TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 366, Number 11, November 2014, Pages 5821–5851
S 0002-9947(2014)06042-X
Article electronically published on July 29, 2014

THE CLASSIFICATION OF ORTHOGONALLY RIGID G_2-LOCAL SYSTEMS AND RELATED DIFFERENTIAL OPERATORS

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Abstract. We prove a criterion for a general self-adjoint differential operator of rank 7 to have its monodromy group inside the exceptional algebraic group $G_2(\mathbb{C})$. We then classify orthogonally rigid local systems of rank 7 on the punctured projective line whose monodromy is dense in the exceptional algebraic group $G_2(\mathbb{C})$. We obtain differential operators corresponding to these local systems under Riemann-Hilbert correspondence.

1. Introduction

It is well known that the exceptional simple algebraic group G_2 can be seen as a subgroup of $\mathrm{GL}(V)$, where V is a 7-dimensional vector space with basis $x_0, x_1, y_1, x_2, y_2, x_3, y_3$, which stabilizes the Dickson alternating trilinear form

$$x_0x_1y_1 + x_0x_2y_2 + x_0x_3y_3 + x_1x_2x_3 + y_1y_2y_3$$

and the quadratic form $-2x_0^2 + x_1y_1 + x_2y_2 + x_3y_3$; cf. [1]. Especially, the group $G_2(\mathbb{C})$ can be seen as a subgroup of the orthogonal group $O_7(\mathbb{C})$. *Orthogonal rigidity* for an irreducible orthogonally self-dual complex rank-n local system $\mathcal{L}$ on the punctured complex projective line $\mathbb{P}^1 \setminus \{x_1, \dots, x_{r+1}\}$ means that the following dimension formula holds:

$$\sum_{i=1}^{r+1} \mathrm{codim}(C_{O_n}(g_i)) = 2\dim(O_n), \tag{1.0.1}$$

where $C_{O_n}(g_i)$ denotes the centralizer of the local monodromy generator g_i in the orthogonal group O_n. It is the aim of this article to classify the orthogonally rigid local systems $\mathcal{L}$ of rank 7 whose monodromy group is Zariski dense in $G_2(\mathbb{C})$. The dimension formula (1.0.1) is equivalent to the vanishing of the parabolic cohomology of $\pi_1(\mathbb{P}^1 \setminus \{x_1, \dots, x_{r+1}\})$ with values in the Lie algebra of O_n (acting adjointly via the monodromy representation of $\mathcal{L}$) and is hence closely related to the dimension of the tangent space of the component of the space of representations of $\pi_1(\mathbb{P}^1 \setminus \{x_1, \dots, x_{r+1}\})$ with given local monodromy data; cf. [21]. The dimension formula is also a necessary condition for the condition that there exist only finitely many equivalence classes of irreducible orthogonally self-dual local systems $\mathcal{L}$ with given local monodromy data [20]. Hence, for such local systems, the notion of orthogonal rigidity is weaker than the notion of (physical) rigidity used in [13] (which

Received by the editors September 27, 2012 and, in revised form, November 13, 2012 and November 27, 2012.

2010 *Mathematics Subject Classification.* Primary 32S40, 20G41.

Key words and phrases. Ordinary differential equation, exceptional algebraic group, local system, middle convolution.

can be seen as rigidity relative to the larger group GL_n) but still strong enough to impose a lot of structure on $\mathcal{L}$.

By the work of N. Katz on the middle convolution functor MC_χ, all rigid irreducible local systems $\mathcal{L}$ on the punctured line can be constructed by applying iteratively MC_χ and tensor products with rank-1-sheaves to a rank-1-sheaf. For orthogonally rigid local systems with G_2-monodromy we prove that there is a similar method of construction (cf. Theorem 6.1 below):

Theorem 1.1. *Let $\mathcal{L}$ be an orthogonally rigid $\mathbb{C}$-local system on a punctured projective line $\mathbb{P}^1 \setminus \{x_1, \dots, x_r\}$ of rank 7 whose monodromy group is dense in the exceptional simple group G_2. If $\mathcal{L}$ has nontrivial local monodromy at $x_1, \dots, x_r$, then $r = 3, 4$ and $\mathcal{L}$ can be constructed by applying iteratively a sequence of the following operations to a rank-1-system:*

- *Middle convolutions MC_χ, with varying χ.*
- *Tensor products with rank-1-local systems.*
- *Tensor operations like symmetric or alternating products.*
- *Pullbacks along rational functions.*

Especially, each such local system which has quasi-unipotent monodromy is motivic, i.e., it arises from the variation of periods of a family of varieties over the punctured projective line.

A list of the occurring cases together with the local monodromy data is given in Theorem 6.1. Rigid local systems on the punctured line with G_2-monodromy were classified in [8], and our classification contains these as special cases.

We remark that the verification that the monodromy group is inside the group $G_2(\mathbb{C})$ cannot be decided by looking at the local monodromy data alone. To prove this, we make use of recent results of Bogner and Reiter in [4] on the interpretation of MC_χ at the level of differential operators, related to the Hadamard product. Using these results, the differential operators which belong to the local systems of Theorem 1.1 under Riemann-Hilbert correspondence can easily be determined. Using a general criterion for a self-adjoint differential operator of rank 7 to have its monodromy contained in $G_2(\mathbb{C})$ (given by Theorem 2.1), it can then be proven that in each listed case the differential operators under consideration have the property that they stabilize a nontrivial alternating trilinear form, implying that the monodromy is contained in the group G_2.

Motivated by the results of Theorem 1.1 one may ask the question of whether any irreducible orthogonally rigid local system can be obtained by a sequence of tensor operations, middle convolutions MC_χ, and rational pullbacks applied to a local system of rank one.

2. Differential operators with differential Galois group in G_2

Throughout the article, let $\partial = \frac{d}{dx}$ and $\vartheta = x\partial$. Let $L = \sum_{i=0}^n a_i(x)\partial^i \in \mathbb{C}(x)[\partial]$ be a differential operator. Recall that the *adjoint* L^* of L is defined as $L^* = \sum_{i=0}^n (-\partial)^i a_i(x)$ and that L is called *self-adjoint* if $L = (-1)^n L^*$. If L is self-adjoint, then the differential Galois group of L, and hence the monodromy of L, is contained in the orthogonal group $O_n(\mathbb{C})$ if n is odd ([12]). Recall that by [16, Table 5], the exceptional algebraic group $G_2(\mathbb{C})$ can be seen as a subgroup of $O_7(\mathbb{C})$ stabilizing a nontrivial trilinear form.

Theorem 2.1. *Let $L = \sum_{i=0}^{7} a_i(x)\partial^i \in \mathbb{C}(x)[\partial]$ be monic, i.e. $a_7(x) = 1$. If L is self-adjoint, then*

$$a_6(x) = 0, \tag{2.1.1}$$

$$a_4(x) = \frac{5}{2}\frac{d}{dx}a_5(x), \tag{2.1.2}$$

$$a_2(x) = -\frac{5}{2}\frac{d^3}{dx^3}a_5(x) + \frac{3}{2}\frac{d}{dx}a_3(x), \tag{2.1.3}$$

$$a_0(x) = \frac{1}{2}\frac{d}{dx}a_1(x) - \frac{1}{4}\frac{d^3}{dx^3}a_3(x) + \frac{1}{2}\frac{d^5}{dx^5}a_5(x). \tag{2.1.4}$$

Moreover, if the above conditions hold and if further

$$a_3(x) = 3\frac{d^2}{dx^2}a_5(x) + \frac{1}{4}a_5(x)^2, \tag{2.1.5}$$

then the differential Galois group and the monodromy group of L is contained in $G_2(\mathbb{C})$.

Proof. The equations in (2.1.1)–(2.1.4) are an immediate consequence of the self-adjointness of L. Let $M := \mathbb{C}(x)[\partial]/(\mathbb{C}(x)[\partial]L)$ be the corresponding D-module with basis $e_0, \ldots, e_6$, where ∂ acts by the rule

$$\partial e_i = e_{i+1}, \quad i = 0, \ldots, 5, \qquad \partial e_6 = -\sum_{i=0}^{6} a_i(x)e_i.$$

By the above discussion on G_2 as a subgroup of O_7, it suffices to show that the D-module $\Lambda^3 M$ has a trivial 1-dimensional D-submodule. A completely elementary computation shows that under the assumption of (2.1.5), the following element in $\Lambda^3 M$ is annihilated by ∂ and hence generates the required trivial rank-one D-submodule of $\Lambda^3 M$:

$$\begin{aligned}
&e_0 \wedge e_4 \wedge e_5 + e_2 \wedge e_3 \wedge e_4 + 2e_1 \wedge e_2 \wedge e_6 - e_1 \wedge e_3 \wedge e_5 - e_0 \wedge e_3 \wedge e_6 \\
&+\frac{1}{2}a_5(x)(-e_0 \wedge e_1 \wedge e_6 + e_0 \wedge e_2 \wedge e_5 - 3e_0 \wedge e_3 \wedge e_4 + 3e_1 \wedge e_2 \wedge e_4) \\
&+\frac{1}{2}a_5'(x)(e_0 \wedge e_1 \wedge e_5 - 2e_0 \wedge e_2 \wedge e_4 + 9e_1 \wedge e_2 \wedge e_3) \\
&-(\frac{1}{2}a_5''(x) + \frac{1}{2}a_5(x)^2)e_0 \wedge e_1 \wedge e_4 + (\frac{3}{2}a_5''(x) + \frac{1}{2}a_5(x)^2)e_0 \wedge e_2 \wedge e_3 \\
&+(\frac{1}{2}a_5'''(x) - \frac{1}{4}a_5'(x)a_5(x))e_0 \wedge e_1 \wedge e_3 \\
&+(a_1(x) - \frac{1}{2}a_5(x)a_3(x) - \frac{1}{2}a_5''''(x) + \frac{1}{4}a_5''(x)a_5(x) + \frac{1}{4}a_5'(x)a_5'(x))e_0 \wedge e_1 \wedge e_2 .
\end{aligned}$$

□

Remark 2.2. We remark that a special case of the previous theorem is proved by Katz [12, Theorem 2.10.6], where the differential operator $L = \partial^7 - a_1(x)\partial - \frac{1}{2}a_1'(x)$ is considered.

3. Preliminaries on convolution operations

Recall the construction of the middle convolution from [13]: Consider the addition map

$$\pi : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1, (x, y) \mapsto x + y.$$

Let $\mathcal{L}$ be a complex valued local system on $\mathbb{A}^1 \setminus \{x_1, \ldots, x_r\}$ and let $L = j_*\mathcal{L}[1]$, viewed as a perverse sheaf on $\mathbb{A}^1$ (j denoting the inclusion of $\mathbb{A}^1 \setminus \{x_1, \ldots, x_r\}$ into $\mathbb{A}^1$). Let $\mathcal{L}_\chi$ be a local system on $\mathbb{G}_m$, defined by a nontrivial character $\chi : \pi_1(\mathbb{G}_m) \to \mathbb{C}^\times$. We call $\mathcal{L}_\chi$ a *Kummer sheaf*. Let $L_\chi = (k_*\mathcal{L}_\chi)[1]$, where k denotes the natural inclusion of $\mathbb{G}_m$ to $\mathbb{A}^1$. Sometimes we need the following variant: using the isomorphism $\mathbb{A}^1 \setminus \{y\} \to \mathbb{G}_m, x \mapsto x - y$, we can view $\mathcal{L}_\chi$ as a local system on $\mathbb{A}^1 \setminus \{y\}$. This local system is then denoted $\mathcal{L}_{\chi(x-y)}$.

Following Katz [13], one can define the *middle convolution of $\mathcal{L}$ with the Kummer sheaf $\mathcal{L}_\chi$* as

$$\mathrm{MC}_\chi(\mathcal{L}) := \mathrm{im}\,(R\pi_!(L \boxtimes L_\chi) \to R\pi_*(L \boxtimes L_\chi))\,[-1]|_{\mathbb{A}^1\setminus\{x_1,\ldots,x_r\}}. \tag{3.0.1}$$

Remark 3.1. Since we restrict ourselves to $\mathbb{A}^1 \setminus \{x_1, \ldots, x_r\}$, the 0-th and the 2-th higher direct image vanish by the nontriviality of $\mathcal{L}_\chi$, so (3.0.1) is equivalent to

$$\mathrm{MC}_\chi(\mathcal{L}) = \mathrm{im}\left(R^1\pi_!(j_*\mathcal{L} \boxtimes k_*\mathcal{L}_\chi) \to R^1\pi_*(j_*\mathcal{L} \boxtimes k_*L_\chi)\right)|_{\mathbb{A}^1\setminus\{x_1,\ldots,x_r\}}. \tag{3.1.1}$$

Hence, the middle convoluted local system $\mathrm{MC}_\chi(\mathcal{L})$ can be seen as a variation of the parabolic cohomology groups $H^1(\mathbb{P}^1, i_*(\mathcal{L} \otimes \mathcal{L}_{\chi(x-y)}))$ over $\mathbb{A}^1_y \setminus \{x_1, \ldots, x_r\}$, where i is the inclusion of $\mathbb{A}^1 \setminus \{x_1, \ldots, x_r, y\}$ into $\mathbb{P}^1$ and the local systems $\mathcal{L}$ and $\mathcal{L}_{\chi(x-y)}$ are viewed as local systems on $\mathbb{A}^1 \setminus \{x_1, \ldots, x_r, y\}$ via restriction (cf. [13] and [9]).

In the usual way we fix a set of generators $\gamma_1, \ldots, \gamma_{r+1}$ of $\pi_1(\mathbb{A}^1 \setminus \mathbf{x})$, where γ_i ($i = 1, \ldots, r$) is a simple loop which moves counterclockwise around x_i, where γ_{r+1} moves around ∞, such that the product relation $\gamma_1 \cdots \gamma_{r+1} = 1$ holds. Hence, every local system on $\mathbb{A}^1 \setminus \mathbf{x}$ gives, via its monodromy representation

$$\rho_\mathcal{L} : \pi_1(\mathbb{A}^1 \setminus \mathbf{x}, x_0) \to \mathrm{GL}(\mathcal{L}_{x_0}) \simeq \mathrm{GL}_n(\mathbb{C}),$$

rise to its monodromy tuple $(A_1, \ldots, A_{r+1})$, where $A_i = \rho_\mathcal{L}(\gamma_i)$. The following result is a consequence of the numerology of the middle convolution (cf. [13, Cor. 3.3.6]):

Lemma 3.2. *Let $\mathcal{L}$ be an irreducible local system with monodromy tuple $\mathcal{A} = (A_1, \ldots, A_{r+1}) \in \mathrm{GL}(V)^{r+1}$, s.t. at least two $A_i, A_j, 1 \le i < j \le r$, are nontrivial. Let $\chi : \pi_1(\mathbb{G}_m) \to \mathbb{C}^\times$ be the character which sends a counterclockwise generator of $\pi_1(\mathbb{G}_m)$ to $\lambda \in \mathbb{C}^\times \setminus \{1\}$. Let $(\tilde{B}_1, \ldots, \tilde{B}_{r+1})$ be the monodromy tuple of $\mathrm{MC}_\chi(\mathcal{L})$. Then the following hold:*

(i) *The rank m of $\mathrm{MC}_\chi(\mathcal{L})$ is*

$$m = \sum_{i=1}^{r} \mathrm{rk}(A_i - 1) + \mathrm{rk}(\lambda^{-1}A_{r+1} - 1) - \mathrm{rk}(\mathcal{L}).$$

(ii) *Every Jordan block $\mathrm{J}(\alpha, l)$ occurring in the Jordan decomposition of A_i contributes a Jordan block $\mathrm{J}(\alpha\lambda, l')$ to the Jordan decomposition of $\tilde{B}_i$, where*

$$l' := \begin{cases} l, & \text{if } \alpha \neq 1, \lambda^{-1}, \\ l - 1 & \text{if } \alpha = 1, \\ l + 1, & \text{if } \alpha = \lambda^{-1}. \end{cases}$$

The only other Jordan blocks which occur in the Jordan decomposition of $\tilde{B}_i$ *are blocks of the form* $\mathrm{J}(1,1)$.

(iii) *Every Jordan block* $\mathrm{J}(\alpha^{-1},l)$ *occurring in the Jordan decomposition of* A_{r+1} *contributes a Jordan block* $\mathrm{J}(\alpha^{-1}\lambda^{-1},l')$ *to the Jordan decomposition of* $\tilde{B}_{r+1}$*, where*

$$l' := \begin{cases} l, & \text{if } \alpha \neq 1, \lambda^{-1}, \\ l+1 & \text{if } \alpha = 1, \\ l-1, & \text{if } \alpha = \lambda^{-1}. \end{cases}$$

The only other Jordan blocks which occur in the Jordan decomposition of $\tilde{B}_{r+1}$ *are blocks of the form* $\mathrm{J}(\lambda^{-1},1)$.

By the Riemann-Hilbert correspondence, each local system $\mathcal{L} \in \mathrm{LS}(\mathbb{A}^1 \setminus \mathbf{x})$ corresponds to an ordinary differential equation with regular singularities in $\mathbf{x}$. Let us first describe the tensor operations needed below (cf. [17, Chapter 2] and [4]).

Definition 3.3. (i) Let M_1, M_2 be two differential $\mathbb{C}(x)$-modules. The tensor product of M_1 and M_2 over $\mathbb{C}(x)$ is given by the $\mathbb{C}(x)$-vector space $M_1 \otimes_{\mathbb{C}(x)} M_2$ together with the derivation

$$\partial_{M_1\otimes M_2}(m_1 \otimes m_2) := \partial_{M_1}(m_1) \otimes m_2 + m_1 \otimes \partial_{M_2}(m_2).$$

(ii) Let $L_1, L_2 \in \mathbb{C}(x)[\partial]$ be two monic differential operators with corresponding differential modules

$$(M_i, \partial_{M_i}), \; M_i = \mathbb{C}(x)[\partial]/(\mathbb{C}(x)[\partial](L_i)), \; (i = 1, 2)$$

and cyclic vectors $\Omega_i \in M_i$ $(i = 1, 2)$. The tensor product $L_1 \otimes L_2 \in \mathbb{C}(z)[\partial]$ of L_1 and L_2 over $\mathbb{C}(x)$ is the minimal monic annihilating operator of $\Omega_1 \otimes \Omega_2 \in M_1 \otimes_{\mathbb{C}(x)} M_2$.

Remark 3.4. (i) By [17, Corollary 2.19], the solution space of $L_1 \otimes L_2$ in the Picard-Vessiot field $K \supset \mathbb{C}(x)$ of the operator is spanned by the set

$$\{y_1 y_2 \mid L_1(y_1) = L_2(y_2) = 0\}.$$

In particular, $L_1 \otimes L_2$ is the monic operator of minimal order, whose solution space is spanned by this set.

(ii) Symmetric and exterior powers of differential modules and differential operators are defined similarly. If $L \in \mathbb{C}(x)[\partial]$ is monic, by [17, Corollary 2.23] and [17, Corollary 2.28] $\mathrm{Sym}^2(L)$ is the monic operator of minimal degree whose solution space is spanned by the set

$$\{y_1 y_2 \mid L(y_i) = 0 \text{ for } i = 1, 2\}$$

and $\Lambda^2(L)$ the monic operator of minimal degree whose solution space is spanned by the set of Wronskians

$$\{\det \begin{pmatrix} y_1 & y_2 \\ \partial y_1 & \partial y_2 \end{pmatrix} \mid L(y_i) = 0 \text{ for } i = 1, 2\}.$$

Let $L \in \mathbb{C}(x)[\vartheta]$ be a differential operator which has only regular singularities and suppose that L is smooth on $\mathbb{A}^1 \setminus \{x_1, \ldots, x_r\}$. Let f be a solution of L, viewed as a section of the local system $\mathcal{L}$ on $\mathbb{A}^1 \setminus \{x_1, \ldots, x_r\}$ formed by the solutions of

L, and let $a \in \mathbb{Q} \setminus \mathbb{Z}$. For two simple loops γ_p, γ_q, based at $x_0 \in \mathbb{A}^1 \setminus \{x_1, \ldots, x_r\}$, moving counterclockwise around p, resp. q, we define the *Pochhammer contour*

$$[\gamma_p, \gamma_q] := \gamma_p^{-1}\gamma_q^{-1}\gamma_p\gamma_q.$$

For $y \in \mathbb{A}^1 \setminus \{x_1, \ldots, x_r\}$, the integral

$$C_a^p(f)(y) := \int_{[\gamma_p,\gamma_y]} f(x)(y-x)^a \frac{dx}{y-x} \tag{3.4.1}$$

is called the *convolution* of f and x^a with respect to the Pochhammer contour $[\gamma_p, \gamma_y]$.

Remark 3.5. If x^a is a local section of the Kummer sheaf $\mathcal{L}_\chi$, then the integral $\int_{[\gamma_p,\gamma_y]} f(x)(y-x)^a \frac{dx}{y-x}$ represents an element in

$$H^1(\mathbb{A}^1 \setminus \{x_1, \ldots, x_r, y\}, \mathcal{L} \otimes \mathcal{L}_{\chi(x-y)})$$

in the usual way; cf. [3] (where we view $\mathcal{L}$ and $\mathcal{L}_\chi$ as local systems on $\mathbb{A}^1 \setminus \{x_1, \ldots, x_r, y\}$ by restriction). Under certain conditions (made explicit in [7]), the analytic continuation of the integral (3.4.1) near the singularities is in the image of the local monodromy and therefore contained in the parabolic cohomology group $H^1(\mathbb{P}^1, k_*(\mathcal{L} \otimes \mathcal{L}_{\chi(x-y)})) \leq H^1(\mathbb{A}^1 \setminus \{x_1, \ldots, x_r, y\}, \mathcal{L} \otimes \mathcal{L}_{\chi(x-y)})$; cf. [9]. By Remark 3.1, for varying y, the integral $C_a^p(f)(y)$ can hence be viewed as a section of $\mathrm{MC}_\chi(\mathcal{L})$.

In a similar way as for $C_a^p(f)(y)$, define

$$H_a^p(f)(y) := \int_{[\gamma_p,\gamma_y]} f(x) \left(1 - \frac{y}{x}\right)^{-a} \frac{dx}{x}.$$

The integral $H_a^p(f)$ is called the *Hadamard product* of f and $(1-x)^{-a}$ with respect to the Pochhammer contour $[\gamma_p, \gamma_y]$. We have the obvious relations:

$$C_a^p(f) = (-1)^{a-1} H_{1-a}^p(x^a f), \quad H_a^p(f) = (-1)^{-a} C_{1-a}^p\left(x^{a-1} f\right). \tag{3.5.1}$$

In [4], the following is proved:

Proposition 3.6. *Let $L = \sum_{i=0}^m x^i P_i(\vartheta) \in \mathbb{C}[x, \vartheta]$ be Fuchsian, f a solution of L and $a \in \mathbb{Q} \setminus \mathbb{Z}$. Then $C_a^p(f)$ is a solution of*

$$\mathcal{C}_a(L) := \sum_{i=0}^m y^i \prod_{j=0}^{i-1} (\vartheta + i - a - j) \prod_{k=0}^{m-i-1} (\vartheta - k) P_i(\vartheta - a) \in \mathbb{C}[y, \vartheta] \tag{3.6.1}$$

for each $p \in \mathbb{P}^1$ and $H_a^p(f)$ is a solution of

$$\mathcal{H}_a(L) := \sum_{i=0}^m y^i \prod_{j=0}^{i-1} (\vartheta + a + j) \prod_{k=0}^{m-i-1} (\vartheta - k) P_i(\vartheta) \in \mathbb{C}[y, \vartheta]. \tag{3.6.2}$$

Remark 3.7. (i) Note that the differential operator $\vartheta - a$ corresponds under the Riemann-Hilbert correspondence to the Kummer sheaf $\mathcal{L}_\chi$. It is shown in [4, Cor. 4.16] that the operator $\mathcal{C}_a(L)$ from (3.6.1) has a right factor $(\vartheta - a) *_C L$ that coincides with the differential operator associated to the middle convolution $\mathrm{MC}_\chi(\mathcal{L})$ via the Riemann-Hilbert correspondence (where $\mathcal{L}$ corresponds to L under the Riemann-Hilbert correspondence).

(ii) By the second equation in (3.5.1), the operator $\mathcal{H}_a(L)$ from (3.6.2) can be written as $\mathcal{H}_a(L) = \mathcal{C}_{1-a}(L \otimes (\vartheta - (a-1)))$. It is shown in [4, Prop. 4.17] that the operator $\mathcal{H}_a(L)$ from (3.6.2) has a right factor $L_a *_H L$, where $L_a = \vartheta - x(\vartheta - a))$, that coincides with the differential operator associated to the middle convolution $\mathrm{MC}_{\chi^{-1}}(\mathcal{L}\otimes\mathcal{L}_\chi)$ via the Riemann-Hilbert correspondence.

4. Jordan forms in G_2 and exceptional isomorphisms

Let us collect the information on the conjugacy classes of the simple algebraic group G_2. Below, we list the possible Jordan canonical forms of elements of the group $G_2(\mathbb{C}) \leq \mathrm{GL}_7(\mathbb{C})$ together with the dimensions of the centralizers in the groups $G_2(\mathbb{C})$, $\mathrm{SO}_7(\mathbb{C})$ and in the group $\mathrm{GL}_7(\mathbb{C})$. The list exhausts all possible cases; cf. [8, Section 1.3]. We use the following conventions: $1_n \in \mathbb{C}^{n\times n}$ denotes the identity matrix, $\mathbf{J}(n)$ denotes a unipotent Jordan block of size n, $\omega \in \mathbb{C}^\times$ denotes a primitive 3-rd root of unity, and $i \in \mathbb{C}^\times$ denotes a primitive 4-th root of unity. Moreover, an expression like $(x\mathbf{J}(2), x^{-1}\mathbf{J}(2), x^2, x^{-2}, 1)$ denotes a matrix in Jordan canonical form in $\mathrm{GL}_7(\mathbb{C})$ with one Jordan block of size 2 having eigenvalue x, one Jordan block of size 2 having eigenvalue x^{-1}, and three Jordan blocks of size 1 having eigenvalues $x^2, x^{-2}, 1$ (resp.). The table of GL_7 conjugacy classes of G_2 is as follows:

Jordan form	Centralizer dimension in			Conditions
	G_2	SO_7	GL_7	
$(1,1,1,1,1,1,1)$	14	21	49	
$(\mathbf{J}(2),\mathbf{J}(2),1,1,1)$	8	13	29	
$(\mathbf{J}(3),\mathbf{J}(2),\mathbf{J}(2))$	6	9	19	
$(\mathbf{J}(3),\mathbf{J}(3),1)$	4	7	17	
$\mathbf{J}(7)$	2	3	7	
$(-1,-1,-1,-1,1,1,1)$	6	9	25	
$(-\mathbf{J}(2),-\mathbf{J}(2),1,1,1)$	4	7	17	
$(-\mathbf{J}(2),-\mathbf{J}(2),\mathbf{J}(3))$	4	5	11	
$(-\mathbf{J}(3),-1,\mathbf{J}(3))$	2	3	9	
$(\omega,\omega,\omega,1,\omega^{-1},\omega^{-1},\omega^{-1})$	8	9	19	
$(\omega\mathbf{J}(2),\omega^{-1}\mathbf{J}(2),\omega,\omega^{-1},1)$	4	5	11	
$(\omega\mathbf{J}(3),\omega^{-1}\mathbf{J}(3),1)$	2	3	7	
$(i,i,-1,1,i^{-1},i^{-1},-1)$	4	5	13	
$(i\mathbf{J}(2),i^{-1}\mathbf{J}(2),-1,-1,1)$	2	3	9	
$(x,x,x^{-1},x^{-1},1,1,1)$	4	7	17	$x^2 \neq 1$
$(x,x,x^2,1,x^{-1},x^{-1},x^{-2})$	4	5	11	$x^4 \neq 1 \neq x^3$
$(x,-1,-x,1,-x^{-1},-1,x^{-1})$	2	3	9	$x^4 \neq 1$
$(x\mathbf{J}(2),x^{-1}\mathbf{J}(2),x^2,x^{-2},1)$	2	3	7	$x^4 \neq 1 \neq x^3$
$(x\mathbf{J}(2),x^{-1}\mathbf{J}(2),\mathbf{J}(3))$	2	3	7	$x^2 \neq 1$
$(x,y,xy,1,(xy)^{-1},y^{-1},x^{-1})$	2	3	7	pairw. diff. eigenvalues

Later, we will need information on how Jordan forms are transformed under the exceptional isomorphisms $\mathrm{SO}_6 = \Lambda^2\mathrm{SL}_4$ and $\mathrm{SO}_5 = \Lambda^2\mathrm{SP}_4$. Under the exceptional isomorphism $\mathrm{SO}_6 = \Lambda^2\mathrm{SL}_4$ selected Jordan forms are transformed as follows:

Jordan form in SO_6	$\leftrightarrow$	Jordan form $\Lambda^2\mathrm{SL}_4$
$(\mathbf{J}(2), \mathbf{J}(2), 1, 1)$	$\leftrightarrow$	$\Lambda^2(\mathbf{J}(2), 1, 1)$
$(\mathbf{J}(5), 1)$	$\leftrightarrow$	$\Lambda^2(\mathbf{J}(4))$
$(\mathbf{J}(3), 1, -1, -1)$	$\leftrightarrow$	$\Lambda^2(i\mathbf{J}(2), -i\mathbf{J}(2))$
$(\omega\mathbf{J}(3), \omega^{-1}\mathbf{J}(3))$	$\leftrightarrow$	$\Lambda^2(\omega\mathbf{J}(3), 1)$
$(x\mathbf{J}(2), x^{-1}\mathbf{J}(2), x^2, x^{-2})$	$\leftrightarrow$	$\Lambda^2(x\mathbf{J}(2), 1, x^{-2})$
$(\mathbf{J}(3), 1, x^2, x^{-2})$	$\leftrightarrow$	$\Lambda^2(x\mathbf{J}(2), x^{-1}\mathbf{J}(2))$
$(x, y, xy, (xy)^{-1}, y^{-1}, x^{-1})$	$\leftrightarrow$	$\Lambda^2(x, y, (xy)^{-1}, 1)$

Under the exceptional isomorphism $\mathrm{SO}_5 = \Lambda^2\mathrm{SP}_4$ the Jordan forms are transformed as follows:

Jordan form in SO_5	$\leftrightarrow$	Jordan form $\Lambda^2\mathrm{SP}_4$
$\mathbf{J}(5)$	$\leftrightarrow$	$\Lambda^2(\mathbf{J}(4))$
$(-\mathbf{J}(3), -1, 1)$	$\leftrightarrow$	$\Lambda^2(-\mathbf{J}(2), \mathbf{J}(2))$
$(\mathbf{J}(3), x^2, x^{-2})$	$\leftrightarrow$	$\Lambda^2(x\mathbf{J}(2), x^{-1}\mathbf{J}(2))$
$(x\mathbf{J}(2), x^{-1}\mathbf{J}(2), 1)$	$\leftrightarrow$	$\Lambda^2(\mathbf{J}(2), x, x^{-1})$
$(xy, xy^{-1}, x^{-1}y, xy^{-1}, 1)$	$\leftrightarrow$	$\Lambda^2(x, y, y^{-1}, x^{-1})$

The proof of the above statements is a straightforward computation using bases and is hence omitted.

5. The possible cases

Recall the following result of Scott [19]:

Lemma 5.1. *Let K be an algebraically closed field and let V be an n-dimensional K-vector space. Let $(T_1, \ldots, T_{r+1}) \in \mathrm{GL}(V)^{r+1}$ with $T_1 \cdots T_{r+1} = 1$ such that $\langle T_1, \ldots, T_{r+1}\rangle$ is irreducible. Then the following statements hold:*

$$\sum_{i=1}^{r+1} \mathrm{rk}(T_i - 1) \geq 2n \qquad \textit{(Scott Formula)},$$

$$\sum_{i=1}^{r+1} \dim(C_{\mathrm{GL}(V)}(T_i)) \leq (r-1)n^2 + 2 \qquad \textit{(Dimension count)},$$

where $\dim(C_{\mathrm{GL}(V)}(T_i))$ denotes the dimension of the centralizer of T_i in $\mathrm{GL}(V)$.

Let $(J_1, \ldots, J_{r+1})$ be a tuple of matrices in Jordan form which occur in the group G_2 (these Jordan forms are listed in the first table in Section 4). Ultimately, we want to determine all tuples of elements $(T_1, \ldots, T_{r+1}) \in G_2(\mathbb{C})^{r+1}$ having Jordan forms $J_1, \ldots, J_{r+1}$ (resp.) which are monodromy tuples of an orthogonally rigid local system $\mathcal{L}$ whose monodromy group is Zariski dense in the group $G_2(\mathbb{C})$ (especially this implies that $\mathcal{L}$ is irreducible), if such tuples $(T_1, \ldots, T_{r+1})$ exist. We call such a local system $\mathcal{L}$ a G_2-*local system.* Our strategy is first to narrow the possibilities by considering the associated centralizer dimensions

$$N_i^{O_7} := \dim(C_{O_7}(J_i)) \quad \text{and} \quad N_i^{\mathrm{GL}_7} := \dim(C_{\mathrm{GL}_7}(J_i)).$$

In this way one obtains to each tuple $(J_1, \ldots, J_{r+1})$ as above the associated tuples of centralizer dimensions

$$(N_1^{O_7}, \ldots, N_{r+1}^{O_7}) \quad \text{and} \quad (N_1^{\mathrm{GL}_7}, \ldots, N_{r+1}^{\mathrm{GL}_7})$$

in the underlying orthogonal, resp., general linear group. The following table lists all tuples of centralizer dimensions which formally satisfy the condition for orthogonal rigidity,

$$\sum_{i=1}^{r+1} \operatorname{codim}(C_{O_7}(J_i)) = \sum_{i=1}^{r+1} (21 - \dim(C_{O_7}(J_i))) = 2\dim(O_7) = 42. \tag{5.1.1}$$

In the case that the associated centralizer dimensions in the underlying general linear group conflict with the Scott Formula of Lemma 5.1, necessary for irreducibility, this is denoted in the last column by red:

case	$(N_1^{O_7}, \ldots, N_{r+1}^{O_7})$	$(N_1^{\mathrm{GL}_7}, \ldots, N_{r+1}^{\mathrm{GL}_7})$	remarks
P_1	(13, 5, 3)	(29, 13, 9)	red.
		(29, 13, 7)	
		(29, 11, 9)	
		(29, 11, 7)	
P_2	(9, 9, 3)	(25, 25, 9/7)	red.
		(25, 19, 9)	red.
		(25, 19, 7)	lin. rigid, s. [8]
		(19, 19, 9)	
		(19, 19, 7)	
P_3	(9, 7, 5)	(25, 17, 13/11)	red.
		(19, 17, 13)	
		(19, 17, 11)	
P_4	(7, 7, 7)	(17, 17, 17)	red., s. [8]
P_5	(13, 13, 9, 7)	(29, 29, 19, 17)	

Remark 5.2. That the list exhausts all possible combinations (up to permutation of the entries) follows from a simple case-by-case check using the first table in Section 4.

6. The main result

Theorem 6.1. *Let $\mathcal{L}$ be a complex orthogonally rigid G_2-local system on a punctured projective line $\mathbb{P}^1 \setminus \{x_1, \dots, x_{r+1}\}$ of rank 7; i.e., the monodromy group of $\mathcal{L}$ is dense in the exceptional simple group G_2. Then the following holds:*

(i) *If $\mathcal{L}$ has nontrivial local monodromy at $x_1, \dots, x_{r+1}$, then $r = 2, 3$ and $\mathcal{L}$ can be constructed by applying iteratively a sequence of tensor operations, middle convolutions MC_χ, and rational pullbacks, applied to a local system of rank one.*

(ii) *The Jordan canonical forms of the local monodromy of $\mathcal{L}$ are given in the tables below. We use the notation introduced in Section 4: the numbers ω, resp. i, denote primitive roots of unity of order 3, resp. 4, and we impose the additional conditions on the eigenvalues of the following Jordan forms:*

Jordan form	*Conditions*
$(x, x, x^{-1}, x^{-1}, 1, 1, 1)$	$x^2 \neq 1$
$(x, x, x^2, 1, x^{-1}, x^{-1}, x^{-2})$	$x^4 \neq 1 \neq x^3$
$(x, -1, -x, 1, -x^{-1}, -1, x^{-1})$	$x^4 \neq 1$
$(x\mathbf{J}(2), x^{-1}\mathbf{J}(2), x^2, x^{-2}, 1)$	$x^4 \neq 1 \neq x^3$
$(x\mathbf{J}(2), x^{-1}\mathbf{J}(2), \mathbf{J}(3))$	$x^2 \neq 1$
$(x, y, xy, 1, (xy)^{-1}, y^{-1}, x^{-1})$	*pairwise different eigenvalues*

Moreover, the cardinality of isomorphy classes of G_2-local systems with given local monodromy data is listed under #:

- *The case P_1 (in each case we have $\# = 1$):*

P_1			
1	$(\mathbf{J}(2), \mathbf{J}(2), 1, 1, 1)$	$(i, i, -1, 1, -1, -i, -i)$	$\mathbf{J}(7)$
2	$(\mathbf{J}(2), \mathbf{J}(2), 1, 1, 1)$	$(i, i, -1, 1, -1, -i, -i)$	$(\omega\mathbf{J}(3), \omega^{-1}\mathbf{J}(3), 1)$
3	$(\mathbf{J}(2), \mathbf{J}(2), 1, 1, 1)$	$(i, i, -1, 1, -1, -i, -i)$	$(x\mathbf{J}(2), x^{-1}\mathbf{J}(2), x^2, x^{-2}, 1)$
4	$(\mathbf{J}(2), \mathbf{J}(2), 1, 1, 1)$	$(i, i, -1, 1, -1, -i, -i)$	$(x\mathbf{J}(2), x^{-1}\mathbf{J}(2), \mathbf{J}(3))$
5	$(\mathbf{J}(2), \mathbf{J}(2), 1, 1, 1)$	$(i, i, -1, 1, -1, -i, -i)$	$(x, y, xy, 1, (xy)^{-1}, y^{-1}, x^{-1})$
			$\pm i \notin \{x, y, xy\}$
6	$(\mathbf{J}(2), \mathbf{J}(2), 1, 1, 1)$	$(-\mathbf{J}(2), -\mathbf{J}(2), \mathbf{J}(3))$	$\mathbf{J}(7)$
7	$(\mathbf{J}(2), \mathbf{J}(2), 1, 1, 1)$	$(-\mathbf{J}(2), -\mathbf{J}(2), \mathbf{J}(3))$	$(\omega\mathbf{J}(3), \omega^{-1}\mathbf{J}(3), 1)$
8	$(\mathbf{J}(2), \mathbf{J}(2), 1, 1, 1)$	$(-\mathbf{J}(2), -\mathbf{J}(2), \mathbf{J}(3))$	$(x\mathbf{J}(2), x^{-1}\mathbf{J}(2), x^2, x^{-2}, 1)$
9	$(\mathbf{J}(2), \mathbf{J}(2), 1, 1, 1)$	$(-\mathbf{J}(2), -\mathbf{J}(2), \mathbf{J}(3))$	$(x\mathbf{J}(2), x^{-1}\mathbf{J}(2), \mathbf{J}(3))$
10	$(\mathbf{J}(2), \mathbf{J}(2), 1, 1, 1)$	$(-\mathbf{J}(2), -\mathbf{J}(2), \mathbf{J}(3))$	$(x, y, xy, 1, (xy)^{-1}, y^{-1}, x^{-1})$
11	$(\mathbf{J}(2), \mathbf{J}(2), 1, 1, 1)$	$(z, z, z^{-1}, z^{-1}, z^2, z^{-2}, 1)$	$(-\mathbf{J}(3), -1, \mathbf{J}(3))$

P_1			
12	$(\mathbf{J}(2),\mathbf{J}(2),1,1,1)$	$(z,z,z^{-1},z^{-1},z^2,z^{-2},1)$	$(i\mathbf{J}(2),i^{-1}\mathbf{J}(2),-1,-1,1)$
13	$(\mathbf{J}(2),\mathbf{J}(2),1,1,1)$	$(z,z,z^{-1},z^{-1},z^2,z^{-2},1)$	$(x,-1,-x,1,-x^{-1},-1,x^{-1})$ $x\neq \pm z^{\pm 1}$
14	$(\mathbf{J}(2),\mathbf{J}(2),1,1,1)$	$(z,z,z^{-1},z^{-1},z^2,z^{-2},1)$	$\mathbf{J}(7)$
15	$(\mathbf{J}(2),\mathbf{J}(2),1,1,1)$	$(z,z,z^{-1},z^{-1},z^2,z^{-2},1)$	$(\omega\mathbf{J}(3),\omega^{-1}\mathbf{J}(3),1)$
16	$(\mathbf{J}(2),\mathbf{J}(2),1,1,1)$	$(z,z,z^{-1},z^{-1},z^2,z^{-2},1)$	$(x\mathbf{J}(2),x^{-1}\mathbf{J}(2),x^2,x^{-2},1)$ $x\neq z^{\pm 1},x^2\neq z^{\pm 1}$
17	$(\mathbf{J}(2),\mathbf{J}(2),1,1,1)$	$(z,z,z^{-1},z^{-1},z^2,z^{-2},1)$	$(x\mathbf{J}(2),x^{-1}\mathbf{J}(2),\mathbf{J}(3))$ $xz^{\pm 1}\neq 1$
18	$(\mathbf{J}(2),\mathbf{J}(2),1,1,1)$	$(z,z,z^{-1},z^{-1},z^2,z^{-2},1)$	$(x,y,xy,1,(xy)^{-1},y^{-1},x^{-1})$ $z\notin\{x,y,xy,1,(xy)^{-1},y^{-1},x^{-1}\}$
19	$(\mathbf{J}(2),\mathbf{J}(2),1,1,1)$	$(\omega\mathbf{J}(2),\omega^{-1}\mathbf{J}(2),\omega,\omega^{-1},1)$	$(-\mathbf{J}(3),-1,\mathbf{J}(3))$
20	$(\mathbf{J}(2),\mathbf{J}(2),1,1,1)$	$(\omega\mathbf{J}(2),\omega^{-1}\mathbf{J}(2),\omega,\omega^{-1},1)$	$(i\mathbf{J}(2),i^{-1}\mathbf{J}(2),-1,-1,1)$
21	$(\mathbf{J}(2),\mathbf{J}(2),1,1,1)$	$(\omega\mathbf{J}(2),\omega^{-1}\mathbf{J}(2),\omega,\omega^{-1},1)$	$(x,-1,-x,1,-x^{-1},-1,x^{-1})$ $x^6\neq 1$
22	$(\mathbf{J}(2),\mathbf{J}(2),1,1,1)$	$(\omega\mathbf{J}(2),\omega^{-1}\mathbf{J}(2),\omega,\omega^{-1},1)$	$\mathbf{J}(7)$
23	$(\mathbf{J}(2),\mathbf{J}(2),1,1,1)$	$(\omega\mathbf{J}(2),\omega^{-1}\mathbf{J}(2),\omega,\omega^{-1},1)$	$(x\mathbf{J}(2),x^{-1}\mathbf{J}(2),x^2,x^{-2},1)$ $x^6\neq 1$
24	$(\mathbf{J}(2),\mathbf{J}(2),1,1,1)$	$(\omega\mathbf{J}(2),\omega^{-1}\mathbf{J}(2),\omega,\omega^{-1},1)$	$(x\mathbf{J}(2),x^{-1}\mathbf{J}(2),\mathbf{J}(3))$ $x^3\neq 1$
25	$(\mathbf{J}(2),\mathbf{J}(2),1,1,1)$	$(\omega\mathbf{J}(2),\omega^{-1}\mathbf{J}(2),\omega,\omega^{-1},1)$	$(x,y,xy,1,(xy)^{-1},y^{-1},x^{-1})$ $x^3\neq 1,y^3\neq 1,(xy)^3\neq 1$

- *The case* P_2 : *The linearly rigid case* $P_2(25,19,7)$ *is settled in* [8] *and is therefore omitted.*

$P_2(19,19,9)$				#
2	$(\mathbf{J}(2),\mathbf{J}(2),\mathbf{J}(3))$	$(\mathbf{J}(2),\mathbf{J}(2),\mathbf{J}(3))$	$(i\mathbf{J}(2),i^{-1}\mathbf{J}(2),1,-1_2)$	2
3	$(\mathbf{J}(2),\mathbf{J}(2),\mathbf{J}(3))$	$(\mathbf{J}(2),\mathbf{J}(2),\mathbf{J}(3))$	$(x,-x,-x^{-1},x^{-1},1,-1_2)$	4
$P_2(19,19,7)$				
1	$(\mathbf{J}(2),\mathbf{J}(2),\mathbf{J}(3))$	$(\mathbf{J}(2),\mathbf{J}(2),\mathbf{J}(3))$	$\mathbf{J}(7)$	1
2	$(\mathbf{J}(2),\mathbf{J}(2),\mathbf{J}(3))$	$(\mathbf{J}(2),\mathbf{J}(2),\mathbf{J}(3))$	$(\omega\mathbf{J}(3),\omega^{-1}\mathbf{J}(3),1)$	1
3	$(\mathbf{J}(2),\mathbf{J}(2),\mathbf{J}(3))$	$(\mathbf{J}(2),\mathbf{J}(2),\mathbf{J}(3))$	$(x\mathbf{J}(2),x^{-1}\mathbf{J}(2),x^2,x^{-2},1)$	2
4	$(\mathbf{J}(2),\mathbf{J}(2),\mathbf{J}(3))$	$(\mathbf{J}(2),\mathbf{J}(2),\mathbf{J}(3))$	$(x\mathbf{J}(2),x^{-1}\mathbf{J}(2),\mathbf{J}(3))$	2
5	$(\mathbf{J}(2),\mathbf{J}(2),\mathbf{J}(3))$	$(\mathbf{J}(2),\mathbf{J}(2),\mathbf{J}(3))$	$(x,y,xy,1,(xy)^{-1},y^{-1},x^{-1})$	4

- *The case* P_3 :

P_3				#
5	$(\mathbf{J}(2),\mathbf{J}(2),\mathbf{J}(3))$	$(-\mathbf{J}(2),-\mathbf{J}(2),1_3)$	$(i,i,-1,1,i^{-1},i^{-1},-1)$	2
6	$(\mathbf{J}(2),\mathbf{J}(2),\mathbf{J}(3))$	$(-\mathbf{J}(2),-\mathbf{J}(2),1_3)$	$(-\mathbf{J}(2),-\mathbf{J}(2),\mathbf{J}(3))$	1
7	$(\mathbf{J}(2),\mathbf{J}(2),\mathbf{J}(3))$	$(-\mathbf{J}(2),-\mathbf{J}(2),1_3)$	$(x,x,x^2,1,x^{-1},x^{-1},x^{-2})$	2
8	$(\mathbf{J}(2),\mathbf{J}(2),\mathbf{J}(3))$	$(-\mathbf{J}(2),-\mathbf{J}(2),1_3)$	$(\omega\mathbf{J}(2),\omega^{-1}\mathbf{J}(2),\omega,\omega^{-1},1)$	2
9	$(\mathbf{J}(2),\mathbf{J}(2),\mathbf{J}(3))$	$(z,z,z^{-1},z^{-1},1_3)$	$(i,i,-1,1,i^{-1},i^{-1},-1)$	2
		$z^4\neq 1$		
10	$(\mathbf{J}(2),\mathbf{J}(2),\mathbf{J}(3))$	$(z,z,z^{-1},z^{-1},1_3)$	$(-\mathbf{J}(2),-\mathbf{J}(2),\mathbf{J}(3))$	2
11	$(\mathbf{J}(2),\mathbf{J}(2),\mathbf{J}(3))$	$(z,z,z^{-1},z^{-1},1_3)$	$(x,x,x^2,1,x^{-1},x^{-1},x^{-2})$	2
		$x^{\pm 3}z^{\pm 1}\neq 1, x^{\pm 1}z^{\pm 1}\neq 1$		
11	$(\mathbf{J}(2),\mathbf{J}(2),\mathbf{J}(3))$	$(z,z,z^{-1},z^{-1},1_3)$	$(x,x,x^2,1,x^{-1},x^{-1},x^{-2})$	1
		$x^{\pm 3}z^{\pm 1}= 1, x^{\pm 1}z^{\pm 1}= 1$		
12a	$(\mathbf{J}(2),\mathbf{J}(2),\mathbf{J}(3))$	$(z,z,z^{-1},z^{-1},1_3)$	$(\omega\mathbf{J}(2),\omega^{-1}\mathbf{J}(2),\omega,\omega^{-1},1)$	2
		$z^3\neq 1$		
12b	$(\mathbf{J}(2),\mathbf{J}(2),\mathbf{J}(3))$	$(z,z,z^{-1},z^{-1},1_3)$	$(\omega\mathbf{J}(2),\omega^{-1}\mathbf{J}(2),\omega,\omega^{-1},1)$	1
		$z^3= 1$		

- *The case* P_5 *(in each case we have* $\# = 1$*):*

P_5				
1	$(\mathbf{J}(2),\mathbf{J}(2),1_3)$	$(\mathbf{J}(2),\mathbf{J}(2),1_3)$	$(\omega,\omega,\omega,1,\omega^{-1},\omega^{-1},\omega^{-1})$	$(\mathbf{J}(3),\mathbf{J}(3),1)$
2	$(\mathbf{J}(2),\mathbf{J}(2),1_3)$	$(\mathbf{J}(2),\mathbf{J}(2),1_3)$	$(\omega,\omega,\omega,1,\omega^{-1},\omega^{-1},\omega^{-1})$	$(-\mathbf{J}(2),-\mathbf{J}(2),1_3)$
3	$(\mathbf{J}(2),\mathbf{J}(2),1_3)$	$(\mathbf{J}(2),\mathbf{J}(2),1_3)$	$(\omega,\omega,\omega,1,\omega^{-1},\omega^{-1},\omega^{-1})$	$(x,x,x^{-1},x^{-1},1_3)$

Proof. The proof is divided into the cases P_1, P_2, P_3, P_5, where each case is dealt with in one of the following subsections.

6.1. **The case P_1.** We can assume $x_1 = 0, x_2 = 1, x_3 = \infty$. If $\phi, \phi' : \pi_1(\mathbb{G}_m) \to \mathbb{C}^\times$ are characters, there exists a unique local system $\mathcal{L}(\phi, \phi')$ of rank one on $\mathbb{A}^1 \setminus \{0, 1\}$ whose local monodromy data at $0, 1$ are ϕ, resp. ϕ'. We will often identify a character $\phi : \pi_1(\mathbb{G}_m) \to \mathbb{C}^\times$ with the value $\phi(\gamma) = \lambda \in \mathbb{C}^\times$ of a counterclockwise oriented generator γ of $\pi_1(\mathbb{G}_m)$. The functor which sends a local system $\mathcal{L}$ on $\mathbb{A}^1 \setminus \{0, 1\}$ to $\mathcal{L} \otimes \mathcal{L}(\phi, \phi')$ is denoted $\mathrm{MT}_{\mathcal{L}(\phi,\phi')}$. The irreducibility condition and the deduced Scott Formula (Lemma 5.1) imply that only the possibilities listed in the case P_1 of Theorem 6.1 occur (this follows again by a simple case-by-case check using the first table of Section 4).

Let us first prove that in each of the listed cases, if there exists an orthogonally rigid local system with the given Jordan forms and G_2-monodromy, then it can be reduced to a rank-one system via the middle convolution and tensor products: Applying the functor

$$M_\phi = \mathrm{MC}_\phi \circ \mathrm{MT}_{\mathcal{L}(1,\phi^{-1})} \circ \mathrm{MC}_{\phi^{-1}} \circ \mathrm{MT}_{\mathcal{L}(1,\phi)}, \tag{6.1.1}$$

where

$$\phi = \begin{cases} i & \text{in cases} \quad 1) - 5), \\ -1 & \text{in cases} \quad 6) - 10), \\ z & \text{in cases} \quad 11) - 18), \\ \omega & \text{in cases} \quad 19) - 26), \end{cases}$$

we obtain a local system of rank 5 or 6 whose monodromy is contained in the orthogonal group by [7, Thm. 2.4.(i)] or [6, Cor. 5.15]. The change of the local monodromy data in each step can be traced via Lemma 3.2. The result is

nr.	rk			
1	5	$(\mathbf{J}(2), \mathbf{J}(2), 1)$	$(\mathbf{J}(3), -1, -1)$	$\mathbf{J}(5)$
2	6	$(\mathbf{J}(2), \mathbf{J}(2), 1, 1)$	$(\mathbf{J}(3), 1, -1, -1)$	$(\omega\mathbf{J}(3), \omega^{-1}\mathbf{J}(3))$
3	6	$(\mathbf{J}(2), \mathbf{J}(2), 1, 1)$	$(\mathbf{J}(3), 1, -1, -1)$	$(x\mathbf{J}(2), x^{-1}\mathbf{J}(2), x^2, x^{-2})$
4	5	$(\mathbf{J}(2), \mathbf{J}(2), 1)$	$(\mathbf{J}(3), -1, -1)$	$(x\mathbf{J}(2), x^{-1}\mathbf{J}(2), 1)$
5	6	$(\mathbf{J}(2), \mathbf{J}(2), 1, 1)$	$(\mathbf{J}(3), 1, -1, -1)$	$(x, y, xy, (xy)^{-1}, y^{-1}, x^{-1})$
6	5	$(\mathbf{J}(2), \mathbf{J}(2), 1)$	$\mathbf{J}(5)$	$\mathbf{J}(5)$
7	6	$(\mathbf{J}(2), \mathbf{J}(2), 1, 1)$	$(\mathbf{J}(5), 1)$	$(\omega\mathbf{J}(3), \omega^{-1}\mathbf{J}(3))$
8	6	$(\mathbf{J}(2), \mathbf{J}(2), 1, 1)$	$(\mathbf{J}(5), 1)$	$(x\mathbf{J}(2), x^{-1}\mathbf{J}(2), x^2, x^{-2})$
9	5	$(\mathbf{J}(2), \mathbf{J}(2), 1)$	$\mathbf{J}(5)$	$(x\mathbf{J}(2), x^{-1}\mathbf{J}(2), 1)$
10	6	$(\mathbf{J}(2), \mathbf{J}(2), 1, 1)$	$(\mathbf{J}(5), 1)$	$(x, y, xy, (xy)^{-1}, y^{-1}, x^{-1})$
11	5	$(\mathbf{J}(2), \mathbf{J}(2), 1)$	$(\mathbf{J}(3), z^2, z^{-2})$	$(-\mathbf{J}(3), -1, 1)$
12	6	$(\mathbf{J}(2), \mathbf{J}(2), 1, 1)$	$(\mathbf{J}(3), 1, z^2, z^{-2})$	$(i\mathbf{J}(2), i^{-1}\mathbf{J}(2), -1, -1)$
13	6	$(\mathbf{J}(2), \mathbf{J}(2), 1, 1)$	$(\mathbf{J}(3), 1, z^2, z^{-2})$	$(x, -1, -x, -x^{-1}, -1, x^{-1})$
14	5	$(\mathbf{J}(2), \mathbf{J}(2), 1)$	$(\mathbf{J}(3), z^2, z^{-2})$	$\mathbf{J}(5)$
15	6	$(\mathbf{J}(2), \mathbf{J}(2), 1, 1)$	$(\mathbf{J}(3), 1, z^2, z^{-2})$	$(\omega\mathbf{J}(3), \omega^{-1}\mathbf{J}(3))$
16	6	$(\mathbf{J}(2), \mathbf{J}(2), 1, 1)$	$(\mathbf{J}(3), 1, z^2, z^{-2})$	$(x\mathbf{J}(2), x^{-1}\mathbf{J}(2), x^2, x^{-2})$
17	5	$(\mathbf{J}(2), \mathbf{J}(2), 1)$	$(\mathbf{J}(3), z^2, z^{-2})$	$(x\mathbf{J}(2), x^{-1}\mathbf{J}(2), 1)$
18	6	$(\mathbf{J}(2), \mathbf{J}(2), 1, 1)$	$(\mathbf{J}(3), 1, z^2, z^{-2})$	$(x, y, xy, (xy)^{-1}, y^{-1}, x^{-1})$
19	5	$(\mathbf{J}(2), \mathbf{J}(2), 1)$	$(\mathbf{J}(3), \omega, \omega^{-1})$	$(-\mathbf{J}(3), -1, 1)$
20	6	$(\mathbf{J}(2), \mathbf{J}(2), 1, 1)$	$(\mathbf{J}(3), 1, \omega, \omega^{-1})$	$(i\mathbf{J}(2), i^{-1}\mathbf{J}(2), -1, -1)$
21	6	$(\mathbf{J}(2), \mathbf{J}(2), 1, 1)$	$(\mathbf{J}(3), 1, \omega, \omega^{-1})$	$(x, -1, -x, -x^{-1}, -1, x^{-1})$
22	5	$(\mathbf{J}(2), \mathbf{J}(2), 1)$	$(\mathbf{J}(3), \omega, \omega^{-1})$	$\mathbf{J}(5)$
23	6	$(\mathbf{J}(2), \mathbf{J}(2), 1, 1)$	$(\mathbf{J}(3), 1, \omega, \omega^{-1})$	$(x\mathbf{J}(2), x^{-1}\mathbf{J}(2), x^2, x^{-2})$
24	5	$(\mathbf{J}(2), \mathbf{J}(2), 1)$	$(\mathbf{J}(3), \omega, \omega^{-1})$	$(x\mathbf{J}(2), x^{-1}\mathbf{J}(2), 1)$
25	6	$(\mathbf{J}(2), \mathbf{J}(2), 1, 1)$	$(\mathbf{J}(3), 1, \omega, \omega^{-1})$	$(x, y, xy, (xy)^{-1}, y^{-1}, x^{-1})$

It is because the fundamental group of a punctured sphere is a free group that a homomorphism from it to SO_5 (respectively to SO_6) can always be lifted to Sp_4 (respectively to SL_4). Using the tensor identities $\Lambda^2\mathrm{SL}_4 = \mathrm{SO}_6$ and $\Lambda^2\mathrm{SP}_4 = \mathrm{SO}_5$

and the effect on the Jordan forms listed in Section 4, we hence obtain up to two rigid local systems of rank 4, which, under the above tensor identities, give rise to the corresponding tuples of Jordan forms in SO_5, resp. SO_6, in the above list. These are given in the list below. The reason that there can be more than one possibility comes from the fact that the exterior square identifies dual local systems and transforms the scalar endomorphism -1_4 into the identity. The list below should hence be understood up to a suitable tensor product with quadratic local systems (multiplication of two local monodromies by -1) and taking duals. In some cases, however, we get up to a tensor product with quadratic local systems and taking duals only one local system of rank 4 because all the other choices lead to a reducible local system due to the Scott Formula.

nr.	rk			
1	4	$(\mathbf{J}(2),1,1)$	$(i\mathbf{J}(2),-i\mathbf{J}(2))$	$\mathbf{J}(4)$
2	4	$(\mathbf{J}(2),1,1)$	$(i\mathbf{J}(2),-i\mathbf{J}(2))$	$(\omega\mathbf{J}(3),1)$
3	4	$(\mathbf{J}(2),1,1)$	$(i\mathbf{J}(2),-i\mathbf{J}(2))$	$(x\mathbf{J}(2),1,x^{-2})$
4	4	$(\mathbf{J}(2),1,1)$	$(i\mathbf{J}(2),-i\mathbf{J}(2))$	$(\mathbf{J}(2),x,x^{-1})$
5	4	$(\mathbf{J}(2),1,1)$	$(i\mathbf{J}(2),-i\mathbf{J}(2))$	$(x,y,(xy)^{-1},1)$
6	4	$(\mathbf{J}(2),1,1)$	$\mathbf{J}(4)$	$-\mathbf{J}(4)$
7	4	$(\mathbf{J}(2),1,1)$	$\mathbf{J}(4)$	$(-\omega\mathbf{J}(3),-1)$
8	4	$(\mathbf{J}(2),1,1)$	$\mathbf{J}(4)$	$(-x\mathbf{J}(2),-1,-x^{-2})$
9	4	$(\mathbf{J}(2),1,1)$	$\mathbf{J}(4)$	$(-\mathbf{J}(2),-x,-x^{-1})$
10	4	$(\mathbf{J}(2),1,1)$	$\mathbf{J}(4)$	$(-x,-y,-(xy)^{-1},-1)$
11	4	$(\mathbf{J}(2),1,1)$	$(z\mathbf{J}(2),z^{-1}\mathbf{J}(2))$	$(-\mathbf{J}(2),\mathbf{J}(2))$
12	4	$(\mathbf{J}(2),1,1)$	$(z\mathbf{J}(2),z^{-1}\mathbf{J}(2))$	$\pm(i\mathbf{J}(2),1,-1)$
13	4	$(\mathbf{J}(2),1,1)$	$(z\mathbf{J}(2),z^{-1}\mathbf{J}(2))$	$\pm(x,-x^{-1},1,-1)$
14	4	$(\mathbf{J}(2),1,1)$	$(z\mathbf{J}(2),z^{-1}\mathbf{J}(2))$	$\pm\mathbf{J}(4)$
15	4	$(\mathbf{J}(2),1,1)$	$(z\mathbf{J}(2),z^{-1}\mathbf{J}(2))$	$(\omega\mathbf{J}(3),1)$
	4	$(\mathbf{J}(2),1,1)$	$(z\mathbf{J}(2),z^{-1}\mathbf{J}(2))$	$-(\omega\mathbf{J}(3),1)$
				$z^6\neq 1$
16	4	$(\mathbf{J}(2),1,1)$	$(z\mathbf{J}(2),z^{-1}\mathbf{J}(2))$	$(x\mathbf{J}(2),1,x^{-2})$
	4	$(\mathbf{J}(2),1,1)$	$(z\mathbf{J}(2),z^{-1}\mathbf{J}(2))$	$-(x\mathbf{J}(2),1,x^{-2})$
				$-zx^{\pm 2}\neq 1, -zx^{\pm 1}\neq 1$
17	4	$(\mathbf{J}(2),1,1)$	$(z\mathbf{J}(2),z^{-1}\mathbf{J}(2))$	$(\mathbf{J}(2),x,x^{-1})$
	4	$(\mathbf{J}(2),1,1)$	$(z\mathbf{J}(2),z^{-1}\mathbf{J}(2))$	$-(\mathbf{J}(2),x,x^{-1})$
				$-xz^{\pm 1}\neq 1$
18	4	$(\mathbf{J}(2),1,1)$	$(z\mathbf{J}(2),z^{-1}\mathbf{J}(2))$	$(x,y,(xy)^{-1},1)$
	4	$(\mathbf{J}(2),1,1)$	$(z\mathbf{J}(2),z^{-1}\mathbf{J}(2))$	$-(x,y,(xy)^{-1},1)$
				$-z^{\pm 1}\notin\{x,y,xy\}$

$nr.$	rk			
19	4	$(\mathbf{J}(2),1,1)$	$(\omega\mathbf{J}(2),\omega^{-1}\mathbf{J}(2))$	$(\mathbf{J}(2),-\mathbf{J}(2))$
20	4	$(\mathbf{J}(2),1,1)$	$(\omega\mathbf{J}(2),\omega^{-1}\mathbf{J}(2))$	$\pm(i\mathbf{J}(2),1,-1)$
21	4	$(\mathbf{J}(2),1,1)$	$(\omega\mathbf{J}(2),\omega^{-1}\mathbf{J}(2))$	$\pm(x,-x^{-1},1,-1)$
22	4	$(\mathbf{J}(2),1,1)$	$(\omega\mathbf{J}(2),\omega^{-1}\mathbf{J}(2))$	$\pm\mathbf{J}(4)$
23	4	$(\mathbf{J}(2),1,1)$	$(\omega\mathbf{J}(2),\omega^{-1}\mathbf{J}(2))$	$(x\mathbf{J}(2),1,x^{-2})$
	4	$(\mathbf{J}(2),1,1)$	$(\omega\mathbf{J}(2),\omega^{-1}\mathbf{J}(2))$	$-(x\mathbf{J}(2),1,x^{-2})$
				$x^{12}\neq 1$
24	4	$(\mathbf{J}(2),1,1)$	$(\omega\mathbf{J}(2),\omega^{-1}\mathbf{J}(2))$	$(\mathbf{J}(2),x,x^{-1})$
	4	$(\mathbf{J}(2),1,1)$	$(\omega\mathbf{J}(2),\omega^{-1}\mathbf{J}(2))$	$-(\mathbf{J}(2),x,x^{-1})$
				$x^{6}\neq 1$
25	4	$(\mathbf{J}(2),1,1)$	$(\omega\mathbf{J}(2),\omega^{-1}\mathbf{J}(2))$	$(x,y,(xy)^{-1},1)$
	4	$(\mathbf{J}(2),1,1)$	$(\omega\mathbf{J}(2),\omega^{-1}\mathbf{J}(2))$	$-(x,y,(xy)^{-1},1)$
				$x^6\neq 1, y^6\neq 1, (xy)^6\neq 1$

It follows from Katz' existence algorithm for rigid local systems given in [13] (cf. [6]) or, alternatively, from the results in [2] that all these local monodromy data arise from hypergeometric irreducible local systems. By reversing the above steps (using that the middle convolution operation is reversible by [13], (5.1.5)) we therefore obtain an upper bound on the isomorphy classes of G_2-local systems: there exist at most two orthogonally rigid local systems having G_2-monodromy with the same local monodromy data. We remark that since the monodromy of the so obtained rank-7 local systems lies a priori in the orthogonal group but not necessarily in G_2, we only obtain upper bounds.

We now reduce the monodromy tuples modulo ℓ (in the sense of Lemma A.2 below) in order to show that there exists at most one such local system for each type of local monodromy in $G_2:$ Via Proposition A.1 we can compute the normalized structure constant $n(\mathrm{cl}(\sigma_1),\dots,\mathrm{cl}(\sigma_{r+1}))$ (the definition is recalled in the Appendix) corresponding to the reduced monodromy tuple $(\sigma_1,\dots,\sigma_{r+1})$ via the generic character table of the group $G_2(q)$. Using the computer algebra system CHEVIE [10], we obtain in the notation of Chang and Ree (cf. Remark A.5) the following list:

$n(u_1,k_{2,2},u_6)$	$=$	1	$n(u_1,k_{3,1},u_6)$	$=$	1	$n(u_1,h_{1b},u_6)$	$=$	1
$n(u_1,k_{2,2},k_{2,3})$	$=$	$2-\frac{3}{2q}$	$n(u_1,k_{3,1},k_{2,3})$	$=$	1	$n(u_1,h_{1b},k_{2,3})$	$=$	1
$n(u_1,k_{2,2},k_{24})$	$=$	$2-\frac{1}{2q}$	$n(u_1,k_{3,1},k_{24})$	$=$	0	$n(u_1,h_{1b},k_{24})$	$=$	0
$n(u_1,k_{2,2},k_{3,2})$	$=$	1	$n(u_1,k_{3,1},k_{3,2})$	$<$	1	$n(u_1,h_{1b},k_{3,2})$	$=$	1
$n(u_1,k_{2,2},k_{3,3,1})$	$=$	0	$n(u_1,k_{3,1},k_{3,3,1})$	$<$	1	$n(u_1,h_{1b},k_{3,3,1})$	$=$	0
$n(u_1,k_{2,2},k_{3,3,2})$	$=$	0	$n(u_1,k_{3,1},k_{3,3,2})$	$<$	1	$n(u_1,h_{1b},k_{3,3,2})$	$=$	0
$n(u_1,k_{2,2},h_{1a,1})$	$=$	1	$n(u_1,k_{3,1},h_{1a,1})$	$=$	1	$n(u_1,h_{1b},h_{1a,1})$	$=$	1
$n(u_1,k_{2,2},h_{1b,1})$	$=$	1	$n(u_1,k_{3,1},h_{1b,1})$	$=$	1	$n(u_1,h_{1b},h_{1b,1})$	$=$	1
$n(u_1,k_{2,2},h_1)$	$=$	1	$n(u_1,k_{3,1},h_1)$	$=$	1	$n(u_1,h_{1b},h_1)$	$=$	1

Note that the output of [10] comes along with a list of possible exceptions depending on the eigenvalues of the conjugacy classes. In all cases but

$$n(u_1, h_{1b}, h_{1a,1}), \quad n(u_1, h_{1b}, h_{1b,1}), \quad n(u_1, h_{1b}, h_1)$$

these exceptions correspond to those obtained from the Scott Formula or we are in the case where the upper bound is already 1 given by the above mentioned construction of the respective tuples using middle convolution and tensor operations. We sketch the arguments in the remaining three cases: The characters of $G_2(q)$ fall into finitely many families $\mathcal{F}_j$. Using the character table of $G := G_2(q)$ in [10] or [11, Anhang B], one easily sees that the contribution of most of these families $\mathcal{F}_j$,

$$\left| \frac{\mid G \mid}{\prod_{i=1}^3 \mid C_G(\sigma_i) \mid} \sum_{\chi \in \mathcal{F}_j} \frac{\prod_{i=1}^3 \chi(\sigma_i)}{\chi(1)} \right|,$$

to the normalized structure constant $n(\mathcal{C}(q))$ is bounded by c/q, where c is constant. This then implies that $\lim_k(\lfloor n(\mathcal{C}(q^k))\rfloor) < 2$. By Theorem A.3 we have hence at most one orthogonally rigid local system having G_2-monodromy with the given local monodromy data in the above list for the case P_1.

To show the containment of the monodromy group in $G_2(\mathbb{C})$ we construct a differential operator by translating the middle convolution operations and tensor product operations to the level of differential operators. To simplify the construction we would rather work with the Hadamard product than with MC_χ, and we change the singularities, together with their local monodromy, from $0, 1, \infty$ to $1, \infty, 0$, respectively (which can be achieved by a Möbius transformation). Then the sequence of middle convolutions given in (6.1.1) is changed into

$$\mathrm{MT}_{\mathcal{L}(\phi^{-1},1)} \circ \mathrm{MC}_\phi \circ \mathrm{MT}_{\mathcal{L}(\phi,1)} \circ \mathrm{MC}_{\phi^{-1}}, \tag{6.1.2}$$

as can be checked e.g. by looking at the respective transformations of local monodromy data. Let $L = \vartheta(\vartheta - c)(\vartheta - d)(\vartheta + (c + d)) - x(\vartheta + a)^2(\vartheta + 1 - a)^2$ be a hypergeometric differential operator with Riemann scheme,

$$\mathcal{R}(L) = \left\{ \begin{array}{ccc} 0 & 1 & \infty \\ \hline 0 & 0 & a \\ c & 1 & a \\ d & 1 & 1-a \\ -c-d & 2 & 1-a \end{array} \right\},$$

corresponding to the list of local monodromy data of the above hypergeometric rank-4 systems (with permuted singularities as mentioned above). Let $L_b = \vartheta - x(\vartheta + b)$, $b \in \{a, 1 - a\}$. Using Remark 3.7(ii) we compute an operator using a sequence of Hadamard products which is inverse to the convolution sequence in (6.1.2),

$$\begin{aligned} P_1 &:= ((\vartheta - a) *_C ((\vartheta + a) \otimes ((\vartheta + a - 1) *_C ((\vartheta - a) \otimes \Lambda^2(L))))) \\ &\qquad\qquad\qquad\qquad\qquad\qquad (\text{with } \phi = \exp(2\pi i a)) \\ &= L_{1-a} *_H L_a *_H ((\vartheta - 1) \otimes \Lambda^2(L)), \end{aligned}$$

explicitly as

$$\begin{aligned}P_1 &= \vartheta\,(\vartheta-d)\,(\vartheta+d)\,(\vartheta-c)\,(\vartheta+c)\,(c+d+\vartheta)\,(-c-d+\vartheta)\\ &\quad - x\,(2\,\vartheta+1)\,(\vartheta+a)\,(\vartheta+1-a)\cdot(\vartheta^4+2\,\vartheta^3+(2-c^2-d^2-2\,a^2+2\,a-cd)\,\vartheta^2\\ &\quad +(1-c^2-d^2-2\,a^2+2\,a-cd)\,\vartheta-2\,a\,(a-1)\left(a^2-a-cd-d^2-c^2+1\right))\\ &\quad + x^2\,(\vartheta+1)\,(\vartheta+2-2\,a)\,(\vartheta+1+a)\,(\vartheta+a)\,(\vartheta+2-a)\,(\vartheta+1-a)\,(\vartheta+2\,a)\,.\end{aligned}$$

The associated Riemann scheme is

$$\mathcal{R}(P_1)=\left\{\begin{array}{ccc} 0 & 1 & \infty \\ \hline 0 & 0 & 1\\ c & 1 & a\\ d & 1 & a+1\\ c+d & 2 & 2a\\ -c-d & 3 & 2-2a\\ -d & 3 & 1-a\\ -c & 4 & 2-a \end{array}\right\},$$

which fits into the local monodromy of type P_1. Note that if $P=\sum x^i p_i(\vartheta)$ is any differential operator, then it follows from the identities $\vartheta x^i = x^i(\vartheta+i)$ and $\vartheta^* = -\vartheta-1$ in the ring $\mathbb{C}(x)[\vartheta]$ that $P^*=\sum x^i p_i(-\vartheta-i-1)$. Hence we obtain for the above differential operator P_1 that

$$\left(\frac{1}{x}\cdot P_1\right)^* = \frac{1}{x}\cdot\sum_{i=0}^{2} x^i p_i(-\vartheta-i) = -\frac{1}{x}P_1,$$

meaning that the operator $\frac{1}{x}P_1$ is self-adjoint. Using a gauge transformation with respect to $x^3(x-1)$ we derive from the monic operator $x^{-6}(x-1)^{-2}\frac{1}{x}P_1$ the following differential operator:

$$L := x^3(x-1)(x^{-6}(x-1)^{-2}\cdot(\frac{1}{x}P_1))\cdot x^{-3}(x-1)^{-1} = x^{-3}(x-1)^{-1}\cdot x^{-1}P_1\cdot x^{-3}(x-1)^{-1}.$$

The right hand expression of L and the formula $(L_1L_2)^* = L_2^*L_1^*$ immediately imply that L is self-adjoint *and* monic. Computing the coefficients, we can write $L=\sum_{i=0}^{7} a_i(x)\partial^i$ with

$$\begin{aligned}a_5(x) &= ((-6\,A+12)\,x^2+(-18+6\,A+2\,B)\,x+14-2\,B)/(x(x-1))^2,\\ a_3(x) &= (\left(9\,A^2-144\,A+252\right)x^4+\left(-18\,A^2+(306-6\,B)\,A-756+84\,B\right)x^3\\ &\quad +\left(9\,A^2+(-240+12\,B)\,A+1437+B^2-198\,B\right)x^2\\ &\quad +\left((-6\,B+78)\,A+164\,B-1074-2\,B^2\right)x+301+B^2-50\,B)/(x(x-1))^4,\end{aligned}$$

where

$$A=a^2-a,\quad B=c^2+cd+d^2.$$

It follows that

$$a_3(x) \;=\; 3\frac{d^2}{dx}a_5(x)+\frac{1}{4}a_5(x)^2. \tag{6.1.3}$$

Hence, all the conditions of Theorem 2.1 are fulfilled for the operator L, and therefore the monodromy of L and that of P_1 is contained in the group $G_2(\mathbb{C})$. Finally it follows from [18] and [14, Theorem 1] that an irreducible algebraic subgroup of $G_2 \subseteq O_7$ of positive dimension containing a nontrivial unipotent element different from J(7) is G_2. This settles the case P_1.

6.2. **The case P_2.** By [8], Theorem 1.3.1, the possible Jordan forms for a triple $(A, B, C) \in G_2(\mathbb{C})^3$ in the rigid case $P_2(25, 19, 7)$ are (up to permutation)

$$\mathrm{J}(A) = (-1_4, 1_3), \quad \mathrm{J}(B) = (\mathbf{J}(2), \mathbf{J}(2), \mathbf{J}(3))$$

and

	J(C)	condition
1)	$\mathbf{J}(7)$	
2)	$(\omega\mathbf{J}(3), \omega^{-1}\mathbf{J}(3), 1)$	$\lvert\omega\rvert = 3$
3)	$(\tilde{y}\mathbf{J}(2), \tilde{y}^{-1}\mathbf{J}(2), \tilde{y}^2, 1, \tilde{y}^{-2})$	$\tilde{y}^4 \neq 1 \neq \tilde{y}^6$
4)	$(\tilde{y}\mathbf{J}(2), \tilde{y}^{-1}\mathbf{J}(2), \mathbf{J}(3))$	$\tilde{y}^4 \neq 1$
5)	$(\tilde{x}, \tilde{y}, \tilde{x}\tilde{y}, 1, (\tilde{x}\tilde{y})^{-1}, \tilde{y}^{-1}, \tilde{x}^{-1})$	eigenvalues pairwise different

In the case $P_2(19, 19, 9)$, the list of possible local monodromy data is

$nr.$	rk			
1	7	$(\mathbf{J}(2), \mathbf{J}(2), \mathbf{J}(3))$	$(\mathbf{J}(2), \mathbf{J}(2), \mathbf{J}(3))$	$(-\mathbf{J}(3), -1, \mathbf{J}(3))$
2	7	$(\mathbf{J}(2), \mathbf{J}(2), \mathbf{J}(3))$	$(\mathbf{J}(2), \mathbf{J}(2), \mathbf{J}(3))$	$(i\mathbf{J}(2), i^{-1}\mathbf{J}(2), -1, -1, 1)$
3	7	$(\mathbf{J}(2), \mathbf{J}(2), \mathbf{J}(3))$	$(\mathbf{J}(2), \mathbf{J}(2), \mathbf{J}(3))$	$(x, -x, -x^{-1}, x^{-1}, 1, -1, -1)$

In the case $P_2(19, 19, 7)$ the list of possible local monodromy data is as follows:

$nr.$	rk			
1	7	$(\mathbf{J}(2), \mathbf{J}(2), \mathbf{J}(3))$	$(\mathbf{J}(2), \mathbf{J}(2), \mathbf{J}(3))$	$\mathbf{J}(7)$
2	7	$(\mathbf{J}(2), \mathbf{J}(2), \mathbf{J}(3))$	$(\mathbf{J}(2), \mathbf{J}(2), \mathbf{J}(3))$	$(\omega\mathbf{J}(3), \omega^{-1}\mathbf{J}(3), 1)$
3	7	$(\mathbf{J}(2), \mathbf{J}(2), \mathbf{J}(3))$	$(\mathbf{J}(2), \mathbf{J}(2), \mathbf{J}(3))$	$(x\mathbf{J}(2), x^{-1}\mathbf{J}(2), x^2, x^{-2}, 1)$
4	7	$(\mathbf{J}(2), \mathbf{J}(2), \mathbf{J}(3))$	$(\mathbf{J}(2), \mathbf{J}(2), \mathbf{J}(3))$	$(x\mathbf{J}(2), x^{-1}\mathbf{J}(2), \mathbf{J}(3))$
5	7	$(\mathbf{J}(2), \mathbf{J}(2), \mathbf{J}(3))$	$(\mathbf{J}(2), \mathbf{J}(2), \mathbf{J}(3))$	$(x, y, xy, 1, (xy)^{-1}, y^{-1}, x^{-1})$

Rigid local systems $\mathcal{L}$ and their associated monodromy triples (A, B, C) from the above rigid case $P_2(25, 19, 7)$ yield triples in the cases $P_2(19, 19, 9)(2), (3)$ and $P_2(19, 19, 7)$ as follows: Consider the quadratic pullback of the rigid local system $\mathcal{L}$ along the map $f : \mathbb{P}^1 \to \mathbb{P}^1$ which sends the coordinate x of the standard affine chart of $\mathbb{P}^1$ to x^2. This induces a transformation of monodromy tuples via the association

$$(A, B, C) \mapsto (B^A, B, C^2)$$

with (B^A, B, C^2) visibly belonging to $P_2(19, 19, 9)(2), (3)$ or the various subcases of $P_2(19, 19, 7)$. Note that (B^A, B, C^2) now reflects the monodromy at $1, -1, \infty$ of $f^*\mathcal{L}$, whereas the local monodromy of $f^*\mathcal{L}$ at 0 vanishes since A is an involution. This proves the existence of at least one G_2-local system in the cases $P_2(19, 19, 7)(1)$ and $P_2(19, 19, 7)(2)$. Using both solutions of the equation $\tilde{y}^2 = x$ one sees that there exist at least two isomorphism classes of G_2-local systems in the cases $P_2(19,19,7)(3)$, $P_2(19,19,7)(4)$ and $P_2(19,19,9)(2)$. Using all the solutions of the equations $\tilde{x}^2 = x$ and $\tilde{y}^2 = y$ one obtains at least four isomorphism classes of G_2-local systems in the cases $P_2(19,19,9)(3)$ and $P_2(19,19,7)(5)$.

For the case $P_2(19,19,9)(1)$ we disprove the existence as follows: Applying the functor

$$M_\phi = \mathrm{MT}_{\mathcal{L}(-1,-1)} \circ \mathrm{MC}_{-1} \circ \mathrm{MT}_{\mathcal{L}(-1,-1)} \circ \mathrm{MC}_{-1},$$

we obtain in each $P_2(19,19,9)$-case the following tuple of Jordan forms:

nr.	rk				
1	5	$(-\mathbf{J}(2), -\mathbf{J}(2), 1)$	$(\mathbf{J}(2), -\mathbf{J}(2), 1)$	$(-\mathbf{J}(3), \mathbf{J}(3))$	*red.*
2	5	$(-\mathbf{J}(2), -\mathbf{J}(2), 1)$	$(-\mathbf{J}(2), -\mathbf{J}(2), 1)$	$(i\mathbf{J}(2), i^{-1}\mathbf{J}(2), 1)$	
3	5	$(-\mathbf{J}(2), -\mathbf{J}(2), 1)$	$(-\mathbf{J}(2), -\mathbf{J}(2), 1)$	$(x, -x, -x^{-1}, x^{-1}, 1)$	

Nr. 1 of the previous table leads to a contradiction to the Jordan form of the local monodromy at ∞, and hence we deduce that the case $P_2(19,19,9)(1)$ cannot occur. We proceed in order to prove an upper bound for the cases $P_2(19,19,9)(2)$ and $P_2(19,19,9)(3)$): Note that in the above cases nr. 2 and nr. 3 the monodromy is contained in the group SO_5 by [6, Cor. 5.15]. Using the isomorphism

$$\Lambda^2\mathrm{SP}_4 \cong \mathrm{SO}_5$$

we get

nr.	rk			
2	4	$(-\mathbf{J}(2), 1, 1)$	$(-\mathbf{J}(2), 1, 1)$	$-(\mathbf{J}(2), i, i^{-1})$
3	4	$(-\mathbf{J}(2), 1, 1)$	$(-\mathbf{J}(2), 1, 1)$	$\pm(ix, i, -i, (ix)^{-1})$

Applying MC_{-1} we get an orthogonally rigid local system of rank $3, 4$ resp., with the following local monodromy data:

nr.	rk			
2	3	$\mathbf{J}(3)$	$\mathbf{J}(3)$	$(1, i, i^{-1})$
3	4	$(\mathbf{J}(3), 1)$	$(\mathbf{J}(3), 1)$	$\mp(ix, i, -i, (ix)^{-1})$

Via the isomorphisms

$$\mathrm{sym}^2\mathrm{SP}_2 = \mathrm{SO}_3, \quad \mathrm{SL}_2 \otimes \mathrm{SL}_2 = \mathrm{SO}_4$$

we can decompose it into linearly rigid irreducible local systems $\mathcal{L}_1$ and $\mathcal{L}_2$ of rank 2 with the following tuple of Jordan forms:

nr.	$\mathcal{L}_i$				
2	$\mathcal{L}_1 = \mathcal{L}_2$	$\mathbf{J}(2)$	$\mathbf{J}(2)$	$\pm(\zeta_8, \zeta_8^{-1})$	
3	$\mathcal{L}_1$	$\mathbf{J}(2)$	$\mathbf{J}(2)$	(y, y^{-1})	$y^2 = x$
	$\mathcal{L}_2$	$\mathbf{J}(2)$	$\mathbf{J}(2)$	$\pm(iy, i^{-1}y^{-1})$	

This also shows the existence of at most two, resp. four, orthogonally rigid local systems with G_2-monodromy and the same local monodromy in the cases $P_2(19,19,9)(2)$, resp. $P_2(19,19,9)(3)$, implying that the above lower bound is sharp in the cases $P_2(19,19,9)(2), (3)$.

For an upper bound case $P_2(19,19,7)$ we determine the corresponding normalized structure constant of the reduced monodromy tuple. This can again be computed via Lemma A.1 and by the help of [10]:

1) $\mathbf{n}(u_2,u_2,u_6)=(3q-2)/q$,

2) $\mathbf{n}(u_2,u_2,k_{3,2})=\frac{4q-1}{3q}$, $\quad \mathbf{n}(u_2,u_2,k_{3,3,i})=\frac{q-1}{3q}$, $\quad i=1,2$,

3) $\mathbf{n}(u_2,u_2,h_{1a,1})=\begin{cases} 3(q-1)/q, & h_{1a,1}=h'^2_{1a,1},\\ (q-1)/q, & h_{1a,1}\neq h'^2_{1a,1},\end{cases}$

4) $\mathbf{n}(u_2,u_2,h_{1b,1})=\begin{cases} (3q-1)/q, & h_{1b,1}=h'^2_{1b,1},\\ (q-1)/q, & h_{1b,1}\neq h'^2_{1b,1},\end{cases}$

5) $\mathbf{n}(u_2,u_2,h_1)=\begin{cases} 4, & h_1=h'^2_1,\\ 0, & h_1\neq h'^2_1.\end{cases}$

This implies that in all cases with the possible exception of $P_2(19,19,7)(1)$ the lower bound given above is sharp. It remains to show that in the case $P_2(19,19,7)(1)$, the above proved lower bound 1 is also an upper bound. For this we argue as follows: By the above proof of the lower bound, we know that there is a pullback G_2-local system with monodromy tuple (B^A,B,C^2) in the case $P_2(19,19,7)(1)$. Suppose that there exists another G_2-local system in the case $P_2(19,19,7)(1)$ with monodromy tuple $(g_1,g_2,g_3)\in G_2(\mathbb{C})^3$ which is inequivalent to (B^A,B,C^2) under diagonal conjugation in $\mathrm{GL}_7(\mathbb{C})$. Then, two possibilities arise: Either the braided tuple $(g_2,g_1^{g_2},g_3)$ is equivalent to (g_1,g_2,g_3) (under diagonal conjugation in $\mathrm{GL}_7(\mathbb{C})$) or it is not. If $(g_2,g_1^{g_2},g_3)$ is not equivalent to (g_1,g_2,g_3), then we claim that neither $(g_2,g_1^{g_2},g_3)$ nor (g_1,g_2,g_3) can arise from a quadratic pullback as above: the braiding of the first two entries of a pullback triple (B^A,B,C^2) is the equivalent triple

$$(B,B^{(AB)},C^2)=(B^A,B,C^2)^{B^{-1}A},$$

since $C=(AB)^{-1}$ and since A is an involution. This implies that the existence of an inequivalent triple (g_1,g_2,g_3) with an inequivalent braided triple $(g_2,g_1^{g_2},g_3)$ conflicts the above upper bound of two inequivalent monodromy triples in the case $P_2(19,19,7)(1)$.

Suppose now that (g_1,g_2,g_3) is equivalent to $(g_2,g_1^{g_2},g_3)$, i.e., there exists an element $h\in\mathrm{GL}_7(\mathbb{C})$ with

$$(g_1,g_2,g_3)^h=(g_1^h,g_2^h,g_3^h)=(g_2,g_1^{g_2},g_3).$$

The equalities

$$g_1^h=g_2,\quad g_2^h=g_1^{g_2},\quad g_3^h=g_3$$

imply

$$(g_1,g_2,g_3)^{hg_2^{-1}}=(g_2,g_1,g_3^{hg_2^{-1}}),$$

and hence $(hg_2^{-1})^2\in Z(G_2)=1$. This implies that hg_2^{-1} is an involution and

$$h^2=(g_2h^{-1})^2h^2=g_2g_2^h=g_1g_2.$$

Hence the quadratic pullback of (hg_2^{-1},g_2,h^{-1}) is

$$(g_2^{(hg_2^{-1})},g_2,h^{-2}) \quad=\quad (g_1,g_2,g_3),$$

meaning that (g_1, g_2, g_3) belongs to a quadratic pullback of a rigid local system (this follows from the associated triple of Jordan forms), contradicting our assumption on (g_1, g_2, g_3). This finishes the proof of the case P_2.

For completeness we list the associated differential operators: Using the same arguments as in the P_1-case together with the results of [8], we get the following operators in the linearly rigid $P_2(25, 19, 7)$-case (the other differential operators are then deduced by pullbacks):

$$\begin{aligned}
L := {} & 8\,(\vartheta-1)\,(\vartheta-2)\,(\vartheta-3)\,(2\,\vartheta-1)\,(2\,\vartheta-3)\,(2\,\vartheta-5)\,(2\,\vartheta-7)\\
& - 8x\,(2\,\vartheta-5)\,(\vartheta-1)\,(\vartheta-2)\,(2\,\vartheta-1)\,(2\,\vartheta-3)\left(8\,\vartheta^2-24\,\vartheta+25-4(p^2+q^2+pq)\right)\\
& + 2x^2(\vartheta-1)(2\vartheta-1)(2\vartheta-3)(96\vartheta^4-384\vartheta^3+(720-96\,(q^2+p^2+pq))\vartheta^2\\
& + (192\,(q^2+p^2+pq)-672)\vartheta+141+8\,\left(p^2-1+qp+q^2\right)\left(2\,p^2-15+2\,qp+2\,q^2\right))\\
& + x^3(2\vartheta-1)(-256\vartheta^6+768\vartheta^5+(-1312+384\,(p^2+q^2+qp))\vartheta^4\\
& + (1344-768(p^2+q^2+qp))\vartheta^3-(160+32\left(4(p^2+q^2+pq)-21\right)\left(p^2-1+qp+q^2\right))\vartheta^2\\
& + (32\,\left(4\,(p^2+q^2+pq)-9\right)\left(p^2-1+qp+q^2\right))\vartheta\\
& + (64\,p^2q^2\left(q^2+2\,qp+p^2\right)-3)-8\,\left(6\,(p^2+q^2+pq)-5\right)\left(p^2-1+qp+q^2\right))\\
& + 128x^4\vartheta(\vartheta-q)(\vartheta+q)(\vartheta-p)(\vartheta+p)(\vartheta+p+q)(\vartheta-p-q),
\end{aligned}$$

$$\mathcal{R}(L) = \left\{ \begin{array}{ccc} 0 & 1 & \infty \\ \hline 1/2 & 0 & 0 \\ 1 & 0 & q \\ 3/2 & 1 & p \\ 2 & 1 & p+q \\ 5/2 & 1 & -p-q \\ 3 & 2 & -q \\ 7/2 & 2 & -p \end{array} \right\}.$$

6.3. **The P_3 case.** In the case P_3 the list of possible local monodromy data is as follows:

P_3	rk			
1	7	$(\mathbf{J}(2), \mathbf{J}(2), \mathbf{J}(3))$	$(\mathbf{J}(3), \mathbf{J}(3), 1)$	$(i, i, -1, 1, i^{-1}, i^{-1}, -1)$
2	7	$(\mathbf{J}(2), \mathbf{J}(2), \mathbf{J}(3))$	$(\mathbf{J}(3), \mathbf{J}(3), 1)$	$(-\mathbf{J}(2), -\mathbf{J}(2), \mathbf{J}(3))$
3	7	$(\mathbf{J}(2), \mathbf{J}(2), \mathbf{J}(3))$	$(\mathbf{J}(3), \mathbf{J}(3), 1)$	$(x, x, x^2, 1, x^{-1}, x^{-1}, x^{-2})$
4	7	$(\mathbf{J}(2), \mathbf{J}(2), \mathbf{J}(3))$	$(\mathbf{J}(3), \mathbf{J}(3), 1)$	$(\omega\mathbf{J}(2), \omega^{-1}\mathbf{J}(2), \omega, \omega^{-1}, 1)$
5	7	$(\mathbf{J}(2), \mathbf{J}(2), \mathbf{J}(3))$	$(-\mathbf{J}(2), -\mathbf{J}(2), 1, 1, 1)$	$(i, i, -1, 1, i^{-1}, i^{-1}, -1)$
6	7	$(\mathbf{J}(2), \mathbf{J}(2), \mathbf{J}(3))$	$(-\mathbf{J}(2), -\mathbf{J}(2), 1, 1, 1)$	$(-\mathbf{J}(2), -\mathbf{J}(2), \mathbf{J}(3))$
7	7	$(\mathbf{J}(2), \mathbf{J}(2), \mathbf{J}(3))$	$(-\mathbf{J}(2), -\mathbf{J}(2), 1, 1, 1)$	$(x, x, x^2, 1, x^{-1}, x^{-1}, x^{-2})$
8	7	$(\mathbf{J}(2), \mathbf{J}(2), \mathbf{J}(3))$	$(-\mathbf{J}(2), -\mathbf{J}(2), 1, 1, 1)$	$(\omega\mathbf{J}(2), \omega^{-1}\mathbf{J}(2), \omega, \omega^{-1}, 1)$

P_3	rk			
9	7	$(\mathbf{J}(2),\mathbf{J}(2),\mathbf{J}(3))$	$(z,z,z^{-1},z^{-1},1,1,1)$	$(i,i,-1,1,i^{-1},i^{-1},-1)$
10	7	$(\mathbf{J}(2),\mathbf{J}(2),\mathbf{J}(3))$	$(z,z,z^{-1},z^{-1},1,1,1)$	$(-\mathbf{J}(2),-\mathbf{J}(2),\mathbf{J}(3))$
11	7	$(\mathbf{J}(2),\mathbf{J}(2),\mathbf{J}(3))$	$(z,z,z^{-1},z^{-1},1,1,1)$	$(x,x,x^2,1,x^{-1},x^{-1},x^{-2})$
12	7	$(\mathbf{J}(2),\mathbf{J}(2),\mathbf{J}(3))$	$(z,z,z^{-1},z^{-1},1,1,1)$	$(\omega\mathbf{J}(2),\omega^{-1}\mathbf{J}(2),\omega,\omega^{-1},1)$

Applying the functor

$$M_\phi = \mathrm{MT}_{\mathcal{L}(\phi^{-1},1)} \circ \mathrm{MC}_\phi \circ \mathrm{MT}_{\mathcal{L}(\phi,1)} \circ \mathrm{MC}_{\phi^{-1}},$$

where

$$\phi = \begin{cases} i & \text{in cases} \quad 1),5),9), \\ -1 & \text{in cases} \quad 2),6),10), \\ x & \text{in cases} \quad 3),7),11), \\ \omega & \text{in cases} \quad 4),8),12), \end{cases}$$

we obtain an orthogonally rigid local system of rank 5 with the following local monodromy data. The contradiction of the rank being 5 and having Jordan form of type $(\mathbf{J}(3),\mathbf{J}(3))$ in the cases (1)-(4) shows their nonexistence (reducibility).

nr.	rk				
1	5	$(i,i,1,-i,-i)$	$(\mathbf{J}(3),\mathbf{J}(3))$	$(\mathbf{J}(3),-1,-1)$	*red.*
2	5	$(1,-\mathbf{J}(2),-\mathbf{J}(2))$	$(\mathbf{J}(3),\mathbf{J}(3))$	$\mathbf{J}(5)$	*red.*
3	5	$(x,x,1,x^{-1},x^{-1})$	$(\mathbf{J}(3),\mathbf{J}(3))$	$(x^2,\mathbf{J}(3),x^{-2})$	*red.*
4	5	$(\omega,\omega,1,\omega^{-1},\omega^{-1})$	$(\mathbf{J}(3),\mathbf{J}(3))$	$(\omega,\mathbf{J}(3),\omega^{-1})$	*red.*
5	5	$(i,i,1,-i,-i)$	$(-\mathbf{J}(2),-\mathbf{J}(2),1)$	$(\mathbf{J}(3),-1,-1)$	
6	5	$(1,-\mathbf{J}(2),-\mathbf{J}(2))$	$(-\mathbf{J}(2),-\mathbf{J}(2),1)$	$\mathbf{J}(5)$	
7	5	$(x,x,1,x^{-1},x^{-1})$	$(-\mathbf{J}(2),-\mathbf{J}(2),1)$	$(x^2,\mathbf{J}(3),x^{-2})$	
8	5	$(\omega,\omega,1,\omega^{-1},\omega^{-1})$	$(-\mathbf{J}(2),-\mathbf{J}(2),1)$	$(\omega,\mathbf{J}(3),\omega^{-1})$	
9	5	$(i,i,1,-i,-i)$	$(z,z,z^{-1},z^{-1},1)$	$(\mathbf{J}(3),-1,-1)$	$z^4 \neq 1$
10	5	$(1,-\mathbf{J}(2),-\mathbf{J}(2))$	$(z,z,z^{-1},z^{-1},1)$	$\mathbf{J}(5)$	
11	5	$(x,x,1,x^{-1},x^{-1})$	$(z,z,z^{-1},z^{-1},1)$	$(x^2,\mathbf{J}(3),x^{-2})$	
12	5	$(\omega,\omega,1,\omega^{-1},\omega^{-1})$	$(z,z,z^{-1},z^{-1},1)$	$(\omega,\mathbf{J}(3),\omega^{-1})$	

Using the isomorphism $\Lambda^2\mathrm{SP}_4 \cong \mathrm{SO}_5$ and the Scott Formula we get

$nr.$	rk			
5	4	$(i,1,1,-i)$	$(-\mathbf{J}(2),1,1)$	$(i\mathbf{J}(2),-i\mathbf{J}(2))$
6	4	$(-\mathbf{J}(2),1,1)$	$(-\mathbf{J}(2),1,1)$	$-\mathbf{J}(4)$
7	4	$(x,1,1,x^{-1})$	$(-\mathbf{J}(2),1,1)$	$\pm(x\mathbf{J}(2),x^{-1}\mathbf{J}(2))$
8	4	$(\omega,1,1,\omega^{-1})$	$(-\mathbf{J}(2),1,1)$	$\pm(\omega\mathbf{J}(2),\omega^{-1}\mathbf{J}(2))$
9	4	$(i,1,1,-i)$	$(z,1,1,z^{-1})$	$(i\mathbf{J}(2),-i\mathbf{J}(2)$
10	4	$(-\mathbf{J}(2),1,1)$	$(z,1,1,z^{-1})$	$-\mathbf{J}(4)$
11	4	$(x,1,1,x^{-1})$	$(z,1,1,z^{-1})$	$\pm(x\mathbf{J}(2),x^{-1}\mathbf{J}(2))$
12	4	$(\omega,1,1,\omega^{-1})$	$(z,1,1,z^{-1})$	$\pm(\omega\mathbf{J}(2),\omega^{-1}\mathbf{J}(2))$

Applying MC_{-1} we obtain an orthogonally rigid local system of rank 3 or 4 with the following local monodromy data:

$nr.$	rk			
5	4	$(i,1,1,-i)$	$(\mathbf{J}(3),1)$	$(i\mathbf{J}(2),-i\mathbf{J}(2))$
6	3	$\mathbf{J}(3)$	$\mathbf{J}(3)$	$\mathbf{J}(3)$
7	4	$(-x,1,1,-x^{-1})$	$(\mathbf{J}(3),1)$	$\mp(x\mathbf{J}(2),x^{-1}\mathbf{J}(2))$
8	4	$(-\omega,1,1,-\omega^{-1})$	$(\mathbf{J}(3),1)$	$\mp(\omega\mathbf{J}(2),\omega^{-1}\mathbf{J}(2))$
9	4	$(i,1,1,-i)$	$(-z,1,1,-z^{-1})$	$(i\mathbf{J}(2),-i\mathbf{J}(2)$
10	3	$\mathbf{J}(3)$	$(-z,1,-z^{-1})$	$\mathbf{J}(3)$
11	4	$(-x,1,1,-x^{-1})$	$(-z,1,1,-z^{-1})$	$\mp(x\mathbf{J}(2),x^{-1}\mathbf{J}(2))$
12	4	$(-\omega,1,1,-\omega^{-1})$	$(-z,1,1,-z^{-1})$	$\mp(\omega\mathbf{J}(2),\omega^{-1}\mathbf{J}(2))$

Via the isomorphisms $\mathrm{sym}^2\mathrm{SP}_2 \cong \mathrm{SO}_3$ and $\mathrm{SL}_2 \otimes \mathrm{SL}_2 \cong \mathrm{SO}_4$ we can decompose the above rank 3 and rank 4 local systems into linearly rigid irreducible local systems $\mathcal{L}_1$ and $\mathcal{L}_2$ of rank 2 with the following tuple of Jordan forms. The conditions for a product of eigenvalues not being 1 is due to the irreducibility condition from the Scott Formula.

$nr.$	$\mathcal{L}_i$					
5	$\mathcal{L}_1$	(ζ_8, ζ_8^{-1})	$\mathbf{J}(2)$	$\pm\mathbf{J}(2)$		
	$\mathcal{L}_2$	(ζ_8, ζ_8^{-1})	$\mathbf{J}(2)$	$(i, -i)$		
6	$\mathcal{L}_1 = \mathcal{L}_2$	$\mathbf{J}(2)$	$\mathbf{J}(2)$	$-\mathbf{J}(2)$		
7	$\mathcal{L}_1$	$(\tilde{x}, \tilde{x}^{-1})$	$\mathbf{J}(2)$	$\mathbf{J}(2)$	$\tilde{x}^2 = -x$	
	$\mathcal{L}_2$	$(\tilde{x}, \tilde{x}^{-1})$	$\mathbf{J}(2)$	(x, x^{-1})		
8	$\mathcal{L}_1$	$(\zeta_{12}, \zeta_{12}^{-1})$	$\mathbf{J}(2)$	$\pm\mathbf{J}(2)$		
	$\mathcal{L}_2$	$(\zeta_{12}, \zeta_{12}^{-1})$	$\mathbf{J}(2)$	$\pm(\omega, \omega^{-1})$		
9	$\mathcal{L}_1$	(ζ_8, ζ_8^{-1})	$(\tilde{z}, \tilde{z}^{-1})$	$\mathbf{J}(2)$	$\tilde{z}^2 = -z$	$\zeta_8^{\pm 1}\tilde{z}^{\pm 1} \neq 1$
	$\mathcal{L}_2$	(ζ_8, ζ_8^{-1})	$(\tilde{z}, \tilde{z}^{-1})$	$(i, -i)$	$\tilde{z}^2 = -z$	$\zeta_8^{\pm 1}\tilde{z}^{\pm 1}i^{\pm 1} \neq 1$
10	$\mathcal{L}_1 = \mathcal{L}_2$	$\mathbf{J}(2)$	$(\tilde{z}, \tilde{z}^{-1})$	$\mathbf{J}(2)$	$\tilde{z}^2 = -z$	
11	$\mathcal{L}_1$	$(\tilde{x}, \tilde{x}^{-1})$	$(\tilde{z}, \tilde{z}^{-1})$	$\mathbf{J}(2)$	$\tilde{x}^2 = -x, \tilde{z}^2 = -z$	$\tilde{x}^{\pm 1}\tilde{z}^{\pm 1} \neq 1$
	$\mathcal{L}_2$	$(\tilde{x}, \tilde{x}^{-1})$	$(\tilde{z}, \tilde{z}^{-1})$	(x, x^{-1})	$\tilde{x}^2 = -x, \tilde{z}^2 = -z$	$\tilde{x}^{\pm 1}\tilde{z}^{\pm 1}x^{\pm 1} \neq 1$
12	$\mathcal{L}_1$	$(\zeta_{12}, \zeta_{12}^{-1})$	$(\tilde{z}, \tilde{z}^{-1})$	$\mathbf{J}(2)$	$\tilde{z}^2 = -z$	$\zeta_{12}^{\pm 1}\tilde{z}^{\pm 1} \neq 1$
	$\mathcal{L}_2$	$(\zeta_{12}, \zeta_{12}^{-1})$	$(\tilde{z}, \tilde{z}^{-1})$	(ω, ω^{-1})	$\tilde{z}^2 = -z$	$\zeta_{12}^{\pm 1}\tilde{z}^{\pm 1}\omega^{\pm 1} \neq 1$

Thus there exist at most four orthogonally rigid local systems having G_2-monodromy with the same local monodromy data. Computing the normalized structure constant of the reduced monodromy tuple we show that there exist at most two such local systems:

$$
\begin{array}{lcl}
n(u_2, k_{2,1}, k_{2,2}) & = & \frac{3(q-1)}{q} \\
n(u_2, k_{2,1}, k_{3,1}) & = & 2 \\
n(u_2, k_{2,1}, h_{1b}) & = & \begin{cases} 2 & o(h_{1b}) \mid (q-1)/2 \\ 0 & o(h_{1b}) \nmid (q-1)/2 \end{cases} \\
n(u_2, h_{1a}, k_{2,2}) & = & \begin{cases} 2 & o(h_{1a}) \mid (q-1)/2 \\ 0 & o(h_{1a}) \nmid (q-1)/2 \end{cases} \\
n(u_2, h_{1a}, k_{3,1}) & = & \begin{cases} 2 & o(h_{1a}) \mid (q-1)/2 \\ 0 & o(h_{1a}) \nmid (q-1)/2 \end{cases} \\
n(u_2, h_{1a}, h_{1b}) & = & \begin{cases} 2 & o(h_{1a}) \mid (q-1)/2,\ o(h_{1b}) \mid (q-1)/2 \\ 2 & o(h_{1a}) \nmid (q-1)/2,\ o(h_{1b}) \nmid (q-1)/2 \\ 0 & \text{else} \end{cases}
\end{array}
$$

Replacing q by q^2 we see that there are at most two such local systems in the case P_3 with the same local monodromy data. The existence follows from the construction of the corresponding differential operators. Let

$$P_3 := L_{2c+1/2} *_H L_{-2c+1/2} *_H (L_3 \otimes (\Lambda^2 (L_{1/2} *_H (L_0 \otimes (L_1 \otimes L_2))))),$$

where

$$\mathcal{R}(L_3) = \left\{ \begin{array}{ccc} 0 & 1 & \infty \\ \hline 1 & 0 & -1 \end{array} \right\}, \quad \mathcal{R}(L_\alpha) = \left\{ \begin{array}{ccc} 0 & 1 & \infty \\ \hline 0 & -\alpha & \alpha \end{array} \right\}, \alpha \in \{1/2, \pm 2c + 1/2\},$$

$$L_0 = \vartheta - 1/2, \quad \mathcal{R}(L_0) = \left\{ \begin{array}{ccc} 0 & 1 & \infty \\ \hline 1/2 & 0 & -1/2 \end{array} \right\}$$

and

$$\begin{aligned} L_1 &:= 4\,(\vartheta - c)\,(\vartheta + c) + z\left(-8\,\vartheta^2 - 4\,\vartheta - 1 - 4\,b^2 + 4\,c^2\right) + z^2\,(2\,\vartheta + 1)^2, \\ L_2 &:= 4\,(\vartheta - c)\,(\vartheta + c) + z\left(-8\,\vartheta^2 - 4\,\vartheta - 1 - 4\,b^2 + 20\,c^2\right) \\ &\quad + z^2\,(2\,\vartheta + 1 + 4\,c)\,(2\,\vartheta + 1 - 4\,c) \end{aligned}$$

with

$$\mathcal{R}(L_1) = \left\{ \begin{array}{ccc} 0 & 1 & \infty \\ \hline c & b & 1/2 \\ -c & -b & 1/2 \end{array} \right\}, \quad \mathcal{R}(L_2) = \left\{ \begin{array}{ccc} 0 & 1 & \infty \\ \hline c & b & 2c + 1/2 \\ -c & -b & -2c + 1/2 \end{array} \right\}.$$

Then

$$\begin{aligned} P_3 &= 16\,\vartheta^2\,(\vartheta - 2)^2\,(\vartheta - 1)^3 \\ &- 8\,x\vartheta^2\,(2\,\vartheta - 1)\,(\vartheta - 1)^2\left(4\,\vartheta^2 - 4\,\vartheta + 8\,b^2 + 5 - 24\,c^2\right) \\ &+ 4\,x^2\vartheta^3(24\,\vartheta^4 + (38 - 288c^2 + 64\,b^2)\vartheta^2 + 16\,b^2 + 64\,b^4 - 144\,c^2 + 7 + 576\,c^4 - 384\,c^2b^2) \\ &- 2\,x^3(2\,\vartheta + 1)\,(2\,\vartheta + 1 + 4\,c)\,(2\,\vartheta + 1 - 4\,c)\,(4\,\vartheta^4 + 8\,\vartheta^3 + 11\,\vartheta^2 + 8\,\vartheta^2b^2 \\ &- 56\,\vartheta^2c^2 + 7\,\vartheta - 56\,\vartheta\,c^2 + 8\,\vartheta\,b^2 + 64\,c^4 + 2 + 4\,b^2 - 36\,c^2 - 64\,c^2b^2) \\ &+ x^4\,(\vartheta + 1)\,(2\,\vartheta + 3 - 4\,c)\,(2\,\vartheta + 1 - 4\,c)\,(\vartheta + 1 - 4\,c)\,(\vartheta + 1 + 4\,c) \\ &\cdot (2\,\vartheta + 3 + 4\,c)\,(2\,\vartheta + 1 + 4\,c) \end{aligned}$$

with

$$\mathcal{R}(P_3) = \left\{ \begin{array}{ccc} 0 & 1 & \infty \\ \hline 0 & 0 & 1 \\ 0 & 1 & 2c + 1/2 \\ 1 & 2 & 2c + 3/2 \\ 1 & 1/2 + 2b & 4c + 1 \\ 1 & 3/2 + 2b & -4c + 1 \\ 2 & 3/2 - 2b & -2c + 3/2 \\ 2 & 1/2 - 2b & -2c + 1/2 \end{array} \right\}.$$

Thus if we replace b by $b + 1/2$ (or c by $c + 1/2$) in the construction we get the same local monodromy data for $P_3(b)$ and $P_3(b + 1/2)$. However, if L_1 is reducible, i.e. if $\pm b \pm c + 1/2 \in \mathbb{Z}$ or $L_1(b) \sim L_1(b + 1/2)$, i.e. $2b \in 1/2 + \mathbb{Z}$, then there is only one P_3 with the given local monodromy data.

We can now use the same arguments as in the P_1-case to show that each local system belonging to the P_3-case has G_2-monodromy.

6.4. **The case P_5.** We start with the possible local monodromy data of orthogonally rigid quadruples with G_2-monodromy.

nr.	rk				
1	7	$(\mathbf{J}(2),\mathbf{J}(2),1_3)$	$(\mathbf{J}(2),\mathbf{J}(2),1_3)$	$(\omega,\omega,\omega,1,\omega^{-1},\omega^{-1},\omega^{-1})$	$(\mathbf{J}(3),\mathbf{J}(3),1)$
2	7	$(\mathbf{J}(2),\mathbf{J}(2),1_3)$	$(\mathbf{J}(2),\mathbf{J}(2),1_3)$	$(\omega,\omega,\omega,1,\omega^{-1},\omega^{-1},\omega^{-1})$	$(-\mathbf{J}(2),-\mathbf{J}(2),1_3)$
3	7	$(\mathbf{J}(2),\mathbf{J}(2),1_3)$	$(\mathbf{J}(2),\mathbf{J}(2),1_3)$	$(\omega,\omega,\omega,1,\omega^{-1},\omega^{-1},\omega^{-1})$	$(x,x,x^{-1},x^{-1},1_3)$
					$x^3\neq 1$

In each case, for fixed t, there is at least one local system $\mathcal{L}$ on $\mathbb{P}^1\setminus\{0,1,t,\infty\}$ with these given local monodromy data since for $t=-1$, the local system $\mathcal{L}$ arises from a quadratic pullback of a P_1-case with the following tuples of Jordan forms:

nr.			
1	$(\mathbf{J}(2),\mathbf{J}(2),1_3)$	$(-\omega,-\omega,\omega,1,\omega^{-1},-\omega,-\omega^{-1})$	$(\mathbf{J}(3),-\mathbf{J}(3),-1)$
2	$(\mathbf{J}(2),\mathbf{J}(2),1_3)$	$(-\omega,-\omega,\omega,1,\omega^{-1},-\omega,-\omega^{-1})$	$(i\mathbf{J}(2),-i\mathbf{J}(2),-1,-1,1)$
3	$(\mathbf{J}(2),\mathbf{J}(2),1_3)$	$(-\omega,-\omega,\omega,1,\omega^{-1},-\omega^{-1},-\omega^{-1})$	$(y,-y^{-1},-1,1,-1,-y,y^{-1})$
			$y^2=x$

The generic character table of the group $G_2(q)$ shows that there exists at most one such local system: the normalized structure constant of the reduced monodromy tuple gives

$$\begin{array}{|lclclclc|}\hline n(u_2,u_2,k_3,u_3) & = & 1 & n(u_2,u_2,k_3,u_4) & = & n(u_2,u_2,k_3,u_5) & = & 0\\ n(u_2,u_2,k_3,k_{2,1}) & = & 1 & & & n(u_2,u_2,k_3,h_{1a}) & = & 1\\ \hline\end{array}$$

which proves the uniqueness in the case P_5.

It remains to show that the above local systems in the P_5-case can be obtained by middle convolution, tensor operations and rational pullbacks: applying the functor

$$M_\phi=\mathrm{MT}_{\mathcal{L}(1,1,\phi^{-1})}\circ\mathrm{MC}_{\phi\omega^{-1}}\circ\mathrm{MT}_{\mathcal{L}(1,1,\phi\omega)}\circ\mathrm{MC}_{\phi^{-1}\omega}\circ\mathrm{MT}_{\mathcal{L}(1,1,\omega^{-1})},$$

where

$$\phi=\begin{cases}1 & \text{in case } 1),\\ -1 & \text{in case } 2),\\ x & \text{in case } 3),\end{cases}$$

we obtain an orthogonally rigid local system of rank 4 or 5 with the following local monodromy data:

nr.	rk				
1	4	$(\mathbf{J}(2),\mathbf{J}(2))$	$(\mathbf{J}(2),\mathbf{J}(2))$	$(\mathbf{J}(3),1)$	$(\omega,\omega^{-1},1_2)$
2	5	$(\mathbf{J}(2),\mathbf{J}(2),1)$	$(\mathbf{J}(2),\mathbf{J}(2),1)$	$(-\mathbf{J}(2),-\mathbf{J}(2),1)$	$(\omega,\omega^{-1},1_3)$
3	5	$(\mathbf{J}(2),\mathbf{J}(2),1)$	$(\mathbf{J}(2),\mathbf{J}(2),1)$	$(x,x,x^{-1},x^{-1},1)$	$(\omega,\omega^{-1},1_3)$

Applying the functor

$$M_\phi=\mathrm{MT}_{\mathcal{L}(1,\phi^{-1},1)}\circ\mathrm{MC}_{\phi}\circ\mathrm{MT}_{\mathcal{L}(1,\phi,\phi^{-1})}\circ\mathrm{MC}_{\phi^{-1}}\circ\mathrm{MT}_{\mathcal{L}(1,1,\phi)},$$

where

$$\phi = \begin{cases} 1 & \text{in case} \quad 1), \\ -1 & \text{in case} \quad 2), \\ x & \text{in case} \quad 3), \end{cases}$$

we obtain an orthogonally rigid local system of rank 4 with the following local monodromy data:

nr.	rk				
1	4	$(\mathbf{J}(2), \mathbf{J}(2))$	$(\mathbf{J}(2), \mathbf{J}(2))$	$(\mathbf{J}(3), 1)$	$(\omega, \omega^{-1}, 1, 1)$
2	4	$(\mathbf{J}(2), \mathbf{J}(2))$	$(-\mathbf{J}(2), -\mathbf{J}(2))$	$(\mathbf{J}(3), 1)$	$(\omega, \omega^{-1}, 1, 1)$
3	4	$(\mathbf{J}(2), \mathbf{J}(2))$	(x, x, x^{-1}, x^{-1})	$(\mathbf{J}(3), 1)$	$(\omega, \omega^{-1}, 1, 1)$

Via the isomorphism

$$\mathrm{SL}_2 \otimes \mathrm{SL}_2 = \mathrm{SO}_4$$

we can decompose it into linearly rigid irreducible local systems $\mathcal{L}_1$ and $\mathcal{L}_2$ of rank 2 with the following local monodromy data:

nr.	$\mathcal{L}_1$				$\mathcal{L}_2$				
1	$\mathbf{J}(2)$	1_2	$\mathbf{J}(2)$	(ω, ω^{-1})	1_2	$\mathbf{J}(2)$	$\mathbf{J}(2)$	(ω, ω^{-1})	
	$\mathbf{J}(2)$	1_2	$\mathbf{J}(2)$	$-(\omega, \omega^{-1})$	1_2	$\mathbf{J}(2)$	$\mathbf{J}(2)$	$-(\omega, \omega^{-1})$	
2	$\mathbf{J}(2)$	1_2	$\mathbf{J}(2)$	(ω, ω^{-1})	1_2	$-\mathbf{J}(2)$	$\mathbf{J}(2)$	(ω, ω^{-1})	
	$\mathbf{J}(2)$	1_2	$\mathbf{J}(2)$	$-(\omega, \omega^{-1})$	1_2	$-\mathbf{J}(2)$	$\mathbf{J}(2)$	$-(\omega, \omega^{-1})$	
3	$\mathbf{J}(2)$	1_2	$\mathbf{J}(2)$	(ω, ω^{-1})	1_2	(x, x^{-1})	$\mathbf{J}(2)$	(ω, ω^{-1})	
	$\mathbf{J}(2)$	1_2	$\mathbf{J}(2)$	$-(\omega, \omega^{-1})$	1_2	(x, x^{-1})	$\mathbf{J}(2)$	$-(\omega, \omega^{-1})$	$x^6 \neq 1$

This finishes the proof of the case P_5 and also the proof of Theorem 6.1. □

Appendix A. Generic character tables and structure constants

Let $\mathcal{C} = (C_1, \ldots, C_{r+1})$ be a tuple of conjugacy classes of a group G and

$$\Sigma(\mathcal{C}) = \{\sigma \in G^{r+1} \mid \sigma_i \in C_i, \sigma_1 \cdots \sigma_{r+1} = 1\}.$$

Then the *normalized structure constant* $n(\mathcal{C})$ is defined as

$$n(\mathcal{C}) = \frac{\mid \Sigma(\mathcal{C}) \mid}{\mid Inn(G) \mid}.$$

The following result is well known (cf. [15, Chap. I, Thm. 5.8]):

Proposition A.1. *Let G be a finite group, let $\mathrm{Irr}(G)$ denote the set of irreducible characters of G and let $\mathcal{C} = (C_1, \ldots, C_{r+1})$ be a tuple of conjugacy classes of G with representatives $\sigma_1, \ldots, \sigma_{r+1}$. Then*

$$n(\mathcal{C}) = \frac{\mid Z(G) \mid \cdot \mid G \mid^{r-1}}{\prod_i \mid C_G(\sigma_i) \mid} \sum_{\chi \in \mathrm{Irr}(G)} \frac{\prod_i \chi(\sigma_i)}{\chi(1)^{r-1}}.$$

In order to find an upper bound for the number of local systems with the same tuple of local monodromy data we use reduction modulo l and derive the bound from the normalized structure constant:

Let G be a reductive algebraic group defined over $\mathbb{Z}$ which is an irreducible subgroup of GL_n (e.g. $G_2 \leq \mathrm{GL}_7$) and let $\mathcal{C} = (C_1, \ldots, C_{r+1})$ be a tuple of conjugacy classes in G. Consider the map

$$\pi : C_1 \times \cdots \times C_{r+1} \to G, (g_1, \ldots, g_{r+1}) \mapsto g_1 \cdots g_{r+1}$$

and let $X := \pi^{-1}(1)$ (with $1 \in G$ the neutral element). The variety X decomposes into irreducible components $X_1, \ldots, X_k$. The following result is the content of [13], Lemma 5.9.3, and will be useful below:

Lemma A.2. *Let R be a subring of $\mathbb{C}$ which is finitely generated as a $\mathbb{Z}$-algebra. Then there exists an $N \in \mathbb{N}_{>0}$ such that for any prime number ℓ which does not divide N there exists a finite extension K_ν of $\mathbb{Q}_\ell$ with valuation ring O_ν and an isomorphism $\iota : \mathbb{C} \to \bar{\mathbb{Q}}_\ell$ under which R is mapped into O_ν.*

The idea of the proof is as follows: Using Noether normalization, R is an integral extension of $\mathbb{Z}[\frac{1}{N}][x_1, \ldots, x_{r+1}]$, where $x_1, \ldots, x_{r+1}$ are algebraically independent. By the axiom of choice, for any algebraically independent set $\{y_1, \ldots, y_{r+1}\} \subseteq \mathbb{Z}_\ell$ (where ℓ does not divide N), there exists an isomorphism $\iota : \mathbb{C} \to \bar{\mathbb{Q}}_\ell$ which maps x_i to y_i, $i = 1, \ldots, r+1$.

Lemma A.2 implies that for any tuple $\mathcal{C}$ of conjugacy classes there exists an $M \in \mathbb{N}_{>0}$ such that for any prime number ℓ which does not divide M, there exists a finite extension K_ν of $\mathbb{Q}_\ell$ with valuation ring O_ν such that $C_1 \times \cdots \times C_{r+1}$ and $X = \pi^{-1}(1)$ is defined over O_ν. Similarly, for any $\mathbf{g} = (g_1, \ldots, g_{r+1}) \in X$ there exists an $N \in \mathbb{N}_{>0}$ such that for any prime number ℓ which does not divide N, there exists a finite extension K_ν of $\mathbb{Q}_\ell$ with valuation ring O_ν such that the coefficients of all elements of $\mathbf{g}$ are contained in O_ν. Hence, for almost all ℓ we find $\nu \mid \ell$ such that we can reduce the entries of $\mathbf{g}$ modulo the valuation ideal $m_\nu \subseteq \mathcal{O}_\nu$. In this way we obtain the *reduced tuple* $\bar{\mathbf{g}} = (\bar{g}_1, \ldots, \bar{g}_{r+1}) \in (\bar{C}_1, \ldots, \bar{C}_{r+1})$, where $\bar{C}_i$ is the conjugacy class of $\bar{g}_i$ in $G(\mathbb{F}_q)$ with $\mathbb{F}_q = O_\nu/m_\nu$. For positive natural numbers k, let $\mathcal{C}(q^k)$ denote the tuple of conjugacy classes of $\bar{g}_1, \ldots, \bar{g}_{r+1}$ in the group $G(\mathbb{F}_{q^k})$.

Theorem A.3. *Suppose that G is an irreducible simple algebraic subgroup of GL_n which is defined over $\mathbb{Z}$ and suppose that there exists an $s \in \mathbb{N}> 0$ such that*

$$\sup_q(\lim_k \lfloor n(\mathcal{C}(q^k)) \rfloor) = s,$$

where the supremum is taken over all prime powers q which are cardinalities of the residue fields of ν as above. Then, up to diagonal $G(\mathbb{C})$-conjugation, there exist at most s tuples

$$\mathbf{g}_i := (g_{i,1}, \ldots, g_{i,r+1}) \in C_1 \times \cdots \times C_{r+1} \quad (i = 1, \ldots, s)$$

with $g_{i,1} \cdots g_{i,r+1} = 1$ and such that the generated subgroup $\langle g_{i1}, \ldots, g_{i,r+1} \rangle$ is irreducible.

Proof. Assume that there exist $t > s$ different equivalence classes (w.r. to diagonal $G(\mathbb{C})$-conjugation) of tuples

$$\mathbf{g}_i = (g_{i,1}, \ldots, g_{i,r+1}) \in C_1 \times \cdots \times C_{r+1} \quad (i = 1, \ldots, t)$$

with $g_{i,1} \cdots g_{i,r+1} = 1$ and such that the generated subgroup $\langle g_{i1}, \dots, g_{i,r+1} \rangle$ is irreducible. We have the following two cases:

Case 1. The tuples $\mathbf{g}_i = (g_{i,1}, \dots, g_{i,r+1})\,(i = 1, \dots, t)$ lie in t different irreducible components X_i of X. By Lemma A.2, for almost all ℓ there exists a finite extension K_ν of $\mathbb{Q}_\ell$ such that $\mathbf{g}_i \in X_i(O_\nu)\,(i = 1, \dots, t)$. If $\ell \gg 0$ and $k \gg 0$, then the reductions modulo m_ν of the components X_i remain different. Hence reduction modulo the maximal ideal m_ν of O_ν leads to t different equivalence classes (under diagonal conjugation with elements in $G(\mathbb{F}_q)$) $\bar{\mathbf{g}}_i \in (\bar{C}_1, \dots, \bar{C}_{r+1})$, contrary to $t > s = \sup_q(\lim_k \lfloor n(\mathcal{C}(q^k)) \rfloor)$.

Case 2. Two of the tuples, say $\mathbf{g}_1$ and $\mathbf{g}_2$, lie in the same irreducible component X_1. Since $\langle \mathbf{g}_1 \rangle$ is irreducible the $G(\mathbb{C})$-stabilizer of $\mathbf{g}_1 \in C_1 \times \cdots \times C_{r+1}$ under diagonal conjugation is equal to the centralizer of $\langle \mathbf{g}_1 \rangle$ and hence coincides with the (finite) center $Z(G)$ of G. This implies that the dimension of the component X_1 of X with $\mathbf{g}_1 \in X_1$ is $\geq \dim G$. Therefore, by the assumption in Case 2, $\dim X_1 > \dim G$ and, by dimension reasons, there exist infinitely many $G(\mathbb{C})$-orbits $V_j\,(j \in J)$ in X_1. Pick $u > s$ different orbits $V_1, \dots, V_u$ and representatives

$$v_k \in V_k \quad (k = 1, \dots, u).$$

Suppose that the $V_1, \dots, V_u$ are defined over R, where we see R as a subring of O_ν $(\nu | \ell)$ as above. We claim that for $\ell \gg 0$, the reductions modulo-ν are different. This can be seen inductively as follows: The orbits are (quasi-)affine varieties inside an ambient affine space $\mathbb{A}^s$. Pick functions f_j in the vanishing ideals of V_j with the property that for $j \neq j'$, there exists $v_{j'} \in V_{j'}$ with $f_j(v_{j'}) \neq 0$. Extending scalars and assuming ℓ large enough, we can assume that the functions f_j and the v_j are defined over R and hence over O_ν. If $f_j(v_{j'})$ is algebraic, then for almost all ℓ the inequality $f_j(v_{j'}) \neq 0$ will hold modulo ν for all pairs of j, j' where $j \neq j'$. If $f_j(v_{j'})$ is transcendent, then with the freedom to choose the isomorphism $\iota : \mathbb{C} \to \bar{\mathbb{Q}}_\ell$ (see the remark following Lemma A.2) in a way that the inequality $f_j(v_{j'}) \neq 0$ will hold modulo ν for all pairs of j, j' where $j \neq j'$. Therefore, the orbits remain different modulo ν, and hence $\sup_q(\lim_k \lfloor n(\mathcal{C}(q^k)) \rfloor) \geq u$, a contradiction to $u > s$.

Remark A.4. Recall that there are character tables for $G(\mathbb{F}_q)$ which compute the character values of the elements of $G(\mathbb{F}_q)$ as functions depending on q, the so-called *generic character tables*. For groups with small Lie rank, these generic character tables are implemented in [10], especially, the case $G = G_2$ (cf. [5] and [11, Anhang B]) can be found there. Using the generic character table of $G_2(q)$ we can determine $n(\bar{\mathcal{C}}(q^k))$ and also $\sup_q(\lim_k \lfloor n(\mathcal{C}(q^k)) \rfloor)$ in many cases.

Remark A.5. We give an overview of the class representatives c_j in $G_2(q)$ taken from Chang and Ree. In order to determine $\lim_k \lfloor n(\mathcal{C}(q^k)) \rfloor$ we can assume that the eigenvalues of all class representatives of $C_1, \dots, C_{r+1}$ are in $\mathbb{F}_q$. Otherwise we replace q by q^4. The generic character table depends on the congruence of q mod 12. Thus we can also assume that $q \equiv 1 \mod 12$ (by extending the ground field). We list (in the notation of Chang and Ree [5]) for a class representative c_j having eigenvalues in $\mathbb{F}_q$ its corresponding Jordan form. The order of the centralizer of c_j in $G \in \{G_2, O_7, \mathrm{GL}_7\}$ is a polynomial in q of degree $d_G := \dim C_{G(\overline{\mathbb{F}_q})}(g_j)$.

class rep.	Jordan form	d_{G_2}	d_{O_7}	d_{GL_7}	conditions
1	1	14	21	49	
u_1	$(J(2), J(2), 1, 1, 1)$	8	13	29	
u_2	$(J(3), J(2), J(2))$	6	9	19	
u_3	$(J(3), J(3), J(1))$	4	7	17	
u_4	$(J(3), J(3), J(1))$	4	7	17	
u_5	$(J(3), J(3), J(1))$	4	7	17	
u_6	$J(7)$	2	3	7	
k_2	$(-1_4, 1_3)$	6	9	25	
$k_{2,1}$	$(-J(2), -J(2), 1, 1, 1)$	4	7	17	
$k_{2,2}$	$(-J(2), -J(2), J(3))$	4	5	11	
$k_{2,3}$	$(-J(3), -J(1), J(3))$	2	3	9	
$k_{2,4}$	$(-J(3), -J(1), J(3))$	2	3	9	

class rep.	Jordan form	d_{G_2}	d_{O_7}	d_{GL_7}	conditions
k_3	$(\omega, \omega, \omega, 1, \omega^{-1}, \omega^{-1}, \omega^{-1})$	8	9	19	$\omega^3 = 1$
$k_{3,1}$	$(\omega J(2), \omega^{-1} J(2), \omega, \omega^{-1}, 1)$	4	5	11	
$k_{3,2}$	$(\omega J(3), \omega^{-1} J(3), 1)$	2	3	7	
$k_{3,3,i}$	$(\omega J(3), \omega^{-1} J(3), 1)$	2	3	7	$i = 1, 2,\ k_{3,3,1}^{-1} \sim k_{3,3,2}$
h_{1a}	$(x, x, x^{-1}, x^{-1}, 1, 1, 1)$	4	7	17	$x^{q-1} = 1,\ x^2 \neq 1$
$h_{1a,1}$	$(xJ(2), x^{-1}J(2), J(3))$	2	3	7	
h_{1b}	$(x, x, x^2, 1, x^{-1}, x^{-1}, x^{-2})$	4	5	11	$x^{q-1} = 1,\ x^3 \neq 1, x^4 \neq 1$
	$(i, i, -1, 1, -1, i^{-1}, i^{-1})$	4	5	13	
$h_{1b,1}$	$(xJ(2), x^{-1}J(2), x^2, x^{-2}, 1)$	2	3	7	
	$(iJ(2), i^{-1}J(2), -1, -1, 1)$	2	3	9	
h_1	$(x, y, xy, 1, (xy)^{-1}, y^{-1}, x^{-1})$	2	3	7	$x^{q-1} = y^{q-1} = 1$
					pairw. diff. eigenvalues
	$(x, -1, -x, 1, -x^{-1}, -1, x^{-1})$	2	3	9	

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