THE GROTHENDIECK RING OF THE STRUCTURE GROUP
OF THE GEOMETRIC FROBENIUS MORPHISM

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Abstract. The geometric Frobenius morphism on smooth varieties is an fppf-fiber bundle. We study representations of the structure group scheme. In particular, we describe irreducible representations and compute its Grothendieck ring of finite dimensional representations.

1. Introduction

Let $k$ be a field of characteristic $p > 0$. Then for all smooth $k$-varieties $X$ of dimension $n$, the $r$-th geometric Frobenius morphism

$$F^r : X \to X^{(r)}$$

is an fppf-fiber bundle with fibers $k^n_r$, the $r$-th Frobenius kernel of the affine space of dimension $n$ considered as $G^n_a$. Now denote

$$R(n, r) := k[A^n_r] = k[x_1, \ldots, x_n]/(x_1^{p^r}, \ldots, x_n^{p^r}).$$

Let $G(n, r)$ be the automorphism group scheme of $R(n, r)$. Then for each $G(n, r)$-representation $V$, there is an associated canonical $X^{(r)}$-vector bundle obtained by twisting the fiber bundle $F^r$ with $V$. In order to understand these bundles K-theoretically one needs to understand the Grothendieck ring of $G(n, r)$. The latter one is the purpose of this paper. The topic arose from a correspondence between Pierre Deligne and Markus Rost where Deligne suggested this setting for $n = r = 1$.

We will give a description of the irreducible $G(n, r)$-representations whose classes form a $\mathbb{Z}$-basis of $K_0(G(n, r)-\text{rep})$ as an abelian group. Note that $\text{Lie} G(n, r) = \text{Der}_k(R(n, r))$. That is, for $r = 1$, this is the restricted Witt algebra of Cartan type. In fact, the description and the involved computations we give generalize results of [Nak92] for the simple restricted modules of $W_n = \text{Der}_k(R(n, 1))$. In order to get the description of irreducible representations, we need a triangular decomposition $G(n, r) = G^-G^0G^+$ with $G^0 = \text{GL}_n$. In fact, most of the arguments involved are quite general. Hence we will give them in the abstract notion of triangulated groups. Also, this notion turns out to be useful in order to describe the recursive part of our description which passes from $G(n, r)$ to $G(n, r + 1)$. Furthermore, this notion also covers Jantzen’s groups $G_sT$ and $G_sB$. Note that the description of irreducible $G(n, 1)$-representations was already given by Abrams [Abr96, Abr97]. Unfortunately, one part of our description only works if we exclude the case $\text{char}(k) = 2$.

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Our main goal is to describe $K_0(G(n,r)\text{-rep})$ as a surjective image of $r + 1$ copies of $K_0(\text{GL}_n\text{-rep})$ as an abelian group. This involves the de Rham complex of $R(n,r)$ over $k$ and Cartier’s Theorem which computes its cohomology. We will also compute the kernel elements of our surjective map

$$K_0(\text{GL}_n\text{-rep})^{\oplus r+1} \rightarrow K_0(G(n,r)\text{-rep})$$

and introduce a ring structure of the left hand side.

2. Basic properties

Let us first fix some notions we are using. As already said, $k$ is a field of characteristic $p > 0$. By an algebraic $k$-group $G$ we understand a group scheme $G$, that is, a functor from commutative $k$-algebras to groups, which is representable by a finitely generated Hopf algebra. When we denote $g \in G$ we understand a choice of a commutative $k$-algebra $A$ and $g \in G(A)$. For readability, we will suppress $A$.

Furthermore, for $r \geq 0$, we denote by $G^{(r)}$ the $r$-th Frobenius twist and by $G_r$ the kernel of the $r$-th Frobenius morphism $F^r : G \rightarrow G^{(r)}$.

Now the group scheme

$$G(n,r) = \text{Aut}(R(n,r))$$

with

$$R(n,r) := k[\mathbb{A}_n^r] = k[x_1, \ldots, x_n]/(x_1^{p^r}, \ldots, x_n^{p^r})$$

is defined to be

$$G(n,r)(A) := \text{Aut}_{A_{\text{alg}}}(R(n,r)_A)$$

with

$$R(n,r)_A := R(n,r) \otimes_k A = A[x_1, \ldots, x_n]/(x_1^{p^r}, \ldots, x_n^{p^r}).$$

Each element $g \in G(n,r)$ is determined by the images of $x_i$. In fact, a choice $g_i \in R(n,r)_A$ for each $i = 1, \ldots, n$ defines an element $g \in G(n,r)(A)$ by $g(x_i) = g_i$ if and only if $g_i(0)^{p^r} = 0$ for all $i$ and $J_g \in \text{GL}_n(A)$ where

$$J_g := \left(\frac{\partial g_j}{\partial x_i}(0)\right)_{ij}$$

is the Jacobian matrix of $g$.

As $\text{GL}_n$ acts linearly on $\mathbb{A}_n^r$, we get

$$G^0 := \text{GL}_n \subset G(n,r)$$

as a subgroup. Furthermore, $\mathbb{G}_a^n_{a,r}$, the $r$-th Frobenius kernel of $\mathbb{G}_a^n$, acts on $\mathbb{A}_n^r$ by translation. That is, we get

$$G^- := \mathbb{G}_a^n_{a,r} \subset G(n,r)$$

as a subgroup. Now denote

$$G^+ := \{g \in G(n,r) \mid \forall i : g(x_i)(0) = 0, \ J_g = \text{id}\} = \{g \in G(n,r) \mid \forall i : g(x_i) = x_i + \sum_{I, \deg(I) \geq 2} \lambda_I x^I,\}$$

where $I \in \{0, \ldots, p^r - 1\}^n$ is a multi-index with the usual degree $\deg(I)$ and $x_I \in R(n,r)$ is the corresponding monomial. Note that $G^+ \cong \mathbb{A}_N$ for an $N \in \mathbb{N}$ as a scheme. These three subgroups provide a triangular decomposition

$$G(n,r) = G^- G^0 G^+.$$
That is, the multiplication $m : G^- \times G^0 \times G^+ \to G$ is an isomorphism of $k$-schemes.

Moreover, for all $1 \leq i \leq n$, denote

$$U_i = U_i(n, r) := \{ g \in G(n, r) \mid \forall j : g(x_j)(0)^p = 0 \} \subset G(n, r).$$

These subschemes are in fact subgroups who afford the triangular decomposition

$$U_i = G^-_i G^0 G^+.$$

Note that $G^-_i = \mathbb{G}^n_{a,i}$.

As we already noticed,

$$\text{Lie}(G(n, r)) = \text{Der}_k(R(n, r)),$$

the self-derivations of $R(n, r)$ by [DG80 II§4.2.3 Proposition]. A canonical basis of this Lie algebra is given by the operators

$$\delta_{(i,x^I)} := x^I \frac{\partial}{\partial x^I}, \quad I \in \{0, \ldots, p^r - 1\}^n.$$

**3. Triangulated groups**

Now we will introduce the notion of triangulated groups and triangulated morphisms. It turned out to be a convenient notion in order to study the group schemes $G(n, r)$. Triangular decompositions are a standard tool in algebraic Lie theory. This notion is meant to catch some of their properties in an abstract way.

**Definition 3.1.** Let $H$ be an algebraic $k$-group. A **pretriangulation** of $H$ is a collection of three subgroups $(H^-, H^0, H^+)$ such that the multiplication map

$$m : H^- \times H^0 \times H^+ \to H$$

is an isomorphism of $k$-schemes. We shortly denote this by

$$H = H^- H^0 H^+.$$

As an example, we have

$$G(n, r) = G^- G^0 G^+$$

as well as $U_i(n, r) = G^-_i G^0 G^+$.

Furthermore, for each split reductive group $G$, we get for the $r$-th Frobenius kernel

$$G_r = U^-_r T_r U^+_r,$$

where $T \subset G$ is a maximal torus and $U^{\pm} \subset G$ the unipotent subgroups.

**Definition 3.2.** Let $G = G^- G^0 G^+$ and $H = H^- H^0 H^+$ be pretriangulated and $f : G \to H$ a group homomorphism. Then $f$ is said to be **triangulated** if for all $\alpha \in \{-, 0, +\}$ the restriction of $f$ to $G^\alpha$ factors through $H^\alpha$.

We denote this factorization by $f^\alpha : G^\alpha \to H^\alpha$ and we write

$$f = f^- f^0 f^+.$$

As an example consider a pretriangulation $H = H^- H^0 H^+$. Then the $r$-th Frobenius twist of $H$ is pretriangulated by $H^{(r)} = (H^-)^{(r)} (H^0)^{(r)} (H^+)^{(r)}$ and the $r$-th Frobenius morphism

$$F^r_H : H \to H^{(r)}$$

is triangulated with $(F^r_H)^\alpha = F^r_{H^\alpha}$.
The following lemma is straightforward.

**Lemma 3.3.** Let \( f : G \to H \) be triangulated. Then the following hold:

1. The kernel of \( f \) is pretriangulated by
   \[
   \text{Ker}(f) = \text{Ker}(f^-) \text{Ker}(f^0) \text{Ker}(f^+) \text{.}
   \]
2. The image of \( f \) is pretriangulated by
   \[
   \text{Im}(f) = \text{Im}(f^-) \text{Im}(f^0) \text{Im}(f^+) \text{.}
   \]
3. The closed immersion \( \text{Im}(f) \hookrightarrow H \) is triangulated.

Note that under the isomorphism \( G/\text{Ker}(F) \cong \text{Im}(f) \), we also obtain a pretriangulation
\[
G/\text{Ker}(f) = (G^-/\text{Ker}(f^-))(G^0/\text{Ker}(f^0))(G^+/\text{Ker}(f^+)) \text{.}
\]

As an example we take a pretriangulation \( H = H^-H^0H^+ \) and the \( r \)-th Frobenius morphism \( F^r_H : H \to H^{(r)} \). Then we get that the \( r \)-th Frobenius kernel of \( H \) is pretriangulated by
\[
H_r = H^-H^0H^+ \text{.}
\]

Our next aim is to describe the irreducible representation of a pretriangulated group \( H \) in terms of \( H^0 \). In order to do that, we need the following.

**Definition 3.4.** A pretriangulation \( H = H^-H^0H^+ \) is called a triangulation if the following statements hold:

1. The following products are semidirect by conjugation
   \[
   B^- := H^- \ltimes H^0 \text{ and } B^+ := H^0 \ltimes H^+ \text{.}
   \]
2. \( H^- \) and \( H^+ \) are unipotent.
3. \( H^- \) is finite.

Note that this definition is not symmetric. As an example, the pretriangulation \( G(n,r) = G^-G^0G^+ \) is also a triangulation.

For a triangulation \( H = H^-H^0H^+ \), we denote the group homomorphisms

1. the projections \( \pi^\pm : B^\pm \to H^0 \text{,} \)
2. the inclusions \( j^\pm : B^\pm \hookrightarrow H \text{,} \)

and the functor
\[
\mathcal{I} := (j^+)_*(\pi^+)^* : H^0\text{-rep} \to H\text{-rep}
\]

between categories of finite dimensional representations. Here
\[
(\pi^+)^* : H^0\text{-rep} \to B^+\text{-rep} \text{.}
\]

and
\[
(j^+)_* = \text{Mor}_{B^+}(H,-) : B^+\text{-rep} \to H\text{-rep}
\]
is induction. That is,
\[
(j^+)_*(V) = \{ f \in \text{Mor}(H,V) \mid f(hg) = hf(g) \forall g \in H, \forall h \in B^+ \},
\]
and the \( H \)-action is induced by right translation on \( H \). The functor \((j^+)_*\) preserves finiteness as
\[
\text{Mor}_{B^+}(H,-) \cong \text{Mor}(H^-,-)
\]
by using the decomposition \( H = B^+H^- \) and the finiteness of \( H^- \). In order to express the \( H \)-action, let us denote for \( h \in H \) the decomposition
\[
h = h_+h_0h_-\]
Let us consider the group scheme $G$ according to the decomposition $H = B^+H^- = H^+H^0H^-$. The following lemma is straightforward.

**Lemma 3.5.** Let $H = H^-H^0H^+$ be a triangulation and $V$ an $H^0$-representation.

1. Under the isomorphism
   $\mathcal{I}(V) \cong \text{Mor}(H^-, V)$
   the $H$-action translates as follows: For all $h \in H$, $a \in H^-$, and $f : H^- \to V$
   $$(hf)(a) = (ah)_0f((ah)_-)$$

2. If we restrict to $B^-$, we get
   $$(j^-)^*\mathcal{I}(V) \cong k[H^-] \otimes_k V.$$
   Here $H^-$ acts on $k[H^-]$ by the right regular representation and trivially on $V$ and $H^0$ acts on $k[H^-]$ by conjugation and as given on $V$.

In particular, we get

$\mathcal{I}(V)^{H^-} \cong (k[H^-] \otimes_k V)^{H^-} = k[H^-] \otimes_k V \cong V$

as $H^0$-representations.

**Example 3.6.** Let us consider the group scheme $G(n, r)$ and the triangulation $G(n, r) = G^-G^0G^+$. Recall that $G^- = (\mathbb{G}_{a,r})^n$ and $G^0 = \text{GL}_n$. Then for all $g \in G(n, r)$, we get $g_- = g(0) \in (\mathbb{G}_{a,r})^n$ and $g_0 = J_g \in \text{GL}_n$. As $R(n, r) = k[(\mathbb{G}_{a,r})^n]$, we get for each $\text{GL}_n$-representation $V$ that

$\mathcal{I}(V) \cong R(n, r) \otimes_k V$

as $k$-vector spaces. The $G(n, r)$-action reads as follows:

$g(f \otimes v) = \left(\frac{\partial g(x_j)}{\partial x_i}\right)_{ij}(g(f) \otimes v)$

for all $g \in G(n, r)$, $f \in R(n, r)$, and $v \in V$. Note that this uses the fact that for $g \in G(n, r)(A)$ we get $\left(\frac{\partial g(x_j)}{\partial x_i}\right)_{ij} \in \text{GL}_n(R(n, r)_A)$ which acts on $R(n, r)_A \otimes_k V$.

In particular, take $V = k$ with the trivial $\text{GL}_n$-action. Then we get $\mathcal{I}(k) \cong R(n, r)$ together with the standard action of $G(n, r)$ on $R(n, r)$. That is, the standard action can be recovered from the triangulation.

Now the functor $\mathcal{I}$ gives rise to the following description of irreducible $H$-representations.

**Proposition 3.7.** Let $H = H^-H^0H^+$ be a triangulation. Then the maps

$soc\mathcal{I} : \{\text{irred. } H^0\text{-rep}\} / \cong \to \{\text{irred. } H^-\text{-rep}\} / \cong$

and

$(-)^{H^-} : \{\text{irred. } H^-\text{-rep}\} / \cong \to \{\text{irred. } H^0\text{-rep}\} / \cong$

are well defined and inverse to each other.

**Proof.** Let $V$ be an irreducible $H^0$-representation. As $H^-$ is unipotent, a standard argument by taking $H^-$-invariants and using $\mathcal{I}(V)^{H^-} \cong V$ shows that $soc\mathcal{I}(V)$ is an irreducible $H$-representation. Also, $soc\mathcal{I}(V)^{H^-} \cong V$ follows. Now let $W$ be an irreducible $H$-representation. As $H^+$ is unipotent, $(W^\vee)^{H^+} \neq 0$. Thus there
is an irreducible $H^0$-representation $V \neq 0$ such that $V^\vee \subset (W^\vee)^{H^+}$. Hence by dualization
\[ 0 \neq \text{Hom}_{B^+}((j^+)^*W, (\pi^+)^*V) \cong \text{Hom}_H(W, \mathcal{I} V). \]
This shows that $W \cong \text{soc} \mathcal{I}(V)$ and finishes the proof. \hfill $\square$

Finally, we extend the notion of triangulations as follows.

**Definition 3.8.** A triangulation $H = H^- H^0 H^+$ is called an $r$-triangulation if $H^-$ is of height $\leq r$. That is, $H^-$ equals its $r$-th Frobenius kernel:
\[ H^- = (H^-)_r. \]

Note that an $r$-triangulation is also an $(r + 1)$-triangulation. As an example, the triangulation $G(n, r) = G^- G^0 G^+$ is also an $r$-triangulation as $G^- = G^{n}_{a, r}$.

Moreover, $U_i(n, r) = G_i^1 G^0 G^+$ is an $i$-triangulation.

Now for an $r$-triangulation $H = H^- H^0 H^+$, the $r$-th Frobenius factors as
\[ F^r_H : H \to (B^+)^{(r)} \subset H^{(r)}, \]
which provides the group homomorphism
\[ P_r := \pi^+ \circ F^r_H : H \to (H^0)^{(r)}. \]

In the example $G(n, r)$, this computes as
\[ P_r(g) = F^r_{\text{GL}_n}(J_g) = \left( \left( \frac{\partial g(x_j)}{\partial x_i}(0) \right)^{p^r} \right)_{ij}. \]

If we consider $(H^0)^{(r)}$ triangulated with trivial $\pm$-factors, $P_r$ is triangulated with $P_r^\pm$ trivial and $P_r^0 = F^r_{H^0}$. Thus
\[ \text{Ker}(P_r) = H^- H^0 H^+. \]

Moreover, $P_r$ has the following very useful property.

**Lemma 3.9.** Let $H = H^- H^0 H^+$ be an $r$-triangulation such that $H^0$ is reduced. Then $P_r$ induces an isomorphism
\[ H/\text{Ker}(P_r) \cong (H^0)^{(r)}. \]

**Proof.** This follows from the triangulated structure of $P_r$, Lemma 3.3 and the fact that $F^r_{H^0}$ induces an isomorphism
\[ H^0/(H^0)_r \cong (H^0)^{(r)} \]
if $H^0$ is reduced [Jan03, I.9.5]. \hfill $\square$

As an immediate consequence, we get that for $H^0$ reduced, the functor
\[ P^*_r : (H^0)^{(r)}-\text{rep} \to H-\text{rep} \]
preserves irreducible representations.

**Notation 3.10.** For any algebraic $k$-group $G$ and a $G^{(r)}$-representation $W$, we denote the $r$-th Frobenius twist by
\[ W^{[r]} := (F^r_G)^*W. \]
Then we get the following computational rule.

**Lemma 3.11.** Let $H = H^0 H^+$ be an $r$-triangulation, $V$ an $H^0$-representation, and $W$ an $(H^0)^{(r)}$-representation. Then

$$\mathcal{I}(V \otimes_k W^{[r]}) \cong \mathcal{I}(V) \otimes_k P^*_r W$$

as $H$-representations.

**Proof.** The claim follows from the Tensor Identity [Jan03 I.3.6] for induction and the fact that $P^*_r W|_{H^0} = W^{[r]}$. \qed

This provides the following proposition which reads as and uses Steinberg’s Tensor Product Theorem [Jan03 II.3.16, II.3.17]. For this, we need the following notation where for split reductive groups we follow [Jan03].

**Notation 3.12.** Let $H = H^0 H^+$ be a triangulation such that $H^0$ is split reductive. Denote by $T \subset H^0$ a split maximal torus and by $X(T)_+$ the dominant weights. Furthermore, denote by $S$ the simple roots and for $r \geq 1$

$$X_r(T) := \{ \lambda \in X(T) \mid \forall \alpha \in S : 0 \leq \langle \lambda, \alpha^\vee \rangle < p^r \}.$$  

For $\lambda \in X(T)_+$, we denote the associated irreducible $H^0$-representation by $L(\lambda)$. Then, according to Proposition 3.7 we denote the associated irreducible $H$-representation by

$$L(\lambda, H) := \text{soc} \mathcal{I}(L(\lambda)).$$

Note that for an $r$-triangulation $H = H^0 H^+$ with $H^0$ split reductive, we get for all $\lambda \in X(T)_+$

$$L(p^r \lambda, H) \cong P^*_r L(\lambda),$$

as $P^*_r$ preserves irreducible representations and

$$(P^*_r L(\lambda))^{H^-} = P^*_r L(\lambda)|_{H^0} = L(\lambda)^{[r]} \cong L(p^r \lambda).$$

**Proposition 3.13.** Let $H = H^0 H^+$ be an $r$-triangulation such that $H^0$ is split reductive, $\lambda \in X_r(T)$ and $\mu \in X(T)_+$. Then

$$L(\lambda + p^r \mu, H) \cong L(\lambda, H) \otimes_k P^*_r L(\mu) \cong L(\lambda, H) \otimes_k L(p^r \mu, H).$$

**Proof.** By Steinberg’s Tensor Product Theorem

$$L(\lambda + p^r \mu) \cong L(\lambda) \otimes_k L(\mu)^{[r]}.$$  

Then the previous lemma provides

$$\mathcal{I}(L(\lambda + p^r \mu)) \cong \mathcal{I}(L(\lambda)) \otimes_k P^*_r L(\mu).$$

As $P^*_r L(\mu) = (P^*_r L(\mu))^{H^-}$, the result follows by taking socles. \qed

Note that examples of such $r$-triangulations with reductive $H^0$ are given by Jantzen’s groups $G_r T$ and $G_r B$. The proposition just generalizes results which are already known for these groups.
4. TRANSFER HOMOMORPHISMS

In order to prepare the description of the irreducible $G(n,r)$-representations, we will introduce several transfer homomorphisms. They will be between $G(n,r)$, $U_i(n,r)$, and $G(n,r)^0 \cong \text{GL}_n$ and their Frobenius twists respectively.

In the previous section, we already introduced the morphism
$$P_r : G(n,r) \rightarrow (\text{GL}_n)^{(r)}$$
arising from the $r$-triangulation $G(n,r) = G^{-G^0G^+}$. We also saw that for all $\lambda \in X(T)_+$ we have
$$P_r^\ast L(\lambda) \cong L(p^r \lambda, G(n,r)).$$

Now, for $r \geq 2$ we also introduce a transfer morphism
$$T_r : G(n,r) \rightarrow G(n,r-1)^{(1)}$$
as follows: Let $S \subset R(n,r)$ be the subalgebra generated by $x_1^{p^{r-1}}, \ldots, x_n^{p^{r-1}}$. Now let $g \in G(n,r) = \text{Aut}(R(n,r))$. Then $S$ is invariant under $g$. Thus we get an induced automorphism on
$$R(n,r) \otimes_{S,(1)^p} k \cong R(n,r-1) \otimes_{k,(1)^p} k = R(n,r-1)^{(1)}$$
which defines $T_r(g)$. Note that if $F_a : R(n,r) \rightarrow R(n,r)$ denotes the arithmetic Frobenius, then
$$T_r(g)(x_i \otimes 1) = g(x_i) \otimes 1 = (F_a(g(x_i)))(x_i \otimes 1).$$
Now $T_r$ is triangulated as $(T_r)^-$ is just the first Frobenius morphism
$$F^1 : G^a_n \rightarrow (G^a_{a,r-1})^{(1)}$$
and $(T_r)^0$ is also the first Frobenius morphism
$$F^1 : \text{GL}_n \rightarrow (\text{GL}_n)^{(1)}.$$
Note that $(T_r)^+$ is not just the first Frobenius as we set $x_i^{p^{r-1}} = 0$ for all $i = 1, \ldots, n$.

**Lemma 4.1.** For all $r \geq 2$, the morphism $T_r : G(n,r) \rightarrow G(n,r-1)^{(1)}$ induces an isomorphism
$$G(n,r)/\text{Ker}(T_r) \xrightarrow{\cong} G(n,r-1)^{(1)}.$$

**Proof.** By the triangulated structure of $T_r$ and Lemma 3.3 its suffices to show the claim for $(T_r)^-, (T_r)^0, (T_r)^+$ separately. For $(T_r)^-$ it follows by [Jan03, I.9.5] which states that it induces an isomorphism
$$G^a_{n,r}/G^a_{n,1} \cong (G^a_{a,r-1})^{(1)}.$$  
Also by [Jan03, I.9.5], it follows for $(T_r)^0$: It induces an isomorphism
$$\text{GL}_n/(\text{GL}_n)_1 \cong (\text{GL}_n)^{(1)}.$$ 
It is left to show that the closed immersion
$$T_r^+ : G(n,r)^+ / \text{Ker}(T_r^+) \hookrightarrow (G(n,r-1)^+)^{(1)}$$
is an isomorphism. The describing ideal of this immersion is the kernel of the morphism
$$(T_r^+)^{\#} : k[G(n,r-1)^+^{(1)}] \rightarrow k[G(n,r)^+].$$
As the parameters for $G(n,r)^+$ are free and $T_r$ acts as the $p$-th power on the parameters, $(T_r^+)^{\#}$ is injective, which shows the claim. \qed
As an immediate consequence, we get that
\[ T_r^* : G(n, r-1)^{(1)}\text{-rep} \rightarrow G(n, r)\text{-rep} \]
preserves irreducible representations. More concretely,
\[ T_r^* L(\lambda, G(n, r-1)^{(1)}) \cong L(p\lambda, G(n, r)) \]
for all \( \lambda \in X(T)_+ \). This follows by Proposition 3.7 and the observation that
\[ L(p\lambda) = L(\lambda)^{(1)} = (F_{\text{GL}}^1)^* \left( L(\lambda, G(n, r-1)^{(1)}(G(n,r-1)^{(1)})^- \right) \]
\subset (T_r^* U(\lambda, G(n, r-1)^{(1)})^G(n,r)^-).

Furthermore, we introduce the transfer homomorphisms
\[ t_{r,i} : U_i(n, r) \rightarrow G(n, i) \]
for all \( 1 \leq i \leq r \). Recall that
\[ U_i = \{ f \in G(n, r) \mid f(0) \in \mathbb{C}^n_{a,i} \}, \]
so let \( g \in U_i \). Then it induces an isomorphism on \( R(n, i) \) which we denote by \( t_{r,i}(g) \). Note that
\[ t_{r,i}(g)(x_i) \equiv g(x_i) \mod (x_1^{p_1}, \ldots, x_n^{p_i}). \]
Also \( t_{r,i} \) is triangulated: The restriction to \( G_i^- \) and \( G^0 \) is just the identity.

Finally, we discuss how the maps \( P_r, T_r, t_{r,i} \) are related. First note that for all \( r \geq 2 \), the diagram
\[
\begin{array}{ccc}
G(n, r) & \xrightarrow{T_r} & G(n, r-1)^{(1)} \\
\downarrow{P_r} & & \downarrow{P_{r-1}} \\
(G^0)^{(r)} & & (G^0)^{(r)}
\end{array}
\]
commutes. Furthermore, for all \( 1 \leq i \leq r \), the diagram
\[
\begin{array}{ccc}
U_i(n, r) & \xrightarrow{t_{r,i}} & G(n, i) \\
\downarrow{P_i} & & \downarrow{P_i} \\
(G^0)^{(i)} & & (G^0)^{(i)}
\end{array}
\]
commutes.

We again denote \( G_i^- = G(n, r)^- \) and \( U_i = U_i(n, r) \subset G(n, r) \). Recall that \( U_i^- = G_i^- \). Our next aim is to study the induction functor \( \text{ind}_{U_i}^{G(n,r)} \) and its relation to the induced functors of the three morphism types.

**Lemma 4.2.** For all \( 1 \leq i \leq r \), we get for the induction functor
\[ \text{res}_{G_i^-}^{G(n,r)} \circ \text{ind}_{U_i}^{G(n,r)} = \text{ind}_{G_i^-}^{G_i^-} \circ \text{res}_{G_i^-}^{U_i}. \]
Furthermore, \( \text{ind}_{U_i}^{G(n,r)} \) is exact.

**Proof.** We use the morphism description of the induction \( \text{ind}_{U_i}^{G(n,r)} \). Then we obtain for all \( U_i \)-representations \( V \) that
\[ \text{ind}_{U_i}^{G(n,r)} V = \{ f \in \text{Mor}(G(n, r), V) \mid f(u g) = u f(g) \ \forall u \in U_i \} \]
\[ = \{ f \in \text{Mor}(G^-, V) \mid f(c g) = c f(g) \ \forall c \in G_i^- \}. \]
by using the decomposition $G(n, r) = B^+ \times G^-$ and $U_i = B^+ \times G_i^-$, where $B^+ = G(n, r)^0 \times G(n, r)^+$.

The restriction of this to $G^-$ coincides with

$$\text{ind}_{G^-}^G \text{res}_{G^-}^U V,$$

as the $G^-$-action is given by right translation. This shows the first claim. Now $\text{ind}_{G^-}^G$ is exact by \cite[I.5.13, I.9.5]{Jan03}. This implies the exactness of $\text{ind}_{G^-}^G U_i$. □

We start with the relation of $\text{ind}_{G^-}^G U_i$ to the $\mathcal{I}$-functors. Recall that $G(n, r)$ is $r$-triangulated and $U_i$ is $i$-triangulated. So let us denote the $\mathcal{I}$-functor for a $j$-triangulated group as $\mathcal{I}_j$.

**Lemma 4.3.** For all $1 \leq i \leq r$, both triangles of the diagram

\[
\begin{array}{ccc}
G(n, i) \text{-rep} & \xrightarrow{\mathcal{I}_i} & G^0 \text{-rep} \\
\downarrow & & \downarrow \\
U_i(n, r) \text{-rep} & \xleftarrow{\mathcal{I}_i} & G(n, r) \text{-rep}
\end{array}
\]

commute.

**Proof.** The commutativity of the upper triangle follows immediately from $G(n, i)^- = U_i(n, r)^-$ and Lemma 3.5.

The commutativity of the lower triangle follows from $G(n, r) = B^+ \times G^-$, $U_i = B^+ \times G_i^-$, and the transitivity of induction \cite[I.3.5]{Jan03}.

Finally, there is a more complicated relation of the induction $\text{ind}_{U_i}^G(n, r)$ to the functors $P_i^*$, $T_i^*$, and $\mathcal{I}$:

**Lemma 4.4.** For all $1 \leq i \leq r$, the triangle and the square of the following diagram commute up to functor isomorphism:

\[
\begin{array}{ccc}
G(n, i) \text{-rep} & \xrightarrow{P_i^*} & (G^0)^{(i)} \text{-rep} \\
\downarrow & & \downarrow \text{P}_i^* \\
U_i(n, r) \text{-rep} & \text{ind}_{U_i}^G(n, r) & G(n, r) \text{-rep}
\end{array}
\]

Here $T^i : G(n, r) \to G(n, r - i)^{(i)}$ is the composition

$$T^i := T_{r-(i-1)} \circ \cdots \circ T_r.$$

**Proof.** The commutativity of the triangle follows from $P_i \circ t_{r,i} = P_i$.

For the commutativity of the square, note that the morphism

$$T^i = T_{r-(i-1)} \circ \cdots \circ T_r$$

is triangulated with

$$(T^i)^- = F_{G^-}^i : G^- \to (G^-)^{(i)}_{r-i}$$
and

\[(T^i)^0 = \Phi^i_{G^0} : G^0 \to (G^0)^{(i)}.\]

Recall the morphism description of the functor $\mathcal{I}$ from Lemma 3.5. Let $V$ be a $(G^0)^{(i)}$-representation. On one hand

\[(T^i)^* \mathcal{I}_{r-i}(V) = \text{Mor}((G^-)^{(i)}_{r-i}, V)\]

as $(G(n, r - i)^{(i)})^- = (G^-)^{(i)}_{r-i}$. On the other hand, $G_i^-$ operates trivially on $P_i^*V$, which implies

\[
\text{ind}_{U_i}^{G(n, r)} P_i^*V = \{ f \in \text{Mor}(G^-, P_i^*V) \mid f(cg) = cf(g) \forall c \in G_i^- \}
\]

(\text{cf. the proof of Lemma 4.2}). As the $i$-th Frobenius $F^i_{i_G^-} : G^- \to (G^-)^{(i)}_{r-i}$ induces an isomorphism $G^- / G_i^- \cong (G^-)^{(i)}_{r-i}$, it induces a natural linear isomorphism

\[
(T^i)^* \mathcal{I}_{r-i}(V) \xrightarrow{(F^i)^*} \text{Mor}_{G_i^-}(G^-, P_i^*V) = \text{ind}_{U_i}^{G(n, r)} P_i^*V.
\]

This isomorphism is in fact $G(n, r)$-equivariant, which can be seen by using the triangulated structure of $T^i$. □

5. DIFFERENTIALS AND CARTIER’S THEOREM

We are now going to introduce some concrete $G(n, r)$-representations which play a major role in the description of the irreducible representations. We consider the Kähler-differentials

\[
\Omega_r := \Omega_{R(n, r), k} = \bigoplus_{i=1}^{n} R(n, r)dx_i.
\]

We claim that this is a canonical $G(n, r)$-representation.

**Notation 5.1.** For any $g \in G(n, r)$ and any $R(n, r)$-module $M$, we denote by $M^{(g)}$ the module twisted by $g$, that is,

\[
x^{(g)} m := g(x)m.
\]

We obtain that

\[
R(n, r) \xrightarrow{\partial_g} R(n, r)^{(g)} \xrightarrow{d} \Omega_r^{(g)}
\]

is an $R(n, r)$-derivation which induces an $R(n, r)$-module isomorphism

\[
\partial_g : \Omega_r \to \Omega_r^{(g)}.
\]

This reads as

\[
\partial_g(fdx_i) = g(f)dg(x_i)
\]

and provides a canonical $G(n, r)$-action on $\Omega_r$ as a $k$-vector space. Moreover, this operation extends to exterior and symmetric powers over $R(n, r)$ as well as tensor products. In particular, we obtain a representation by the $i$-th higher differentials

\[
\Omega_r^i := \wedge^i_{R(n, r)} \Omega_r = \bigoplus_{j_1 < \ldots < j_i} R(n, r)dx_{j_1} \wedge \ldots \wedge dx_{j_i}.
\]

They are connected by the de Rham complex

\[
0 \to R(n, r) \xrightarrow{d_1} \Omega_r^1 \xrightarrow{d_2} \cdots \xrightarrow{d_n} \Omega_r^n \to 0.
\]
The differential maps are defined by
\[ d_i(fdx_{j_1} \wedge \ldots \wedge dx_{j_i}) := df \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_i}. \]
In fact the maps \( d_i \) are \( G(n,r) \)-equivariant, which can be shown by induction on \( i \).

**Remark 5.2.** Note that with \( U = k^n \), we canonically get
\[ \Omega_i \cong R(n,r) \otimes_k \Lambda^i U \cong \mathcal{I}_r(\Lambda^i U) \]
according to Example 3.6. Hence, for all \((GL_n)^{(r)}\)-representations \( V \), we obtain
\[ \mathcal{I}_r(\Lambda^i U \otimes_k V^{[r]}) \cong \Omega_i \otimes_k P_r^* V \]
by Lemma 3.11. That is, we also have a twisted de Rham complex \( \Omega_* \otimes_k P_r^* V \).

Now we use the transfer morphism \( t_{r,j} : U_j(n,r) \to G(n,j) \) in order to compare the de Rham complexes \( \Omega_j^* \) and \( \Omega_r^* \) by the functor
\[ \text{ind}_{U_j}^{G(n,r)}(t_{r,j}^* \Omega_j^*) \cong \Omega_r^* \]

for all \( 0 \leq i \leq n \).

**Proposition 5.3.** For all \( 1 \leq j < r \), we get
\[ \text{ind}_{U_j}^{G(n,r)}(t_{r,j}^* \Omega_j^*) \cong \Omega_r^* \]
as complexes. Furthermore,
\[ \text{ind}_{U_j}^{G(n,r)}(t_{r,j}^* H^i(\Omega_j^*)) \cong H^i(\Omega_r^*) \]
for all \( 0 \leq i \leq n \).

**Proof.** By Lemma 4.3 and the previous remark, we get canonical isomorphisms
\[ \text{ind}_{U_j}^{G(n,r)}(t_{r,j}^* \Omega_j^*) \cong \text{ind}_{U_j}^{G(n,r)}(t_{r,j}^* \mathcal{I}_j \Lambda^i U) \cong \mathcal{I}_r(\Lambda^i U) \cong \Omega_i^r. \]

Similarly, one can check by a tedious exercise that also
\[ \text{ind}_{U_j}^{G(n,r)}(t_{r,j}^* (d_i : \Omega_j^{i-1} \to \Omega_j^i)) \cong (d_i : \Omega_r^{i-1} \to \Omega_r^i). \]
The claim about the cohomology follows from the exactness of \( \text{ind}_{U_j}^{G(n,r)} \) in Lemma 4.2. \( \square \)

That is, we can compute the cohomology of the complex \( \Omega_r^* \) by the cohomology of the complex \( \Omega_1^* \).

Before we do this, we need an additional observation. Let \( f^r : R(n,r) \to R(n,r) \) be the \( r \)-th power of the absolute Frobenius. It factors as
\[ R(n,r) \xrightarrow{f^r} k \hookrightarrow R(n,r) \]
as \( P^r = P^0 \circ f^r \) for all \( P \in R(n,r) \). This provides induced \( G(n,r) \)-representations
\[ \Omega_i^r \otimes_{R(n,r), f^r} k \cong \Lambda^i U \otimes_{k, f^r} k = \Lambda^i U^{(r)} \]
where again \( U = k^n \).

**Lemma 5.4.** For all \( 1 \leq i \leq n \), we get
\[ \Omega_i^r \otimes_{R(n,r), f^r} k \cong P_r^* \Lambda^i U^{(r)}. \]
Proof. Observe that the group homomorphism
\[ G(n, r) \to GL(U^{(r)}) = (GL_n)^{(r)} \]
corresponding to the \( G(n, r) \)-representation \( \Omega_r \otimes R(n, r), f \cdot k \) coincides with \( P_r \). This provides the claim for \( i = 1 \). The claim for \( i \geq 2 \) follows from this by compatibility with exterior powers. \( \square \)

We get a representation-theoretic reformulation of Cartier’s famous theorem about the cohomology of the de Rham complex. It follows from its proof in [Kat70, Theorem 7.2] and the previous lemma.

**Theorem 5.5** (Cartier). There is a unique collection of isomorphisms of \( G(n, 1) \)-representations
\[ C^{-1} : P_1^* \Lambda^i U^{(1)} \to H^i(\Omega^*_{\bullet}) \]
which satisfies
1. \( C^{-1}(1) = 1 \),
2. \( C^{-1}(\omega \wedge \tau) = C^{-1}(\omega) \wedge C^{-1}(\tau) \),
3. \( C^{-1}(df \otimes 1) = [f^{p-1} df] \in H^1(\Omega^*_{\bullet}) \).

**Remark 5.6.** In fact, the proof in [Kat70] does not provide the property that the \( C^{-1} \) are \( G(n, 1) \)-equivariant. But by property (2), it suffices to check it for \( i = 1 \), which follows from property (3).

As announced before, we can deduce a computation of the cohomology of \( \Omega^*_{\bullet} \) for \( r \geq 2 \) with the help of the transfer morphism \( T_r : G(n, r) \to G(n, r - 1)^{(1)} \).

**Corollary 5.7.** For all \( r \geq 2 \) and \( 1 \leq i \leq n \), we get an isomorphism
\[ H^i(\Omega^*_{\bullet}) \cong T_r^*((\Omega^*_{\bullet})^{(1)}) \]
of \( G(n, r) \)-representations.

Proof. According to Proposition 5.3, Cartier’s Theorem, and Lemma 4.4, we obtain
\[ H^i(\Omega^*_{\bullet}) \cong \text{ind}_{U_1} G(n, r) (t_{r, 1}^* (H^i(\Omega^*_{\bullet}))) \]
\[ \cong \text{ind}_{U_1} G(n, r) (t_{r, 1}^* (P_1^* \Lambda^i U^{(1)}))) \]
\[ \cong T_r^* (I_{r-1} (\Lambda^i U^{(1)}))) \]
\[ \cong T_r^* (\Omega^*_{r-1} (1)), \]
whence the claim. \( \square \)

Finally we want to twist the de Rham complex \( \Omega^*_{\bullet} \) with a \((GL_n)^{(1)}\)-representation \( V \). For \( r = 1 \), we already introduced the twist
\[ \Omega^*_{\bullet} \otimes_k P_1^* V \cong I_1 (\Lambda^i U \otimes_k V^{[1]}). \]

By Cartier’s Theorem, its cohomology computes as
\[ H^i(\Omega^*_{\bullet}) \otimes_k P_1^* V \cong P_1^* (\Lambda^i U^{(1)} \otimes_k V). \]

Again, we consider the functor
\[ \text{ind}_{U_1} G(n, r) \circ t_{r, 1}^* : G(n, 1) - \text{rep} \to G(n, r) - \text{rep}. \]

According to Lemma 4.3, it provides a complex
\[ \text{ind}_{U_1} G(n, r) (t_{r, 1}^* I_1 (\Lambda^i U \otimes_k V^{[1]})) = I_r (\Lambda^i U \otimes_k V^{[1]}). \]
of \(G(n,r)\)-representations. As \(\text{GL}_n\)-representations, this complex reads as
\[
\Omega^{i-1}_r \otimes_k V^{[1]} \xrightarrow{d_i \otimes \text{id}} \Omega^i_r \otimes_k V^{[1]}.
\]

Similarly as in the previous corollary, we obtain
\[
H^i(\Omega^r \otimes_k V^{[1]}) \cong T_r^*((\Omega^r_{r-1})^{(1)} \otimes_k V)
\]
for its cohomology.

6. IRREDUCIBLE REPRESENTATIONS

We are now going to compute the irreducible \(G(n,r)\)-representations
\[
L(\lambda, G(n,r)) = \text{soc} \mathcal{I}(L(\lambda)) = G(n,r)L(\lambda) \subset \mathcal{I}(L(\lambda))
\]
for all \(\lambda \in X(T)_+\) with respect to their associated irreducible \(\text{GL}_n\)-representations. For this, we take as a split maximal torus the diagonal matrices. This torus affords canonical projections \(\varepsilon_i \in X(T)\) for \(1 \leq i \leq n\) which are a free \(\mathbb{Z}\)-basis of the character group \(X(T)\).

According to Proposition 3.13 there is a mod \(p^r\)-periodicity for the dominant weights and one can restrict to the \(L(\lambda, G(n,r))\) with \(\lambda \in X_r(T)\). This will cover the case \(r = 1\). The case \(r \geq 2\) is more subtle.

We restrict to the following subset \(X'_1(T) \subset X_1(T)\):
\[
X'_1(T) := \left\{ \lambda = \sum_{i=1}^n m_i(\epsilon_1 + \ldots + \epsilon_i) \in X(T) \mid \forall 1 \leq i \leq n: 0 \leq m_i < p \right\}.
\]

**Notation 6.1.** As \(X'_1(T)\) is a set of representatives for \(X(T)/pX(T)\), we get a unique decomposition
\[
\lambda = r(\lambda) + ps(\lambda)
\]
for all \(\lambda \in X(T)_+\) with \(r(\lambda) \in X'_1(T)\) and \(s(\lambda) \in X(T)_+\). We call \(r(\lambda)\) the mod \(p\)-reduction of \(\lambda\).

The following proposition covers the dominant weights \(\lambda\) with \(r(\lambda) = 0\).

**Proposition 6.2.** Let \(\lambda \in X(T)_+\). Then we obtain
\[
L(p\lambda, G(n,1)) \cong P_1^p L(\lambda),
\]
and for \(r \geq 2\) we get
\[
L(p\lambda, G(n,r)) \cong T^*_r L(\lambda, G(n, r-1)^{(1)}).
\]

**Proof.** The claim for \(r = 1\) is just a special case of Proposition 3.13 and the claim for \(r \geq 2\) follows by the discussion after Lemma 4.1. \(\square\)

The idea for the case \(r(\lambda) \neq 0\) is the following: The \(G(n,r)^{-}\)-invariants of the socle of \(\mathcal{I}(L(\lambda)) = L(\lambda) \otimes R(n,r)\) are \(L(\lambda) \otimes k \cong L(\lambda)\). That is, the socle is generated by this subspace as a \(G(n,r)\)-representation.

**Notation 6.3.** For \(\lambda \in X'_1(T)\) write \(\lambda = \sum_{i=1}^n m_i(\epsilon_1 + \ldots + \epsilon_i)\), where \(0 \leq m_i < p\), and consider the \(\text{GL}_n\)-representation
\[
\text{Sym}^{m_1}(U) \otimes_k \text{Sym}^{m_2}(\Lambda^2 U) \otimes_k \ldots \otimes_k \text{Sym}^{m_n}(\Lambda^n U),
\]
where \(U = k^n\) with canonical basis \(e_1, \ldots, e_n\). Now consider the vector
\[
v(\lambda) = e_1^{m_1} \otimes (e_1 \wedge e_2)^{m_2} \otimes \ldots \otimes (e_1 \wedge \ldots \wedge e_n)^{m_n}
\]
in this representation. We define \( W(\lambda) \) to be the \( \GL_n \)-subrepresentation generated by this vector.

For a general \( \lambda \in X(T)_+ \) set
\[
W(\lambda) := W(r(\lambda)) \otimes L(s(\lambda))^{[1]}.
\]

Note that \( W(r(\lambda)) \) has highest weight \( r(\lambda) \) of multiplicity 1. That is, there is a subrepresentation \( V \subset W(r(\lambda)) \) such that
\[
L(r(\lambda)) \cong W(r(\lambda))/V.
\]
Thus
\[
L(\lambda) \cong W(\lambda)/(V \otimes_k L(s(\lambda))^{[1]})
\]
by Steinberg’s Tensor Product Theorem. For example, if \( \lambda = \varepsilon_1 + \ldots + \varepsilon_i \) is a fundamental weight, then
\[
L(\varepsilon_1 + \ldots + \varepsilon_i) = \Lambda^i V = W(\lambda).
\]

As \( \mathcal{I} \) is exact, we get
\[
\mathcal{I}(L(\lambda)) = \mathcal{I}(W(\lambda))/\mathcal{I}(V \otimes_k L(s(\lambda))^{[1]}).
\]
Since \( v(r(\lambda)) \) is a highest weight vector, it follows that
\[
\text{soc} \mathcal{I}(L(\lambda)) = G(n, r) L(\lambda)
\]
\[
= G(n, r)(v(r(\lambda)) \otimes_k L(s(\lambda))^{[1]})/\mathcal{I}(V \otimes_k L(s(\lambda))^{[1]}).
\]

Finally, note that
\[
\mathcal{I}(\text{Sym}_{m_1}^n (U) \otimes_k \text{Sym}_{m_2}^n (\Lambda^2 U) \otimes_k \ldots \otimes_k \text{Sym}_{m_n}^n (\Lambda^n U)) = R(n, r) \otimes_k \text{Sym}_{m_1}^n (U) \otimes_k \text{Sym}_{m_2}^n (\Lambda^2 U) \otimes_k \ldots \otimes_k \text{Sym}_{m_n}^n (\Lambda^n U)
\]
\[
\cong \text{Sym}_{m_1}^{R(n, r)}(\Omega^1_t) \otimes_{R(n, r)} \text{Sym}_{m_2}^{R(n, r)}(\Omega^2_t) \otimes_{R(n, r)} \ldots \otimes_{R(n, r)} \text{Sym}_{m_n}^{R(n, r)}(\Omega^n_t)
\]
as \( G(n, r) \)-representations, and for \( \lambda = r(\lambda) \), the vector \( v(\lambda) \) corresponds to
\[
v = (dx_1)^{m_1} \otimes (dx_1 \wedge dx_2)^{m_2} \otimes \ldots \otimes (dx_1 \wedge \ldots \wedge dx_n)^{m_n}.
\]

In order to compute \( G(n, r)v \subset \mathcal{I}(W(\lambda)) \), we will use the Lie algebra operators \( \delta_{(i, x^j)} \). The following lemma describes their action on \( \mathcal{I}(V) \).

**Lemma 6.4.** Let \( \delta_{(i, x^j)} \in \text{Lie } G(n, r) \) be a canonical basis element. Then for all \( \GL_n \)-representations \( V \), the induced action on
\[
\mathcal{I}(V) = R(n, r) \otimes_k V
\]
reads as
\[
\delta_{(i, x^j)}(f \otimes v) = \left( x^j \frac{\partial}{\partial x_i} f \right) \otimes v + \sum_{j=1}^n f \frac{\partial}{\partial x_j} x^j \otimes E_{ji}(v),
\]
where \( E_{ji} \in M_n(k) = \text{Lie } \GL_n \) is the \( (j, i) \)-th standard matrix.

**Proof.** First recall the \( G(n, r) \)-action on \( \mathcal{I}(V) = R(n, r) \otimes_k V 
\]
\[
g(f \otimes v) = \left( \frac{\partial g(x_s)}{\partial x_k} \right)_{ks} (g(f) \otimes v)
\]
for all \( g \in G(n, r), f \in R(n, r) \), and \( v \in V \).
In order to compute the action of \( \delta_{(i,x')} = x' \frac{\partial}{\partial x'_i} \), we consider the corresponding element \( g_{(i,I)} = 1 + \delta_{(i,x')} \epsilon \in G(n,r)(k[\epsilon]) \), where \( k[\epsilon] \) are the dual numbers. That is,

\[
g_{(i,I)}(x_s) = \begin{cases} x_s + x'_i \epsilon, & s = i, \\ x_s, & s \neq i. \end{cases}
\]

Now we get

\[
\delta_{(i,x')} (f \otimes v) = \left. \frac{\partial}{\partial \epsilon} \left( \left( \frac{\partial g_{(i,I)}(x_s)}{\partial x_k} \right)_{ks} (g_{(i,I)}(f) \otimes v) \right) \right|_{\epsilon = 0}.
\]

The product rule provides

\[
\delta_{(i,x')} (f \otimes v) = \delta_{(i,x')} (f) \otimes v + f \left. \frac{\partial}{\partial \epsilon} \left( \left( \frac{\partial g_{(i,I)}(x_s)}{\partial x_k} \right)_{ks} (1 \otimes v) \right) \right|_{\epsilon = 0}.
\]

As

\[
\frac{\partial g_{(i,I)}(x_s)}{\partial x_k} = \begin{cases} 1 + \frac{\partial x'_i}{\partial x'_k} \epsilon, & s = i = k, \\ \frac{\partial x'_i}{\partial x'_k} \epsilon, & s = i \neq k, \\ 1, & s = k \neq i, \\ 0, & s \neq k, s \neq i, \end{cases}
\]

we get

\[
\left. \left. \frac{\partial}{\partial \epsilon} \left( \frac{\partial}{\partial x_k} g_{(i,I)}(x_s) \right) \right|_{\epsilon = 0} = \begin{cases} \frac{\partial x'_i}{\partial x'_k}, & s = i, \\ 0, & s \neq i. \end{cases}
\]

whence the claim. \( \square \)

**Remark 6.5.** Note that the action of \( \delta_{(i,x')} \) on \( W(\lambda) = W(r(\lambda)) \otimes L(s(\lambda))[1] \) is \(-1 \otimes k \text{id}_{L(s(\lambda))[1]} \) applied to the action on \( W(r(\lambda)) \) as \( \text{Lie GL}_n \) acts trivially on the Frobenius twists \( V[1] \).

Now we are ready to treat the case where the mod \( p \)-reduction of \( \lambda \) is a fundamental weight \( \epsilon_1 + \ldots + \epsilon_i \). By Steinberg’s Tensor Product Theorem, we know that

\[
L(\lambda) \cong L(\epsilon_1 + \ldots + \epsilon_i) \otimes_k L(s(\lambda))[1] \cong \Lambda^i U \otimes_k L(s(\lambda))[1] = W(\lambda)
\]

with \( U = k^n \). That is,

\[
\mathcal{I}(L(\lambda)) \cong \Omega^i_r \otimes_k L(s(\lambda))[1],
\]

which is part of the twisted de Rham complex as introduced at the end of the previous section. Recall that the differentials read as

\[
\Omega^{i-1}_r \otimes_k L(s(\lambda))[1] \xrightarrow{d_i \otimes \text{id}} \Omega^i_r \otimes_k L(s(\lambda))[1],
\]

where \( d_i : \Omega^{i-1}_r \to \Omega^i_r \) is the de Rham-differential.

**Proposition 6.6.** Let \( \lambda \in X(T)_+ \) with \( r(\lambda) = \epsilon_1 + \ldots + \epsilon_i \). Then

\[
L(\lambda, G(n,r)) \cong \text{soc}(\Omega^i_r \otimes_k L(s(\lambda))[1]) = \text{Im}(d_i \otimes \text{id}),
\]

where \( d_i : \Omega^{i-1}_r \to \Omega^i_r \) is the de Rham-differential.
Proof. We will use Lie($G(n, r)$)-operators to prove the claim. Recall that $f \in \text{Lie} G(n, r)$ acts on $\Omega^k \otimes_k L(s(\lambda))[1]$ as $f \otimes \text{id}$.

We already noticed that the socle is generated by the $G(n, r)$-invariants as a $G(n, r)$-representation. A generating system of these invariants is given by

$$(dx_{j_1} \wedge \ldots \wedge dx_{j_i}) \otimes v$$

for all $j_1 < \ldots < j_i$ and $v \in L(s(\lambda))[1]$. Now the inclusion

$$\text{soc}(\Omega^k \otimes L(s(\lambda))[1]) \subset \text{Im}(d_i \otimes \text{id})$$

follows from the fact that the generators lie in the image of $d_i \otimes \text{id}$. For the inclusion $\text{Im}(d_i \otimes \text{id}) \subset \text{soc}(\Omega^k \otimes L(s(\lambda))[1])$ note that $\text{Im}(d_i \otimes \text{id})$ is as a $k$-vector space generated by the elements

$$(dx^l \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_{i-1}}) \otimes v,$$

where $v \in L(s(\lambda))[1]$, $x^l = x_1^{m_1} \ldots x_n^{m_n} \in R(n, r)$, and $j_1 < \ldots < j_{i-1}$. As $i - 1 < n$, there is an index $l \notin \{j_1, \ldots, j_{i-1}\}$. Then we get that the Lie algebra operator $\delta_{(l,x^l)} \in \text{Lie}(G(n, r))$ acts as

$$\delta_{(l,x^l)}((dx_i \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_{i-1}}) \otimes v) = (dx^l \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_{i-1}}) \otimes v$$

according to Lemma 6.4. This provides all image elements from the generators. \(\square\)

The next proposition covers the case where the mod $p$-reduction of $\lambda$ is neither 0 nor a fundamental weight if we assume char($k$) $\neq 2$.

**Proposition 6.7.** Assume that char($k$) $\neq 2$. Let $\lambda \in X(T)_+$ be a dominant weight with $r(\lambda) \neq 0$ and $r(\lambda) \neq \epsilon_1 + \ldots + \epsilon_i$ for all $i = 1, \ldots, n$. Then

$$L(\lambda, G(n, r)) = \mathcal{I}(L(\lambda)).$$

Before we give the proof, we need some technical lemmas.

**Lemma 6.8.** Let $V$ be a $\text{GL}_n$-representation, $v \in V$, and $1 \leq s_j \leq p^{r-1}$. If

$$x_1^{ps_1-1} \ldots x_n^{ps_n-1} v \in G(n, r)v \subset R(n, r) \otimes_k V = \mathcal{I}(V),$$

then

$$x^J v \in G(n, r)v \subset \mathcal{I}(V)$$

for all $J = (j_1, \ldots, j_n)$ with $p(s_k - 1) \leq j_k < ps_k$ for all $1 \leq k \leq n$.

**Proof.** This follows by the gradual application of the Lie algebra operators

$$\delta_i = \frac{\partial}{\partial x_i} \otimes \text{id} : R(n, r) \otimes V \to R(n, r) \otimes V.$$

\(\square\)

**Lemma 6.9.** Let $V$ be a $\text{GL}_n$-representation, $v \in V$, and $1 \leq s_j \leq p^{r-1}$. Assume that $x^J v \in G(n, r)v \subset R(n, r) \otimes_k V = \mathcal{I}(V)$ for $J = (j_1, \ldots, j_n)$.

1. If $j_k = ps$, then for all $j \neq k$ we get

$$x^J E_{jk} v \in G(n, r)v \subset \mathcal{I}(V).$$

2. If $j_i = ps$ and $x_j \frac{\partial}{\partial x_k} x^J v \in G(n, r)v$ we get

$$x^J E_{ki} E_{jk} v \in G(n, r)v \subset \mathcal{I}(V).$$
Proof. The first part follows from
\[ x^J E_{jk} v = \delta_{(k,x)}(x^J v) \]
as \( j_k = ps \).

The second part follows from
\[
x^J E_{ki} E_{jk} v = \delta_{(i,x)}(x^J E_{jk} v) = \delta_{(i,x)} \left( \delta_{(k,x)}(x^J v) - x_j \frac{\partial}{\partial x_k} x^J v \right)
\]
as \( j_i = ps \). \(\square\)

Lemma 6.10. Let \( \lambda \in X'(T) \) and write \( \lambda = \sum_{i=1}^n m_i (\epsilon_1 + \ldots + \epsilon_i) \). Let \( v = v(\lambda) \in W(\lambda) \) be the vector from above. Let \( k \) be the highest index such that \( m_k \neq 0 \) and \( i < k \) be the highest index such that \( m_i \neq 0 \).

1. For all \( j \leq k \), we get
\[ E_{jk} v = \delta_{jk} m_k v, \]
where \( \delta_{jk} \) is the Kronecker-\( \delta \).

2. For \( j > k \), we get
\[ E_{kk} E_{jk} v = (m_k - 1) E_{jk} v. \]

3. If \( m_k = 1 \), we get
\[ E_{ii} v = (m_i + 1) v. \]

4. If \( m_k = 1 \), we get for all \( j > k \)
\[ E_{ii} E_{ki} E_{jk} v = m_i E_{ki} v \]
and
\[ E_{ii} E_{ki} E_{jk} v = m_i E_{ki} E_{jk} v. \]

Proof. Recall that
\[ v = e_1^{m_1} \otimes (e_1 \wedge e_2)^{m_2} \otimes \ldots \otimes (e_1 \wedge \ldots \wedge e_k)^{m_k}. \]
The first claim follows by
\[ E_{jk} v = m_k e_1^{m_1} \otimes (e_1 \wedge e_2)^{m_2} \otimes \ldots \otimes (e_1 \wedge \ldots \wedge e_k - 1 \wedge e_j)(e_1 \wedge \ldots \wedge e_k)^{m_k - 1}. \]
The second follows from the first by
\[ E_{kk} E_{jk} v = E_{jk} E_{kk} v + (E_{kk} E_{jk} - E_{jk} E_{kk}) v, \]
using \( E_{kk} E_{jk} = 0 \) and \( E_{jk} E_{kk} = E_{jk} \).

Now let \( m_k = 1 \). Then the third claim follows as \( E_{ii} \) acts precisely on the factors \((e_1 \wedge \ldots \wedge e_i)^{m_i}\) and \((e_1 \wedge \ldots \wedge e_k)\) of \( v \).

The claim \( E_{ii} E_{ki} v = m_i E_{ki} v \) follows from the third in the same fashion as the second follows from the first.

Finally, for the last claim using \( E_{ii} E_{jk} = E_{jk} E_{ii} = 0 \) and the third claim, we get
\[ E_{ii} E_{jk} v = E_{jk} E_{ii} v = (m_i + 1) E_{jk} v. \]

Hence
\[
E_{ii} E_{ki} E_{jk} v = E_{ki} E_{ii} E_{jk} v - E_{ki} E_{jk} v = m_i E_{ki} E_{jk} v,
\]
which finishes the proof. \(\square\)
**Proof of Proposition 6.7.** By previous discussions, it suffices to prove
\[ G(n, r)(v(r(\lambda))) \otimes_k L(s(\lambda))^{[1]} = \mathcal{I}(W(r(\lambda))) \otimes_k L(s(\lambda))^{[1]} = \mathcal{I}(W(\lambda)). \]

Again we will use $\text{Lie}(G(n, r))$-operators to prove the claim. As again $f \in \text{Lie} G(n, r)$ acts as $f \otimes \text{id}$, we can assume that $\lambda = r(\lambda)$.

Let $v = v(\lambda)$ as above. It suffices to show
\[ R(n, r) \otimes_k kv(\lambda) \subset G(n, r)v(\lambda) \subset \mathcal{I}(W(\lambda)). \]

Now again write
\[ \lambda = \sum_{i=1}^{n} m_i(\epsilon_1 + \ldots + \epsilon_i). \]

As $\lambda = r(\lambda)$, we have $0 \leq m_i \leq p - 1$ for all $i = 1, \ldots, n$.

By Lemma 6.8, it suffices to prove
\[ x_1^{ps_1-1} \ldots x_n^{ps_n-1}v \in G(n, r)v \]
for all choices $1 \leq s_j \leq p^{r-1}$.

By assumption we have $\lambda \neq 0$. That is, there is a highest index $k$ such that $m_k \neq 0$.

**Case 1 ($m_k \geq 2$).** Let us assume that $m_k \geq 2$. We argue by descending induction on $s_k$.

Take $I = (ps_1 - 1, \ldots, ps_n - 1)$. Note that for all $J = (j_1, \ldots, j_n)$ with $j_k \geq ps_k$ we get
\[ x^J v \in G(n, r)v. \]

For $s_k = p^{r-1}$ this is clear as in this case $x^J = 0$. The case $s_k < p^{r-1}$ follows by the induction hypothesis and Lemma 6.8.

By Lemma 6.3 we get
\[ \delta_{(k, x^J)}(v) = m_k \frac{\partial}{\partial x_k} x^J v + \sum_{j > k} \frac{\partial}{\partial x_j} x^J E_{jk} v \]
since $E_{jk} v = \delta_{jk} m_k v$ for $j \leq k$ by Lemma 6.10 (1).

Now we apply the operator $\delta_{(k, x^J)}$ to $\delta_{(k, x^J)}(v)$. Let $j > k$. Using $E_{kk} E_{jk} v = (m_k - 1) E_{jk} v$ in Lemma 6.10 (2) we obtain
\[ \delta_{(k, x^J)} \left( \frac{\partial}{\partial x_k} x^J E_{jk} v \right) = x_k^2 \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_j} x^J E_{jk} v + 2x_k \frac{\partial}{\partial x_j} x^J E_{kk} E_{jk} v \]
\[ = x_k^2 \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_j} x^J E_{jk} v + 2(m_k - 1)x_k \frac{\partial}{\partial x_j} x^J E_{jk} v \]
\[ \in G(n, r)v \]
by (1) and Lemma 6.9 (1).

As $\delta_{(k, x^J)}(\delta_{(k, x^J)}(v)) \in G(n, r)v$, we obtain
\[ \delta_{(k, x^J)} \left( \frac{\partial}{\partial x_k} x^J v \right) \in G(n, r)v \]
by (2). Using $E_{kk}v = m_kv$, we get
\[
\delta_{(k,x_k^2)} \left( \frac{\partial}{\partial x_k} x^I v \right) = x_k^2 \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_k} x^I v + 2x_k \frac{\partial}{\partial x_k} x^I E_{kk}v \\
= \left( x_k^2 \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_k} x^I + 2m_k x_k \frac{\partial}{\partial x_k} x^I \right) v \\
= (ps_k - 1)(ps_k - 2 + 2m_k)x^I v \\
= 2(1 - m_k)x^I v \\
\in G(n, r)v.
\]
As $2 \leq m_k \leq p - 1$ and char$(k) = p \neq 0$, we have $2(1 - m_k) \neq 0$. Hence
\[
x_1^{ps_1-1} \cdots x_n^{ps_n-1} v = x^I v \in G(n, r)v
\]
for all choices $1 \leq s_j \leq p^{r-1}$, which finishes the proof for the case $m_k \geq 2$.

**Case 2 ($m_k = 1$).** In the second case, we assume that $m_k = 1$. As $r(\lambda) = \lambda$ is not a fundamental weight by assumption, there is a highest index $i < k$ with $m_i \neq 0$. Here, we will argue by descending induction on $s_i + s_k$.

Again take $I = (ps_1 - 1, \ldots, ps_n - 1)$. Note that for all $J = (j_1, \ldots, j_n)$ with $j_k \geq ps_k \land j_i \geq p(s_i - 1)$ or $j_k \geq p(s_k - 1) \land j_i \geq ps_i$, we get
\[(3) \quad x^I v \in G(n, r)v.
\]
For $s_k = p^{r-1}$ or $s_i = p^{r-1}$, this is clear, as in this case $x^I = 0$. In the case $j_k \geq ps_k$ and $j_i \geq p(s_i - 1)$ and $s_k < p^{r-1}$ it follows by the induction hypothesis and Lemma 6.8. The case $j_k \geq p(s_k - 1)$ and $j_i \geq ps_i$ is analogous.

Again by Lemma 6.4 we obtain
\[(4) \quad \delta_{(k,x^I)}(v) = \frac{\partial}{\partial x_k} x^I v + \sum_{j > k} \frac{\partial}{\partial x_j} x^I E_{jk}v,
\]
as $E_{kk}v = m_kv = v$.

For $j > k$ we get
\[
x_k \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_j} x^I E_{jk}v \in G(n, r)v
\]
by (3) and Lemma 6.9(1).

Now we apply the operator $\delta_{(i,x^2)} \circ \delta_{(i,x_k)}$ to $\delta_{(k,x^I)}(v)$. For $j > k$, we get
\[
\delta_{(i,x^2)} \left( \delta_{(i,x_k)} \left( \frac{\partial}{\partial x_j} x^I E_{jk}v \right) \right) \\
= \delta_{(i,x^2)} \left( x_k \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_j} x^I E_{jk}v \right) + \delta_{(i,x^2)} \left( \frac{\partial}{\partial x_j} x^I E_{ki}E_{jk}v \right).
\]
But using $E_{ii}E_{ki}E_{jk}v = m_i E_{ki}E_{jk}v$ in Lemma 6.10(4), we get for $j > k$
\[
\delta_{(i,x^2)} \left( \frac{\partial}{\partial x_j} x^I E_{ki}E_{jk}v \right) = \frac{x_i^2}{x_i} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} x^I E_{ki}E_{jk}v + \frac{x_i^2}{x_j} \frac{\partial}{\partial x_j} x^I E_{ki}E_{jk}v \\
= \frac{x_i^2}{x_i} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} x^I E_{ki}E_{jk}v + 2m_i x_i \frac{\partial}{\partial x_j} x^I E_{ki}E_{jk}v \\
\in G(n, r)v
\]
by (3) and Lemma 6.9(2).
As \( \delta(i,x^2_1)\delta(i,x_k)(\delta(k,x^I)(v))) \in G(n, r)v \), we obtain

\[
\delta(i,x^2_1)\left( \delta(i,x_k) \left( \frac{\partial}{\partial x_k} x^I v \right) \right) \in G(n, r)v
\]

by (4). That is, by using

\[
x_k \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_k} x^I = (p s_k - 1) \frac{\partial}{\partial x_i} x^I = - \frac{\partial}{\partial x_i} x^I
\]

we get

\[
\delta(i,x^2_1) \left( - \frac{\partial}{\partial x_i} x^I v + \frac{\partial}{\partial x_k} x^I E_{ki} v \right) \in G(n, r)v.
\]

Using \( E_{ii} E_{ki} v = m_i E_{ki} v \) in Lemma [6.10]. (4), we obtain

\[
\delta(i,x^2_1) \left( \frac{\partial}{\partial x_k} x^I E_{ki} v \right) = x^2_i \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_k} x^I E_{ki} v + 2 x_i \frac{\partial}{\partial x_k} x^I E_{ii} E_{ki} v
\]

\[
= x^2_i \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_k} x^I E_{ki} v + 2 m_i x_i \frac{\partial}{\partial x_k} x^I E_{ki} v
\]

\[
\in G(n, r)v
\]

by (3) and Lemma 6.9(1). Hence

\[
\delta(i,x^2_1) \left( \frac{\partial}{\partial x_i} x^I v \right) \in G(n, r)v.
\]

Using \( E_{ii} v = (m_i + 1)v \) in Lemma [6.10]. (3), we obtain

\[
\delta(i,x^2_1) \left( \frac{\partial}{\partial x_i} x^I v \right) = x^2_i \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_i} x^I v + 2 x_i \frac{\partial}{\partial x_i} x^I E_{ii} v
\]

\[
= x^2_i \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_i} x^I v + 2 m_i x_i \frac{\partial}{\partial x_i} x^I E_{ii} v
\]

\[
= 2 x^I v - 2 (m_i + 1) x^I v
\]

\[
= -2 m_i x^I v
\]

\[
\in G(n, r)v.
\]

That is, we get

\[
2 m_i x^I v \in G(n, r)v,
\]

which is nonzero as \( \text{char}(k) = p \neq 2 \) and \( 1 \leq m_i \leq p - 1 \). Hence

\[
x_1^{ps_1-1} \cdots x_n^{ps_n-1} v = x^I v \in G(n, r)v,
\]

which finishes the proof for \( m_k = 1 \).

Finally, we proved the proposition. \( \square \)

7. The Grothendieck ring

For a \( k \)-group scheme \( G \) denote by

\[
\text{Rep}(G) := K_0(G-\text{rep})
\]

the Grothendieck ring of finite dimensional representations. This is a free abelian group where a \( \mathbb{Z} \)-basis is given by the classes of the irreducible representations.

Using the description of irreducible \( G(n, r) \)-representations of the previous section, we are now ready to describe \( \text{Rep}(G(n, r)) \). Recall that the functor \( \mathcal{I} \) is exact.
Theorem 7.1. Assume that char(k) ≠ 2. Then the maps
\[ \left( \frac{T}{P_1^*} \right) : \text{Rep}(\text{GL}_n) \oplus \text{Rep}((\text{GL}_n)^{(1)}) \to \text{Rep}(\text{G}(n, 1)) \]
and for \( r \geq 2 \)
\[ \left( \frac{I}{T_r^*} \right) : \text{Rep}(\text{GL}_n) \oplus \text{Rep}(\text{G}(n, r - 1)^{(1)}) \to \text{Rep}(\text{G}(n, r)) \]
are surjective morphisms of abelian groups.

Proof. Recall that a \( Z \)-basis of \( \text{Rep}(\text{G}(n, r)) \) is given by \( \text{L}(\lambda, \text{G}(n, r)) \) with \( \lambda \in X(T)^+ \). That is, it suffices to show that these classes lie in the respective image.

We start with the case \( r = 1 \) and the map \( \left( \frac{I}{P_1^*} \right) \). For \( r(\lambda) = 0 \), we know by Proposition 6.2 that
\[ L(\lambda, \text{G}(n, 1)) = P_1^* L(s(\lambda)), \]
which shows the claim. For \( r(\lambda) = \varepsilon_1 + \ldots + \varepsilon_i \), we know by Proposition 6.6 that
\[ L(\lambda, \text{G}(n, 1)) \cong \text{Im}(d_i \otimes \text{id} : \Omega_i^{i-1} \otimes_k L(s(\lambda))^{[1]} \to \Omega_i^i \otimes_k L(s(\lambda))^{[1]}), \]
the images in the twisted de Rham complex. We also know that its cohomology computes as
\[ H^i(\Omega_i^i \otimes_k L(s(\lambda))^{[1]} \cong P_1^*(\Lambda^iU^{(1)} \otimes_k L(s(\lambda))). \]
That is, the cohomology classes lie in the image of \( \left( \frac{I}{P_1^*} \right) \). As
\[ \Omega_i^i \otimes_k L(s(\lambda))^{[1]} = \text{I}(\Lambda^iU \otimes_k L(s(\lambda))^{[1]}), \]
also the classes of the objects of the complex lie in this image. As
\[ [\text{Im}(d_n \otimes \text{id})] = [\Omega_n^i \otimes_k L(s(\lambda))^{[1]}] + [H^n(\Omega_i^i \otimes_k L(s(\lambda))^{[1]})] \]
we get the claim for \( r(\lambda) = \varepsilon_1 + \ldots + \varepsilon_i \). Now let \( i < n \). Then
\[ [\text{Im}(d_i \otimes \text{id})] = [\Omega_i^i \otimes_k L(s(\lambda))^{[1]}] - [\text{Im}(d_{i+1} \otimes \text{id})] - [H^i(\Omega_i^i \otimes_k L(s(\lambda))^{[1]})] \]
inductively provides the claim for \( r(\lambda) = \varepsilon_1 + \ldots + \varepsilon_i \). Finally, in the case that \( r(\lambda) \) is neither 0 nor a fundamental weight, Proposition 6.7 provides
\[ L(\lambda, \text{G}(n, r)) = \text{I}(L(\lambda)), \]
which finishes the case \( r = 1 \).

Now let \( r \geq 2 \). That is, we consider the map \( \left( \frac{I}{T_r^*} \right) \). Then for \( r(\lambda) = 0 \), we get
\[ L(\lambda, \text{G}(n, r)) = T_r^* L(s(\lambda), \text{G}(n, r - 1)^{(1)}) \]
by Proposition 6.2 which shows the claim. For the case \( r(\lambda) = \varepsilon_1 + \ldots + \varepsilon_i \) we use the twisted de Rham complex
\[ \Omega_r^i \otimes_k L(s(\lambda))^{[1]} = \text{I}(\Lambda^iU \otimes_k L(s(\lambda))^{[1]}). \]
As the image of the \( i \)-th differential is \( L(\lambda, \text{G}(n, r)) \) by Proposition 6.6 and its cohomology computes as
\[ H^i(\Omega_r^i \otimes_k L(s(\lambda))^{[1]} \cong T_r^*((\Omega_{r-1}^i)^{(1)} \otimes_k L(s(\lambda))), \]
the claim follows in the same fashion as in the case \( r = 1 \). Again the case where \( r(\lambda) \) is neither 0 nor fundamental follows from Proposition 6.7. \( \square \)
Remark 7.2. Let $\text{char}(k) = 2$. Then for $n = r = 1$, the case where $r(\lambda)$ is neither $0$ nor fundamental does not occur. But this is the only case where the assumption $\text{char}(k) \neq 2$ is necessary in order to apply Proposition 6.7. That is, for $n = r = 1$, we also get a surjection
\[
\begin{pmatrix} I \\ P_1^* \end{pmatrix} : \text{Rep}(\mathbb{G}_m) \oplus \text{Rep}((\mathbb{G}_m)^{(1)}) \to \text{Rep}(G(1, 1)).
\]
In the case that $r \geq 2$ or $n \geq 2$, the author does not know whether the maps in question are surjective.

So far, we only described the abelian group structure of $\text{Rep}(G(n, r))$. We also want to understand its ring structure and the kernels of the maps of the theorem. For both topics, the main tool is the restriction map
\[
\text{res} : \text{Rep}(G(n, r)) \to \text{Rep}(\text{GL}_n).
\]
In fact, this map is injective.

Lemma 7.3. The restriction map
\[
\text{res} : \text{Rep}(G(n, r)) \to \text{Rep}(\text{GL}_n)
\]
is injective.

Proof. As $\mathbb{G}_m \subset \text{GL}_n \subset G(n, r)$, we can study $\mathbb{G}_m$-weight spaces for $G(n, r)$-representations. Furthermore, the $\mathbb{G}_m$-weight space filtration of each $\text{GL}_n$-representation is $\text{GL}_n$-invariant. Denote by $\text{deg} : X(T) \to \mathbb{Z}$ the degree map which is induced by $\text{deg}(\epsilon_i) = 1$ for all $i = 1, \ldots, n$. Then for all $n \in \mathbb{Z}$ and $\text{GL}_n$-representations $V$, we get for the $n$-th $\mathbb{G}_m$-weight space
\[
V_n = \bigoplus_{\lambda \in \text{deg}^{-1}(n)} V_\lambda.
\]
Now consider $L(\lambda, G(n, r))$ for $\lambda \in X(T)_+$, whose classes form a $\mathbb{Z}$-basis of $\text{Rep}(G(n, r))$. Its lowest $\mathbb{G}_m$-weight space is $L(\lambda)$ of weight $s = \text{deg}(\lambda)$. Hence, we obtain
\[
[\text{res} L(\lambda, G(n, r))] = [L(\lambda)] + \sum_{\substack{\mu \in X(T)_+ \\ \text{deg}(\mu) > s}} m_\mu [L(\mu)] \in \text{Rep}(\text{GL}_n),
\]
where $m_\mu$ is the multiplicity of $L(\mu)$ in $L(\lambda, G(n, r))$. As $[L(\lambda)]$ with $\lambda \in X(T)_+$ form a $\mathbb{Z}$-basis of $\text{Rep}(\text{GL}_n)$, res maps a $\mathbb{Z}$-basis of $\text{Rep}(G(n, r))$ to a linearly independent set. Hence res is injective. □

Now $\text{Rep}(\text{GL}_n)$ is a $\text{Rep}((\text{GL}_n)^{(1)})$-algebra by the pullback of the first Frobenius
\[
F^* : \text{Rep}((\text{GL}_n)^{(1)}) \to \text{Rep}(\text{GL}_n).
\]
Note that under the identification $\text{GL}_n^{(1)} \cong \text{GL}_n$ the ring homomorphism $F^*$ is the $p$-th Adams operation $\psi^p$ on the $\lambda$-ring $\text{Rep}(\text{GL}_n)$. We fix an $r$ and endow the direct sum
\[
\text{Rep}(\text{GL}_n) \oplus \text{Rep}((\text{GL}_n)^{(1)})
\]
with a ring structure by assigning
\[
(b, a) \cdot (b', a') := (F^*(a)b' + F^*(a')b + [R(n, r)]bb', aa'),
\]
where \([R(n, r)]\) is the class of the \(G(n, r)\)-representation \(R(n, r)\) restricted to the subgroup \(\text{GL}_n\). Now the inclusion
\[
\text{Rep}((\text{GL}_n)^{(1)}) \hookrightarrow \text{Rep}((\text{GL}_n) \oplus \text{Rep}((\text{GL}_n)^{(1)}))
\]
gives a \(\text{Rep}((\text{GL}_n)^{(1)})\)-algebra structure. Furthermore, the map
\[
\left(\frac{[R(n, r)] \cdot (-)}{F^*}\right) : \text{Rep}((\text{GL}_n) \oplus \text{Rep}((\text{GL}_n)^{(1)})) \to \text{Rep}(\text{GL}_n)
\]
is a \(\text{Rep}((\text{GL}_n)^{(1)})\)-algebra morphism. Finally note that
\[
\text{res} \circ I = [R(n, r)] \cdot (-)
\]
on \(\text{Rep}(\text{GL}_n)\).

Let us consider the case \(r = 1\). In order to show that
\[
\left(\frac{I}{P^*_1}\right) : \text{Rep}(\text{GL}_n) \oplus \text{Rep}((\text{GL}_n)^{(1)}) \to \text{Rep}(G(n, 1))
\]
is a ring homomorphism, it suffices to show this after composition with \(\text{res}\). But this is precisely the ring homomorphism \(\left(\frac{[R(n, 1)] \cdot (-)}{F^*}\right)\) considered above.

Now consider the case \(r \geq 2\). Then we introduce a ring structure on
\[
\text{Rep}(\text{GL}_n) \oplus \text{Rep}(G(n, r-1)(1))
\]
by
\[
(b, a) \cdot (b', a') := (F^*(\text{res}(a))b' + F^*(\text{res}(a'))b + [R(n, r)]bb', aa'),
\]
where we use the restriction
\[
\text{res} : \text{Rep}(G(n, r-1)(1)) \hookrightarrow \text{Rep}((\text{GL}_n)^{(1)}).
\]
This makes
\[
\text{Rep}(\text{GL}_n) \oplus \text{Rep}(G(n, r-1)(1)) \overset{\text{id} \oplus \text{res}}{\longrightarrow} \text{Rep}(\text{GL}_n) \oplus \text{Rep}((\text{GL}_n)^{(1)})
\]
into a ring injection. In order to show that
\[
\left(\frac{I}{T^*_r}\right) : \text{Rep}(\text{GL}_n) \oplus \text{Rep}(G(n, r-1)(1)) \to \text{Rep}(G(n, r))
\]
is a ring homomorphism, we compose with \(\text{res}\) again. But this composition coincides with the ring injection \(\text{id} \oplus \text{res}\) followed by the ring homomorphism
\[
\left(\frac{[R(n, r)] \cdot (-)}{F^*}\right) : \text{Rep}(\text{GL}_n) \oplus \text{Rep}((\text{GL}_n)^{(1)})) \to \text{Rep}(\text{GL}_n)
\]
considered above.

Now we come to kernel elements. For all \(r \geq 0\), let us consider the element
\[
\delta_r := \sum_{i=0}^n (-1)^{n-i} \Lambda^i U^{(r)} \in \text{Rep}((\text{GL}_n)^{(r)}),
\]
where \(U = k^n\), and for \(r \geq 1\) the \((r-1)\)-th Frobenius
\[
F^{r-1} : (\text{GL}_n)^{(1)} \to (\text{GL}_n)^{(r)}.
\]

**Proposition 7.4.** For all \(r \geq 1\), the kernel of
\[
\left(\frac{[R(n, r)] \cdot (-)}{F^*}\right) : \text{Rep}(\text{GL}_n) \oplus \text{Rep}((\text{GL}_n)^{(1)}) \to \text{Rep}(\text{GL}_n)
\]
is generated by \((\delta_0, -(F^{r-1})^*(\delta_r))\) as a \(\text{Rep}((\text{GL}_n)^{(1)})\)-module.
For \( r = 1 \), \( \text{Rep}(G(n, 1)) \) is a \( \text{Rep}((\text{GL}_n)^{(1)}) \)-algebra by \( P^*_1 \). Then

\[
\left( \begin{array}{c} \mathcal{I} \\ P^*_1 \end{array} \right) : \text{Rep}(\text{GL}(U)) \oplus \text{Rep}(\text{GL}(U)^{(1)}) \to \text{Rep}(G(n, 1))
\]

is a \( \text{Rep}((\text{GL}_n)^{(1)}) \)-algebra map. The proposition implies that its kernel is generated by \( (\delta_0, -\delta_1) \) as a \( \text{Rep}((\text{GL}_n)^{(1)}) \)-module as

\[
\text{res} \circ \left( \begin{array}{c} \mathcal{I} \\ P^*_1 \end{array} \right) = \left( \begin{array}{c} [R(n, 1)] \cdot (-) \\ F^* \end{array} \right)
\]

and as \( \text{res} : \text{Rep}(G(n, 1)) \to \text{Rep}(\text{GL}_n) \) is an injective \( \text{Rep}((\text{GL}_n)^{(1)}) \)-algebra map.

For \( r \geq 2 \), we do not even have a \( \text{Rep}((\text{GL}_n)^{(1)}) \)-module structure on \( \text{Rep}(G(n, r)) \). But we can study the commutative diagram

\[
\text{Rep}(\text{GL}_n) \oplus \text{Rep}(G(n, r - 1)^{(1)}) \xrightarrow{(\mathcal{I} + T^*_r)^{(1)}} \text{Rep}(G(n, r))
\]

\[
\text{id} \oplus \text{res} \\
\text{Res}
\]

\[
\text{Res}
\]

and get the following.

**Corollary 7.5.** For all \( r \geq 2 \), the image

\[
(\text{id} \oplus \text{res})(\text{Ker}(\mathcal{I} + T^*_r)) \subset \text{Rep}(\text{GL}_n) \oplus \text{Rep}((\text{GL}_n)^{(1)})
\]

coincides with the kernel of

\[
\begin{pmatrix} [R(n, r)] \cdot (-) \\ F^* \end{pmatrix} : \text{Rep}(\text{GL}_n) \oplus \text{Rep}((\text{GL}_n)^{(1)}) \to \text{Rep}(\text{GL}_n).
\]

Furthermore,

\[
\text{Ker}\left( \begin{array}{c} \mathcal{I} \\ T^*_r \end{array} \right) = \{(\delta_0 F^*(a), -\mathcal{I}_{r-1}(\delta_1 a)) \mid a \in \text{Rep}((\text{GL}_n)^{(1)})\}.
\]

Before we prove the proposition and the corollary, we need another tool, the character map

\[
\text{Rep}(\text{GL}_n) \xrightarrow{\text{ch}} \mathbb{Z}[X(T)]
\]

\[
[V] \mapsto \sum_{\lambda \in X(T)} \dim(V_\lambda)e(\lambda)
\]

where \( e(\lambda) \) is the basis element of \( \mathbb{Z}[X(T)] \) corresponding to \( \lambda \in X(T) \). Due to the highest weight characterization of \( L(\lambda) \), the character map maps this \( \mathbb{Z} \)-basis of \( \text{Rep}(\text{GL}_n) \) to a linearly independent set in \( \mathbb{Z}[X(T)] \). Hence it is injective. It is well known that its image is precisely \( \mathbb{Z}[X(T)]^\mathcal{W} \), where \( \mathcal{W} = S_n \) is the Weyl group.

Now let us write the indeterminant \( t_i \) for \( e(\varepsilon_i) \) and denote by \( s_i \) the \( i \)-th elementary symmetric polynomial in the \( t_i \). Then

\[
\mathbb{Z}[X(T)] = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]
\]

and

\[
\mathbb{Z}[X(T)]^\mathcal{W} = \mathbb{Z}[s_1, \ldots, s_n, s_n^{-1}].
\]
As \( F^*: \text{Rep}(\text{GL}_n^{(1)}) \to \text{Rep}(\text{GL}_n) \) acts on the \( T \)-weights by the \( p \)-th power, it corresponds to the \( p \)-th Adams operation \( \psi^p \) on \( \mathbb{Z}[X(T)] \) which is given by \( \psi^p(t_i) = t_i^p \). Note that

\[
\text{ch}(\Lambda^i U) = s_i,
\]

\[
\text{ch}(\delta_r) = \sum_{i=0}^{n} (-1)^{n-i}s_i = \prod_{i=1}^{n}(t_i - 1) =: \delta,
\]

\[
\text{ch}([R(n, r)]) = \prod_{i=1}^{n} \frac{t_i^p - 1}{t_i - 1} =: U_r.
\]

Hence

\[
\text{ch}\left(F^*\right)^*(\delta_r) = (\psi^p)^*(\delta) = U_r \delta = \text{ch}([R(n, r)]\delta_0),
\]

which implies that \( (\delta_0, -(F^*-1)^*(\delta_r)) \) lies in the kernel of

\[
\begin{pmatrix}
[R(n, r)] \\
F^*
\end{pmatrix}
: \text{Rep}(\text{GL}_n) \oplus \text{Rep}(\text{GL}_n^{(1)}) \to \text{Rep}(\text{GL}_n).
\]

Before we prove the whole proposition, we first deduce the corollary.

**Proof of Corollary 7.5** According to the proposition, the kernel of \( \begin{pmatrix} [R(n, r)] \cdot (-) \\ F^* \end{pmatrix} \) is generated by \( (\delta_0, -(F^*-1)^*(\delta_r)) \) as a \( \text{Rep}(\text{GL}_n^{(1)}) \)-module. Hence the kernel elements are those of the form

\[
(\delta_0 F^*(a), -(F^*-1)^*(\delta_r)a)
\]

for all \( a \in \text{Rep}(\text{GL}_n^{(1)}) \). Furthermore, \( (\psi^p)^r(\delta) = U_r \delta \). Hence

\[
(F^*-1)^*(\delta_r)a = [R(n, r-1)^{(1)}] \delta_1 a = \text{res} \mathcal{I}_{r-1}(\delta_1 a)
\]

lies in the image of

\[
\text{res} : \text{Rep}(G(n, r-1)^{(1)}) \to \text{Rep}(\text{GL}_n^{(1)}).
\]

That is,

\[
\text{Ker} \begin{pmatrix}
[R(n, r)] \\
F^*
\end{pmatrix} \subset \text{Im}(\text{id} \oplus \text{res}).
\]

Now the assertion follows from the commutativity of

\[
\begin{array}{ccc}
\text{Rep}(\text{GL}_n) \oplus \text{Rep}(G(n, r-1)^{(1)}) & \xrightarrow{\text{id} \oplus \text{res}} & \text{Rep}(G(n, r)) \\
& \xrightarrow{\begin{pmatrix} \tau^p \end{pmatrix}} & \\
\text{Rep}(\text{GL}_n) \oplus \text{Rep}(\text{GL}_n^{(1)}) & \xrightarrow{\begin{pmatrix} [R(n, r)] \cdot (-) \\ F^* \end{pmatrix}} & \text{Rep}(\text{GL}_n)
\end{array}
\]

and the injectivity of \( \text{res} \).

\[\square\]

Now we give the proof of the proposition.

**Proof of Proposition 7.3** We will prove the proposition in terms of \( \mathbb{Z}[X(T)] \). In fact, we prove the following claim: The kernel of the map

\[
\begin{pmatrix}
U_r \\
\psi^p
\end{pmatrix}: \mathbb{Z}[X(T)] \oplus \mathbb{Z}[X(T)] \to \mathbb{Z}[X(T)]
\]

coincides with

\[
\{ (\delta \psi^p(a), -(\psi^p)^{-1}(\delta)a) \mid a \in \mathbb{Z}[X(T)] \}.
\]
Then the proposition follows by passing to $S_n$-invariants as $\mathbb{Z}[X(T)]$ is factorial.

We start with the case $r = 1$. As $\psi^p$ and $U_1 \cdot (-)$ are injective, it suffices to show that

$$\mathbb{Z}[t_1^{\pm p}, \ldots, t_n^{\pm p}] \cap U_1 \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}] = (U_1 \delta)\mathbb{Z}[t_1^{\pm p}, \ldots, t_n^{\pm p}].$$

For this, define the ideals

$$P_i := \langle t_i^p - 1 \rangle \subset \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$$

and

$$Q_i := \langle t_i^p - 1 \rangle \mathbb{Z}[t_1^{\pm p}, \ldots, t_n^{\pm p}].$$

Both $P_i$ and $Q_i$ are prime ideals of height 1, as they are generated by irreducible elements in factorial rings. We claim that the inclusion

$$Q_i \subset P_i \cap \mathbb{Z}[t_1^{\pm p}, \ldots, t_n^{\pm p}]$$

is an equality. As $\mathbb{Z}[t_1^{\pm p}, \ldots, t_n^{\pm p}]$ is factorial, it is integrally closed. Thus the “going-down” theorem [Mat86, Theorem 9.4] implies that $P_i \cap \mathbb{Z}[t_1^{\pm p}, \ldots, t_n^{\pm p}]$ is also a prime ideal of height 1, whence the equality. As

$$(U_1) = P_1 \cdots P_n = P_1 \cap \ldots \cap P_n \subset \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$$

and

$$(U_1 \delta) = Q_1 \cdots Q_n = Q_1 \cap \ldots \cap Q_n \subset \mathbb{Z}[t_1^{\pm p}, \ldots, t_n^{\pm p}],$$

the case $r = 1$ follows.

Now we consider the case $r \geq 2$. Then we can factor our map as follows since $U_1 \psi^p(U_{r-1}) = U_r$:

$$\begin{array}{ccc}
\mathbb{Z}[X(T)] \oplus \mathbb{Z}[X(T)] & \xrightarrow{(U_r \cdot (-))} & \mathbb{Z}[X(T)] \\
\psi^p(U_{r-1}) \cdot (-) \oplus \text{id} & & \downarrow (U_1 \cdot (-)) \psi^p \\
\mathbb{Z}[X(T)] \oplus \mathbb{Z}[X(T)] & & \\
\end{array}$$

By the case $r = 1$ the kernel of the map $(U_r \cdot (-))$ consists of the elements of the form

$$(\delta \psi^p(a), -\delta a)$$

for $a \in \mathbb{Z}[X(T)]$. As $\psi^p(U_{r-1}) \cdot (-) \oplus \text{id}$ is injective and no prime factor $t_i - 1$ of $\delta$ divides $\psi^p(U_{r-1})$, the images of the elements of the kernel of $(U_r \cdot (-))$ are those of the above type where $U_{r-1}$ divides $a$. As $\delta U_{r-1} = (\psi^p)^{r-1}(\delta)$, the claim for $r \geq 2$ follows. 

\[ \square \]

References


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