 \leq i \leq 7\}$.

By the above algorithm, we show that each of the seven lines in L_8 consisting of an $A_6^8(1)$-vertex in 4_{21} has seven tied $A_3^8(1)$-vertices and the choice is unique. This gives the lemma. □

By applying the argument in Theorem 25 and Lemma 26 to an $A_6^8(1)$-divisor in 4_{21}, we obtain the following theorem. We leave the details to the reader.

Theorem 27. *Let D be the center of an $A_6^8(1)$-vertex in 4_{21}. The center D corresponds to a skew 7-line $\sum_{i=1}^{7} l_{a_i}$ given by $2D+7K_{S_8} = \sum_{i=1}^{7} l_{a_i}$. Furthermore, each $A_6^8(1)$-vertex in 4_{21} with center D has seven uniquely determined tied $A_3^8(1)$-vertices where they are in the vertex figure of $l_{a_i}, 1 \leq i \leq 7$, respectively.*

In Lemma 26, we observe that when we have four lines for a tied $A_3^8(1)$-vertex, the remaining three lines in the $A_6^8(1)$-vertex give an $A_2^8(1)$-vertex which is not contained in any tied $A_3^8(1)$-vertex in the $A_6^8(1)$-vertex. The converse is also true by the following proposition.

Proposition 28. *For seven lines in L_8 consisting of an $A_6^8(1)$-vertex in 4_{21}, if three lines in the $A_6^8(1)$-vertex form an $A_2^8(1)$-vertex which is not contained in any tied $A_3^8(1)$-vertex in the $A_6^8(1)$-vertex, the remaining four lines in the 7-simplex give a tied $A_3^8(1)$-vertex.*

Proof. Let $l_i, 1 \leq i \leq 7$, be the given seven lines in L_8 consisting of an $A_6^8(1)$-vertex in 4_{21}. Suppose $\{l_5, l_6, l_7\}$ gives an $A_2^8(1)$-vertex which is not contained in any tied $A_3^8(1)$-vertex. We want to show that $\{l_1, l_2, l_3, l_4\}$ gives a tied $A_3^8(1)$-vertex by following the algorithm in Lemma 26.

First, we show that the intersection between the subset of lines producing a tied $A_3^8(1)$-vertex and $\{l_5, l_6, l_7\}$ must have two or no lines. By the assumption, the intersection cannot be three lines. Furthermore, if it is a line, the union of a tied $A_3^8(1)$-vertex and $\{l_5, l_6, l_7\}$ must be a set of six lines and have a tied $A_3^8(1)$-vertex containing $\{l_5, l_6, l_7\}$ by Lemma 24.

We consider a subset $\{l_i \; 2 \leq i \leq 7\}$ of the given seven lines. From the above, there are three tied $A_3^8(1)$-vertices and each of them contains two lines from $\{l_5, l_6, l_7\}$. Thus a tied $A_3^8(1)$-vertex in $\{l_i \; 2 \leq i \leq 7\}$ consists of two lines from $\{l_2, l_3, l_4\}$ and two lines from $\{l_5, l_6, l_7\}$.Without losing generality, we assume a tied $A_3^8(1)$-vertex is given by $\{l_2, l_3, l_5, l_6\}$. Now, we consider $\{l_i \; 1 \leq i \leq 6\}$. Since $\{l_2, l_3, l_5, l_6\}$ produces a tied $A_3^8(1)$-vertex, by the above claim, we obtain that two other tied $A_3^8(1)$-vertices are given by $\{l_1, l_4, l_5, l_6\}$ and $\{l_1, l_2, l_3, l_4\}$. Thus $\{l_1, l_2, l_3, l_4\}$ must produce a tied $A_3^8(1)$-vertex. This gives the proposition. □

Remark and Definition. We call the 1-degree 2-simplex in an $A_6^8(1)$-vertex in the above proposition a *Fano block* in the $A_6^8(1)$-vertex which appears in the following theorem.

From the theorem and the proposition, we have the following theorem.

Theorem 29. *Let $\{l_i \; 1 \leq i \leq 7\}$ be a set of lines in L_8 which forms an $A_6^8(1)$-vertex in 4_{21}. The set of Fano blocks in the $A_6^8(1)$-vertex consists of seven $A_2^8(1)$-vertices in* *the $A_6^8(1)$-vertex. Furthermore, the seven lines and Fano blocks in the $A_6^8(1)$-vertex produce a Steiner system $S(2,3,7)$ which is known as a Fano plane.*

Proof. By Proposition 28 and Theorem 27, there are only seven Fano blocks in the given set of lines. By Lemma 26, we can label the seven Fano blocks as

$$\{l_5, l_6, l_7\}, \{l_3, l_4, l_7\}, \{l_1, l_2, l_7\}, \{l_2, l_4, l_6\},$$
$$\{l_1, l_3, l_6\}, \{l_2, l_3, l_5\} \text{ and } \{l_1, l_4, l_5\}.$$

Furthermore, one can directly check that the seven lines $\{l_i \ 1 \le i \le 7\}$ and Fano blocks form a Steiner system $S(2,3,7)$. □

Remark. As we explained at the beginning of this subsection, the set of $A_6^8(1)$-divisors in 4_{21} is the subset of the union of $W(E_8)$-orbits (in fact, two) consisting of 82560 E_8-lattice points with length 14. On the other hand, by Theorem 27 the subset of $A_6^8(1)$-vertices in 4_{21} correspond to a subset of 6-faces in 4_{21} which is the union of two $W(E_8)$-orbits consisting of 69120 and 138240 elements respectively. Thus the set of $A_6^8(1)$-vertices is a $W(E_8)$-orbit consisting of 69120 elements.

4.2.4. *Rulings and* 1*-degree* 7*-simplexes.* Again, the set of $A_7^8(1)$-divisors in 4_{21} is a subset of the set of vectors with norm 16 in the E_8-lattice which has more than one E_8-orbit. But we find out that the configuration of $A_7^8(1)$-vertices in 4_{21} is close to that of $A_4^8(1)$-vertices. Here each center of $A_7^8(1)$-vertices gives a ruling in 4_{21}. Furthermore, each center of $A_7^8(1)$-vertices can be written as the sum of two centers of tied $A_3^8(1)$-vertices that are in the vertex figures of the antipodal pair of lines in the corresponding ruling. The details are as follows.

Theorem 30. *Let D be an $A_7^8(1)$-divisor in 4_{21}. The $A_7^8(1)$-divisor D corresponds to a ruling f by $D/2+2K_{S_8}=f$. Furthermore, the $A_7^8(1)$-divisor D can be written as a sum of two divisors of tied $A_3^8(1)$-vertices in the 7-simplex. There are seven pairs of $A_7^8(1)$-divisors of tied $A_3^8(1)$-simplexes whose sum is D, and each pair corresponds to the antipodal pairs of lines in the 7-crosspolytope given by the ruling f.*

Proof. Let $\{l_i \ 1 \le i \le 8\}$ be a set of lines in L_8 producing an $A_7^8(1)$-vertex in 4_{21} whose center is D.

We claim that if a subset of four lines in $\{l_i \ 1 \le i \le 8\}$ gives a tied $A_3^8(1)$-vertex with center D_1, the complement subset also forms another tied $A_3^8(1)$-vertex with center $D_2 := D - D_1$.

Proof of the Claim. Without losing generality, suppose D_1 is given as $l_1+l_2+l_3+l_4$. If we consider $\{l_i \ 1 \le i \le 7\}$, $\{l_5, l_6, l_7\}$ must form a Fano block in $\{l_i \ 1 \le i \le 7\}$ by Proposition 28 and Theorem 29. Now we consider an $A_6^8(1)$-vertex given by $\{l_i \ 2 \le i \le 8\}$. By Proposition 11, $\{l_2, l_3, l_4\}$ is a Fano block in $\{l_i \ 2 \le i \le 8\}$, and the complement $\{l_5, l_6, l_7, l_8\}$ gives a tied $A_3^8(1)$-vertex. This gives the claim.

Here $D_1 \cdot K_{S_8} = D_2 \cdot K_{S_8} = -4$ and $D_1 \cdot D_2 = 16$. Since D_1 and D_2 are the centers of tied $A_3^8(1)$-vertices, these divisor classes correspond to l_{D_1} and l_{D_2} by Corollary 9. Moreover, the intersection $l_{D_1} \cdot l_{D_2}$ is

$$l_{D_1} \cdot l_{D_2} = \left(\frac{1}{2}D_1 + K_{S_8}\right) \cdot \left(\frac{1}{2}D_2 + K_{S_8}\right) = 1,$$

and $l_{D_1} + l_{D_2} := f$ is a ruling. In fact, we obtain

$$f = l_{D_1} + l_{D_2} = \left(\frac{1}{2}D_1 + K_{S_8}\right) + \left(\frac{1}{2}D_2 + K_{S_8}\right) = \frac{1}{2}D + 2K_{S_8}.$$

For a fixed line l_8, each tied $A_3^8(1)$-vertex containing l_8 gives a Fano block in $\{l_i \ 1 \le i \le 7\}$. Thus there are seven tied $A_3^8(1)$-vertices containing l_8 with centers $D_1^i, 1 \le i \le 7$, by Theorem 29, and moreover there are seven pairs of centers D_2^i, $1 \le i \le 7$, of tied $A_3^8(1)$-vertices such that $D_1^i + D_2^i = D$ for $1 \le i \le 7$. Thus there are seven pairs of lines which produce the ruling f. In fact, these are all the possible pairs in the ruling because the ruling f corresponds to a 7-crosspolytope in 4_{21} which has seven pairs of antipodal lines with 1-intersection. □

As a direct application of the above theorem, we obtain the following corollary which shows how to construct an $A_7^8(1)$-vertex from an $A_6^8(1)$-vertex.

Corollary 31. *Let $S := \{l_m \ 1 \le m \le 7\}$ be an $A_6^8(1)$-vertex where $\{l_i, l_j, l_k\}$ is a given Fano block. There is a line l_8 whose union with S (resp. $\{l_i, l_j, l_k\}$) gives an $A_7^8(1)$-vertex (resp. a tied $A_3^8(1)$-vertex). Furthermore, if l is a line whose vertex figure contains the tied $A_3^8(1)$-vertex given by $S - \{l_i, l_j, l_k\}$, then an $A_3^8(1)$-vertex given by $\{l, l_i, l_j, l_k\}$ is untied.*

Proof. By Proposition 28, the $A_3^8(1)$-vertex with the center D given by $S - \{l_i, l_j, l_k\}$ is tied, and the line l is determined by $l = D/2 + K_{S_8}$. By Lemma 19, there is a line l_8 such that $\{l_i, l_j, l_k, l_8\}$ forms a tied $A_3^8(1)$-vertex. Since $\{l_i, l_j, l_k\}$ is a Fano block in the $A_3^8(1)$-vertex, the line l_8 is not in S. Furthermore, by the remark of Theorem 20, $S \cup \{l_8\}$ gives an $A_7^8(1)$-vertex. Since $l \cdot l_a = (D/2 + K_{S_8}) \cdot l_a = 1$ for $a = i, j, k$, $\{l, l_i, l_j, l_k\}$ gives an $A_3^8(1)$-vertex, and this is untied by Proposition 11. This gives the corollary. □

Remark. Here, $l \ne l_8$ since $l \cdot l_8 = (D/2 + K_{S_8}) \cdot l_8 = 5/2$.

4.3. **2- and 3-degree simplexes.** As in the $A_m^r(1)$-vertices $m(\le 3)$ in $(r-4)_{21}$, the $A_m^r(b)$-divisors with higher (i.e. > 1) degree are either uniquely determined or correspond to the lines for the tied issues. Thus the configurations of $A_m^r(b)$-vertices with higher degree in $(r-4)_{21}$ are naturally related to the k-Steiner systems along the monoidal transform.

A. $A_m^r(b)$-vertex 2-degree 1-simplex $r = 7, 8$.

The $A_1^r(2)$-vertices exist when $r = 7, 8$.

(a) 2-degree 1-simplex in 3_{21}.

By definition, the set of $A_1^7(2)$-vertices is the same with the 2-Steiner system $\mathcal{S}_A(2, S_7)$. Furthermore, the set of an $A_1^7(2)$-divisor, $\mathcal{D}_1^7(2)$, is $\{-K_{S_7}\}$.

(b) 2-degree 1-simplex in 4_{21}.

The center of each $A_1^8(2)$-vertex in $Pic\ S_8$ is a divisor $D := l_1 + l_2$ where $l_1 \cdot l_2 = 2$. As in Theorem 2, the divisor D satisfies $D^2 = 2$ and $D \cdot K_{S_8} = -2$, and equivalently $D + K_{S_8}$ is a line in $Pic\ S_8$. Thus the set $\mathcal{D}_1^8(2)$ of $A_1^8(2)$-divisors in 4_{21} is bijective to the L_8.

For a fixed center D of an $A_1^8(2)$-vertex in 4_{21}, the configuration of all the $A_1^8(2)$-vertices with a common center D is described as follows.

Let l_3 and l_4 be two lines of an $A_1^8(2)$-vertex with a center $D = l_3 + l_4$, and from the above we have a line $l := D + K_{S_8}$ which also satisfies $(D + K_{S_8}) \cdot l_i = (l_3 + l_4 + K_{S_8}) \cdot l_i = 0$ for $i = 3, 4$. Therefore, l_3 and l_4 are in $N_0(l, S_8)$, namely, the vertex figure of l. The map $\pi_l^8 : S_8 \to S_7$, the blowing down of line l, sends l_3 and l_4 to lines in S_7 with intersection 2. Thus the configuration of $A_1^8(2)$-vertices with a fixed center D is the same as the configuration of $A_1^7(2)$-vertices in 3_{21} which is the 2-Steiner system $\mathcal{S}_A(2, S_7)$.

Note by Lemma 19 that G_l, the Gieser transforms of l, transfer l_1 and l_2 to each other.

The configuration to inscribe 2-degree 1-simplexes is summarized as follows.

Theorem 32. *The 2-Steiner system $\mathcal{S}_A(2, S_7)$ in S_7 determines the configuration of $A_1^8(2)$-vertices in 4_{21} with a common center and the configuration of $A_1^7(2)$-simplexes in 3_{21}.*

B. 2-degree 2-simplex in 4_{21}.

The $A_2^8(2)$-vertices exist when $r = 8$.

By definition, the configuration of the $A_2^8(2)$-vertices in 4_{21} is exactly the 3-Steiner system $\mathcal{S}_B(3, S_8)$, and the set of $A_2^8(2)$-divisors is $\{-3K_{S_8}\}$.

C. 3-degree 1-simplexes in 4_{21}.

The $A_1^8(3)$-vertices exist when $r = 8$. Again by definition, the configuration of the $A_1^8(3)$-vertices in 4_{21} is exactly the 3-Steiner system $\mathcal{S}_A(2, S_8)$, and the set of $A_2^8(2)$-divisors is $\{-2K_{S_8}\}$.

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Department of Mathematics, Ewha Womans University, Seodaemun-Gu Daehyun-dong, Seoul, Korea

E-mail address: `jaehyoukl@ewha.ac.kr`