COHOMOLOGY OF COXETER ARRANGEMENTS AND SOLOMON'S DESCENT ALGEBRA

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ABSTRACT. We refine a conjecture by Lehrer and Solomon on the structure of the Orlik-Solomon algebra of a finite Coxeter group W and relate it to the descent algebra of W . As a result, we claim that both the group algebra of W and the Orlik-Solomon algebra of W can be decomposed into a sum of induced one-dimensional representations of element centralizers, one for each conjugacy class of elements of W . We give a uniform proof of the claim for symmetric groups. In addition, we prove that a relative version of the conjecture holds for every pair (W, W_L) , where W is arbitrary and W_L is a parabolic subgroup of W , all of whose irreducible factors are of type A .

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1. INTRODUCTION

Suppose V is a finite-dimensional, complex vector space. A linear transformation t in $\mathrm{GL}(V)$ is called a *reflection* if it has finite order and the fixed point set of t is a hyperplane in V , or equivalently, if t is diagonalizable with eigenvalues 1, with multiplicity $\dim V - 1$, and ζ , where ζ is a root of unity, with multiplicity 1. Suppose that W is a finite subgroup of $\mathrm{GL}(V)$ generated by a set of reflections T . For each t in T , let $H_t = \mathrm{Fix}(t)$ denote the fixed point set of t in V and set $\mathcal{A} = \{H_t \mid t \in T\}$ and $M_W = V \setminus \bigcup_{t \in T} H_t$. Then $(V, \mathcal{A})$ is a hyperplane arrangement, and the complement M_W is an open, W -stable subset of V .

The action of W on M_W determines a representation of W on the singular cohomology of M_W . For $p \geq 0$ let $H^p(M_W)$ denote the p^{th} singular cohomology space of M_W with complex coefficients and let $H^*(M_W) = \bigoplus_{p \geq 0} H^p(M_W)$ denote the total cohomology of M_W , a graded $\mathbb{C}$ -vector space. It follows from a result

Received by the editors July 16, 2012 and, in revised form, December 4, 2012.

2010 *Mathematics Subject Classification.* Primary 20F55; Secondary 05E10, 52C35.

The authors would like to thank their charming wives for their unwavering support during the preparation of this paper.

of Brieskorn [11] that $\dim H^*(M_W) = |W|$, and so a naive guess would be that the representation of W on $H^*(M_W)$ is the regular representation of W . A simple computation for the symmetric group S_3 shows that this is not the case.

In 1987, Lehrer [25] determined the character of the representation of W on $H^*(M_W)$ when $W = S_n$ is a symmetric group by explicitly computing the “equivariant Poincaré polynomials” $P_{S_n}(g, t) = \sum_{p \geq 0} \text{trace}(g, H^p(M_W)) t^p$, for g in S_n (here, t is an indeterminate). Subsequently, equivariant Poincaré polynomials were computed case by case for other groups by various authors. In 2001, Blair and Lehrer [9] showed that for any complex reflection group, the equivariant Poincaré polynomials have the form $P_W(g, t) = \sum_{x \in Z_W(g)} f_g(x) (-t)^{c(x)}$, where $f_g: Z_W(g) \rightarrow \mathbb{C}$ is explicitly given and $c(x)$ is the codimension of the fixed point space of x in V . Felder and Veselov [14] have found an elegant description of the character of $H^*(M_W)$ when W is a Coxeter group that precisely describes how the character of $H^*(M_W)$ differs from the regular character ρ of W . Specifically, Felder and Veselov show that the character of $H^*(M_W)$ is given as

$$\sum_t (2 \text{Ind}_{(t)}^W(1) - \rho),$$

where the sum runs over a set of “special” involutions t in W .

In contrast, while the representation of W on $H^*(M_W)$ is well understood, much less is known about the representations of W on the individual graded pieces $H^p(M_W)$ for $p \geq 0$. When W is a symmetric group, Lehrer and Solomon [26] have described these representations as sums of representations induced from linear characters of centralizers of elements in W . They conjecture that a similar decomposition exists in general.

For the symmetric group S_n , Barcelo and Bergeron [1] construct an explicit S_n -stable subspace of the exterior algebra of the free Lie algebra on n letters that affords the character of $H^*(M_W)$ tensored with the sign character. Their construction could be used to study the characters of the individual cohomology spaces $H^p(M_W)$.

For the hyperoctahedral group $W(B_n)$, Douglass [12] extended the construction of Lehrer and Solomon and expressed each $H^p(M_W)$ as a sum of representations induced from linear characters of subgroups. However, the subgroups appearing are not always centralizers of elements of $W(B_n)$. At the same time, Bergeron used the free Lie algebra on $2n$ letters to construct a representation of $W(B_n)$ analogous to the representation of S_n constructed in [1]. The character of this representation of $W(B_n)$ is again the character of $H^*(M_W)$ tensored with the sign character. He then uses this construction to study the characters of the individual cohomology spaces $H^p(M_W)$.

In this paper we state a conjecture for a finite Coxeter group W (Conjecture 2.1) that both refines the conjecture of Lehrer and Solomon [26, Conjecture 1.6] and directly relates the representation of W on $H^p(M_W)$ to a subrepresentation of the right regular representation of W . It is straightforward to see that Conjecture 2.1 holds for W if and only if it holds for every irreducible factor of W . Thus, to prove the conjecture we may assume that W is irreducible. Conjecture 2.1 is proved for symmetric groups in this paper (Theorem 6.3). The conjecture has been proven for all rank two Coxeter groups [13] and has been checked using the computer algebra system GAP [32] and the package CHEVIE [16] for all Coxeter groups with rank eight or less ([6], [7], [8]). More generally, in §7 we extend the constructions used in the proof of Theorem 6.3 and prove a “relative” version of Conjecture 2.1 for pairs

(W, W_L) , where now W is any finite Coxeter group and W_L is a parabolic subgroup of W , all of whose irreducible factors are of type A . If the conclusion of Theorem 7.1 were to hold for every parabolic subgroup W , not just those that are products of symmetric groups, then Conjecture 2.1 would hold for W . The statement of the conjecture, along with an expository review of the background material from the theories of Coxeter groups and hyperplane arrangements we use later in the paper, is given in §2.

The subrepresentations of the right regular representation of W that we consider arise from a decomposition of a subalgebra of the group algebra of W known as “Solomon’s descent algebra”, into projective, indecomposable modules. Projective, indecomposable modules in an artinian $\mathbb{C}$ -algebra are generated by idempotents. The idempotents in the descent algebra we use in this paper were discovered by Bergeron, Bergeron, Howlett, and Taylor [4]; we call them *BBHT idempotents*. In §3 we study the relationships between BBHT idempotents for W and BBHT idempotents for parabolic subgroups. We also compare the BBHT idempotents for W with those of its irreducible factors when W is reducible. In §4 we show that the right ideals in the group algebra of W generated by BBHT idempotents afford induced representations. It then follows that the BBHT idempotents give rise to a decomposition of the group algebra that is the direct analog of the decomposition of $H^*(M_W)$ given by Brieskorn’s Lemma [11, Lemme 3].

The proof of Conjecture 2.1 for symmetric groups is given in §5 and §6. As a consequence, we obtain a decomposition of the group algebra $\mathbb{C}S_n$ as a direct sum of representations induced from one-dimensional representations of centralizers, one for each conjugacy class. Similar decompositions of the group algebra of the symmetric group have been proven independently by Bergeron, Bergeron, and Garsia [3], Hanlon [19], and more recently, Schocker [33], all using different methods.

For readers familiar with the literature on the free Lie algebra and its connections with the descent algebras and group algebras of symmetric groups, Theorems 5.1(a) and 6.3(a) will seem familiar. Indeed, Theorem 5.1(a) was likely known in some form to Brandt [10], Wever [39], and Klyachko [22], and a proof of Theorem 6.3(a) can be extracted from results in [3, §4], [31, Theorem 8.24], and [15, §4]. In contrast with these references, where the methods are combinatorial and the emphasis is on the connections between the group algebra $\mathbb{C}S_n$ and the free Lie algebra on n letters, our methods are mainly group- and representation-theoretic, using the theory of Coxeter groups in conjunction with special features of symmetric groups, and our focus is on the connections between the group algebra $\mathbb{C}S_n$ and the representations of S_n on the cohomology spaces $H^p(M_W)$. Moreover, our line of reasoning, which is motivated by the arguments in Lehrer and Solomon [26], demonstrates a striking parallelism between the Orlik-Solomon algebras and group algebras of symmetric groups that to our knowledge has not been observed before. We hope that the approach taken in this paper will lead to a deeper understanding of the topological and geometric properties of general Coxeter arrangements. For example, as we show in §7, the constructions in §5 and §6 have natural extensions when the focus is shifted from considering not just a single symmetric group to considering a product of symmetric groups embedded as a parabolic subgroup in a larger finite Coxeter group W , where the type of W is arbitrary.

Bergeron and Bergeron conjecture in [2] that there might be a decomposition of the group algebra $\mathbb{C}W(B_n)$ analogous to the decomposition of $\mathbb{C}S_n$ studied by

Bergeron-Bergeron-Garsia [3] and Hanlon [19]. In [5] Bergeron gives a decomposition of the group algebra of a hyperoctahedral group as a direct sum of representations induced from linear characters of subgroups. However, this decomposition is not the decomposition proposed in Conjecture 2.1; it is the analog for group algebras of hyperoctahedral groups of the decomposition of $H^*(M_W)$ found in [12].

2. THE ORLIK-SOLOMON ALGEBRA AND SOLOMON'S DESCENT ALGEBRA

In the rest of this paper we assume that V is a finite-dimensional, complex vector space and $W \subseteq \mathrm{GL}(V)$ is a finite Coxeter group with Coxeter generating set S . Then each s in S acts on V as a reflection with order two, and W is generated by S subject to the relations $(st)^{m_{s,t}} = 1$, where $m_{s,s} = 1$ and $m_{s,t} = m_{t,s} > 1$ for $s \neq t$ in S . Let T denote the set of all reflections in W .

We assume also that a positive definite, Hermitian form $\langle \cdot, \cdot \rangle$ on V is given and that W is a subgroup of the unitary group of V with respect to this form. It is known that $\mathrm{Fix}(W)^\perp$, the orthogonal complement of the space of fixed points of W on V , has a basis $\Delta = \{\alpha_s \mid s \in S\}$ so that $\langle \alpha_s, \alpha_t \rangle = -\cos(\pi/m_{s,t})$ for s and t in S . Then s acts on V as the reflection through the hyperplane orthogonal to α_s , and $\Phi = \{w(\alpha_s) \mid w \in W, s \in S\}$ is a root system in $\mathrm{Fix}(W)^\perp$ with base Δ .

2.1. Shapes and conjugacy classes. We begin by recalling a parameterization of the conjugacy classes in W due to Geck and Pfeiffer (see [18, §3.2]) in a form compatible with the arrangement $(V, \mathcal{A})$ of W .

The *lattice of $\mathcal{A}$* , denoted by $L(\mathcal{A})$, is the set of subspaces of V that arise as intersections of hyperplanes in $\mathcal{A}$:

$$L(\mathcal{A}) = \{H_{t_1} \cap H_{t_2} \cap \cdots \cap H_{t_p} \mid t_1, t_2, \dots, t_p \in T\}.$$

For X in $L(\mathcal{A})$ define

$$W_X = \{w \in W \mid X \subseteq \mathrm{Fix}(w)\}$$

to be the pointwise stabilizer of X in W . It follows from Steinberg's Theorem [37] that W_X is generated by $\{t \in T \mid X \subseteq H_t\}$. It then follows that $X = \mathrm{Fix}(W_X)$, and so the assignment $X \mapsto W_X$ defines an injection from $L(\mathcal{A})$ to the set of subgroups of W . Notice that W_X is again a Coxeter group.

The action of W on $\mathcal{A}$ induces an action of W on $L(\mathcal{A})$. Obviously, $wW_Xw^{-1} = W_{w(X)}$, and so for X and Y in $L(\mathcal{A})$ the subgroups W_X and W_Y are conjugate if and only if X and Y lie in the same W -orbit. Thus, the assignment $X \mapsto W_X$ induces a bijection between the set of orbits of W on $L(\mathcal{A})$ and the set of conjugacy classes of subgroups W_X .

By a *shape of W* we mean a W -orbit in $L(\mathcal{A})$. We denote the set of shapes of W by Λ . For example, if W is the symmetric group S_n , then Λ is in bijection with the set of partitions of n and with the set of Young diagrams with n boxes. When λ is a shape and X is a subspace in λ , we say λ is the *shape of W_X* .

It is shown in [30, §6.2] that the assignment $w \mapsto \mathrm{Fix}(w)$ defines a surjection from W to $L(\mathcal{A})$. Composing with the map that sends an element X in $L(\mathcal{A})$ to its W -orbit, we get a map

$$\mathrm{sh}: W \rightarrow \Lambda.$$

We say $\mathrm{sh}(w)$ is the *shape of w* . Thus, $\mathrm{sh}(w)$ is the W -orbit of $\mathrm{Fix}(w)$ in $L(\mathcal{A})$. Clearly, sh is constant on conjugacy classes, and so we can define the *shape* of a conjugacy class to be the shape of any of its elements.

An element w in W , or its conjugacy class, is called *cuspidal* if $\text{Fix}(w) = \text{Fix}(W)$. For example, if W is the symmetric group S_n , then the conjugacy class consisting of n -cycles is the only cuspidal class. In general, there is more than one cuspidal conjugacy class. Cuspidal elements and conjugacy classes are called *elliptic* by some authors.

Suppose that λ is a shape, X in $L(\mathcal{A})$ has shape λ , and C is a conjugacy class in W with shape λ . If w is in C , then $\text{Fix}(w)$ is in the W -orbit of X , and so $C \cap W_X$ is a non-empty union of cuspidal W_X -conjugacy classes. Geck and Pfeiffer [18, §3.2] have shown that in fact $C \cap W_X$ is a single W_X -conjugacy class. It follows that $C \mapsto C \cap W_X$ defines a bijection between the set of conjugacy classes in W with shape λ and the set of cuspidal conjugacy classes in W_X .

Fix a set $\{X_\lambda \mid \lambda \in \Lambda\}$ of W -orbit representatives in $L(\mathcal{A})$. Summarizing the preceding discussion we see that conjugacy classes in W are parametrized by pairs (λ, C_λ) , where λ is a shape and C_λ is a cuspidal conjugacy class in W_{X_λ} .

2.2. The Orlik-Solomon algebra. Next, consider the cohomology ring $H^*(M_W)$. Arnold and Brieskorn (see [30, §1.1]) have computed $H^*(M_W)$. In the following we use the presentation of this algebra given by Orlik and Solomon [28].

Recall that the set T of reflections in W parametrizes the hyperplanes in $\mathcal{A}$. The *Orlik-Solomon algebra* of W is the $\mathbb{C}$ -algebra, $A = A(\mathcal{A})$, with generators $\{a_t \mid t \in T\}$ and relations

- $a_{t_1}a_{t_2} = -a_{t_2}a_{t_1}$ for t_1 and t_2 in T and
- $\sum_{i=1}^p (-1)^i a_{t_1} \cdots \widehat{a_{t_i}} \cdots a_{t_p} = 0$ whenever $\{H_{t_1}, \dots, H_{t_p}\}$ is a linearly dependent subset of $\mathcal{A}$.

The algebra A is a skew-commutative, graded, connected $\mathbb{C}$ -algebra that is isomorphic as a graded algebra to $H^*(M_W)$. Let A^p denote the degree p subspace of A . Then

- $A^0 \cong \mathbb{C}$,
- for $1 \leq p \leq |S|$ the subspace A^p is spanned by the set of all $a_{t_1} \cdots a_{t_p}$, where $\text{codim } H_{t_1} \cap \cdots \cap H_{t_p} = p$, and
- $A^p = 0$ for $p > |S|$.

See [30, §3.1] for details.

The action of W on $\mathcal{A}$ extends to an action of W on A as algebra automorphisms. An element w in W acts on a generator a_t of A by $wa_t = a_{wtw^{-1}}$. With this W -action A is isomorphic to $H^*(M_W)$ as graded W -algebras.

Orlik and Solomon [29] have shown that the normalizer of W_X in W is the setwise stabilizer of X in W ; that is,

$$N_W(W_X) = \{w \in W \mid w(X) = X\}.$$

For X in $L(\mathcal{A})$, define A_X to be the span of $\{a_{t_1} \cdots a_{t_r} \mid H_{t_1} \cap \cdots \cap H_{t_r} = X\}$. Proofs of the following statements may be found in [30, Corollary 3.27 and Theorem 6.27]:

- If $\text{codim } X = p$, then $A_X \subseteq A^p$.
- There are vector space decompositions $A^p \cong \bigoplus_{\text{codim } X=p} A_X$ and $A \cong \bigoplus_{X \in L(\mathcal{A})} A_X$.
- For w in W , $wA_X = A_{w(X)}$. Thus, A_X is an $N_W(W_X)$ -stable subspace of A .

For a shape λ in Λ , set $A_\lambda = \bigoplus_{X \in \lambda} A_X$. Suppose X is a fixed subspace in λ and that $\text{codim } X = p$. Then A_λ is a W -stable subspace of A^p and we have isomorphisms of $\mathbb{C}W$ -modules

$$A_\lambda \cong \text{Ind}_{N_W(W_X)}^W(A_X) \quad \text{and} \quad A \cong \bigoplus_{\lambda \in \Lambda} A_\lambda$$

(see [26]).

2.3. Solomon's descent algebra. In contrast with the Orlik-Solomon algebra A , which is defined for every complex reflection group, Solomon's descent algebra is defined using the Coxeter generating set S of W and so has no immediate analog for complex reflection groups that are not Coxeter groups.

Suppose that I is a subset of S . Define

$$X_I = \bigcap_{s \in I} H_s \quad \text{and} \quad W_I = \langle I \rangle.$$

Then X_I is in $L(\mathcal{A})$ and $\text{codim } X_I = |I|$. It follows from Steinberg's Theorem [37] that $W_I = W_{X_I}$. Recall that $\Delta = \{\alpha_s \mid s \in S\}$ is a base of Φ . For $I \subseteq S$ define

$$\Delta_I = \{\alpha_s \mid s \in I\}.$$

Then X_I is the orthogonal complement of the span of Δ_I .

Orlik and Solomon (see [30, §6.2]) have shown that each orbit of W on $L(\mathcal{A})$ contains a subspace X_I for some subset I of S . For subsets I and J of S define $I \sim J$ if there is a w in W with $w(\Delta_I) = \Delta_J$. Then $\sim$ is an equivalence relation. It is well known that W_I and W_J are conjugate if and only if $I \sim J$ (see [35]). It follows that the assignment $I \mapsto X_I$ induces a bijection between the set of $\sim$ -equivalence classes of subsets of S and the set of shapes of W . We use this bijection to identify shapes of W with subsets of the power set of S .

Next, let ℓ denote the length function of W determined by the generating set S and define

$$W^I = \{w \in W \mid \ell(ws) > \ell(w) \forall s \in I\}.$$

Then W^I is a set of left coset representatives of W_I in W . Also, define

$$x_I = \sum_{w \in W^I} w$$

in the group algebra $\mathbb{C}W$. Solomon [34] has shown that the span of $\{x_I \mid I \subseteq S\}$ is in fact a subalgebra of $\mathbb{C}W$. This subalgebra is denoted by $\Sigma(W)$ and is called the *descent algebra* of W . It is not hard to see that $\{x_I \mid I \subseteq S\}$ is linearly independent, and so $\dim \Sigma(W) = 2^{|S|}$. Notice that $x_S = 1$ is the identity in both $\mathbb{C}W$ and its subalgebra $\Sigma(W)$.

Bergeron, Bergeron, Howlett, and Taylor [4] have defined a basis $\{e_I \mid I \subseteq S\}$ of $\Sigma(W)$ that consists of quasi-idempotents and is compatible with the set of shapes of W . For λ in Λ define

$$S_\lambda = \{I \subseteq S \mid X_I \in \lambda\} \quad \text{and} \quad e_\lambda = \sum_{I \in S_\lambda} e_I.$$

Then each e_λ is idempotent and $\{e_\lambda \mid \lambda \in \Lambda\}$ is a complete set of primitive, orthogonal idempotents in $\Sigma(W)$. (See §3 for more details.) In particular, $\sum_{\lambda \in \Lambda} e_\lambda = 1$ in $\mathbb{C}W$.

Define $E_\lambda = e_\lambda \mathbb{C}W$. In §3 we show that $e_I \mathbb{C}W_I$ affords an action of $N_W(W_I)$ and that if I is in S_λ , then E_λ is induced from $e_I \mathbb{C}W_I$. Thus, in analogy with the decomposition in §2.2 of the Orlik-Solomon algebra A , we have isomorphisms of $\mathbb{C}W$ -modules

$$E_\lambda \cong \operatorname{Ind}_{N_W(W_I)}^W(e_I \mathbb{C}W_I) \quad \text{and} \quad \mathbb{C}W \cong \bigoplus_{\lambda \in \Lambda} E_\lambda.$$

2.4. Centralizers and complements. The last ingredient we need in order to state the conjecture is a certain set of characters of centralizers of elements of W . These characters, together with the sign character, should quantify the difference between the representation of W on $H^p(M_W)$ and a subrepresentation of the regular representation. They naturally arise in work of Howlett and Lehrer [21] and in recent results of Konvalinka, Pfeiffer, and Röver [23] that describe the structure of the centralizer of an element in W .

Suppose that I is a subset of S and C is a conjugacy class in W such that $C \cap W_I$ is a cuspidal conjugacy class in W_I . Howlett [20] has shown that W_I has a complement, N_I , in $N_W(W_I)$. Moreover, it is shown in [23] that if c is in $C \cap W_I$, then $Z_W(c) \subseteq N_W(W_I)$ and $Z_W(c)W_I = N_W(W_I)$. It follows that

- (1) for any X in $L(\mathcal{A})$, W_X has a complement, say N_X , in $N_W(W_X)$ such that

$$N_W(W_X) \cong W_X \rtimes N_X,$$

and

- (2) for c in W_X cuspidal, $Z_W(c) \subseteq N_W(W_X)$ and $Z_W(c)/Z_{W_X}(c) \cong N_X$.

Recall $N_W(W_X)$ is the setwise stabilizer of X in W . Define $\alpha_X: N_W(W_X) \rightarrow \mathbb{C}$ by $\alpha_X(n) = \det n|_X$. For c in W with $X = \operatorname{Fix}(c)$, define $\alpha_c = \alpha_X|_{Z_W(c)}$. Then

$$\alpha_c(z) = \det z|_{\operatorname{Fix}(c)}$$

for z in $Z_W(c)$.

2.5. Relating the Orlik-Solomon algebra and the descent algebra. We now have all the concepts we need in order to state the conjecture.

Let ϵ denote the sign character of W . For c in W , define $X_c = \operatorname{Fix}(c)$, $\operatorname{rk}(c) = \operatorname{codim} X_c$, and $W_c = W_{X_c}$. Notice that c is a cuspidal element in W_c .

Associated with each λ in Λ we have the W -stable subspace A_λ of the Orlik-Solomon algebra A , the right ideal E_λ in $\mathbb{C}W$, and the set of conjugacy classes with shape λ . We conjecture that A_λ and E_λ are related to the set of conjugacy classes with shape λ as follows.

Conjecture 2.1. *Choose a set $\mathcal{C}$ of conjugacy class representatives in W . For λ in Λ , set $\mathcal{C}_\lambda = \{c \in \mathcal{C} \mid \operatorname{sh}(c) = \lambda\}$. Then, for each conjugacy class representative c in $\mathcal{C}$, there is a linear character φ_c of $Z_W(c)$ such that for every λ in Λ ,*

- (a) *the character of E_λ is*

$$\sum_{c \in \mathcal{C}_\lambda} \operatorname{Ind}_{Z_W(c)}^W(\varphi_c)$$

and

(b) the character of A_λ is

$$\sum_{c \in \mathcal{C}_\lambda} \text{Ind}_{Z_W(c)}^W(\epsilon_c \alpha_c \varphi_c),$$

where ϵ_c denotes the restriction of ϵ to $Z_W(c)$.

In particular,

$$H^p(M_W) \cong A^p \cong \bigoplus_{\text{rk}(c)=p} \text{Ind}_{Z_W(c)}^W(\epsilon_c \alpha_c \varphi_c)$$

for $0 \leq p \leq |S|$,

$$\mathbb{C}W \cong \bigoplus_{c \in \mathcal{C}} \text{Ind}_{Z_W(c)}^W(\varphi_c), \quad \text{and} \quad H^*(M_W) \cong \bigoplus_{c \in \mathcal{C}} \text{Ind}_{Z_W(c)}^W(\epsilon_c \alpha_c \varphi_c).$$

As stated in the introduction, we prove Conjecture 2.1 for symmetric groups in §5 and §6. The conjecture is known to be true for all Coxeter groups with rank up to eight ([13], [6], [7], [8]).

We in fact prove more than is stated in the conjecture. First, we show that the character φ_c of $Z_W(c)$ may be chosen to be a trivial extension of a character of $Z_{W_c}(c)$. Second, we construct explicit $\mathbb{C}W$ -module isomorphisms

$$E_\lambda \cong \text{Ind}_{Z_W(c)}^W(\varphi_c) \quad \text{and} \quad A_\lambda \cong \text{Ind}_{Z_W(c)}^W(\epsilon_c \alpha_c \varphi_c).$$

In §7 we extend the constructions in §5 and §6 and show that if W is any finite Coxeter group, λ is in Λ , c is in W with $\text{sh}(c) = \lambda$, and the irreducible components of W_c are all of type A , then the character φ_c of $Z_{W_c}(c)$ constructed in §6 extends to a character $\tilde{\varphi}_c$ of $Z_W(c)$. Moreover, we construct explicit $\mathbb{C}W$ -module isomorphisms

$$E_\lambda \cong \text{Ind}_{Z_W(c)}^W(\tilde{\varphi}_c) \quad \text{and} \quad A_\lambda \cong \text{Ind}_{Z_W(c)}^W(\epsilon_c \alpha_c \tilde{\varphi}_c).$$

Note that with the given assumptions we have $|\mathcal{C}_\lambda| = 1$, and so the sums in Conjecture 2.1 (a) and (b) reduce to a single summand. Also, in contrast with the case when the ambient group W is a symmetric group and $\tilde{\varphi}_c$ is the trivial extension, in the general case $\tilde{\varphi}_c$ may not be the trivial extension of φ_c .

A direct proof of the conjecture involves finding suitable linear characters φ_c of $Z_W(c)$. The computations in [6] and [7], as well as calculations in type B , show that some natural guesses about the characters φ_c are not true. For example, $Z_W(c)$ acts on the eigenspaces of c in V , and so the powers of the determinant character of $Z_W(c)$ acting on an eigenspace of c are linear characters of $Z_W(c)$. An example in [6, §4] shows that it is not always possible to choose φ_c to be one of these characters for any eigenspace.

On the other hand, suppose that c is an involution. Then $Z_W(c) = N_W(W_{X_c}) \cong W_{X_c} \rtimes N_{X_c}$. Let ϵ_{X_c} be the sign character of W_{X_c} . Then in all the cases that have been computed so far, it turns out that φ_c may be chosen to be the “trivial” extension of ϵ_{W_c} to $Z_W(c)$ in the sense that $\varphi_c(wn) = \epsilon_{X_c}(w)$ for w in W_{X_c} and n in N_{X_c} . The representations $\text{Ind}_{Z_W(c)}^W \varphi_c$ play a role in understanding the characters of finite reductive groups ([24], [17]), and the corresponding representations of the Iwahori-Hecke algebra of W in [27] play a role in the representation theory of complex reductive groups. The fact that these representations seem to be closely related to the representation of W on $H^*(M_W)$ is quite mysterious.

3. BBHT IDEMPOTENTS

In this section we collect several preliminary results about the descent algebra of W and the BBHT idempotents.

For subsets I, J , and K of S define

$$W^{IJ} = (W^I)^{-1} \cap W^J \quad \text{and} \quad W^{IJK} = \{w \in W^{IJ} \mid w^{-1}(\Delta_I) \cap \Delta_J = \Delta_K\}.$$

Then W^{IJ} is the set of minimal length (W_I, W_J) -double coset representatives in W . Solomon [34] has shown that $x_I x_J = \sum_K a_{IJK} x_K$, where $a_{IJK} = |W^{IJK}|$.

Let 2^S denote the power set of S and fix a function $\sigma: 2^S \rightarrow \mathbb{R}_{>0}$. Notice that $W^K = \{w \in W \mid w(\Delta_K) \subseteq \Phi^+\}$, where Φ^+ is the positive system determined by Δ . Thus, for $J \subseteq K$ and w in W^K we have $w(\Delta_J) \subseteq \Phi^+$. For subsets J and K of S define

$$m_{JK}^\sigma = \sum_{\substack{w \in W^K \\ w(\Delta_J) \subseteq \Delta}} \sigma(wJ) \quad \text{if } J \subseteq K, \quad \text{and} \quad m_{JK}^\sigma = 0 \quad \text{if } J \not\subseteq K.$$

Because $\sigma(I) > 0$ for all subsets I of S , we have $m_{JJ}^\sigma \neq 0$ for all J , and so the system of equations

$$(3.1) \quad x_K = \sum_{J \subseteq S} m_{JK}^\sigma e_J^\sigma, \quad K \subseteq S,$$

can be solved uniquely for $\{e_J^\sigma \mid J \subseteq S\}$. Define n_{JK}^σ and e_K^σ by

$$e_K^\sigma = \sum_{J \subseteq S} n_{JK}^\sigma x_J.$$

Then e_K^σ is in $\Sigma(W)$, $n_{KK}^\sigma = (m_{KK}^\sigma)^{-1}$ for all subsets K of S , and $n_{JK}^\sigma = 0$ when $J \not\subseteq K$.

Bergeron, Bergeron, Howlett, and Taylor [4, §7] have shown that e_I^σ is a quasi-idempotent in $\Sigma(W)$. Precisely, for λ in Λ define $\sigma(\lambda) = \sum_{I \in S_\lambda} \sigma(I)$. Then

$$(3.2) \quad e_I^\sigma e_J^\sigma = \sigma(\lambda)^{-1} e_J^\sigma$$

when I and J are in S_λ . Thus, if we set

$$e_\lambda^\sigma = \sum_{I \in S_\lambda} \sigma(I) e_I^\sigma,$$

it follows from (3.2) that e_λ^σ is an idempotent in $\Sigma(W)$ and hence an idempotent in $\mathbb{C}W$. We call the quasi-idempotents e_I^σ *BBHT quasi-idempotents* and the idempotents e_λ^σ *BBHT idempotents*.

By definition we have $1 = x_S = \sum_{J \subseteq S} n_{JS}^\sigma e_J^\sigma$, and so $1 = \sum_{\lambda \in \Lambda} e_\lambda^\sigma$ in $\Sigma(W)$ and $\mathbb{C}W$. It follows that $\{e_\lambda^\sigma \mid \lambda \in \Lambda\}$ is a set of pairwise orthogonal idempotents in $\mathbb{C}W$ and that

$$(3.3) \quad e_\lambda^\sigma e_I^\sigma = e_I^\sigma \quad \text{and} \quad e_I^\sigma e_\lambda^\sigma = \sigma(\lambda)^{-1} e_\lambda^\sigma$$

for $I \in S_\lambda$.

In the special case when the function $\sigma(I) = 1$ for all subsets I of S we do not include σ in the notation. Thus,

$$(3.4) \quad m_{JK} = |\{w \in W^K \mid w(\Delta_J) \subseteq \Delta\}| \quad \text{for } J \subseteq K, \\ e_K = \sum_{J \subseteq S} n_{JK} x_J, \quad \text{and} \quad e_\lambda = \sum_{I \in S_\lambda} e_I.$$

Notice that the quantities m_{JK}^σ , e_J^σ , m_{JK} , e_J , ... are defined relative to an ambient Coxeter system (W, S) . Below we also consider the analogous quantities defined relative to a parabolic subsystem (W_L, L) . To help keep things straight, in this section and the next we use the following conventions:

- σ always denotes a function from 2^S to $\mathbb{R}_{>0}$.
- When $\sigma(I) = 1$ for all $I \subseteq S$, the BBHT quasi-idempotents in $\mathbb{C}W$ defined with respect to σ are denoted by e_J .
- τ always denotes a function from 2^L to $\mathbb{R}_{>0}$, where L is a subset of S .
- When $\tau(I) = 1$ for all $I \subseteq L$, the BBHT quasi-idempotents in $\mathbb{C}W_L$ defined with respect to τ are denoted by e_J^L . Thus, $e_J^S = e_J$.

For example, $\{S\}$ is a shape of W and $\{L\}$ is a shape of W_L . Then $e_{\{S\}}^\sigma = \sigma(S)e_S^\sigma$, $e_{\{S\}} = e_S$, and $e_L^L = e_L^\tau = e_{\{L\}}^\tau$ when $\tau(I) = 1$ for all subsets I of L .

The following lemmas give some translation properties for the quantities defined above.

Lemma 3.1. *Suppose that $K \subseteq S$ and d is in W with $d^{-1}(\Delta_K) \subseteq \Delta$. Then*

- (a) $x_{K^d} = x_K d$;
- (b) $m_{I^d J^d}^\sigma = m_{IJ}^\sigma$ for $I \subseteq J \subseteq K$; and
- (c) $e_{L^d}^\sigma = e_L^\sigma d$ for $L \subseteq K$.

Proof. It is shown in [4, Lemma 2.4] that $W^{K^d} = W^K d$. Statement (a) follows immediately.

Suppose that $I \subseteq J \subseteq K$. Clearly, $\Delta_{L^d} = d^{-1}(\Delta_L)$ for all $L \subseteq K$, and so

$$m_{I^d J^d}^\sigma = \sum_{\substack{w \in W^{J^d} \\ w(\Delta_{I^d}) \subseteq \Delta}} \sigma(w(I^d)) = \sum_{\substack{wd^{-1} \in W^J \\ wd^{-1}(\Delta_I) \subseteq \Delta}} \sigma(wd^{-1}I) = \sum_{\substack{y \in W^J \\ y(\Delta_I) \subseteq \Delta}} \sigma(yI) = m_{IJ}^\sigma.$$

This proves (b).

Using (a) and (b) we see that for $J \subseteq K$,

$$\sum_I m_{IJ}^\sigma(e_I^\sigma d) = x_J d = x_{J^d} = \sum_I m_{I^d J^d}^\sigma e_I^\sigma = \sum_I m_{IJ}^\sigma e_I^\sigma = \sum_I m_{IJ}^\sigma e_{I^d}^\sigma.$$

Thus, $\sum_I m_{IJ}^\sigma(e_I^\sigma d) = \sum_I m_{IJ}^\sigma e_{I^d}^\sigma$. Now fix a subset L of K , multiply both sides by $n_{J^L}^\sigma$, and sum over J , to get $e_L^\sigma d = e_{L^d}^\sigma$. (Note that $n_{J^L}^\sigma = 0$ unless $J \subseteq L$.) This proves (c). $\square$

Recall that we have fixed a positive, definite, Hermitian form on V such that W is a subgroup of $U(V)$, the unitary group of V . Define

$$N(W) = \{n \in U(V) \mid n(\Delta) = \Delta\}.$$

Notice that if n is in $N(W)$, then $nSn^{-1} = S$. Thus, $N(W)$ acts on S and on 2^S , and $WN(W) \cong W \rtimes N(W)$.

Lemma 3.2. *Suppose that n is in $N(W)$ and that $\sigma(I^n) = \sigma(I)$ for all $I \subseteq S$. Then $e_{I^n}^\sigma = n^{-1}e_I^\sigma n$ for $I \subseteq S$. In particular, n centralizes e_S^σ in $\mathbb{C}W$.*

Proof. Because $n(\Delta) = \Delta$, it follows that $\ell(nwn^{-1}) = \ell(w)$ for all w in W . Therefore, $n^{-1}W^I n = W^{I^n}$, and hence $n^{-1}x_I n = x_{I^n}$ for all $I \subseteq S$.

Suppose $I \subseteq J \subseteq S$. Then

$$\sum_{\substack{w \in W^{J^n} \\ w(\Delta_{I^n}) \subseteq \Delta}} \sigma(w(I^n)) = \sum_{\substack{nw n^{-1} \in W^J \\ nwn^{-1}(\Delta_I) \subseteq \Delta}} \sigma(I^{nw^{-1}}) = \sum_{\substack{y \in W^J \\ y(\Delta_I) \subseteq \Delta}} \sigma((yI)^n) = \sum_{\substack{y \in W^J \\ y(\Delta_I) \subseteq \Delta}} \sigma(yI),$$

and so $m_{I^n J^n}^\sigma = m_{IJ}^\sigma$. Thus, $n_{I^n J^n}^\sigma = n_{IJ}^\sigma$ and it follows from (3.1) that $n^{-1}e_I^\sigma n = e_{I^n}^\sigma$. $\square$

Lemma 3.3. *Let w_0 denote the longest element in W .*

(a) *The element w_0 is in $\Sigma(W)$ and*

$$w_0 = \sum_{L \subseteq S} (-1)^{|L|} e_L = \sum_{\lambda \in \Lambda} (-1)^{d_\lambda} e_\lambda,$$

where $d_\lambda = |L|$ for L in S_λ .

(b) *For $J \subseteq S$, $w_0 e_J = (-1)^{|J|} e_J$, $w_0 e_J w_0 = e_J w_0$, and $e_J w_0 = (-1)^{|J|} e_{J w_0}$.*

Proof. For $J \subseteq S$ define $Y^J = \{w \in W \mid \{s \in S \mid ws > w\} = J\}$. Then $W^J = \coprod_{J \subseteq K} Y^K$, and so $W^J \cap P = \coprod_{J \subseteq K} (Y^K \cap P)$ for every subset P of W . For $P \subseteq W$ define

$$p_J^P = \sum_{w \in W^J \cap P} w, \quad q_K^P = \sum_{w \in Y^K \cap P} w, \quad f_J^P = |W^J \cap P|, \quad \text{and} \quad g_K^P = |Y^K \cap P|.$$

Then $p_J^P = \sum_{J \subseteq K} q_K^P$ and $f_J^P = \sum_{J \subseteq K} g_K^P$, and so by Möbius inversion,

$$(3.5) \quad q_J^P = \sum_{J \subseteq K} (-1)^{|K| - |J|} p_K^P \quad \text{and} \quad g_J^P = \sum_{J \subseteq K} (-1)^{|K| - |J|} f_K^P.$$

Taking $J = \emptyset$ and $P = W$ in (3.5), we have $Y^\emptyset = \{w_0\}$, and so

$$(3.6) \quad w_0 = q_\emptyset^W = \sum_K (-1)^{|K|} p_K^W = \sum_K (-1)^{|K|} x_K.$$

For $J \subseteq S$, set $W_+^J = \{w \in W \mid w(\Delta_J) \subseteq \Delta\}$. Then for $J \subseteq K$, $m_{JK} = |W^K \cap W_+^J|$. Taking $P = W_+^J$ in (3.5), we have $Y^J \cap W_+^J = \{w_0 w_J\}$, where w_J is the longest element in W_J , and so $g_J^P = 1$ and $f_K^P = m_{JK}$. Therefore,

$$(3.7) \quad (-1)^{|J|} = \sum_{J \subseteq K} (-1)^{|K|} m_{JK}.$$

Using (3.6) and (3.7), we have

$$\begin{aligned} w_0 &= \sum_K (-1)^{|K|} x_K = \sum_K (-1)^{|K|} \sum_J m_{JK} e_J = \sum_J \left(\sum_{J \subseteq K} (-1)^{|K|} m_{JK} \right) e_J \\ &= \sum_J (-1)^{|J|} e_J = \sum_{\lambda \in \Lambda} (-1)^{d_\lambda} e_\lambda. \end{aligned}$$

Now suppose that $J \subseteq S$ and $\mu \in \Lambda$ are such that $J \in S_\mu$. By (3.3), we have $e_\mu e_J = e_J$, and so

$$w_0 e_J = \sum_{\lambda \in \Lambda} (-1)^{d_\lambda} e_\lambda e_\mu e_J = (-1)^{d_\mu} e_\mu e_J = (-1)^{|J|} e_J.$$

For subsets J and K of S we have $m_{JK} = m_{Jw_0 K w_0}$. Thus, $n_{JK} = n_{Jw_0 K w_0}$ and it follows from (3.1) that $w_0 e_J w_0 = e_{Jw_0}$. Finally,

$$e_J w_0 = w_0 (w_0 e_J w_0) = (-1)^{|Jw_0|} e_{Jw_0} = (-1)^{|J|} e_{Jw_0}.$$

This completes the proof of the lemma. $\square$

For a subset L of S , the pair (W_L, L) is a Coxeter system. Because W^L is a complete set of left coset representatives of W_L in W , left multiplication by x_L defines an embedding of $\mathbb{C}W_L$ into $\mathbb{C}W$. For $I \subseteq L$ define $W_L^I = W_L \cap W^I$ and $x_I^L = \sum_{w \in W_L^I} w$. Then $\{x_I^L \mid I \subseteq L\}$ is a basis of $\Sigma(W_L)$. It is well known and easy to prove that $W^L W_L^I = W^I$, and so $x_L x_I^L = x_I$. If n is in N_L , then $n(\Delta_L) = \Delta_L$, and so by Lemma 3.1(a) we have $x_L n = x_L$. It follows that $x_L \mathbb{C}W_L$ is stable under right multiplication by elements of $N_W(W_L)$.

It is straightforward to check that for a in $\mathbb{C}W_L$ and wn in $W_L N_L$, the assignment $(a, wn) \mapsto a \cdot wn = n^{-1}awn$ defines a right action of the group $N_W(W_L) = W_L N_L$ on $\mathbb{C}W_L$. Then

$$x_L(a \cdot wn) = x_L n^{-1}awn = x_L awn,$$

and so left multiplication by x_L defines an $N_W(W_L)$ -equivariant embedding of $\mathbb{C}W_L$ into $\mathbb{C}W$. For later reference we record this fact in the following lemma.

Lemma 3.4. *Suppose that L is a subset of S . Then $N_W(W_L)$ acts on $\mathbb{C}W_L$ by $a \cdot wn = n^{-1}awn$, for $a \in \mathbb{C}W_L$, w in W_L , and $n \in N_L$. Moreover, left multiplication by x_L defines an $N_W(W_L)$ -equivariant embedding of $\mathbb{C}W_L$ into $\mathbb{C}W$.*

Recall that the quasi-idempotents e_J^σ are defined relative to the ambient set S and the function σ . Define

$$\sigma_L : 2^L \rightarrow \mathbb{R}_{>0} \quad \text{by} \quad \sigma_L(I) = m_{I,L}^\sigma.$$

Then for $J \subseteq L$ we have the quasi-idempotent $e_J^{\sigma_L} = \sum_{I \subseteq L} n_{IJ}^{\sigma_L} x_I^L$ in $\mathbb{C}W_L$ defined relative to the set L and the function σ_L .

Lemma 3.5. *Suppose that I , J , and L are subsets of S with $I, J \subseteq L$. Then*

- (a) $m_{IJ}^{\sigma_L} = m_{IJ}^\sigma$ and $n_{IJ}^{\sigma_L} = n_{IJ}^\sigma$;
- (b) $x_L e_J^{\sigma_L} = e_J^\sigma$; and
- (c) $n^{-1} e_J^{\sigma_L} n = e_{Jn}^{\sigma_L}$ for n in N_L .

Proof. It is shown in [4, Theorem 7.5] that $m_{IJ}^{\sigma_L} = m_{IJ}^\sigma$. It then follows from the definitions that $n_{IJ}^{\sigma_L} = n_{IJ}^\sigma$. This proves (a). Now

$$x_L e_J^{\sigma_L} = x_L \sum_{I \subseteq J} n_{IJ}^{\sigma_L} x_I^L = \sum_{I \subseteq J} n_{IJ}^{\sigma_L} x_L x_I^L = \sum_{I \subseteq J} n_{IJ}^\sigma x_I = e_J^\sigma,$$

and so (b) holds.

Suppose n is in N_L . Then using (b) and Lemma 3.1 we have

$$e_J^\sigma n = x_L e_J^{\sigma_L} n = (x_L n)(n^{-1} e_J^{\sigma_L} n) = x_L (n^{-1} e_{Jn}^{\sigma_L} n)$$

and

$$e_J^\sigma n = e_{J^n}^\sigma = x_{L^n} e_{J^n}^{\sigma_L} = x_L e_{J^n}^{\sigma_L}.$$

Now $n^{-1} e_{J^n}^{\sigma_L} n$ and $e_{J^n}^{\sigma_L}$ are both in $\mathbb{C}W_L$ and $x_L(n^{-1} e_{J^n}^{\sigma_L} n) = x_L e_{J^n}^{\sigma_L}$, so we conclude from Lemma 3.4 that $n^{-1} e_{J^n}^{\sigma_L} n = e_{J^n}^{\sigma_L}$. This proves (c). $\square$

We conclude this section with a description of the quasi-idempotents e_K^σ in the case when W is reducible.

Suppose that W is reducible, say $W \cong W_1 \times W_2$. Then $S = S_1 \amalg S_2$ is the disjoint union of S_1 and S_2 , where $W_1 = \langle S_1 \rangle$ and $W_2 = \langle S_2 \rangle$, and $\Delta = \Delta_1 \amalg \Delta_2$, where $\Delta_i = \{ \alpha \mid s_\alpha \in S_i \}$ for $i = 1, 2$.

Suppose K is a subset of S . Then $K = K_1 \amalg K_2$, where $K_i = K \cap S_i$ for $i = 1, 2$. Every element in W has a unique expression as a product $w_1 w_2$ with w_1 in W_1 and w_2 in W_2 . Then $w_1 w_2$ is in W_K if and only if w_1 is in W_{K_1} and w_2 is in W_{K_2} . Moreover, $w_1 w_2$ is in W^K if and only if w_1 is in W^{K_1} and w_2 is in W^{K_2} . It follows that $x_K = x_{K_1} x_{K_2}$. If J is a subset of K , then $\Delta_J = \Delta_{J_1} \amalg \Delta_{J_2}$ and $w_1 w_2(\Delta_J) \subseteq \Delta$ if and only if $w_1(\Delta_{J_1}) \subseteq \Delta_1$ and $w_2(\Delta_{J_2}) \subseteq \Delta_2$. Similarly, $w_1 w_2(J_1 \amalg J_2) = w_1 J_1 \amalg w_2 J_2$.

Now suppose that $\sigma: 2^S \rightarrow \mathbb{R}_{>0}$ has the property that $\sigma(J_1 \amalg J_2) = \sigma(J_1)\sigma(J_2)$ for $J_1 \subseteq S_1$ and $J_2 \subseteq S_2$. Then for $J \subseteq K \subseteq S$ we have

$$\begin{aligned} m_{JK}^\sigma &= \sum_{\substack{w \in W^K \\ w(\Delta_J) \subseteq \Delta}} \sigma({}^w J) \\ &= \sum_{\substack{w_1 \in W^{K_1}, w_2 \in W^{K_2} \\ w_1(\Delta_{J_1}) \subseteq \Delta_1, w_2(\Delta_{J_2}) \subseteq \Delta_2}} \sigma({}^{w_1} J_1 \amalg {}^{w_2} J_2) \\ &= \sum_{\substack{w_1 \in W^{K_1} \\ w_1(\Delta_{J_1}) \subseteq \Delta_1}} \sum_{\substack{w_2 \in W^{K_2} \\ w_2(\Delta_{J_2}) \subseteq \Delta_2}} \sigma({}^{w_1} J_1) \sigma({}^{w_2} J_2) \\ &= m_{J_1 K_1}^\sigma m_{J_2 K_2}^\sigma \\ &= m_{J_1 K_1}^{\sigma_1} m_{J_2 K_2}^{\sigma_2}, \end{aligned}$$

where σ_i is the restriction of σ to 2^{S_i} for $i = 1, 2$. Conversely, if we are given functions $\sigma_i: 2^{S_i} \rightarrow \mathbb{R}_{>0}$ for $i = 1, 2$ and define $\sigma: 2^S \rightarrow \mathbb{R}_{>0}$ by $\sigma(K) = \sigma_1(K_1)\sigma_2(K_2)$, then $m_{J_1 K_1}^{\sigma_1} m_{J_2 K_2}^{\sigma_2} = m_{J_1 K_1}^\sigma m_{J_2 K_2}^\sigma = m_{JK}^\sigma$.

With σ as above, set $f_J^\sigma = e_{J_1}^\sigma e_{J_2}^\sigma$ for $J = J_1 \amalg J_2 \subseteq S$. Then

$$\sum_{J \subseteq K} m_{JK}^\sigma f_J^\sigma = \sum_{\substack{J_1 \subseteq K_1 \\ J_2 \subseteq K_2}} m_{J_1 K_1}^\sigma m_{J_2 K_2}^\sigma e_{K_1}^\sigma e_{K_2}^\sigma = x_{K_1} x_{K_2} = x_K,$$

and so $f_J^\sigma = e_J^\sigma$. This proves the following proposition.

Proposition 3.6. *Suppose $W \cong W_1 \times W_2$ is reducible and $S = S_1 \amalg S_2$, where $W_1 = \langle S_1 \rangle$ and $W_2 = \langle S_2 \rangle$. Suppose in addition that $\sigma: 2^S \rightarrow \mathbb{R}_{>0}$ has the property that for $J \subseteq S$, $\sigma(J) = \sigma(J \cap S_1)\sigma(J \cap S_2)$. Then*

$$e_J^\sigma = e_{J \cap S_1}^\sigma e_{J \cap S_2}^\sigma \quad \text{and} \quad e_J = e_{J \cap S_1} e_{J \cap S_2}.$$

4. E_λ IS AN INDUCED REPRESENTATION

Suppose λ is in Λ and $\sigma: 2^S \rightarrow \mathbb{R}_{>0}$. Define

$$E_\lambda^\sigma = e_\lambda^\sigma \mathbb{C}W$$

to be the right ideal in $\mathbb{C}W$ generated by e_λ^σ . Similarly, using the notation in (3.4), define

$$E_\lambda = e_\lambda \mathbb{C}W.$$

We have seen in §2.2 that $A_\lambda \cong \text{Ind}_{N_W(W_X)}^W(A_X)$ for X in λ . In this section we show that E_λ^σ has a similar description as an induced representation, and we analyze how E_λ^σ depends on the choice of σ . In particular, it is shown in Corollary 4.8 that for L in S_λ ,

$$E_\lambda \cong \text{Ind}_{N_W(W_L)}^W(e_L \mathbb{C}W_L) \cong \text{Ind}_{N_W(W_L)}^W(e_L^L \mathbb{C}W_L).$$

We begin with a lemma which follows immediately from (3.3).

Lemma 4.1. *Suppose that λ is in Λ and I is in S_λ . Then $E_\lambda^\sigma = e_I^\sigma \mathbb{C}W$.*

The next proposition shows that up to isomorphism, E_λ^σ does not depend on σ .

Proposition 4.2. *Suppose that λ is in Λ and that σ and σ_1 are functions from 2^S to $\mathbb{R}_{>0}$. Then there is a unit u in $\Sigma(W)$ such that left multiplication by u defines an isomorphism of right $\mathbb{C}W$ -modules $E_\lambda^\sigma \cong E_\lambda^{\sigma_1}$.*

Proof. Let $\text{rad}(\Sigma(W))$ denote the Jacobson radical of $\Sigma(W)$ and let θ denote the natural projection from $\Sigma(W)$ to $\Sigma(W)/\text{rad}(\Sigma(W))$. Bergeron, Bergeron, Howlett, and Taylor [4, §7] have shown that $\theta(e_\lambda^\sigma) = \theta(e_\lambda^{\sigma_1})$ is a primitive idempotent in $\Sigma(W)/\text{rad}(\Sigma(W))$. Thus, it follows from [38, Theorem 3.1] that there is a unit u in $1 + \text{rad}(\Sigma(W))$ such that $ue_\lambda^\sigma = e_\lambda^{\sigma_1}u$. Then left multiplication by u defines an isomorphism of right $\mathbb{C}W$ -modules $e_\lambda^\sigma \mathbb{C}W \cong e_\lambda^{\sigma_1} \mathbb{C}W$. $\square$

Suppose that N is a subgroup of $N(W)$, so WN is a subgroup of $N_{U(V)}(W)$. Then as in Lemma 3.4, WN acts on $\mathbb{C}W$ on the right by $a \cdot wn = n^{-1}awn$ for a in $\mathbb{C}W$, w in W , and n in N . If N centralizes e_λ^σ , then clearly E_λ^σ is a WN -submodule of $\mathbb{C}W$. It follows from Lemma 3.2 that if σ is constant on N -orbits in 2^S , then N centralizes $e_{\{S\}}^\sigma = e_S^\sigma$. More generally, if σ is constant on N -orbits in 2^S and S_λ is N -stable, then N centralizes e_λ^σ .

Let $\Sigma(W)^N$ denote the algebra of N -invariants in $\Sigma(W)$.

Proposition 4.3. *Suppose that λ is in Λ and that σ and σ_1 are functions from 2^S to $\mathbb{R}_{>0}$ such that S_λ is N -stable, N centralizes e_λ^σ , and σ_1 is constant on N -orbits in 2^S . Then there is a unit v in $\Sigma(W)^N$ such that left multiplication by v defines an isomorphism of right $\mathbb{C}WN$ -modules E_λ^σ and $E_\lambda^{\sigma_1}$.*

Proof. We saw in the proof of Proposition 4.2 that there is a unit u in $1 + \text{rad}(\Sigma(W))$ such that $ue_\lambda^\sigma = e_\lambda^{\sigma_1}u$, and we observed in the proof of Lemma 3.2 that $n^{-1}x_I n = x_{I^n}$ for $I \subseteq S$ and n in N . It follows that $\Sigma(W)$ and $\text{rad}(\Sigma(W))$ are stable under conjugation by N , and so $v = \frac{1}{|N|} \sum_{n \in N} n^{-1}un$ is in $1 + \text{rad}(\Sigma(W))$ and hence is a unit. By assumption, N centralizes e_λ^σ and $e_\lambda^{\sigma_1}$ and it follows that $ve_\lambda^\sigma = e_\lambda^{\sigma_1}v$. Therefore, left multiplication by v defines an $N_{U(V)}(W)$ -module isomorphism $e_\lambda^\sigma \mathbb{C}W \cong e_\lambda^{\sigma_1} \mathbb{C}W$. $\square$

In the next proposition, ℓ_x denotes left multiplication by x .

Proposition 4.4. *Suppose L is a subset of S , $\sigma: 2^S \rightarrow \mathbb{R}_{>0}$, $\tau: 2^L \rightarrow \mathbb{R}_{>0}$, and τ is constant on N_L -orbits. Then there is a unit v in $\Sigma(W_L)^{N_L}$ such that if $\mu_v = \ell_{x_L} \ell_v \ell_{x_L}^{-1}$, then the diagram*

$$\begin{array}{ccc} e_L^{\sigma_L} \mathbb{C}W_L & \xrightarrow{\ell_v} & e_L^{\tau} \mathbb{C}W_L \\ \ell_{x_L} \downarrow & & \downarrow \ell_{x_L} \\ e_L^{\sigma} \mathbb{C}W_L & \xrightarrow{\mu_v} & x_L e_L^{\tau} \mathbb{C}W_L \end{array}$$

is a commutative diagram of right $N_W(W_L)$ -modules and $N_W(W_L)$ -isomorphisms.

Proof. It follows from Lemma 3.5(c) and Lemma 3.2 that $e_L^{\sigma_L} \mathbb{C}W_L$ and $e_L^{\tau} \mathbb{C}W_L$ are $N_W(W_L)$ -stable right ideals of $\mathbb{C}W_L$, where $N_W(W_L)$ acts on $\mathbb{C}W_L$ as in Lemma 3.4. By Lemma 3.5(b) we have $e_L^{\sigma} = x_L e_L^{\sigma_L}$, and so it follows from Lemma 3.1(a) that $e_L^{\sigma} \mathbb{C}W_L$ and $x_L e_L^{\tau} \mathbb{C}W_L$ are stable under right multiplication by $N_W(W_L)$. It now follows from Lemma 3.4 that the vertical maps are isomorphisms of $N_W(W_L)$ -modules.

The hypotheses of Proposition 4.3 are satisfied with L , $\{L\}$, and σ_L in place of S , S_λ , and σ . Thus, there is a unit v in $\Sigma(W_L)^{N_L}$ such that $\ell_v: e_L^{\sigma_L} \mathbb{C}W_L \rightarrow e_L^{\tau} \mathbb{C}W_L$ is an isomorphism. The conclusion of the proposition is now clear, as $\mu_v = \ell_{x_L} \ell_v \ell_{x_L}^{-1}$. $\square$

The decomposition $1 = \sum_{\lambda \in \Lambda} e_\lambda^\sigma$ gives a decomposition

$$\mathbb{C}W \cong \bigoplus_{\lambda \in \Lambda} E_\lambda^\sigma$$

of $\mathbb{C}W$ into right ideals. It is shown in [4, §7] that $\dim E_\lambda^\sigma = |\text{sh}^{-1}(\lambda)|$, the number of elements in W with shape λ . In the next lemma we compute $|\text{sh}^{-1}(\lambda)|$ in terms of cuspidal elements in a parabolic subgroup of W with shape λ .

Lemma 4.5. *Suppose that λ is a shape in Λ , X is a subspace in λ , and C is a conjugacy class in W with shape λ . Then*

- (a) $|C| = |W : N_W(W_X)| |C \cap W_X|$ and
- (b) $|\text{sh}^{-1}(\lambda)| = |W : N_W(W_X)| |\text{sh}^{-1}(\lambda) \cap W_X|$.

Proof. Notice that $C \cap W_X$ is a cuspidal conjugacy class in W_X . Thus, it follows from (1) and (2) in §2.4 that $|N_W(W_X) : Z_W(c)| = |W_X : Z_{W_X}(c)|$ for c in $C \cap W_X$. Therefore

$$|C| = |W : N_W(W_X)| |N_W(W_X) : Z_W(c)| = |W : N_W(W_X)| |C \cap W_X|.$$

This proves (a). Statement (b) follows from (a) and the observation that $\text{sh}^{-1}(\lambda)$ is the union of those conjugacy classes in W whose intersection with W_X is a cuspidal conjugacy class in W_X . $\square$

Corollary 4.6. *Suppose λ is in Λ , J is in S_λ , $\sigma: 2^S \rightarrow \mathbb{R}_{>0}$, and $\tau: 2^J \rightarrow \mathbb{R}_{>0}$. Then*

- (a) $\dim A_{X_J} = \dim e_J^\sigma \mathbb{C}W_J = \dim e_J^\tau \mathbb{C}W_J = |\text{sh}^{-1}(\lambda) \cap W_J|$ and
- (b) $\dim A_\lambda = \dim E_\lambda^\sigma = |\text{sh}^{-1}(\lambda)|$.

Proof. It is clear that $\{w \in W \mid \text{Fix}(w) = X_J\} = \text{sh}^{-1}(\lambda) \cap W_J$ is the set of all cuspidal elements in W_J , and it is shown in [12, Proposition 2.4] that $\dim A_{X_J} = |\{w \in W \mid \text{Fix}(w) = X_J\}|$. Therefore, $\dim A_{X_J} = |\text{sh}^{-1}(\lambda) \cap W_J|$, and so it follows from Lemma 4.5(b) that $\dim A_\lambda = |\text{sh}^{-1}(\lambda)|$. It is shown in [4, Theorem 7.15] that $\dim E_\lambda^\sigma = |\text{sh}^{-1}(\lambda)|$. This proves (b).

It remains to show that $\dim e_J^\sigma \mathbb{C}W_J = \dim e_J^\tau \mathbb{C}W_J = |\text{sh}^{-1}(\lambda) \cap W_J|$. Using Lemma 3.5(b), Lemma 3.4, Proposition 4.2, and (b) applied to the shape $\{J\}$ of W_J , we have

$$\dim e_J^\sigma \mathbb{C}W_J = \dim x_J e_J^{\sigma_J} \mathbb{C}W_J = \dim e_J^{\sigma_J} \mathbb{C}W_J = \dim e_J^\tau \mathbb{C}W_J = |\text{sh}^{-1}(\lambda) \cap W_J|,$$

as desired. $\square$

The next proposition and its corollary are the main results in this section.

Proposition 4.7. *Suppose that $\sigma: 2^S \rightarrow \mathbb{R}_{>0}$, λ is in Λ , X is in λ , and L is in S_λ .*

(a) $N_W(W_L)$ acts on $e_L^\sigma \mathbb{C}W_L$ by right multiplication and

$$E_\lambda^\sigma \cong \text{Ind}_{N_W(W_L)}^W (e_L^\sigma \mathbb{C}W_L).$$

(b) $N_W(W_X)$ acts on A_X and

$$A_\lambda \cong \text{Ind}_{N_W(W_X)}^W (A_X).$$

Proof. Statement (b) is proved in [26, §2]. We prove (a).

It was shown in Proposition 4.4 that $e_L^\sigma \mathbb{C}W_L$ is stable under right multiplication by $N_W(W_L)$. By Lemma 4.1, $E_\lambda^\sigma = e_L^\sigma \mathbb{C}W$. Therefore, to prove that $E_\lambda^\sigma \cong \text{Ind}_{N_W(W_L)}^W (e_L^\sigma \mathbb{C}W_L)$ it is enough to show that the multiplication map $e_L^\sigma \mathbb{C}W_L \otimes_{\mathbb{C}N_W(W_L)} \mathbb{C}W \rightarrow E_\lambda^\sigma$ is a bijection. This map is obviously a surjection. Using Lemma 4.5 and Corollary 4.6, we have

$$\begin{aligned} \dim E_\lambda^\sigma &= |\text{sh}^{-1}(\lambda)| \\ &= |W : N_W(W_L)| |\text{sh}^{-1}(\lambda) \cap W_L| \\ &= |W : N_W(W_L)| \dim e_L^\sigma \mathbb{C}W_L \\ &= \dim e_L^\sigma \mathbb{C}W_L \otimes_{\mathbb{C}N_W(W_L)} \mathbb{C}W, \end{aligned}$$

and so the multiplication map is indeed a bijection. $\square$

We saw in Proposition 4.4 that $e_L^\sigma \mathbb{C}W_L$, $e_L^{\sigma_L} \mathbb{C}W_L$, $e_L^\tau \mathbb{C}W_L$, and $x_L e_L^\tau \mathbb{C}W_L$ all afford equivalent representations of $N_W(W_L)$ when $\tau: 2^L \rightarrow \mathbb{R}_{>0}$ is chosen only subject to the restriction that it is constant on N_L -orbits. Thus, Proposition 4.7(a) implies the following corollary.

Corollary 4.8. *Suppose λ is in Λ , L is in S_λ , $\sigma: 2^S \rightarrow \mathbb{R}_{>0}$, and $\tau: 2^L \rightarrow \mathbb{R}_{>0}$ is constant on N_L -orbits. Then*

$$E_\lambda^\sigma \cong \text{Ind}_{N_W(W_L)}^W (e_L^{\sigma_L} \mathbb{C}W_L) \cong \text{Ind}_{N_W(W_L)}^W (e_L^\tau \mathbb{C}W_L) \cong \text{Ind}_{N_W(W_L)}^W (x_L e_L^\tau \mathbb{C}W_L).$$

In particular, if $\sigma(I) = 1$ and $\tau(J) = 1$ for all $I \subseteq S$ and $J \subseteq L$, then

$$E_\lambda \cong \text{Ind}_{N_W(W_L)}^W (e_L^L \mathbb{C}W_L).$$

5. SYMMETRIC GROUPS: $\lambda = (n)$

In this section and the next we prove Conjecture 2.1 for symmetric groups. In these two sections we take W to be the symmetric group on n letters with $n \geq 2$ and we identify W with the subgroup of $\mathrm{GL}_n(\mathbb{C})$ that acts on the basis $\{v_1, v_2, \dots, v_n\}$ as permutations. Here, v_i is the column vector whose j^{th} entry is 0 for $j \neq i$ and 1 for $j = i$. For $1 \leq i \leq n-1$ let s_i denote the matrix in W that interchanges v_i and v_{i+1} and fixes v_j for $j \neq i, i+1$. Then $S = \{s_1, s_2, \dots, s_{n-1}\}$ is a Coxeter generating set for W .

By a *partition of n* we mean a non-increasing finite sequence of positive integers whose sum is n . Say $\lambda = (\lambda_1, \dots, \lambda_p)$ is a partition of n . Then $\lambda_1 \geq \dots \geq \lambda_p > 0$ and $\sum_{k=1}^p \lambda_k = n$. The integers λ_i are called the *parts of λ* .

It is well known that for $W = S_n$ we may identify Λ with the set of partitions of n . We make this identification precise as follows. Suppose that λ is a partition of n with p parts. Define partial sums τ_i for $i = 0, 1, \dots, p$ by $\tau_0 = 0$ and $\tau_i = \lambda_1 + \dots + \lambda_i$ for $1 \leq i \leq p$. Define

$$I_\lambda = S \setminus \{s_{\tau_1}, s_{\tau_2}, \dots, s_{\tau_{p-1}}\} \quad \text{and} \quad W_\lambda = \langle I_\lambda \rangle.$$

Then W_λ is isomorphic to the product of symmetric groups $S_{\lambda_1} \times \dots \times S_{\lambda_p}$, where the factor S_{λ_i} acts on the subset $\{v_{\tau_{i-1}+1}, v_{\tau_{i-1}+2}, \dots, v_{\tau_i}\}$ of $\{v_1, v_2, \dots, v_n\}$. Next, define

$$X_\lambda = \mathrm{Fix}(W_\lambda).$$

Then X_λ is in $L(\mathcal{A})$ and $W_{X_\lambda} = W_\lambda$. We have seen in Proposition 4.7 that

$$E_\lambda \cong \mathrm{Ind}_{N_W(W_\lambda)}^W(e_{I_\lambda} \mathbb{C} W_\lambda) \quad \text{and} \quad A_\lambda \cong \mathrm{Ind}_{N_W(W_\lambda)}^W(A_{X_\lambda}).$$

It is well known and straightforward to check that $\{X_\lambda \mid \lambda \text{ is a partition of } n\}$ is a complete set of orbit representatives for the action of W on $L(\mathcal{A})$ and that $\{I_\lambda \mid \lambda \text{ is a partition of } n\}$ is a complete set of representatives for the $\sim$ -equivalence classes of subsets of S .

Notice that in the extreme case when all parts of λ are equal to 1 we have $I_\lambda = \emptyset$ and $W_\lambda = W_\emptyset = \{1\}$. At the other extreme, when $\lambda = (n)$, we have $I_\lambda = S$ and $W_\lambda = W_S = W$. We first prove Conjecture 2.1 when $\lambda = (n)$.

For the rest of this section we take $\lambda = (n)$. Then $W_\lambda = N_W(W_\lambda) = W$, and so $E_{(n)} = e_{I_{(n)}} \mathbb{C} W_{(n)}$ and $A_{(n)} = A_{X_{(n)}}$. Moreover, $A_{X_{(n)}} = A^{n-1}$ is the top, non-zero graded piece of A . To simplify the notation, we denote $A_{(n)}$, $E_{(n)}$, and $e_{I_{(n)}}$ by A_n , E_n , and e_n , respectively.

Define $c_1 = 1$ in W and for $1 < i \leq n$ define $c_i = s_{i-1} \dots s_2 s_1$ so that c_i acts on the basis $\{v_1, v_2, \dots, v_n\}$ as an i -cycle. Also, set $c = c_n$. Then:

- c is a cuspidal element in W ,
- the set of cuspidal elements in W is precisely the conjugacy class of c , and
- $Z_W(c) = \langle c \rangle$ is the cyclic group of order n generated by c .

Set $\zeta = e^{2\pi i/n}$ in $\mathbb{C}$ and define $\varphi: Z_W(c) \rightarrow \mathbb{C}$ by $\varphi(c^{-1}) = \zeta$. The elements we have denoted by c_i are denoted by c_i^{-1} by Lehrer and Solomon [26]. However, the character φ of $Z_W(c)$ is the same as in [26].

Theorem 5.1. *With the preceding notation we have that*

- the character of W on E_n is $\mathrm{Ind}_{Z_W(c)}^W(\varphi)$ and*
- the character of W on A_n is $\epsilon \mathrm{Ind}_{Z_W(c)}^W(\varphi)$.*

Statement (b) has been proven by Stanley [36, Theorem 7.2] and by Lehrer and Solomon [26, Theorem 3.9]. As mentioned in the introduction, a proof of (a) may be extracted from classical results about the representation of S_n on the free Lie algebra on n letters. In contrast, our proof below that the character of W on E_n is $\text{Ind}_{Z_W(c)}^W(\varphi)$ follows the Lehrer-Solomon argument and demonstrates a parallelism between the group algebra and the Orlik-Solomon algebra that we expect will apply in some form to all finite Coxeter groups. In addition, our argument is valid not only for E_n and e_n , but more generally for $E_{(n)}^\sigma$ and $e_{(n)}^\sigma$ for any function $\sigma: 2^S \rightarrow \mathbb{R}_{>0}$.

To emphasize and differentiate the parallel arguments, we use the convention that the superscript $+$ denotes quantities associated with E_n and the superscript $-$ denotes quantities associated with A_n . Notice that with the notation of §2 we have $X_c = \{0\}$, and so $\alpha_c = \det|_{X_c}$ is the trivial character.

Suppose t is an indeterminate. For $0 \leq k \leq n$, define elements $b^+(n, k)$ and $b^-(n, k)$ in $\mathbb{C}W$ by

$$(1 - c_1 t)(1 - c_2 t) \cdots (1 - c_n t) = \sum_{k=0}^n b^+(n, k) t^k$$

and

$$(1 + c_n t)(1 - c_{n-1} t) \cdots (1 + (-1)^{n-1} c_1 t) = \sum_{k=0}^n b^-(n, k) t^k$$

respectively (the k^{th} factor in the product on the left-hand side of the last equation is $(1 + (-1)^{k-1} c_{n-k+1} t)$).

Set $W_{n-1} = \langle s_1, s_2, \dots, s_{n-2} \rangle$. Then $W_{n-1} \cong S_{n-1}$. The analog of the idempotent e_n in E_n is the basis element $a_n = a_{s_1} a_{s_2} \cdots a_{s_{n-1}}$ in $A_n = A^{n-1}$. Lehrer and Solomon [26, §3] prove the following statements:

- (i) $A_n = \mathbb{C}W a_n$.
- (ii) $c^{-k} a_n = b^-(n-1, k) a_n$ for $0 \leq k \leq n-1$. In particular, $A_n = \mathbb{C}W_{n-1} a_n$.
- (iii) Consider the homomorphism of left $\mathbb{C}W$ -modules from $\mathbb{C}W$ to A_n given by right multiplication by a_n . The kernel of this mapping is the left $\mathbb{C}W_{n-1}$ -module generated by $\{c^{-k} - b^-(n-1, k) \mid 1 \leq k \leq n-1\}$.
- (iv) $\{w a_n \mid w \in W_{n-1}\}$ is a $\mathbb{C}$ -basis of A_n , and A_n is the left regular $\mathbb{C}W_{n-1}$ -module.

Next we show that the analogous statements hold with A_n replaced by E_n and $b^-(n, k)$ replaced by $b^+(n, k)$.

For $k = 1, 2, \dots, n-1$, define $x_k = x_{S \setminus \{s_k\}}$ and $w_k = c_1 c_2 \cdots c_k$. Then w_k is the longest element in $\langle s_1, s_2, \dots, s_{k-1} \rangle$.

Lemma 5.2. *Suppose $1 \leq k \leq n-1$. Then*

$$W^{S \setminus \{s_k\}} w_k = \{c_{i_1} \cdots c_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\}.$$

Proof. It suffices to show that if $1 \leq i_1 < \cdots < i_k \leq n$, then $c_{i_1} \cdots c_{i_k} w_k$ is in $W^{S \setminus \{s_k\}}$. For this, we consider elements in W as acting on $\{1, \dots, n\}$. That is, we identify the vector v_j with j for $1 \leq j \leq n$. Then

$$W^{S \setminus \{s_k\}} = \{w \in W \mid w(j) < w(j+1) \forall j \in \{1, \dots, n-1\} \setminus \{k\}\}$$

and

$$w_k(j) = \begin{cases} k+1-j, & 1 \leq j \leq k, \\ j, & k+1 \leq j \leq n. \end{cases}$$

Fix $i_1, \dots, i_k$ with $1 \leq i_1 < \dots < i_k \leq n$ and set $w = c_{i_1} \cdots c_{i_k} w_k$. If $1 \leq j \leq k$, then $w(j) = i_j < i_{j+1} = w(j+1)$. If $j \geq i_k$, then $w(j) \leq j < j+1 = w(j+1)$. Suppose that $k < j < i_k$. Choose r minimal such that

$$j+1 \leq i_k, \quad j+1-1 \leq i_{k-1}, \quad \dots, \quad j+1-r \leq i_{k-r}, \quad \text{and} \quad j+1-r-1 > i_{k-r-1}.$$

Then $w(j) \leq j-r-1 < j-r = j+1-r-1 = w(j+1)$. $\square$

Corollary 5.3. *For $1 \leq k \leq n-1$, we have $b^+(n, k) = (-1)^k x_k w_k$.*

Proof. Using the definition and Lemma 5.2 we have

$$b^+(n, k) = (-1)^k \sum_{1 \leq i_1 < \dots < i_k \leq n} c_{i_1} \cdots c_{i_k} = (-1)^k x_k w_k.$$

$\square$

Proposition 5.4. *The following analogs of (i)–(iv) above hold:*

- (a) $E_n = e_n \mathbb{C}W$.
- (b) $e_n c^k = e_n b^+(n-1, k)$ for $0 \leq k \leq n-1$. In particular, $E_n = e_n \mathbb{C}W_{n-1}$.
- (c) Consider the endomorphism of $\mathbb{C}W$ considered as a right $\mathbb{C}W$ -module given by left multiplication by e_n . The kernel of this mapping is the free, right $\mathbb{C}W_{n-1}$ -module with basis $\{c^k - b^+(n-1, k) \mid 1 \leq k \leq n-1\}$.
- (d) $\{e_n w \mid w \in W_{n-1}\}$ is a $\mathbb{C}$ -basis of E_n and E_n is the right regular $\mathbb{C}W_{n-1}$ -module.

Proof. The first statement follows immediately from the definitions.

We prove (b) by recursion. It is clear that $e_n c^k = e_n b^+(n-1, k)$ for $k = 0$, since $b^+(n-1, 0) = 1 = c^0$. Suppose $e_n c^{k-1} = e_n b^+(n-1, k-1)$. It follows from [4, Theorem 7.8] that $e_n x_J = 0$ unless $J = S$. Thus, it follows from Corollary 5.3 that $e_n b^+(n, k) = (-1)^k e_n x_k w_k = 0$ for $1 \leq k \leq n-1$. On the other hand, it follows from the definition that

$$\sum_{k=0}^n b^+(n, k) t^k = \left(\sum_{k=0}^{n-1} b^+(n-1, k) t^k \right) (1 - c_n t),$$

and hence $b^+(n, k) = b^+(n-1, k) - b^+(n-1, k-1)c$ for $1 \leq k \leq n-1$. Therefore,

$$e_n c^k = e_n c^{k-1} c = e_n b^+(n-1, k-1) c = e_n b^+(n-1, k).$$

Next, consider the endomorphism of $\mathbb{C}W$ given by $x \mapsto e_n x$. Let K denote the kernel of this mapping and let K_1 denote the $\mathbb{C}W_{n-1}$ -submodule of $\mathbb{C}W$ generated by $\{c^k - b^+(n-1, k) \mid 1 \leq k \leq n-1\}$. It follows from (b) that $K_1 \subseteq K$. Moreover, $\{c^k - b^+(n-1, k) \mid 1 \leq k \leq n-1\}$ is a $\mathbb{C}W_{n-1}$ -basis of K_1 because the cyclic subgroup generated by c is a left transversal of W_{n-1} in W . Therefore, $\dim_{\mathbb{C}} K_1 = (n-1)(n-1)!$. However,

$$\dim K = \dim \mathbb{C}W - \dim E_n = n! - |W : Z_W(c)| = (n-1)(n-1)! = \dim K_1.$$

Therefore, $K_1 = K$. This proves (c).

Because $b^+(n-1, k)$ is in $\mathbb{C}W_{n-1}$ for $1 \leq k \leq n-1$, it follows from (b) that the image of the mapping $x \mapsto e_n x$ is $e_n \mathbb{C}W_{n-1}$. Therefore, $E_n = e_n \mathbb{C}W_{n-1}$. Since $\dim E_n = (n-1)!$, it follows that $\{e_n w \mid w \in W_{n-1}\}$ is a $\mathbb{C}$ -basis of E_n . This proves (d). $\square$

Finally, define idempotents f^+ and f^- in $\mathbb{C}Z_W(c)$ by

$$f^+ = \frac{1}{n} \sum_{k=0}^{n-1} \varphi(c^k) c^{-k} = \frac{1}{n} \sum_{k=0}^{n-1} \zeta^{-k} c^{-k}$$

and

$$f^- = \frac{1}{n} \sum_{k=0}^{n-1} \epsilon(c)^k \varphi(c^k) c^{-k} = \frac{1}{n} \sum_{k=0}^{n-1} \epsilon(c)^k \zeta^{-k} c^{-k}.$$

Obviously, the lines $\mathbb{C}f^+$ and $\mathbb{C}f^-$ in $\mathbb{C}W$ are stable under left and right multiplication by $Z_W(c)$ and afford the characters φ and $\epsilon\varphi$ of $Z_W(c)$, respectively. Moreover, $\text{Ind}_{Z_W(c)}^W(\varphi)$ is afforded by the right $\mathbb{C}W$ -module $f^+\mathbb{C}W$, and $\epsilon\text{Ind}_{Z_W(c)}^W(\varphi) = \text{Ind}_{Z_W(c)}^W(\epsilon\varphi)$ is afforded by the left $\mathbb{C}W$ -module $\mathbb{C}Wf^-$. Thus, to prove Theorem 5.1 it is enough to find $\mathbb{C}W$ -isomorphisms $E_n \cong f^+\mathbb{C}W$ and $A_n \cong \mathbb{C}Wf^-$.

Lemma 5.5. *The idempotent f^+ acts invertibly by right multiplication on e_n , and the idempotent f^- acts invertibly by left multiplication on a_n .*

Proof. Lehrer and Solomon [26, §3] show that f^- acts invertibly on a_n . Their argument is easily modified to show that f^+ acts invertibly by right multiplication on e_n as follows.

We have $(1 - c_1\zeta) \cdots (1 - c_{n-1}\zeta) = \sum_{k=0}^{n-1} b^+(n-1, k) \zeta^k$. Multiply both sides on the left by $\frac{1}{n}e_n$ and use Proposition 5.4(b) to get

$$\frac{1}{n}e_n(1 - \zeta c_1) \cdots (1 - \zeta c_{n-1}) = \frac{1}{n} \sum_{k=0}^{n-1} \zeta^k e_n b^+(n-1, k) = \frac{1}{n} \sum_{k=0}^{n-1} \zeta^k e_n c^k = e_n f^+.$$

If $1 \leq k \leq n-1$, then

$$1 - \zeta^k = 1 - \zeta^k c_k^k = (1 - \zeta c_k)(1 + \zeta c_k + \cdots + \zeta^{k-1} c_k^{k-1}).$$

Since ζ is a primitive n^{th} root of unity, $1 - \zeta^k \neq 0$ in $\mathbb{C}$. Thus, $1 - \zeta c_k$ acts invertibly on e_n for $1 \leq k \leq n-1$, and so f^+ acts invertibly on e_n . $\square$

Proof of Theorem 5.1. (See [26, §3].) Consider the mapping from $f^+\mathbb{C}W$ to E_n given by $x \mapsto e_n x$. It follows from Lemma 5.5 and the discussion preceding it that $e_n f^+ \neq 0$, that $Z_W(c)$ acts on the line $\mathbb{C}e_n f^+$ in E_n as the character φ , and that the mapping is a surjection. Since $\dim f^+\mathbb{C}W = |W : Z_W(c)| = (n-1)! = \dim E_n$, the mapping is also an injection. Thus, we have an isomorphism of right $\mathbb{C}W$ -modules, $E_n \cong f^+\mathbb{C}W$.

As in [26, §3], similar reasoning applies to the mapping from $\mathbb{C}Wf^-$ to A_n given by $x \mapsto x a_n$ and shows that $A_n \cong \mathbb{C}Wf^-$. $\square$

6. SYMMETRIC GROUPS: ARBITRARY λ

In this section we consider the case of an arbitrary partition of n and complete the proof of Conjecture 2.1 for symmetric groups.

Suppose $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_p)$ is a partition of n . Recall from §5 that $I_\lambda = S \setminus \{s_{\tau_1}, s_{\tau_2}, \dots, s_{\tau_{p-1}}\}$ and that $W_\lambda = \langle I_\lambda \rangle$ is isomorphic to the product of symmetric groups $S_{\lambda_1} \times \cdots \times S_{\lambda_p}$, where the factor S_{λ_i} acts on $\{v_{\tau_{i-1}+1}, v_{\tau_{i-1}+2}, \dots, v_{\tau_i}\}$. For $1 \leq i \leq p$ define $g_{\lambda_i} = s_{\tau_{i-1}} \cdots s_{\tau_{i-1}+2} s_{\tau_{i-1}+1}$. Then g_{λ_i} is the λ_i -cycle in S_{λ_i}

that corresponds to the n -cycle $c = c_n$ in §5. Next, define $c_\lambda = g_{\lambda_1} g_{\lambda_2} \cdots g_{\lambda_p}$ and $Z_\lambda = Z_{W_\lambda}(c_\lambda)$. Then

- c_λ is a cuspidal element in W_λ ,
- the set of cuspidal elements in W_λ is precisely the conjugacy class of c_λ , and
- $Z_\lambda \cong \langle g_{\lambda_1} \rangle \times \langle g_{\lambda_2} \rangle \times \cdots \times \langle g_{\lambda_p} \rangle$.

Notice that $\{c_\lambda \mid \lambda \text{ is a partition of } n\}$ is a complete set of conjugacy class representatives in W .

With λ as above, for $1 \leq i \leq p$, define φ_{λ_i} to be the character of $\langle g_{\lambda_i} \rangle$ with $\varphi_{\lambda_i}(g_{\lambda_i}^{-1}) = e^{2\pi i/\lambda_i}$. Then φ_{λ_i} is the analog of the character φ in §5 for the factor S_{λ_i} of W_λ . Next, define the character φ_λ of $Z_\lambda \cong \langle g_{\lambda_1} \rangle \times \langle g_{\lambda_2} \rangle \times \cdots \times \langle g_{\lambda_p} \rangle$ to be

$$\varphi_\lambda = \varphi_{\lambda_1} \otimes \cdots \otimes \varphi_{\lambda_p}.$$

Note that this notation is not consistent with that of Lehrer and Solomon; our character φ_λ corresponds to the character $\varphi_\lambda \epsilon$ in [26]. Applying the special case $\lambda = (n)$ considered in §5 to each factor S_{λ_i} of W_λ , for $1 \leq i \leq p$ define

$$f_{\lambda_i}^+ = \frac{1}{\lambda_i} \sum_{k=0}^{\lambda_i-1} \varphi_{\lambda_i}(g_{\lambda_i}^k) g_{\lambda_i}^{-k} \quad \text{and} \quad f_{\lambda_i}^- = \frac{1}{\lambda_i} \sum_{k=0}^{\lambda_i-1} \epsilon(g_{\lambda_i}^k) \varphi(g_{\lambda_i}^k) g_{\lambda_i}^{-k}.$$

Finally, define idempotents f_λ^+ and f_λ^- in $\mathbb{C}Z_\lambda$ by

$$f_\lambda^+ = f_{\lambda_1}^+ f_{\lambda_2}^+ \cdots f_{\lambda_p}^+ \quad \text{and} \quad f_\lambda^- = f_{\lambda_1}^- f_{\lambda_2}^- \cdots f_{\lambda_p}^-.$$

Obviously, the lines $\mathbb{C}f_\lambda^+$ and $\mathbb{C}f_\lambda^-$ in CW are stable under left and right multiplication by Z_λ and afford the characters φ_λ and $\epsilon\varphi_\lambda$ of Z_λ , respectively.

Now consider the canonical complement N_{X_λ} of W_λ in $N_W(W_\lambda)$. Set $N_\lambda = N_{X_\lambda}$. If λ has m_j parts equal to j , then N_λ is isomorphic to the product of symmetric groups $\prod_j S_{m_j}$ (see [20] or [26]). In particular, N_λ has one Coxeter generator, say r_i , for each i such that $\lambda_i = \lambda_{i+1}$. The generator r_i acts on the set $\{v_1, v_2, \dots, v_n\}$ by interchanging $v_{\tau_{i-1}+j}$ and v_{τ_i+j} for $1 \leq j \leq \lambda_i$, and fixing v_k for $k \leq \tau_{i-1}$ and $k > \tau_{i+1}$.

It is well known and easy to check ([26], [23]) that $N_\lambda \subseteq Z_W(c_\lambda)$, and so $Z_W(c_\lambda) \cong Z_\lambda \rtimes N_\lambda$.

Lemma 6.1. *The subgroup N_λ of $Z_W(c_\lambda)$ stabilizes the characters φ_λ and $\epsilon\varphi_\lambda$ of Z_λ and centralizes the idempotents f_λ^+ and f_λ^- . In particular, φ_λ extends to a character, also denoted by φ_λ , of $Z_W(c_\lambda)$, with $\varphi_\lambda(nz) = \varphi_\lambda(z)$ for n in N_λ and z in Z_λ .*

Proof. Suppose that i is such that $\lambda_i = \lambda_{i+1}$ and consider the generator r_i of N_λ . Then r_i is an involution and it follows from the description of the action of r_i on the basis $\{v_1, \dots, v_n\}$ of V that

$$r_i g_{\lambda_j} r_i^{-1} = r_i g_{\lambda_j} r_i = \begin{cases} g_{\lambda_{i+1}}, & j = i, \\ g_{\lambda_i}, & j = i + 1, \\ g_{\lambda_j}, & j \neq i, i + 1. \end{cases}$$

Since $\varphi_\lambda(g_{\lambda_i}) = \varphi_\lambda(g_{\lambda_{i+1}})$, it follows that r_i stabilizes φ_λ and $\epsilon\varphi_\lambda$.

The group N_λ is generated by $\{r_i \mid \lambda_i = \lambda_{i+1}\}$, and so N_λ stabilizes the characters φ_λ and $\epsilon\varphi_\lambda$ of Z_λ . Moreover, N_λ acts on $\{g_{\lambda_1}, \dots, g_{\lambda_p}\}$ by conjugation

as a group of permutations. Thus, it follows from the definition of $f_{\lambda_i}^+$ and $f_{\lambda_i}^-$ that conjugation by N_λ permutes $\{f_{\lambda_1}^+, \dots, f_{\lambda_p}^+\}$ and $\{f_{\lambda_1}^-, \dots, f_{\lambda_p}^-\}$. Since the $f_{\lambda_i}^+$'s pairwise commute and the $f_{\lambda_i}^-$'s pairwise commute, we see that N_λ centralizes both $f_{\lambda_1}^+ \cdots f_{\lambda_p}^+ = f_\lambda^+$ and $f_{\lambda_1}^- \cdots f_{\lambda_p}^- = f_\lambda^-$. $\square$

Set $\alpha_\lambda = \alpha_{X_\lambda}$. Then α_λ is a character of $N_W(W_\lambda)$ and $\alpha_\lambda(r_i) = -1$. Note that this notation is not consistent with that of Lehrer and Solomon; our character α_λ corresponds to the character $\alpha_\lambda \epsilon$ in [26] as $\epsilon(r_i) = (-1)^{\lambda_i}$.

Theorem 6.2. *Suppose that λ is a partition of n . Then the $N_W(W_\lambda)$ -modules $e_{I_\lambda} \mathbb{C}W_\lambda$ and A_{X_λ} , and the character φ_λ of $Z_W(c_\lambda)$, are related by*

- (a) *the character of the right $N_W(W_\lambda)$ -module $e_{I_\lambda} \mathbb{C}W_\lambda$ is $\text{Ind}_{Z_W(c_\lambda)}^{N_W(W_\lambda)}(\varphi_\lambda)$ and*
- (b) *the character of the left $N_W(W_\lambda)$ -module A_{X_λ} is $\epsilon \alpha_\lambda \text{Ind}_{Z_W(c_\lambda)}^{N_W(W_\lambda)}(\varphi_\lambda)$.*

Proof. Statement (b) has been proven by Lehrer and Solomon [26, Theorem 4.4]. Their argument may be rephrased as follows. Extending the definition of the element a_n in A_n when $\lambda = (n)$, Lehrer and Solomon define an element a_λ in A_{X_λ} on which f_λ^- acts invertibly. Then:

- (i) $Z_W(c_\lambda)$ acts on the line $\mathbb{C}f_\lambda^- a_\lambda$ in A_{X_λ} via the character $\epsilon_\lambda \alpha_\lambda \varphi_\lambda$.
- (ii) $A_{X_\lambda} = \mathbb{C}N_W(W_\lambda)f_\lambda^- a_\lambda$.
- (iii) The multiplication map $\mathbb{C}N_W(W_\lambda) \otimes_{\mathbb{C}Z_W(c_\lambda)} \mathbb{C}f_\lambda^- a_\lambda \rightarrow A_{X_\lambda}$ is an isomorphism.

Therefore, $A_{X_\lambda} \cong \text{Ind}_{Z_W(c_\lambda)}^{N_W(W_\lambda)}(\mathbb{C}f_\lambda^- a_\lambda)$, and hence the character of A_{X_λ} is indeed $\text{Ind}_{Z_W(c_\lambda)}^{N_W(W_\lambda)}(\epsilon_\lambda \alpha_\lambda \varphi_\lambda)$.

To prove (a) we first note that by Proposition 4.4, $e_{I_\lambda} \mathbb{C}W_\lambda$ and $e_{I_\lambda}^\lambda \mathbb{C}W_\lambda$ are isomorphic right $N_W(W_\lambda)$ -modules, and so it suffices to prove that $e_{I_\lambda}^\lambda \mathbb{C}W_\lambda$ affords the character $\text{Ind}_{Z_W(c_\lambda)}^{N_W(W_\lambda)}(\varphi_\lambda)$. We argue as for A_{X_λ} with $e_{I_\lambda}^\lambda$ in place of a_λ .

For the rest of this proof we fix a partition $\lambda = (\lambda_1, \dots, \lambda_p)$ of n . To simplify the notation, set $I = I_\lambda$ and $e = e_I^\lambda$. It suffices to show that the line $\mathbb{C}ef_\lambda^+$ in the right $N_W(W_\lambda)$ -module $e\mathbb{C}W_\lambda$ satisfies properties analogous to (i), (ii), and (iii) above.

- (i') $Z_W(c_\lambda)$ acts on the line $\mathbb{C}ef_\lambda^+$ via the character φ_λ : We have seen in Lemma 3.2 that N_λ centralizes e and in Lemma 6.1 that N_λ centralizes f_λ^+ . Thus, N_λ centralizes ef_λ^+ and $Z_W(c_\lambda) = N_\lambda Z_\lambda$ acts on the line $\mathbb{C}ef_\lambda^+$ via the character φ_λ if $ef_\lambda^+ \neq 0$.

Let $\tau: 2^I \rightarrow \mathbb{R}_{>0}$ be the function that takes the constant value 1. We have $I = \coprod_{j=1}^p I_j$ where $W_{\lambda_j} = \langle I_j \rangle$, and so $e = e_I^\tau = e_{I_1}^\tau \cdots e_{I_p}^\tau$ by Proposition 3.6. Therefore,

$$ef_\lambda^+ = (e_{I_1}^\tau \cdots e_{I_p}^\tau)(f_{\lambda_1}^+ \cdots f_{\lambda_p}^+) = (e_{I_1}^\tau f_{I_1}^+) \cdots (e_{I_p}^\tau f_{I_p}^+).$$

For $1 \leq j \leq p$, the idempotent $e_{I_j}^\tau$ in $\mathbb{C}W_{\lambda_j}$ is defined by using the partition (λ_j) of λ_j as in §5, and so the idempotent $f_{\lambda_j}^+$ acts as a unit on $e_{I_j}^\tau$ by Lemma 5.5. Therefore, f_λ^+ acts invertibly by right multiplication on e , and so $ef_\lambda^+ \neq 0$.

- (ii') $e\mathbb{C}W_\lambda = ef_\lambda^+ \mathbb{C}N_W(W_\lambda)$: Because f_λ^+ acts invertibly on e and $N_W(W_\lambda) = N_\lambda Z_\lambda W_\lambda$, we have

$$e\mathbb{C}W_\lambda = ef_\lambda^+ \mathbb{C}W_\lambda = ef_\lambda^+ \mathbb{C}N_\lambda Z_\lambda W_\lambda = ef_\lambda^+ \mathbb{C}N_W(W_\lambda).$$

(iii') The multiplication map $\mathbb{C}ef_\lambda^+ \otimes_{\mathbb{C}Z_W(c_\lambda)} \mathbb{C}N_W(W_\lambda) \rightarrow e\mathbb{C}W_\lambda$ is an isomorphism: It follows from (ii') that the mapping is surjective. Moreover, using Corollary 4.6 we have

$$\begin{aligned} \dim e f_\lambda^+ \mathbb{C}N_W(W_\lambda) &= \dim e\mathbb{C}W_\lambda \\ &= |W_\lambda : Z_\lambda| \\ &= |N_W(W_\lambda) : Z_W(c_\lambda)| \\ &= \dim \mathbb{C}ef_\lambda^+ \otimes_{\mathbb{C}Z_W(c_\lambda)} \mathbb{C}N_W(W_\lambda), \end{aligned}$$

and so the mapping is an isomorphism.

This completes the proof of the theorem. $\square$

The proof of Conjecture 2.1 for symmetric groups now follows from Proposition 4.7, Theorem 6.2, and transitivity of induction.

Theorem 6.3. *For each partition λ of n there is a linear character φ_λ of $Z_W(c_\lambda)$ such that*

- (a) *the character of E_λ is $\text{Ind}_{Z_W(c_\lambda)}^W(\varphi_\lambda)$ and*
- (b) *the character of A_λ is $\text{Ind}_{Z_W(c_\lambda)}^W(\epsilon_\lambda \alpha_\lambda \varphi_\lambda)$, where ϵ_λ denotes the restriction of ϵ to $Z_W(c_\lambda)$.*

In particular,

$$H^p(M_W) \cong \bigoplus_{\lambda \vdash n, \text{rk}(c_\lambda)=p} \text{Ind}_{Z_W(c_\lambda)}^W(\epsilon_\lambda \alpha_\lambda \varphi_\lambda)$$

for $0 \leq p \leq n-1$, and

$$\mathbb{C}W \cong \bigoplus_{\lambda \vdash n} \text{Ind}_{Z_W(c_\lambda)}^W(\varphi_\lambda) \quad \text{and} \quad A \cong \bigoplus_{\lambda \vdash n} \text{Ind}_{Z_W(c_\lambda)}^W(\epsilon_\lambda \alpha_\lambda \varphi_\lambda).$$

7. PARABOLIC SUBGROUPS OF TYPE A

In this section we return to the case when W is an arbitrary finite Coxeter group and prove a relative version of Theorem 6.3.

Suppose that λ is in Λ , c is in W with $\text{sh}(c) = \lambda$, and that all the irreducible components of W_c are of type A. Without loss of generality we may assume that $W_c = W_L$ is a standard parabolic subgroup. Suppose that $L = \coprod_{i=1}^p L_i$, where $|L_i| = l_i$ and each L_i is of type A_{l_i} . Define $n_i = l_i + 1$. Then

$$W_L \cong W_{L_1} \times \cdots \times W_{L_p} \cong S_{n_1} \times \cdots \times S_{n_p}.$$

We assume that $l_1 \geq \cdots \geq l_p$, $L_i = \{s_{i,1}, \dots, s_{i,l_i}\}$, and $\Delta_{L_i} = \{\alpha_{i,1}, \dots, \alpha_{i,l_i}\}$, where the labeling is such that $s_{i,j}$ and $s_{i,k}$ commute if $|j-k| > 1$.

Define $c_i = s_{i,l_i} \cdots s_{i,2} s_{i,1}$ in W_{L_i} and $c = c_1 \cdots c_p$. Then c is a cuspidal element in W_L and the set of cuspidal elements in W_L is precisely the W_L -conjugacy class of c . For $1 \leq i \leq p$ define φ_i to be the character of $\langle c_i \rangle$ with $\varphi_i(c_i^{-1}) = \zeta_{n_i}$, where ζ_{n_i} is a fixed primitive n_i th root of unity, and set $\varphi_c = \varphi_1 \otimes \cdots \otimes \varphi_p$. Then φ_c is a character of $Z_{W_L}(c) \cong \langle c_1 \rangle \times \cdots \times \langle c_p \rangle$.

The rest of this section is devoted to the proof of the following theorem.

Theorem 7.1. *The character φ_c of $Z_{W_L}(c)$ extends to a character $\tilde{\varphi}_c$ of $Z_W(c)$ such that*

$$E_\lambda \cong \text{Ind}_{Z_W(c)}^W(\tilde{\varphi}_c) \quad \text{and} \quad A_\lambda \cong \text{Ind}_{Z_W(c)}^W(\epsilon_c \alpha_c \tilde{\varphi}_c).$$

The proof follows the same outline as in §6: We find lines in $e_L^L \mathbb{C}W_L$ and A_{X_L} such that the analogs of statements (i), (ii), and (iii) and (i'), (ii'), and (iii') in the proof of Theorem 6.2 hold, and so

$$e_L^L \mathbb{C}W_L \cong \text{Ind}_{Z_W(c)}^{N_W(W_L)}(\tilde{\varphi}_c) \quad \text{and} \quad A_{X_L} \cong \text{Ind}_{Z_W(c)}^{N_W(W_L)}(\epsilon \alpha_c \tilde{\varphi}_c).$$

We then apply $\text{Ind}_{N_W(W_L)}^W$ to both sides of both equations, and the theorem follows from Proposition 4.7 by transitivity of induction. The argument in this section is complicated by the fact that the subgroup N_L is not necessarily contained in $Z_W(c)$.

In case W is a symmetric group, it was shown in Lemma 6.1 that $\tilde{\varphi}_c$ is the trivial extension of φ_c . In the general case, this is no longer so. Formulas for $\tilde{\varphi}_c$ are given in the proof of Lemma 7.3 below. Notice that c is an involution if and only if $l_1 = 1$ for $1 \leq i \leq p$ and that in this case φ_c is the sign character of $Z_{W_L}(c)$ and $\tilde{\varphi}_c$ is the trivial extension of φ_c to $Z_W(c)$.

Although N_L is not necessarily contained in $Z_W(c)$, Konvalinka, Pfeiffer, and Röver [23] have shown that $Z_{W_L}(c)$ does have a complement, N_c , in $Z_W(c)$, and N_c is also a complement to W_L in $N_W(W_L)$.

By [20], the group N_L is generated by $\{r_i, g_j \mid 1 \leq i \leq p-1, 1 \leq j \leq p\}$, where r_i and g_i act on W_L as follows:

- If $n_i \neq n_{i+1}$, then $r_i = 1$. If $n_i = n_{i+1}$, then r_i acts on L by

$$r_i s_{j,k} r_i^{-1} = \begin{cases} s_{i+1,k} & \text{if } j = i, \\ s_{i,k} & \text{if } j = i+1, \\ s_{j,k} & \text{otherwise.} \end{cases}$$

In particular, r_i is an involution,

$$r_i c_j r_i = \begin{cases} c_{i+1} & \text{if } j = i, \\ c_i & \text{if } j = i+1, \\ c_j & \text{otherwise,} \end{cases}$$

and r_i is in $Z_W(c)$.

- Either $g_i = 1$ or g_i acts on L by

$$g_i s_{j,k} g_i^{-1} = \begin{cases} s_{i, n_i - k} & \text{if } j = i, \\ s_{j,k} & \text{if } j \neq i. \end{cases}$$

In particular, g_i is an involution, and if $g_i \neq 1$, then

$$g_i c_j g_i = \begin{cases} c_i^{-1} & \text{if } j = i, \\ c_j & \text{if } j \neq i. \end{cases}$$

Notice that if W is a symmetric group, then $g_i = 1$ for $1 \leq i \leq p$.

For $1 \leq i \leq p$, let w_i denote the longest element in W_{L_i} and define

$$h_i = \begin{cases} 1 & \text{if } g_i = 1, \\ g_i w_i & \text{if } g_i \neq 1. \end{cases}$$

Then h_i is in $Z_W(c_j)$ for $1 \leq i, j \leq p$. It is shown in [23] that

$$N_c = \langle r_i, h_j \mid 1 \leq i \leq p-1, 1 \leq j \leq p \rangle$$

is a complement to $Z_{W_L}(c)$ in $Z_W(c)$ and to W_L in $N_W(W_L)$.

As in §6 define

$$\begin{aligned} f_i^+ &= \frac{1}{n_i} \sum_{k=0}^{n_i-1} \varphi_i(c_i^k) c_i^{-k}, & f_i^- &= \frac{1}{n_i} \sum_{k=0}^{n_i-1} \epsilon(c_i^k) \varphi_i(c_i^k) c_i^{-k}, \\ f_L^+ &= f_1^+ f_2^+ \cdots f_p^+, & f_L^- &= f_1^- f_2^- \cdots f_p^-. \end{aligned}$$

Then the lines $\mathbb{C}f_L^+$ and $\mathbb{C}f_L^-$ in $\mathbb{C}W$ are stable under left and right multiplication by $Z_{W_L}(c)$ and afford the characters φ_c and $\epsilon\varphi_c$ of $Z_{W_L}(c)$, respectively. Because h_i centralizes c_j for $1 \leq i, j \leq p$, the proof of the second statement in Lemma 6.1 applies word-for-word to N_c and proves the next lemma.

Lemma 7.2. *The subgroup N_c of $Z_W(c)$ centralizes the idempotents f_L^+ and f_L^- in $\mathbb{C}Z_{W_L}(c)$.*

For $1 \leq i \leq p$, define $a_i = a_{s_{i,1}} \cdots a_{s_{i,l_i}}$. Set $a_L = a_1 \cdots a_p$. Then a_L is in A_{X_L} . The next lemma is the analog of statements (i) and (i') in the proof of Theorem 6.2.

Lemma 7.3. *The lines $\mathbb{C}e_L^L f_L^+$ in $e_L^L \mathbb{C}W_L$ and $\mathbb{C}f_L^- a_L$ in A_{X_L} are non-zero and $Z_W(c)$ -stable. Let $\tilde{\varphi}_c$ be the character of $Z_W(c)$ acting by right multiplication on the line $\mathbb{C}e_L^L f_L^+$. Then $\tilde{\varphi}_c$ is an extension of φ_c , and the character of $Z_W(c)$ acting by left multiplication on the line $\mathbb{C}f_L^- a_L$ is $\epsilon\alpha_c \tilde{\varphi}_c$.*

Proof. The argument in statement (i') in §6 applies verbatim to f_L^+ and shows that f_L^+ acts invertibly by right multiplication on e_L^L , and so $e_L^L f_L^+ \neq 0$. Similarly, the argument in Lehrer and Solomon [26, §3] shows that f_i^- acts as a unit on a_i for $1 \leq i \leq p$. It follows that f_L^- acts invertibly by left multiplication on a_L , and so $f_L^- a_L \neq 0$. Therefore, $Z_{W_L}(c)$ acts on $\mathbb{C}e_L^L f_L^+$ and $\mathbb{C}f_L^- a_L$ by φ_c and $\epsilon\varphi_c$, respectively, and $\tilde{\varphi}_c$ extends φ_c .

If $n_i = n_{i+1}$, then as in §6 we have

$$(7.1) \quad (e_L^L f_L^+) \cdot r_i = (e_L^L \cdot r_i) f_L^+ = (r_i e_L^L r_i) f_L^+ = e_L^L f_L^+$$

and

$$(7.2) \quad \begin{aligned} r_i \cdot f_L^- a_L &= f_L^- (r_i \cdot a_L) = f_L^- \cdot a_1 \cdots a_{i-1} a_{i+1} a_{i+2} \cdots a_p \\ &= (-1)^{l_i^2} f_L^- \cdot a_L = (-1)^{l_i} f_L^- \cdot a_L. \end{aligned}$$

If $h_i \neq 1$, then

$$(7.3) \quad e_L^L f_L^+ \cdot h_i = (e_L^L \cdot g_i w_i) f_L^+ = (g_i e_L^L g_i w_i) f_L^+ = (e_L^L w_i) f_L^+ = (-1)^{l_i} e_L^L f_L^+$$

(the last equality follows from Lemma 3.3), and

$$(7.4) \quad h_i \cdot f_L^- a_L = f_L^- (h_i \cdot a_L) = f_L^- a_L$$

since h_i centralizes L . It follows that $Z_W(c)$ acts on the lines $\mathbb{C}e_L^L f_L^+$ and $\mathbb{C}f_L^- a_L$. Moreover, from (7.1) and (7.3) we see that

$$\tilde{\varphi}_c(r_i) = 1 \quad \text{and} \quad \tilde{\varphi}_c(h_i) = (-1)^{l_i}.$$

To complete the proof we need to show that $Z_W(c)$ acts on the line $\mathbb{C}f_L^-a_L$ as $\epsilon\alpha_c\tilde{\varphi}_c$.

For w in $Z_{W_L}(c)$ we have $\alpha_c(w) = 1$ and $\tilde{\varphi}_c(w) = \varphi_c(w)$. Hence

$$w \cdot f_L^-a_L = \epsilon(w)\varphi_c(w) \cdot f_L^-a_L = (\epsilon\alpha_c\tilde{\varphi}_c)(w) f_L^-a_L.$$

For n in N_L it is shown in [13, Lemma 2.1] that $\epsilon(n)\alpha_c(n)$ is the sign of the permutation of L induced by conjugation by n .

Suppose next that $1 \leq i \leq p-1$ and $n_i = n_{i+1}$. Then the sign of the permutation of L induced by conjugation by r_i is $(-1)^{l_i}$, and so $\epsilon(r_i)\alpha_c(r_i) = (-1)^{l_i}$. Therefore, using (7.2) we see that

$$r_i \cdot f_L^-a_L = (-1)^{l_i} f_L^-a_L = (\epsilon\alpha_c\tilde{\varphi}_c)(r_i) f_L^-a_L.$$

Finally, suppose that $1 \leq i \leq p$ and that $g_i \neq 1$. Then the sign of the permutation of L induced by conjugation by g_i is 1 if $l_i \equiv 0, 1 \pmod{4}$ and -1 if $l_i \equiv 2, 3 \pmod{4}$. Thus, $\epsilon(g_i)\alpha_c(g_i) = 1$ if $l_i \equiv 0, 1 \pmod{4}$ and $\epsilon(g_i)\alpha_c(g_i) = -1$ if $l_i \equiv 2, 3 \pmod{4}$. Since $\ell(w_i) = \binom{l_i+1}{2}$, it follows that $\epsilon(w_i) = 1$ if $l_i \equiv 0, 3 \pmod{4}$ and $\epsilon(w_i) = -1$ if $l_i \equiv 1, 2 \pmod{4}$. Hence $\epsilon(g_i)\alpha_c(g_i)\epsilon(w_i) = (-1)^{l_i}$. Because $\alpha_c(w_i) = 1$ we conclude that

$$\epsilon(h_i)\alpha_c(h_i) = \epsilon(g_i)\epsilon(w_i)\alpha_c(g_i)\alpha_c(w_i) = (-1)^{l_i} = \tilde{\varphi}_c(h_i).$$

Therefore, using (7.4) we see that

$$h_i \cdot f_L^-a_L = f_L^-a_L = (\epsilon\alpha_c\tilde{\varphi}_c)(h_i) f_L^-a_L.$$

This completes the proof of the lemma. $\square$

Because f_λ^+ acts invertibly on e_L^L , N_c acts on the line $\mathbb{C}e_L^L f_L^+$ by scalars, and $N_W(W_L) = N_c W_L$, we see that

$$(7.5) \quad e_L^L \mathbb{C}W_L = e_L^L f_L^+ \mathbb{C}W_L = e_L^L f_L^+ \mathbb{C}N_c W_L = e_L^L f_L^+ \mathbb{C}N_W(W_L).$$

Lehrer and Solomon [26, Proposition 4.4(ii)] have shown that $A_{X_L} = \mathbb{C}W_L a_L$. Thus, because f_λ^- acts invertibly on a_L , N_c acts on the line $\mathbb{C}f_L^-a_L$ by scalars, and $N_W(W_L) = W_L N_c$, we see that

$$(7.6) \quad A_{X_L} = \mathbb{C}W_L a_L = \mathbb{C}W_L f_L^-a_L = \mathbb{C}W_L N_c f_L^-a_L = \mathbb{C}N_W(W_L) f_L^-a_L.$$

Equations (7.5) and (7.6) are the analogs of statements (ii) and (ii') in the proof of Theorem 6.2. The dimension computation in the proof of statement (iii') now applies to show that the analogs of statements (iii) and (iii') both hold in the present situation: The multiplication maps

$$(7.7) \quad \mathbb{C}e_L^L f_L^+ \otimes_{\mathbb{C}Z_W(c)} \mathbb{C}N_W(W_L) \rightarrow e_L^L \mathbb{C}W_L \quad \text{and} \quad \mathbb{C}N_W(W_L) \otimes_{\mathbb{C}Z_W(c)} \mathbb{C}f_L^-a_L \rightarrow A_{X_L}$$

are isomorphisms of $\mathbb{C}N_W(W_L)$ -modules.

It follows from (7.7) and Lemma 7.3 that

$$e_L^L \mathbb{C}W_L \cong \text{Ind}_{Z_W(c)}^{N_W(W_L)}(\tilde{\varphi}_c) \quad \text{and} \quad A_{X_L} \cong \text{Ind}_{Z_W(c)}^{N_W(W_L)}(\epsilon\alpha_c\tilde{\varphi}_c).$$

Apply $\text{Ind}_{N_W(W_L)}^W$ to both sides of these last two equations and use transitivity of induction to get

$$\text{Ind}_{N_W(W_L)}^W (e_L^I \mathbb{C}W_L) \cong \text{Ind}_{Z_W(c)}^W (\tilde{\varphi}_c)$$

and

$$\text{Ind}_{N_W(W_L)}^W (A_{X_L}) \cong \text{Ind}_{Z_W(c)}^W (\epsilon \alpha_c \tilde{\varphi}_c).$$

By Proposition 4.7 and Corollary 4.8 we have

$$E_\lambda \cong \text{Ind}_{Z_W(c)}^W (\tilde{\varphi}_c) \quad \text{and} \quad A_\lambda \cong \text{Ind}_{Z_W(c)}^W (\epsilon \alpha_c \tilde{\varphi}_c).$$

This completes the proof of the theorem.

ACKNOWLEDGEMENTS

The authors would like to acknowledge support from the DFG-priority programme SPP1489 “Algorithmic and Experimental Methods in Algebra, Geometry, and Number Theory”. Part of the research for this paper was carried out while the authors were staying at the Mathematical Research Institute Oberwolfach supported by the “Research in Pairs” programme. The second author wishes to acknowledge support from Science Foundation Ireland.

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