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SEQUENTIAL FORMULAE FOR THE NORMAL CONE TO SUBLEVEL SETS

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ABSTRACT. Let X be a reflexive Banach space and let Φ be an extended real-valued lower semicontinuous convex function on X. Given a real λ and the sublevel set $S = [\Phi \le \lambda]$, we establish at $\overline{x} \in S$ the following formula for the normal cone to S:

$$(\star) \qquad N_S(\overline{x}) = \limsup_{x \to \overline{x}} \mathbb{R}_+ \, \partial\Phi(x) \quad \text{if} \quad \Phi(\overline{x}) = \lambda,$$

without any qualification condition. The case $\Phi(\overline{x}) < \lambda$ is also studied. Here $\mathbb{R}_+ := [0, +\infty[$ and $\partial\Phi$ stands for the subdifferential of Φ in the sense of convex analysis. The proof is based on the sequential convex subdifferential calculus developed previously by the second author. Formula $(\star)$ is extended to nonreflexive Banach spaces via the use of nets. The normal cone to the intersection of finitely many sublevel sets is also examined, thus leading to new formulae without a qualification condition. Our study goes beyond the convex framework: when $\dim X < +\infty$, we show that the inclusion of the left member of $(\star)$ into the right one still holds true for a locally Lipschitz continuous function. Finally, an application of formula $(\star)$ is given to the study of the asymptotic behavior of some gradient dynamical system.

1. Introduction

In classical analysis, it is well known that the gradient of a function $\Phi \in \mathcal{C}^1(\mathbb{R}^N, \mathbb{R})$ at a noncritical[1] point $\overline{x} \in \mathbb{R}^N$ is orthogonal to the level surface of Φ through $\overline{x}$. An analogous result for normals to convex sublevel sets can be stated in terms of subgradients. More precisely, given a real $\lambda \in \mathbb{R}$ and a continuous convex function $\Phi : X \to \mathbb{R}$ defined on a normed space X, let us define the sublevel set $S := [\Phi \le \lambda]$. Assuming the Slater condition

$$(1.1) \qquad [\Phi < \lambda] \ne \emptyset,$$

the normal cone $N_S(\overline{x})$ to the set S at $\overline{x} \in S$ is given by

$$(1.2) \qquad N_S(\overline{x}) = \mathbb{R}_+ \, \partial\Phi(\overline{x}) \quad \text{if} \quad \Phi(\overline{x}) = \lambda;$$

see for example [4, Proposition 9.6.1] or [8, Lemma V.6]. The subdifferential $\partial\Phi(\overline{x})$ and the normal cone $N_S(\overline{x})$ in the sense of convex analysis are recalled at the beginning of the next section. For an extended real-valued convex function $\Phi : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ such that $\partial\Phi(\overline{x}) \ne \emptyset$, Theorem 23.7 of [31] shows that

$$(1.3) \qquad N_S(\overline{x}) = \mathrm{cl}\,(\mathbb{R}_+ \, \partial\Phi(\overline{x})) \quad \text{if} \quad \Phi(\overline{x}) = \lambda,$$

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[1]The terminology of noncritical point means that $\nabla\Phi(\overline{x}) \ne 0$.

provided that the Slater condition (1.1) is satisfied. Other formulae for $N_S(\overline{x})$ are established in the framework of nonsmooth analysis; see for instance [20, 28, 32]. By particularizing Proposition 10.3 of [32] to the convex case, we obtain that

$$N_S(\overline{x}) = \mathbb{R}_+ \, \partial\Phi(\overline{x}) \cup N_{\operatorname{dom}\Phi}(\overline{x}) \quad \text{if} \quad \Phi(\overline{x}) = \lambda, \tag{1.4}$$

for a lower semicontinuous convex function $\Phi : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ satisfying the Slater condition (1.1). Observe that the assumption $\partial\Phi(\overline{x}) \neq \emptyset$ has been dropped in formula (1.4). However, the common point of formulae (1.2)-(1.4) is that they may fail to be true if the qualification condition (1.1) is not satisfied.

The first aim of the present paper is to establish, without any qualification condition, a general description of $N_S(\overline{x})$ in terms of the subdifferential of Φ. For an extended real-valued lower semicontinuous convex function $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ defined on a reflexive Banach space X, we show the following formula:

$$N_S(\overline{x}) = \begin{cases} \limsup\limits_{x \to \overline{x}} \mathbb{R}_+ \, \partial\Phi(x) & \text{if} \quad \Phi(\overline{x}) = \lambda, \\ \limsup\limits_{x \to \overline{x},\, \mu \downarrow 0} \mu \, \partial\Phi(x) = N_{\operatorname{dom}\Phi}(\overline{x}) & \text{if} \quad \Phi(\overline{x}) < \lambda. \end{cases} \tag{1.5}$$

The technique of the proof is based on sequential convex subdifferential calculus; see [33–35]. These papers were originally motivated by the seminal contributions [3, 25]. For other references on sequential convex subdifferential calculus, one may consult [24, 26, 30]. The approach is strong enough to allow us to consider the situation where S is the intersection of finitely many sublevel sets and to study the setting of general nonreflexive Banach spaces. The extension to a nonconvex framework is also examined. It is shown for locally Lipschitz functions that the Mordukhovich limiting normal cone to S at $\overline{x}$ is still included in some upper limit expressed with subgradients of Φ.

In the last part of the paper, we apply formula (1.5) to the investigation of the asymptotic behavior of the following dynamical system:

$$\ddot{x}(t) + \gamma(t)\,\dot{x}(t) + \partial\Phi(x(t)) \ni 0, \qquad t \geq 0, \tag{E}$$

where $\gamma \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$ and $\Phi : \mathbb{R}^N \to \mathbb{R}$ is a convex function with a nonempty set S of minima. The case of a constant coefficient $\gamma > 0$ gives rise to the so-called "Heavy Ball with Friction" system. The asymptotic properties of the HBF equation were intensively studied; see for example [1, 6] in the case of a smooth potential and [5, 19] in a nonsmooth framework. The case of a vanishing damping parameter $\gamma(t) \to 0$ has been recently investigated in [15, 16] for a function Φ of class $\mathcal{C}^1$. See also [18] for a related study in the framework of semilinear hyperbolic equations. If the quantity $\gamma(t)$ tends to 0 too rapidly as $t \to +\infty$, we show that nonstationary solutions of (E) cannot converge toward a minimum point $\overline{x} \in S$ satisfying the obtuseness condition $-N_S(\overline{x}) \subset \operatorname{int}(T_S(\overline{x})) \cup \{0\}$.

The paper is organized as follows. Section 2 is concerned with preliminary results, and among them, sequential convex subdifferential calculus plays a major role. Section 3 is the core of the paper: in a reflexive Banach space, the normal cone $N_S(\overline{x})$ to the convex sublevel set S is expressed as an upper limit involving the subdifferential of Φ at nearby points, as written above. An extension of this formula in nonreflexive Banach spaces is considered in Section 4. The case of a finite intersection of convex sublevel sets is examined in Section 5. The extension to a nonconvex framework is studied in Section 6. Finally, in Section 7, the sequential formula for the normal cone is applied to the asymptotic study of the system (E).

2. Generalities and preliminary results

In this preliminary section, we gather several results that will be useful in the sequel. Given a normed space X, let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a convex function. The effective domain of f is defined by $\operatorname{dom} f = \{x \in X,\, f(x) < +\infty\}$. The subdifferential of f at $x \in \operatorname{dom} f$ is the set $\partial f(x)$ defined by

$$\partial f(x) = \{\xi^* \in X^*,\, f(y) \geq f(x) + \langle \xi^*, y - x\rangle \text{ for every } y \in X\}.$$

Given a set $S \subset X$, we denote by δ_S the indicator function of S, i.e., $\delta_S(x) = 0$ if $x \in S$ and $\delta_S(x) = +\infty$ if $x \notin S$. The normal cone of S at $x \in S$ is defined by

$$N_S(x) = \partial\delta_S(x) = \{\xi^* \in X^*,\, \langle \xi^*, y - x\rangle \leq 0 \text{ for every } y \in S\}.$$

We now recall fundamental results with respect to sequential convex subdifferential calculus. The next one concerns the subdifferential of a sum of convex functions; see [33, Theorem 3], [34, Theorem 2.1] and [35, Theorem 1].

Theorem 2.1. *Let X be a reflexive Banach space (resp. a general Banach space) and let $f_1, f_2 : X \to \mathbb{R}\cup\{+\infty\}$ be two proper lower semicontinuous convex functions. Then, for any $\overline{x} \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$, one has $x^* \in \partial(f_1 + f_2)(\overline{x})$ if and only if there are sequences (resp. nets) $(x_{i,n})_n$ in X and $(x^*_{i,n})_n$ in X^*, for $i = 1, 2$, with $x^*_{i,n} \in \partial f_i(x_{i,n})$ such that*

$$x^* = \lim_{\|\,\|}(x^*_{1,n} + x^*_{2,n}) \quad (\textit{resp. } x^* = \lim_{w^*}(x^*_{1,n} + x^*_{2,n}))$$

and such that one of the following properties holds:

(*a*) $(x_{i,n}, f_i(x_{i,n})) \to (\overline{x}, f_i(\overline{x}))$, $\langle x^*_{i,n}, x_{i,n} - \overline{x}\rangle \to 0$ *and* $(\|x^*_{1,n}\| + \|x^*_{2,n}\|)(\|x_{1,n} - x_{2,n}\|) \to 0$;

(*b*) $x_{i,n} \to \overline{x}$ *and* $f_i(x_{i,n}) - \langle x^*_{i,n}, x_{i,n} - \overline{x}\rangle \to f_i(\overline{x})$.

Before giving the corresponding statement for the composition of convex functions, let us recall the notion of a vector-valued convex mapping. Let Y be a Banach space and Y_+ a convex cone of Y inducing a preorder $\preccurlyeq$ on Y, defined by $y_1 \preccurlyeq y_2$ if and only if $y_2 - y_1 \in Y_+$. Let $+\infty$ be an abstract maximal element adjoined to Y. A mapping $F : X \to Y \cup \{+\infty\}$ is said to be convex if for all x, $x' \in X$, and $t \in]0, 1[$, one has

$$F(tx + (1-t)x') \preccurlyeq t\,F(x) + (1-t)\,F(x').$$

The set $\operatorname{dom} F = \{x \in X,\, F(x) \in Y\}$ is the effective domain of F, $\operatorname{Im} F = F(X)$ is the effective image of F and $\operatorname{epi} F = \{(x, y) \in X \times Y,\, F(x) \preccurlyeq y\}$ is the epigraph of F. A function $f : Y \to \mathbb{R} \cup \{+\infty\}$ is Y_+ -nondecreasing on a subset Σ of Y if $f(y_1) \leq f(y_2)$ for all y_1, $y_2 \in \Sigma$ satisfying $y_1 \preccurlyeq y_2$. By convention one puts $f(+\infty) = +\infty$ and $y^*(+\infty) = +\infty$ for all $y^* \in Y^*_+ := \{y^* \in Y^* : \langle y^*, y\rangle \geq 0\ \forall y \in Y_+\}$. So, if f is the null linear functional 0_{Y^*}, that is, the zero of Y^*, then

$$(2.1) \qquad (0_{Y^*} \circ F)(x) = 0 \text{ if } x \in \operatorname{dom} F \quad \text{and} \quad (0_{Y^*} \circ F)(x) = +\infty \text{ if } x \notin \operatorname{dom} F;$$

otherwise stated, $0_{Y^*} \circ F = \delta_{\operatorname{dom} F}$, the indicator of dom F. In particular, if $Y = \mathbb{R}$, $Y_+ = [0, +\infty[$ and $F : X \to \mathbb{R} \cup \{+\infty\}$, we have $0F = \delta_{\operatorname{dom} F}$, or in other words, $0(+\infty) := +\infty$. If f is convex and Y_+ -nondecreasing over $\operatorname{Im} F + Y_+$, then $f \circ F$ is convex. The following result gives the subdifferential of the composition of a vector-valued convex function with a nondecreasing convex function. The assertion (a) with (ii_a) of the following theorem has been proved in [34]. Below we show how the proof of [34, Theorem 3.1] can be adapted to a more general framework.

Theorem 2.2. *Let X and Y be two normed spaces and Y_+ be a convex cone of Y. Assume that $F : X \to Y \cup \{+\infty\}$ is a convex mapping with closed epigraph and that $f : Y \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous convex function which is nondecreasing over $\operatorname{Im} F + Y_+$. Let $\overline{x} \in \operatorname{dom}(f \circ F)$.*

(a) *If X and Y are reflexive Banach spaces, then $x^* \in \partial(f \circ F)(\overline{x})$ if and only if there exist sequences $x_n \to \overline{x}$, $y_n \to F(\overline{x})$, $x_n^* \to x^*$, $e_n^* \to 0$, and $y_n^* \in Y_+^*$ such that*

$$y_n^* + e_n^* \in \partial f(y_n) \quad \text{and} \quad x_n^* \in \partial(y_n^* \circ F)(x_n)$$

and such that either (i_a) or (ii_a) holds:

(i_a) *$f(y_n) \to f(F(\overline{x}))$, $\langle y_n^*, y_n - F(\bar{x})\rangle \to 0$, $\langle y_n^*, F(x_n) - F(\overline{x})\rangle \to 0$, and $\|y_n^*\| \, \|y_n - y_n'\| \to 0$ for some $y_n' \in F(x_n) + Y_+$ with $y_n' \to F(\overline{x})$ and $\langle y_n^*, y_n'\rangle = \langle y_n^*, F(x_n)\rangle$;*

(ii_a) *$f(y_n) - \langle y_n^*, y_n - F(\bar{x})\rangle \to f(F(\overline{x}))$ and $\langle y_n^*, F(x_n) - F(\overline{x})\rangle \to 0$.*

(b) *If X and Y are general Banach spaces, then $x^* \in \partial(f \circ F)(\overline{x})$ if and only if there exist nets $x_n \to \overline{x}$, $y_n \to F(\overline{x})$, $x_n^* \overset{w^*}{\to} x^*$, $e_n^* \overset{w^*}{\to} 0$, and $y_n^* \in Y_+^*$ such that*

$$y_n^* + e_n^* \in \partial f(y_n) \quad \text{and} \quad x_n^* \in \partial(y_n^* \circ F)(x_n)$$

and such that either (i_b) or (ii_b) holds:

(i_b) *$f(y_n) \to f(F(\overline{x}))$, $\langle y_n^* + e_n^*, y_n - F(\bar{x})\rangle \to 0$, $\langle y_n^*, F(x_n) - F(\overline{x})\rangle - \langle x_n^*, x_n - \overline{x}\rangle \to 0$, $(\|y_n^*\| + \|e_n^*\|)\|y_n - y_n'\| \to 0$ for some $y_n' \in F(x_n) + Y_+$ with $y_n' \to F(\overline{x})$ and $\langle y_n^*, y_n'\rangle = \langle y_n^*, F(x_n)\rangle$;*

(ii_b) *$f(y_n) - \langle y_n^* + e_n^*, y_n - F(\bar{x})\rangle \to f(F(\overline{x}))$ and $\langle y_n^*, F(x_n) - F(\overline{x})\rangle - \langle x_n^*, x_n - \overline{x}\rangle \to 0$.*

(c) *If X is a general Banach space and Y is finite dimensional, then $x^* \in \partial(f \circ F)(\overline{x})$ if and only if there exist nets $x_n \to \overline{x}$, $y_n \to F(\overline{x})$, $x_n^* \overset{w^*}{\to} x^*$, $e_n^* \to 0$, and $y_n^* \in Y_+^*$ such that*

$$y_n^* + e_n^* \in \partial f(y_n) \quad \text{and} \quad x_n^* \in \partial(y_n^* \circ F)(x_n)$$

and such that either (i_c) or (ii_c) holds:

(i_c) *$f(y_n) \to f(F(\overline{x}))$, $\langle y_n^*, y_n - F(\bar{x})\rangle \to 0$, $\langle y_n^*, F(x_n) - F(\overline{x})\rangle \to 0$, $\langle x_n^*, x_n - \overline{x}\rangle \to 0$, $\|y_n^*\| \, \|y_n - y_n'\| \to 0$ for some $y_n' \in F(x_n) + Y_+$ with $y_n' \to F(\overline{x})$ and $\langle y_n^*, y_n'\rangle = \langle y_n^*, F(x_n)\rangle$;*

(ii_c) *$f(y_n) - \langle y_n^*, y_n - F(\bar{x})\rangle \to f(F(\overline{x}))$, $\langle y_n^*, F(x_n) - F(\overline{x})\rangle \to 0$, and $\langle x_n^*, x_n - \overline{x}\rangle \to 0$.*

Proof. Set $\overline{y} := F(\overline{x})$ and, as usual, also set $f_1(x, y) := f(y)$ and $f_2(x, y) := \delta_{\operatorname{epi} F}(x, y)$ for all $x \in X$ and $y \in Y$. It is easily checked that $x^* \in \partial(f \circ F)(\overline{x})$ if and only if $(x^*, 0) \in \partial(f_1 + f_2)(\overline{x}, \overline{y})$. By the theorem above, this amounts to saying, whenever both Banach spaces X and Y are reflexive (resp. either X or Y is nonreflexive), that there exist sequences (resp. nets)

$$(0, y_n^* + e_n^*) + (x_n^*, -y_n^*) \to (x^*, 0) \quad \text{strongly (resp. weakly-*)} \tag{2.2}$$

with

$$y_n^* + e_n^* \in \partial f(y_n), \; (x_n^*, -y_n^*) \in \partial \delta_{\operatorname{epi} F}(x_n, y_n'),$$

$$y_n \to \overline{y}, \; (x_n, y_n') \to (\overline{x}, \overline{y}), \tag{2.3}$$

$$f(y_n) \to f(\overline{y}), \; \langle y_n^* + e_n^*, y_n - \overline{y}\rangle \to 0, \tag{2.4}$$

$$\langle y_n^*, y_n' - \overline{y}\rangle - \langle x_n^*, x_n - \overline{x}\rangle \to 0, \tag{2.5}$$

$$(\|y_n^* + e_n^*\| + \|x_n^*\| + \|y_n^*\|)\|y_n - y_n'\| \to 0. \tag{2.6}$$

We observe that (2.2) is equivalent to $x_n^* \to x^*$ and $e_n^* \to 0$ strongly (resp. weakly-*) and that the inclusion $(x_n^*, -y_n^*) \in \partial\delta_{\operatorname{epi} F}(x_n, y_n')$ means that

$$\langle x_n^*, x - x_n\rangle - \langle y_n^*, y - y_n'\rangle \le 0 \quad \text{for all } (x, y) \in \operatorname{epi} F. \tag{2.7}$$

On one hand, for any $y' \in Y_+$, taking in the latter inequality $x = x_n$ and $y = y_n' + y'$ yields $\langle y_n^*, y'\rangle \ge 0$ thus $y_n^* \in Y_+^*$. On the other hand, taking $x = x_n$ and $y = F(x_n)$ in (2.7) we obtain $\langle y_n^*, y_n' - F(x_n)\rangle \le 0$, and hence $\langle y_n^*, y_n' - F(x_n)\rangle = 0$ since $y_n^* \in Y_+^*$ and $y_n' - F(x_n) \in Y_+$. Consequently, (2.7) can be rewritten as

$$\langle x_n^*, x - x_n\rangle \le \langle y_n^*, y\rangle - \langle y_n^*, F(x_n)\rangle \quad \text{for all } (x, y) \in \operatorname{epi} F,$$

which, according to the inclusion $y_n^* \in Y_+^*$, is equivalent to

$$\langle x_n^*, x - x_n\rangle \le y_n^* \circ F(x) - y_n^* \circ F(x_n) \quad \text{for all } x \in \operatorname{dom} F.$$

The function $y_n^* \circ F$ being convex (since $y_n^* \in Y_+^*$), the inequality (2.7) is then equivalent to $x_n^* \in \partial(y_n^* \circ F)(x_n)$.

(a) Suppose first that X and Y are reflexive Banach spaces, so the properties above hold true for sequences $(x_n)_n$, $(x_n^*)_n$, etc. Since $x_n^* \overset{\|\,\|}{\to} x^*$, we observe that (2.5) is equivalent to $\langle y_n^*, y_n' - \overline{y}\rangle \to 0$, which is equivalent to $\langle y_n^*, F(x_n) - \overline{y}\rangle \to 0$ thanks to the equality $\langle y_n^*, y_n' - F(x_n)\rangle = 0$. Further, since $e_n^* \overset{\|\,\|}{\to} 0$, the second convergence in (2.4) is equivalent to $\langle y_n^*, y_n - \overline{y}\rangle \to 0$, and the convergence in (2.6) is equivalent to $\|y_n^*\|\,\|y_n - y_n'\| \to 0$. So, the conditions in (a) with (i_a) hold true and (i_a) obviously entails (ii_a).

On the other hand, taking sequences as given by (a) with (ii_a), we have, for each $x \in \operatorname{dom} F$,

$$\begin{aligned}\langle x_n^*, x - x_n\rangle &\le y_n^*(F(x)) - y_n^*(F(x_n))\\ &= \langle y_n^* + e_n^*, F(x) - y_n\rangle - \langle e_n^*, F(x) - y_n\rangle + \langle y_n^*, y_n - \overline{y}\rangle\\ &\quad - \langle y_n^*, F(x_n) - \overline{y}\rangle\\ &\le f(F(x)) - \{f(y_n) - \langle y_n^*, y_n - \overline{y}\rangle\} - \langle e_n^*, F(x) - y_n\rangle\\ &\quad - \langle y_n^*, F(x_n) - \overline{y}\rangle,\end{aligned}$$

and passing to the limit gives $\langle x^*, x - \overline{x}\rangle \le f(F(x)) - f(F(\overline{x}))$, and hence $x^* \in \partial(f \circ F)(\overline{x})$. The equivalences in assertion (a) are then established.

(b) Suppose that X and Y are general Banach spaces. Consider the nets obtained in the part preceding the arguments of (a). From (2.6) we deduce $(\|y_n^*\| + \|e_n^*\|)$ $\|y_n' - y_n\| \to 0$. Further, since $\langle y_n^*, y_n' - F(x_n)\rangle = 0$, it results from (2.5) that $\langle y_n^*, F(x_n) - F(\overline{x})\rangle - \langle x_n^*, x_n - \overline{x}\rangle \to 0$. All the properties in (b) with (i_b) are then justified and the ones concerning (ii_b) readily follow.

Now let us show that (b) with (ii_b) implies $x^* \in \partial(f \circ F)(\overline{x})$. Considering the nets as given by (b) with (ii_b), we have, for each $x \in \operatorname{dom} F$,

$$\begin{aligned}\langle x_n^*, x-\overline{x}\rangle &= \langle x_n^*, x-x_n\rangle + \langle x_n^*, x_n-\overline{x}\rangle\\ &\le y_n^* \circ F(x) - y_n^* \circ F(x_n) + \langle x_n^*, x_n-\overline{x}\rangle\\ &= \langle y_n^*+e_n^*, F(x)-y_n\rangle - \langle e_n^*, F(x)-\overline{y}\rangle + \langle y_n^*+e_n^*, y_n-\overline{y}\rangle\\ &\quad - \langle y_n^*, F(x_n)-\overline{y}\rangle + \langle x_n^*, x_n-\overline{x}\rangle\\ &\le f(F(x)) - \{f(y_n) - \langle y_n^*+e_n^*, y_n-\overline{y}\rangle\} - \langle e_n^*, F(x)-\overline{y}\rangle\\ &\quad - \{\langle y_n^*, F(x_n)-\overline{y}\rangle - \langle x_n^*, x_n-\overline{x}\rangle\},\end{aligned}$$

and this gives $\langle x^*, x-\overline{x}\rangle \le f(F(x)) - f(F(\overline{x}))$, and hence $x^* \in \partial(f \circ F)(\overline{x})$. The equivalences in assertion (b) are then proved.

(c) Consider the nets given by (b) with (i_b). From the finite dimensional property of Y the net $(e_n^*)_n$ strongly converges to zero. The convergence $\langle y_n^*+e_n^*, y_n-\overline{y}\rangle \to 0$ from (i_b) then ensures $\langle y_n^*, y_n-\overline{y}\rangle \to 0$. Combining this with the convergence $\|y_n^*\|\,\|y_n'-y_n\| \to 0$ and with the equality $\langle y_n^*, y_n'\rangle = \langle y_n^*, F(x_n)\rangle$ from (i_b), we obtain $\langle y_n^*, F(x_n)-\overline{y}\rangle \to 0$. It results from this and from the convergence $\langle y_n^*, F(x_n)-\overline{y}\rangle - \langle x_n^*, x_n-\overline{x}\rangle \to 0$ that $\langle x_n^*, x_n-\overline{x}\rangle \to 0$. All the properties in (i_c) are then deduced from the ones in (i_b), and evidently (ii_c) follows from (i_c).

Finally, as in (a) above, one also shows that (c) with (ii_c) entails that $x^* \in \partial(f \circ F)(\overline{x})$. □

By taking a continuous affine mapping F and the cone $Y_+ = \{0\}$, the above theorem implies the following statement; see [34, Corollary 3.3].

Corollary 2.3. *Let X and Y be two reflexive Banach spaces and $F : X \to Y$ be a continuous affine mapping with linear part $A \in \mathcal{L}(X,Y)$. Suppose that $f : Y \to \mathbb{R} \cup \{+\infty\}$ is a proper convex lower semicontinuous function. Then for $\overline{x} \in \operatorname{dom}(f \circ F)$ one has $x^* \in \partial(f \circ F)(\overline{x})$ if and only if there exist*

$$y_n^* \in \partial f(y_n) \quad \textit{with} \quad y_n^* \circ A \to x^*,$$

$$y_n \to F(\overline{x}) \quad \textit{and} \quad f(y_n) - \langle y_n^*, y_n - F(\overline{x})\rangle \to f(F(\overline{x})).$$

Let us now recall the notion of *Painlevé-Kuratowski limits*. Given a subset T of the normed space X, a set-valued mapping $M : T \rightrightarrows Z$ from T into a topological space Z, and $\overline{x} \in \operatorname{cl}_X(T)$, we recall that the *upper and lower limits* as $T \ni x \to \overline{x}$ are defined by

$$\limsup_{T\ni x\to\overline{x}} M(x) := \{z \in Z : \forall W \in \mathcal{N}_Z(z), \forall V \in \mathcal{N}_X(\overline{x}), \exists x \in V \cap T, M(x) \cap W \neq \emptyset\}$$

and

$$\liminf_{T\ni x\to\overline{x}} M(x) := \{z \in Z : \forall W \in \mathcal{N}_Z(z), \exists V \in \mathcal{N}_X(\overline{x}), \forall x \in V \cap T, M(x) \cap W \neq \emptyset\}.$$

Above $\mathcal{N}_Z(z)$ stands for the set of neighborhoods of z in Z. If both semilimits coincide, the common set is called the *Painlevé-Kuratowski limit* of $M(x)$ as $T \ni x \to \overline{x}$ and one writes $\lim_{T\ni x\to\overline{x}} M(x)$.

It is known that $z \in \limsup_{T\ni x\to\overline{x}} M(x)$ if and only if there are a net $(x_i)_{i\in I}$ in T converging to $\overline{x}$ and a net $(z_i)_{i\in I}$ in Z converging to z such that $z_i \in M(x_i)$ for all $i \in I$. When the topology of Z is metrizable, $(x_i)_i$ and $(z_i)_i$ can be taken as sequences.

One also knows that $z \in \liminf_{T\ni x\to\overline{x}} M(x)$ if and only if for any net $(x_i)_{i\in I}$ in T converging to $\overline{x}$ there exist a subnet $(x_{s(j)})_{j\in J}$ and a net $(z_j)_{j\in J}$ in Z converging to z such that $z_j \in M(x_{s(j)})$. When the topology of Z is metrizable, it is also necessary and sufficient that for any sequence $(x_n)_{n\in\mathbb{N}}$ in T converging to $\overline{x}$ there exists a sequence $(z_n)_{n\in\mathbb{N}}$ in Z converging to z such that $z_n \in M(x_n)$ for all n large enough.

When Z is a normed space E (resp. the topological dual of E), that is, $Z = E$ (resp. $Z = E^*$), both strong and weak (resp. weak-star) topologies on E (resp. E^*) can be considered. As said above, only sequences are needed to describe the upper and lower limits whenever E (resp. E^*) is endowed with the norm-topology, but *nets* are required provided that E (resp. E^*) is equipped with the weak (resp. weak-star) topology. Below, the upper and lower limits with respect to the strong topology of E (resp. E^*) are denoted by $\limsup_{T\ni x\to\overline{x}} M(x)$ and $\liminf_{T\ni x\to\overline{x}} M(x)$, and those limits with respect to the weak (resp. weak-star) topology of E (resp. E^*) are denoted respectively by

$$^{w}\limsup_{T\ni x\to\overline{x}} M(x),\ \ ^{w}\liminf_{T\ni x\to\overline{x}} M(x)\ \ \left(\text{resp. } ^{w^*}\limsup_{T\ni x\to\overline{x}} M(x),\ \ ^{w^*}\liminf_{T\ni x\to\overline{x}} M(x)\right).$$

Sometimes, even for E (resp. E^*) endowed with the weak (resp. weak-star) topology, one needs to work only with the above sequential convergence properties. Otherwise stated, for E (resp. E^*) endowed with the weak (resp. weak-star) topology, one needs to consider the *sequential upper and lower limits*

$$^{\text{seq}}\limsup_{T\ni x\to\overline{x}} M(x) \quad \text{and} \quad ^{\text{seq}}\liminf_{T\ni x\to\overline{x}} M(x)$$

defined as follows: An element $u\in E$ (resp. $u^*\in E^*$) belongs to $^{\text{seq}}\limsup_{T\ni x\to\overline{x}} M(x)$ provided there exist sequences $(x_n)_{n\in\mathbb{N}}$ in T converging strongly to $\overline{x}$ and $(u_n)_{n\in\mathbb{N}}$ in E (resp. $(u_n^*)_{n\in\mathbb{N}}$ in E^*) converging weakly (resp. weakly star) to u (resp. u^*) with $u_n \in M(x_n)$ (resp. $u_n^* \in M(x_n)$) for all $n \in \mathbb{N}$; similarly, the sequential lower limit $^{\text{seq}}\liminf_{T\ni x\to\overline{x}} M(x)$ is the set of $u \in E$ (resp. $u^* \in E^*$) such that for any sequence $(x_n)_{n\in\mathbb{N}}$ in T converging strongly to $\overline{x}$ there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in E (resp. $(u_n^*)_{n\in\mathbb{N}}$ in E^*) converging weakly (resp. weakly star) to u (resp. u^*) with $u_n \in M(x_n)$ (resp. $u_n^* \in M(x_n)$) for all n large enough.

In the case of a sequence of sets $(C_n) \subset X$, the upper and lower limits are then translated as

$$\limsup_{n\to+\infty} C_n = \left\{x \in X,\, \exists\, n_k \to +\infty,\, \exists\, x_{n_k} \in C_{n_k} \ \text{with}\ x_{n_k} \underset{k\to+\infty}{\longrightarrow} x\right\}$$

and

$$\liminf_{n\to+\infty} C_n = \left\{x \in X,\, \exists\, x_n \in C_n \ \text{for large}\ n,\ \text{with}\ x_n \underset{n\to+\infty}{\longrightarrow} x\right\}.$$

The *Painlevé-Kuratowski limit* of the sequence (C_n), when the upper and lower limits coincide, is denoted by $\lim_{n\to+\infty} C_n$.

When the sequence $(C_n)_{n\in\mathbb{N}}$ Painlevé-Kuratowski converges to C with respect to both the strong and the weak topologies of X, one says that the sequence $(C_n)_{n\in\mathbb{N}}$ *Mosco-converges* to C. It is easily seen that this is equivalent to

$$^{\text{seq}}\limsup_{n\to+\infty} C_n \subset C \subset \liminf_{n\to+\infty} C_n.$$

The graph of the set-valued mapping $M : X \rightrightarrows Z$ is defined by

$$\operatorname{gph} M = \{(x,z) \in X \times Z,\ z \in M(x)\}.$$

Let us now state the Attouch theorem (see [2]) applied to the indicator functions of sets.

Theorem 2.4. *Let X be a reflexive Banach space. Let C and C_n, $n \in \mathbb{N}$, be nonempty closed convex sets of X. Assume that the sequence (C_n) converges toward C in the sense of Mosco. Then the mappings N_{C_n} converge graphically to N_C. This means equivalently that*

(*i*) $\limsup_{n\to+\infty} (\operatorname{gph} N_{C_n}) \subset \operatorname{gph} N_C$.

(*ii*) $\operatorname{gph} N_C \subset \liminf_{n\to+\infty} (\operatorname{gph} N_{C_n})$.

When the sets C_n are sublevel sets, Mosco-convergence of the sequence (C_n) can be easily obtained, as shown by the following statement. In the sequel, we denote by $[\Phi \le \lambda]$ the sublevel set $\{x \in X,\ \Phi(x) \le \lambda\}$.

Lemma 2.5. *Let X be a normed space and let $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous convex function. Given a sequence $(\lambda_n) \subset \mathbb{R}$, let us define the set C_n by $C_n = [\Phi \le \lambda_n]$.*

(*i*) *If the sequence (λ_n) is nonincreasing and satisfies $\lim_{n\to+\infty} \lambda_n = \lambda \in \mathbb{R} \cup \{-\infty\}$, then the sequence (C_n) Mosco-converges to the set $C = [\Phi \le \lambda]$.*

(*ii*) *If the sequence (λ_n) is nondecreasing and satisfies $\lim_{n\to+\infty} \lambda_n = \lambda \in \mathbb{R} \cup \{+\infty\}$ with $C_n \neq \emptyset$ for n large enough, then the sequence (C_n) Mosco-converges to the set C defined by*

$$C = \begin{cases} [\Phi \le \lambda] & \textit{if} \quad \lambda < +\infty, \\ \operatorname{cl}(\operatorname{dom} \Phi) & \textit{if} \quad \lambda = +\infty. \end{cases}$$

Proof. (*i*) The sequence (C_n) is nonincreasing, hence

$$\lim_{n\to+\infty} C_n = \bigcap_{n\in\mathbb{N}} \operatorname{cl}(C_n) \quad \text{and} \quad {}^w\lim_{n\to+\infty} C_n = \bigcap_{n\in\mathbb{N}} \operatorname{cl}_w(C_n).$$

The sets C_n are closed by the lower semicontinuity of Φ. Since the sets C_n are convex, they are also weakly closed. It ensues that

$$\lim_{n\to+\infty} C_n = {}^w\lim_{n\to+\infty} C_n = \bigcap_{n\in\mathbb{N}} C_n = [\Phi \le \lambda],$$

and the sequence (C_n) Mosco-converges to the set $C := [\Phi \le \lambda]$.

(*ii*) If $\lambda_{n_0} = \lambda$ for some $n_0 \in \mathbb{N}$, then $\lambda_n = \lambda$ for every $n \ge n_0$ and the previous case applies. Hence we can suppose without loss of generality that $\lambda_n < \lambda$ for every $n \in \mathbb{N}$. The sequence (C_n) is nondecreasing, hence

$$\begin{aligned} \lim_{n\to+\infty} C_n &= \operatorname{cl}\left(\bigcup_{n\in\mathbb{N}} C_n\right) = \operatorname{cl}([\Phi < \lambda]), \\ {}^w\lim_{n\to+\infty} C_n &= \operatorname{cl}_w\left(\bigcup_{n\in\mathbb{N}} C_n\right) = \operatorname{cl}_w([\Phi < \lambda]). \end{aligned}$$

Since the set $[\Phi < \lambda]$ is convex, we have $\operatorname{cl}([\Phi < \lambda]) = \operatorname{cl}_w([\Phi < \lambda])$. It ensues that the sequence (C_n) Mosco-converges to the set $C := \operatorname{cl}([\Phi < \lambda])$. If $\lambda = +\infty$, we have $C = \mathrm{cl}(\mathrm{dom}\,\Phi)$. If $\lambda < +\infty$, taking $n_0 \in \mathbb{N}$ with $C_{n_0} \neq \emptyset$ we observe that $[\Phi < \lambda] \supset C_{n_0} \neq \emptyset$, so we can fix $a \in [\Phi < \lambda]$. For any $x \in [\Phi \le \lambda]$, we have

$$\Phi\left(\frac{1}{n}a + \left(1 - \frac{1}{n}\right)x\right) \le \frac{1}{n}\Phi(a) + \left(1 - \frac{1}{n}\right)\Phi(x) < \lambda;$$

hence $\frac{1}{n}a + \left(1 - \frac{1}{n}\right)x \in [\Phi < \lambda]$, which entails $x \in \mathrm{cl}\,([\Phi < \lambda])$. Hence we have proved that $[\Phi \le \lambda] \subset \mathrm{cl}\,([\Phi < \lambda])$. Since the reverse inclusion follows from the lower semicontinuity of Φ, we infer that $C = \mathrm{cl}\,([\Phi < \lambda]) = [\Phi \le \lambda]$ when $\lambda < +\infty$. The proof is complete. □

Let us end these preliminary results with the notion of a horizon subdifferential. Let X be a normed space and let $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous convex function. The horizon subdifferential of Φ at $\overline{x} \in \mathrm{dom}\,\Phi$ is defined by

$$\partial^\infty \Phi(\overline{x}) := {}^{\mathrm{seq}} \limsup_{x \xrightarrow[\Phi]{} \overline{x},\, \mu \downarrow 0} \mu\, \partial\Phi(x),$$

that is, $x^* \in \partial^\infty\Phi(\overline{x})$ provided there exist sequences $(x_n) \subset X$, $(x_n^*) \subset X^*$, $(\mu_n) \downarrow 0$ such that $x_n \to \overline{x}$, $\Phi(x_n) \to \Phi(\overline{x})$, $x_n^* \in \partial\Phi(x_n)$ and $\mu_n x_n^* \xrightarrow{w^*} x^*$.

Lemma 2.6. *Let X be a Banach space and let $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Then we have, for every $\overline{x} \in \mathrm{dom}\,\Phi$,*

$$\begin{aligned}\partial^\infty \Phi(\overline{x}) &= \limsup_{x \xrightarrow[\Phi]{} \overline{x},\, \mu \downarrow 0} \mu\, \partial\Phi(x) = \limsup_{x \to \overline{x},\, \mu \downarrow 0} \mu\, \partial\Phi(x) = {}^{\mathrm{seq}} \limsup_{x \to \overline{x},\, \mu \downarrow 0} \mu\, \partial\Phi(x) \\ &= N_{\mathrm{dom}\,\Phi}(\overline{x}) = \{x^* \in X^* : (x^*, 0) \in N_{\mathrm{epi}\, f}\big((\overline{x}, f(\overline{x}))\big)\}.\end{aligned}$$

Proof. Consider any $\overline{x} \in \mathrm{dom}\,\Phi$. The inclusions of the second member into the third and the third into the fourth are obvious. Let us show that the fourth member is included into the fifth, that is,

$$(2.8) \qquad {}^{\mathrm{seq}} \limsup_{x \to \overline{x},\, \mu \downarrow 0} \mu\, \partial\Phi(x) \subset N_{\mathrm{dom}\,\Phi}(\overline{x}).$$

Let $x^* \in {}^{\mathrm{seq}} \limsup_{x \to \overline{x},\, \mu \downarrow 0} \mu\, \partial\Phi(x)$. By definition, there exist $(x_n) \subset X$, $(x_n^*) \subset X^*$ and $(\mu_n) \downarrow 0$ such that $x_n \to \overline{x}$, $x_n^* \in \partial\Phi(x_n)$ and $\mu_n x_n^* \xrightarrow{w^*} x^*$. Take a real $\varepsilon > 0$ and by the lower semicontinuity of Φ choose some integer n_0 such that $\Phi(\overline{x}) - \varepsilon \le \Phi(x_n)$ for all $n \ge n_0$. Fixing any $x \in \mathrm{dom}\,\Phi$, the subdifferential inequality implies that, for every $n \ge n_0$,

$$(2.9) \qquad \langle \mu_n x_n^*, x - x_n \rangle \le \mu_n\left(\Phi(x) - \Phi(x_n)\right) \le \mu_n(\Phi(x) - \Phi(\overline{x}) + \varepsilon).$$

Taking the limit as $n \to +\infty$ in both extreme members of (2.9), we find $\langle x^*, x - \overline{x}\rangle \le 0$ because $x_n \to \overline{x}$, $\mu_n x_n^* \xrightarrow{w^*} x^*$ and $\mu_n \to 0$ as $n \to +\infty$. Since this is true for every $x \in \mathrm{dom}\,\Phi$, we infer that $x^* \in N_{\mathrm{dom}\,\Phi}(\overline{x})$, which shows the inclusion (2.8).

To prove the inclusion of the fifth member into the second, fix any $x^* \in N_{\mathrm{dom}\,\Phi}(\overline{x})$. For each real $\eta > 0$, we know from the lower semicontinuity of the convex function Φ (see, e.g., [8] or [23]) that $\partial_{\eta^2}\Phi(\overline{x}) \neq \emptyset$, so we choose some $y_\eta^* \in \partial_{\eta^2}\Phi(\overline{x})$. Putting $\mu(\eta) := \eta/(1+\|y_\eta^*\|)$, it is clear from the definitions of an approximate subdifferential and a normal cone that $z_\eta^* := y_\eta^* + \mu(\eta)^{-1}x^* \in \partial_{\eta^2}\Phi(\overline{x})$. Since X is a Banach space, the Borwein version[2] of the Brønsted-Rockafellar theorem [14] yields some $x_\eta \in X$

[2]See [10, Theorem 1] or also [8, Theorem IV.52].

and $x^*_\eta \in \partial\Phi(x_\eta)$ such that

$$\|x_\eta - \overline{x}\| \leq \eta,\ |\Phi(x_\eta) - \Phi(\overline{x})| \leq \eta(\eta+1),\ \|x^*_\eta - z^*_\eta\| \leq \eta(1+\|z^*_\eta\|).$$

Observing that the latter inequality entails

$$\|\mu(\eta)x^*_\eta - \mu(\eta)y^*_\eta - x^*\| \leq \eta\big(\mu(\eta) + \mu(\eta)\|y^*_\eta\| + \|x^*\|\big)$$

and noting that $\mu(\eta)\|y^*_\eta\| \leq \eta \to 0$ as $\eta \downarrow 0$, we see that $\mu(\eta)x^*_\eta \to x^*$ with $\mu(\eta) \downarrow 0$ as $\eta \downarrow 0$. This justifies the desired inclusion $N_{\mathrm{dom}\,\Phi}(\overline{x}) \subset \limsup_{x \to_\Phi \overline{x}, \mu\downarrow 0} \mu\,\partial\Phi(x)$. So, we have established the equality between the second, third, fourth and fifth members. Further, the second member is obviously included into the first and the first is also obviously included into the fourth, so we can add the first member into the preceding chain of equalities.

Finally, the equality $N_{\mathrm{dom}\,\Phi}(\overline{x}) = \big\{x^* : (x^*, 0) \in N_{\mathrm{epi}\,\Phi}\big((\overline{x}, \Phi(\overline{x}))\big)\big\}$ follows easily from the definition of a normal cone to a convex set, so the proof of the lemma is completed. □

3. A sequential formula for the normals to sublevel sets

Let X be a reflexive Banach space and let $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous convex function. In the next theorem, we establish a sequential formula for the normal cone to the sublevel set, which is valid without any qualification condition. For convenience, throughout the remainder of the paper we denote as usual $\mathbb{R}_- :=]-\infty, 0]$ and $\mathbb{R}_+ := [0, +\infty[$.

Theorem 3.1. *Let X be a reflexive Banach space and let $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous convex function. Given $\lambda \in \mathbb{R}$, let us define the set S by $S = [\Phi \leq \lambda]$. If $\overline{x} \in S$, we have*

$$N_S(\overline{x}) = \limsup_{\substack{\{x \to \overline{x},\ \mu \geq 0 \\ \mu(\Phi(x) - \lambda) \to 0}} \partial(\mu\Phi)(x) \tag{3.1}$$

$$= \limsup_{\substack{\{x \to \overline{x},\ \mu \geq 0 \\ \mu(\Phi(x) - \lambda) \to 0}} \mu\,\partial\Phi(x) \tag{3.2}$$

$$= \begin{cases} \limsup_{x \to \overline{x}} \mathbb{R}_+\,\partial\Phi(x) & \text{if } \Phi(\overline{x}) = \lambda, \\ \limsup_{x \to \overline{x},\ \mu\downarrow 0} \mu\,\partial\Phi(x) = N_{\mathrm{dom}\,\Phi}(\overline{x}) & \text{if } \Phi(\overline{x}) < \lambda. \end{cases} \tag{3.3}$$

(Concerning the second member of the first equality we recall that $0\Phi = \delta_{\mathrm{dom}\,\Phi}$.)

Proof. Let us start with the inclusion

$$N_S(\overline{x}) \subset \limsup_{\substack{\{x \to \overline{x},\ \mu \geq 0 \\ \mu(\Phi(x) - \lambda) \to 0}} \partial(\mu\Phi)(x). \tag{3.4}$$

Define the lower semicontinuous convex function $F : X \to \mathbb{R} \cup \{+\infty\}$ by $F(x) = \Phi(x) - \lambda$. Observe that $S = \{x \in X,\ F(x) \in \mathbb{R}_-\}$ and hence $\delta_S = \delta_{\mathbb{R}_-} \circ F$. This implies that

$$N_S(\overline{x}) = \partial\delta_S(\overline{x}) = \partial\left(\delta_{\mathbb{R}_-} \circ F\right)(\overline{x}).$$

The function $\delta_{\mathbb{R}_-}$ is lower semicontinuous convex and nondecreasing. Let us fix $x^* \in N_S(\overline{x})$ and apply Theorem 2.2 (a) with $Y = \mathbb{R}$, $Y_+ = \mathbb{R}_+$, $f = \delta_{\mathbb{R}_-}$ and the function F defined above. We obtain the existence of $(x_n) \subset X$, $(y_n) \subset \mathbb{R}$,

$(x_n^*) \subset X^*$, $(e_n^*) \subset \mathbb{R}$ and $(y_n^*) \subset \mathbb{R}_+$ such that $x_n \to \overline{x}$, $y_n \to \Phi(\overline{x}) - \lambda$, $x_n^* \to x^*$, $e_n^* \to 0$ and

$$y_n^* + e_n^* \in N_{\mathbb{R}_-}(y_n), \tag{3.5}$$

$$x_n^* \in \partial(y_n^* \Phi)(x_n), \tag{3.6}$$

$$y_n^* \left(\Phi(x_n) - \Phi(\overline{x})\right) \to 0 \quad \text{as } n \to +\infty. \tag{3.7}$$

Let us assume that $\Phi(\overline{x}) < \lambda$. Since $y_n \to \Phi(\overline{x}) - \lambda$ as $n \to +\infty$, we have $y_n < 0$ for n large enough. It ensues that $N_{\mathbb{R}_-}(y_n) = \{0\}$ and formula (3.5) then implies that

$$y_n^* = -e_n^* \to 0 \quad \text{as } n \to +\infty, \quad \text{if } \Phi(\overline{x}) < \lambda. \tag{3.8}$$

Now observe that

$$y_n^*(\Phi(x_n) - \lambda) = y_n^*(\Phi(x_n) - \Phi(\overline{x})) + y_n^*(\Phi(\overline{x}) - \lambda).$$

In view of (3.7)-(3.8), we immediately deduce that

$$y_n^*(\Phi(x_n) - \lambda) \to 0 \quad \text{as } n \to +\infty. \tag{3.9}$$

If $\Phi(\overline{x}) = \lambda$, formula (3.9) also holds true as a direct consequence of (3.7). Finally, we have built sequences $x_n \to \overline{x}$, $x_n^* \to x^*$, $(y_n^*) \subset \mathbb{R}_+$ satisfying (3.6) and (3.9), which clearly shows that

$$x^* \in \limsup_{\substack{\{x \to \overline{x},\ \mu \geq 0 \\ \mu\,(\Phi(x) - \lambda) \to 0}} \partial(\mu\Phi)(x).$$

Inclusion (3.4) is proved.

Let us now establish the inclusion

$$\limsup_{\substack{\{x \to \overline{x},\ \mu \geq 0 \\ \mu\,(\Phi(x) - \lambda) \to 0}} \partial(\mu\Phi)(x) \subset \limsup_{\substack{\{x \to \overline{x},\ \mu \geq 0 \\ \mu\,(\Phi(x) - \lambda) \to 0}} \mu\,\partial\Phi(x). \tag{3.10}$$

Let $x^* \in \limsup_{\substack{\{x \to \overline{x},\ \mu \geq 0 \\ \mu\,(\Phi(x) - \lambda) \to 0}} \partial(\mu\Phi)(x)$. By definition, there exist sequences $x_n \to \overline{x}$, $x_n^* \to x^*$, $(\mu_n) \subset \mathbb{R}_+$ satisfying $x_n^* \in \partial(\mu_n\Phi)(x_n)$ and $\mu_n(\Phi(x_n) - \lambda) \to 0$. First assume that there exists a subsequence of (μ_n), still denoted by (μ_n), such that $\mu_n > 0$ for every $n \in \mathbb{N}$. We then have $x_n^* \in \mu_n\partial\Phi(x_n)$, and hence

$$x^* \in \limsup_{\substack{\{x \to \overline{x},\ \mu \geq 0 \\ \mu\,(\Phi(x) - \lambda) \to 0}} \mu\,\partial\Phi(x). \tag{3.11}$$

Now assume that $\mu_n = 0$ for n large enough. Recalling that $0\Phi = \delta_{\operatorname{dom}\Phi}$, we obtain $x_n^* \in N_{\operatorname{dom}\Phi}(x_n)$ for n large enough. Since the operator $N_{\operatorname{cl}(\operatorname{dom}\Phi)}$ is graph-closed with respect to the norms in X and X^*, we deduce at the limit as $n \to +\infty$ that $x^* \in N_{\operatorname{dom}\Phi}(\overline{x})$. In view of Lemma 2.6, we have $N_{\operatorname{dom}\Phi}(\overline{x}) = \limsup_{x \xrightarrow[\Phi]{} \overline{x},\ \mu \downarrow 0} \mu\,\partial\Phi(x)$; hence

$$x^* \in \limsup_{\substack{\{x \to \overline{x},\ \mu \downarrow 0 \\ \Phi(x) \to \Phi(\overline{x})}} \mu\,\partial\Phi(x) \subset \limsup_{\substack{\{x \to \overline{x},\ \mu \geq 0 \\ \mu\,(\Phi(x) - \lambda) \to 0}} \mu\,\partial\Phi(x).$$

Therefore the inclusion (3.11) is satisfied in both cases, which proves (3.10).

Let us now show the inclusion

$$\limsup_{\substack{\begin{cases} x \to \overline{x},\ \mu \geq 0 \\ \mu\,(\Phi(x) - \lambda) \to 0 \end{cases}}} \mu\,\partial\Phi(x) \subset N_S(\overline{x}). \tag{3.12}$$

Let $x^* \in \limsup_{\substack{\begin{cases} x \to \overline{x},\ \mu \geq 0 \\ \mu\,(\Phi(x) - \lambda) \to 0 \end{cases}}} \mu\,\partial\Phi(x)$. By definition, there exist $(x_n) \subset X$, $(x_n^*) \subset X^*$ and $(\mu_n) \subset \mathbb{R}_+$ such that $x_n^* \in \partial\Phi(x_n)$, $x_n \to \overline{x}$, $\mu_n\, x_n^* \to x^*$ and $\mu_n(\Phi(x_n) - \lambda) \to 0$ as $n \to +\infty$. Let us fix $x \in S$. The subdifferential inequality gives

$$\langle x_n^*, x - x_n \rangle \leq \Phi(x) - \Phi(x_n) \leq \lambda - \Phi(x_n).$$

Multiplying by μ_n, we find $\langle \mu_n\, x_n^*, x - x_n \rangle \leq \mu_n(\lambda - \Phi(x_n))$. Taking the limit as $n \to +\infty$, we obtain that $\langle x^*, x - \overline{x} \rangle \leq 0$ because $x_n \to \overline{x}$, $\mu_n\, x_n^* \to x^*$ and $\mu_n(\lambda - \Phi(x_n)) \to 0$ as $n \to +\infty$. Since this is true for every $x \in S$, we conclude that $x^* \in N_S(\overline{x})$, which shows the inclusion (3.12). By combining inclusions (3.4), (3.10) and (3.12), we obtain formulae (3.1)-(3.2).

To prove formula (3.3), let us start with the following lemma.

Lemma 3.2. *Assume that there exist sequences $(x_n) \subset X$, $(x_n^*) \subset X^*$ and $(\mu_n) \subset \mathbb{R}_+$ such that $x_n^* \in \partial\Phi(x_n)$, $x_n \to \overline{x}$ and $\mu_n\, x_n^* \to x^*$ as $n \to +\infty$. Then we have*

$$\lim_{n \to +\infty} \mu_n\,(\Phi(x_n) - \Phi(\overline{x})) = 0.$$

Proof of Lemma 3.2. Since $x_n^* \in \partial\Phi(x_n)$, the subdifferential inequality gives

$$\Phi(x_n) - \Phi(\overline{x}) \leq \langle x_n^*, x_n - \overline{x} \rangle.$$

Multiplying by μ_n and taking the upper limit as $n \to +\infty$, we obtain that

$$\limsup_{n \to +\infty} \mu_n\,(\Phi(x_n) - \Phi(\overline{x})) \leq 0, \tag{3.13}$$

because $x_n \to \overline{x}$ and $\mu_n\, x_n^* \to x^*$. To complete the proof, we have to show that

$$\liminf_{n \to +\infty} \mu_n\,(\Phi(x_n) - \Phi(\overline{x})) \geq 0. \tag{3.14}$$

First assume that the sequence (μ_n) is bounded from above, say by $M > 0$. We then have

$$\mu_n\,(\Phi(x_n) - \Phi(\overline{x})) \geq -M\,(\Phi(x_n) - \Phi(\overline{x}))_-,$$

where $\alpha_- := \max\{0, -\alpha\}$ denotes the negative part of the real α. The lower semicontinuity of Φ implies that $\liminf_{n\to+\infty} \Phi(x_n) \geq \Phi(\overline{x})$, hence $\lim_{n\to+\infty}(\Phi(x_n) - \Phi(\overline{x}))_- = 0$. Taking the lower limit as $n \to +\infty$ in the above inequality, we obtain (3.14).

Now assume that the sequence (μ_n) is not bounded from above. There exists a subsequence of (μ_n), still denoted by (μ_n) such that $\lim_{n\to+\infty} \mu_n = +\infty$. Observe that $x_n^* = (\mu_n\, x_n^*)/\mu_n \to 0$ because $\mu_n\, x_n^* \to x^*$ as $n \to +\infty$. Since $x_n \to \overline{x}$ as $n \to +\infty$ and $x_n^* \in \partial\Phi(x_n)$, we deduce that $0 \in \partial\Phi(\overline{x})$, due to the graph-closedness of the operator $\partial\Phi$. Therefore $\overline{x}$ is a minimizer of Φ over X. This implies that $\Phi(x_n) \geq \Phi(\overline{x})$ for every $n \in \mathbb{N}$, hence formula (3.14) is also satisfied in this case. By combining inequalities (3.13) and (3.14), we conclude that $\lim_{n\to+\infty} \mu_n\,(\Phi(x_n) - \Phi(\overline{x})) = 0$. □

We are now able to prove formula (3.3). First assume that $\Phi(\overline{x}) = \lambda$. Lemma 3.2 shows that

$$\limsup_{x\to\overline{x}} \mathbb{R}_+ \,\partial\Phi(x) \;\subset\; \limsup_{\begin{cases} x\to\overline{x},\ \mu\geq 0 \\ \mu\,(\Phi(x)-\Phi(\overline{x}))\to 0\end{cases}} \mu\,\partial\Phi(x),$$

and since the reverse inclusion is obviously true, both members are equal. It suffices then to use formula (3.2).

Now assume that $\Phi(\overline{x}) < \lambda$. Observe that

$$\begin{aligned} N_S(\overline{x}) &= \limsup_{\begin{cases} x\to\overline{x},\ \mu\geq 0 \\ \mu\,(\Phi(x)-\lambda)\to 0\end{cases}} \mu\,\partial\Phi(x) && \text{in view of formula (3.2)} \\ &\subset \limsup_{\begin{cases} x\to\overline{x},\ \mu\geq 0 \\ \mu\,(\Phi(x)-\lambda)\to 0 \\ \mu\,(\Phi(x)-\Phi(\overline{x}))\to 0\end{cases}} \mu\,\partial\Phi(x) && \text{from Lemma 3.2} \\ &\subset \limsup_{x\to\overline{x},\,\mu\downarrow 0} \mu\,\partial\Phi(x) && \text{because } \Phi(\overline{x})\neq\lambda \\ &= N_{\operatorname{dom}\Phi}(\overline{x}) && \text{from Lemma 2.6.} \end{aligned}$$

Since $S\subset\operatorname{dom}\Phi$, the inclusion $N_{\operatorname{dom}\Phi}(\overline{x})\subset N_S(\overline{x})$ is obvious and we conclude that

$$N_S(\overline{x}) = \limsup_{x\to\overline{x},\,\mu\downarrow 0} \mu\,\partial\Phi(x) = N_{\operatorname{dom}\Phi}(\overline{x}).$$

□

Alternative proof of formula (3.1). Let us define the affine continuous function $F : X\to X\times\mathbb{R}$ by $F(x) = (x,\lambda)$ for every $x\in X$ and let $f : X\times\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$ be the lower semicontinuous convex function given by $f = \delta_{\operatorname{epi}\Phi}$. Observe that

$$x\in S \iff \Phi(x)\leq\lambda \iff F(x)\in\operatorname{epi}\Phi.$$

It ensues that $\delta_S = \delta_{\operatorname{epi}\Phi}\circ F = f\circ F$, and hence

$$N_S(\overline{x}) = \partial\delta_S(\overline{x}) = \partial(f\circ F)(\overline{x}).$$

By applying Corollary 2.3, we obtain that $x^*\in N_S(\overline{x})$ if and only if

$$\text{(3.15)}\qquad \begin{cases} \text{there exist sequences } (x_n^*,t_n^*)\subset X^*\times\mathbb{R}, \quad (x_n,t_n)\subset X\times\mathbb{R} \\ \text{such that } (x_n^*,t_n^*)\in N_{\operatorname{epi}\Phi}(x_n,t_n), \quad (x_n,t_n)\longrightarrow(\overline{x},\lambda), \\ x_n^*\longrightarrow x^* \quad\text{and}\quad \langle (x_n^*,t_n^*),(x_n,t_n)-(\overline{x},\lambda)\rangle\longrightarrow 0 \quad \text{as } n\to+\infty. \end{cases}$$

Observe that $\langle (x_n^*,t_n^*),(x_n,t_n)-(\overline{x},\lambda)\rangle = \langle x_n^*, x_n-\overline{x}\rangle + t_n^*\,(t_n-\lambda)$, so whenever $x_n\to\overline{x}$ and $x_n^*\to x^*$ as $n\to+\infty$, we have the equivalence

$$\text{(3.16)}\qquad \langle (x_n^*,t_n^*),(x_n,t_n)-(\overline{x},\lambda)\rangle \underset{n\to+\infty}{\longrightarrow} 0 \iff t_n^*\,(t_n-\lambda)\underset{n\to+\infty}{\longrightarrow} 0.$$

On the other hand, from a classical property of epigraphs, the real number t_n^* satisfies $t_n^*\leq 0$. By setting $\mu_n = -t_n^*\geq 0$, assertion (3.15) can be reformulated as

$$\text{(3.17)}\qquad \begin{cases} \text{there exist sequences } (x_n)\subset X,\ (x_n^*)\subset X^*,\ (\mu_n)\subset\mathbb{R}_+,\ (t_n)\subset\mathbb{R} \\ \text{such that } (x_n^*,-\mu_n)\in N_{\operatorname{epi}\Phi}(x_n,t_n), \quad x_n\longrightarrow\overline{x}, \quad t_n\longrightarrow\lambda, \\ x_n^*\longrightarrow x^* \quad\text{and}\quad \mu_n\,(t_n-\lambda)\longrightarrow 0 \quad\text{as } n\to+\infty. \end{cases}$$

Let us now distinguish the cases $\mu_n > 0$ and $\mu_n = 0$. If $\mu_n > 0$, the inclusion $(x_n^*,-\mu_n)\in N_{\operatorname{epi}\Phi}(x_n,t_n)$ ensures that $t_n = \Phi(x_n)$, so the following equivalences hold:

$$\begin{aligned}(x_n^*, -\mu_n) \in N_{\operatorname{epi}\Phi}(x_n, t_n) &\iff (x_n^*/\mu_n, -1) \in N_{\operatorname{epi}\Phi}(x_n, \Phi(x_n)) \\ &\iff x_n^*/\mu_n \in \partial\Phi(x_n) \\ &\iff x_n^* \in \mu_n\, \partial\Phi(x_n) = \partial(\mu_n\Phi)(x_n).\end{aligned}$$

If $\mu_n = 0$ we have the equivalences

$$\begin{aligned}(x_n^*, 0) \in N_{\operatorname{epi}\Phi}(x_n, t_n) &\iff x_n^* \in N_{\operatorname{dom}\Phi}(x_n) \\ &\iff x_n^* \in \partial(0\Phi)(x_n),\end{aligned}$$

according to the equality $0\Phi = \delta_{\operatorname{dom}\Phi}$. We immediately deduce from the above discussion that for every $n \in \mathbb{N}$,

$$\mu_n\,(t_n - \lambda) = \mu_n\,(\Phi(x_n) - \lambda) \tag{3.18}$$

and

$$(x_n^*, -\mu_n) \in N_{\operatorname{epi}\Phi}(x_n, t_n) \iff x_n^* \in \partial(\mu_n\Phi)(x_n). \tag{3.19}$$

By using (3.18)-(3.19), we obtain that assertion (3.17) is equivalent to

$$\begin{cases} \text{there exist sequences } (x_n) \subset X,\ (x_n^*) \subset X^*,\ (\mu_n) \subset \mathbb{R}_+, \\ \text{such that } x_n^* \in \partial(\mu_n\Phi)(x_n), \quad x_n \longrightarrow \overline{x}, \quad x_n^* \longrightarrow x^* \text{ and} \\ \mu_n\,(\Phi(x_n) - \lambda) \longrightarrow 0 \quad \text{as } n \to +\infty, \end{cases} \tag{3.20}$$

which is in turn equivalent to $x^* \in \limsup\limits_{\substack{x \to \overline{x},\ \mu \geq 0 \\ \mu\,(\Phi(x) - \lambda) \to 0}} \partial(\mu\Phi)(x)$. We conclude that

$$N_S(\overline{x}) = \limsup_{\substack{x \to \overline{x},\ \mu \geq 0 \\ \mu\,(\Phi(x) - \lambda) \to 0}} \partial(\mu\Phi)(x). \qquad \square$$

Remark 3.3. The set $\limsup\limits_{x\to\overline{x}} \mathbb{R}_+\, \partial\Phi(x)$ arising in formula (3.3) can be described as follows:

$$x^* \in \limsup_{x\to\overline{x}} \mathbb{R}_+\, \partial\Phi(x) \iff x^* = 0 \ \text{ or } \ \exists\, x_n \to \overline{x},\ \exists\, x_n^* \in \partial\Phi(x_n) \setminus \{0\},\ \frac{x_n^*}{\|x_n^*\|} \to \frac{x^*}{\|x^*\|}.$$

To prove the first implication, let us fix $x^* \in \left[\limsup\limits_{x\to\overline{x}} \mathbb{R}_+\, \partial\Phi(x)\right] \setminus \{0\}$. There exist a sequence (x_n) such that $x_n \to \overline{x}$ and a sequence $(\mu_n) \subset \mathbb{R}_+$ along with a sequence (x_n^*) such that $x_n^* \in \partial\Phi(x_n)$ and $\mu_n\, x_n^* \to x^*$. It ensues that $\mu_n\, \|x_n^*\| \to \|x^*\|$ as $n \to +\infty$. Since $\|x^*\| \neq 0$, we have $\mu_n \neq 0$ and $\|x_n^*\| \neq 0$ for n large enough, and hence $x_n^*/\|x_n^*\| \to x^*/\|x^*\|$ as $n \to +\infty$. The reverse implication is immediate.

We now show how equality (3.3) allows us to recover classical formulae for the normals to sublevel sets under the Slater condition.[3] Recall that, given a closed convex set $C \subset X$ and $x \in C$, the recession cone C^∞ is defined by

$$C^\infty = \{u \in X, \quad x + tu \in C \ \text{ for all } t \geq 0\}.$$

The set C^∞ does not depend on $x \in C$ and is also given by

$$C^\infty = \{u \in X,\ u + C \subset C\}.$$

For more details on recession analysis, see, for example, [4, 7, 23, 31].

[3]See formulae (1.2)-(1.4) of the introduction.

Corollary 3.4. *Let X be a reflexive Banach space and let $\Phi : X \to \mathbb{R}\cup\{+\infty\}$ be a lower semicontinuous convex function. Given $\overline{x} \in \operatorname{dom}\Phi$, let us consider the level set $S = [\Phi \le \Phi(\overline{x})]$. Then we have*

(*i*) $N_S(\overline{x}) = \mathbb{R}_+\,\partial\Phi(\overline{x}) \cup N_{\operatorname{dom}\Phi}(\overline{x}) \cup \limsup\limits_{x\to\overline{x},\,\mu\to+\infty} \mu\,\partial\Phi(x)$.

Assuming in addition the Slater condition $[\Phi < \Phi(\overline{x})] \neq \emptyset$, we obtain

(*ii*) $N_S(\overline{x}) = \mathbb{R}_+\,\partial\Phi(\overline{x}) \cup N_{\operatorname{dom}\Phi}(\overline{x})$.

(*iii*) *If additionally $\partial\Phi(\overline{x}) \neq \emptyset$, then $N_S(\overline{x}) = \operatorname{cl}(\mathbb{R}_+\,\partial\Phi(\overline{x}))$.*

(*iv*) *Finally, if $\overline{x} \in \operatorname{int}(\operatorname{dom}\Phi)$, we have $N_S(\overline{x}) = \mathbb{R}_+\,\partial\Phi(\overline{x})$.*

Proof. (*i*) In view of formula (3.3), it suffices to prove the equality

$$\limsup_{x\to\overline{x}} \mathbb{R}_+\,\partial\Phi(x) = \mathbb{R}_+\,\partial\Phi(\overline{x}) \cup N_{\operatorname{dom}\Phi}(\overline{x}) \cup \limsup_{x\to\overline{x},\,\mu\to+\infty} \mu\,\partial\Phi(x). \tag{3.21}$$

Let $x^* \in \limsup\limits_{x\to\overline{x}} \mathbb{R}_+\,\partial\Phi(x)$. By definition, there exist a sequence $(x_n) \subset X$, a sequence $(x_n^*) \subset X^*$ and a sequence $(\mu_n) \subset \mathbb{R}_+$ such that $x_n \to \overline{x}$, $x_n^* \to x^*$ as $n \to +\infty$ and $x_n^* \in \mu_n\,\partial\Phi(x_n)$. There exists a subsequence of (μ_n), still denoted by (μ_n), such that $\lim_{n\to+\infty} \mu_n = \overline{\mu} \in \mathbb{R}_+ \cup \{+\infty\}$. If $\overline{\mu} \in \mathbb{R}_+ \setminus \{0\}$, we deduce from the graph-closedness of the operator $\partial\Phi$ that $x^*/\overline{\mu} \in \partial\Phi(\overline{x})$, hence $x^* \in \mathbb{R}_+\,\partial\Phi(\overline{x})$. Now assume that $\lim_{n\to+\infty} \mu_n = 0$. We then obtain that $x^* \in \limsup\limits_{x\to\overline{x},\,\mu\downarrow 0} \mu\,\partial\Phi(x)$, and therefore $x^* \in N_{\operatorname{dom}\Phi}(\overline{x})$ in view of Lemma 2.6. Finally, if $\lim_{n\to+\infty} \mu_n = +\infty$, we find $x^* \in \limsup\limits_{x\to\overline{x},\,\mu\to+\infty} \mu\,\partial\Phi(x)$. The above arguments show that

$$x^* \in \mathbb{R}_+\,\partial\Phi(\overline{x}) \cup N_{\operatorname{dom}\Phi}(\overline{x}) \cup \limsup_{x\to\overline{x},\,\mu\to+\infty} \mu\,\partial\Phi(x),$$

hence the first inclusion in formula (3.21) is proved. Since the reverse inclusion is obvious, the proof of (*i*) is complete.

(*ii*) Let us now assume the Slater condition, thus implying that $0 \notin \partial\Phi(\overline{x})$. By using the graph-closedness of the operator $\partial\Phi$, we immediately obtain that the set $\limsup\limits_{x\to\overline{x},\,\mu\to+\infty} \mu\,\partial\Phi(x)$ is empty. Formula (*ii*) then follows from (*i*).

(*iii*) If $\partial\Phi(\overline{x}) \neq \emptyset$, then we have $N_{\operatorname{dom}\Phi}(\overline{x}) = (\partial\Phi(\overline{x}))^\infty$; see for example [9, Prop. 2.126]. Recalling that $\operatorname{cl}(\mathbb{R}_+ C) = \mathbb{R}_+ C \cup C^\infty$ for any nonempty closed convex set $C \subset X$ such that $0 \notin C$ and applying this fact with $C = \partial\Phi(\overline{x})$, we obtain (*iii*) from (*ii*).

(*iv*) If $\overline{x} \in \operatorname{int}(\operatorname{dom}\Phi)$, then we have $N_{\operatorname{dom}\Phi}(\overline{x}) = \{0\}$, hence $N_S(\overline{x}) = \mathbb{R}_+\,\partial\Phi(\overline{x})$ in view of (*ii*). □

If we assume that $\overline{x} \in \operatorname{int}(\operatorname{dom}\Phi)$, then formula (3.3) can be slightly improved, as shown by the following theorem.

Theorem 3.5. *Let X be a reflexive Banach space and let $\Phi : X \to \mathbb{R}\cup\{+\infty\}$ be a lower semicontinuous convex function. Given $\overline{x} \in \operatorname{int}(\operatorname{dom}\Phi)$, let us define the set S by $S = [\Phi \le \Phi(\overline{x})]$. If $\overline{x} \notin \operatorname{int}(S)$ we have*[4]

$$N_S(\overline{x}) = \limsup_{x\to\overline{x},\,x\notin S} \mathbb{R}_+\,\partial\Phi(x). \tag{3.22}$$

---

[4]The assumption $\overline{x} \notin \operatorname{int}(S)$ ensures that $\overline{x} \in \operatorname{cl}(X\setminus S)$, hence the upper limit arising in (3.22) is well defined. On the contrary, if $\overline{x} \in \operatorname{int}(S)$, we obviously have $N_S(\overline{x}) = \{0\}$. It is easy to check that the assumption $\overline{x} \in \operatorname{int}(S)$ implies that Φ is constant on some neighborhood of $\overline{x}$, hence $\partial\Phi(\overline{x}) = \{0\}$. We deduce that $\overline{x} \in \operatorname{argmin}\Phi$ and therefore $S = \operatorname{argmin}\Phi$ in this case.

Proof. First observe that

$$\begin{aligned}\limsup_{x\to\overline{x},\, x\notin S} \mathbb{R}_+\,\partial\Phi(x) &\subset \limsup_{x\to\overline{x}} \mathbb{R}_+\,\partial\Phi(x)\\ &= N_S(\overline{x}) \quad \text{from Theorem 3.1.}\end{aligned}$$

Hence it suffices to prove the inclusion $N_S(\overline{x}) \subset \limsup_{x\to\overline{x},\, x\notin S} \mathbb{R}_+\,\partial\Phi(x)$. Let us fix $x^* \in N_S(\overline{x})\setminus\{0\}$ and let us define the closed convex set $S_n \subset X$ by $S_n = [\Phi \le \Phi(\overline{x})+1/n]$ for every $n \ge 1$. By applying Lemma 2.5 (i) with $\lambda_n = \Phi(\overline{x}) + 1/n$, we obtain that the sequence (S_n) Mosco-converges to the set $S = [\Phi \le \Phi(\overline{x})]$. Theorem 2.4 (ii) then shows that there exist a sequence $(x_n) \subset X$ and a sequence $(x_n^*) \subset X^*$ such that $x_n \in S_n$, $x_n^* \in N_{S_n}(x_n)$, $x_n \to \overline{x}$ and $x_n^* \to x^*$ as $n \to +\infty$. By assumption, we have $\overline{x} \in \operatorname{int}(\operatorname{dom}\Phi)$, hence there exists $n_0 \in \mathbb{N}$ such that $x_n \in \operatorname{int}(\operatorname{dom}\Phi)$ for every $n \ge n_0$. On the other hand, since $x_n^* \to x^*$ as $n \to +\infty$ and $x^* \ne 0$, there exists $n_1 \ge n_0$ such that $x_n^* \ne 0$ for every $n \ge n_1$. Without loss of generality, we suppose in the sequel that $n \ge n_1$.

Let us now prove that $\Phi(x_n) = \Phi(\overline{x}) + 1/n$. Let us argue by contradiction and assume that $\Phi(x_n) < \Phi(\overline{x})+1/n$. From a classical result, the lower semicontinuous convex function Φ is continuous on the set $\operatorname{int}(\operatorname{dom}\Phi)$, hence Φ is continuous at x_n. Since $\Phi(x_n) < \Phi(\overline{x})+1/n$, we deduce that $x_n \in \operatorname{int}(S_n)$ and hence $N_{S_n}(x_n) = \{0\}$. This implies that $x_n^* = 0$, which gives the contradiction. We have shown that $\Phi(x_n) = \Phi(\overline{x}) + 1/n$, hence in particular $x_n \notin S$. Since $x_n \in \operatorname{int}(\operatorname{dom}\Phi)$ and since $[\Phi < \Phi(x_n)] \ne \emptyset$ because $\Phi(\overline{x}) < \Phi(x_n)$, Corollary 3.4 (iv) can be applied with the point x_n and the set $S_n = [\Phi \le \Phi(x_n)]$. We deduce that $N_{S_n}(x_n) = \mathbb{R}_+\,\partial\Phi(x_n)$ and therefore $x_n^* \in \mathbb{R}_+\,\partial\Phi(x_n)$. As a conclusion, we have built a sequence (x_n) such that $x_n \notin S$ for every $n \ge n_1$ and $x_n \to \overline{x}$ as $n \to +\infty$, along with a sequence (x_n^*) such that $x_n^* \in \mathbb{R}_+\,\partial\Phi(x_n)$ for every $n \ge n_1$ and $x_n^* \to x^*$ as $n \to +\infty$. This shows that $x^* \in \limsup_{x\to\overline{x},\, x\notin S} \mathbb{R}_+\,\partial\Phi(x)$, which proves the inclusion $N_S(\overline{x}) \setminus \{0\} \subset \limsup_{x\to\overline{x},\, x\notin S} \mathbb{R}_+\,\partial\Phi(x)$. It remains to establish that $0 \in \limsup_{x\to\overline{x},\, x\notin S} \mathbb{R}_+\,\partial\Phi(x)$. Since $\overline{x} \notin \operatorname{int}(S)$ by assumption, there exists a sequence $(x_n) \subset X$ such that $x_n \notin S$ and $x_n \to \overline{x}$ as $n \to +\infty$. Recalling that $\overline{x} \in \operatorname{int}(\operatorname{dom}\Phi)$, we have $x_n \in \operatorname{int}(\operatorname{dom}\Phi)$ for n large enough, thus Φ is continuous at x_n. It ensues that $\partial\Phi(x_n) \ne \emptyset$, hence $0 \in \mathbb{R}_+\,\partial\Phi(x_n)$ for n large enough and we conclude that $0 \in \limsup_{x\to\overline{x},\, x\notin S} \mathbb{R}_+\,\partial\Phi(x)$. □

Remark 3.6. If $\overline{x} \notin \operatorname{int}(\operatorname{dom}\Phi)$, formula (3.22) of the above theorem may fail. Take for example $X = \mathbb{R}$, $\Phi = \delta_{\mathbb{R}_+}$ and $\overline{x} = 0$. We have $N_S(\overline{x}) = N_{\mathbb{R}_+}(0) = \mathbb{R}_-$, while

$$\limsup_{x\to\overline{x},\, x\notin S} \mathbb{R}_+\,\partial\Phi(x) = \limsup_{x\to 0,\, x<0} \mathbb{R}_+\,\partial\Phi(x) = \emptyset.$$

4. Extension to nonreflexive Banach spaces

This section is concerned with the normal cone to convex sublevel set of nonreflexive Banach space. In such a framework we obtain the following result similar to Theorem 3.1, but here the upper limit has to be taken with respect to the weak-star topology of X^* and an additional condition involving the bracket $\langle\cdot, x - \overline{x}\rangle$ needs to be required.

Theorem 4.1. *Let X be a Banach space and let $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous convex function. For $\lambda \in \mathbb{R}$, let $S = [\Phi \le \lambda]$ and let $\overline{x} \in S$. Then*

$$N_S(\overline{x}) = {}^{w^*} \limsup_{\begin{cases} x \to \overline{x}, \mu \ge 0 \\ \mu(\Phi(x) - \lambda) \to 0 \\ \mu\langle \cdot, x - \overline{x}\rangle \to 0 \end{cases}} \mu\, \partial\Phi(x),$$

where the second member stands for the set of weak-star limits ${}^{w^*}\lim \mu_i u_i^*$ such that $\mu_i \ge 0$, $u_i^* \in \partial\Phi(x_i)$, the net $(x_i)_i$ strongly converges to $\overline{x}$, $\mu_i(\Phi(x_i) - \lambda) \to 0$ and $\mu_i\langle u_i^*, x_i - \overline{x}\rangle \to 0$.

Further, if $\Phi(\overline{x}) < \lambda$ one also has

$$N_S(\overline{x}) = {}^{w^*} \limsup_{\begin{cases} x \to \overline{x}, \mu \downarrow 0 \\ \mu\langle \cdot, x - \overline{x}\rangle \to 0 \end{cases}} \mu\, \partial\Phi(x) = N_{\operatorname{dom}\Phi}(\overline{x}). \tag{4.1}$$

Proof. First we note that in the proof of Theorem 3.1, according to Theorem 2.2 (c), $(x_n)_n$, $(y_n)_n$, $(x_n^*)_n$, $(y_n^*)_n$, and (e_n^*) have to be taken as *nets* instead of sequences, the strong convergence $x_n^* \to x^*$ has to be replaced by the weak-star convergence $x_n^* \overset{w^*}{\to} x^*$, and we have in addition $\langle x_n^*, x_n - \overline{x}\rangle \to 0$. With those elements at hand, one can show with arguments in the line of Theorem 3.1 that

$$N_S(\overline{x}) \subset {}^{w^*} \limsup_{\begin{cases} x \to \overline{x}, \mu \ge 0 \\ \mu(\Phi(x) - \lambda) \to 0 \\ \langle \cdot, x - \overline{x}\rangle \to 0 \end{cases}} \partial(\mu\, \Phi)(x),$$

where the second member denotes the set of weak-star limits of nets $(z_n^*)_n$ for which there are a net $(x_n)_n$ converging to $\overline{x}$ and a net $(\mu_n)_n$ in $\mathbb{R}_+$ such that $z_n^* \in \partial(\mu_n\Phi)(x_n)$, $\mu_n(\Phi(x_n) - \lambda) \to 0$ and $\langle z_n^*, x_n - \overline{x}\rangle \to 0$. Let us show that the latter second member is included in the same upper limit but with $\mu\, \partial\Phi(x)$ in place of $\partial(\mu\Phi)(x)$. Let x^* be in the upper limit with $\partial(\mu\Phi)(x)$. By definition, there exist nets $x_n \to \overline{x}$, $x_n^* \overset{w^*}{\to} x^*$, $(\mu_n) \subset \mathbb{R}_+$ satisfying $x_n^* \in \partial(\mu_n\Phi)(x_n)$, $\mu_n(\Phi(x_n) - \lambda) \to 0$ and $\langle x_n^*, x_n - \overline{x}\rangle \to 0$. First suppose that there exists a subnet of (μ_n), still denoted by (μ_n), such that $\mu_n > 0$ for every n. We then have $x_n^* \in \mu_n\partial\Phi(x_n)$ and hence

$$x^* \in {}^{w^*} \limsup_{\begin{cases} x \to \overline{x}, \mu \ge 0 \\ \mu(\Phi(x) - \lambda) \to 0 \\ \mu\langle \cdot, x - \overline{x}\rangle \to 0 \end{cases}} \mu\, \partial\Phi(x). \tag{4.2}$$

Now suppose, for some element n_0 of the set of indices, that $\mu_n = 0$ for all $n \succeq n_0$. Recalling that $0\Phi = \delta_{\operatorname{dom}\Phi}$, we obtain $x_n^* \in N_{\operatorname{dom}\Phi}(x_n)$ for n large enough. Writing, for every $x \in \operatorname{dom}\Phi$,

$$\langle x_n^*, x - \overline{x}\rangle = \langle x_n^*, x - x_n\rangle + \langle x_n^*, x_n - \overline{x}\rangle \le \langle x_n^*, x_n - \overline{x}\rangle,$$

and taking the convergence $\langle x_n^*, x_n - \overline{x}\rangle \to 0$ into account, we see at the limit on n that $x^* \in N_{\operatorname{dom}\Phi}(\overline{x})$. In view of Lemma 2.6, we have $N_{\operatorname{dom}\Phi}(\overline{x}) = \limsup_{x \underset{\Phi}{\to} \overline{x},\, \mu\downarrow 0} \mu\, \partial\Phi(x)$, hence

$$x^* \in \limsup_{\begin{cases} x \to \overline{x},\ \mu \downarrow 0 \\ \Phi(x) \to \Phi(\overline{x}) \end{cases}} \mu\, \partial\Phi(x) \subset {}^{w^*} \limsup_{\begin{cases} x \to \overline{x}, \mu \ge 0 \\ \mu(\Phi(x) - \lambda) \to 0 \\ \mu\langle \cdot, x - \overline{x}\rangle \to 0 \end{cases}} \mu\, \partial\Phi(x).$$

Therefore the inclusion (4.2) is satisfied in both cases, which proves the desired inclusion.

Now let us show the inclusion

$$
{}^{w^*}\limsup_{\substack{x \to \overline{x}, \mu \geq 0 \\ \mu(\Phi(x) - \lambda) \to 0 \\ \mu\langle \cdot, x - \overline{x}\rangle \to 0}} \mu\, \partial\Phi(x) \subset N_S(\overline{x}). \tag{4.3}
$$

Fix x^* in the first member. By definition, there are nets $(x_n) \subset X$, $(x_n^*) \subset X^*$ and $(\mu_n) \subset \mathbb{R}_+$ such that $x_n^* \in \partial\Phi(x_n)$, $x_n \to \overline{x}$, $\mu_n\, x_n^* \overset{w^*}{\to} x^*$, $\mu_n(\Phi(x_n) - \lambda) \to 0$, and $\mu_n\langle x_n^*, x_n - \overline{x}\rangle \to 0$. For any fixed element $x \in S$, the subdifferential inequality gives

$$
\langle \mu_n\, x_n^*, x - x_n\rangle \leq \mu_n\big(\Phi(x) - \Phi(x_n)\big) \leq \mu_n(\lambda - \Phi(x_n));
$$

thus $\langle x^*, x - \overline{x}\rangle \leq 0$ because $\mu_n\, x_n^* \overset{w^*}{\to} x^*$, $\mu_n\langle x_n^*, x_n - \overline{x}\rangle \to 0$, and $\mu_n(\lambda - \Phi(x_n)) \to 0$. We deduce that $x^* \in N_S(\overline{x})$, which shows inclusion (4.3) and finishes the proof of the first equality of the theorem.

Finally, assume that $\Phi(\overline{x}) < \lambda$. Under this additional hypothesis, for the nets at the beginning of the proof of the theorem, we obtain as in (3.8) that $y_n^* \downarrow 0$ or equivalently $\mu_n \downarrow 0$, since $\mu_n = y_n^*$. The inclusion

$$
N_S(\overline{x}) \subset {}^{w^*}\limsup_{\substack{x \to \overline{x}, \mu \downarrow 0 \\ \mu\langle \cdot, x - \overline{x}\rangle \to 0}} \partial(\mu\, \Phi)(x)
$$

follows immediately, and by arguing as above we deduce that

$$
N_S(\overline{x}) \subset {}^{w^*}\limsup_{\substack{x \to \overline{x}, \mu \downarrow 0 \\ \mu\langle \cdot, x - \overline{x}\rangle \to 0}} \mu\, \partial\Phi(x). \tag{4.4}
$$

Let us now show the inclusion

$$
{}^{w^*}\limsup_{\substack{x \to \overline{x}, \mu \downarrow 0 \\ \mu\langle \cdot, x - \overline{x}\rangle \to 0}} \mu\, \partial\Phi(x) \subset N_{\operatorname{dom}\Phi}(\overline{x}). \tag{4.5}
$$

Fix any x^* in the left member. By definition there exist nets $(\mu_n)_n$ in $\mathbb{R}_+$ with $\mu_n \to 0$, $(x_n)_n$ in X converging strongly to $\overline{x}$, $(u_n^*)_n$ in X^* with $u_n^* \in \partial\Phi(x_n)$ such that $\mu_n\, u_n^* \overset{w^*}{\to} x^*$ and $\mu_n\langle u_n^*, x_n - \overline{x}\rangle \to 0$. Consider any $x \in \operatorname{dom}\Phi$ and any real $\varepsilon > 0$. From the lower semicontinuity of Φ, choose some index element n_0 such that for any $n \succeq n_0$ we have $\Phi(x_n) \geq \Phi(\overline{x}) - \varepsilon$, where $\preceq$ denotes the directed preorder of the set of elements n. Then, for every $n \succeq n_0$ we have

$$
\langle u_n^*, x - x_n\rangle \leq \Phi(x) - \Phi(x_n) \leq \Phi(x) - \Phi(\overline{x}) + \varepsilon,
$$

hence

$$
\begin{aligned}
\langle \mu_n u_n^*, x - \overline{x}\rangle &= \langle \mu_n u_n^*, x - x_n\rangle + \langle \mu_n u_n^*, x_n - \overline{x}\rangle \\
&\leq \mu_n(\Phi(x) - \Phi(\overline{x}) + \varepsilon) + \mu_n\langle u_n^*, x_n - \overline{x}\rangle,
\end{aligned}
$$

so passing to the limit gives $\langle x^*, x - \overline{x}\rangle \leq 0$. Since this is true for every $x \in \operatorname{dom}\Phi$, we obtain $x^* \in N_{\operatorname{dom}\Phi}(\overline{x})$, which confirms the inclusion (4.5). Finally, since $\overline{x} \in S \subset \operatorname{dom}\Phi$, it is easily seen that

$$
N_{\operatorname{dom}\Phi}(\overline{x}) \subset N_S(\overline{x}). \tag{4.6}
$$

By combining the inclusions (4.4), (4.5) and (4.6), we find the equalities in (4.1). The proof is then complete. □

5. Extension to the intersection of finitely many sublevel sets

In this section, we consider k functions $\Phi_1, \ldots, \Phi_k : X \to \mathbb{R} \cup \{+\infty\}$, which are assumed to be lower semicontinuous and convex. For $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$, let us define the set S by

$$\begin{aligned} S &= \left\{x \in X,\ \Phi_1(x) \le \lambda_1, \ldots, \Phi_k(x) \le \lambda_k\right\} \\ &= \bigcap_{i=1}^{k} [\Phi_i \le \lambda_i]. \end{aligned}$$

Given $\overline{x} \in S$, our aim is to give a sequential formula for the normal cone $N_S(\overline{x})$, without resorting to any qualification condition. The key ingredients in that direction are Theorems 2.1 and 2.2. We focus our attention on the case where the Banach space X is reflexive; of course, adaptations to the case of a nonreflexive Banach space can be realized as in Section 4. First a suitable application of Theorem 2.2 leads to the following result.

Theorem 5.1. *Let X be a reflexive Banach space and let $\Phi_1, \ldots, \Phi_k : X \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous convex functions. Given $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$, let us define the set S by $S = \bigcap_{i=1}^{k} [\Phi_i \le \lambda_i]$. If $\overline{x} \in S$, we have*

$$N_S(\overline{x}) = \limsup_{\begin{cases} x \to \overline{x},\ \mu_i \ge 0 \\ \sum_{i=1}^{k} \mu_i\, (\Phi_i(x) - \lambda_i) \to 0 \end{cases}} \partial \left[\sum_{i=1}^{k} \mu_i\, \Phi_i\right](x). \tag{5.1}$$

Proof. Let us start with the inclusion $\subset$. Analogously to the proof of Theorem 3.1, define the function $F : X \to \mathbb{R}^k \cup \{+\infty\}$ by

$$F(x) = \begin{cases} (\Phi_1(x) - \lambda_1, \ldots, \Phi_k(x) - \lambda_k) & \text{if } x \in \bigcap_{i=1}^{k} \operatorname{dom} \Phi_i, \\ +\infty & \text{otherwise.} \end{cases}$$

The space $Y = \mathbb{R}^k$ is endowed with the preorder $\preccurlyeq$ via the convex cone $Y_+ = (\mathbb{R}_+)^k$, that is, $y_1 \preccurlyeq y_2 \Leftrightarrow y_2 - y_1 \in (\mathbb{R}_+)^k$. The abstract maximal element $\{+\infty\}$ is adjoined to $\mathbb{R}^k$. The convexity of each function Φ_i implies the convexity of F in the vector sense. The closedness of epi F is easily obtained from the closedness of each epigraph epi Φ_i, $i \in \{1, \ldots, k\}$. Now observe that $S = \left\{x \in X,\ F(x) \in (\mathbb{R}_-)^k\right\}$ and hence $\delta_S = \delta_{(\mathbb{R}_-)^k} \circ F$. This implies that

$$N_S(\overline{x}) = \partial \delta_S(\overline{x}) = \partial \left(\delta_{(\mathbb{R}_-)^k} \circ F\right)(\overline{x}).$$

The function $\delta_{(\mathbb{R}_-)^k}$ is lower semicontinuous convex and nondecreasing with respect to the preorder $\preccurlyeq$. Let us fix $x^* \in N_S(\overline{x})$ and apply Theorem 2.2 (a) with $Y = \mathbb{R}^k$, $Y_+ = (\mathbb{R}_+)^k$, $f = \delta_{(\mathbb{R}_-)^k}$ and the function F defined above. We obtain the existence of $(x_n) \subset X$, $(y_n) \subset \mathbb{R}^k$, $(x_n^*) \subset X^*$, $(e_n^*) \subset \mathbb{R}^k$ and $(y_n^*) \subset (\mathbb{R}_+)^k$ such that $x_n \to \overline{x}$, $y_n \to F(\overline{x})$, $x_n^* \to x^*$, $e_n^* \to 0$ and

$$y_n^* + e_n^* \in N_{(\mathbb{R}_-)^k}(y_n),\ x_n^* \in \partial(y_n^* \circ F)(x_n) \text{ and } \langle y_n^*, F(x_n) - F(\overline{x})\rangle \to 0.$$

Let us denote respectively by $(y_{1,n}, \ldots, y_{k,n})$, $(y_{1,n}^*, \ldots, y_{k,n}^*)$ and $(e_{1,n}^*, \ldots, e_{k,n}^*)$ the coordinates of the vectors y_n, y_n^* and e_n^* in the canonical basis of $\mathbb{R}^k$. Since $y_n^* \circ F = \sum_{i=1}^{k} y_{i,n}^*\, (\Phi_i - \lambda_i)$ according to (2.1), we obtain

$$x_n^* \in \partial \left[\sum_{i=1}^{k} y_{i,n}^*\, \Phi_i\right](x_n) \tag{5.2}$$

and

$$\sum_{i=1}^{k} y_{i,n}^*(\Phi_i(x_n) - \Phi_i(\overline{x})) \to 0 \quad \text{as } n \to +\infty. \tag{5.3}$$

Let us denote by $I(\overline{x})$ the set of active indices: $I(\overline{x}) = \{i \in \{1,\dots,k\},\ \Phi_i(\overline{x}) = \lambda_i\}$. Let us fix $i \notin I(\overline{x})$. Since $y_{i,n} \to \Phi_i(\overline{x}) - \lambda_i$ as $n \to +\infty$ and since $\Phi_i(\overline{x}) < \lambda_i$, we have $y_{i,n} < 0$ for n large enough. It ensues that $N_{\mathbb{R}_-}(y_{i,n}) = \{0\}$, and formula $y_{i,n}^* + e_{i,n}^* \in N_{\mathbb{R}_-}(y_n)$ then implies that $y_{i,n}^* = -e_{i,n}^* \to 0$ as $n \to +\infty$. Hence we have proved that

$$\forall i \notin I(\overline{x}), \quad y_{i,n}^* \to 0 \ \text{ as } n \to +\infty. \tag{5.4}$$

Now observe that

$$\begin{aligned}\sum_{i=1}^{k} y_{i,n}^*(\Phi_i(x_n) - \lambda_i) &= \sum_{i=1}^{k} y_{i,n}^*(\Phi_i(x_n) - \Phi_i(\overline{x})) + \sum_{i=1}^{k} y_{i,n}^*(\Phi_i(\overline{x}) - \lambda_i)\\ &= \sum_{i=1}^{k} y_{i,n}^*(\Phi_i(x_n) - \Phi_i(\overline{x})) + \sum_{i\notin I(\overline{x})} y_{i,n}^*(\Phi_i(\overline{x}) - \lambda_i). \qquad (5.5)\end{aligned}$$

In view of (5.3), (5.4) and (5.5), we immediately deduce that

$$\sum_{i=1}^{k} y_{i,n}^*(\Phi_i(x_n) - \lambda_i) \to 0 \ \text{ as } n \to +\infty. \tag{5.6}$$

Finally, we have built sequences $x_n \to \overline{x}$, $x_n^* \to x^*$, $(y_{i,n}^*) \subset \mathbb{R}_+$ satisfying (5.2) and (5.6), which clearly shows that

$$x^* \in \limsup_{\substack{\begin{cases} x \to \overline{x},\ \mu_i \geq 0 \\ \sum_{i=1}^k \mu_i\,(\Phi_i(x) - \lambda_i) \to 0\end{cases}}} \partial\left[\sum_{i=1}^{k} \mu_i\,\Phi_i\right](x). \tag{5.7}$$

Conversely, assume that x^* satisfies (5.7) and let us prove that $x^* \in N_S(\overline{x})$. By definition, there exist sequences $x_n \to \overline{x}$, $x_n^* \to x^*$, $(\mu_{i,n}) \subset \mathbb{R}_+$, $i \in \{1,\dots,k\}$, verifying

$$x_n^* \in \partial\left[\sum_{i=1}^{k} \mu_{i,n}\,\Phi_i\right](x_n) \quad \text{and} \quad \sum_{i=1}^{k} \mu_{i,n}(\Phi_i(x_n) - \lambda_i) \to 0 \ \text{ as } n \to +\infty.$$

Let us fix $x \in S$. By using the first relation above and the convexity of the function $\sum_{i=1}^k \mu_{i,n}\,\Phi_i$, we find

$$\begin{aligned}\langle x_n^*, x - x_n\rangle &\leq \sum_{i=1}^{k} \mu_{i,n}(\Phi_i(x) - \Phi_i(x_n))\\ &\leq \sum_{i=1}^{k} \mu_{i,n}(\lambda_i - \Phi_i(x_n)) \quad \text{since } x \in S.\end{aligned}$$

Recalling that $x_n \to \overline{x}$, $x_n^* \to x^*$ and $\sum_{i=1}^k \mu_{i,n}(\lambda_i - \Phi_i(x_n)) \to 0$ as $n \to +\infty$, and taking the limit as $n \to +\infty$ in the above inequality, we obtain that $\langle x^*, x - \overline{x}\rangle \leq 0$. Since this is true for every $x \in S$, we conclude that $x^* \in N_S(\overline{x})$. □

Remark 5.2. Condition $\sum_{i=1}^k \mu_i\,(\Phi_i(x) - \lambda_i) \to 0$ can be interpreted as a relaxed complementary slackness condition.

Theorem 5.3. *Let X be a reflexive Banach space and let $\Phi_1, \dots, \Phi_k : X \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous convex functions. Given $\lambda_1, \dots, \lambda_k \in \mathbb{R}$, let us define the set S by $S = \bigcap_{i=1}^k [\Phi_i \le \lambda_i]$. If $\overline{x} \in S$, we have*

$$N_S(\overline{x}) \subset \limsup_{\substack{x \to \overline{x},\ \mu_i \ge 0 \\ \mu_i \to 0 \ \text{if} \ i \notin I(\overline{x})}} \partial \left[\sum_{i=1}^k \mu_i \, \Phi_i \right] (x), \tag{5.8}$$

with $I(\overline{x}) = \{i \in \{1, \dots, k\},\ \Phi_i(\overline{x}) = \lambda_i\}$. If moreover the following Slater qualification condition is satisfied,

$$\bigcap_{i=1}^k [\Phi_i < \lambda_i] \ne \emptyset, \tag{5.9}$$

then the above inclusion holds as an equality.

Proof. The inclusion (5.8) is obtained as a by-product of the proof of Theorem 5.1; see formula (5.4) in particular. The proof of the reverse inclusion is based on the following lemma.

Lemma 5.4. *Assume that there exist sequences $(x_n) \subset X$, $(x_n^*) \subset X^*$ and $(\mu_{i,n}) \subset \mathbb{R}_+$ for $i = 1, \dots, k$ such that $x_n \to \overline{x}$, $x_n^* \to x^*$, $\mu_{i,n} \to 0$ if $i \notin I(\overline{x})$ and $x_n^* \in \partial \left[\sum_{i=1}^k \mu_{i,n} \, \Phi_i \right] (x_n)$. Assume moreover that the Slater condition* (5.9) *holds. Then the following properties are satisfied:*

(*i*) *There exists $M > 0$ such that $\sum_{i=1}^k \mu_{i,n} \le M$ for every $n \in \mathbb{N}$.*
(*ii*) $\lim_{n \to +\infty} \sum_{i=1}^k \mu_{i,n} (\Phi_i(x_n) - \Phi_i(\overline{x})) = 0$.

Proof of Lemma 5.4. (*i*) Let us argue by contradiction and assume that the sequence (M_n) defined by $M_n = \sum_{i=1}^k \mu_{i,n}$ is not bounded. There exists a subsequence of (M_n), still denoted by (M_n), such that $\lim_{n \to +\infty} M_n = +\infty$. For every $i \in \{1, \dots, k\}$, let us define the sequence $(\nu_{i,n})$ by $\nu_{i,n} = \mu_{i,n}/M_n$. Each sequence $(\nu_{i,n})$ satisfies $\nu_{i,n} \in [0, 1]$ for every $n \in \mathbb{N}$. Hence we can extract a subsequence of $(\nu_{i,n})$, still denoted by $(\nu_{i,n})$, such that $\lim_{n \to +\infty} \nu_{i,n} = \overline{\nu}_i \in [0, 1]$, for every $i \in \{1, \dots, k\}$. The real numbers $\overline{\nu}_i$ satisfy $\sum_{i=1}^k \overline{\nu}_i = 1$ and $\overline{\nu}_i = 0$ if $i \notin I(\overline{x})$ because $\mu_{i,n} \to 0$ in this case. Let us now fix $x \in \bigcap_{i=1}^k [\Phi_i < \lambda_i]$, which is nonempty by assumption. Observing that $x_n^*/M_n \in \partial \left[\sum_{i=1}^k \nu_{i,n} \, \Phi_i \right] (x_n)$, the subdifferential inequality yields

$$\sum_{i=1}^k \nu_{i,n} \, \Phi_i(x) \ge \sum_{i=1}^k \nu_{i,n} \, \Phi_i(x_n) + \langle x_n^*/M_n, x - x_n \rangle . \tag{5.10}$$

Since $x_n \to \overline{x}$, $x_n^* \to x^*$ and $M_n \to +\infty$ as $n \to +\infty$, we immediately obtain

$$\lim_{n \to +\infty} \langle x_n^*/M_n, x - x_n \rangle = 0. \tag{5.11}$$

On the other hand, we have

$$\lim_{n \to +\infty} \sum_{i=1}^k \nu_{i,n} \, \Phi_i(x) = \sum_{i=1}^k \overline{\nu}_i \, \Phi_i(x), \tag{5.12}$$

because $\lim_{n\to+\infty} \nu_{i,n} = \overline{\nu}_i$. Finally, the lower semicontinuity of each function Φ_i implies that

$$\liminf_{n\to+\infty} \nu_{i,n}\, \Phi_i(x_n) \geq \overline{\nu}_i\, \Phi_i(\overline{x}),$$

hence

$$\liminf_{n\to+\infty} \sum_{i=1}^{k} \nu_{i,n}\, \Phi_i(x_n) \geq \sum_{i=1}^{k} \liminf_{n\to+\infty} \nu_{i,n}\, \Phi_i(x_n) \geq \sum_{i=1}^{k} \overline{\nu}_i\, \Phi_i(\overline{x}). \tag{5.13}$$

By combining (5.10), (5.11), (5.12) and (5.13), we find $\sum_{i=1}^{k} \overline{\nu}_i \Phi_i(x) \geq \sum_{i=1}^{k} \overline{\nu}_i \Phi_i(\overline{x})$. Recalling that $\overline{\nu}_i = 0$ if $i \notin I(\overline{x})$, we infer that

$$\sum_{i\in I(\overline{x})} \overline{\nu}_i\, \Phi_i(x) \geq \sum_{i\in I(\overline{x})} \overline{\nu}_i\, \Phi_i(\overline{x}) = \sum_{i\in I(\overline{x})} \overline{\nu}_i\, \lambda_i. \tag{5.14}$$

On the other hand, since $x \in \bigcap_{i=1}^{k} [\Phi_i < \lambda_i]$ and since $\overline{\nu}_i > 0$ for at least one $i \in I(\overline{x})$, we have

$$\sum_{i\in I(\overline{x})} \overline{\nu}_i\, \Phi_i(x) < \sum_{i\in I(\overline{x})} \overline{\nu}_i\, \lambda_i,$$

which yields a contradiction with (5.14).

(ii) From (i), there exists $M > 0$ such that $\sum_{i=1}^{k} \mu_{i,n} \leq M$ for every $n \in \mathbb{N}$. We then have

$$\mu_{i,n}\, (\Phi_i(x_n) - \Phi_i(\overline{x})) \geq -M\, (\Phi_i(x_n) - \Phi_i(\overline{x}))_-.$$

The lower semicontinuity of Φ_i implies that $\liminf_{n\to+\infty} \Phi_i(x_n) \geq \Phi_i(\overline{x})$, hence $\lim_{n\to+\infty} (\Phi_i(x_n) - \Phi_i(\overline{x}))_- = 0$. Taking the lower limit as $n \to +\infty$ in the above inequality, we obtain $\liminf_{n\to+\infty} \mu_{i,n}\, (\Phi_i(x_n) - \Phi_i(\overline{x})) \geq 0$. It ensues that

$$\liminf_{n\to+\infty} \sum_{i=1}^{k} [\mu_{i,n}\, (\Phi_i(x_n) - \Phi_i(\overline{x}))] \geq \sum_{i=1}^{k} \liminf_{n\to+\infty} [\mu_{i,n}\, (\Phi_i(x_n) - \Phi_i(\overline{x}))] \geq 0. \tag{5.15}$$

By using the assumption $x_n^* \in \partial \left[\sum_{i=1}^{k} \mu_{i,n}\, \Phi_i\right] (x_n)$, the subdifferential inequality gives

$$\sum_{i=1}^{k} [\mu_{i,n}\, (\Phi_i(x_n) - \Phi_i(\overline{x}))] \leq \langle x_n^*, x_n - \overline{x} \rangle.$$

Since $\lim_{n\to+\infty} \langle x_n^*, x_n - \overline{x} \rangle = 0$, we deduce that

$$\limsup_{n\to+\infty} \sum_{i=1}^{k} [\mu_{i,n}\, (\Phi_i(x_n) - \Phi_i(\overline{x}))] \leq 0. \tag{5.16}$$

By combining inequalities (5.15) and (5.16), we conclude that

$$\lim_{n\to+\infty} \sum_{i=1}^{k} [\mu_{i,n}\, (\Phi_i(x_n) - \Phi_i(\overline{x}))] = 0. \qquad \square$$

Let us come back to the proof of Theorem 5.3. Lemma 5.4 shows that

$$\limsup_{\begin{cases} x \to \overline{x},\ \mu_i \geq 0 \\ \mu_i \to 0 \text{ if } i \notin I(\overline{x}) \end{cases}} \partial \left[\sum_{i=1}^{k} \mu_i \, \Phi_i \right](x) \subset \limsup_{\begin{cases} x \to \overline{x},\ \mu_i \geq 0 \\ \mu_i \to 0 \text{ if } i \notin I(\overline{x}) \\ \sum_{i=1}^{k} \mu_i (\Phi_i(x) - \Phi_i(\overline{x})) \to 0 \end{cases}} \partial \left[\sum_{i=1}^{k} \mu_i \, \Phi_i \right](x)$$

$$\subset \limsup_{\begin{cases} x \to \overline{x},\ \mu_i \geq 0 \\ \sum_{i=1}^{k} \mu_i (\Phi_i(x) - \lambda_i) \to 0 \end{cases}} \partial \left[\sum_{i=1}^{k} \mu_i \, \Phi_i \right](x)$$

$$= N_S(\overline{x}) \quad \text{in view of Theorem 5.1.}$$

The proof is complete. □

We now recover as a corollary the classical formula for the normal cone $N_S(\overline{x})$ by assuming the continuity of the functions Φ_i, $i = 1, \ldots, k$, along with the Slater condition (5.9).

Corollary 5.5. *Let X be a reflexive Banach space and let $\Phi_1, \ldots, \Phi_k : X \to \mathbb{R}$ be continuous convex functions. Given $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$, let us define the set S by $S = \bigcap_{i=1}^{k} [\Phi_i \leq \lambda_i]$. Assume that the Slater condition* (5.9) *holds true. Then, for $\overline{x} \in S$, we have*

$$N_S(\overline{x}) = \sum_{i \in I(\overline{x})} \mathbb{R}_+ \, \partial \Phi_i(\overline{x}),$$

with $I(\overline{x}) = \{ i \in \{1, \ldots, k\},\ \Phi_i(\overline{x}) = \lambda_i \}$ and the convention that the sum over an empty set of indices equals zero.

Proof. In view of Theorem 5.3, the following equality holds:

$$N_S(\overline{x}) = \limsup_{\begin{cases} x \to \overline{x},\ \mu_i \geq 0 \\ \mu_i \to 0 \text{ if } i \notin I(\overline{x}) \end{cases}} \partial \left[\sum_{i=1}^{k} \mu_i \, \Phi_i \right](x).$$

On the other hand, since each function Φ_i is continuous, we have for every $x \in X$,

$$\partial \left[\sum_{i=1}^{k} \mu_i \, \Phi_i \right](x) = \sum_{i=1}^{k} \mu_i \, \partial \Phi_i(x).$$

Hence, we must prove that

$$\limsup_{\begin{cases} x \to \overline{x},\ \mu_i \geq 0 \\ \mu_i \to 0 \text{ if } i \notin I(\overline{x}) \end{cases}} \sum_{i=1}^{k} \mu_i \, \partial \Phi_i(x) = \sum_{i \in I(\overline{x})} \mathbb{R}_+ \, \partial \Phi_i(\overline{x}).$$

Let $x^* \in \limsup_{\begin{cases} x \to \overline{x},\ \mu_i \geq 0 \\ \mu_i \to 0 \text{ if } i \notin I(\overline{x}) \end{cases}} \sum_{i=1}^{k} \mu_i \, \partial \Phi_i(x)$. By definition, there exist sequences $(x_n) \subset X$, $(x^*_{i,n}) \subset X^*$, $(\mu_{i,n}) \subset \mathbb{R}_+$ such that $x^*_{i,n} \in \partial \Phi_i(x_n)$, $x_n \underset{n \to +\infty}{\longrightarrow} \overline{x}$, $\sum_{i=1}^{k} \mu_{i,n} \, x^*_{i,n} \underset{n \to +\infty}{\longrightarrow} x^*$ and $\mu_{i,n} \underset{n \to +\infty}{\longrightarrow} 0$ if $i \notin I(\overline{x})$. Since each function Φ_i is continuous, the sequence $(x^*_{i,n})$ is bounded and hence has a convergent subsequence with respect to the weak-star topology of X^*. On the other hand, Lemma 5.4

shows that each sequence $(\mu_{i,n})$ is bounded. Therefore, by iteratively extracting subsequences, we can build an increasing map $\varphi : \mathbb{N} \to \mathbb{N}$ such that

$$\forall i \in \{1, \dots, k\}, \quad x^*_{i,\varphi(n)} \underset{n \to +\infty}{\overset{w^*}{\longrightarrow}} x^*_i \quad \text{and} \quad \mu_{i,\varphi(n)} \underset{n \to +\infty}{\longrightarrow} \overline{\mu}_i, \tag{5.17}$$

for some $x^*_1, \dots, x^*_k \in X^*$ and $\overline{\mu}_1, \dots, \overline{\mu}_k \in \mathbb{R}_+$. From this and the sequential $\| \, \| \times w^*$ graph-closedness property of each operator $\partial\Phi_i$, we immediately obtain that $x^*_i \in \partial\Phi_i(\overline{x})$. Since $\sum_{i=1}^k \mu_{i,n} \, x^*_{i,n} \underset{n \to +\infty}{\longrightarrow} x^*$, we deduce from (5.17) that $\sum_{i=1}^k \overline{\mu}_i \, x^*_i = x^*$. Recalling that $\overline{\mu}_i = 0$ for $i \notin I(\overline{x})$, this implies that $x^* = \sum_{i \in I(\overline{x})} \overline{\mu}_i \, x^*_i \in \sum_{i \in I(\overline{x})} \mathbb{R}_+ \, \partial\Phi_i(\overline{x})$. Therefore the inclusion

$$\limsup_{\substack{\{ x \to \overline{x}, \; \mu_i \geq 0 \\ \mu_i \to 0 \text{ if } i \notin I(\overline{x})}} \sum_{i=1}^k \mu_i \, \partial\Phi_i(x) \subset \sum_{i \in I(\overline{x})} \mathbb{R}_+ \, \partial\Phi_i(\overline{x})$$

is proved, and since the reverse inclusion is immediate, the proof is complete. □

In Theorems 5.1 and 5.3 the normal cone $N_S(\overline{x})$ is described through the subdifferential of positively linear combinations of the functions Φ_i, $i = 1, \dots, k$. The next theorem provides an additional description via the separate subdifferentials of the functions Φ_i. For such a description we use the set

$$\limsup_{\substack{\{ x_i \to \overline{x}, \; \mu_i \geq 0 \\ \mu_i \, (\Phi_i(x_i) - \lambda_i) \to 0 \\ \mu_i \langle x^*_i, x_i - \overline{x} \rangle \to 0}} \sum_{i=1}^k \mu_i \, \partial\Phi_i(x_i)$$

defined as the set of limits $\lim_{n \to +\infty} \sum_{i=1}^k \mu_{i,n} \, x^*_{i,n}$ such that $\mu_{i,n} \geq 0$, $x_{i,n} \underset{n \to +\infty}{\longrightarrow} \overline{x}$, $x^*_{i,n} \in \partial\Phi_i(x_{i,n})$, $\mu_{i,n}(\Phi_i(x_{i,n}) - \lambda_i) \underset{n \to +\infty}{\longrightarrow} 0$ and $\mu_{i,n} \langle x^*_{i,n}, x_{i,n} - \overline{x} \rangle \underset{n \to +\infty}{\longrightarrow} 0$ for $i = 1, \dots, k$. In the same vein, we denote by

$$\limsup_{\substack{\{ x_i \to \overline{x}, \; \mu_i \geq 0 \\ \mu_i \, (\Phi_i(x_i) - \lambda_i - \langle x^*_i, x_i - \overline{x} \rangle) \to 0}} \sum_{i=1}^k \mu_i \, \partial\Phi_i(x_i)$$

and respectively

$$\limsup_{\substack{\{ x_i \to \overline{x}, \; \mu_i \geq 0 \\ \sum_{i=1}^k \mu_i \, (\Phi_i(x_i) - \lambda_i - \langle x^*_i, x_i - \overline{x} \rangle) \to 0}} \sum_{i=1}^k \mu_i \, \partial\Phi_i(x_i)$$

the sets, where the last conditions of the above definition are replaced respectively by

$$\mu_{i,n} \left(\Phi_i(x_{i,n}) - \lambda_i - \langle x^*_{i,n}, x_{i,n} - \overline{x} \rangle \right) \underset{n \to +\infty}{\longrightarrow} 0 \quad \text{for } i = 1, \dots, k$$

and

$$\sum_{i=1}^k \mu_{i,n} \left(\Phi_i(x_{i,n}) - \lambda_i - \langle x^*_{i,n}, x_{i,n} - \overline{x} \rangle \right) \underset{n \to +\infty}{\longrightarrow} 0.$$

By combining Theorems 2.1 and 3.1, we show in the next theorem that the normal cone $N_S(\overline{x})$ to the set $S = \bigcap_{i=1}^k [\Phi_i \leq \lambda_i]$ coincides with the above upper limits.

Theorem 5.6. *Let X be a reflexive Banach space and let $\Phi_1, \ldots, \Phi_k : X \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous convex functions. Given $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$, let us define the set S by $S = \bigcap_{i=1}^k [\Phi_i \leq \lambda_i]$. For $\overline{x} \in S$, we have*

$$\begin{aligned} N_S(\overline{x}) &= \limsup_{\begin{cases} x_i \to \overline{x},\ \mu_i \geq 0 \\ \mu_i\,(\Phi_i(x_i) - \lambda_i) \to 0 \\ \mu_i \langle x_i^*, x_i - \overline{x}\rangle \to 0 \end{cases}} \sum_{i=1}^k \mu_i\, \partial\Phi_i(x_i) \\ &= \limsup_{\begin{cases} x_i \to \overline{x},\ \mu_i \geq 0 \\ \mu_i\,(\Phi_i(x_i) - \lambda_i - \langle x_i^*, x_i - \overline{x}\rangle) \to 0 \end{cases}} \sum_{i=1}^k \mu_i\, \partial\Phi_i(x_i) \\ &= \limsup_{\begin{cases} x_i \to \overline{x},\ \mu_i \geq 0 \\ \sum_{i=1}^k \mu_i\,(\Phi_i(x_i) - \lambda_i - \langle x_i^*, x_i - \overline{x}\rangle) \to 0 \end{cases}} \sum_{i=1}^k \mu_i\, \partial\Phi_i(x_i). \end{aligned}$$

Proof. Let $x^* \in N_S(\overline{x})$. First observe that $\delta_S = \sum_{i=1}^k \delta_{[\Phi_i \leq \lambda_i]}$, hence

$$N_S(\overline{x}) = \partial\delta_S(\overline{x}) = \partial\left(\sum_{i=1}^k \delta_{[\Phi_i \leq \lambda_i]}\right)(\overline{x}).$$

From Theorem 2.1, there exist sequences $(u_{i,n})_n \subset X$, $(u_{i,n}^*)_n \subset X^*$, $i \in \{1, \ldots, k\}$, such that $u_{i,n}^* \in N_{[\Phi_i \leq \lambda_i]}(u_{i,n})$ and

$$u_{i,n} \underset{n\to+\infty}{\longrightarrow} \overline{x}, \quad \sum_{i=1}^k u_{i,n}^* \underset{n\to+\infty}{\longrightarrow} x^* \quad \text{and} \quad \langle u_{i,n}^*, u_{i,n} - \overline{x}\rangle \underset{n\to+\infty}{\longrightarrow} 0. \tag{5.18}$$

In view of Theorem 3.1, we have

$$N_{[\Phi_i \leq \lambda_i]}(u_{i,n}) = \limsup_{\begin{cases} x \to u_{i,n},\ \nu \geq 0 \\ \nu\,(\Phi_i(x) - \lambda_i) \to 0 \end{cases}} \nu\, \partial\Phi_i(x).$$

Hence there exist sequences $(\nu_{i,n,m})_m \subset \mathbb{R}$, $(v_{i,n,m})_m \subset X$ and $(v_{i,n,m}^*)_m \subset X^*$ such that

$$\nu_{i,n,m} \geq 0, \quad v_{i,n,m}^* \in \partial\Phi_i(v_{i,n,m}), \tag{5.19}$$

$$v_{i,n,m} \underset{m\to+\infty}{\longrightarrow} u_{i,n}, \quad \nu_{i,n,m}\, v_{i,n,m}^* \underset{m\to+\infty}{\longrightarrow} u_{i,n}^*, \quad \nu_{i,n,m}\,(\Phi_i(v_{i,n,m}) - \lambda_i) \underset{m\to+\infty}{\longrightarrow} 0. \tag{5.20}$$

From (5.20), there exists an increasing function $\varphi : \mathbb{N} \to \mathbb{N}$ such that for every $i \in \{1, \ldots, k\}$ and every $n \in \mathbb{N}$,

$$\|v_{i,n,\varphi(n)} - u_{i,n}\| \leq \frac{1}{n}, \quad \left|\langle u_{i,n}^*, v_{i,n,\varphi(n)} - u_{i,n}\rangle\right| \leq \frac{1}{n}, \tag{5.21}$$

$$\|\nu_{i,n,\varphi(n)}\, v_{i,n,\varphi(n)}^* - u_{i,n}^*\| \leq \frac{1}{n} \quad \text{and} \quad \left|\nu_{i,n,\varphi(n)}\,(\Phi_i(v_{i,n,\varphi(n)}) - \lambda_i)\right| \leq \frac{1}{n}. \tag{5.22}$$

Let us then set $\mu_{i,n} = \nu_{i,n,\varphi(n)}$, $x_{i,n} = v_{i,n,\varphi(n)}$ and $x^*_{i,n} = v^*_{i,n,\varphi(n)}$. In view of (5.19), we have $\mu_{i,n} \geq 0$ and $x^*_{i,n} \in \partial\Phi_i(x_{i,n})$. By using (5.18), (5.21) and (5.22), we find

$$\begin{aligned}\|x_{i,n} - \overline{x}\| &\leq \|x_{i,n} - u_{i,n}\| + \|u_{i,n} - \overline{x}\| \\ &\leq \frac{1}{n} + \|u_{i,n} - \overline{x}\| \underset{n\to+\infty}{\longrightarrow} 0,\end{aligned}$$

$$\begin{aligned}\left\|\sum_{i=1}^{k} \mu_{i,n}\, x^*_{i,n} - x^*\right\| &\leq \left\|\sum_{i=1}^{k} \mu_{i,n}\, x^*_{i,n} - \sum_{i=1}^{k} u^*_{i,n}\right\| + \left\|\sum_{i=1}^{k} u^*_{i,n} - x^*\right\| \\ &\leq \sum_{i=1}^{k} \left\|\mu_{i,n}\, x^*_{i,n} - u^*_{i,n}\right\| + \left\|\sum_{i=1}^{k} u^*_{i,n} - x^*\right\| \\ &\leq \frac{k}{n} + \left\|\sum_{i=1}^{k} u^*_{i,n} - x^*\right\| \underset{n\to+\infty}{\longrightarrow} 0,\end{aligned}$$

$$\begin{aligned}|\langle \mu_{i,n}\, x^*_{i,n}, x_{i,n} - \overline{x}\rangle| &\leq |\langle \mu_{i,n}\, x^*_{i,n} - u^*_{i,n}, x_{i,n} - u_{i,n}\rangle| \\ &\quad + |\langle \mu_{i,n}\, x^*_{i,n} - u^*_{i,n}, u_{i,n} - \overline{x}\rangle| \\ &\quad + |\langle u^*_{i,n}, x_{i,n} - u_{i,n}\rangle| + |\langle u^*_{i,n}, u_{i,n} - \overline{x}\rangle| \\ &\leq \|\mu_{i,n}\, x^*_{i,n} - u^*_{i,n}\|\, \|x_{i,n} - u_{i,n}\| \\ &\quad + \|\mu_{i,n}\, x^*_{i,n} - u^*_{i,n}\|\, \|u_{i,n} - \overline{x}\| \\ &\quad + |\langle u^*_{i,n}, x_{i,n} - u_{i,n}\rangle| + |\langle u^*_{i,n}, u_{i,n} - \overline{x}\rangle| \\ &\leq \frac{1}{n^2} + \frac{1}{n}\|u_{i,n} - \overline{x}\| + \frac{1}{n} + |\langle u^*_{i,n}, u_{i,n} - \overline{x}\rangle| \underset{n\to+\infty}{\longrightarrow} 0,\end{aligned}$$

and

$$\left|\mu_{i,n}\,(\Phi_i(x_{i,n}) - \lambda_i)\right| \leq \frac{1}{n}.$$

Finally, we have built sequences $(x_{i,n})_n \subset X$, $(x^*_{i,n})_n \subset X^*$, $(\mu_{i,n})_n \subset \mathbb{R}_+$, $i \in \{1, \ldots, k\}$, such that $x^*_{i,n} \in \partial\Phi_i(x_{i,n})$ with $x_{i,n} \underset{n\to+\infty}{\longrightarrow} \overline{x}$, $\sum_{i=1}^k \mu_{i,n}\, x^*_{i,n} \underset{n\to+\infty}{\longrightarrow} x^*$, $\mu_{i,n}\, \langle x^*_{i,n}, x_{i,n} - \overline{x}\rangle \underset{n\to+\infty}{\longrightarrow} 0$ and $\mu_{i,n}\,(\Phi_i(x_{i,n}) - \lambda_i) \underset{n\to+\infty}{\longrightarrow} 0$. Hence we obtain that $x^* \in \limsup\limits_{\begin{cases} x_i \to \overline{x},\ \mu_i \geq 0 \\ \mu_i\,(\Phi_i(x_i) - \lambda_i) \to 0 \\ \mu_i \langle x^*_i, x_i - \overline{x}\rangle \to 0 \end{cases}} \sum_{i=1}^{k} \mu_i\, \partial\Phi_i(x_i)$, which proves the inclusion

$$N_S(\overline{x}) \subset \limsup_{\begin{cases} x_i \to \overline{x},\ \mu_i \geq 0 \\ \mu_i\,(\Phi_i(x_i) - \lambda_i) \to 0 \\ \mu_i \langle x^*_i, x_i - \overline{x}\rangle \to 0 \end{cases}} \sum_{i=1}^{k} \mu_i\, \partial\Phi_i(x_i).$$

The inclusions

$$\limsup_{\begin{cases} x_i \to \overline{x},\ \mu_i \geq 0 \\ \mu_i\,(\Phi_i(x_i) - \lambda_i) \to 0 \\ \mu_i \langle x_i^*, x_i - \overline{x}\rangle \to 0 \end{cases}} \sum_{i=1}^{k} \mu_i\, \partial\Phi_i(x_i)$$

$$\subset \limsup_{\begin{cases} x_i \to \overline{x},\ \mu_i \geq 0 \\ \mu_i\,(\Phi_i(x_i) - \lambda_i - \langle x_i^*, x_i - \overline{x}\rangle) \to 0 \end{cases}} \sum_{i=1}^{k} \mu_i\, \partial\Phi_i(x_i)$$

$$\subset \limsup_{\begin{cases} x_i \to \overline{x},\ \mu_i \geq 0 \\ \sum_{i=1}^{k} \mu_i\,(\Phi_i(x_i) - \lambda_i - \langle x_i^*, x_i - \overline{x}\rangle) \to 0 \end{cases}} \sum_{i=1}^{k} \mu_i\, \partial\Phi_i(x_i)$$

are obvious. It remains to prove that

$$\limsup_{\begin{cases} x_i \to \overline{x},\ \mu_i \geq 0 \\ \sum_{i=1}^{k} \mu_i\,(\Phi_i(x_i) - \lambda_i - \langle x_i^*, x_i - \overline{x}\rangle) \to 0 \end{cases}} \sum_{i=1}^{k} \mu_i\, \partial\Phi_i(x_i) \subset N_S(\overline{x}).$$

For that purpose, let us assume that $x^* \in X^*$ is such that $\sum_{i=1}^{k} \mu_{i,n}\, x_{i,n}^* \underset{n\to+\infty}{\longrightarrow} x^*$ with $\mu_{i,n} \geq 0,\ x_{i,n}^* \in \partial\Phi_i(x_{i,n}),\ \ x_{i,n} \underset{n\to+\infty}{\longrightarrow} \overline{x}$ and

$$\sum_{i=1}^{k} \mu_{i,n} \left(\Phi_i(x_{i,n}) - \lambda_i - \langle x_{i,n}^*, x_{i,n} - \overline{x}\rangle\right) \underset{n\to+\infty}{\longrightarrow} 0. \tag{5.23}$$

Let us fix $x \in S$. Since $x_{i,n}^* \in \partial\Phi_i(x_{i,n})$ and since Φ_i is convex, we have

$$\begin{aligned} \langle x_{i,n}^*, x - x_{i,n}\rangle &\leq \Phi_i(x) - \Phi_i(x_{i,n}) \\ &\leq \lambda_i - \Phi_i(x_{i,n}) \quad \text{because } x \in S. \end{aligned}$$

This can be rewritten as

$$\left\langle x_{i,n}^*, x - \overline{x}\right\rangle \leq \lambda_i - \Phi_i(x_{i,n}) + \langle x_{i,n}^*, x_{i,n} - \overline{x}\rangle.$$

After multiplication by $\mu_{i,n}$ and summation from $i = 1$ to k, we find

$$\left\langle \sum_{i=1}^{k} \mu_{i,n}\, x_{i,n}^*, x - \overline{x} \right\rangle \leq \sum_{i=1}^{k} \mu_{i,n}\,(\lambda_i - \Phi_i(x_{i,n}) + \langle x_{i,n}^*, x_{i,n} - \overline{x}\rangle).$$

Recalling formula (5.23) and the fact that $\sum_{i=1}^{k} \mu_{i,n}\, x_{i,n}^* \underset{n\to+\infty}{\longrightarrow} x^*$, we obtain at the limit as $n \to +\infty$ that $\langle x^*, x - \overline{x}\rangle \leq 0$. Since this is true for every $x \in S$, we conclude that $x^* \in N_S(\overline{x})$. The proof is complete. □

6. Extension to the nonconvex framework

In this section, we assume that $X = \mathbb{R}^N$ and that the function $\Phi : \mathbb{R}^N \to \mathbb{R}$ is locally Lipschitz continuous. Our purpose is to find a suitable extension of the normal cone formula (3.3) in the nonconvex framework. It is worthwhile noticing at first that equality (3.3) may no longer be true for nonconvex functions. Take for example the function $\Phi : \mathbb{R} \to \mathbb{R}$ defined by $\Phi(x) = -x^2$ and let $\overline{x} = 0$. We have

$S = [\Phi \le 0] = \mathbb{R}$ and $N_S(\overline{x}) = N_{\mathbb{R}}(0) = \{0\}$, while

$$\limsup_{x \to \overline{x}} \mathbb{R}_+ \, \partial\Phi(x) = \limsup_{x \to 0} \mathbb{R}_+ \, \Phi'(x) = \mathbb{R},$$

because $\Phi'(x) < 0$ (resp. > 0) for $x > 0$ (resp. $x < 0$).

Consider a function $\Phi : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ and a point $\overline{x}$ with $\Phi(\overline{x})$ finite. We recall that, for a vector $x^* \in \mathbb{R}^N$, one says that

(i) x^* is a *Fréchet subgradient*[5] of Φ at $\overline{x}$, written $x^* \in \partial_{\mathcal{F}}\Phi(\overline{x})$ if

$$\liminf_{x \to \overline{x},\, x \neq \overline{x}} \frac{\Phi(x) - \Phi(\overline{x}) - \langle x^*, x - \overline{x}\rangle}{\|x - \overline{x}\|} \ge 0.$$

(ii) x^* is a *Mordukhovich limiting subgradient* of Φ at $\overline{x}$, written $x^* \in \partial_L\Phi(\overline{x})$ if there are sequences $x_n \to_\Phi \overline{x}$ and $x_n^* \in \partial_{\mathcal{F}}\Phi(x_n)$ with $x_n^* \to x^*$.

Let $C \subset \mathbb{R}^N$ and $\overline{x} \in C$. The Fréchet (resp. limiting) normal cone of C at $\overline{x}$ denoted by $N^{\mathcal{F}}(C;\overline{x})$ (resp. $N^L(C;\overline{x})$) is defined by $N^{\mathcal{F}}(C;\overline{x}) = \partial_{\mathcal{F}}\delta_C(\overline{x})$ (resp. $N^L(C;\overline{x}) = \partial_L\delta_C(\overline{x})$). The set C is said to be *normally regular* at $\overline{x}$ if it is locally closed at $\overline{x}$ and $N^{\mathcal{F}}(C;\overline{x}) = N^L(C;\overline{x})$. In finite dimensions, the normal regularity of a set coincides with the Clarke regularity, requiring the equality between the Clarke tangent cone and the Bouligand tangent cone (see [32]). The concepts are different in any infinite dimensional space (see [12, 28]). A function $\Phi : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ is called *subdifferentially subregular* at $\overline{x} \in \operatorname{dom}\Phi$ if $\operatorname{epi}\Phi$ is normally regular at $(\overline{x}, \Phi(\overline{x}))$ as a subset of $\mathbb{R}^N \times \mathbb{R}$. When the function Φ is locally Lipschitz continuous, the latter condition amounts to the equality $\partial_L\Phi(\overline{x}) = \partial_{\mathcal{F}}\Phi(\overline{x})$. For standard references on nonsmooth analysis, see for example [9, 11, 20, 21, 28, 32].

A powerful sum rule (see [28, 32]) is available for the Fréchet subgradients: If $\Phi_1, \Phi_2 : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ are lower semicontinuous and finite at $\overline{x}$ and if $x^* \in \partial_{\mathcal{F}}(\Phi_1 + \Phi_2)(\overline{x})$, then for every real $\varepsilon > 0$ there exist $x_i \in \mathbb{R}^N$ with $\|x_i - \overline{x}\| \le \varepsilon$ and $|\Phi_i(x_i) - \Phi(\overline{x})| \le \varepsilon$, $i = 1, 2$, such that

$$x^* \in \partial_{\mathcal{F}}\Phi_1(x_1) + \partial_{\mathcal{F}}\Phi_2(x_2) + \varepsilon\mathbb{B}, \tag{6.1}$$

where $\mathbb{B}$ denotes the closed unit ball of $\mathbb{R}^N$. From the latter fuzzy sum rule, we deduce a chain rule for the composition of a locally Lipschitz mapping F with a lower semicontinuous function f. The next theorem is strongly related[6] to [29, Theorem 4.10], and it could be derived as a corollary of this result. For the sake of completeness we provide a self-contained proof.

Theorem 6.1. *Let F be a locally Lipschitz mapping from $\mathbb{R}^N$ into $\mathbb{R}^m$ and $f : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function which is finite at $F(\overline{x})$, where $\bar{x} \in \mathbb{R}^N$. Then for any $x^* \in \partial_L(f \circ F)(\overline{x})$, there exist sequences $(x_n) \subset \mathbb{R}^N$, $(y_n) \subset \mathbb{R}^m$, $(x_n^*) \subset \mathbb{R}^N$ and $(y_n^*) \subset \mathbb{R}^m$ such that $x_n \to \overline{x}$, $y_n \to F(\overline{x})$, $f(y_n) \to f(F(\overline{x}))$, $x_n^* \to x^*$ and*

$$y_n^* \in \partial_{\mathcal{F}} f(y_n) \quad \text{and} \quad x_n^* \in \partial_{\mathcal{F}}(y_n^* \circ F)(x_n).$$

Proof. Consider any $x^* \in \partial_L(f \circ F)(\bar{x})$ and take an open neighborhood V_0 of $\overline{x}$ over which F is Lipschitz with $\gamma \ge 0$ as a Lipschitz constant. Fix any real $\varepsilon > 0$. Taking, if necessary, another positive real less than ε, we may suppose that $B(\overline{x}, 4\varepsilon) \subset V_0$, where $B(\overline{x}, 4\varepsilon)$ denotes the open ball centered at $\overline{x}$ with radius 4ε. There exist $z \in \mathbb{R}^N$ with $\|z - \overline{x}\| \le \varepsilon$ and $|f \circ F(z) - f \circ F(\overline{x})| \le \varepsilon$, and $z^* \in \partial_{\mathcal{F}}(f \circ F)(z)$

[5]It is also called a *lower* or *regular subgradient* in [28, 32].

[6]We thank the referee who indicated to us the reference [29].

with $\|z^* - x^*\| \le \varepsilon$. Put $f_1(x,y) := f(y)$ and $f_2(x,y) := \delta_{\mathrm{gph}\,F}(x,y)$ for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^m$. It is not difficult to see that $(z^*,0) \in \partial_{\mathcal{F}}(f_1+f_2)(z,F(z))$. Both functions f_1 and f_2 are lower semicontinuous. The inclusion (6.1) above combined with the equality $\partial_{\mathcal{F}} f_1(x,y) = \{0\} \times \partial_{\mathcal{F}} f(y)$ says that there exist

$$(x,y) \in B\big((z,F(z)),\varepsilon\big) \text{ with } |f(y) - f(F(z))| \le \varepsilon,$$

$$(z',F(z')) \in B\big((z,F(z)),\varepsilon\big),$$

and

$$(0,y^*) \in \partial_{\mathcal{F}} f_1(x,y), \quad (u^*,-v^*) \in N^{\mathcal{F}}\big(\mathrm{gph}\,F;(z',F(z'))\big)$$

such that

$$(z^*,0) \in (0,y^*) + (u^*,-v^*) + \varepsilon\mathbb{B}, \tag{6.2}$$

hence $z^* \in u^* + \varepsilon\mathbb{B}$ and $v^* \in y^* + \varepsilon\mathbb{B}$. Consequently, we have the inclusion $y^* \in \partial_{\mathcal{F}} f(y)$ with $\|y - F(\overline{x})\| \le \varepsilon(1+\gamma)$ and $|f(y) - f(F(\overline{x}))| \le 2\varepsilon$, along with the equality $v^* = y^* + e^*$ for some $e^* \in \varepsilon\mathbb{B}$. The inclusion $(u^*,-v^*) \in N^{\mathcal{F}}\big(\mathrm{gph}\,F,(z',F(z'))\big)$ means that, for any real $\eta > 0$, there exists some open neighborhood $V \subset V_0$ of z' such that, for all $x' \in V$,

$$\langle u^*, x'-z'\rangle - \langle v^*, F(x') - F(z')\rangle \le \eta\big(\|x'-z'\| + \|F(x') - F(z')\|\big),$$

so

$$\langle u^*, x'-z'\rangle \le (v^* \circ F)(x') - (v^* \circ F)(z') + \eta(1+\gamma)\|x'-z'\|.$$

This guarantees that $u^* \in \partial_{\mathcal{F}}(v^* \circ F)(z')$ and hence $u^* \in \partial_{\mathcal{F}}(y^* \circ F + e^* \circ F)(z')$. Then, the inclusion (6.1) gives $\xi,\zeta \in \mathbb{R}^N$ with $\|\xi - z'\| \le \varepsilon$ and $\|\zeta - z'\| \le \varepsilon$ such that

$$u^* \in \partial_{\mathcal{F}}(y^* \circ F)(\xi) + \partial_{\mathcal{F}}(e^* \circ F)(\zeta) + \varepsilon\mathbb{B}.$$

The inequality $\|\xi - z'\| \le \varepsilon$ combined with the inequalities $\|z' - z\| \le \varepsilon$ and $\|z - \overline{x}\| \le \varepsilon$ entails that $\|\xi - \overline{x}\| \le 3\varepsilon$, and in the same way $\|\zeta - \overline{x}\| \le 3\varepsilon$. Observing that the function $e^* \circ F$ is Lipschitz continuous on $B(\overline{x},4\varepsilon)$ with $\varepsilon\gamma$ as a Lipschitz constant, we have $\partial_{\mathcal{F}}(e^* \circ F)(\zeta) \subset \varepsilon\gamma\,\mathbb{B}$, hence

$$u^* \in \partial_{\mathcal{F}}(y^* \circ F)(\xi) + \varepsilon(1+\gamma)\mathbb{B}.$$

Therefore there exists $\xi^* \in \partial_{\mathcal{F}}(y^* \circ F)(\xi)$ such that $\|\xi^* - u^*\| \le \varepsilon(1+\gamma)$. Since $\|u^* - z^*\| \le \varepsilon$ and $\|z^* - x^*\| \le \varepsilon$, we deduce $\|\xi^* - x^*\| \le \varepsilon(3+\gamma)$. Finally, we have proved that for every $\varepsilon > 0$, there exist $\xi \in \mathbb{R}^N$ with $\|\xi - \overline{x}\| \le 3\varepsilon$, $y \in \mathbb{R}^m$ with $\|y - F(\overline{x})\| \le \varepsilon(1+\gamma)$ and $|f(y) - f(F(\overline{x}))| \le 2\varepsilon$, $y^* \in \partial_{\mathcal{F}} f(y)$ and $\xi^* \in \partial_{\mathcal{F}}(y^* \circ F)(\xi)$ such that $\|\xi^* - x^*\| \le \varepsilon(3+\gamma)$. By taking $\varepsilon = 1/n$, we easily build the sequences satisfying the requirements of the statement. ☐

We can now establish our first nonconvex normal cone formula for the sublevel set.

Theorem 6.2. *Let $\Phi : \mathbb{R}^N \to \mathbb{R}$ be a locally Lipschitz continuous function and let $\overline{x} \in \mathbb{R}^N$ and $S = [\Phi \le \Phi(\overline{x})]$. Then one has*

$$N^L(S;\overline{x}) \subset \limsup_{x \to \overline{x}} \mathbb{R}_+ \partial_{\mathcal{F}} \Phi(x).$$

Proof. As in the proof of Theorem 3.1, put $\lambda := \Phi(\overline{x})$ and define the locally Lipschitz continuous function $F : \mathbb{R}^N \to \mathbb{R}$ by $F(x) = \Phi(x) - \lambda$. The equality $S = \{x \in \mathbb{R}^N,\, F(x) \in \mathbb{R}_-\}$ tells us that $\delta_S = \delta_{\mathbb{R}_-} \circ F$, so

$$N^L(S;\overline{x}) = \partial_L \delta_S(\overline{x}) = \partial_L\left(\delta_{\mathbb{R}_-} \circ F\right)(\overline{x}).$$

Further, the function $\delta_{\mathbb{R}_-}$ is of course lower semicontinuous. Therefore, fixing $x^* \in N^L(S;\overline{x})$, we can apply the above chain rule with $f = \delta_{\mathbb{R}_-}$ and the function F defined above. We obtain the existence of sequences $(x_n) \subset \mathbb{R}^N$, $(y_n) \subset \mathbb{R}$, $(x_n^*) \subset \mathbb{R}^N$, and $(y_n^*) \subset \mathbb{R}$ such that $x_n \to \overline{x}$, $y_n \to \Phi(\overline{x}) - \lambda$, $x_n^* \to x^*$ and

$$\begin{aligned} y_n^* &\in N^{\mathcal{F}}(\mathbb{R}_-; y_n) = N_{\mathbb{R}_-}(y_n), \quad \text{hence} \quad y_n^* \geq 0,\\ x_n^* &\in \partial_{\mathcal{F}}(y_n^*\Phi)(x_n) = y_n^*\, \partial_{\mathcal{F}}\Phi(x_n). \end{aligned}$$

This clearly justifies the desired inclusion. □

When the locally Lipschitz continuous function Φ is subdifferentially subregular, the above result can be strengthened by requiring that the points x converging to $\overline{x}$ stay outside the sublevel set S. To achieve that property, let us start with a result that follows immediately from a nonsmooth version of the Sard theorem.

Lemma 6.3. *Let $\Phi : \mathbb{R}^N \to \mathbb{R}$ be a locally Lipschitz continuous function. Given any $\overline{\lambda} \in \mathbb{R}$, there exists a decreasing sequence $(\lambda_n) \subset \mathbb{R}$ such that $\lim_{n\to+\infty} \lambda_n = \overline{\lambda}$ and such that for every $x \in \Phi^{-1}(\lambda_n)$, Φ is differentiable at x with $\nabla\Phi(x) \neq 0$.*

Proof. Let $\Omega \subset \Phi(\mathbb{R}^N)$ consist of the values $\lambda \in \mathbb{R}$ such that, for every $x \in \Phi^{-1}(\lambda)$, Φ is differentiable at x and $\nabla\Phi(x) \neq 0$. Let us also define the set $\widehat{\Omega}$ by

$$\begin{aligned} \widehat{\Omega} &= \Omega \cup \left[\mathbb{R} \setminus \Phi(\mathbb{R}^N)\right]\\ &= \{\lambda \in \mathbb{R},\, \forall x \in \Phi^{-1}(\lambda),\, \Phi \text{ is differentiable at } x \text{ and } \nabla\Phi(x) \neq 0\}. \end{aligned}$$

Notice that the set $\Phi^{-1}(\lambda)$ may be empty in the definition of $\widehat{\Omega}$. From a Lipschitzian version of the Sard theorem, we know that $\Phi(\mathbb{R}^N) \setminus \Omega$ is negligible; see [27] or [32, Theorem 9.65]. It ensues that $\operatorname{int}\left(\Phi(\mathbb{R}^N) \setminus \Omega\right) = \emptyset$, or equivalently $\operatorname{cl}(\widehat{\Omega}) = \mathbb{R}$. Given any $\overline{\lambda} \in \mathbb{R}$, it is then easy to build a decreasing sequence $(\lambda_n) \subset \widehat{\Omega}$ such that $\lim_{n\to+\infty} \lambda_n = \overline{\lambda}$. □

Under the qualification condition $0 \notin \partial_L\Phi(x)$ and the subdifferential subregularity of the locally Lipschitz continuous function Φ, the following was already known; see for example [20, Theorem 2.4.7] or [32, Proposition 10.3].

Lemma 6.4. *Let $\Phi : \mathbb{R}^N \to \mathbb{R}$ be a locally Lipschitz continuous function. Given $x \in \mathbb{R}^N$, let us consider the level set $S = [\Phi \leq \Phi(x)]$. If Φ is subdifferentially subregular at x with $0 \notin \partial_L\Phi(x)$, then the set S is normally regular at x and $N^L(S;x) = \mathbb{R}_+\, \partial_L\Phi(x)$.*

Let us finally recall a result of approximation for normals; see [32, Exercise 6.18]. It can be seen as a nonconvex version of (ii) in Theorem 2.4.

Lemma 6.5. *Let C and C_n, $n \in \mathbb{N}$, be nonempty closed sets of $\mathbb{R}^N$. Assume that the sequence (C_n) converges toward C in the sense of Painlevé-Kuratowski. Then, one has*

$$\operatorname{gph} N^L(C;\cdot) \subset \liminf_{n\to+\infty}\left(\operatorname{gph} N^L(C_n;\cdot)\right).$$

We can now state the second main result of this section.

Theorem 6.6. *Let $\Phi : \mathbb{R}^N \to \mathbb{R}$ be a locally Lipschitz continuous function, which is subdifferentially subregular near $\overline{x} \in \mathbb{R}^N$. Let us consider the set $S = [\Phi \le \Phi(\overline{x})]$. If $\overline{x} \notin \operatorname{int}(S)$, then we have*[7]

$$N^L(S;\overline{x}) \subset \limsup_{x\to\overline{x},\, x\in D\setminus S} \mathbb{R}_+ \nabla\Phi(x), \tag{6.3}$$

where D denotes the set of $x \in \mathbb{R}^N$ where Φ is differentiable at x.

Proof. Since Φ is locally Lipschitz continuous, Rademacher's theorem asserts that Φ is differentiable almost everywhere on the open set $\mathbb{R}^N \setminus S$. This implies that $D\setminus S$ is dense in $\mathbb{R}^N \setminus S$, i.e. $\operatorname{cl}(D\setminus S) \supset \mathbb{R}^N\setminus S$, and hence $\operatorname{cl}(D\setminus S) \supset \operatorname{cl}(\mathbb{R}^N\setminus S)$. From the assumption $\overline{x} \notin \operatorname{int}(S)$, we have $\overline{x} \in \operatorname{cl}(\mathbb{R}^N \setminus S)$, and therefore $\overline{x} \in \operatorname{cl}(D\setminus S)$. This tells us that the upper limit arising in (6.3) is well defined.

Let us now prove formula (6.3). Observing that $0 \in \limsup_{x\to\overline{x},\, x\in D\setminus S} \mathbb{R}_+ \nabla\Phi(x)$, it suffices to show that

$$N^L(S;\overline{x}) \setminus \{0\} \subset \limsup_{x\to\overline{x},\, x\in D\setminus S} \mathbb{R}_+ \nabla\Phi(x). \tag{6.4}$$

Let us fix $x^* \in N^L(S;\overline{x}) \setminus \{0\}$. Let us set $\overline{\lambda} = \Phi(\overline{x})$ and consider the decreasing sequence $(\lambda_n) \subset \mathbb{R}$ given by Lemma 6.3. Let us define the set $S_n \subset \mathbb{R}^N$ by $S_n = [\Phi \le \lambda_n]$ for every $n \in \mathbb{N}$. The sequence (S_n) is nonincreasing and the sets S_n are closed, hence

$$\lim_{n\to+\infty} S_n = \bigcap_{n\in\mathbb{N}} \operatorname{cl}(S_n) = \bigcap_{n\in\mathbb{N}} S_n = [\Phi \le \Phi(\overline{x})] = S.$$

Lemma 6.5 then shows that there exist a sequence $(x_n) \subset \mathbb{R}^N$ and a sequence $(x_n^*) \subset \mathbb{R}^N$ such that $x_n \in S_n$, $x_n^* \in N^L(S_n;x_n)$, $x_n \to \overline{x}$ and $x_n^* \to x^*$ as $n \to +\infty$. Since $x^* \neq 0$, there exists $n_0 \in \mathbb{N}$ such that $x_n^* \neq 0$ for every $n \ge n_0$. Without loss of generality, we suppose in the sequel that $n \ge n_0$. Let us now prove that $\Phi(x_n) = \lambda_n$. Let us argue by contradiction and assume that $\Phi(x_n) < \lambda_n$. Since Φ is continuous at x_n, we deduce that $x_n \in \operatorname{int}(S_n)$ and hence $N^L(S_n;x_n) = \{0\}$. This implies that $x_n^* = 0$, which gives the contradiction. We have shown that $\Phi(x_n) = \lambda_n$, hence in particular $x_n \notin S$. From the definition of the sequence (λ_n), we have $x_n \in D$ and $\nabla\Phi(x_n) \neq 0$; see Lemma 6.3. Since the function Φ is subdifferentially subregular near $\overline{x}$ by assumption, we infer that, for n large enough, say $n \ge n_1$ with $n_1 \ge n_0$,

$$\partial_L\Phi(x_n) = \partial_{\mathcal{F}}\Phi(x_n) = \{\nabla\Phi(x_n)\}, \quad \text{hence} \quad 0 \notin \partial_L\Phi(x_n).$$

For each $n \ge n_1$, the hypotheses of Lemma 6.4 are satisfied with the point $x = x_n$ and the set $S_n = [\Phi \le \Phi(x_n)]$; thus we obtain that $N^L(S_n;x_n) = \mathbb{R}_+ \nabla\Phi(x_n)$ and hence $x_n^* \in \mathbb{R}_+ \nabla\Phi(x_n)$. As a conclusion, we have built a sequence (x_n) such that $x_n \in D \setminus S$ for every $n \ge n_1$ and $x_n \to \overline{x}$ as $n \to +\infty$, along with a sequence (x_n^*) such that $x_n^* \in \mathbb{R}_+ \nabla\Phi(x_n)$ for every $n \ge n_1$ and $x_n^* \to x^*$ as $n \to +\infty$. This shows that $x^* \in \limsup_{x\to\overline{x},\, x\in D\setminus S} \mathbb{R}_+ \nabla\Phi(x)$, which ends the proof of the inclusion (6.4). □

[7] If $\overline{x} \in \operatorname{int}(S)$, we obviously have $N_S(\overline{x}) = \{0\}$. The assumption $\overline{x} \in \operatorname{int}(S)$ means that $\overline{x}$ is a local maximum for the function Φ.

Remark 6.7. The inclusion (6.3) may be strict if Φ is not convex. Take for example the function $\Phi : \mathbb{R} \to \mathbb{R}$ defined by

$$\Phi(x) = \begin{cases} x^3(1+\sin^2(1/x)) & \text{if} \quad x > 0, \\ 0 & \text{if} \quad x \leq 0, \end{cases}$$

and let $\overline{x} = 0$. It is easy to check that the function Φ is of class $\mathcal{C}^1$ on $\mathbb{R}$, hence Φ is locally Lipschitz continuous and subdifferentially subregular on $\mathbb{R}$. We have $S = \mathbb{R}_-$ and $N^L(S;\overline{x}) = N_{\mathbb{R}_-}(0) = \mathbb{R}_+$. On the other hand, we find

$$\limsup_{x\to\overline{x},\, x\in D\setminus S} \mathbb{R}_+ \nabla\Phi(x) = \limsup_{x\to 0,\, x>0} \mathbb{R}_+ \Phi'(x) = \mathbb{R},$$

since the sign of the derivative $\Phi'(x)$ changes infinitely many times as $x \to 0^+$.

7. Application to the asymptotic study of some gradient-like system

7.1. **On a strict obtuseness condition.** In this section, we assume that $X = \mathbb{R}^N$ and that $S \subset \mathbb{R}^N$ is a nonempty closed convex set. Given $\overline{x} \in S$, we focus our attention on the condition

$$-N_S(\overline{x}) \subset \operatorname{int}(T_S(\overline{x})) \cup \{0\}, \tag{7.1}$$

where $T_S(\overline{x})$ is the tangent cone defined by $T_S(\overline{x}) = \operatorname{cl}[\cup_{\lambda>0}\lambda\,(S-\overline{x})]$. The convex cones $N_S(\overline{x})$ and $T_S(\overline{x})$ are polar to each other, i.e. $N_S(\overline{x}) = [T_S(\overline{x})]^*$ and $T_S(\overline{x}) = [N_S(\overline{x})]^*$. The polar cone K^* of a cone $K \subset \mathbb{R}^N$ is defined by

$$K^* = \{x^* \in \mathbb{R}^N, \quad \forall x \in K, \quad \langle x^*, x\rangle \leq 0\}.$$

A closed convex cone $K \subset \mathbb{R}^N$ is said to be obtuse if $K \supset -K^*$. The condition (7.1) amounts to saying that the cone $T_S(\overline{x})$ is strictly obtuse, which means that the set S is locally strictly obtuse at $\overline{x}$. Assumption (7.1) entails that $\operatorname{int}(S) \neq \emptyset$. Indeed, if $\operatorname{int}(S) = \emptyset$, then we have $\operatorname{int}(T_S(\overline{x})) = \emptyset$. Condition (7.1) then implies that $N_S(\overline{x}) = \{0\}$, which implies in our context that $\overline{x} \in \operatorname{int}(S)$, a contradiction. Condition (7.1) is satisfied in particular if $\overline{x}$ is an interior point of S or if $\overline{x}$ is a boundary point of S such that S is smooth[8] at $\overline{x}$. Before stating the main result, let us establish some preliminary lemmas.

Lemma 7.1. *Let $S \subset \mathbb{R}^N$ be a nonempty closed convex set. Given $\overline{x} \in S$, assume that condition* (7.1) *is satisfied. Then there exists a convex cone $K \subset \mathbb{R}^N$ which is closed and pointed,*[9] *such that*

$$K \subset \operatorname{int}(T_S(\overline{x})) \cup \{0\} \tag{7.2}$$

and

$$-N_S(\overline{x}) \subset \operatorname{int}(K) \cup \{0\}. \tag{7.3}$$

Proof. If $\overline{x} \in \operatorname{int}(S)$, it suffices to take $K = \{0\}$. Now assume that $\overline{x} \in \operatorname{bd}(S)$. Let us define the set K by

$$K = \{x \in \mathbb{R}^N, \quad d(x, -N_S(\overline{x})) \leq d(x, \mathbb{R}^N \setminus T_S(\overline{x}))\}.$$

[8] A convex set S is smooth at $\overline{x} \in \operatorname{bd}(S)$ if there exists $d \neq 0$ such that $N_S(\overline{x}) = \mathbb{R}_+ d$.

[9] A convex cone K is said to be pointed if $K \cap (-K) = \{0\}$.

It is immediate to check that the set K is a closed cone satisfying (7.2)-(7.3). Since $\overline{x} \in \mathrm{bd}\,(S)$, there exists $u \in \mathbb{R}^N \setminus \{0\}$ such that $\mathbb{R}_+ u \subset N_S(\overline{x})$. By polarity, we have $T_S(\overline{x}) \subset \{x \in \mathbb{R}^N,\ \langle x, u\rangle \leq 0\}$, hence

$$K \subset \{x \in \mathbb{R}^N, \quad \langle x, u\rangle < 0\} \cup \{0\}.$$

It ensues that the cone K is pointed. To prove the convexity of the set K, we resort to the following claim.

Claim 7.2. Let $C \subset \mathbb{R}^N$ be a nonempty convex set. Then the following hold:

(i) The function $d(\cdot, C)$ is convex on $\mathbb{R}^N$.

(ii) If $C \neq \mathbb{R}^N$, the function $d(\cdot, \mathbb{R}^N \setminus C)$ is concave on C.

The first point is elementary. The second one is given as an exercise by N. Bourbaki [13, Exercise 18, p. 150]; see also [22]. We deduce from this claim that the function $\Delta = d(., -N_S(\overline{x})) - d(., \mathbb{R}^N \setminus T_S(\overline{x}))$ is convex on $T_S(\overline{x})$. In view of formula (7.2), we have $K \subset T_S(\overline{x})$ and we infer that the set $K = \{x \in T_S(\overline{x}), \Delta(x) \leq 0\}$ is convex as a sublevel set of the convex function Δ. □

Lemma 7.3. *Let $S \subset \mathbb{R}^N$ be a nonempty closed convex set and let $\overline{x} \in S$. Let $K \subset \mathbb{R}^N$ be a closed cone such that $K \subset \mathrm{int}\,(T_S(\overline{x})) \cup \{0\}$. Then there exists a scalar $\lambda > 0$ such that $K \cap \mathbb{B} \subset \lambda\,(\mathrm{int}\,(S) - \overline{x}) \cup \{0\}$.*

Proof. Let us denote by $\mathbb{S}$ the closed unit sphere and let us show that $K \cap \mathbb{S} \subset \lambda\,(\mathrm{int}\,(S) - \overline{x})$ for some $\lambda > 0$. Let us argue by contradiction and assume that there exists a sequence (d_n) such that $d_n \in K \cap\ \mathbb{S}$ and $d_n \notin n\,(\mathrm{int}\,(S) - \overline{x})$. Since the set $K \cap\ \mathbb{S}$ is compact, there exist $d \in K \cap \mathbb{S}$ along with a subsequence of (d_n), still denoted by (d_n), such that $\lim_{n\to+\infty} d_n = d$. By assumption, we have $K \subset \mathrm{int}\,(T_S(\overline{x})) \cup \{0\}$, whence $d \in K \cap \mathbb{S} \subset \mathrm{int}\,(T_S(\overline{x}))$. Recalling that $\mathrm{int}\,(T_S(\overline{x})) = \bigcup_{\lambda>0} \lambda\,(\mathrm{int}\,(S) - \overline{x})$, there exists $\lambda > 0$ such that $d \in \lambda\,(\mathrm{int}\,(S) - \overline{x})$. Since the set $\lambda\,(\mathrm{int}\,(S) - \overline{x})$ is open, there exists $n_0 \in \mathbb{N}$ such that $d_n \in \lambda\,(\mathrm{int}\,(S) - \overline{x})$ for every $n \geq n_0$. By taking $m \geq \max(n_0, \lambda)$, we obtain $d_m \in m\,(\mathrm{int}\,(S) - \overline{x})$, a contradiction. The conclusion follows immediately. □

From now on, we assume that the set S coincides with the set of minima of a lower semicontinuous convex function $\Phi : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$.

Lemma 7.4. *Let $\Phi : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function such that $S = \mathrm{argmin}\,\Phi \neq \emptyset$. Let $\overline{x} \in S$ and assume that there exists an open cone K_0 such that $N_S(\overline{x}) \subset K_0 \cup \{0\}$. Then there exists a neighborhood V of $\overline{x}$ such that $\partial\Phi(x) \subset K_0 \cup \{0\}$ for every $x \in V$.*

Proof. Let us argue by contradiction and assume that there exist a sequence (x_n) tending toward $\overline{x}$ as $n \to +\infty$, along with a sequence (x_n^*) such that $x_n^* \in \partial\Phi(x_n)$ and $x_n^* \notin K_0 \cup \{0\}$ for every $n \in \mathbb{N}$. From the sequence $(x_n^*/\|x_n^*\|)$ we can extract a subsequence, still denoted by $(x_n^*/\|x_n^*\|)$, such that $\lim_{n\to+\infty} x_n^*/\|x_n^*\| = x^*$. We clearly have $x^* \in \limsup_{x\to\overline{x}} \mathbb{R}_+\,\partial\Phi(x)$ and hence $x^* \in N_{[\Phi\leq\Phi(\overline{x})]}(\overline{x}) = N_S(\overline{x})$, in view of the formula (3.3) of Theorem 3.1. Recalling that $N_S(\overline{x}) \subset K_0 \cup \{0\}$ by assumption and observing that $x^* \neq 0$, we deduce that $x^* \in K_0$. On the other hand, since K_0 is a cone, we have $x_n^*/\|x_n^*\| \in \mathbb{R}^N \setminus K_0$ for every $n \in \mathbb{N}$. Taking the limit as $n \to +\infty$, we infer that $x^* \in \mathrm{cl}\,(\mathbb{R}^N \setminus K_0) = \mathbb{R}^N \setminus K_0$, a contradiction. □

By gathering Lemmas 7.1, 7.3 and 7.4, we obtain the following statement.

Theorem 7.5. *Let $\Phi : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function such that $S = \operatorname{argmin} \Phi \neq \emptyset$. Let $\overline{x} \in S$ and assume that*

$$-N_S(\overline{x}) \subset \operatorname{int}(T_S(\overline{x})) \cup \{0\}.$$

Then there exist a scalar $\lambda > 0$ and a convex cone $K \subset \mathbb{R}^N$ which is closed and pointed, along with a neighborhood V of $\overline{x}$ such that

$$K \cap \mathbb{B} \subset \lambda\,(\operatorname{int}(S) - \overline{x}) \cup \{0\} \quad \text{and} \quad -\partial\Phi(x) \subset \operatorname{int}(K) \cup \{0\} \qquad \text{for every } x \in V. \tag{7.4}$$

Proof. Lemma 7.1 furnishes a closed convex cone $K \subset \mathbb{R}^N$ which is pointed and satisfies conditions (7.2)-(7.3). Lemma 7.3 then gives some $\lambda > 0$ such that the first assertion of (7.4) is fulfilled. We finally use Lemma 7.4 with $K_0 = -\operatorname{int}(K)$. We then obtain the existence of a neighborhood V of $\overline{x}$ such that $\partial\Phi(x) \subset -\operatorname{int}(K) \cup \{0\}$ for every $x \in V$, which proves the second assertion of (7.4). ☐

The result of Theorem 7.5 can already be found in [17, Lemma 4.2], but the proof given here is slightly different and relies on the tools that are developed above. In particular, Lemma 7.4 is based on the expression of $N_S(\overline{x})$ as an upper limit of subgradients near $\overline{x}$; see formula (3.3).

7.2. **A second-order in time gradient system with vanishing damping.** Given a convex function $\Phi : \mathbb{R}^N \to \mathbb{R}$ and a map $\gamma \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$, we consider the following differential inclusion:

$$\ddot{x}(t) + \gamma(t)\,\dot{x}(t) + \partial\Phi(x(t)) \ni 0, \qquad t \geq 0. \tag{E}$$

Our purpose is not to develop the questions of existence and uniqueness for solutions to this dynamical system. We will assume in the sequel that there exists a solution $x \in W^{2,1}_{loc}(\mathbb{R}_+, \mathbb{R}^N)$ satisfying (E) almost everywhere.

The decay properties of the function γ play a central role in the asymptotic behavior of (E). In particular, if the quantity $\gamma(t)$ tends to 0 too rapidly as $t \to +\infty$, convergence of the trajectory may fail (think about the extreme case of $\gamma \equiv 0$ for instance). When $\int_0^{+\infty} \gamma(t)\,dt = +\infty$ and $\operatorname{argmin} \Phi \neq \emptyset$, it can be easily proved that $\lim_{t\to+\infty} \Phi(x(t)) = \min \Phi$; see [15, 18] where energy-like arguments are used. In the case of a unique minimum $\overline{x}$, this immediately implies that the solution $x(t)$ converges toward $\overline{x}$ as $t \to +\infty$. The situation is much more complicated when the function Φ has a continuum of minima. Let us first consider the particular case $\Phi \equiv 0$. The differential inclusion (E) then becomes $\ddot{x}(t) + \gamma(t)\,\dot{x}(t) = 0$ and a double integration immediately shows that its solution is given by

$$x(t) = x(0) + \dot{x}(0) \int_0^t e^{-\int_0^s \gamma(u)\,du} ds. \tag{7.5}$$

If $\dot{x}(0) \neq 0$, the solution $x(\cdot)$ converges if and only if the quantity $\int_0^\infty e^{-\int_0^s \gamma(u)\,du} ds$ is finite.

Coming back to the general case of a convex function $\Phi : \mathbb{R}^N \to \mathbb{R}$ with a nonempty set S of minima, the next result shows that if $\int_0^\infty e^{-\int_0^s \gamma(u)\,du} ds = +\infty$, then nonstationary solutions of (E) cannot converge toward a minimum point $\overline{x} \in S$ satisfying the obtuseness condition

$$-N_S(\overline{x}) \subset \operatorname{int}(T_S(\overline{x})) \cup \{0\}. \tag{7.6}$$

This is a generalization of [15, Theorem 4.1].

Theorem 7.6. *Let $\Phi : \mathbb{R}^N \to \mathbb{R}$ be a convex function and let $\overline{x} \in S = \operatorname{argmin} \Phi$ satisfying the obtuseness condition* (7.6)*. Let $\gamma \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$ be a function such that $\int_0^\infty e^{-\int_0^s \gamma(u)\, du} ds = +\infty$. Assume that there exists a solution $x \in W^{2,1}_{loc}(\mathbb{R}_+, \mathbb{R}^N)$ satisfying equation (E) almost everywhere on $[0, +\infty[$. If the map $x(\cdot)$ converges toward $\overline{x}$, then it is stationary.*

Proof. Assume that $\lim_{t\to+\infty} x(t) = \overline{x}$. From Theorem 7.5, there exist a convex cone $K \subset \mathbb{R}^N$, which is closed and pointed, along with $\lambda > 0$ and $t_0 \geq 0$ such that

$$(7.7)\quad K \cap \frac{1}{\lambda}\mathbb{B} \subset (\operatorname{int}(S) - \overline{x}) \cup \{0\} \quad \text{and} \quad -\partial\Phi(x(t)) \subset K \quad \text{for every } t \geq t_0.$$

Let $v \in K^*$. Observing that for almost every $t \geq t_0$

$$\ddot{x}(t) + \gamma(t)\,\dot{x}(t) \in -\partial\Phi(x(t)) \subset K,$$

we deduce that

$$\langle \ddot{x}(t) + \gamma(t)\,\dot{x}(t), v\rangle \leq 0 \qquad \text{a.e. on } [t_0, +\infty[.$$

Let us define the function $p \in W^{2,1}_{loc}(\mathbb{R}_+, \mathbb{R})$ by $p(t) = \langle x(t), v\rangle$. In view of the above inequality, the function p satisfies

$$(7.8)\qquad \ddot{p}(t) + \gamma(t)\,\dot{p}(t) \leq 0 \qquad \text{a.e. on } [t_0, +\infty[.$$

Let us prove that $\dot{p}(t) = \langle \dot{x}(t), v\rangle \geq 0$ for every $t \geq t_0$. Let us argue by contradiction and assume that there exists $t_1 \geq t_0$ such that $\dot{p}(t_1) < 0$. Let us multiply inequality (7.8) by $e^{\int_{t_1}^t \gamma(s)\, ds}$ and integrate on $[t_1, t]$. Recalling that $\dot{p} \in W^{1,1}_{loc}(\mathbb{R}_+, \mathbb{R})$, we obtain

$$\forall t \geq t_1, \quad \dot{p}(t) \leq \dot{p}(t_1)\, e^{-\int_{t_1}^t \gamma(s)\, ds}.$$

By integrating again, we find

$$\forall t \geq t_1, \quad p(t) \leq p(t_1) + \dot{p}(t_1) \int_{t_1}^t e^{-\int_{t_1}^s \gamma(u)\, du} ds.$$

Since $\int_0^{+\infty} e^{-\int_0^s \gamma(u)\, du} ds = +\infty$ and $\dot{p}(t_1) < 0$ by assumption, we deduce that $\lim_{t\to+\infty} p(t) = -\infty$. This contradicts the boundedness of the map $x(\cdot)$, hence we conclude that

$$(7.9)\qquad \forall t \geq t_0, \quad \langle \dot{x}(t), v\rangle \geq 0.$$

By integrating the above inequality on $[t, +\infty[$ and using the fact that $\lim_{t\to+\infty} x(t) = \overline{x}$, we obtain $\langle x(t) - \overline{x}, v\rangle \leq 0$ for every $t \geq t_0$. Since this is true for every $v \in K^*$, we derive that $x(t) - \overline{x} \in K^{**}$ for every $t \geq t_0$. Recalling that $K^{**} = K$ for every closed convex cone K, we conclude that

$$(7.10)\qquad \forall t \geq t_0, \quad x(t) - \overline{x} \in K.$$

By again using $\lim_{t\to+\infty} x(t) = \overline{x}$, we find the existence of $t_1 \geq t_0$ such that $x(t) - \overline{x} \in \frac{1}{\lambda}\mathbb{B}$ for every $t \geq t_1$. In view of (7.7), we infer that $x(t) \in \operatorname{int}(S) \cup \{\overline{x}\}$ for every $t \geq t_1$. Let us now distinguish the following cases:

- (a) For every $t \geq t_1$, we have $x(t) \in \operatorname{int}(S)$.
- (b) There exists $t_2 \geq t_1$ such that $x(t_2) = \overline{x}$.

Case (a). We then have $\partial\Phi(x(t)) = \{0\}$ for every $t \geq t_1$, so that equation (E) becomes

$$\ddot{x}(t) + \gamma(t)\,\dot{x}(t) = 0 \qquad \text{a.e. on } [t_1, +\infty[.$$

Let us prove that $\dot{x}(t) = 0$ for every $t \geq t_1$. Let us argue by contradiction and assume that there exists $t_1^* \geq t_1$ such that $\dot{x}(t_1^*) \neq 0$. The same computation as in (7.5) shows that

$$\forall t \geq t_1^*, \quad x(t) = x(t_1^*) + \dot{x}(t_1^*) \int_{t_1^*}^{t} e^{-\int_{t_1^*}^{s} \gamma(u)\,du} ds.$$

Since $\int_0^{+\infty} e^{-\int_0^s \gamma(u)\,du} ds = +\infty$ and $\dot{x}(t_1^*) \neq 0$ by assumption, we deduce that $\lim_{t\to+\infty} |x(t)| = +\infty$. This contradicts the boundedness of the map $x(\cdot)$, hence we conclude that $\dot{x}(t) = 0$ for every $t \geq t_1$, thus implying that $x(t) = x(t_1)$ for every $t \geq t_1$.

Case (b). Since $x(t_2) = \overline{x}$ by assumption, the integration of inequality (7.9) on $[t_2, t]$ gives $\langle x(t) - \overline{x}, v\rangle \geq 0$ for every $t \geq t_2$. By arguing as above, we obtain that $x(t) - \overline{x} \in -K$ for every $t \geq t_2$. In view of (7.10), we infer that

$$\forall t \geq t_2, \quad x(t) - \overline{x} \in K \cap (-K).$$

Recalling that the cone K is pointed, we have $K \cap (-K) = \{0\}$, hence $x(t) = \overline{x}$ for every $t \geq t_2$.

As a conclusion, we have proved in both cases (a) and (b) that the solution $x(\cdot)$ is stationary, which ends the proof. □

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