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A C^2 GENERIC TRICHOTOMY FOR DIFFEOMORPHISMS: HYPERBOLICITY OR ZERO LYAPUNOV EXPONENTS OR THE C^1 CREATION OF HOMOCLINIC BIFURCATIONS

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ABSTRACT. Palis conjectured that densely in $\mathrm{Diff}^r(M)$, $r \geq 1$, diffeomorphisms are either hyperbolic or exhibit homoclinic bifurcations. We prove a generic trichotomy for C^2 diffeomorphisms: an Axiom A diffeomorphism with no cycles or Kupka-Smale ones admitting zero Lyapunov exponents or the C^1 creation of homoclinic bifurcations (i.e., the creation of homoclinic tangencies or heterodimensional cycles by some C^1 small perturbations).

INTRODUCTION

Since the discovery of examples of open sets consisting of nonhyperbolic diffeomorphisms in the end of the 1960s by Abraham and Smale [2] and Newhouse [24] on compact manifolds, it turned out that hyperbolicity is not a universal property and the study of nonhyperbolic diffeomorphisms is indispensable for the global understanding of dynamics in the space of diffeomorphisms. Motivated by this, Palis proposed the following conjecture in the late 1980s.

C^r **Palis Conjecture 1** ([27], [28], [29])**.** *The diffeomorphisms on a compact manifold exhibiting either a homoclinic tangency or a heterodimensional cycle are C^r dense in the complement of the closure of the hyperbolic ones for any $r \geq 1$.*

Pujals and Sambarino [38] proved the C^1 Palis Conjecture for surface diffeomorphisms about ten years ago. Since this remarkable achievement, extending their result to higher dimensions has been one of the main motivations for the study of nonhyperbolic diffeomorphisms. There has been a lot of progress, yielding quite a number of results (see for instance [1], [8], [9], [35], [36], [44] and notably [10]). The general strategy on this type of result is the following:

(I) Diffeomorphisms C^1 far away from those exhibiting a homoclinic tangency have dominated splittings (a weak form of hyperbolicity).
(II) Get more information about the dominated splittings if diffeomorphisms are C^1 far away from those exhibiting a heterodimensional cycle.
(III) Remove zero Lyapunov exponents or prove the nonexistence of zero Lyapunov exponents.

Our novelty here concerns (III). We have obtained the existence of C^2 zero Lyapunov exponents for some ergodic measure as an obstruction to the C^1 Palis

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Conjecture. As a consequence, one can use structure theories for C^2 diffeomorphisms such as Pesin Theory ([3], [19], [32]) or Pujals-Sambarino Theory ([38], [39], [40]) in order to remove the zero Lyapunov exponents (although this is apparently a difficult step) and prove the conjecture. Moreover, our result can be thought of as a C^2 generic property in the direction of the C^2 Palis Conjecture, which has not appeared so far up to Pujals-Sambarino Theory (see Remark 3 below). Its advantage is that to know the obstruction to the C^2 hyperbolicity leads to knowing nice ergodic properties for C^2 diffeomorphisms (see [5] for instance). On the other hand, as indicated in [37], the difference between the C^1 and C^2 topologies is enormous, even for surface diffeomorphisms in an attempt to understand generic dynamics. Indeed, the main difficulty in this paper is that we cannot use C^1 generic properties that have been highly developed, while C^2 generic properties obtained so far are very restricted. We here rely on the structure based on Pesin Theory and C^2 generic properties recently obtained by [16] in addition to classical ones.

Let M be a smooth compact manifold without boundary, and let $\mathrm{Diff}^r(M)$, $r \geq 1$, be the space of C^r diffeomorphisms endowed with the C^r topology. We define, for every hyperbolic periodic point p, its index $\mathrm{Ind}\,(p)$ by the dimension of the stable subspace. Let $\mathrm{Per}(f)$ be the set of periodic points of f and let $P^i(f)$, $0 \leq i \leq \dim M$, be that of hyperbolic periodic points with index i. A *heterodimensional cycle* is a geometric configuration between two hyperbolic periodic saddles with different indices such that their stable and unstable manifolds have mutual nonempty intersection; i.e., p, $q \in \mathrm{Per}(f)$ with $\mathrm{Ind}\,(p) \neq \mathrm{Ind}\,(q)$ satisfy $W^s(p,f) \cap W^u(q,f) \neq \emptyset$ and $W^u(p,f) \cap W^s(q,f) \neq \emptyset$. Note that one of the intersections is nontransversal. In particular, we say that f exhibits a heterodimensional cycle in U if there are points $x \in W^s(p,f) \cap W^u(q,f)$ and $y \in W^u(p,f) \cap W^s(q,f)$ such that the closures of the full orbit $\mathcal{O}_f(x)(= \{f^n(x) : n \in \mathbf{Z}\})$ of x and that of y are both contained in U. A *homoclinic tangency* is a nontransversal homoclinic point. We say that f exhibits a homoclinic tangency in U if the closure of a nontransversal homoclinic orbit is contained in U. Let $\mathcal{HT}(M)$ be the set of diffeomorphisms in $\mathrm{Diff}^1(M)$ exhibiting a homoclinic tangency. For a hyperbolic periodic point $p \in \mathrm{Per}(f)$, define the *homoclinic class* of p by

$$H_f(p) = \overline{W^s(\mathcal{O}_f(p), f) \pitchfork W^u(\mathcal{O}_f(p), f)}.$$

Note that $H_f(p)$ is either $\mathcal{O}_f(p)$ or the closure of transversal homoclinic points associated to $\mathcal{O}_f(p)$. For $G \subset M$ and $\rho > 0$, define $U_\rho(G) = \{x \in M : d(x, G) < \rho\}$.

The main result is the following theorem:

Theorem A. *There exists a residual subset $\mathcal{R} \subset \mathrm{Diff}^2(M)$ such that if $f \in \mathcal{R}$, then one of the following properties hold:*

(a) *f is a hyperbolic diffeomorphism (Axiom A with no cycles).*
(b) *f is a Kupka-Smale diffeomorphism admitting zero Lyapunov exponents for some ergodic measure.*
(c) *f can be C^1 perturbed to obtain a homoclinic tangency or a heterodimensional cycle; more precisely,*
   (c-1) *given a C^1 neighborhood $\mathcal{U}$ of f and $\varepsilon > 0$, there exist $g \in \mathcal{U}$ and $p \in \mathrm{Per}(f)$ such that g exhibits a homoclinic tangency associated to $\mathcal{O}_f(p)$ in $U_\varepsilon(\mathcal{O}_f(p))$;*

(c-2) *given a C^1 neighborhood $\mathcal{U}$ of f and $\varepsilon > 0$, there exist $g \in \mathcal{U}$ and $p \in \mathrm{Per}(f)$ with $H_f(p) \neq \mathcal{O}_f(p)$ such that g exhibits a heterodimensional cycle in $U_\varepsilon(H_f(p))$.*

Remark 1. This result is already known in the C^1 topology. (See [9] or [44] for instance.) So, this result is not surprising from a conceptual viewpoint under the knowledge of C^1 dynamics. However, the technical difference between the proofs for the C^1 and C^2 topologies is not small because it amounts to that of quantities of C^1 and C^2 generic properties.

Remark 2. The "trichotomy" in the title means that there are two kinds of three disjoint subsets of $\mathcal{R}$; that is, respectively, the set of $f \in \mathcal{R}$ satisfying property (a), that of $f \in \mathcal{R}$ satisfying property (c) (resp. property (b)) and that of $f \in \mathcal{R}$ satisfying property (b) (resp. property (c)) but not satisfying property (c) (resp. property (b)). If the last subset is absorbed in the second one, it is the C^1 Palis Conjecture (resp. a conjecture of Díaz and Gorodetski mentioned below for the C^2 topology).

The C^1 creation of homoclinic bifurcations from C^2 diffeomorphisms has been considered in [14] for heterodimensional cycles and in [39], [40] for homoclinic tangencies. In [14], we considered C^2 partially hyperbolic diffeomorphisms with one-dimensional center bundles, and proved that either (a) or (b) or (c-2) of Theorem A holds. Since homoclinic tangencies never appear in this partially hyperbolic case, Theorem A is a C^2 generic extension of this result to the general case. In [39], Pujals and Sambarino proved that if a C^2 diffeomorphism has infinitely many sinks or sources with unbounded periods, it can be C^1 approximated by one exhibiting a homoclinic tangency. This can be regarded as a partial converse of a famous Newhouse phenomena ([25], [26]); that is, a C^2 homoclinic tangency can cause an open set (a so-called Newhouse domain) containing both a dense subset of diffeomorphisms exhibiting homoclinic tangencies and a residual subset of those exhibiting infinitely many sinks or sources with unbounded periods. On the other hand, an example in [7] shows that the presence of a C^2 homoclinic tangency does not always imply the existence of zero Lyapunov exponents for some ergodic measure.

In order to drop C^1 perturbations in property (c-2), as seen from the way we create heterodimensional cycles in the proof of Theorem A, we can replace property (c-2) by the following:

(d) Given $\varepsilon > 0$, there exists $p \in \mathrm{Per}(f)$ with $H_f(p) \neq \mathcal{O}_f(p)$ such that f exhibits a pseudo-heterodimensional cycle in $U_\varepsilon(H_f(p))$.

Here, we say that f exhibits a *pseudo-heterodimensional cycle* if there exist two hyperbolic periodic saddles $p, q \in \mathrm{Per}(f)$ with $\mathrm{Ind}\,(p) \neq \mathrm{Ind}\,(q)$ such that $W^u(p,f) \pitchfork W^s(q,f) \neq \emptyset$ and for every $\varepsilon > 0$ the forward orbit of a point in $W^s(p,f)$ and the backward orbit of a point in $W^u(q,f)$ are parts of some ε-pseudo-orbit. Moreover, property (d) requires that the closure of the full orbit of some point in $W^u(p,f) \pitchfork W^s(q,f)$ and the closure of the above ε-pseudo-orbit are both contained in $U_\varepsilon(H_f(p))$.

The following theorem has a statement without involving C^1 topology. One can observe that even the corresponding statement without option (c) in the C^1 topology is easily proved by already standard arguments, including the application of the C^1 Connecting Lemma [12]. However, the argument to obtain a transversal intersection in the pseudo-heterodimensional cycle, C^2 generically, is accompanied by a difficulty stemming from the lack of the general C^2 Connecting Lemma.

Theorem B. *There is a residual subset* $\mathcal{R} \subset \mathrm{Diff}^2(M)$ *such that if* $f \in \mathcal{R}$ *admits a dominated splitting over* $\overline{P^{i_0}(f)}$ *for some* $0 < i_0 < \dim M$ *that extends hyperbolic splittings over periodic orbits with index* i_0*; i.e., a* Df*-invariant continuous splitting* $TM|\overline{P^{i_0}(f)} = E \oplus F$ *with* $\dim E(x) = i_0$ *for all* $x \in \overline{P^{i_0}(f)}$*, satisfying that for some* $\ell > 0$*,*

$$\|(Df^\ell)|E(x)\| \cdot \|(Df^{-\ell})|F(f^\ell(x))\| < 1/2, \qquad \forall x \in \overline{P^{i_0}(f)},$$

and has a neighborhood $\mathcal{U}(f)$ *with a residual subset* $\mathcal{R}(f)$ *such that* $\bigcup_{0<i_0\neq i<\dim M} \overline{P^i(g)}$ *is a countable union of disjoint basic sets (i.e., transitive isolated hyperbolic sets) for all* $g \in \mathcal{R}(f)$*, then* f *satisfies one of the following properties:*

(a) *f is a hyperbolic diffeomorphism (Axiom A with no cycles).*
(b) *f is a Kupka-Smale diffeomorphism admitting zero Lyapunov exponents for some ergodic measure.*
(c) *Given* $\varepsilon > 0$*, there exists* $p \in P^{i_0}(f)$ *with* $H_f(p) \neq \mathcal{O}_f(p)$ *such that* f *exhibits a pseudo-heterodimensional cycle in* $U_\varepsilon(H_f(p))$ *involving a hyperbolic periodic saddle with index* i_0*.*

Clearly, the surface diffeomorphism case provides the following corollary of Theorem B (although it is essentially a corollary of [16, Theorem C]). Díaz and Gorodetski conjectured (as a weak version of [11, Conjecture 1]) that, C^r generically in $\mathrm{Diff}^r(M)$, diffeomorphisms are either hyperbolic or admit nonhyperbolic ergodic measures for any $r \geq 1$. The corollary shows that this conjecture is true for C^2 surface diffeomorphisms as long as they have dominated splittings over the closures of hyperbolic periodic saddles.

Corollary C. *Let* $\dim M = 2$*. There is a residual subset* $\mathcal{R} \subset \mathrm{Diff}^2(M)$ *such that if* $f \in \mathcal{R}$ *admits a dominated splitting over* $\overline{P^1(f)}$*, then* f *satisfies one of the following properties:*

(a) *f is a hyperbolic diffeomorphism (Axiom A with no cycles).*
(b) *f is a Kupka-Smale diffeomorphism admitting zero Lyapunov exponents for some ergodic measure.*

Remark 3. By virtue of the structure theorems for C^2 diffeomorphisms with dominated splittings by Pujals and Sambarino [38], we can remove zero Lyapunov exponents under an additional assumption. Indeed, if one assumes the existence of a dominated splitting over the closure of limit points in $M \setminus (P^0(f) \cup P^2(f))$ for $f \in \mathcal{R}$ in Corollary C, Pujals and Sambarino have already proved in [38, p. 966] that f is an Axiom A diffeomorphism with no cycle. Using [16, Theorem C], we see that having a dominated splitting over the closure of points $x \in M \setminus (P^0(f) \cup P^2(f))$ at which the Lyapunov exponents for some ergodic measure exist is sufficient to have the same conclusion as above. The two closures C^1 generically coincide by Pugh's General Density Theorem ([33], [34]), but we do not have the corresponding C^2 generic result yet. As mentioned in [29], C^2 Axiom A diffeomorphisms with no cycles satisfy the properties of the Main Global Conjecture by Palis (see also [27], [28]). So one can see that the conjecture is supported in this setting. (See [42] for the noninvertible case.)

In Section I, several lemmas that were proved elsewhere are prepared. In Section II, Theorems II.1 and II.2 are given and reduce the proof of Theorem A to proving the two theorems. More precisely, Theorem II.1 provides that a C^2 locally generic sufficient condition for a subbundle over the closure of supports of some ergodic measures with the same index is contracting. Applying Theorem II.1 to each homoclinic class for C^2 locally generic diffeomorphisms and then Theorem II.2, we see that another subbundle is expanding and therefore all the homoclinic classes are hyperbolic, which is C^2 generically a sufficient condition for the full hyperbolicity (Axiom A with no cycles). Theorem II.1 is proved in Section III and Theorem II.2 is proved in Section V after giving two preliminary lemmas in Section IV. Finally, we prove Theorem B in Section VI by referring to the proof of Theorem A.

I. Preliminaries

In this section, we give definitions and lemmas that were proved elsewhere. A *dominated splitting* over a compact invariant set Λ of $f \in \mathrm{Diff}^1(M)$ is a continuous, f-invariant (i.e., invariant under the derivative of f) splitting

$$TM|\Lambda = E \oplus F$$

such that there exist $m \in \mathbf{Z}^+$ and $0 < \lambda < 1$ satisfying

$$\|(Df^m)|E(x)\| \cdot \|(Df^{-m})|F(f^m(x))\| < \lambda$$

for all $x \in \Lambda$ (which is an equivalent condition given in the hypothesis of Theorem B); the dominated splitting with such m and λ is called an (m, λ)-*dominated splitting*. In particular, when $\dim E(x)$ is constant for all $x \in \Lambda$, we say that the dominated splitting is *homogeneous* and then one may write it as $\dim E$. It is well known that the dominated splitting $E \oplus F$ carries a continuous family of C^1 disks $\{D^E_\rho(x) : x \in \Lambda\}$ (resp. $\{D^F_\rho(x) : x \in \Lambda\}$) tangent to $E(x)$ (resp. $F(x)$) at x with size $\rho > 0$, which is locally invariant in the sense that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$f(D^E_\delta(x)) \subset D^E_\varepsilon(f(x)) \quad \big(\text{resp. } f^{-1}(D^F_\delta(x)) \subset D^F_\varepsilon(f^{-1}(x))\big)$$

for all $x \in \Lambda$. The dominated splitting $E \oplus F$ has its continuous extension $\widehat{E} \oplus \widehat{F}$ over a neighborhood U_0 of Λ, and there are neighborhoods $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$ and $U (\subset U_0)$ of Λ in M for which we have a g-invariant dominated splitting

$$TM|M(g, U) = \widetilde{E}_g \oplus \widetilde{F}_g$$

for all $g \in \mathcal{U}$, where

$$M(g, U) = \bigcap_{n \in \mathbf{Z}} g^n(\overline{U}).$$

This satisfies $\max\{d(\widetilde{E}_g(y), \widetilde{E}_f(x)),\ d(\widetilde{F}_g(y), \widetilde{F}_f(x))\} \to 0$ $(g \to f,\ y \to x)$ with $\widetilde{E}_f = \widehat{E}|M(f, U)$ and $\widetilde{F}_f = \widehat{F}|M(f, U)$. When there exist constants $K > 0$ and $0 < \gamma < 1$ such that

$$\|(Df^n)|E(x)\| < K\gamma^n$$
$$(\text{resp. } \|(Df^{-n})|F(x)\| < K\gamma^n\ )$$

for all $x \in \Lambda$ and $n \geq 1$, we say that E (resp. F) is *contracting* (resp. *expanding*). When E is contracting and F is expanding simultaneously, $E \oplus F$ is called a *hyperbolic splitting* and then Λ is said to be a *hyperbolic set*. If $\dim E(x)$ is constant for all $x \in \Lambda$, the index of the hyperbolic set Λ is defined by $\dim E$. A hyperbolic set is a *basic set* if it is in particular transitive and isolated; i.e., $\omega_f(x) = \Lambda$ for some $x \in \Lambda$ and $M(f, U) = \Lambda$ for some neighborhood U of Λ, respectively.

Let $\mathcal{M}(M)$ denote the set of probabilities on the Borel σ-algebra of M endowed with its usual topology; i.e., the unique metrizable topology such that $\mu_k \to \mu$ if and only if $\int \varphi d\mu_k \to \int \varphi d\mu$ for every continuous function $\varphi : M \to \mathbf{R}$. For $f \in \mathrm{Diff}^1(M)$, denote by $\mathcal{M}_f(M)$ the set of f-invariant elements of $\mathcal{M}(M)$, $\mathcal{M}_e(f)$ the set of ergodic elements of $\mathcal{M}_f(M)$ and $\mathcal{M}_e^\infty(f)$ the set of elements of $\mathcal{M}_e(f)$ supported on infinitely many points. For $f \in \mathrm{Diff}^1(M)$, $x \in M$ and $n \in \mathbf{Z}^+$, define a probability $\mu(x, n, f) \in \mathcal{M}(M)$ by

$$\mu(x, n, f) = \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)}.$$

Let $\mathcal{M}(x, f)$ denote the set of $\mu \in \mathcal{M}(M)$ such that there exists a sequence of positive integers $n_1 < n_2 < \dots$ satisfying

$$\mu = \lim_{i \to +\infty} \mu(x, n_i, f).$$

Note that $\mathcal{M}(x, f) \subset \mathcal{M}_f(M)$. If $f \in \mathrm{Diff}^1(M)$, let $\Lambda(f)$ be the set of regular points; i.e., the set of points $x \in M$ satisfying the following properties: there exists a splitting $T_xM = \bigoplus_{i=1}^s E_i(x)$ (the *Lyapunov splitting* at x) and numbers $\lambda_1(x) > \cdots > \lambda_s(x)$ (the *Lyapunov exponents* at x) such that

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|(D_x f^n)v\| = \lambda_i(x)$$

for every $1 \le i \le s$ and $0 \ne v \in E_i(x)$. By Oseledets' theorem, $\Lambda(f)$ is a total probability set; i.e., $\mu(\Lambda(f)) = 1$ for every $\mu \in \mathcal{M}_f(M)$. Define, for $x \in \Lambda(f)$,

$$E^-(x) = \bigoplus_{\lambda_i(x)<0} E_i(x) \quad \text{and} \quad E^+(x) = \bigoplus_{\lambda_i(x)>0} E_i(x).$$

If $\mu \in \mathcal{M}_e(f)$ admits no zero Lyapunov exponents, then we say that μ is *hyperbolic*. When $\mu \in \mathcal{M}_e(f)$ is hyperbolic,

$$TM|\mathrm{supp}\,(\mu) = E_\mu^- \oplus E_\mu^+$$

is called a *dominated splitting associated to the Lyapunov splitting* if it is a dominated splitting such that $E_\mu^-(x) = E^-(x)$ and $E_\mu^+(x) = E^+(x)$ at μ-a.e. x. For hyperbolic $\mu \in \mathcal{M}_e(f)$, define its index as $\mathrm{Ind}\,(\mu) = \dim E^-(x)$ at μ-a.e. x. By the Ergodic Decomposition Theorem, a Borel set $\Gamma(f)$ defined as the set of $x \in M$ for which we have $\mu_x \in \mathcal{M}_e(f)$ and $x \in \mathrm{supp}\,(\mu_x)$ is a total probability set, where μ_x is the unique probability measure on the Borel σ-algebra of M such that, for every continuous $\varphi : M \to \mathbf{R}$,

$$\int_M \varphi \, d\mu_x = \lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x))$$

holds, which comes from the Riesz Representation Theorem. (See [21, Chapter II.6].) For $0 \le i \le \dim M$, define

$$\Gamma^i(f) = \{x \in \Lambda(f) \cap \Gamma(f) \cap \Gamma(f^{-1}) : \ \mathrm{Ind}\,(\mu_x) = i\}.$$

It is well known that $\Gamma^i(f)$ is a Borel set for every $0 \le i \le \dim M$. If $f \in \mathrm{Diff}^2(M)$ and any Lyapunov exponent of any ergodic measure of f is nonzero, then the Pesin set denoted by $\bigcup_{\kappa=1}^{\infty}\Lambda_\kappa$ is a total probability set and the local stable and unstable manifolds $W^\sigma_{\delta_\kappa}(x,f)$ ($\sigma\in\{s,u\}$, $x\in\Lambda_\kappa$) are defined for some $\delta_\kappa>0$ independent of $x\in\Lambda_\kappa$ (see [3], [32]). In particular, letting

$$\Gamma^i_\kappa(f)=\Gamma^i(f)\cap\Lambda_\kappa,$$

we have $\dim W^s_{\delta_\kappa}(x,f)=i$ for all $x\in\Gamma^i_\kappa(f)$ and $\bigcup_{i=0}^{\dim M}\bigcup_{\kappa=1}^{\infty}\Gamma^i_\kappa(f)$ is also a total probability set. For $0\le i\le\dim M$, define

$$S^i(f)=\overline{\Gamma^i(f)},$$

which satisfies that

$$S^i(f)=\overline{\{x\in\mathrm{supp}\,(\mu):\mu\in\mathcal{M}_e(f),\ \mathrm{Ind}\,(\mu)=i\}}.$$

Define

$$S(f)=\overline{\{x\in\mathrm{supp}\,(\mu):\mu\in\mathcal{M}_f(M)\}}.$$

If any Lyapunov exponent of any ergodic measure of f is nonzero, then

$$S(f)=\bigcup_{i=0}^{\dim M}S^i(f).$$

To work in the C^2 topology, we need the following three lemmas dealing with C^2 perturbations and C^2 generic properties.

Lemma I.1 ([16, Theorem A]). *Let $f\in\mathrm{Diff}^2(M)$ and let Λ be an isolated hyperbolic set of f with $\Omega(f|\Lambda)=\Lambda$. Suppose that there exists a sequence $\{p_k\}$ of periodic points of f with $p_k\notin\Lambda$ such that the probabilities $\mu(p_k,\ell_k,f)$ with ℓ_k, the period of p_k, $k\ge1$, converge to a probability μ that satisfies $\mu(\Lambda)>0$. Then, for every neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^2(M)$, there exists $g\in\mathcal{U}$ coinciding with f in a neighborhood of Λ and exhibiting a homoclinic point associated to a basic component of Λ.*

Lemma I.2 ([16, Theorem B]). *Let $f\in\mathrm{Diff}^2(M)$ have a hyperbolic set Λ and let $\mathcal{U}$ be a neighborhood of f such that the continuation $\Lambda(g)$ of Λ is defined for all $g\in\mathcal{U}$. Then, when $\Lambda(g)$ is a finite union of homoclinic classes for all g in a residual subset of $\mathcal{U}$, there exists an open and dense subset $\mathcal{O}(\mathcal{U})$ of $\mathcal{U}$ such that given $x\in M$, if $\mu(\Lambda(g))>0$ for all $\mu\in\mathcal{M}(x,g)$ with $g\in\mathcal{O}(\mathcal{U})$, then $x\in W^s(\Lambda(g),g)$.*

Lemma I.3 ([16, Theorem C]). *There exists a residual subset $\mathcal{R}^2_0$ of $\mathrm{Diff}^2(M)$ such that if $f\in\mathcal{R}^2_0$ and $S(f)$ is a countable uninon of disjoint basic sets, then f is an Axiom A diffeomorphism with no cycles.*

The next lemma is used for the C^1 creation of heterodimensional cycles, and it is a direct consequence of the C^1 Connecting Lemma for pseudo-orbits [4, Théoréme 1.2] by Bonatti and Crovisier. In this particular situation, the proof is also contained in [14, Section I] and [13, Case 2]. Let us give a few notation for the statement. Denote by $(x,y;f)$ with $y=f^n(x)$, $n\ge0$, a finite part of the forward orbit $\{f^j(x):0\le j\le n\}$, which is called a *string*. For a neighborhood U of $\mathcal{O}_f(p)$ with a hyperbolic periodic point p, denote by $H_f(p,U)$ the closure of transversal homoclinic points associated to $\mathcal{O}_f(p)$ whose orbits are contained in U when $H_f(p)\ne\mathcal{O}_f(p)$; otherwise $H_f(p,U)=H_f(p)=\mathcal{O}_f(p)$. It is easy to see that given points x, $y\in H_f(p,U)$ and $\varepsilon>0$, there exists a string $(z,f^n(z);f)$ contained in U such that $d(x,z)<\varepsilon$ and $d(y,f^n(z))<\varepsilon$.

Lemma I.4. *Let $f \in \mathrm{Diff}^1(M)$ have only hyperbolic periodic points and let p, $q \in \mathrm{Per}(f)$ be distinct hyperbolic periodic saddles. Given a neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$, neighborhoods U_p and U_q of $\mathcal{O}_f(p)$ and $\mathcal{O}_f(q)$ in M, respectively, and a positive number $\rho > 0$, if there exists a sequence of strings $(z_n, y_n; f)$, $n \geq 1$, that accumulates on both $H_f(p, U_p)$ and $H_f(q, U_q)$, then there exist $g \in \mathcal{U}$ (resp. $h \in \mathcal{U}$) coinciding with f in a neighborhood of $\mathcal{O}_f(p) \cup \mathcal{O}_f(q)$, and*

$$x \in W^s(p, g) \cap W^u(q, g) \qquad \big(\textit{resp. } y \in W^u(p, h) \cap W^s(q, h)\big)$$

such that

$$\overline{\mathcal{O}_g(x)} \subset U_\rho((z_n, y_n; f)) \cup U_p \cup U_q \qquad \big(\textit{resp. } \overline{\mathcal{O}_h(y)} \subset U_\rho((z_n, y_n; f)) \cup U_p \cup U_q\big)$$

for some arbitrarily large n.

Toward the full hyperbolicity, we start with a weak hyperbolicity over the supports of ergodic measures. We can observe that $m \in \mathbf{Z}^+$ and $0 < \lambda < 1$ in the next three lemmas are thought of as the same constants by taking a common multiple of the integers and the maximum of the positive numbers less than 1, and by setting these as m and λ again. When $x \in \mathrm{Per}(g)$ with $g \in \mathrm{Diff}^1(M)$, denote by $E_g^-(x)$ (resp. $E_g^+(x)$) the direct sum of the generalized eigenspaces for eigenvalues with modulus < 1 (resp. > 1), and by $\pi(x)$ the g-period of x.

Lemma I.5 (Mañé [20, Proposition II.1], [15, Lemma 3.3])**.** *Given a neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$ and a family $\mathcal{A} \subset \{(x, g) : x \in \mathrm{Per}(g),\ g \in \mathcal{U}\}$ with $E_g^\sigma(x) \neq \{0\}$ for all $\sigma \in \{-, +\}$ and $(x, g) \in \mathcal{A}$, the following properties hold.*

If there exist a C^1 neighborhood $\mathcal{V}$ of f and $\varepsilon > 0$ such that, for every $(x, g) \in \mathcal{A}$ with $g \in \mathcal{V}$ satisfying

$$(*) \qquad \inf\{\angle(E_g^-(x)\,, E_g^+(x)) : (x, g) \in \mathcal{A}\} > 0,$$

the isomorphism $(Dh^{\pi(x)})|E_g^-(x)$ is hyperbolic for all $h \in \mathrm{Diff}^1(M)$ ε-close to g and satisfying the properties:

(i) *$\mathcal{O}_h(x) = \mathcal{O}_g(x)$,*
(ii) *$(Dh^{\pi(x)})E_g^\sigma(x) = E_g^\sigma(x)$ for each $\sigma \in \{-, +\}$,*

then there exist constants $K > 0$, $m \in \mathbf{Z}^+$ and $0 < \lambda < 1$ such that

$$\prod_{j=0}^{[\pi(x)/m]-1} \|(Dg^m)|E_g^-(g^{mj}(x))\| \leq K\lambda^{[\pi(x)/m]}$$

for all $(x, g) \in \mathcal{A}$ with $g \in \mathcal{V}$ satisfying $(*)$ and $\pi(x) \geq m$.

Finally, we pick three lemmas from [15] that deal with diffeomorphisms that cannot be C^1 approximated by one exhibiting a homoclinic tangency. The presence of dominated splittings for such diffeomorphisms is proved by Wen [43]. Indeed, for every $\varepsilon > 0$, the absence of dominated splittings over the closure of hyperbolic periodic saddles causes a homoclinic tangency in the ε-neighborhood of some periodic orbit (to which the homoclinic tangency is associated) by a C^1 ε-perturbation. The C^1 creation of homoclinic tangencies in this paper is always of this type, creating a homoclinic tangency by a local perturbation along a periodic orbit. So, if $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathcal{HT}(M)}$, then such a C^1 creation of homoclinic tangencies does not occur for f. The first lemma from [15] deals with atomic hyperbolic ergodic measures.

Lemma I.6 ([15, Proposition 3.5]). *If $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathcal{HT}(M)}$, then there exist a neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$, constants $m \in \mathbf{Z}^+$ and $0 < \lambda < 1$ such that, for every hyperbolic $\nu \in \mathcal{M}_e(g) \setminus \mathcal{M}_e^\infty(g)$ with $g \in \mathcal{U}$ and $0 < \mathrm{Ind}\,(\nu) < \dim M$ supported on a periodic orbit of period larger than m, there is an (m, λ)-dominated splitting $TM|\mathrm{supp}\,(\nu) = E_\nu^- \oplus E_\nu^+$ associated to the Lyapunov splitting satisfying that*

$$\inf\big\{\angle(E_\nu^-(x)\,, E_\nu^+(x)) : x \in \mathrm{supp}\,(\nu),\, \nu \in \mathcal{M}_e(g) \setminus \mathcal{M}_e^\infty(g),\ g \in \mathcal{U}\big\} > 0.$$

The second one deals with nonatomic hyperbolic ergodic measures.

Lemma I.7 ([15, Theorem B]). *If $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathcal{HT}(M)}$, then there exist a neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$, constants $m \in \mathbf{Z}^+$ and $0 < \lambda < 1$ such that for every hyperbolic $\nu \in \mathcal{M}_e^\infty(g)$ with $g \in \mathcal{U}$ and $0 < \mathrm{Ind}\,(\nu) < \dim M$, there is an (m, λ)-dominated splitting $TM|\mathrm{supp}\,(\nu) = E_\nu^- \oplus E_\nu^+$ associated to the Lyapunov splitting satisfying that*

$$\inf\{\angle(E_\nu^-(x)\,, E_\nu^+(x)) : x \in \mathrm{supp}\,(\nu),\, \nu \in \mathcal{M}_e^\infty(g),\ g \in \mathcal{U}\} > 0.$$

The third one deals with general hyperbolic ergodic measures.

Lemma I.8 ([15, Proposition 3.8]). *Let $f \in \mathrm{Diff}^1(M)$ and let $\mu \in \mathcal{M}_e(f)$ be hyperbolic. Then there exist $\ell \in \mathbf{Z}^+$ and $0 < \gamma < 1$ depending on μ such that*

$$\liminf_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|(Df^\ell)|E^-(f^{\ell j}(x))\| < \log \gamma$$

and

$$\liminf_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|(Df^{-\ell})|E^+(f^{\ell j}(x))\| < \log \gamma$$

at μ-a.e. x.

II. Reduction of the proof of Theorem A

In this section, we show that the proof of Theorem A is reduced to proving the two theorems given below. Let $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathcal{HT}(M)}$. Then, by Lemmas I.6 and I.7, there exist constants $m \in \mathbf{Z}^+$ and $0 < \lambda < 1$ such that if $\mu \in \mathcal{M}_e(g)$ is a hyperbolic measure with $\mathrm{Ind}\,(\mu) = i$ and $g \in \mathrm{Diff}^1(M) \setminus \overline{\mathcal{HT}(M)}$ is sufficiently C^1 close to f, then we have a homogeneous (m, λ)-dominated splitting associated to the Lyapunov splitting:

$$TM|\mathrm{supp}\,(\mu) = E_\mu^- \oplus E_\mu^+$$

with $\dim E_\mu^- = i$ except for μ supported on a periodic orbit with period $\le m$. If in addition g has only hyperbolic periodic points, since the number of periodic orbits of g with period $\le m$ is finite, changing m with a multiple of m if necessary, we may suppose that there exists such a splitting over $\mathrm{supp}\,(\mu)$ for all hyperbolic $\mu \in \mathcal{M}_e(g)$ with index i. Then, by continuity, the splittings are extended to a homogeneous (m, λ)-dominated splitting over $S^i(g)$ for every $1 \le i \le \dim M - 1$:

$$TM|S^i(g) = E_g^i \oplus F_g^i \tag{1}$$

with $\dim E_g^i = i$.

Let $\mathcal{R}_1^2$ be the set of Kupka-Smale diffeomorphisms $f \in \mathrm{Diff}^2(M)$ at which the set-valued function $f \mapsto H_f(p)$ is continuous for every $p \in \mathrm{Per}(f)$, which is a C^2 residual set in $\mathrm{Diff}^2(M)$. Then define

$$\mathcal{R}^2 = \mathcal{R}_0^2 \cap \mathcal{R}_1^2 \quad \text{and} \quad \mathcal{D}^2 = \mathcal{R}^2 \setminus \overline{\mathcal{H}T(M)}$$

with $\mathcal{R}_0^2$ given by Lemma I.3. Our aim for the proof of Theorem A is to prove that every f in a dense subset of $\mathcal{D}^2$ has a neighborhood with a residual subset $\mathcal{R}(f)$ satisfying the following property. We say that *g does not admit zero Lyapunov exponents* or *g admits no zero Lyapunov exponents* if every element of $\mathcal{M}_e(g)$ is hyperbolic. The property is that if $g \in \mathcal{R}(f)$ neither admits zero Lyapunov exponents nor exhibits a pseudo-heterodimensional cycle as in property (d) leading to property (c-2) of Theorem A, then g is an Axiom A diffeomorphism with no cycles. Indeed, the residual subset $\mathcal{R}$ of $\mathrm{Diff}^2(M)$ in Theorem A is obtained by taking the union of $\mathcal{R}(f)$ over all f in the dense subset of $\mathcal{D}^2$ together with the interior in $\mathcal{R}^2$ of the set of diffeomorphisms satisfying property (c-1) of Theorem A.

Now let us state the two theorems.

Theorem II.1. *Let $f \in \mathcal{D}^2$ and $1 < i_0 < \dim M$. Suppose that there exist a C^2 neighborhood $\mathcal{U}_*^2$ of f and its residual subset $\mathcal{R}_*^2$ with f such that $\bigcup_{i=0}^{i_0-1} \Gamma^i(g)$ is contained in a countable union of hyperbolic homoclinic classes with indices less than i_0 for all $g \in \mathcal{R}_*^2 \cap \mathcal{D}^2$ neither admitting zero Lyapunov exponents nor exhibiting a pseudo-heterodimensional cycle. Then there exists a residual subset $\mathcal{R}_{**}^2 (\subset \mathcal{R}_*^2)$ of a C^2 neighborhood $\mathcal{U}_{**}^2$ of f such that if $g \in \mathcal{R}_{**}^2 \cap \mathcal{D}^2$ neither admits zero Lyapunov exponents nor exhibits a pseudo-heterodimensional cycle, and a g-invariant set*

$$\Lambda_g^{i_0} = \overline{\{x \in \mathrm{supp}\,(\mu) : \mu \in \mathcal{M}_g,\ \mathrm{Ind}\,(\mu) = i_0\}}$$

with some $\mathcal{M}_g \subset \mathcal{M}_e(g)$ is contained in a transitive set, then $E_g^{i_0}|\Lambda_g^{i_0}$ is contracting.

Theorem II.2. *Let $f \in \mathrm{Diff}^2(M) \setminus \overline{\mathcal{H}T(M)}$ neither admit zero Lyapunov exponents nor exhibit a pseudo-heterodimensional cycle. If an f-invariant set*

$$\Lambda^{i_0} = \overline{\{x \in \mathrm{supp}\,(\mu) : \mu \in \mathcal{M},\ \mathrm{Ind}\,(\mu) = i_0\}}$$

with some $\mathcal{M} \subset \mathcal{M}_e(f)$ is contained in a transitive set and $E_f^{i_0}|\Lambda^{i_0}$ is contracting, then $F_f^{i_0}|\Lambda^{i_0}$ is expanding (and therefore Λ^{i_0} is a hyperbolic set of f).

To see how Theorem A is proved from these theorems, we let $f \in \mathcal{D}^2$. Note that for every $\mu \in \mathcal{M}_e(f)$ if $\mathrm{Ind}\,(\mu) = 0$ (resp. $\mathrm{Ind}\,(\mu) = \dim M$), then $\mathrm{supp}\,(\mu)$ is a hyperbolic periodic source (resp. sink) (see [32] for instance). Since f has only a countable number of periodic points, the hypothesis of Theorem II.1 is satisfied for $i_0 = 1$ with

$$\mathcal{R}_*^2 = \mathcal{R}^2 \quad \text{and} \quad \mathcal{U}_*^2 = \mathrm{Diff}^2(M).$$

To proceed by induction, let us suppose that the hypothesis is satisfied for some $i_0 \geq 1$. Then, setting $\{p_n : n \geq 1\} = P^{i_0}(g)$ for $g \in \mathcal{R}_*^2 \cap \mathcal{D}^2$, we have

$$\Gamma^{i_0}(g) \subset \bigcup_{n \geq 1} H_g(p_n) \tag{2}$$

since every $x \in \Gamma^{i_0}(g)$ can be approximated by some $p \in P^{i_0}(g)$ with $x \in H_g(p)$ by the Pesin Theory including the Katok Closing Lemma (see [3], [19], [32]). It is well known that for every $p \in P^{i_0}(g)$ the closure of $\Gamma^{i_0}(g) \cap H_g(p)$ coincides with $H_g(p)$, and $H_g(p)$ is transitive. Therefore, for each $n \geq 1$ we can apply

Theorem II.1 to $\Lambda_g^{i_0} = H_g(p_n)$ in order to obtain that $E_g^{i_0}|H_g(p_n)$ is contracting for all $g \in \mathcal{R}^2_{**} \cap \mathcal{D}^2$ neither admitting zero Lyapunov exponents nor exhibiting a pseudo-heterodimensional cycle. Then, by Theorem II.2, $H_g(p_n)$ is a hyperbolic set for all $n \geq 1$, and hence (2) implies that $\Gamma^{i_0}(g)$ is contained in a countable union of hyperbolic homoclinic classes. Thinking of this g as f, we obtain the hypothesis with respect to $i_0 + 1$ for $f \in \mathcal{R}^2_{**} \cap \mathcal{D}^2$ with $\mathcal{R}^2_{**} \subset \mathcal{U}^2_{**}$ corresponding to $\mathcal{R}^2_* \subset \mathcal{U}^2_*$ in the previous case of i_0. Since $\mathcal{R}^2_{**} \cap \mathcal{D}^2$ is dense in $\mathcal{U}^2_{**} \cap \mathcal{D}^2$, by induction on i, we obtain a dense subset of $\mathcal{D}^2$ in which every f has a neighborhood with a residual subset $\mathcal{R}(f)$ such that if $g \in \mathcal{R}(f)$ neither admits zero Lyapunov exponents nor exhibits a pseudo-heterodimensional cycle, then

$$\bigcup_{i=0}^{\dim M} \Gamma^i(g) \subset \bigcup_{k \geq 1} \Lambda_k(g) \tag{3}$$

for a sequence $\Lambda_k(g)$, $k \geq 1$, of disjoint hyperbolic homoclinic classes for g. To simplify the notation, let us write $g = f$ and $\Lambda_k(g) = \Lambda_k$. Now we can see that this countable union is actually a finite union by the same argument as in the proof of [16, Lemma III.1] (just replacing $S(f)$ in the proof by $\bigcup_{i=0}^{\dim M} \Gamma^i(f)$). Since every homoclinic class of f is contained in $S(f)$, we have

$$\bigcup_{i=0}^{\dim M} \Gamma^i(f) \subset \bigcup_{k=1}^{s} \Lambda_k \subset S(f)$$

for some $s \geq 1$, and then take the closures to get

$$S(f) = \bigcup_{k=1}^{s} \Lambda_k.$$

Thus, by Lemma I.3, f is an Axiom A diffeomorphism with no cycles. As mentioned above Theorem II.1, this proves Theorem A if the pseudo-heterodimensional cycles that appeared in the proofs of Theorems II.1 and II.2 satisfy property (d) and lead to property (c-2) of Theorem A, which will be checked in the proofs that follow.

III. Proof of Theorem II.1

In this section, we prove Theorem II.1 using lemmas given in Section I. Before starting the proof of Theorem II.1, a few results that will be needed in the proof are prepared. We always consider f admitting no zero Lyapunov exponents.

The first lemma is conceptually an already well-known nature for such f, stated as the lack of contraction for subbundles of dominated splittings over compact invariant sets causes the presence of hyperbolic ergodic measures supported on them with lower indices.

Lemma III.1. *Let $f \in \mathrm{Diff}^1(M)$ admit no zero Lyapunov exponents and let Λ_k be a compact f_k-invariant set for $f_k \to f$ $(k \to +\infty)$ in $\mathrm{Diff}^1(M)$. Suppose that there exist constants $m \in \mathbf{Z}^+$ and $0 < \lambda < 1$ such that f_k-invariant homogeneous (m, λ)-dominated splittings $TM|\Lambda_k = E_k \oplus F_k$ with $\dim E_k = i_0$, $k \geq 1$, are admitted for some $0 < i_0 < \dim M$, satisfying*

$$\inf\{\angle(E_k(x), F_k(x)) : x \in \Lambda_k,\ k \geq 1\} > 0.$$

For every multiple l of m, if there is a sequence $\mu(x_k, \ell_k, f_k) \in \mathcal{M}(M)$, $k \geq 1$, of probabilities with $\ell_k \to +\infty$ $(k \to +\infty)$ and $x_k \in \Lambda_k$ satisfying

$$\liminf_{k \to +\infty} \frac{1}{[\ell_k/l]} \sum_{j=0}^{[\ell_k/l]-1} \log \|(Df_k^l)|E_k(f_k^{lj}(x_k))\| \geq 0,$$

then there exists an accumulation point $\bar{\mu}_l \in \mathcal{M}_f(M)$ of $\{\mu(x_k, \ell_k, f_k) : k \geq 1\}$ in $\mathcal{M}(M)$ with arbitrarily large l such that

$$\bar{\mu}_l\left(\bigcup_{i=0}^{i_0-1} \Gamma^i(f)\right) > 0.$$

Proof. Let Λ be the set of accumulation points of $\{x \in \Lambda_k : k \geq 1\}$. By continuity, we have an f-invariant homogeneous (m, λ)-dominated splitting

$$TM|\Lambda = E \oplus F$$

with $\dim E = i_0$ by the accumulation of $E_k \oplus F_k$, $k \geq 1$. Take neighborhoods V and U of Λ in M with $\overline{V} \subset U$ and $\mathcal{V}$ of f in $\mathrm{Diff}^1(M)$ such that there exist a continuous extension of $E \oplus F$ over U and a homogeneous (m, λ)-dominated splitting over $M(g, V)$:

$$TM|U = \widehat{E} \oplus \widehat{F} \quad \text{and} \quad TM|M(g, V) = E_g \oplus F_g,$$

with $\dim E_g = i_0$ for all $g \in \mathcal{V}$. Set $h = f^l$, $h_k = f_k^l$, $\widetilde{E}_k = E_{f_k}$ and $\widetilde{F}_k = F_{f_k}$ for a multiple l of m and $f_k \in \mathcal{V}$. Then define a sequence of probabilities $\nu_k \in \mathcal{M}(M)$, $k \geq 1$, by

$$\nu_k = \mu(x_k, [\ell_k/l], h_k),$$

and continuous functions $\varphi_k : M(f_k, V) \to \mathbf{R}$ $(f_k \in \mathcal{V})$ and $\varphi : \overline{V} \to \mathbf{R}$ by

$$\varphi_k(x) = \log \|(Dh_k)|\widetilde{E}_k(x)\| \quad \text{and} \quad \varphi(x) = \log \|(Dh)|\widehat{E}(x)\|.$$

Note that $\widetilde{E}_k|\Lambda_k = E_k$ and $\widetilde{F}_k|\Lambda_k = F_k$ if $\Lambda_k \subset \overline{V}$. Let $\bar{\nu} \in \mathcal{M}_h(M)$ be an accumulation point of $\{\nu_k : k \geq 1\}$ in $\mathcal{M}(M)$, which is supported in Λ. To simplify the notation, we omit taking a subsequence of $k = 1, 2, \ldots$ and assume that

$$\bar{\nu} = \lim_{k \to +\infty} \nu_k \in \mathcal{M}_h(M)$$

and

$$\lim_{k \to +\infty} \frac{1}{[\ell_k/l]} \sum_{j=0}^{[\ell_k/l]-1} \log \|(Dh_k)|E_k(h_k^j(x_k))\| \geq 0.$$

Then, since

$$\begin{aligned} \left| \int_\Lambda \varphi \, d\bar{\nu} - \int_{M(f_k,V)} \varphi_k \, d\nu_k \right| &\leq \left| \int_\Lambda \varphi \, d\bar{\nu} - \int_{\overline{V}} \varphi \, d\nu_k + \int_{\overline{V}} \varphi \, d\nu_k - \int_{M(f_k,V)} \varphi_k \, d\nu_k \right| \\ &\leq \left| \int_{\overline{V}} \varphi \, d\bar{\nu} - \int_{\overline{V}} \varphi \, d\nu_k \right| + \left\| \varphi|_{M(f_k,V)} - \varphi_k \right\| \end{aligned}$$

for large k, we have

$$\int_\Lambda \varphi \, d\bar{\nu} = \lim_{k \to +\infty} \int_{M(f_k,V)} \varphi_k \, d\nu_k \geq 0.$$

By Birkhoff's Ergodic Theorem and the total probability of $\Lambda(f)\cap\Gamma(f)$,

$$0 \le \int_{\Lambda\cap\Lambda(f)\cap\Gamma(f)} \varphi\, d\bar\nu = \int_{\Lambda\cap\Lambda(f)\cap\Gamma(f)} \lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(h^j(x))\, d\bar\nu(x).$$

Hence, there exists a Borel set $B\subset \Lambda\cap\Lambda(f)\cap\Gamma(f)$ with $\bar\nu(B)>0$ such that

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(h^j(x)) \ge 0 \tag{1}$$

for all $x\in B$. Given $0<\eta<1$, if $x\in B$, then, using the (m,λ)-domination on Λ, we have

$$\prod_{j=0}^{n-1} \|(Dh^{-1})|F(h^{j+1}(x))\| \le \lambda^{(l/m)n} \prod_{j=0}^{n-1} \|(Dh)|E(h^j(x))\|^{-1} \le (\lambda^{l/m}/\eta)^n$$

for all n sufficiently large. Picking η with $\lambda^{l/m}<\eta<1$, we see from this inequality that

$$B\subset \bigcup_{i=0}^{i_0} \Gamma^i(f). \tag{2}$$

Note that $\bar\nu$ depends on the choice of l. For every neighborhood W of $\bigcup_{i=0}^{i_0-1}\Gamma^i(f)$ with $\bar\nu(\partial W)=\bar\mu_l(\partial W)=0$, taking a subsequence of $k=1,2,\dots$ further if necessary, we have

$$\bar\nu(W) = \lim_{k\to+\infty} \nu_k(W) \le \lim_{k\to+\infty} l\mu(x_k,\ell_k,f_k)(W) = l\bar\mu_l(W),$$

implying that $\bar\mu_l(W)\ge (1/l)\bar\nu(W)$, where $\bar\mu_l\in\mathcal{M}_f(M)$ is any accumulation point of $\{\mu(x_k,\ell_k,f_k) : k\ge 1\}$ in $\mathcal{M}(M)$. Therefore, by the regularity of $\bar\mu_l$, the proof of this lemma is reduced to proving that there exists an arbitrarily large multiple l of m such that

$$\bar\nu\left(\bigcup_{i=0}^{i_0-1}\Gamma^i(f)\right) > 0.$$

To argue by contradiction, let us assume that this is not true. Then, using (2), we have

$$0<\bar\nu(B) = \bar\nu\left(B\cap\bigcup_{i=0}^{i_0}\Gamma^i(f)\right) = \bar\nu\big(B\cap\Gamma^{i_0}(f)\big)$$

for all sufficiently large multiples l of m. In what follows in this proof, we omit mentioning that l is a multiple of m and for simplicity treat l as if it were just a positive integer. By the Ergodic Decomposition Theorem (see [21] for instance) applied to h, we can choose $x_l\in B\cap\Gamma^{i_0}(f)$ for which $\bar\nu_{x_l}\in\mathcal{M}_e(h)$ satisfies

$$\int_M \varphi\, d\bar\nu_{x_l} = \lim_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n-1}\varphi(h^j(x_l)) \ge 0, \tag{3}$$

where the last inequality comes from (1). Define $\mu_{*,l}\in\mathcal{M}_f(M)$ by

$$\mu_{*,l} = \frac{1}{l}\sum_{j=0}^{l-1} f_*^j\bar\nu_{x_l}.$$

It is easy to observe that $\mu_{*,l} \in \mathcal{M}_e(f)$. Let $\bar{\mu}_* \in \mathcal{M}_f(M)$ be an accumulation point of $\{\mu_{*,l} : l \geq 1\}$ in $\mathcal{M}(M)$. For simplicity again, let us suppose that $\bar{\mu}_* = \lim_{l\to+\infty} \mu_{*,l}$. Then we claim that

$$\bar{\mu}_*(\Gamma^{i_0}(f)) = 1.$$

Assume that this is false. Then, using the regularity of $\bar{\mu}_*$, we would have $\beta > 0$ and a neighborhood V_0 of $\Gamma^{i_0}(f)^c$ with $\bar{\mu}_*(\partial V_0) = 0$ such that

$$\bar{\mu}_*(\Gamma^{i_0}(f)^c) = \beta \quad \text{ and } \quad \bar{\mu}_*(V_0 \cap \Gamma^{i_0}(f)) < \beta/4,$$

implying that

$$\mu_{*,l}(V_0) > \beta/2 \quad \text{ and } \quad \mu_{*,l}(V_0 \cap \Gamma^{i_0}(f)) < \beta/4$$

for all l sufficiently large. By the Ergodic Decomposition Theorem and the ergodicity of $\mu_{*,l}$ with $x_l \in \Gamma^{i_0}(f)$, there is $x_* \in \Gamma^{i_0}(f)$ for which $\mu_{*,l} = \mu_{x_*}$. Therefore, taking a C^1 bump function $\psi : M \to \mathbf{R}$ supported in a neighborhood $\widetilde{V}_0$ of $\overline{V}_0$ such that $\psi(x) = 1$ if $x \in V_0$ and $\mu_{x_*}(\widetilde{V}_0 \setminus V_0) < \beta/4$, we have

$$\begin{aligned}\frac{\beta}{2} < \mu_{*,l}(V_0) \leq \int_M \psi \, d\mu_{x_*} &\leq \int_{V_0\cap\Gamma^{i_0}(f)} \psi \, d\mu_{x_*} + \mu_{x_*}(\widetilde{V}_0 \setminus V_0) \\ &\leq \mu_{*,l}(V_0 \cap \Gamma^{i_0}(f)) + \frac{\beta}{4} < \frac{\beta}{2},\end{aligned}$$

which is a contradiction and proves the claim. Hence, using Lemma I.8, we can take some Borel set $B_0 \subset \Gamma^{i_0}(f)$ with $\bar{\mu}_*(B_0) < 1/2$ for which there exist $\bar{\ell} \geq 1$ and $0 < \bar{\gamma} < 1$ such that if $x \in \Gamma^{i_0}(f) \setminus B_0$, then

$$x \in \bigcup_{\ell\leq\bar{\ell}} \bigcup_{\gamma\leq\bar{\gamma}} \Gamma^{i_0}(\ell, \gamma),$$

where $\Gamma^{i_0}(\ell, \gamma)$ ($\ell \in \mathbf{Z}^+$, $0 < \gamma < 1$) is the set of $x \in \Gamma^{i_0}(f)$ satisfying the inequalities of Lemma I.8 for these ℓ and γ. Take a neighborhood U_0 of B_0 with

$$\bar{\mu}_*(\partial U_0) = 0 \quad \text{ and } \quad \bar{\mu}_*(U_0) < 1/2,$$

implying that

$$\mu_{*,l}(U_0^c) \geq 1/2$$

for all l sufficiently large. For such large l, taking a subsequence of $\{l\}$ and replacing some positive h-iterate of x_l by x_l if necessary, we can find $1 \leq \ell \leq \bar{\ell}$ and $0 < \gamma \leq \bar{\gamma}$ independent of the choice of large l such that

$$f^t(x_l) \in \Gamma^{i_0}(\ell, \gamma)$$

for some $0 \leq t \leq l - 1$, where we may assume that l is much larger than ℓ. By the f^ℓ-invariance of $\Gamma^{i_0}(\ell, \gamma)$ there exists $\bar{x}_l \in \Gamma^{i_0}(\ell, \gamma)$ with $\bar{x}_l = f^{-u}(x_l)$ for some $0 \leq u \leq \ell - 1$. Then there is a subsequence $n_i \to +\infty$ such that

$$\frac{1}{n_i} \sum_{j=0}^{n_i-1} \log \|(Df^\ell)|E(f^{\ell j}(\bar{x}_l))\| < \log \gamma. \tag{4}$$

On the other hand, by (3), for any $\gamma < \hat{\gamma} < 1$ there is $n_0 \geq 1$ such that

$$\frac{1}{n} \sum_{j=0}^{n-1} \log \|(Df^l)|E(f^{lj}(x_l))\| > \log \hat{\gamma} \tag{5}$$

for all $n > n_0$. Choose as n a multiple of ℓ, $n = s\ell$ with large $s \in \mathbf{Z}^+$, and define

$$J = \{1 \leq j \leq sl : -u + j\ell < il < -u + (j+1)\ell \;\text{ for some }\; 1 \leq i \leq s\ell\}.$$

Since $\#J \leq s\ell$, if s is large enough to satisfy $s\ell > n_0$, setting $C_0 = \sup_{x\in M} \|D_x f\|$ and using (5), we have that

(6)

$$\begin{aligned}
\log \hat{\gamma} &< \frac{1}{s\ell} \sum_{j=0}^{s\ell-1} \log \|(Df^l)|E(f^{lj}(x_l))\| \\
&\leq \frac{1}{sl}\left(\sum_{0\neq j \in J^c} \log \|(Df^\ell)|E(f^{\ell j}(\bar{x}_l))\| + \log \|(Df^{\ell-u})|E(x_l)\| + \#J\ell \log C_0 \right) \\
&\leq \frac{1}{sl}\left(\sum_{j=0}^{sl-1} \log \|(Df^\ell)|E(f^{\ell j}(\bar{x}_l))\| + \ell \log C_0 + 2\#J\ell \log C_0 \right) \\
&= \frac{1}{sl}\left(\sum_{j=0}^{sl-1} \log \|(Df^\ell)|E(f^{\ell j}(\bar{x}_l))\| + \ell \log C_0 \right) + \frac{1}{l}(2\ell^2 \log C_0).
\end{aligned}$$

Putting $s_i = [n_i/l]$ and using (4), we obtain that

$$\begin{aligned}
&\frac{1}{s_i l}\left(\sum_{j=0}^{s_i l-1} \log \|(Df^\ell)|E(f^{\ell j}(\bar{x}_l))\| + \ell \log C_0 \right) \\
&\qquad \leq \frac{1}{s_i l}\left(\sum_{j=0}^{n_i-1} \log \|(Df^\ell)|E(f^{\ell j}(\bar{x}_l))\| + 2l\ell \log C_0 \right) \\
&\qquad = \frac{n_i}{s_i l}\frac{1}{n_i} \sum_{j=0}^{n_i-1} \log \|(Df^\ell)|E(f^{\ell j}(\bar{x}_l))\| + \frac{2\ell}{s_i} \log C_0 < \log \gamma
\end{aligned}$$

for all i sufficiently large. From this, we can use (6) for $s = s_i$ with large i to have

$$\log \hat{\gamma} < \log \gamma + (2\ell^2 \log C_0)/l.$$

Since $\ell \in \mathbf{Z}^+$ and $0 < \gamma < 1$ are independent of the arbitrarily large choice of l, we can choose l so large that this inequality is absurd. □

In [22, Lemma I.5], Mañé proved that if a continuous invariant subbundle $E \subset TM|\Lambda$ for a compact f-invariant set Λ is not contracting, then for every $n > 0$ there exists $\mu_n \in \mathcal{M}_e(f^n|\Lambda)$ such that $\int \log \|(Df^n)|E\|\, d\mu_n \geq 0$, which implies that $\int \log \|(Df^l)|E\|\, d\mu_l \geq 0$ for every multiple l of m. Then, if Λ admits a homogeneous dominated splitting $TM|\Lambda = E \oplus F$, by Birkhoff's Ergodic Theorem, we can find a sequence of probabilities $\mu(x, lk, f)$, $k \geq 1$, with $x \in \Lambda$ satisfying the hypothesis of Lemma III.1 as $\Lambda_k = \Lambda$, $f_k = f$, $x_k = x$, $\ell_k = lk$ and $E_k \oplus F_k = E \oplus F$ for all $k \geq 1$. Thus, we get an alternative proof of a result of Cao in [6]:

Corollary III.2 ([6, Cao]). *Let $f \in \mathrm{Diff}^1(M)$ admit no zero Lyapunov exponents and let Λ be a compact f-invariant set with a homogeneous dominated splitting*

$$TM|\Lambda = E \oplus F.$$

If E is not contracting (resp. F is not expanding), then

$$\bigcup_{i=0}^{\dim E-1} \Gamma^i(f) \cap \Lambda \neq \emptyset \qquad \left(\textit{resp.} \quad \bigcup_{i=\dim E+1}^{\dim M} \Gamma^i(f) \cap \Lambda \neq \emptyset \right).$$

The following lemma tells us a condition under which the constants as in Lemma I.8 can be taken uniformly for some ergodic measures with the same index.

Lemma III.3. *Let $f \in \mathcal{R}_1^2$ and let Λ be an f-invariant set written as*

$$\Lambda = \overline{\{x \in \operatorname{supp}(\mu) : \mu \in \mathcal{M},\ \operatorname{Ind}(\mu) = i_0\}}$$

for some $\mathcal{M} \subset \mathcal{M}_e(f)$, admitting a homogeneous dominated splitting $TM|\Lambda = E \oplus F$ with $\dim E = i_0$. If f admits no zero Lyapunov exponents and $\bigcup_{i=0}^{i_0-1} \Gamma^i(f)$ is contained in a countable union of hyperbolic homoclinic classes, then there exist $\ell \in \mathbf{Z}^+$ and $0 < \gamma < 1$ such that

$$\liminf_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|(Df^\ell)|E(f^{\ell j}(x))\| \leq \log \gamma$$

and

$$\liminf_{n \to +\infty} \frac{1}{n} \sum_{j=1}^{n} \log \|(Df^\ell)|E(f^{-\ell j}(x))\| \leq \log \gamma$$

at μ-a.e. x for all $\mu \in \mathcal{M}$.

Proof. By the Poincaré Recurrent Theorem, for μ-a.e. x ($\mu \in \mathcal{M}$, $\operatorname{Ind}(\mu) = i_0$) at which $E(x) = E^-(x)$, there are $\kappa = \kappa(x) \geq 1$ and $n_i = n_i(x) \to +\infty$ such that x, $f^{n_i}(x) \in \Gamma_\kappa^{i_0}(f)$ and $\lim_{i \to +\infty} f^{n_i}(x) = x$. To prove the first inequality of this lemma, it is enough to show that

$$\liminf_{i \to +\infty} \frac{1}{[n_i/\ell]} \sum_{j=0}^{[n_i/\ell]-1} \log \|(Df^\ell)|E^-(f^{\ell j}(x))\| \leq \log \gamma$$

at such x. If this property does not hold, given sequences $\ell_k \to +\infty$ and $\gamma_k \to 1$, there exist $\mu_k \in \mathcal{M}$ with $\operatorname{Ind}(\mu_k) = i_0$, μ_k-a.e. point x_k and i_k such that, using the notation n_i as $n_i(x_k)$,

$$\frac{1}{[n_i/\ell_k]} \sum_{j=0}^{[n_i/\ell_k]-1} \log \|(Df^{\ell_k})|E^-(f^{\ell_k j}(x_k))\| > \log \gamma_k$$

for all $i \geq i_k$ and $k \geq 1$. Then we can apply the Katok Closing Lemma for large i to obtain a periodic point $p_k \in P^{i_0}(f)$ with period n_i satisfying

$$\frac{1}{[n_i/\ell_k]} \sum_{j=0}^{[n_i/\ell_k]-1} \log \|(Df^{\ell_k})|\widehat{E}(f^{\ell_k j}(p_k))\| > \log \gamma_k, \tag{7}$$

where $\widehat{E} \oplus \widehat{F}$ is a homogeneous dominated splitting over $M(f, U)$ for some neighborhood U of Λ that extends $E \oplus F$. From this property, it follows that there are no constants $K > 0$, $\ell \in \mathbf{Z}^+$ and $0 < \gamma < 1$ such that

$$\prod_{j=0}^{[n_i/\ell]-1} \|(Df^\ell)|\widehat{E}(f^{\ell j}(p_k))\| \leq K\gamma^{[n_i/\ell]}$$

holds for all $i \geq i_k$ and $k \geq 1$ over all choices of $\{p_k\}$. In fact, if there were such constants K, ℓ and γ, by an easy calculation, the choice of $\{p_k\}$ with respect to $\ell_k = \ell k$ would contradict (7) for large i when k is so large that $\gamma_k > \gamma$. Now applying Lemma I.5 to

$$\mathcal{A} = \big\{(p,f) : p \in \{p_k : k \geq 1\} \text{ for some choice of } \{p_k\}\big\},$$

we can find sequences $f_k \in \mathrm{Diff}^1(M)$, $q_k \in \mathrm{Per}(f_k)$, $k \geq 1$, with $(q_k, f) \in \mathcal{A}$ and $\lim_{k\to+\infty} f_k = f$ such that $\mathcal{O}_{f_k}(q_k) = \mathcal{O}_f(q_k)$, $(Df_k^{\pi_k})\widehat{E}(q_k) = \widehat{E}(q_k)(= E_f^-(q_k))$, $(Df_k^{\pi_k})\widehat{F}(q_k) = \widehat{F}(q_k)(= E_f^+(q_k))$ and $(Df_k^{\pi_k})|\widehat{E}(q_k)$ is a nonhyperbolic isomorphism for all $k \geq 1$, where π_k is the period of q_k. Note that $\pi_k \to +\infty$ $(k \to +\infty)$ since f has only hyperbolic periodic points. Take neighborhoods $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$ and U_0 of Λ in U for which there exist $m \in \mathbf{Z}^+$ and $0 < \lambda < 1$ such that a homogeneous (m,λ)-dominated splitting

$$TM|M(g,U_0) = \widetilde{E}_g \oplus \widetilde{F}_g$$

is admitted for all $g \in \mathcal{U}$. In particular, set $E_k = \widetilde{E}_{f_k}$ and $F_k = \widetilde{F}_{f_k}$; i.e.,

$$TM|M(f_k,U_0) = E_k \oplus F_k.$$

Since $E_k(q_k) = \widehat{E}(q_k)$ over which $Df_k^{\pi_k}$ is nonhyperbolic, we have, for every $l \geq 1$,

$$\liminf_{k\to+\infty} \frac{1}{[\pi_k/l]} \sum_{j=0}^{[\pi_k/l]-1} \log \|(Df_k^l)|E_k(f_k^{lj}(q_k))\| \geq 0.$$

Then we can apply Lemma III.1 for $\mu(q_k,\pi_k,f_k)$, $k \geq 1$, to $\Lambda_k = M(f_k,U_0)$ in order to obtain that

$$\nu\bigg(\bigcup_{i=0}^{i_0-1} \Gamma^i(f)\bigg) > 0$$

for some accumulation point $\nu \in \mathcal{M}_f(M)$ of $\{\mu(q_k,\pi_k,f_k) : k \geq 1\}$ in $\mathcal{M}(M)$. Since $\bigcup_{i=0}^{i_0-1} \Gamma^i(f)$ is contained in a countable union of hyperbolic homoclinic classes by hypothesis, there is a hyperbolic homoclinic class $H_f(p)$ (which is a basic set) with $p \in \bigcup_{i=1}^{i_0-1} P^i(f)$ such that

$$\nu(H_f(p)) > 0. \tag{8}$$

Take $\delta_0 > 0$ so small that $\overline{U_{\delta_0}(H_f(p))}$ is an isolating block of $H_f(p)$ not intersecting some fundamental domains of the stable and unstable sets of $H_f(p)$, and $M(g, U_{\delta_0}(H_f(p)))$ is a hyperbolic set with index less than i_0 for all g sufficiently C^1 close to f. Then, letting

$$\nu = \lim_{i\to+\infty} \mu(q_{k_i}, \pi_{k_i}, f_{k_i})$$

and setting $r_i = q_{k_i}$, $\tau_i = \pi_{k_i}$ and $g_i = f_{k_i}$, we have from the nonhyperbolicity of $r_i \in \mathrm{Per}(g_i)$ that

$$[\,U_{\delta_0}(H_f(p))\,]^c \cap \mathcal{O}_{g_i}(r_i) \neq \emptyset$$

for all i sufficiently large. Then, by (8) and the fact that

$$\mu(r_i,\tau_i,g_i) = \mu(r_i,\tau_i,f),$$

Lemma I.1 can be applied to obtain a homoclinic orbit associated to $H_f(p)$ for some g arbitrarily C^2 close to f and coinciding with f in a neighborhood of $H_f(p)$, which intersects outside $U_{\delta_0}(H_f(p))$ and may be supposed to be transversal. However, since $f \in \mathcal{R}^2_1$, there is a C^2 neighborhood $\mathcal{U}^2$ of f such that

$$H_g(p) \subset U_{\delta_0}(H_f(p))$$

for all $g \in \mathcal{U}^2$. This is a contradiction and proves the first inequality of Lemma III.3. The second inequality can be proved similarly for some $\ell' \in \mathbf{Z}^+$ and $0 < \gamma' < 1$ using the backward recurrence of x. Finally, after taking $0 < \max\{\gamma, \gamma'\} < 1$ and a common multiple of ℓ and ℓ', denoting them by ℓ and γ again, we finish the proof of Lemma III.3. $\square$

Now, to start the proof of Theorem II.1, for $g \in \mathcal{R}^2_* \cap \mathcal{D}^2$ given in the hypothesis of Theorem II.1, let us consider a g-invariant set $\Lambda^{i_0}_g$ contained in a transitive set written as:

$$\Lambda^{i_0}_g = \overline{\{x \in \mathrm{supp}\,(\mu) : \mu \in \mathcal{M}_g,\ \mathrm{Ind}\,(\mu) = i_0\}}$$

with some $\mathcal{M}_g \subset \mathcal{M}_e(g)$, which admits a homogeneous (m, λ)-dominated splitting, the restriction of the splitting (1) in Section II with $i = i_0$ to $\Lambda^{i_0}_g$. By the property of g, we can apply Lemma III.3 to $f = g$, $\Lambda = \Lambda^{i_0}_g$ and $\mathcal{M} = \mathcal{M}_g$ in order to obtain $\ell \in \mathbf{Z}^+$ and $0 < \gamma_g < 1$ for which the inequalities of Lemma III.3 hold. Take a monotone decreasing sequence $\{\varepsilon^{(\iota)}\}$ with $\lim_{\iota \to +\infty} \varepsilon^{(\iota)} = 0$. For any pair (γ, ε) with $0 < \gamma < 1$ and $\varepsilon \in \{\varepsilon^{(\iota)}\}$, we use an argument in [14], which is a slightly modified one in the proof of [22, Lemma II.6]. For any multiple l of m, set $h = g^{\ell l}$. Then we say that $(x, h^{-n}(x); h^{-1})$ in $\Lambda^{i_0}_g$ is a *γ-string* if

$$\prod_{j=1}^{n} \|(Dh)|E^{i_0}_g(h^{-j}(x))\| \le \gamma^n.$$

In particular, if $(h^{-j}(x), h^{-n}(x); h^{-1})$ is a γ-string for all $0 \le j < n$, we call $(x, h^{-n}(x); h^{-1})$ a *uniform γ-string.*

Let us state here the so-called Pliss Lemma due to Pliss [31] (see also [21, Lemma 11.8]):

Lemma III.4 (Pliss Lemma)**.** *For all $0 < \gamma < \bar\gamma < 1$ there exist $N(\gamma, \bar\gamma) > 0$ and $0 < c(\gamma, \bar\gamma) < 1$ such that if $(x, h^{-n}(x); h^{-1})$ is a γ-string and $n \ge N(\gamma, \bar\gamma)$, there exist $0 < n_1 < \cdots < n_k \le n$, $k > 1$, such that $k \ge nc(\gamma, \bar\gamma)$ and $(x, h^{-n_i}(x); h^{-1})$ is a uniform $\bar\gamma$-string for all $1 \le i \le k$.*

A compact set $\Sigma \subset \Lambda^{i_0}_g$ is an *(s, γ)-set* ($s \in \mathbf{Z}^+$, $0 < \gamma < 1$) if for every $x \in \Sigma$ there exists $-s < s_0 < s$ such that $(h^{s_0+n}(x), h^{s_0}(x); h^{-1})$ is a γ-string for all $n > 0$. Then $E^{i_0}_g|\Sigma$ is contracting. Take constants η, γ_1, γ_2, $\bar\gamma_2$ and γ_3 with

$$\max\{\gamma_g, \lambda^{\ell(l/m)}\} < \eta < \gamma_1 < \gamma_2 < \bar\gamma_2 < \gamma_3 < 1.$$

Denote by $\Lambda(N)$ the set of points $y \in \Lambda^{i_0}_g$ such that $(y, h^{-n}(y); h^{-1})$ is an (N, γ_2)-obstruction (i.e., $(y, h^{-j}(y); h^{-1})$ is not a γ_2-string for all $N \le j \le n$) for all $n > N$ with $N = N(\bar\gamma_2, \gamma_3)$ given by Lemma III.4, satisfying the following property:

($*$) Given $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbf{Z}^+$ with $N(\varepsilon) > N$ such that if $(y, h^{-n}(y); h^{-1})$ is an $(N, \bar\gamma_2)$-obstruction and $n > N(\varepsilon)$, then $d(y, \Lambda(N)) < \varepsilon$.

Note that if $y \in \overline{\Lambda(N)}$, then $(y, h^{-n}(y); h^{-1})$ is an (N, γ_1)-obstruction for all $n > N$. For $\mu \in \mathcal{M}_g$ with $\mathrm{Ind}\,(\mu) = i_0$, let $\Sigma_\mu(\varepsilon, \gamma_3)$ be the union of all $(N(\varepsilon), \gamma_3)$-sets in $\mathrm{supp}\,(\mu)$. Then $\overline{\Sigma_\mu(\varepsilon, \gamma_3)}$ is still an $(N(\varepsilon), \gamma_3)$-set. For $n \ge 1$ and μ-a.e. $x \in \Gamma^{i_0}(g)$, denote by $\mathcal{L}(x,n)$ the set of $j \geq n$ for which $(x, h^{-j}(x); h^{-1})$ is a uniform γ_3-string. By the second inequaity of Lemma III.3, an easy calculation shows that

$$\liminf_{n\to+\infty} \frac{1}{n}\sum_{j=1}^{n} \log \|(Dh)|E_g^{i_0}(h^{-j}(x))\| \leq \liminf_{n\to+\infty} \frac{1}{n}\sum_{j=1}^{n} \log \|(Dg^{\ell})|E_g^{i_0}(g^{-\ell j}(x))\| \leq \log \gamma_g.$$

Therefore, using Lemma III.4, we see that $\#\mathcal{L}(x,n) = \infty$ for any $n \geq 1$. Let

$$\mathcal{L}(x,n) = \{m_1 < m_2 < \dots\}.$$

For every $\mu \in \mathcal{M}_g$ with $\mathrm{Ind}\,(\mu) = i_0$, either $\mathrm{supp}\,(\mu) = \overline{\Sigma_\mu(\varepsilon,\gamma_3)}$ is an $(N(\varepsilon),\gamma_3)$-set or $\mathrm{supp}\,(\mu) \setminus \overline{\Sigma_\mu(\varepsilon,\gamma_3)} \neq \emptyset$. In the latter case, since $\mathrm{supp}\,(\mu) = \alpha_g(x)$, we can find m_i, $m_{i+1} \in \mathcal{L}(x,n)$ with $m_{i+1} - m_i > N(\varepsilon)$ and then $(h^{-m_i}(x), h^{-m_{i+1}}(x); h^{-1})$ is an $(N, \bar{\gamma}_2)$-obstruction, implying from $(*)$ that

$$d(h^{-m_i}(x), \Lambda(N)) < \varepsilon. \tag{9}$$

Now we vary a pair of positive numbers (η, ε) as follows. Fix η first and then take γ_1, γ_2, $\bar{\gamma}_2$ and γ_3 larger than η as above. Next, vary $\varepsilon = \varepsilon^{(\iota)} \to 0$ $(\iota \to +\infty)$. If there is $\iota \geq 1$ such that $\mathrm{supp}\,(\mu)$ is an $(N(\varepsilon^{(\iota)}), \gamma_3)$-set for all $\mu \in \mathcal{M}_g$ with $\mathrm{Ind}\,(\mu) = i_0$, then $E_g^{i_0}|\Lambda_g^{i_0}$ is contracting as required. So let us assume that this does not hold, that is, for arbitrarily large choice of ι, there is $\mu \in \mathcal{M}_g$ with $\mathrm{Ind}\,(\mu) = i_0$ for which we have $h^{-m_i}(x)$ in (9) with $\varepsilon = \varepsilon^{(\iota)}$; i.e.,

$$d(h^{-m_i}(x), y^{(\iota)}) < \varepsilon^{(\iota)} \tag{10}$$

for some $y^{(\iota)} \in \Lambda(N)$. Then, take a subsequence $\iota_n \to +\infty$ of $\iota = 1, 2, \dots$ with the limit $\bar{y} = \lim_{n\to+\infty} y^{(\iota_n)}$. Since $\Lambda_g^{i_0}$ is compact,

$$\bar{y} \in \overline{\Lambda(N)} \subset \Lambda_g^{i_0}. \tag{11}$$

Note that $\bar{y}$ and N depend on the choice of η.

Let U_g be a neighborhood of $S^{i_0}(g)$ such that a homogeneous (m,λ)-dominated splitting

$$TM|M(g,U_g) = \widehat{E}_g^{i_0} \oplus \widehat{F}_g^{i_0}$$

that extends $TM|S^{i_0}(g) = E_g^{i_0} \oplus F_g^{i_0}$ is admitted, and let $D_\delta^s(y)$ (resp. $D_\delta^u(y)$), $y \in M(g,U_g)$, be a locally g-invariant disk tangent to $\widehat{E}_g^{i_0}(y)$ (resp. $\widehat{F}_g^{i_0}(y)$) at y with size $\delta > 0$. For $\epsilon > 0$ put $U_{\epsilon,\mu} = U_\epsilon(\mathrm{supp}\,(\mu))$.

Lemma III.5. *For any choice of η above, there exists $\rho_0 = \rho_0(\eta) > 0$ independent of $\iota \geq 1$ such that for every $\epsilon > 0$ and $h^{-m_i}(x)$ $(m_i \in \mathcal{L}(x,n))$ in* (10) *with sufficiently large n, there exists $p^{(\iota)} \in P^{i_0}(g)$ satisfying the following properties:*

(a) $d(p^{(\iota)}, h^{-m_i}(x)) < \epsilon$;
(b) $\mathcal{O}_g(p^{(\iota)}) \subset U_{\epsilon,\mu}$;
(c) $h^{-m_i}(x) \in H_g(p^{(\iota)}, U_{\epsilon,\mu})$ *(and therefore* $\mathrm{supp}\,(\mu) \subset H_g(p^{(\iota)}, U_{\epsilon,\mu})$*)*;
(d) $D_{\rho_0}^s(p^{(\iota)}) \subset W^s(p^{(\iota)}, g)$.

Proof. Let $\kappa \geq 1$ be such that $x \in \Gamma_\kappa^{i_0}(g)$ and let $\delta_\kappa > 0$ be a positive number with $\delta_\kappa \leq \delta$ for which $W_{\delta_\kappa}^s(x,g)$ is defined. Then

$$D_{\delta_\kappa}^s(x) \subset W_{\delta_\kappa}^s(x,g).$$

For $j \geq 1$ let

$$\rho_j = \sup\{\rho > 0 : h^{m_j}(D^s_\rho(h^{-m_j}(x)) \subset W^s_{\delta_\kappa/2}(x, g)\}.$$

Since $(x, h^{-m_j}(x); h^{-1})$ is a uniform γ_3-string for all $j \geq 1$, it is easy to see that a positive number $\rho_0 \leq \delta$ (depending on the choice of η but) independent of the choice of x and ι in (10) is defined by

$$\rho_0 = \inf_{j\geq 1} \rho_j > 0$$

for large n depending on x. Using the Katok Closing Lemma, given $0 < \tilde{\varepsilon} < \min\{\epsilon, \delta_\kappa\}$ we can find $p \in P^{i_0}(g)$ approximating x such that $\mathcal{O}_g(p) \subset U_{\tilde{\varepsilon},\mu}$, $x \in H_g(p, U_{\tilde{\varepsilon},\mu})$, $D^s_{\tilde{\delta}}(p) \subset W^s_{\tilde{\delta}}(p, g)$ with some $\tilde{\delta} > 0$ satisfying $\tilde{\delta} > \delta_\kappa - \tilde{\varepsilon}$, and

$$d(h^{-m_i}(p), h^{-m_i}(x)) < \tilde{\varepsilon}.$$

Therefore, setting $p^{(\iota)} = h^{-m_i}(p)$, we get properties (a) and (b). If $\tilde{\varepsilon} > 0$ is small enough, our choice of m_i implies that

$$h^{m_i}(D^s_{\rho_0}(p^{(\iota)})) \subset W^s_{2\delta_\kappa/3}(p, g).$$

Then, by the g-invariance of $H_g(p, U_{\tilde{\varepsilon},\mu}) = H_g(p^{(\iota)}, U_{\tilde{\varepsilon},\mu})$ and $W^s(\mathcal{O}_g(p), g)$, we obtain properties (c) and (d). $\square$

By properties (a) and (d) of Lemma III.5 and (10), for every $\rho > 0$ with $\rho \leq \rho_0$, there exists $p^{(\iota)} \in P^{i_0}(g)$ approximating $h^{-m_i}(x)$ in (10) with arbitrarily large ι such that

$$W^s_\rho(p^{(\iota)}, g) \pitchfork D^u_\rho(\bar{y}) \neq \emptyset. \tag{12}$$

Using the (m, λ)-domination and (11), we have

$$\prod_{j=0}^{n-1} \|(Dh^{-1})|F^{i_0}_g(h^{-j}(\bar{y}))\| \leq \prod_{j=0}^{n-1} \lambda^{\ell(l/m)} \|(Dh)|E^{i_0}_g(h^{-(j+1)}(\bar{y}))\|^{-1} < (\lambda^{\ell(l/m)}/\gamma_1)^n$$

for all $n > N$. Since $\lambda^{\ell(l/m)}/\gamma_1 < 1$, there is $\rho_1 > 0$ (depending on N and therefore on η) such that

$$\lim_{n\to+\infty} \operatorname{diam} g^{-n}(D^u_{\rho_1}(\bar{y})) = 0. \tag{13}$$

Let us assume that there exists a basic set Λ_1 with index less than i_0 such that

$$\bar{y} \in W^u(\Lambda_1, g). \tag{14}$$

This and (13) imply

$$D^u_{\rho_1}(\bar{y}) \subset W^u(\Lambda_1, g),$$

and then, by (12), for $p^{(\iota)}$ with arbitrarily large ι,

$$W^s(p^{(\iota)}, g) \pitchfork W^u(\Lambda_1, g) \neq \emptyset.$$

From (11) and (14), we have

$$\alpha_g(\bar{y}) \subset \Lambda^{i_0}_g \cap \Lambda_1.$$

Hence, by the Anosov Closing Lemma and the continuity of invariant manifolds (see [41] for instance), for every $\epsilon > 0$ there is $q \in \Lambda_1 \cap \mathrm{Per}(g)$ with $\Lambda^{i_0}_g \cap H_g(q, U_\epsilon(\Lambda^{i_0}_g)) \neq \emptyset$ such that

$$W^s(p^{(\iota)}, g) \pitchfork W^u(q, g) \neq \emptyset.$$

By (11), (12) and property (b) of Lemma III.5, this transversal intersection contains a point whose g-orbit is in an arbitrarily small neighborhood of $\Lambda_g^{i_0}$ for some large ι and small $\epsilon > 0$. Moreover, property (c) of Lemma III.5 implies

$$\Lambda_g^{i_0} \cap H_g(p^{(\iota)}, U_{\epsilon,\mu}) \neq \emptyset.$$

Then, if $\Lambda_g^{i_0}$ is contained in a transitive set, given a neighborhood U of the transitive set we have a pseudo-heterodimensional cycle in U between $\mathcal{O}_g(p^{(\iota)})$ and $\mathcal{O}_g(q)$ from which, using Lemma I.4, a heterodimensional cycle in U between them is created by an arbitrarily small C^1 perturbation of g. Hence, property (14) cannot occur for g under consideration.

In the following lemma, we always use the notation $\bar{y}$ for $\bar{y} = \bar{y}(\eta, g)$ without specifying the choice of η and g, but suppose that the choice has been made as follows. Let $\{\mathcal{U}_n^2\}$ be a basis of C^2 neighborhoods of f and let $\{\eta_n\}$ be a monotone increasing sequence converging to 1. If $\bar{y} = \bar{y}(\eta, g)$ has $g \in \mathcal{U}_n^2 \cap \mathcal{R}_*^2 \cap \mathcal{D}^2$, then η satisfies

$$\max\{\eta_n, \gamma_g, \lambda^{\ell(l/m)}\} < \eta < 1.$$

Then, by (11), $(\bar{y}(\eta, g), h^{-j}(\bar{y}(\eta, g)); h^{-1})$ with $h = g^{\ell l}$ is not an η_n-string for all j sufficiently large. Moreover, $\bar{y}$ may vary over all choices of $\Lambda_g^{i_0}$.

Lemma III.6. *Let $f \in \mathcal{D}^2$ given in Theorem II.1 have a countable number of hyperbolic homoclinic classes Λ_t, $t = 1, 2, \ldots$, with indices less than i_0 whose union contains $\bigcup_{i=0}^{i_0-1} \Gamma^i(f)$. Then there exist a C^2 neighborhood $\mathcal{U}^2$ of $f \in \mathcal{D}^2$ and $s \in \mathbf{Z}^+$ such that, if every $\Lambda_t(g)$ ($1 \le t \le s$), the continuation of Λ_t for g, is defined for all $g \in \mathcal{U}^2$,*

$$\inf_{\substack{\nu \in \mathcal{M}(\bar{y}, g^{-1}) \\ g \in \mathcal{U}^2 \cap \mathcal{R}_*^2 \cap \mathcal{D}^2}} \nu\left(\bigcup_{t=1}^{s} \Lambda_t(g)\right) > 0.$$

Before proving Lemma III.6, let us see that this lemma together with Lemma I.2 leads to Theorem II.1. As observed above, for g given in the hypothesis of Theorem II.1, if $E_g^{i_0}|\Lambda_g^{i_0}$ is not contracting, then property (14) cannot occur. For $f \in \mathcal{D}^2$ given in Theorem II.1, take its neighborhood $\mathcal{U}^2$ and $s \in \mathbf{Z}^+$ obtained by Lemma III.6. For $1 \le t \le s$, let $\mathcal{U}_t^2$ be a C^2 neighborhood of f in $\mathcal{U}^2$ such that $\Lambda_t(g)$ is defined for all $g \in \mathcal{U}_t^2$, and let $\mathcal{O}^2$ be the open and dense subset of $\bigcap_{t=1}^{s} \mathcal{U}_t^2$ obtained as $\mathcal{O}^2 = \mathcal{O}(\bigcap_{t=1}^{s} \mathcal{U}_t^2)$ by Lemma I.2. Given $g \in \mathcal{O}^2 \cap \mathcal{R}_*^2 \cap \mathcal{D}^2$ satisfying the hypothesis of Theorem II.1, if $E_g^{i_0}|\Lambda_g^{i_0}$ is not contracting, from Lemma III.6 and Lemma I.2 it follows that

$$\bar{y} \in W^u(\Lambda_t(g), g)$$

for some $1 \le t \le s$. This means property (14), which is a contradiction. Thus we have proved that $E_g^{i_0}|\Lambda_g^{i_0}$ is contracting for such g. That is, Theorem II.1 holds as $\mathcal{U}_{**}^2 = \bigcap_{t=1}^{s} \mathcal{U}_t^2 \cap \mathcal{U}_*^2$ and $\mathcal{R}_{**}^2 = \mathcal{O}^2 \cap \mathcal{R}_*^2$.

Proof of Lemma III.6. If this lemma does not hold, there exist sequences $f_n \in \mathcal{U}_n^2 \cap \mathcal{R}_*^2 \cap \mathcal{D}^2$ and $s_n \in \mathbf{Z}^+$, $n \ge 1$, with $s_n \to +\infty$ such that $\Lambda_t(f_n)$ is defined for all $1 \le t \le s_n$, and for some choice of $\bar{y}_n = \bar{y}(\eta, f_n) \in \Lambda_{f_n}^{i_0}$ and $\mu^{(n)} \in \mathcal{M}(\bar{y}_n, f_n^{-1})$, we have

$$\lim_{n \to +\infty} \mu^{(n)}\left(\bigcup_{t=1}^{s_n} \Lambda_t(f_n)\right) = 0. \tag{15}$$

Take a subsequence $n_k \to +\infty$ of $n = 1, 2, \dots$ such that $\bar{\mu} \in \mathcal{M}_f(M)$ is defined by $\bar{\mu} = \lim_{k\to+\infty} \mu^{(n_k)}$. By (15), if k is large enough,

$$\bar{y}_{n_k} \notin \Lambda_t(f_{n_k}) \tag{16}$$

for all $1 \le t \le s_{n_k}$. Set $y_k = \bar{y}_{n_k}$, $g_k = f_{n_k}$, $\bar{\mu}_k = \mu^{(n_k)}$, $E_k^{i_0} = E_{g_k}^{i_0}|\Lambda_{g_k}^{i_0}$, $\tilde{\eta}_k = \eta_{n_k}$ and $h_k = g_k^{\ell_k l}$ with a multiple l of m and $\ell_k > 0$ given as ℓ in Lemma III.3 for $f = g_k$ and $\Lambda = \Lambda_{g_k}^{i_0}$. Then $(y_k, h_k^{-j}(y_k); h_k^{-1})$ is not an $\tilde{\eta}_k$-string for all j sufficiently large. Take a sequence $m_k \to +\infty$ such that

$$d\big(\mu(y_k, m_k, g_k^{-1}),\ \bar{\mu}_k\big) < 1/k.$$

Note that m_k can be taken as large as we wish for each k with ℓ_k fixing. Therefore we may assume that

$$\bar{\mu} = \lim_{k\to+\infty} \mu(y_k, m_k, g_k^{-1}) = \lim_{k\to+\infty} \mu\big(g_k^{-\ell_k l[m_k/(\ell_k l)]}(y_k),\ \ell_k l[m_k/(\ell_k l)],\ g_k\big).$$

If m_k has been chosen large enough,

$$\prod_{j=1}^{[m_k/(\ell_k l)]} \|(Dh_k)|E_k^{i_0}(h_k^{-j}(y_k))\| > \tilde{\eta}_k^{[m_k/(\ell_k l)]},$$

which implies that

$$\liminf_{k\to+\infty} \frac{1}{\ell_k[m_k/(\ell_k l)]} \sum_{j=1}^{\ell_k[m_k/(\ell_k l)]} \log \|(Dg_k^l)|E_k^{i_0}(g_k^{-lj}(y_k))\| \ge 0.$$

Applying Lemma III.1 to g_k, $\Lambda_{g_k}^{i_0}$, $g_k^{-\ell_k l[m_k/(\ell_k l)]}(y_k)$ and $\ell_k l[m_k/(\ell_k l)]$ as f_k, Λ_k, x_k and ℓ_k in Lemma III.1, respectively, we obtain

$$\bar{\mu}\left(\bigcup_{i=0}^{i_0-1} \Gamma^i(f)\right) > 0$$

for some choice of l above. Then there are $t_0 \ge 1$ and $\alpha > 0$ such that

$$\bar{\mu}(\Lambda_{t_0}) = \alpha. \tag{17}$$

On the other hand, by (15), setting $\Lambda_{t_0,k} = \Lambda_{t_0}(g_k)$, we have

$$\bar{\mu}_k(\Lambda_{t_0,k}) \ < \alpha/4 \tag{18}$$

for all k sufficiently large, which implies that $y_k \notin W^u(\Lambda_{t_0,k}, g_k)$. Let $\rho > 0$ be an arbitrarily small positive number satisfying $\bar{\mu}(\partial U_\rho(\Lambda_{t_0})) = 0$. By (17),

$$\alpha \le \bar{\mu}(U_\rho(\Lambda_{t_0})) = \lim_{k\to+\infty} \bar{\mu}_k(U_\rho(\Lambda_{t_0})),$$

implying that

$$\bar{\mu}_k(U_\rho(\Lambda_{t_0})) > \alpha/2 \tag{19}$$

for all k sufficiently large.

Now here we need an argument of Mañé introduced in [23]. In order to fit our situation to his, let us put $f_k = g_k^{-1}$ (which is not f_n with $n = k$ but $f_{n_k}^{-1}$). Since $\Lambda_{t_0,k}$ is an isolated hyperbolic set of f_k, we can define a compact set $S_{n,k}$ $(n, k \ge 1)$ associated to $\Lambda_{t_0,k}$ as the set corresponding to S_n associated to Λ in [23]. More precisely, let r_n, $n \geq 0$, be the sequence of positive numbers converging to 0 given in [23] and let

$$V_{n,k} = \{x \in M : d(x, V_k^+) \leq r_n, \quad d(x, V_k^-) \leq r_n\},$$

where

$$V_k^+ = W_\varepsilon^s(\Lambda_{t_0,k}, f_k) \quad \text{and} \quad V_k^- = W_\varepsilon^u(\Lambda_{t_0,k}, f_k)$$

for some $\varepsilon > 0$. Here, denoting $\mu_k = \mu(y_k, m_k, f_k)$, we can suppose that

$$\mu_k(\partial V_{n,k}) = \bar{\mu}_k(\partial V_{n,k}) = 0$$

for all $n \geq 0$ and $k \geq 1$. Then $S_{n,k}$ is the set of points $x \in V_{0,k}$ that can be written as $x = f_k^m(z_n)$, $m \in \mathbf{Z}$, with $z_n \in V_{n,k}$ and $f_k^j(z_n) \in V_{0,k}$ for all $0 \leq j \leq m$ if $m \geq 0$, or for all $m \leq j \leq 0$ if $m \leq 0$. Note that $S_{n,k}$ can be regarded as the "continuation" of S_n associated to $\Lambda = \Lambda_{t_0}$, and $S_{n,k}$ converges to S_n as $k \to +\infty$ in the Hausdorff metric. Moreover, since both μ_k and $S_{n,k}$ are defined for f_k, we see that $\mu_k(S_{n,k})$ exactly corresponds to $\mu_k(S_n^{t_0})$ in [16]. Using (18), for fixed large k, we have

$$\alpha/4 > \bar{\mu}_k(\Lambda_{t_0,k}) = \lim_{n \to +\infty} \bar{\mu}_k(S_{n,k}),$$

implying that

$$\bar{\mu}_k(S_{n,k}) < \alpha/4 \tag{20}$$

if n is large enough depending on k. Given $m \in \mathbf{Z}^+$, if $\rho > 0$ has been chosen sufficiently small depending on m, then

$$U_{2\rho}(\Lambda_{t_0,k}) \subset S_{m,k}.$$

Therefore, also using (19), we obtain that

$$\bar{\mu}_k(S_{m,k}) \geq \bar{\mu}_k(U_{2\rho}(\Lambda_{t_0,k})) \geq \bar{\mu}_k(U_\rho(\Lambda_{t_0})) > \alpha/2 \tag{21}$$

for all k sufficiently large. By (20) and (21), it is possible to choose $n_2(k) > m$ as the smallest integer $n > m$ such that $\bar{\mu}_k(S_{n,k}) < \alpha/4$. Then, there exist $\varepsilon_0 > 0$ and $k_0 > 0$ such that for all $k \geq k_0$ we have

$$\mu_k(S_{n_2(k),k}) < \frac{\alpha}{4} - 2\varepsilon_0, \tag{22}$$

$$\mu_k(S_{n_2(k)-1,k}) > \frac{\alpha}{4} - \varepsilon_0 \tag{23}$$

and

$$\mu_k(S_{m,k}) > \frac{\alpha}{2}. \tag{24}$$

A *k-string* is a substring σ of $\{f_k^j(y_k) : 0 \leq j \leq m_k\}$ with the form $\sigma = \{f_k^j(y_k), \ldots, f_k^{j+\iota}(y_k)\} \subset V_{0,k}$ such that $f_k^{j-1}(y_k) \notin V_{0,k}$ and $f_k^{j+\iota+1}(y_k) \notin V_{0,k}$ if $(f_k^j(y_k), f_k^{m_k}(y_k); f_k) \cap V_{0,k}^c \neq \emptyset$; otherwise $j + \iota = m_k$. For k-strings σ_1 and σ_2, we write $\sigma_1 < \sigma_2$ if the first element of σ_2 is a strict positive f_k-iterate of the last element of σ_1. We now recall some properties proved in [23]. Given a C^2 neighborhood $\mathcal{U}$, there exists $n_1 \in \mathbf{Z}^+$ (independent of k) such that if the following property (a) holds, then we can find $g \in \mathcal{U}$ coinciding with f_k in a neighborhood of $\Lambda_{t_0,k}$ and exhibiting a homoclinic point associated to $\Lambda_{t_0,k}$.

(a) There exist $n \geq n_1$, $k \geq 1$ and two k-strings $\sigma_1 < \sigma_2 \subset S_{n+1,k}$ such that $\sigma \cap (S_{n,k} \setminus S_{n+1,k}) = \emptyset$ for every k-string $\sigma_1 < \sigma < \sigma_2$.

If property (a) does not hold for the above n_1, the following property (which corresponds to [23, Claim 2]) holds:

(b) There exist constants $C > 0$, $\delta > 0$ and $1 + \delta < \xi < 2$ (all independent of k) such that

$$\mu_k(S_{n,k} \setminus S_{n+1,k}) \leq C\left(\frac{1+\delta}{\xi}\right)^n (1+\delta)\xi^{n_1}$$

for all $k \geq 1$ and $n > n_1$ such that $\#_k(S_{n,k}) > s_0$, where $s_0 \in \mathbf{Z}^+$ is so large that $2s_0 - 1 > \xi s_0$ and $\#_k(S_{n,k})$ is the number of k-strings contained in $S_{n,k}$.

As already considered in the proof of Lemma III.3, since $f \in \mathcal{R}_1^2$, we have the continuous dependence of Λ_{t_0} on C^2 perturbations of f. By this and the C^2 creation of homoclinic points from property (a), we can find $n_1 \geq 1$ independent of k such that property (a) does not hold for large k. Now we need property

$$y_k \notin \mathrm{Per}(f_k) \tag{25}$$

for large k, which is obtained as follows. If this property is false, $y_k \in \mathrm{Per}(f_k)$ is not contained in $\Lambda_{t_0,k}$ by (16), and under the condition that property (a) does not hold, there is $\bar{n} > 0$ independent of k such that

$$\mu_k(S_{\bar{n},k}) < \alpha/2$$

for all k sufficiently large by the same argument as in the proof of [16, Theorem A] because the dependence of $S_{n,k}$ on k is not essential. Here note that μ_k is essentially the same as the probability supported uniformly on the periodic f_k-orbit of y_k for a sufficiently large choice of m_k. Making $\rho > 0$ smaller if necessary so that $S_{\bar{n},k} \supset U_\rho(\Lambda_{t_0})$ for all large k and using (19), we have

$$\alpha/2 < \bar{\mu}_k(U_\rho(\Lambda_{t_0})) \leq \bar{\mu}_k(S_{\bar{n},k}) \leq \alpha/2$$

for large k, which is a contradiction and proves (25). From (22), (23) and (25), we may assume that, if k is large enough, the condition $\#_k(S_{n,k}) > s_0$ in property (b) is satisfied for all $n < n_2(k)$ because m_k can be changed arbitrarily large for each k with $n_2(k)$ fixing. Apply property (b) to all $n = j$ with $n_1 < m \leq j < n_2(k)$ for large k, obtaining that

$$\begin{aligned}\mu_k(S_{m,k}) &= \mu_k(S_{n_2(k),k}) + \sum_{m \leq j < n_2(k)} \mu_k(S_{j,k} \setminus S_{j+1,k}) \\ &\leq \mu_k(S_{n_2(k),k}) + C(1+\delta)\xi^{n_1} \sum_{m \leq j < n_2(k)} \left(\frac{1+\delta}{\xi}\right)^j.\end{aligned}$$

Hence, assuming that m has been chosen so large that

$$C(1+\delta)\xi^{n_1} \sum_{m \leq j < n_2(k)} \left(\frac{1+\delta}{\xi}\right)^j < \frac{\alpha}{4}$$

and k is large enough, together with (22) and (24), we obtain that

$$\frac{\alpha}{2} < \mu_k(S_{m,k}) < \mu_k(S_{n_2(k),k}) + \frac{\alpha}{4} < \frac{\alpha}{2},$$

which is a contradiction and completes the proof of Lemma III.6. $\square$

IV. Two preliminary lemmas

In this section, we give two preliminary lemmas for the proof of Theorem II.2, which involve the C^1 creation of heterodimensional cycles from the lack of hyperbolicity. In both lemmas we suppose the following condition. Let $f \in \mathrm{Diff}^2(M) \setminus \overline{\mathcal{HT}(M)}$ admit no zero Lyapunov exponents and let Λ be a compact f-invariant set admitting a homogeneous dominated splitting

$$TM|\Lambda = E^\iota \oplus F^\iota \tag{1}$$

with $\dim E^\iota = \iota$. Moreover, we suppose that E^ι is contracting and Λ is contained in a transitive set denoted by Θ.

Lemma IV.1. *Under the condition above, let $\iota' = \min\{j > \iota : \Gamma^j(f) \cap \Lambda \neq \emptyset\}$ and let $\Lambda' \subset \Lambda$ be a compact f-invariant set admitting a homogeneous dominated splitting $TM|\Lambda' = E' \oplus F'$ with $\dim E' = \iota'$. If $\Gamma^i(f) \cap \Lambda' \neq \emptyset$ for some $i \neq \iota$, then one of the following properties holds:*

(a) *$E'|\overline{\Gamma^i(f) \cap \Lambda'}$ is contracting.*
(b) *Given neighborhoods $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$ and U of Θ in M, there exist a pseudo-heterodimensional cycle in U for f and a heterodimensional cycle in U for some $g \in \mathcal{U}$.*

Proof. If $E'|\overline{\Gamma^i(f) \cap \Lambda'}$ is not contracting, by Corollary III.2 and the mimimum choice of ι', we have

$$\Gamma^\iota(f) \cap \overline{\Gamma^i(f) \cap \Lambda'} \neq \emptyset.$$

Pick $x \in \Gamma^\iota(f) \cap \overline{\Gamma^i(f) \cap \Lambda'}$ and then choose $y \in \Gamma^i(f) \cap \Lambda'$ so close to x. Take $\rho > 0$ so small that $M(f, U_\rho(\Lambda))$ admits a homogeneous dominated splitting $\widehat{E}^\iota \oplus \widehat{F}^\iota$ that extends (1) with a contracting subbundle $\widehat{E}^\iota$, carrying locally invariant disks $D^\iota_\delta(z)$, $z \in M(f, U_\rho(\Lambda))$, tangent to $\widehat{E}^\iota(z)$ at z with size $\delta > 0$ such that $D^\iota_\delta(z) \subset W^s(z, f)$. Using the Katok Closing Lemma, we can find $p \in P^\iota(f) \cap M(f, U_{\rho/2}(\Lambda'))$ and $q \in P^i(f) \cap M(f, U_{\rho/2}(\Lambda'))$ approximating x and y, respectively, such that

$$x \in H_f(p, U_\rho(\Lambda')) \quad \text{and} \quad y \in H_f(q, U_\rho(\Lambda')). \tag{2}$$

Take a sequence p_n, $n \geq 1$, of such periodic points $p \in P^\iota(f)$ with $\lim_{n\to+\infty} p_n = x$ such that $\operatorname{diam} W^\sigma_{loc}(p_n, f)$, $n \geq 1$, ($\sigma \in \{s, u\}$) are uniformly bounded away from zero for all $n \geq 1$. By continuity and the fact that $E^\iota(x) = E^-(x)$ coming from the domination property of (1), if y has been chosen close enough to x, we can find q above satisfying

$$W^u_{loc}(p_n, f) \pitchfork D^\iota_\delta(q) \neq \emptyset$$

for large n. Since

$$D^\iota_\delta(q) \subset W^s(q, f),$$

we have

$$W^u(p_n, f) \pitchfork W^s(q, f) \neq \emptyset \tag{3}$$

for large n. By (2), (3) and the transitivity of Θ, we see that f exhibits a pseudo-heterodimensional cycle between $\mathcal{O}_f(p_n)$ and $\mathcal{O}_f(q)$. Moreover, we can apply Lemma I.4 to obtain

$$W^s(p_n, g) \cap W^u(q, g) \neq \emptyset$$

for some g arbitrarily C^1 close to f and coinciding with f in a neighborhood of $\mathcal{O}_f(p_n) \cup \mathcal{O}_f(q)$. From (3), we still have

$$W^u(p_n, g) \pitchfork W^s(q, g) \neq \emptyset$$

for all g sufficiently C^1 close to f. Consequently, a heterodimensional cycle between $\mathcal{O}_f(p_n)$ and $\mathcal{O}_f(q)$ for g is obtained. Finally, we can easily check that given neighborhoods $\mathcal{U}$ of f and U of Θ, it is possible to take $\rho > 0$ and $g \in \mathcal{U}$ so that the pseudo-heterodimensional cycle for f is in U and the heterodimensional cycle for g is in U. Thus, we have proved that if property (a) does not hold, then property (b) holds. $\square$

The second lemma is a direct consequence of Lemma IV.1 and Corollary III.2.

Lemma IV.2. *Under the same condition as in Lemma IV.1, if*

$$\bigcup_{j>\iota'} \Gamma^j(f) \cap \overline{\Gamma^{\iota'}(f) \cap \Lambda} = \emptyset,$$

then one of the following properties holds:

(a) $\overline{\Gamma^{\iota'}(f) \cap \Lambda}$ *is a hyperbolic set with index* ι'.

(b) *Given neighborhoods* $\mathcal{U}$ *of* f *in* $\mathrm{Diff}^1(M)$ *and* U *of* Θ *in* M, *there exist a pseudo-heterodimensional cycle in* U *for* f *and a heterodimensional cycle in* U *for some* $g \in \mathcal{U}$.

Proof. By Lemma IV.1 applied to $\Lambda' = \overline{\Gamma^{\iota'}(f) \cap \Lambda}$ and $i = \iota'$, either property (b) holds or a homogeneous dominated splitting

$$TM|\overline{\Gamma^{\iota'}(f) \cap \Lambda} = E^{\iota'} \oplus F^{\iota'},$$

the restriction of (1) in Section II with $i = \iota'$ over $\overline{\Gamma^{\iota'}(f) \cap \Lambda}$, has a contracting subbundle $E^{\iota'}$. In the latter case, when its other subbundle $F^{\iota'}$ is not expanding, Corollary III.2 implies

$$\bigcup_{j>\iota'} \Gamma^j(f) \cap \overline{\Gamma^{\iota'}(f) \cap \Lambda} \neq \emptyset,$$

contradicting our hypothesis. Hence $\overline{\Gamma^{\iota'}(f) \cap \Lambda}$ is a hyperbolic set with index ι'. $\square$

V. Proof of Theorem II.2

In this section, we prove Theorem II.2. Let $f \in \mathrm{Diff}^2(M) \setminus \overline{\mathcal{HT}(M)}$ neither admit zero Lyapunov exponents nor exhibit a pseudo-heterodimensional cycle, and let Λ^{i_0} be the f-invariant set given in the hypothesis of Theorem II.2. Put $E^{i_0} = E_f^{i_0}|\Lambda^{i_0}$ and $F^{i_0} = F_f^{i_0}|\Lambda^{i_0}$ (see (1) in Section II). Suppose that Λ^{i_0} is contained in a transitive set Θ and E^{i_0} is contracting. Our goal of this section is to prove that F^{i_0} is expanding. To argue by contradiction, let us assume that F^{i_0} is not expanding. Set $\Gamma_0 = \Gamma^{i_0}(f) \cap \Lambda^{i_0}$. Then we have $\overline{\Gamma}_0 = \Lambda^{i_0}$ and, by Corollary III.2,

$$\bigcup_{i>i_0} \Gamma^i(f) \cap \overline{\Gamma}_0 \neq \emptyset. \tag{1}$$

Let $i_1 = \min\{i > i_0 : \Gamma^i(f) \cap \overline{\Gamma}_0 \neq \emptyset\}$ and set $\Gamma_1 = \Gamma^{i_1}(f) \cap \overline{\Gamma}_0$. Then there is a homogeneous dominated splitting

$$TM|\overline{\Gamma}_1 = E^{i_1} \oplus F^{i_1} \tag{2}$$

with $\dim E^{i_1} = i_1$, where $E^{i_1} = E_f^{i_1}|\overline{\Gamma}_1$ and $F^{i_1} = F_f^{i_1}|\overline{\Gamma}_1$. Applying Lemma IV.1 to $\Lambda = \overline{\Gamma}_0$, $\Lambda' = \overline{\Gamma}_1$, $\iota = i_0$ and $\iota' = i = i_1$, we see that E^{i_1} is contracting. When $\overline{\Gamma}_1$ is not a hyperbolic set; i.e., F^{i_1} is not expanding, we can restart the process to get (1) from Corollary III.2, finding ergodic measures with a higher index supported in $\overline{\Gamma}_1$, and getting the property corresponding to (2) with i_1 and $\overline{\Gamma}_0$ replaced by some larger integer and $\overline{\Gamma}_1$. Repeat the process if necessary and think of $\overline{\Gamma}_1$ as one to which we made the last restarting process. Then there is $i_2 > i_1$ such that, setting $\Gamma_2 = \Gamma^{i_2} \cap \overline{\Gamma}_1$, we have

$$\emptyset \neq \Gamma_2 = \bigcup_{i_1 < i \leq i_2} \Gamma^i(f) \cap \overline{\Gamma}_1, \tag{3}$$

whose closure admits a homogeneous dominated splitting

$$TM|\overline{\Gamma}_2 = E^{i_2} \oplus F^{i_2} \tag{4}$$

with $\dim E^{i_2} = i_2$, where $E^{i_2} = E_f^{i_2}|\overline{\Gamma}_2$ and $F^{i_2} = F_f^{i_2}|\overline{\Gamma}_2$, and

$$\bigcup_{i > i_2} \Gamma^i(f) \cap \overline{\Gamma}_2 = \emptyset. \tag{5}$$

Then, by (2)-(5), Lemma IV.2 applied to $\Lambda = \overline{\Gamma}_1$, $\iota = i_1$ and $\iota' = i_2$ implies that $\overline{\Gamma}_2$ is a hyperbolic set with index i_2 contained in $\overline{\Gamma}_1$. Note that $\overline{\Gamma}_2 \neq \overline{\Gamma}_1$. Here we include the case where $\overline{\Gamma}_1$ is already a hyperbolic set by thinking of $i_1 > i_0$, $\overline{\Gamma}_1$ and $\overline{\Gamma}_0$ as $i_2 > i_1$, $\overline{\Gamma}_2$ and $\overline{\Gamma}_1$, respectively.

In the next step, we consider a subset Λ_2 in $\overline{\Gamma}_1$ defined as an extension of $\overline{\Gamma}_2$. Define Λ_2 by the set of points $x \in \overline{\Gamma}_1$ such that $\lim_{n\to+\infty} p_n = x$ for some sequence $p_n \in P^{i_2}(f_n)$, $n \geq 1$, with $f_n \in \mathrm{Diff}^1(M)$ converging to f in the C^1 topology, satisfying that for every $\rho > 0$ if n is large enough, then $\mathcal{O}_{f_n}(p_n) \subset U_\rho(\overline{\Gamma}_1)$. By Lemma I.6 and continuity (see also the argument for (1) in Section II), we have a homogeneous dominated splitting

$$TM|\Lambda_2 = E_2^- \oplus E_2^+ \tag{6}$$

with $\dim E_2^- = i_2$. Thus, we have a decreasing sequence of subsets of Θ:

$$\overline{\Gamma}_2 \subset \Lambda_2 \subset \overline{\Gamma}_1 \subset \overline{\Gamma}_0 = \Lambda^{i_0} \subset \Theta,$$

where we need the Anosov Closing Lemma for the first inclusion $\overline{\Gamma}_2 \subset \Lambda_2$.

Lemma V.1. *Λ_2 is not a hyperbolic set.*

Proof. If this lemma is false, Λ_2 is a hyperbolic set with index i_2 contained in $\overline{\Gamma}_1$. Note that $\Lambda_2 \neq \overline{\Gamma}_1$. It is well known ([17, Theorem 4.2]) that, for every hyperbolic set Λ of f, if ε, $\delta > 0$ are small enough, there exists a continuous family $\{\widetilde{W}^\sigma_\delta(x,f) : x \in U_\varepsilon(\Lambda)\}$ ($\sigma \in \{s,u\}$) of disks satisfying the following properties:

$$\begin{aligned} \widetilde{W}^\sigma_\delta(x,f) &= W^\sigma_\delta(x,f), && \forall x \in \Lambda, \\ f^{-1}(\widetilde{W}^u_\delta(x,f)) &\subset \widetilde{W}^u_\delta(f^{-1}(x),f), && \forall x \in \overline{U_\varepsilon(\Lambda) \cap f(U_\varepsilon(\Lambda))}, \\ f(\widetilde{W}^s_\delta(x,f)) &\subset \widetilde{W}^s_\delta(f(x),f), && \forall x \in \overline{U_\varepsilon(\Lambda) \cap f^{-1}(U_\varepsilon(\Lambda))}. \end{aligned}$$

Moreover, for any $p \in \Lambda$,

$$\bigcup_{x \in W^s_\varepsilon(p,f)} \widetilde{W}^u_\delta(x,f) \quad \cap \bigcup_{x \in W^u_\varepsilon(p,f)} \widetilde{W}^s_\delta(x,f)$$

contains a neighborhood of p in M. Although Λ may not be isolated, we use the following notation:

$$W^s_\varepsilon(\Lambda, f) = \bigcup_{x\in\Lambda} W^s_\varepsilon(x, f), \qquad W^u_\varepsilon(\Lambda, f) = \bigcup_{x\in\Lambda} W^u_\varepsilon(x, f).$$

Let us apply these properties to $\Lambda = \Lambda_2$. Choose $0 < \bar\rho < \varepsilon/2$ so small that

$$\overline{f(U_{2\bar\rho}(\Lambda_2)) \cup U_{2\bar\rho}(\Lambda_2) \cup f^{-1}(U_{2\bar\rho}(\Lambda_2))} \subset U_\varepsilon(\Lambda_2) \tag{7}$$

and $M(f, U_{2\bar\rho}(\Lambda_2))$ is a hyperbolic set with index i_2. Let $y_n \in \Gamma_1$, $n \geq 1$, be a sequence converging to $\tilde y$ for some $\tilde y \in \overline{\Gamma}_2$. Since $\overline{\Gamma}_2 \subset \Lambda_2$, there exist $w^\sigma_n \in W^\sigma_\varepsilon(\tilde y, f)$, $\sigma \in \{s, u\}$, such that

$$y_n \in \widetilde{W}^u_\delta(w^s_n, f) \cap \widetilde{W}^s_\delta(w^u_n, f) \tag{8}$$

if n is large enough. Then

$$\lim_{n\to+\infty} \max\bigl\{d(y_n, w^\sigma_n) : \sigma = s, u\bigr\} = 0. \tag{9}$$

By the choice of $\bar\rho$ and $y_n \in \Gamma_1$, for large n we can define $s_n \geq 1$ (resp. $u_n \geq 1$) as the positive integer satisfying

$$f^{-j}(y_n) \in U_{\bar\rho}(\Lambda_2) \qquad \bigl(\text{resp. } f^j(y_n) \in U_{\bar\rho}(\Lambda_2)\bigr)$$

for all $0 \leq j \leq s_n - 1$ (resp. $u_n - 1$) and

$$f^{-s_n}(y_n) \notin U_{\bar\rho}(\Lambda_2) \qquad \bigl(\text{resp. } f^{u_n}(y_n) \notin U_{\bar\rho}(\Lambda_2)\bigr).$$

By (7)-(9), under some small choice of ε, $\delta > 0$ if n is large enough, then

$$f^{-j}(\widetilde{W}^u_\delta(w^s_n), f) \subset \widetilde{W}^u_\delta(f^{-j}(w^s_n), f) \qquad \bigl(\text{resp. } f^j(\widetilde{W}^s_\delta(w^u_n), f) \subset \widetilde{W}^s_\delta(f^j(w^u_n), f)\bigr) \tag{10}$$

for all $0 \leq j \leq s_n$ (resp. u_n), and

$$\lim_{n\to+\infty} \operatorname{diam} f^{-s_n}\widetilde{W}^u_\delta(w^s_n) = 0 \qquad \bigl(\text{resp. } \lim_{n\to+\infty} \operatorname{diam} f^{u_n}\widetilde{W}^s_\delta(w^u_n) = 0\bigr) \tag{11}$$

since s_n, $u_n \to +\infty$ as $n \to +\infty$. Hence, $\{f^{-s_n}(y_n)\}$ (resp. $\{f^{u_n}(y_n)\}$) with $y_n \in \Gamma_1$ accumulates on some

$$x^s \in \overline{\Gamma}_1 \setminus U_{\bar\rho}(\Lambda_2) \qquad \bigl(\text{resp. } x^u \in \overline{\Gamma}_1 \setminus U_{\bar\rho}(\Lambda_2)\bigr) \tag{12}$$

that is also an accumulation point of $\{f^{-s_n}(w^s_n) : n \geq 1\}$ (resp. $\{f^{u_n}(w^u_n) : n \geq 1\}$). Here, we may assume that x^s (resp. x^u) is the limit of them by omitting to take subsequences. From the definition, we can observe that all positive (resp. negative) iterates of x^s (resp. x^u) are in $\overline{U_{\bar\rho}(\Lambda_2)}$. In order to apply the C^1 Connecting Lemma at x^s and x^u, and create a homoclinic point, we need the following two claims:

Claim 1. x^s, $x^u \notin \operatorname{Per}(f)$.

Claim 2. There exist $0 < a < \bar\rho$ and a sequence of $v^\sigma_n \in \bigcup_{y\in\mathcal{O}_f(\tilde y)} W^\sigma(y, f)$ ($\sigma \in \{s, u\}$), $n \geq 1$, converging to x^σ such that

$$\{f^j(v^s_n) : j > 0\} \cap B_a(x^s) = \emptyset \quad \text{and} \quad \{f^{-j}(v^u_n) : j > 0\} \cap B_a(x^u) = \emptyset$$

for all n sufficiently large.

For the proof of Claim 1, let x^σ with either $\sigma = s$ or $\sigma = u$ be a periodic point of f. Then, as observed above, $\mathcal{O}_f(x^\sigma)$ is contained in $U_{2\bar\rho}(\Lambda_2)$ where locally invariant disks as in (7) with hyperbolic behavior exist. Therefore x^σ is a hyperbolic periodic point with index i_2, contradicting (12).

Now let us consider Claim 2. Define

$$v_n^s = f^{-s_n}(w_n^s), \qquad v_n^u = f^{u_n}(w_n^u)$$

and let us prove that they satisfy the required properties. First note from the choice of w_n^σ, $\sigma \in \{s,u\}$, and (9) that we can neglect the forward orbit of w_n^s and the backward orbit of w_n^u with large n. If Claim 2 is not true, for every $0 < a < \bar\rho$ there exist $\sigma \in \{s,u\}$ and arbitrarily large n such that $f^{j_n}(v_n^\sigma) \in B_a(x^\sigma)$ for some $0 < j_n \le s_n$ when $\sigma = s$ and $-u_n \le j_n < 0$ when $\sigma = u$, where $j_n \to +\infty$ as $a \to 0$ by Claim 1. By (8)-(11), the string $(v_n^s, f^{j_n}(v_n^s); f)$ or $(v_n^u, f^{j_n}(v_n^u); f^{-1})$ with large n is contained in $U_\varepsilon(\Lambda_2)$, where locally invariant disks as in (7) with hyperbolic behavior exist. Therefore the same argument as in the proof of the Anosov Closing Lemma shows that x^σ can be approximated by some $q_n^\sigma \in P^{i_2}(f)$ with period j_n. Moreover, recalling that $y_n \in \Gamma_1$, we see that for every $\rho > 0$, if n is large enough, then

$$(v_n^s, w_n^s; f) \cup (v_n^u, w_n^u; f^{-1}) \subset U_\rho(\overline{\Gamma}_1),$$

which enables us to take q_n^σ satisfying $\mathcal{O}_f(q_n^\sigma) \subset U_{2\rho}(\overline{\Gamma}_1)$. Hence $x^\sigma \in \Lambda_2$, contradicting (12) and proving Claim 2.

Since $\overline{\Gamma}_2$ is a hyperbolic set with index i_2 containing a dense recurrent points, use the Anosov Closing Lemma to get a sequence of periodic points $r_n \in P^{i_2}(f)$, $n \ge 1$, converging to $\tilde y \in \overline{\Gamma}_2$ and such that given $\rho > 0$, $\mathcal{O}_f(r_n) \subset U_\rho(\overline{\Gamma}_2)$ if n is large enough. By the continuity of invariant manifolds, changing the indexing of r_n, $n \ge 1$, and making $a > 0$ small if necessary, we can approximate v_n^σ in Claim 2 by some $\bar v_n^\sigma \in W^\sigma(\mathcal{O}_f(r_n), f)$ $(\sigma \in \{s,u\})$ whose sequence converges to x^σ, satisfying

$$(13)\quad \{f^j(\bar v_n^s) : j > 0\} \cap B_{a/2}(x^\sigma) = \emptyset \quad \text{and} \quad \{f^{-j}(\bar v_n^u) : j > 0\} \cap B_{a/2}(x^\sigma) = \emptyset$$

for all $\sigma \in \{s,u\}$ and n sufficiently large. In fact, as for the case where $\sigma = u$ (resp. $\sigma = s$) in the first (resp. second) equality, if this does not hold, for every $\delta > 0$ we can find a δ-pseudo-periodic orbit contained in $U_\varepsilon(\Lambda_2)$ consisting of $\mathcal{O}_f(r_n)$, a finite part of the backward (resp. forward) orbit of $\bar v_n^u$ (resp. $\bar v_n^s$) and that of the forward (resp. backward) orbit of some positive (resp. negative) iterate of $\bar v_n^s$ (resp. $\bar v_n^u$). Then, by the same argument as in the proof of the Shadowing Lemma (see [41] for instance), we have the same contradiction to (12) as in the proof of Claim 2.

On the other hand, since x^s, $x^u \in \overline{\Gamma}_1 \setminus \mathrm{Per}(f)$ by Claim 1, we can apply the C^1 Connecting Lemma [14, Lemma I.1(I)] in a similar way to prove [12, Theorem A] as follows. Recalling that $y_n \in \omega_f(y_n)$, we can take a sequence of strings $(x_n^u, x_n^s; f) \subset \mathcal{O}_f(y_n)$, $n \ge 1$, with $\{x_n^u\}$ and $\{x_n^s\}$ converging to x^u and x^s, respectively. First, when x^s is not on the forward f-orbit of x^u, for every C^1 neighborhood $\mathcal{V}$ of f, there exists $g \in \mathcal{V}$ coinciding with f outside a disjoint union $U_\epsilon\big((f(x^u), f^L(x^u); f)\big) \cup U_\epsilon\big((f^{-1}(x^s), f^{-L}(x^s); f^{-1})\big)$ with some $\epsilon > 0$ and $L \in \mathbf{Z}^+$, satisfying that if n is large enough, there is a substring $(y_n^u, y_n^s; f)$ of $(x_n^u, x_n^s; f)$ such that

$$(y_n^u, y_n^s; g) \cap B_{\epsilon b}(x^u) = \{y_n^u\} \quad \text{and} \quad (y_n^u, y_n^s; g) \cap B_{\epsilon b}(x^s) = \{y_n^s\}$$

for some $0 < b < 1$ depending on $\mathcal{V}$. Here $L \in \mathbf{Z}^+$ also depends on $\mathcal{V}$ but $\epsilon > 0$ can be arbitrarily small with L fixing. By (13), if $\epsilon > 0$ is small enough, then $B_{\epsilon b}(x^s)$, $B_{\epsilon b}(x^u)$, $U_\epsilon\big((f(x^u), f^L(x^u); f)\big)$, $U_\epsilon\big((f^{-1}(x^s), f^{-L}(x^s); f^{-1})\big)$, the forward f-orbit of $\bar{v}_n^s$, the backward f-orbit of $\bar{v}_n^u$ and $\mathcal{O}_f(r_n)$ with large n are all pairwise disjoint. As for another case where $x^s = f^k(x^u)$ for some $k \geq 0$, replacing x^s by $f^{L+1}(x^s)$ and making the previous perturbation only in $U_\epsilon\big((f(x^u), f^L(x^u); f)\big)$, we can find a substring $(y_n^u, y_n^s; f)$ as above such that $\{y_n^u\}$ and $\{f^{L+1}(y_n^s)\}$ with $y_n^s = f^k(y_n^u)$ can play the same roles as $\{y_n^u\}$ and $\{y_n^s\}$ in the previous case, respectively. That is, for some $g \in \mathcal{V}$ coinciding with f outside $U_\epsilon\big((f(x^u), f^L(x^u); f)\big)$ with some $\epsilon > 0$ and $L \in \mathbf{Z}^+$, if n is large enough, then

$$\big(y_n^u, f^{L+1}(y_n^s); g\big) \cap B_{\epsilon b}(x^u) = \{y_n^u\}$$

and

$$\big(y_n^u, f^{L+1}(y_n^s); g\big) \cap B_{\epsilon c}\big(f^{L+1}(x^s)\big) = \big\{f^{L+1}(y_n^s)\big\}$$

for some $0 < c < 1$ satisfying $B_{\epsilon c}(f^{L+1}(x^s)) \subset f^{k+L+1}(B_{\epsilon b}(x^u))$. Moreover, it is easy to see that the equalities in (13) with $\bar{v}_n^s$ and x^s replaced by $f^{L+1}(\bar{v}_n^s)$ and $f^{L+1}(x^s)$, respectively, still hold for some $a > 0$. Hence, it suffices to consider only the first case.

Since $\{\bar{v}_n^\sigma\}$ and $\{y_n^\sigma\}$ with each $\sigma \in \{s, u\}$ both converge to x^σ, we can easily find a diffeomorphism h_n arbitrarily C^1 close to the identity and coinciding with the identity outside $B_{\epsilon b}(x^s) \cup B_{\epsilon b}(x^u)$ such that $h_n(\bar{v}_n^u) = y_n^u$ and $h_n(y_n^s) = \bar{v}_n^s$. For sufficiently small $\epsilon > 0$, define

$$g_n = h_n \circ g.$$

Then $\bar{v}_n^s$ with large n is a homoclinic point for $g_n \in \mathcal{V}$ arbitrarily close to x^s associated to $\mathcal{O}_f(r_n)$. Given $\rho > 0$, the homoclinic orbit $\mathcal{O}_{g_n}(\bar{v}_n^s)$ created with sufficiently small $\epsilon > 0$ is contained in $U_\rho(\overline{\Gamma}_1)$ for large n. In fact, if n is large enough, the forward orbit of $\bar{v}_n^s$ and the backward orbit of $\bar{v}_n^u$ are contained in $U_\rho(\overline{\Gamma}_1)$ because of the corresponding properties for v_n^s and v_n^u in the proof of Claim 2. As for the part $(y_n^u, g_n^{-1}(\bar{v}_n^s); g_n)$ with large n, it is enough to see that

$$\begin{aligned}(y_n^u, g_n^{-1}(\bar{v}_n^s); g_n) \cap \big[U_\epsilon\big((f(x^u), f^L(x^u); f)\big) \cup U_\epsilon\big((f^{-1}(x^s), f^{-L}(x^s); f^{-1})\big)\big]^c \\ \subset (y_n^u, y_n^s; f)\end{aligned}$$

by the perturbation method of the C^1 Connecting Lemma (see [12], [13] for instance). By perturbing a little further if necessary, we may assume that the homoclinic orbit $\mathcal{O}_{g_n}(\bar{v}_n^s)$ is transversal, carrying a hyperbolic periodic point with index i_2 arbitrarily close to $\bar{v}_n^s$, whose periodic orbit is contained in $U_\rho(\overline{\Gamma}_1)$ for given $\rho > 0$ if n is large enough. But, this contradicts (12) again because $\{\bar{v}_n^s\}$ converges to x^s. Thus, the proof of Lemma V.1 is complete. □

Now, in examining the nonhyperbolicity of Λ_2, the simplest case is the following one:

$$\Gamma^{i_2}(f) \cap \Lambda_2 = \bigcup_{i \geq i_2} \Gamma^i(f) \cap \Lambda_2. \tag{14}$$

To include this case in a more general setting, suppose that (14) does not hold. Then there is $i > i_2$ such that $\Gamma^i(f) \cap \Lambda_2 \neq \emptyset$. Let $i_3 = \min\{i > i_2 : \Gamma^i(f) \cap \Lambda_2 \neq \emptyset\}$, and set $\Gamma_3 = \Gamma^{i_3}(f) \cap \Lambda_2$, whose closure admits a homogeneous dominated splitting

$$TM|\overline{\Gamma}_3 = E^{i_3} \oplus F^{i_3}$$

with $\dim E^{i_3} = i_3$, where $E^{i_3} = E_f^{i_3}|\overline{\Gamma}_3$ and $F^{i_3} = F_f^{i_3}|\overline{\Gamma}_3$. Apply Lemma IV.1 to $\Lambda = \overline{\Gamma}_1$, $\Lambda' = \overline{\Gamma}_3$, $E' \oplus F' = E_2^-|\overline{\Gamma}_3 \oplus E_2^+|\overline{\Gamma}_3$, $\iota = i_1$, $\iota' = i_2$ and $i = i_3$ to obtain that $E_2^-|\overline{\Gamma}_3$ is contracting. Moreover, applying Lemma IV.1 again to $\Lambda = \Lambda' = \overline{\Gamma}_3$,

$E^\iota \oplus F^\iota = E_2^-|\overline{\Gamma}_3 \oplus E_2^+|\overline{\Gamma}_3$, $E' \oplus F' = E^{i_3} \oplus F^{i_3}$ and $\iota' = i = i_3$, we see that E^{i_3} is also contracting. When F^{i_3} is not expanding, we can regard $\overline{\Gamma}_3$ as a nonhyperbolic $\overline{\Gamma}_1$ to continue our argument for higher indices than i_1. Note that such a nonhyperbolic case appears only finitely many times less than $\dim M$. So, let us suppose that $\overline{\Gamma}_3$ is a hyperbolic set contained in Λ_2, which has index i_3. Then we define a compact invariant set Λ_3 in Λ_2 as an extension of $\overline{\Gamma}_3$ corresponding to Λ_2 in $\overline{\Gamma}_1$, an extension of $\overline{\Gamma}_2$; i.e., Λ_3 is the set of $x \in \Lambda_2$ such that $\lim_{n\to+\infty} q_n = x$ for some $q_n \in P^{i_3}(f_n)$, $n \geq 1$, with $f_n \in \mathrm{Diff}^1(M)$ converging to f in the C^1 topology, satisfying that for every $\rho > 0$, if n is large enough, then $\mathcal{O}_{f_n}(q_n) \subset U_\rho(\Lambda_2)$. Despite the difference between $\overline{\Gamma}_1$ and Λ_2, the following lemma similar to Lemma V.1 still holds.

Lemma V.2. *Λ_3 is not a hyperbolic set.*

Proof. If this lemma is false, then Λ_3 is a hyperbolic set with index i_3. Pick $\tilde{p} \in \Lambda_3$. Since $\Lambda_3 \subset \Lambda_2$, there exists a sequence of periodic points $p_n \in P^{i_2}(f_n)$, $n \geq 1$, with $\lim_{n\to+\infty} f_n = f$ (in the C^1 topology) such that $\lim_{n\to+\infty} p_n = \tilde{p}$ and $\mathcal{O}_{f_n}(p_n) \subset U_\rho(\Lambda_2)$ for given $\rho > 0$ if n is large enough. Now it is still possible to apply the previous argument using a continuous family $\{\widetilde{W}^\sigma_\delta(x, g) : x \in U_\varepsilon(\Lambda_{3,g})\}$ ($\sigma \in \{s, u\}$) of disks as before with small ε, $\delta > 0$ independent of g C^1 close to f, where $\Lambda_{3,g}$ is the continuation of Λ_3 for g (see [41, Theorem 8.3]). Then, for $\tilde{p}_g$, the continuation of $\tilde{p}$ for g (i.e., $\tilde{p}_g = \Phi(g)(\tilde{p})$ with Φ given in [41, Theorem 8.3]),

$$\bigcup_{x \in W^s_\varepsilon(\tilde{p}_g, g)} \widetilde{W}^u_\delta(x, g) \quad \cap \bigcup_{x \in W^u_\varepsilon(\tilde{p}_g, g)} \widetilde{W}^s_\delta(x, g)$$

contains a neighborhood of $\tilde{p}$ in M for all g sufficiently C^1 close to f. Take $0 < \eta < \varepsilon/2$ so small that, for all g sufficiently C^1 close to f, $M(g, U_{2\eta}(\Lambda_{3,g}))$ is a hyperbolic set with index i_3 and

$$\overline{f(U_{2\eta}(\Lambda_3)) \cup U_{2\eta}(\Lambda_3) \cup f^{-1}(U_{2\eta}(\Lambda_3))} \subset U_\varepsilon(\Lambda_{3,g}).$$

For large n, set $\tilde{p}_n = \tilde{p}_{f_n}$ and choose $z^\sigma_n \in W^\sigma_\varepsilon(\tilde{p}_n, f_n)$, $\sigma \in \{s, u\}$, such that

$$p_n \in \widetilde{W}^u_\delta(z^s_n, f_n) \cap \widetilde{W}^s_\delta(z^u_n, f_n).$$

Note that $\mathcal{O}_{f_n}(p_n)$ intersects outside $U_{2\eta}(\Lambda_{3,f_n})$ because $p_n \in P^{i_2}(f_n)$, and therefore intersects outside $U_\eta(\Lambda_3)$ for large n. Hence, for large n, we can define $\tilde{u}_n \geq 1$ (resp. $\tilde{s}_n \geq 1$) as the positive integer satisfying

$$f^j_n(p_n) \in U_\eta(\Lambda_3) \quad \big(\text{resp. } f^{-j}_n(p_n) \in U_\eta(\Lambda_3)\big)$$

for all $0 \leq j \leq \tilde{u}_n - 1$ (resp. $\tilde{s}_n - 1$) and

$$f^{\tilde{u}_n}_n(p_n) \notin U_\eta(\Lambda_3) \quad \big(\text{resp. } f^{-\tilde{s}_n}_n(p_n) \notin U_\eta(\Lambda_3)\big),$$

which satisfies $\lim_{n\to+\infty} \tilde{u}_n = +\infty$ (resp. $\lim_{n\to+\infty} \tilde{s}_n = +\infty$). Then, by similar properties to (8)-(11), $\{f^{\tilde{u}_n}_n(p_n) : n \geq 1\}$ and $\{f^{\tilde{u}_n}_n(z^u_n) : n \geq 1\}$ (resp. $\{f^{-\tilde{s}_n}_n(p_n) : n \geq 1\}$ and $\{f^{-\tilde{s}_n}_n(z^s_n) : n \geq 1\}$) accumulate on some

$$x^u_0 \in \Lambda_2 \setminus U_\eta(\Lambda_3) \quad \big(\text{resp. } x^s_0 \in \Lambda_2 \setminus U_\eta(\Lambda_3)\big), \tag{15}$$

where the property $x^\sigma_0 \in \Lambda_2$ ($\sigma \in \{s, u\}$) comes from the accumulation of points in $\mathcal{O}_{f_n}(p_n)$, $n \geq 1$. Observe that we still have

$$\{f^{-j}(x^u_0) : j > 0\} \subset \overline{U_\eta(\Lambda_3)} \quad \big(\text{resp. } \{f^j(x^s_0) : j > 0\} \subset \overline{U_\eta(\Lambda_3)}\,\big), \tag{16}$$

similarly to the property mentioned above Claim 1 in the proof of Lemma V.1. Moreover, x^σ_0 ($\sigma \in \{s, u\}$) can be approximated by a point v^σ in $\bigcup_{y \in \mathcal{O}_{f_n}(\tilde{p}_n)} W^\sigma(y, f_n)$

whose all positive f_n-iterates are in $U_\varepsilon(\Lambda_{3,f_n})$ when $\sigma = s$, and so are all negative f_n-iterates when $\sigma = u$ for arbitrarily large n, which corresponds to v_n^σ in the proof of Lemma V.1. In fact, we can choose them as

$$v^u = f_n^{\tilde{u}_n}(z_n^u) \quad \text{and} \quad v^s = f_n^{-\tilde{s}_n}(z_n^s)$$

for large n. Then, as before, it is enough to see that $(v^s, z_n^s; f_n) \cup (v^u, z_n^u; f_n^{-1})$ is contained in $U_\varepsilon(\Lambda_{3,f_n})$ for all n sufficiently large, which is obtained by the choice of z_n^σ, $\sigma \in \{s,u\}$, and $\tilde{s}_n$, $\tilde{u}_n \geq 1$. From these properties, the following Claims 1′ and 2′ can be proved similarly to the proofs of Claims 1 and 2 by thinking of x^σ, v_n^σ and $\tilde{y}$ as x_0^σ, v^σ and $\tilde{p}$, respectively for $\sigma \in \{s,u\}$.

Claim 1′. $x_0^s, x_0^u \notin \mathrm{Per}(f)$.

Claim 2′. There exist $0 < a < \eta$ and $v^\sigma \in \bigcup_{y \in \mathcal{O}_{f_n}(\tilde{p}_n)} W^\sigma(y, f_n)$ $(\sigma \in \{s,u\})$ approximating x_0^σ for arbitrarily large n such that

$$\{f_n^j(v^s) : j > 0\} \cap B_a(x_0^s) = \emptyset \quad \text{and} \quad \{f_n^{-j}(v^u) : j > 0\} \cap B_a(x_0^u) = \emptyset.$$

According to the proof of Lemma V.1, the rest of the proof of Lemma V.2 after Claims 1′ and 2′ have been obtained is almost identical to that of Lemma V.1. By C^1-perturbing f_n with large n, we create a transversal homoclinic point approximating x_0^s, whose orbit is contained in an arbitrarily small neighborhood of Λ_2, which is associated to a periodic orbit $\mathcal{O}_{f_n}(r_n)$ with $r_n \in P^{i_3}(f_n)$. For the creation, we apply [14, Lemma I.1(I)] through a sequence of strings $\{(x_n^u, x_n^s; f_n)\}$ with $x_n^u = f_n^{\tilde{u}_n}(p_n)$ and $x_n^s = f_n^{-\tilde{s}_n}(p_n)$. That is, through the sequence of strings contained in arbitrarily small neighborhoods of Λ_2, x_0^u is forwardly related to x_0^s by $f_n \to f$ using the terminology given in [14], and then applying [14, Lemma I.1(I)] to have a string for a C^1 small perturbation of f_n, from which together with the backward f_n-orbit of v^u and the forward one of v^s or its some positive iterate, we obtain the required homoclinic orbit by the same procedure as in the proof of Lemma V.1. Thus, we have a sequence of periodic orbits of diffeomorphisms converging to f carried by the sequence of created transversal homoclinic orbits. This provides a periodic point with index i_3 approximating x_0^s, whose orbit stays arbitrarily close to Λ_2, contradicting (15) and concluding the proof of Lemma V.2. □

Inductively, we may finally assume that nonhyperbolic $\overline{\Gamma^{i_j}(f) \cap \Lambda_j}$ with $j \geq 2$ never appears, obtaining a sequence of nonhyperbolic compact f-invariant sets Λ_j, $1 \leq j \leq k$, for some $k \geq 2$ such that

$$\Lambda_1 \supset \cdots \supset \Lambda_k$$

and

$$\Gamma^{i_k}(f) \cap \Lambda_k = \bigcup_{i \geq i_k} \Gamma^i(f) \cap \Lambda_k, \tag{17}$$

where each Λ_j $(2 \leq j \leq k)$ is defined as an extension of a hyperbolic set $\overline{\Gamma^{i_j}(f) \cap \Lambda_j}$ in Λ_{j-1} with index i_j and $\Lambda_1 = \overline{\Gamma}_1$. Note that the case (14) is a special case with $k = 2$. Moreover, as in (2) and (6), a homogeneous dominated splitting

$$TM|\Lambda_k = E_j^-|\Lambda_k \oplus E_j^+|\Lambda_k \tag{18}$$

with $\dim E_j^- = i_j$ is admitted for all $1 \leq j \leq k$, where $E_1^- = E^{i_1}$ and $E_1^+ = F^{i_1}$. Here, by (17) and Corollary III.2, $E_k^+|\Lambda_k$ is expanding. Therefore, from the nonhyperbolicity of Λ_k and Corollary III.2 it follows that $\Gamma^{i_j}(f) \cap \Lambda_k \neq \emptyset$ for some $1 \leq j < k$. For $1 \leq j \leq k$ define

$$\Lambda_k^j = \overline{\Gamma^{i_j}(f) \cap \Lambda_k}.$$

Then we have the following properties:

- $\Lambda_k^k \neq \emptyset$.
- If $\Lambda_k^j \neq \emptyset$ with $2 \leq j \leq k$, then Λ_k^j is a hyperbolic set with index i_j.
- There exists $1 \leq j < k$ such that $\Lambda_k^j \neq \emptyset$.

When $\Lambda_k^1 \neq \emptyset$ and

$$\Lambda_k^1 \cap \Lambda_k^j \neq \emptyset \tag{19}$$

for some $2 \leq j \leq k$, letting $\kappa = \max\{2 \leq j \leq k : \Lambda_k^1 \cap \Lambda_k^j \neq \emptyset\}$, we have from Corollary III.2 that $E_\kappa^+|\Lambda_k^1 \cup \Lambda_k^\kappa$ is expanding. Given $\rho > 0$, using the Katok and Anosov Closing Lemmas, we can find

$$p \in P^{i_1}(f) \cap M(f, U_{\rho/2}(\Lambda_k^1)) \quad \text{and} \quad q \in P^{i_\kappa}(f) \cap M(f, U_{\rho/2}(\Lambda_k^\kappa))$$

approximating points in Λ_k^1 and Λ_k^κ, respectively, whose distance can be arbitrarily small by (19), and such that

$$H_f(p, U_\rho(\Lambda_k^1)) \cap \Lambda_k^1 \neq \emptyset \quad \text{and} \quad H_f(q, U_\rho(\Lambda_k^\kappa)) \cap \Lambda_k^\kappa \neq \emptyset. \tag{20}$$

Let $\rho_0 > 0$ be such that a homogeneous dominated splitting denoted by

$$TM|M(f, U_{\rho_0}(\Lambda_k)) = \widehat{E}_j^- \oplus \widehat{E}_j^+$$

that extends (18) is defined for all $1 \leq j \leq k$. For $\sigma \in \{-,+\}$ let $D_{j,\delta}^\sigma(x)$, $x \in M(f, U_{\rho_0}(\Lambda_k))$, denote locally invariant disks tangent to $\widehat{E}_j^\sigma(x)$ at x with size $\delta > 0$ coming from this splitting. Take $\delta_1 > 0$ so small that $D_{\kappa,\delta_1}^+(p) \subset W^u(p, f)$ and $D_{\kappa,\delta_1}^-(q) \subset W^s(q, f)$, which is determined by the domination property of (18) with $j = \kappa$ including the hyperbolicity of Λ_k^κ and the expansion of $E_\kappa^+|\Lambda_k^1 \cup \Lambda_k^\kappa$ when $\rho > 0$ is small enough. If $d(p, q) = \xi$ is chosen sufficiently small depending on δ_1, then

$$D_{\kappa,\delta_1}^+(p) \pitchfork D_{\kappa,\delta_1}^-(q) \neq \emptyset,$$

and hence

$$W^u(p, f) \pitchfork W^s(q, f) \neq \emptyset.$$

From this together with (20) and the transitivity of Θ, given a neighborhood U of Θ, if positive constants ρ, δ_1 and ξ are chosen small enough, we obtain a pseudo-heterodimensional cycle in U between $\mathcal{O}_f(p)$ and $\mathcal{O}_f(q)$. Moreover, we can apply Lemma I.4 to create a heterodimensional cycle in U between $\mathcal{O}_f(p)$ and $\mathcal{O}_f(q)$ by an arbitrarily small C^1 perturbation of f, keeping the transversal intersection continued from above, which contradicts our assumption of f and proves that (19) does not occur.

Since $E_1^-|\Lambda_k^1$ is contracting, the fact that (19) does not occur and Corollary III.2 imply that Λ_k^1 is also a hyperbolic set with index i_1 contained in Λ_k. Let $\Lambda_k^\iota \in \{\Lambda_k^j : 1 \leq j \leq k\}$ be the hyperbolic set whose index $i_\iota \in \{i_j : 1 \leq j \leq k\}$ is the smallest among indices of Λ_k^j, $1 \leq j \leq k$, that are not empty. Note that $\iota < k$

by the last item of the properties of Λ_k^j above (19). Given a small $\rho > 0$ such that $\Lambda_k^\iota \cap U_\rho(\Lambda_k^j) = \emptyset$ for all $\iota < j \le k$, define

$$\tilde{\Lambda}_k^\iota = M\Big(f,\ \Lambda_k \setminus \bigcup_{\iota < j \le k} U_\rho(\Lambda_k^j)\Big).$$

Then $\Lambda_k^\iota \subset \tilde{\Lambda}_k^\iota$ and by Corollary III.2 together with the smallest choice of i_ι, we see that $\tilde{\Lambda}_k^\iota$ is a hyperbolic set with index ι. Making $\rho > 0$ smaller if necessary, we can suppose that $M(g, U_{2\rho}(\tilde{\Lambda}_{k,g}^\iota))$ is still a hyperbolic set with index ι for all g sufficiently C^1 close to f, where $\tilde{\Lambda}_{k,g}^\iota$ is the continuation of $\tilde{\Lambda}_k^\iota$ for g. Then choose

$$x_k \in \Lambda_k \setminus \tilde{\Lambda}_k^\iota$$

similarly to $x_0^u \in \Lambda_2$ in (15) as an accumulation point of hyperbolic periodic saddles with index $i_k (> i_\iota)$ coming from the definition of Λ_k, which are some positive iterates (i.e., each of which is the first positive iterate outside $U_\rho(\tilde{\Lambda}_k^\iota)$) of points converging to a point in $\tilde{\Lambda}_k^\iota$. Then $\alpha_f(x_k)$ is contained in the closure of $U_\rho(\tilde{\Lambda}_k^\iota)$ by the property corresponding to (16), and therefore $\alpha_f(x_k)$ in $U_{2\rho}(\tilde{\Lambda}_k^\iota)$ is a hyperbolic set with index ι. Since $x_k \notin \tilde{\Lambda}_k^\iota$, there exist $\ell \in \mathbf{Z}$ and $\iota < j \le k$ such that

$$f^\ell(x_k) \in U_\rho(\Lambda_k^j).$$

Note that x_k, ℓ and j above all depend on the choice of ρ. Letting ρ_n, $n \ge 1$, be a sequence of this ρ converging to zero, we have $x_k(n)$, $\ell(n)$ and $j(n)$ as x_k, ℓ and j for $\rho = \rho_n$, respectively. Put $y_k(n) = f^{\ell(n)}(x_k(n))$ and let Ξ be the set of accumulation points of

$$\bigcup_{n \ge 1} \alpha_f(x_k(n)) \cup \{f^{-m}(y_k(n)) : m \ge 0\}.$$

Let $\iota' = \max\{\iota < j \le k : \Xi \cap \Lambda_k^j \ne \emptyset\}$ and set

$$\Xi' = \Xi \cup \bigcup_{\iota \le j \le \iota'} \Lambda_k^j.$$

Then Ξ' is a nonhyperbolic compact f-invariant set in Λ_k, containing Λ_k^j, $\iota < j \le \iota'$, and $\Lambda_k^\iota \cup \alpha_f(x_k(n))$, which are hyperbolic sets with distinct indices. From Corollary III.2, we have that $E_{\iota'}^+|\Xi'$ is expanding. Hence, for every $\varepsilon > 0$, there exist $x_k \in \{x_k(n) : n \ge 1\}$ and $\ell \in \mathbf{Z}$ (possibly $\ell \notin \{\ell(n) : n \ge 1\}$) such that

$$f^\ell(x_k) \in U_\varepsilon(\Lambda_k^{\iota'}).$$

Using the Anosov Closing Lemma, we have $\bar{q} \in P^{\iota'}(f) \cap M(f, U_\varepsilon(\Lambda_k^{\iota'}))$ approximating a point in $\Lambda_k^{\iota'}$, whose distance to $f^\ell(x_k)$ is less than ε. Given $\rho > 0$, if $\varepsilon > 0$ has been chosen small enough, then

$$H_f(\bar{q}, U_\rho(\Lambda_k^{\iota'})) \cap \Lambda_k^{\iota'} \ne \emptyset, \tag{21}$$

and, by the hyperbolicity of $\Lambda_k^{\iota'}$ and the expansion of $E_{\iota'}^+|\Xi'$, for an appropriate choice of $\delta > 0$ and $\varepsilon > 0$ similarly to that of δ_1 and ξ before, we have

$$D_{\iota',\delta}^+(f^\ell(x_k)) \pitchfork D_{\iota',\delta}^-(\bar{q}) \ne \emptyset$$

with

$$D_{\iota',\delta}^-(\bar{q}) \subset W^s(\bar{q}, f) \quad \text{and} \quad D_{\iota',\delta}^+(f^\ell(x_k)) \subset W^u(f^\ell(x_k), f).$$

Then, together with the hypebolicity of $\alpha_f(x_k)$, making $\delta > 0$ smaller if necessary, we can find an ι-dimensional disk D^s_l with

$$D^s_l \subset W^s(\bar{q}, f) \tag{22}$$

arbitrarily close to $D^-_{\iota,\delta}(f^{-l}(x_k))$ in the C^1 topology for large $l > 0$ (as in the λ-lemma [30]). For every large l there exists $y_l \in \alpha_f(x_k)$ such that

$$W^u_\delta(y_l, f) \pitchfork D^-_{\iota,\delta}(f^{-l}(x_k)) \neq \emptyset.$$

Therefore

$$W^u_\delta(y_l, f) \pitchfork D^s_l \neq \emptyset$$

for all l sufficiently large. By the Anosov Closing Lemma, given $\rho > 0$, we can approximate y_l with large l by some $\bar{p} \in P^\iota(f) \cap M\big(f, U_{\rho/2}(\alpha_f(x_k))\big)$ satisfying

$$W^u_\delta(\bar{p}, f) \pitchfork D^s_l \neq \emptyset \tag{23}$$

and

$$H_f\big(\bar{p}, U_\rho(\alpha_f(x_k))\big) \cap \alpha_f(x_k) \neq \emptyset. \tag{24}$$

Then (22) and (23) imply

$$W^u(\bar{p}, f) \pitchfork W^s(\bar{q}, f) \neq \emptyset.$$

From this together with (21), (24) and the transitivity of Θ, it follows that given $\rho > 0$ there exists a pseudo-heterodimensional cycle in $U_\rho(\Theta)$ between $\mathcal{O}_f(\bar{p})$ and $\mathcal{O}_f(\bar{q})$ for f. Moreover, using Lemma I.4, we obtain a heterodimensional cycle in $U_\rho(\Theta)$ between $\mathcal{O}_f(\bar{p})$ and $\mathcal{O}_f(\bar{q})$ for some diffeomorphism arbitrarily C^1 close to f. This contradicts our assumption of f and proves that F^{i_0} is expanding. This completes the proof of Theorem II.1.

VI. Proof of Theorem B

The main part of the proof of Theorem B corresponds to the part where the contraction of subbundles over homoclinic classes is proved in the proof of Theorem A. Indeed, the proof for the expansion of another subbundle is just repeating the argument for the inverse by the symmetric condition of Theorem B.

First recall that if $f \in \mathrm{Diff}^2(M)$, then every $x \in \Gamma^i(f)$, $0 < i < \dim M$, can be approximated by some $p \in P^i(f)$ by the Katok Closing Lemma, implying $\Gamma^i(f) \subset \overline{P^i(f)}$. Since $\Gamma^i(f) \supset P^i(f)$ and $\Gamma^i(f) = P^i(f)$ for $i \in \{0, \dim M\}$, we have

$$\overline{\Gamma^i(f)} = \overline{P^i(f)} \tag{1}$$

for all $0 \le i \le \dim M$.

We need the following lemma that corresponds to Theorem II.1.

Lemma VI.1. *Let $f \in \mathcal{R}^2$. Suppose that there exist a C^2 neighborhood $\mathcal{U}^2_*$ of f and its residual subset $\mathcal{R}^2_*$ with f satisfying the following properties:*

(a) *there are constants $m \in \mathbf{Z}^+$, $0 < \lambda < 1$ and $0 < i_0 < \dim M$ such that $\overline{\Gamma^{i_0}(g)}$ admits a homogeneous (m, λ)-dominated splitting*

$$TM|\overline{\Gamma^{i_0}(g)} = E_g \oplus F_g$$

with $\dim E_g = i_0$ for all $g \in \mathcal{R}^2_* \cap \mathcal{R}^2$, and

$$\inf\{\angle(E_g(x), F_g(x)) : x \in \overline{\Gamma^{i_0}(g)},\ g \in \mathcal{R}^2_* \cap \mathcal{R}^2\} > 0;$$

(b) $\bigcup_{i=0}^{i_0-1}\Gamma^i(g)$ *is contained in a countable union of hyperbolic homoclinic classes for all* $g\in\mathcal{R}^2_*\cap\mathcal{R}^2$.

Then there exists a residual subset $\mathcal{R}^2_{**}(\subset\mathcal{R}^2_*)$ *of a* C^2 *neighborhood* $\mathcal{U}^2_{**}$ *of* f *such that if* $g\in\mathcal{R}^2_{**}\cap\mathcal{R}^2$*, neither admits zero Lyapunov exponents nor exhibits a pseudo-heterodimensional cycle, and a* g*-invariant set*

$$\Lambda_g^{i_0}=\overline{\{x\in\mathrm{supp}\,(\mu):\mu\in\mathcal{M}_g,\ \mathrm{Ind}\,(\mu)=i_0\}}$$

for some $\mathcal{M}_g\subset\mathcal{M}_e(g)$ *is contained in a transitive set, then* $E_g|\Lambda_g^{i_0}$ *is contracting.*

Here, the above pseuso-heterodimensional cycle that g does not exhibit can be specified as one in an arbitrarily small neighborhood of the transitive set and involving a periodic saddle with index i_0.

Lemma VI.1 is proved just by following the proof of Theorem II.1 since Lemma VI.1 is essentially a special case of Theorem II.1. The condition not belonging to $\overline{\mathcal{HT}(M)}$ is stronger than property (a) of Lemma VI.1, but we can easily see that the same argument as in the proof of Theorem II.1 works for the proof of Theorem VI.1. Indeed, in Section III we used the condition belonging to $\mathcal{D}^2$ only after the proof of Lemma III.3, and in that part we could replace the condition by the condition that $f\in\mathcal{R}^2$ has a C^2 neighborhood in which every $g\in\mathcal{R}^2$ admits a homogeneous dominated splitting over $\overline{\Gamma^{i_0}(g)}$ that extends hyperbolic splittings over hyperbolic periodic orbits with index i_0, having the uniform domination properties and angles as in property (a) of Lemma VI.1. This uniformity is needed in applications of Lemma III.1.

On the other hand, the argument for the proof of Theorem II.2 (which is not necessary for the proof of Theorem B) needs more properties than this, which guarantees the uniform domination and angles over hyperbolic periodic saddles with large periods including all C^1 small perturbations of f.

To have property (a) of Lemma VI.1, define $\mathcal{Q}^2$ as a C^2 residual subset of $\mathrm{Diff}^2(M)$ at which the set-valued function $f\mapsto\overline{P^i(f)}$ is continuous for all $1<i<\dim M$, and let

$$\widetilde{\mathcal{R}}^2=\mathcal{Q}^2\cap\mathcal{R}^2.$$

Then observe from (1) and continuity that if $f\in\widetilde{\mathcal{R}}^2$ satisfies the hypothesis of Theorem B, for a C^2 neighborhood $\mathcal{U}(f)$ of f with a residual subset $\mathcal{R}(f)$ given in the hypothesis of Theorem B, shrinking $\mathcal{U}(f)$ if necessary, we get a homogeneous (m,λ)-dominated splitting over $\overline{\Gamma^{i_0}(g)}$ for all $g\in\mathcal{U}(f)$ with uniform angles. Hence, all properties for applying Lemma VI.1 are obtained as

$$\mathcal{U}^2_*=\mathcal{U}(f)\quad\text{ and }\quad\mathcal{R}^2_*=\mathcal{R}(f)\cap\widetilde{\mathcal{R}}^2.$$

Note that the hypothesis of Theorem B (and therefore that of Lemma VI.1) is satisfied for every $g\in\mathcal{R}^2_*$. Now set $\{p_n:n\ge 1\}=P^{i_0}(g)$ for $g\in\mathcal{R}^2_*$ and apply Lemma VI.1 to $\Lambda_g^{i_0}=H_g(p_n)$, $n\ge 1$, in order to obtain a residual subset $\mathcal{R}^2_{**}(\subset\mathcal{R}^2_*)$ of $\mathcal{U}^2_{**}$ such that $E_g|H_g(p_n)$, $n\ge 1$, is contracting for all $g\in\mathcal{R}^2_{**}$ satisfying neither property (b) nor property (c) of Theorem B. Then define $\widehat{\mathcal{R}}^2$ by the union of the subset $\mathcal{R}^2_{**}$ of $\mathcal{R}^2_*=\mathcal{R}(f)\cap\widetilde{\mathcal{R}}^2$ over all $f\in\widetilde{\mathcal{R}}^2$ satisfying the hypothesis of Theorem B together with the complement $\widetilde{\mathcal{R}}^2_c$ in $\widetilde{\mathcal{R}}^2$ of the closure of the union. Note that every element in $\widetilde{\mathcal{R}}^2_c$ does not satisfy the hypothesis of Theorem B. To obtain that $F_g|H_g(p_n)$, $n\ge 1$, is expanding, we can repeat this argument to their inverse diffeomorphisms for the new choice of $\mathcal{R}_*^2 \subset \mathcal{U}_*^2$ defined by

$$\mathcal{U}_*^2 = \mathcal{U}(f) \quad \text{and} \quad \mathcal{R}_*^2 = \mathcal{R}(f) \cap \widehat{\mathcal{R}}^2$$

with $f \in \widehat{\mathcal{R}}^2 \setminus \widetilde{\mathcal{R}}_c^2$ and $\mathcal{U}(f)$ shrunk if necessary. Finally, define $\mathcal{R}$ by the union of the subset $\mathcal{R}_{**}^2$ of $\mathcal{R}_*^2 = \mathcal{R}(f) \cap \widehat{\mathcal{R}}^2$ (obtained by Lemma VI.1 for the new choice of $\mathcal{R}_*^2 \subset \mathcal{U}_*^2$ above) over all $f \in \widehat{\mathcal{R}}^2 \setminus \widetilde{\mathcal{R}}_c^2$ together with $\widetilde{\mathcal{R}}_c^2$ that is also the complement in $\widehat{\mathcal{R}}^2$ of the closure of the union. Thus, we obtain a C^2 residual subset $\mathcal{R}$ of $\mathrm{Diff}^2(M)$ (which is the required one in Theorem B) such that if $f \in \mathcal{R}$ satisfies the hypothesis of Theorem B and neither property (b) nor property (c) of Theorem B, then $H_f(p)$ is a hyperbolic set with index i_0 for all $p \in P^{i_0}(f)$. This is the property corresponding to (2) and (3) of Section II in the proof of Theorem A. The final step applying Lemma I.3 in order to conclude that such f is an Axiom A diffeomorphism with no cycles is the same as in the proof of Theorem A. This completes the proof of Theorem B.

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