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SECTIONS OF CALABI-YAU THREEFOLDS WITH K3 FIBRATION

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Abstract. We study sections of a Calabi-Yau threefold fibered over a curve by K3 surfaces. We show that there exist infinitely many isolated sections on certain K3 fibered Calabi-Yau threefolds and that the group of algebraic 1-cycles generated by these sections modulo algebraic equivalence is not finitely generated. We also give examples of K3 surfaces over the function field F of a complex curve with Zariski dense F-rational points, whose geometric model is Calabi-Yau.

1. Introduction

Let Y be a smooth projective variety over the function field F of a smooth projective curve B. Let $Y(F)$ denote the set of F-rational points (or equivalently, sections of a projective model $\mathcal{Y} \to B$ with generic fiber Y).

The Zariski density or potential density (i.e. density after a finite field extension) of $Y(F)$ is expected to relate to the global geometry of Y. For instance, it is known [GHS03] that $Y(F)$ is Zariski dense for rationally connected varieties Y. On the other hand, Lang's conjecture over function fields [Hin98], confirmed in the curve case, predicts that potential density fails for certain classes of general type varieties.

For the intermediate case where the canonical class K_Y is trivial, it is generally expected that $Y(F)$ is potentially dense. This is known for abelian varieties and $K3$ surfaces with additional structures [BT00]. Already, the case for general $K3$ surfaces remains widely open.

If $F = \mathbb{C}(t)$, Hassett and Tschinkel [HT08] proved that a pencil of low degree K3 surfaces has a Zariski dense collection of sections via the study of the deformation of sections in the pencil. The examples in [HT08] correspond to lines in the Hilbert scheme of the K3 surface. It is natural to consider rational curves in that Hilbert scheme of higher degree.

In this paper, we will consider families of K3 surfaces which correspond to conics in the Hilbert scheme. One concrete example is the bidegree $(2,4)$ hypersurface $X \subset \mathbb{P}^1 \times \mathbb{P}^3$, which is a family of quartic surfaces over a conic. Note that X is a Calabi-Yau threefold, so the sections in X are generally expected to be *isolated* or *infinitesimally rigid*, i.e. the space of embedded deformations is reduced and zero dimensional. Our main result is the following:

Theorem 1. *There exist countably many isolated sections ℓ_n on a very general bidegree $(2,4)$ hypersurface $X \subset \mathbb{P}^1 \times \mathbb{P}^3$ with respect to the projection to $\mathbb{P}^1$. Furthermore, the union of these sections is Zariski dense in X.*

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Let $\mathcal{A}^2(X)$ denote the group of algebraic cycles of codimension 2 modulo algebraic equivalence. It is known that this group is not finitely generated for general Calabi-Yau threefolds [Voi00]. An interesting question is to study the subgroup of $\mathcal{A}^2(X)$ generated by these sections ℓ_n. Here we follow the method of Clemens [Cle83a] to obtain the following theorem, which will imply the Zariski density of $\{\ell_n\}$.

Theorem 2. *The subgroup $\mathcal{A}^{sec} \subseteq \mathcal{A}^2(X)$ generated by the sections of $X \to \mathbb{P}^1$ is not finitely generated.*

More generally, we can obtain by induction the density results for higher dimensional Calabi-Yau hypersurfaces in $\mathbb{P}^1 \times \mathbb{P}^N$ for $N \geq 3$.

Corollary 1. *The union of sections on a very general bidegree $(2, N+1)$ hypersurfaces of $\mathbb{P}^1 \times \mathbb{P}^N$ is Zariski dense.*

This paper is organized as follows: In section 2, we recall the Néron model theory on degenerations of intermediate Jacobians. In particular, we state a theorem describing the Néron models coming from geometry. Section 3 is devoted to showing the existence of infinitely many isolated sections on X. We will describe the construction of these sections using specialization. In section 4, we find a useful degeneration of our Calabi-Yau threefolds and study the the deformation theory of curves on the singular fiber of the degeneration. As an application of the result in section 2, we compute the group of components of the Néron model associated to the degeneration. The main theorems are proved in section 5 and section 6. In section 7, we extend our results to higher dimensional cases.

2. Preliminaries on Néron models

In this section, we briefly review some results [GGK10] of Néron model theory for families of intermediate Jacobians coming from a variation of Hodge structure (VHS), which will be used later in this paper. For simplicity, our VHS arises from geometry and is paramatrized by a complex disc.

2.1. **Geometric setting.** Let X be a smooth projective variety of dimension $2k-1$. The k-th intermediate Jacobian $J(X)$ of X is a compact torus defined as

$$J(X) = H^{2k-1}(X, \mathbb{C})/(F^k H^{2k-1}(X) \oplus H^{2k-1}(X, \mathbb{Z})),$$

where $F^{\cdot} H^{2k-1}(X)$ is the Hodge filtration of $H^{2k-1}(X)$.

More generally, let Δ be a complex disc and let $\pi : \mathcal{X} \to \Delta$ be a *semistable degeneration*, that is:

- **(1)** $\mathcal{X}$ is smooth of dimension $2k$;
- **(2)** π is projective, with the restriction $\pi : \mathcal{X}^* = \mathcal{X} \backslash \pi^{-1}(0) \to \Delta^*$ smooth, where $\Delta^* = \Delta \backslash \{0\}$;
- **(3)** the fiber $\mathcal{X}_0 = \pi^{-1}(0)$ is reduced with non-singular components crossing normally; write $\mathcal{X}_0 = \bigcup X_i$.

Consider the VHS associated to the $(2k-1)$-th cohomology along the fibers of $\pi : \mathcal{X} \to \Delta^*$; then there is family of intermediate Jacobians

$$\mathcal{J} \to \Delta^*, \tag{2.1}$$

which forms an analytic fiber space with fiber $\mathcal{J}_s = J(\mathcal{X}_s)$, $s \in \Delta^*$.

Because of the semistability assumption, the Monodromy theorem [Lan73] implies that the monodromy transformation

$$T : H^{2k-1}(\mathcal{X}_s, \mathbb{Z}) \to H^{2k-1}(\mathcal{X}_s, \mathbb{Z})$$

is unipotent. In this situation, Green, Griffiths and Kerr [GGK10] have constructed a slit analytic space $\bar{\mathcal{J}}(\mathcal{X}) \to \Delta$ such that

- the restriction $\bar{\mathcal{J}}(\mathcal{X})|_{\Delta^*}$ is $\mathcal{J} \to \Delta^*$;
- every *admissible normal function* (ANF) extends to a holomorphic section of $\bar{\mathcal{J}}(\mathcal{X}) \to \Delta$; here an ANF is a holomorphic section of (2.1) satisfying the admissibility condition (cf. [Sai96] or [GGK10], II.B);
- the fiber $\bar{\mathcal{J}}_0(\mathcal{X})$ inserted over the origin fits into an exact sequence

$$0 \to \mathcal{J}_0 \to \bar{\mathcal{J}}_0(\mathcal{X}) \to G \to 0, \tag{2.2}$$

  where G is a finite abelian group and $\mathcal{J}_0$ is a connected, complex Lie group, considered as the identity component of $\bar{\mathcal{J}}_0(\mathcal{X})$.

The total space $\bar{\mathcal{J}}(\mathcal{X})$ is called the Néron model associated to $\mathcal{X}$.

Remark 2.2. In fact, every ANF without *singularities* [GG07] extends to the identity component (cf. [GGK10], II. A).

Remark 2.3. Kato, Nakayama and Usui have an alternate approach constructing Néron models via a log mixed Hodge theory [KNU10], which is homeomorphic to the construction in [GGK10]. (cf. [Hay10]).

2.4. **Abel-Jacobi map.** Let $\mathrm{CH}^k(X)_{hom}$ be the subgroup of the Chow group of X consisting of codimension k algebraic cycles which are homologically equivalent to zero. There is an Abel-Jacobi map

$$\mathrm{AJ}_X : \mathrm{CH}^k(X)_{hom} \to J(X) \tag{2.3}$$

introduced by Griffiths [Gri68].

Returning to the semistable degeneration $\mathcal{X} \to \Delta$, given a codimension k algebraic cycle $\mathcal{Z} \subset \mathcal{X}$ with $Z_s = \mathcal{Z} \cdot \mathcal{X}_s \in \mathrm{CH}^k(\mathcal{X}_s)_{hom}$ for $s \neq 0$, there is an associated admissible normal function $\nu_{\mathcal{Z}}$ via the Abel-Jacobi map

$$\nu_{\mathcal{Z}}(s) = \mathrm{AJ}_{\mathcal{X}_s}(Z_s), \ s \in \Delta^* \tag{2.4}$$

(cf. [GGK10] III).

Furthermore, the associated function $\nu_{\mathcal{Z}}$ will extend to the identity component of $\bar{\mathcal{J}}_0(\mathcal{X})$ if $\mathcal{Z}$ is cohomological to zero in $\mathcal{X}$.

2.5. **The case of threefolds.** With the notation above, now we assume that $\mathcal{X} \to \Delta$ is a semistable degeneration of projective threefolds, and denote by $\bar{\mathcal{X}}$ a smooth projective variety containing $\mathcal{X}$ as an analytic open subset.

In this situation, we have a precise description of the group of components G via an intersection computation.

Theorem 2.6 ([GGK10], Thm. III. C.6)**.** *For any multi-index $I = (i_0, \ldots, i_m)$, $|I| = m + 1$, let*

$$\begin{aligned} Y_I &= \bigcap_{i \in I} X_i, \\ Y^{[m]} &= \coprod_{|I|=m+1} Y_I. \end{aligned} \tag{2.5}$$

Assuming that all the cohomology groups of $Y^{[m]}$ are torsion free, then the natural map $j : Y^{[0]} \to \bar{\mathcal{X}}$ induces a sequence of maps

$$H_4(Y^{[0]}, M) \xrightarrow{j_*^M} H_4(\bar{\mathcal{X}}, M) \cong H^4(\bar{\mathcal{X}}, M) \xrightarrow{j_M^*} H^4(Y^{[0]}, M) \cong H_2(Y^{[0]}, M)$$

where $M = \mathbb{Z}$ or $\mathbb{Q}$, and the composition gives the morphism

$$\mu_M : \bigoplus_{i=1}^{m} H_4(X_i, M) \to \bigoplus_{i=1}^{m} H_2(X_i, M).$$

Then there is an identification of the group G in (2.2),

$$G = \frac{(Im\ \mu_{\mathbb{Q}})_{\mathbb{Z}}}{Im\ \mu_{\mathbb{Z}}}. \tag{2.6}$$

Furthermore, the extension of the admissible normal function $\nu_{\mathcal{Z}}$ (2.4) *maps to the component corresponding to the class $[Z_0]$ in $\bigoplus_{i=1}^{m} H_2(X_i, \mathbb{Z})$.*

Remark 2.7. A similar result holds for the case of $\mathcal{X}$ being a family of curves. But when $\dim \mathcal{X}_s > 3$, the identification (2.6) may fail (cf. [GGK10]).

3. Construction of sections on K3-fibered Calabi-Yau threefolds

In this section, our aim is to show the existence of isolated sections on a very general bidegree $(2,4)$ hypersurface in $\mathbb{P}^1 \times \mathbb{P}^3$ with respect to the projection to $\mathbb{P}^1$. We begin with the construction of a hypersurface X_0 with at worst nodes as singularities containing infinitely many isolated sections.

Lemma 3.1. *There exists a hypersurface $X_0 \subset \mathbb{P}^1 \times \mathbb{P}^3$ of bidegree $(2,4)$ with finitely many nodes, such that X_0 admits an infinite collection of sections $\{\ell_n\}$ with respect to the projection $X_0 \to \mathbb{P}^1$. Moreover, each ℓ_n lies in the smooth locus of X_0 and is infinitesimally rigid.*

Proof. Let $S \to \mathbb{P}^1$ be a smooth rational elliptic surface, obtained by blowing up $\mathbb{P}^2$ along nine base points of a general pencil of cubic curves. This was first studied by Nagata [Nag61], who showed there are infinitely many exceptional curves of the first kind, and each of them yields a section ℓ_n of $S \to \mathbb{P}^1$.

We have a natural embedding $S \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^2$ and choose a smooth surface $H \subset \mathbb{P}^1 \times \mathbb{P}^2$ of bidegree $(1,1)$ meeting S transversally in $\mathbb{P}^1 \times \mathbb{P}^2$.

Let $x = (t_0, t_1; x_0, \ldots, x_3)$ be the coordinates of $\mathbb{P}^1 \times \mathbb{P}^3$. Consider $\mathbb{P}^1 \times \mathbb{P}^2$ as a hyperplane of $\mathbb{P}^1 \times \mathbb{P}^3$ defined by $x_3 = 0$. Let $|\mathfrak{L}|$ be the linear system of bidegree $(2,4)$ hypersurfaces in $\mathbb{P}^1 \times \mathbb{P}^3$ containing S and H. Then a general member in $|\mathfrak{L}|$ will be a singular hypersurface with finitely many nodes contained in $S \cap H$.

More explicitly, assume that S is defined by $q(x) = x_3 = 0$, while H is given by the equations $l(x) = x_3 = 0$ for some polynomial $q(x)$ of bidegree $(1,3)$ and $l(x)$ of bidegree $(1,1)$.

Then a hypersurface $X_0 \in |\mathfrak{L}|$ is given by an equation

$$l(x)q(x) + x_3 f(x) = 0 \tag{3.1}$$

for some bidegree $(2,3)$ polynomial $f(x)$. The singularities of X_0 are eighteen nodes defined by

$$l(x) = f(x) = x_3 = q(x) = 0 \tag{3.2}$$

for a generic choice of $f(x)$ by Bertini's theorem.

For each n, the space of X_0 containing a node on ℓ_n is only a finite union of hypersurfaces in $|\mathfrak{L}|$. Then we can ensure no node of X_0 lies on $\{\ell_n\}$ for a generic choice of $f(x)$ avoiding countably many hypesurfaces in $|\mathfrak{L}|$.

Furthermore, since $\ell_n^2 = -1$ in S, then $\mathcal{N}_{\ell_n/S} = \mathcal{O}_{\ell_n}(-1)$. Then the normal bundle exact sequence

$$0 \to \mathcal{O}_{\ell_n}(-1) \to \mathcal{N}_{\ell_n/X_0} \to \mathcal{O}_{\ell_n}(-1) \to 0 \tag{3.3}$$

implies that

$$\mathcal{N}_{\ell_n/X_0} \cong \mathcal{O}_{\ell_n}(-1) \oplus \mathcal{O}_{\ell_n}(-1).$$

This proves the infinitesimally rigidity. □

The result below follows from the above lemma and deformation theory.

Theorem 3.2. *For a general bidegree $(2,4)$ hypersurface X in $\mathbb{P}^1 \times \mathbb{P}^3$, there exist infinitely many sections $\{\ell_n\}$ on X with respect to the projection $X \to \mathbb{P}^1$ such that ℓ_n is infinitesimally rigid in X.*

Proof. From the above lemma, the rational curves ℓ_n in X_0 are stable under deformations by the Kodaira stability theorem [Kod63]. This implies that the relative Hilbert scheme parameterizing the pair $(\ell, X), \ell \subset X$, is smooth over the deformation space of X at (ℓ_n, X_0), and hence dominating. These sections ℓ_n deform to nearby neighborhoods of X_0. Although X_0 is singular, we can restrict everything to the smooth locus of X_0 to ensure the argument still applies.

Furthermore, the fibration $\pi : X_0 \to \mathbb{P}^1$ is given by the linear system $|\pi^*\mathcal{O}_{\mathbb{P}^1}(1)|$. Since $\pi^*\mathcal{O}_{\mathbb{P}^1}(1)$ has no higher cohomology, it deforms with X_0 and the dimension of $|\pi^*\mathcal{O}_{\mathbb{P}^1}(1)|$ is constant by semicontinuity. Thus the fibration will be preserved under deformation. Note that the deformation of ℓ_n meets the generic fiber of π at one point; it follows that the deformation of ℓ_n remains to be a section in a general deformation of X_0. □

Throughout this paper, by abuse of the notation, we continue to denote $\ell_n \subset X$ by the section obtained from the deformation of $\ell_n \subset X_0$.

Remark 3.3. Our method constructs infinitely many isolated rational curves of bidegree $(1, d)$ on a K3-fibered Calabi-Yau threefold in $\mathbb{P}^1 \times \mathbb{P}^N$. For example, the exceptional divisors of S give degree 0 sections on $X_0 \to \mathbb{P}^1$, which are of type $(1, 0)$. See [EJS99] for the existence of isolated rational curves of bidegree $(0, d)$ on K3-fibered Calabi-Yau threefolds in $\mathbb{P}^1 \times \mathbb{P}^N$ for every integer $d \geq 1$.

4. The degeneration of Calabi-Yau threefolds

In this section, we will study the degeneration of our Calabi-Yau threefolds and the deformation theory of sections on the degenerations.

4.1. **An important degeneration.** Remember that there are infinitely many sections $\{\ell_n\}$ constructed in section 3 on a very general bidegree $(2,4)$ hypersurface $X \subset \mathbb{P}^1 \times \mathbb{P}^3$. Our first result is as follows:

Lemma 4.2. *Let X be a very general bidegree $(2,4)$ hypersurface of $\mathbb{P}^1 \times \mathbb{P}^3$. There exists a projective family of bidegree $(2,4)$ hypersurfaces $\mathfrak{X} \to B$, containing X as a generic fiber, such that*

- *$\mathfrak{X}$ is smooth, and the general fiber of $\mathfrak{X} \to B$ is smooth;*

- *$\forall n_0 \in \mathbb{Z}$, there exists a point $b_{n_0} \in B$ such that the fiber $X_{n_0} := \mathfrak{X}_{b_{n_0}}$ is singular with only finitely many nodes and satisfies*
  - *(a) the specialization $\ell_{n_0} \subset X_{n_0}$ passes through exactly one node, while other specializations $\ell_n \subset X_{n_0}$ do not pass through any nodes for $n \neq n_0$;*
  - *(b) all $\ell_n \subset X_{n_0}$ are infinitesimally rigid.*

(The notation $\mathfrak{X}$ is different from $\mathcal{X}$ in section 2.)

Proof. Consider X as a deformation of the X_0 constructed in Lemma 3.1, where ℓ_n does not meet a singular locus of X_0. Let $\mathfrak{L}' \subset |\mathcal{L}|$ be the linear system of bidegree $(2,4)$ hypersurfaces containing S. The idea of the proof comes from an observation that the space of bidegree $(2,4)$ hypersurfaces satisfying condition (a) is a divisor in $\mathfrak{L}'$.

Indeed, we can give an explicit construction as in [Cle83c]. With the notation in Lemma 3.1, we first consider the one parameter family of bidegree (2,4) hypersurfaces defined by the equation

$$l_u(x)q(x) + x_3 f(x) = 0, \tag{4.1}$$

where $l_u(x) = u_0 l_0(x) + u_1 l_1(x), u \in \mathbb{P}^1$, defines a linear pencil of bidegree $(1,1)$ hypersurfaces.

Let C be the curve defined by

$$q(x) = f(x) = 0, \tag{4.2}$$

meeting ℓ_n transversally at distinct points for a generic choice of $f(x)$. Then one can choose $l_u(x)$ outside a countable union of hypersurfaces in the space of all pencils, such that the hyperplane $l_u(x) = 0$ meets S transversely and does not contain more than one point of the countable set

$$C \cap \left(\bigcup_n \ell_n \right).$$

Then the two parameter family

$$\mathfrak{X} = \{l_u(x)q(x) + x_3 f(x) + \lambda F(x) = 0\} \to \mathbb{P}^1 \times \Delta \tag{4.3}$$

will be the desired degeneration for generic $F(x)$. According to our construction, for each integer n_0, one can find a point $u_{n_0} \in \mathbb{P}^1$ such that $\mathfrak{X}_{(u_{n_0},0)}$ satisfies condition (a).

To complete the proof, it remains to show that all ℓ_n are infinitesimally rigid in X_{n_0}. When $n \neq n_0$, the rigidity of ℓ_n comes from the same argument in the proof of Lemma 3.1.

If $n = n_0$, we know that X_{n_0} only has eighteen nodes defined by (3.2) and the section ℓ_{n_0} will pass one of the nodes. Let X'_{n_0} be the blow up of X_{n_0} along H and X''_{n_0} the blow up of X_{n_0} along S, where H and S are the surfaces defined in Lemma 3.1.

One could see that X'_{n_0} and X''_{n_0} are the two small projective resolutions of X_{n_0}. It suffices to show that the strict transforms ℓ'_{n_0} and ℓ''_{n_0} of ℓ_{n_0} in X'_{n_0} and X''_{n_0}, respectively, are infinitesimally rigid.

Note that ℓ'_{n_0} is still contained in $S \subset X'_{n_0}$ as an exceptional curve, so one can conclude that

$$\mathcal{N}_{\ell'_{n_0}/X'_{n_0}} = \mathcal{O}_{\ell'_{n_0}}(-1) \oplus \mathcal{O}_{\ell'_{n_0}}(-1) \tag{4.4}$$

from the exact sequence (3.3).

Next, if one can find a special case of X_{n_0} such that

$$\mathcal{N}_{\ell''_{n_0}/X''_{n_0}} = \mathcal{O}_{\ell''_{n_0}}(-1) \oplus \mathcal{O}_{\ell''_{n_0}}(-1), \tag{4.5}$$

then semicontinuity will ensure that (4.5) holds for the generic case. The existence of such X_{n_0} is known by Lemma 9 in [Par91], which completes the proof. □

4.3. **Deforming the section through a node.** With the notation from the previous section, let $\pi : \mathcal{X} \to \Delta$ be the restriction $\mathfrak{X}|_{\{u_{n_0}\}\times\Delta}$, whose central fiber is $\pi^{-1}(0) = \mathcal{X}_0 = X_{n_0}$.

If $m \neq n_0$, we know that the section $\ell_m \subset X_{n_0}$ deforms to a section $\ell_m(s)$ of $\mathcal{X}_s$ and hence yields a codimension two cycle $\mathcal{L}_m \subset \mathcal{X}$ with

$$\mathcal{L}_m \cdot \mathcal{X}_s = \ell_m(s), \ s \in \Delta. \tag{4.6}$$

However, ℓ_{n_0} in X_{n_0} cannot deform with X_{n_0} in $\mathcal{X}$ since there is a non-trivial obstruction for first order deformations. This obstruction will vanish after a degree two base change. In this subsection, we will show that this is a sufficient condition to deform ℓ_{n_0} with X_{n_0}.

Remark 4.4. Actually, as we will see later, there exist exactly two sections $\ell'_{n_0,s}, \ell''_{n_0,s}$ on the nearby fiber $X_{n_0,s}$ of which both can be specialized to ℓ_{n_0} and the local monodromy action on $\ell'_{n_0,s}$ and $\ell''_{n_0,s}$ is transitive. One cannot deform ℓ_{n_0} with X_{n_0} unless we do a degree two base change.

The following result was inspired by [Cle83a].

Theorem 4.5. *The section $\ell_{n_0} \subset X_{n_0}$ can deform with X_{n_0} in $\mathcal{X}$ only after a degree two base change. In other words, we have the following diagram:*

$$\begin{tikzcd} \mathcal{L}_{n_0} \arrow[r, hook, "i"] \arrow[d] & \widetilde{\mathcal{X}} \arrow[r] \arrow[d, "\tilde{\pi}"'] & \mathcal{X} \arrow[d, "\pi"] \\ \widetilde{\Delta} \arrow[r, "Id"] & \widetilde{\Delta} \arrow[r, "d"] & \Delta \end{tikzcd} \tag{4.7}$$

where $d : \widetilde{\Delta} \to \Delta$ is the double covering map of the disc Δ ramified at the center $0 \in \Delta$, and $\mathcal{L}_{n_0} \cap \tilde{\pi}^{-1}(0) = \ell_{n_0}$.

Before proceeding to the proof, let us fix some notation as follows:

- X_{n_0} is defined by the equation $F_0(x) = 0$, without loss of generality, having a node $p_0 = (1, 0; 1, 0, 0, 0) \in \mathbb{P}^1 \times \mathbb{P}^3$;
- the section $\ell_{n_0} \subset X_{n_0}$ passing through p_0 is parametrized by the morphism

$$\begin{aligned} \phi : \mathbb{P}^1 &\longrightarrow X_{n_0}, \\ t = (t_0, t_1) &\mapsto (t_0, t_1; \phi_0(t), \dots, \phi_3(t)) \end{aligned} \tag{4.8}$$

  with $\phi(1, 0) = p_0$ for some degree d homogenous polynomials $\phi_i(t)$, $i = 0, \dots, 3$;

- the family $\mathcal{X} \to \Delta$ is given by the equation

$$F_0(x) + sF(x) = 0,\ s \in \Delta, \tag{4.9}$$

for some polynomial $F(x)$, with $F(p_0) \neq 0$.

With the notation above, we first give an explicit description of the global sections of the normal sheaf $\mathcal{N}_{\ell_{n_0}/X_{n_0}}$.

Lemma 4.6. *A global section of $\mathcal{N}_{\ell_{n_0}/X_{n_0}}$ can be represented by a set of homogenous polynomials*

$$\{(\sigma_i(t))_{i=0,1}, (\delta_j(t))_{j=0,\dots,3}\} \tag{4.10}$$

with deg$(\sigma_i) = 1$ and deg$(\delta_j) = d$, subject to the condition

$$\sum_{i=0}^{1} \sigma_i(t) \frac{\partial F_0}{\partial t_i}(\phi(t)) + \sum_{j=0}^{3} \delta_j(t) \frac{\partial F_0}{\partial x_j}(\phi(t)) = 0. \tag{4.11}$$

Moreover, (4.10) *is a trivial section of $\mathcal{N}_{\ell_{n_0}/X_{n_0}}$ if and only if it satisfies the condition*

$$\delta_j(t) = \sigma_0(t)\frac{\partial \phi_j}{\partial t_0} + \sigma_1(t)\frac{\partial \phi_j}{\partial t_1},\ j = 0, \dots, 3. \tag{4.12}$$

Proof. Let us denote the invertible sheaf $\pi_1^*\mathcal{O}_{\mathbb{P}^1}(a) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^3}(b)$ by $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^3}(a,b)$, where π_1 and π_2 are natural projections of $\mathbb{P}^1 \times \mathbb{P}^3$. Let $\mathcal{T}_X$ be the tangent sheaf of X. Due to the exact sequence

$$0 \to \mathcal{O}_{X_{n_0}}^{\oplus 2} \to \mathcal{O}_{X_{n_0}}^{\oplus 2}(1,0) \oplus \mathcal{O}_{X_{n_0}}^{\oplus 4}(0,1) \to \mathcal{T}_{\mathbb{P}^1\times\mathbb{P}^3}|_{X_{n_0}} \to 0 \tag{4.13}$$

and

$$0 \to \mathcal{T}_{X_{n_0}} \to \mathcal{T}_{\mathbb{P}^1\times\mathbb{P}^3}|_{X_{n_0}} \to \mathcal{O}_{X_{n_0}}(2,4) \to 0, \tag{4.14}$$

one can express a global section of $\mathcal{T}_{X_{n_0}}$ as a set of bidegree homogenous polynomials $\{(\sigma_i)_{i=0,1}; (\delta_j)_{j=0,\dots,3}\}$ satisfying

$$\sum_{i=0}^{1} \sigma_i \frac{\partial F_0}{\partial t_i} + \sum_{j=0}^{3} \delta_j \frac{\partial F_0}{\partial x_j} = 0, \tag{4.15}$$

where σ_i are of bidegree $(1,0)$, while δ_j are of bidegree $(0,1)$.

Then the statement follows from the following exact sequence:

$$\mathcal{T}_{\ell_{n_0}} \to \mathcal{T}_{X_{n_0}}|_{\ell_{n_0}} \to \mathcal{N}_{\ell_{n_0}/X_{n_0}} \longrightarrow 0, \tag{4.16}$$

where the induced map $g : H^0(\ell_{n_0}, \mathcal{T}_{\ell_{n_0}}) \to H^0(\ell_{n_0}, \mathcal{T}_{X_{n_0}}|_{\ell_{n_0}})$ can be expressed as

$$a_0\frac{\partial}{\partial t_0} + a_1\frac{\partial}{\partial t_1} \longmapsto (a_0, a_1; a_0\frac{\partial \phi_i}{\partial t_0} + a_1\frac{\partial \phi_i}{\partial t_1})_{i=0,\dots,3}. \tag{4.17}$$

□

Proof of Theorem 4.5. Let us make the base change $\tilde{\Delta} \to \Delta$ sending r to r^2, and write

$$\tilde{\mathcal{X}} := \{F_0(x) + r^2F(x) = 0,\ r \in \tilde{\Delta}\}. \tag{4.18}$$

To prove the assertion, it suffices to show the existence of a formal deformation $\Phi(r,t)$ in (4.18), i.e. there is a sequence of maps

$$\phi^{[k]}(t) = (t; \phi_0^{[k]}(t), \dots, \phi_3^{[k]}(t)) \in \mathbb{P}^1 \times \mathbb{P}^3,\ k \geq 0, \tag{4.19}$$

with $deg(\phi_i^{[k]}(t)) = d$ and $\phi^{[0]} = \phi$, such that the power series

$$\Phi(r,t) = (t_0, t_1; \sum_{k=0}^{\infty} r^k \phi_i^{[k]}(t))_{i=0,1\dots,3} \tag{4.20}$$

satisfies the condition

$$F_0(\Phi(r,t)) + r^2 F(\Phi(r,t)) = 0. \tag{4.21}$$

Our proof of the existence of $\Phi(r,t)$ proceeds as follows:

(I) *First order deformation.* The first order deformation of ϕ is determined by $\phi^{[1]}(t)$, which can be solved by differentiating (4.21) with respect to r and setting $r = 0$. Hence we obtain

$$\sum_{i=0}^{3} \frac{\partial F_0}{\partial x_i}(\phi(t))\phi_i^{[1]}(t) = 0. \tag{4.22}$$

Note that (4.22) is a homogenous polynomial of degree $4d+2$, which has $4d+3$ coefficients and the coefficient of the t_0^{4d+2} term is zero by assumption. Then one can consider (4.22) as $(4d+2)$ equations in $4(d+1)$ unknowns and denote $M_{(\phi,\ F_0)}$ by the $(4d+2)\times(4d+4)$ matrix corresponding to the system of these equations.

Our first claim is that the $M_{(\phi,\ F_0)}$ is of full rank, which is equivalent to saying that the dimension of the solution space of $\phi^{[1]}(t)$ is two.

By Lemma 4.6, the set

$$\{(t_0, t_1); \phi_0^{[1]}, \dots, \phi_i^{[1]})\} \tag{4.23}$$

gives a global section of $\mathcal{N}_{\ell_{n_0}/X_{n_0}}$ and is trivial if and only if

$$\phi_i^{[1]} = t_0 \frac{\partial \phi_i}{\partial t_0} + t_1 \frac{\partial \phi_i}{\partial t_1} \tag{4.24}$$

by (4.12). So if rank $M_{(\phi,\ F_0)} \le 4d+1$, then $\dim H^0(\ell_{n_0}, \mathcal{N}_{\ell_{n_0}/X_{n_0}}) \ge 2$.

Let $ev : H^0(\ell_{n_0}, \mathcal{N}_{\ell_{n_0}/X_{n_0}}) \to \mathbb{C}^3$ be the evaluation map at p_0. As in [Cle83a], §3, one can show that there is at most one condition lifting the analytic section of $\mathcal{N}_{\ell_{n_0}/X_{n_0}}$ to a section of $\mathcal{N}_{\ell'_{n_0}/X'_{n_0}}$, because the image of ev is at most two dimensional, while the composition of the sequence of evaluation maps at p_0,

$$\mathcal{N}_{\ell'_{n_0}/X'_{n_0}} \to \mathcal{N}_{\ell_{n_0}/X_{n_0}} \to \mathbb{C}^3, \tag{4.25}$$

only has a one dimensional image.

However, from (4.4), we know that $H^0(\ell'_{n_0}, \mathcal{N}_{\ell'_{n_0}/X'_{n_0}}) = 0$. Thus we prove the first claim by contradiction.

(II) *Higher order.* We continue to solve $\phi^{[2]}(t)$ by differentiating (4.21) twice, and thus obtain

$$\sum_{i=0}^{3} \frac{\partial F_0}{\partial x_i}(\phi(t))\phi_i^{[2]}(t) = -\sum_{i,\ j} \frac{\partial F_0}{\partial x_i \partial x_j}(\phi(t))\phi_i^{[1]}(t)\phi_j^{[1]}(t) + 2F(\phi(t)). \tag{4.26}$$

Obviously, there is a non-trivial obstruction to lift $\phi^{[1]}(t)$ to second order given by the equation

$$\sum_{i,\ j} \frac{\partial F_0}{\partial x_i \partial x_j}(p_0)\phi_i^{[1]}(1,0)\phi_j^{[1]}(1,0) = -2F(p_0) \neq 0. \tag{4.27}$$

Any $\phi^{[1]}(t)$ satisfying (4.27) can be lifted to the second order, because M_{ϕ,F_0} is of full rank.

Our second claim is that there exists a first order deformation which can be lifted to second order. Otherwise, every $\phi^{[1]}(t)$ will satisfy the condition

$$\sum_{i,\ j} \frac{\partial F_0}{\partial x_i \partial x_j}(p_0)\phi_i^{[1]}(1,0)\phi_j^{[1]}(1,0) = 0. \tag{4.28}$$

From the above assumption, there is a non-trivial analytic section

$$\alpha \in H^0(\ell_{n_0}, \mathcal{N}_{\ell_{n_0}/X_{n_0}}),$$

whose image via the evaluation map at p_0 lies in the tangent cone C_{p_0} of X_{n_0} at p_0 and is normal to the tangent direction of ℓ_{n_0}.

As in [Cle83a], §3, this means that $ev(\alpha)$ lies in the union of the images of $\mathcal{N}_{\ell'_{n_0}/X'_{n_0}}$ and $\mathcal{N}_{\ell''_{n_0}/X''_{n_0}}$ in $\mathcal{N}_{\ell_{n_0}/X_{n_0}}$. This is a contradiction, because none of the non-trivial sections of $\mathcal{N}_{\ell_{n_0}/X_{n_0}}$ can lift by Lemma 4.6. Hence there exists $\bar{\phi}^{[1]}(t)$ satisfying (4.27), and the second claim is proved.

Furthermore, set $b_i = \bar{\phi}_i^{[1]}(1,0)$. It is not difficult to see that (4.22) along with

$$\sum_{i,\ j} \frac{\partial F_0}{\partial x_i \partial x_j}(p_0)\phi_i^{[1]}(1,0)b_j = 0 \tag{4.29}$$

only has a one dimensional set of solutions. Hence the associated $(4d+3)\times(4d+4)$ matrix $M'_{\bar{\phi}^{[1]},F_0}$ is of full rank.

For higher orders, $\phi^{[n]}(t)$ is determined by the equation

$$\sum_{i=0}^{3} \frac{\partial F_0}{\partial x_i}(\phi(t))\phi_i^{[n]}(t) = \text{some polynomial given by } \phi^{[k]} \text{ for } k < n, \tag{4.30}$$

while the obstruction to the (n+1)-th order is

$$\sum_{i,\ j} \frac{\partial F_0}{\partial x_i \partial x_j}(p_0)\phi_i^{[n]}(1,0)b_j = \text{some number given by } \phi^{[k]} \text{ for } k < n. \tag{4.31}$$

Then one can solve $\phi^{[n]}(t)$ by induction because of the full rank of $M'_{\bar{\phi}^{[1]},F_0}$. □

4.7. **Semistable degeneration.** In this subsection, we will desingularize the family $\widetilde{\mathcal{X}}$ to obtain a semistable degeneration, and identify the group of components associated to this semistable degeneration.

Let $\mathcal{W}$ be the blow up of $\tilde{\mathcal{X}}$ along all the nodes on $\tilde{\mathcal{X}}_0$. Then we have

- the ambient space $\mathcal{W}$ is smooth, and the generic fiber of $\mathcal{W} \to \tilde{\Delta}$ is smooth;
- the central fiber $\mathcal{W}_0 = \bigcup_{i=0}^{18} W_i$ is strictly normal crossing, where
  (1) W_0 is the blow up of $\tilde{\mathcal{X}}_0$ along all the nodes;
  (2) W_i are disjoint smooth quadratic threefolds in $\mathbb{P}^4$, meeting W_0 transversally at the exceptional divisor $E_i \cong \mathbb{P}^1 \times \mathbb{P}^1$ for $i = 1, \ldots, 18$.

As an application of Theorem 2.6, we shall apply (2.1) to compute the group of components of the Néron model $\bar{J}(\mathcal{W})$ associated to the semistable degeneration $\mathcal{W} \to \tilde{\Delta}$.

In order to give a geometric description of the homology groups of each component of $\mathcal{W}_0$, we first set the following notation:

- P is the strict transform of the bidegree $(1,0)$ hyperplane section of X_{n_0} in W_0, and D is a generic fiber of W_0 over $\mathbb{P}^1$; $\tilde{H}$ is the strict transform of H in W_0;
- E_i is an exceptional divisor of $W_0, i = 1, 2, \ldots, 18$.
- Q_i is the hyperplane section of W_i, $i = 1, 2, \ldots, 18$.
- L is a line on the fiber of W_0 over $\mathbb{P}^1$, and L' is a section of W_0 with respect to the projection; C' is the proper transform of the curve (4.2) in W_0; R_i is one of the rulings of E_i.
- L_i is the line in W_i.

Then the integral basis for these homology groups can be represented by the fundamental class of the algebraic cycles above:

(4.32)

$$H_2(W_0) = \langle [L], [L'], [C'], [R_1], \ldots, [R_{18}] \rangle, \quad H_4(W_0) = \langle [D], [K], [\tilde{P}], [E_1], \ldots, [E_{18}] \rangle;$$
$$H_2(W_i) = \langle [L_i] \rangle, \ i = 1, 2, \ldots, 18; \quad H_4(W_i) = \langle [Q_i] \rangle, \ i = 1, 2, \ldots, 18.$$

By a straightforward computation, we can express the map

$$\mu_{\mathbb{Z}} : \bigoplus_{i=0}^{18} H_4(W_i, \mathbb{Z}) \longrightarrow \bigoplus_{i=0}^{18} H_2(W_i, \mathbb{Z})$$

as a matrix:

	$[P]$	$[D]$	$[\tilde{H}]$	$[E_i]$	$[Q_j]$
$[L]$	1	0	1	0	0
$[L']$	0	1	1	0	0
$[C']$	0	0	18	1	0
$[R_k]$	0	0	1	$2\delta_{ik}$	$2\delta_{jk}$
$[L_l]$	0	0	1	$2\delta_{il}$	$2\delta_{jl}$

Thus the group is computed as

$$G = \frac{\operatorname{Im}(\mu_{\mathbb{Q}})_{\mathbb{Z}}}{\operatorname{Im}(\mu_{\mathbb{Z}})} = \frac{\bigoplus_{k=1}^{18} \mathbb{Z}([R_k] + [L_k])}{\bigoplus_{k=1}^{18} \mathbb{Z}(2[R_k] + 2[L_k]) \oplus \mathbb{Z} \sum_{i=1}^{18} ([R_i] + [L_i])}. \tag{4.33}$$

Furthermore, let $\check{\mathcal{L}}_{n_0}$ denote the strict transform of $\mathcal{L}_{n_0}$ in $\mathcal{W}$. The following lemma is straightforward (cf. [Cle83b]) and will be used in the next section.

Lemma 4.8. *Let E_{i_0} be the exceptional divisor in W_0 corresponding to the node which ℓ_{n_0} passes through, and $\check{\ell}_{n_0}$ the strict transform of ℓ_{n_0} in W_0. Then*

$$\check{\mathcal{L}}_{n_0} \cap W_0 = \check{\ell}_{n_0} + (\textit{one of the rulings of } E_{i_0}). \tag{4.34}$$

5. Infinite generation of the subgroup of the Griffiths group

In this section, we study the Griffiths group of our Calabi-Yau threefolds and prove Theorem 2.

5.1. **The Griffiths group of Calabi-Yau threefolds.** Let X be a smooth projective threefold and $\mathrm{CH}^2(X)_{alg}$ be the subgroup of $\mathrm{CH}^2(X)$ consisting of codimension 2 cycles which are algebraically equivalent to zero. The Abel-Jacobi image

$$\mathrm{AJ}_X(\mathrm{CH}^2(X)_{alg}) = A \subseteq J(X) \tag{5.1}$$

is an abelian variety. The abelian variety A is called the algebraic part of $J(X)$. It lies in the largest complex subtorus $J(X)_{alg} \subset J(X)$, whose tangent space at 0 is contained in $H^{1,2}(X)$ (cf. [Voi07], vI).

The Griffiths group $\mathrm{Griff}^2(X)$ is the quotient $\mathrm{CH}^2(X)_{hom}/\mathrm{CH}^2(X)_{alg}$, which is a natural subgroup of $\mathcal{A}^2(X)$. In the case of Calabi-Yau threefolds, the following result is mentioned in [Voi92] and [Voi00].

Theorem 5.2. *If X is a non-rigid Calabi-Yau threefold, i.e. $h^1(\mathcal{T}_X) \neq 0$, then $J(X_s)_{alg} = 0$ for a general deformation X_s of X. In particular, AJ_{X_s} factors*

$$\mathrm{AJ}_{X_s} : \mathrm{Griff}^2(X_s) \to J(X_s).$$

Now we return to the case of X a generic bidegree $(2,4)$ hypersurface of $\mathbb{P}^1 \times \mathbb{P}^3$, and hence $\mathrm{AJ}_X : \mathrm{Griff}^2(X) \to J(X)$ is well defined. Recalling that the group $\mathcal{A}^{sec} \subset \mathcal{A}^2(X)$ is generated by $\{\ell_n\}$ of different degree, we consider the non-trivial group $\mathcal{A}^{sec} \cap \mathrm{Griff}^2(X)$.

Since the rank of $H_2(X,\mathbb{Z})$ is two by the Lefschetz hyperplane theorem, there exist integers a, a_n, b_n, such that

$$\psi_n := a\ell_n - a_n\ell_0 - b_n\ell_1 \equiv_{hom} 0, \ \forall n \in \mathbb{Z}. \tag{5.2}$$

As in Remark 3.3, the fundamental class of ℓ_n is of type $(1, d_n)$ in $H_2(X,\mathbb{Z})$ and we can assume ℓ_0 is of type $(1,0)$. After a suitable choice of ℓ_1, we can select a to be odd.

In fact, let us denote κ by the largest number such that d_n is divisible by 2^κ for all $n \in \mathbb{Z}$, and denote ℓ_1 by the section of type $(1, d_1)$ satisfying that $d_1/2^\kappa$ is odd; one can choose $a = d_1/2^\kappa$ as desired.

To prove Theorem 2, it suffices to show that $\mathcal{A}^{sec} \cap \mathcal{A}^2(X)$ contains a subgroup which is not finitely generated. Denote $\mathcal{G}$ by the subgroup of $\mathcal{A}^{sec} \cap \mathcal{A}^2(X)$ generated by ψ_n. We will show that $\mathcal{G}$ is the desired subgroup in the next subsection.

5.3. **Infinite generation of $\mathcal{G}$.** In this subsection, we shall prove the following result, which implies Theorem 2.

Theorem 5.4. *The Abel-Jacobi image $\mathrm{AJ}_X(\mathcal{G}) \otimes \mathbb{Q}$ is of infinite rank for generic X.*

Proof. Suppose that there is a relation

$$\sum_{\text{finite}} c_n \mathrm{AJ}_X(\psi_n) = 0 \tag{5.3}$$

for generic X, which gives $\mathrm{AJ}_X(\sum_{\text{finite}} ac_n\ell_n) = 0$. In particular, we assume that (5.3) holds for the generic fiber of the two parameter family $\mathfrak{X} \to \mathbb{P}^1 \times \Delta$ (4.3).

From the construction in Lemma 4.2, for each integer n, there is a point $(u_n, 0) \in \mathbb{P}^1 \times \Delta$ such that the fiber $X_n = \mathfrak{X}_{(u_n,0)}$ over $(u_n, 0)$ satisfies conditions (a) and (b). Let $\mathcal{X} \to \Delta$ be the restriction $\mathfrak{X}|_{\{u_n\}\times\Delta}$ and $\tilde{\mathcal{X}} = \mathcal{X} \times_\Delta \tilde{\Delta}$ for a degree two base change $\tilde{\Delta} \xrightarrow{r\mapsto r^2} \Delta$. Then the family of cycles

$$\mathcal{Z} = ac_n\mathcal{L}_n + \sum_{m\neq n} ac_m\tilde{\mathcal{L}}_m \tag{5.4}$$

satisfies $\mathrm{AJ}_{\tilde{\mathcal{X}}_r}(\mathcal{Z}\cdot\tilde{\mathcal{X}}_r) = 0$, where $\mathcal{L}_n$ is given by (4.7) in Theorem 4.5 and $\tilde{\mathcal{L}}_m$ is the lift of (4.6) in $\tilde{\mathcal{X}}$ for $m \neq n$.

To make use of the Néron model, we blow up $\tilde{\mathcal{X}}$ along all the nodes to get the semistable degeneration $\mathcal{W} \to \tilde{\Delta}$ as in section 4.6. There is an associated Néron model $\bar{J}(\mathcal{W})$. Denote by $\check{\mathcal{Z}}$ the lifting of $\mathcal{Z}$ in $\mathcal{W}$; then the associated admissible normal function $\nu_{\check{\mathcal{Z}}}$ is a zero holomorphic section and naturally extends to the identity component of $\bar{J}(\mathcal{W})$.

On the other hand, write $\check{Z}_0 := \check{\mathcal{Z}}\cdot\mathcal{W}_0$; then $\nu_{\check{\mathcal{Z}}}$ extends to the component corresponding to the class of $[\check{Z}_0]$ in $\bigoplus_{i=0}^{18} H_2(W_i,\mathbb{Z})$ by Theorem 2.6. According to Lemma 4.8, we have

$$[\check{Z}_0] = ac_n([R_n]+[L_n]) + \text{linear combinations of } [L],[L'] \text{ and } [C'], \tag{5.5}$$

which corresponds to $ac_n([R_n]+[L_n])$ in G. Then as indicated in section 4.6, $\nu_{\check{\mathcal{Z}}}$ extends to the identity component if and only if c_n is even, because a is odd.

Repeating the above process for each integer n, one proves that all the coefficients in (5.3) are even. Thus, the elements $\{\mathrm{AJ}_X(\psi_n)\}$ are linearly independent modulo two, which implies that $\mathrm{AJ}_X(\mathcal{G})\otimes\mathbb{Z}_2$ has infinite rank. Then the assertion follows from the lemma below.

Lemma 5.5. *Let M be an abelian group with*

$$M_{torsion} \subseteq (\mathbb{Q}/\mathbb{Z})^r.$$

Then $rank_{\mathbb{Z}_2}(M\otimes\mathbb{Z}_2) \leq rank_{\mathbb{Q}}(M\otimes\mathbb{Q}) + r$. □

Remark 5.6. Note that the monodromy of the degeneration $\mathcal{W} \to \tilde{\Delta}$ satisfies $(T-I)^2 = 0$. One can take an alternate approach by using Clemens' Néron model $\bar{J}^{\text{Clemens}}(\mathcal{W})$ [Cle83b] to extend the associated normal function. Actually, the identity component of $\bar{J}(\mathcal{W})$ is a subspace of Clemens's Néron model (see also [Zuc76]).

Remark 5.7. Let $\iota : \mathcal{W} \to \mathcal{W}$ be the natural involution. The admissible normal function $\nu_{\mathcal{Z}'}$ associated to the family of algebraic cycles $\mathcal{Z}' = \check{\mathcal{L}}_n - \iota(\check{\mathcal{L}}_n)$ extends to the identity component [GGK10]. One can also prove the infinite generation of $\mathcal{G}$ by showing that $\nu_{\mathcal{Z}'}(0)$ is a non-trivial element in the identity component. The proof is similar to our computation of the group of components (cf. [Bar84]).

6. Proof of Theorem 1

Proof of Theorem 1. Assume to the contrary that the union of the sections $\{\ell_n\}$ is not Zariski dense in X. Let Σ be the Zariski closure of the union of these curves, and $\tilde{\Sigma}$ the desingularization of Σ. Then the proper morphism

$$\varphi : \tilde{\Sigma} \to X$$

induces a homomorphism

$$\varphi_* : \mathcal{A}^1(\tilde{\Sigma}) \to \mathcal{A}^2(X),$$

where $\mathcal{A}^1(\tilde{\Sigma}) \cong NS(\tilde{\Sigma})$ is the Néron-Severi group of $\tilde{\Sigma}$. The homomorphism φ_* maps the algebraic cycle ℓ_n in $\tilde{\Sigma}$ to the corresponding 1-cycle in X. So the group $\mathcal{A}^{sec}$ is contained in the image of $NS(\tilde{\Sigma})$ via φ_*.

It is well known that $NS(\tilde{\Sigma})$ is a finitely generated abelian group by the Néron-Severi theorem, which contradicts Theorem 2. This completes the proof. $\square$

Remark 6.1. Our results can be naturally generalized to other Calabi-Yau threefolds fibered by complete intersection K3 surfaces in $\mathbb{P}^n$. For instance, one can find an analogous statement for some Calabi-Yau threefolds in $\mathbb{P}^1 \times \mathbb{P}^4$ fibered by the complete intersection of a quadratic and a cubic in $\mathbb{P}^4$.

Remark 6.2. Our method does not yield examples in $\bar{\mathbb{Q}}(t)$, since all type $(2,4)$ hypersurfaces X over $\bar{\mathbb{Q}}$ might lie in a countable union of "bad" hypersurfaces of the parameterization space.

Furthermore, one can also see [Beĭ87], [Blo84] for conjectures of $\mathrm{CH}_0(Y)_{hom}$ when Y is a surface over a number field or a function field of a curve defined over a finite field.

7. Higher dimensional Calabi-Yau hypersufaces in $\mathbb{P}^1 \times \mathbb{P}^N$

In this section, we consider the case of bidegree $(2, N+1)$ hypersurfaces in $\mathbb{P}^1 \times \mathbb{P}^N$ for $N \geq 3$. The following theorem is obtained via a similar argument as in Lemma 3.1.

Theorem 7.1. *For a very general hypersurface $X^N \subset \mathbb{P}^1 \times \mathbb{P}^N$ of bidegree $(2, N+1)$, there exists an infinite sequence of sections $\{\ell_k\}$ on X^N of different degrees with respect to the projection $X^N \to \mathbb{P}^1$, such that*

$$\mathcal{N}_{\ell_k/X^N} = \mathcal{O}_{\ell_k}(-1) \oplus \mathcal{O}_{\ell_k}(-1) \oplus \overbrace{\mathcal{O}_{\ell_k} \oplus \ldots \oplus \mathcal{O}_{\ell_k}}^{N-3}. \tag{7.1}$$

Furthermore, the subgroup $\mathcal{G}^N \subset \mathcal{A}^2(X^N)$ generated by the algebraic codimesion 2-cycles ψ_k^N, which are swept out by the deformations of ψ_k, is not finitely generated.

Proof. The proof will proceed by induction on N. Suppose our statement holds for $N = m \geq 3$. When $N = m+1$, it suffices to produce a singular one with the desired properties. The construction is as follows,

Let us denote the coordinate of $\mathbb{P}^1 \times \mathbb{P}^{m+1}$ by $x = (s, t; x_0, \ldots, x_{m+1})$. Then we consider the bidegree $(2, m+2)$ hypersurface X_0^{m+1} defined by the equation

$$x_0 g(x) + x_n h(x) = 0 \tag{7.2}$$

for some bidegree $(2, m+1)$ polynomials g, h.

Now, we choose our $g(x), h(x)$ satisfying the following conditions:

(1) X_0^{m+1} is only singular at

$$x_0 = x_{m+1} = g(x) = h(x) = 0; \tag{7.3}$$

(2) the subvariety

$$X^m := \{x_{m+1} = g(x) = 0\} \tag{7.4}$$

satisfies the inductive assumption. Denote by ℓ_k the corresponding sections on X^m;

(3) all the sections ℓ_k lie outside the singular locus (7.3).

Similar to the proof of Lemma 3.1, condition (1) will be satisfied due to Bertini's theorem and condition (3) can be achieved for a generic choice of $g(x)$ outside countably many hypersurfaces of the parametrization space of bidegree $(2, m)$ polynomials.

Next, we compute the normal bundle $\mathcal{N}_{\ell_k/X_0^{m+1}}$ from the following exact sequence:

$$0 \to \mathcal{N}_{\ell_k/X^m} \to \mathcal{N}_{\ell_k/X_0^{m+1}} \to \mathcal{N}_{X^m/X_0^{m+1}}|_{\ell_k} \to 0. \tag{7.5}$$

By assumption, we have $\mathcal{N}_{\ell_k/X^m} = \mathcal{O}_{\ell_k}(-1)^{\oplus 2} \oplus \mathcal{O}_{\ell_k}^{\oplus m-3}$. Since

$$\mathcal{N}_{X^m/X_0^{m+1}}|_{\ell_k} = \mathcal{O}_{\ell_k},$$

it follows that

$$\mathcal{N}_{\ell_k/X_0^{m+1}} = \mathcal{O}_{\ell_k}(-1)^{\oplus 2} \oplus \mathcal{O}_{\ell_k}^{\oplus m-2}. \tag{7.6}$$

Moreover, let $\mathcal{G}^{m+1} \subset \mathcal{A}^2(X_0^{m+1})$ be the subgroup generated by deformations of ψ_k in X_0^{m+1}. There is a morphism

$$i^* : \mathcal{A}^2(X_0^{m+1}) \to \mathcal{A}^2(X^m) \tag{7.7}$$

induced by the inclusion $X^m \xrightarrow{i} X_0^{m+1}$. Then the infinite generation of $\mathcal{G}^{m+1}$ follows from the inductive assumption on X^m. This completes the proof. $\square$

As in section 6, Corollary 1 is deduced from the infinite generation of $\mathcal{G}^N$.

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