

COHOMOLOGICAL SUPPORT LOCI OF VARIETIES OF ALBANESE FIBER DIMENSION ONE

ZHI JIANG AND HAO SUN

ABSTRACT. Let X be a smooth projective variety of Albanese fiber dimension 1 and of general type. We prove that the translates through 0 of all components of $V^0(\omega_X)$ generate $\text{Pic}^0(X)$. We then study the pluricanonical maps of X . We show that $|4K_X|$ induces a birational map of X .

1. INTRODUCTION

In [CH3, Theorem 1], Chen and Hacon proved that if X is of maximal Albanese dimension and of general type, then the translates through 0 of all irreducible components of $V^0(\omega_X)$ generate $\text{Pic}^0(X)$. This is a fundamental result to study pluricanonical maps of varieties of maximal Albanese dimension (cf. [J1, Ti, JLT]). Here we provide a similar result for varieties of general type and of Albanese fiber dimension one:

Theorem 1.1. *Let X be a smooth projective variety of dimension ≥ 2 , of Albanese fiber dimension one, and of general type. Then the translates through 0 of all irreducible components of $V^0(\omega_X)$ generate $\text{Pic}^0(X)$.*

We notice that this kind of result applies only for varieties of Albanese fiber dimension ≤ 1 . Indeed, we take $X = Y \times Z$, where Y is a variety of maximal Albanese dimension and of general type and Z is a variety of general type of dimension $l \geq 2$ with $p_g(Z) = q(Z) = 0$. Then X is of Albanese fiber dimension l but $V^0(\omega_X)$ is empty. As an application of Theorem 1.1, we give an improvement of a result of Chen and Hacon [CH4, Corollary 4.3]:

Theorem 1.2. *Let X be a smooth projective variety of Albanese fiber dimension one and of general type. Then the 4-canonical map $\varphi_{|4K_X|}$ is birational.*

In Section 2, we recall some definitions and results which will be used in this paper. We will prove Theorem 1.1 in Section 3. We then study pluricanonical maps of varieties of Albanese fiber dimension 1 and prove Theorem 1.2 in Section 5.

2. NOTATIONS AND PRELIMINARIES

In this section, we recall some notations and techniques that will be used throughout the paper.

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We will always denote by X a smooth complex projective variety and $a_X : X \rightarrow A_X$ its Albanese morphism. If $l = \dim X - \dim a_X(X)$, then we say that X is of Albanese fiber dimension l . In particular, X is of Albanese fiber dimension 0 if and only if X is of maximal Albanese dimension.

For an abelian variety A , we will frequently denote by $\widehat{A} = \text{Pic}^0(A)$ the dual abelian variety. Moreover, for a morphism $t : A \rightarrow B$ between abelian varieties, we will denote by $\widehat{t} : \widehat{B} \rightarrow \widehat{A}$ the dual morphism between the dual abelian varieties. By abuse of notation, we identify a point of \widehat{A} with the corresponding line bundle of A .

Let $t : X \rightarrow A$ be a morphism from X to an abelian variety and let \mathcal{F} be a coherent sheaf on X , we will denote by $V^i(\mathcal{F}, t) = \{P \in \widehat{A} \mid h^i(X, \mathcal{F} \otimes t^*P) > 0\}$ the i -th cohomological support locus of \mathcal{F} . In particular, if $t = a_X$, we will simply denote $V^i(\mathcal{F}, a_X)$ by $V^i(\mathcal{F})$.

When $\mathcal{F} = \omega_X$, we have the following important theorem of Green-Lazarsfeld and Simpson (see [GL2, Theorem 0.1] and [Si, Theorem 4.2]):

Theorem 2.1. *For any $i \geq 0$, each irreducible component of $V^i(\omega_X, t)$ is a torsion translate of a subtorus of \widehat{A} .*

We recall the following definitions due to Pareschi and Popa ([PP1] and [PP2]).

Definition 2.2. Let \mathcal{F} be a coherent sheaf on a smooth projective variety Y .

- (1) The sheaf \mathcal{F} is said to be a *GV-sheaf* if $\text{codim } V^i(\mathcal{F}) \geq i$ for all $i \geq 0$.
- (2) If Y is an abelian variety, \mathcal{F} is said to be *M -regular* if $\text{codim } V^i(\mathcal{F}) > i$ for every $i > 0$.

Lemma 2.3. *Assume that \mathcal{F} is a GV-sheaf on a smooth projective variety Y . Let W be an irreducible component of $V^0(\mathcal{F})$ such that $k = \text{codim}_{\text{Pic}^0(Y)} W$. Then W is also a component of $V^k(\mathcal{F})$. In particular, $\dim X \geq k$.*

Proof. Please see [PP2, Proposition 3.15]. □

The following definition is also introduced by Pareschi and Popa ([PP1, Definition 2.10]).

Definition 2.4. Let \mathcal{F} be a coherent sheaf on a smooth projective variety Y . We say \mathcal{F} is continuously globally generated at $y \in Y$ (in brief CGG at y) if the natural map

$$\bigoplus_{P \in U} H^0(Y, \mathcal{F} \otimes P) \otimes P^\vee \rightarrow \mathcal{F} \otimes \mathbb{C}(y)$$

is surjective for any non-empty open subset $U \subset \text{Pic}^0(Y)$.

We will need the following theorem ([PP1, Proposition 2.13]):

Theorem 2.5. *Let \mathcal{F} be an M -regular sheaf on an abelian variety A . Then \mathcal{F} is CGG at each point of A .*

The following proposition is a slight generalization of [GL2, Theorem 0.1 (2)].

Proposition 2.6. *Let $\alpha : X \rightarrow A$ be a morphism from a smooth projective variety to an abelian variety such that $\dim X - \dim \alpha(X) = s$. Assume that $V^0(\omega_X, \alpha)$ has an irreducible component $P_0 + \widehat{B}$ of codimension $0 \leq k < \infty$ in \widehat{A} , where P_0 is a*

torsion point and \widehat{B} is an abelian subvariety of \widehat{A} . Then there exists a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & A \\ f \downarrow & & \downarrow \pi \\ X_B & \xrightarrow{g} & B, \end{array}$$

where π is the natural projection, f is a fibration and X_B is a normal variety of dimension $\dim X - s - k$.

Proof. By assumption, $P_0 + \widehat{B}$ is an irreducible component of $V^0(\omega_X, \alpha) = V^0(\alpha_*\omega_X)$. We know that $\alpha_*\omega_X$ is a GV-sheaf on A (see [H, Corollary 4.2]). Hence by Lemma 2.3, $P_0 + \widehat{B}$ is an irreducible component of $V^k(\alpha_*\omega_X)$.

Let $X \xrightarrow{h} X_A \xrightarrow{t} A$ (resp. $X_A \xrightarrow{h_B} X_B \xrightarrow{g} B$) be the Stein factorization of α (resp. the Stein factorization of the composition $\pi \circ t$). We then have the following commutative diagram:

$$\begin{array}{ccccc} & & X & & \\ & & \downarrow h & \searrow \alpha & \\ & & X_A & \xrightarrow{t} & A \\ f \swarrow & & \downarrow h_B & & \downarrow \pi \\ & & X_B & \xrightarrow{g} & B. \end{array}$$

Since t is finite, $H^k(A, \alpha_*\omega_X \otimes Q) = H^k(X_A, h_*\omega_X \otimes t^*Q) \neq 0$, for any $Q \in P_0 + \widehat{B}$. On the other hand, by Kollár's theorem ([Kol2, Theorem 3.4]) we have

$$\begin{aligned} h^k(X_A, h_*\omega_X \otimes t^*(P_0 \otimes P)) &= \sum_{i+j=k} h^i(X_B, R^j h_{B*}(h_*(\omega_X \otimes P_0)) \otimes g^*P) \\ &\neq 0, \end{aligned}$$

for all $P \in \widehat{B}$. For all $j \geq 0$, since P_0 is a torsion line bundle, the sheaves $R^j h_{B*}(h_*(\omega_X \otimes P_0))$ are GV-sheaves (see [HP2, Theorem 2.2]), hence

$$R^k h_{B*}(h_*(\omega_X \otimes P_0)) \neq 0.$$

We conclude again by Kollár's theorem ([Kol2, Theorem 3.4]) that $\dim X_A - \dim X_B \geq k$ and hence the equality holds and $\dim X_B = \dim X - s - k$. \square

3. PROOF OF THEOREM 1.1

Lemma 3.1. *Let Y be a smooth projective variety of general type and of maximal Albanese dimension. Let $t : Y \rightarrow A$ be a morphism to an abelian variety A of dimension ≥ 1 such that $\dim Y - \dim t(Y) = 1$. Then*

$$\dim V^0(\omega_Y, t) \geq 1,$$

where

$$V^0(\omega_Y, t) := \{P \in \widehat{A} \mid H^0(Y, \omega_Y \otimes t^*P) \neq 0\},$$

and $\dim V^0(\omega_Y, t)$ is by definition the maximal dimension of an irreducible component of $V^0(\omega_Y, t)$.

Proof. By the universal property of the Albanese morphism a_Y , we have a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{a_Y} & A_Y \\ & \searrow t & \downarrow \mu \\ & & A. \end{array}$$

Since K_Y is an effective divisor (see [CH2, Lemma 3.1] or [JLT, Lemma 2.3]), we may assume that μ is surjective, otherwise $\ker(\widehat{\mu} : \widehat{A} \rightarrow \widehat{A}_Y)$ is contained in $V^0(\omega_X, t)$ and is of dimension ≥ 1 .

When μ is surjective, $\widehat{\mu} : \widehat{A} \rightarrow \widehat{A}_Y$ is an isogeny onto its image and

$$\dim V^0(\omega_Y, t) = \dim(\widehat{\mu}(\widehat{A}) \cap V^0(\omega_Y)).$$

Hence, if $V^0(\omega_Y) = \widehat{A}_Y$, Lemma 3.1 is clear. Otherwise, $\chi(Y, \omega_Y) = 0$ and $V^0(\omega_Y)$ is a union of torsion translates of proper abelian subvarieties of \widehat{A}_Y and $\dim V^0(\omega_Y) \geq 1$ (see [CH3, Theorem 1]).

Let K be the neutral component of the kernel of $A_Y \xrightarrow{\mu} A$. We then have the exact sequence

$$\widehat{A} \xrightarrow{\widehat{\mu}} \widehat{A}_Y \twoheadrightarrow \widehat{K}.$$

Because an irreducible component of a general fiber of t is a smooth curve of genus ≥ 2 , hence for any $P \in \widehat{A}_Y$, $t_*(\omega_X \otimes P)$ there exists a non-zero GV -sheaf on A by [H, Corollary 4.2]¹. Therefore, for any $P \in \widehat{A}_Y$, there exists a line bundle $P' \in \widehat{A}$ such that

$$H^0(Y, \omega_Y \otimes P \otimes t^*P') \simeq H^0(A, t_*(\omega_Y \otimes P) \otimes P') \neq 0.$$

Hence the composition of morphisms $V^0(\omega_Y) \hookrightarrow \widehat{A}_Y \rightarrow \widehat{K}$ is surjective. There exists an irreducible component $Q + \widehat{B}$ of $V^0(\omega_Y)$, where Q is a torsion point of \widehat{A}_Y and \widehat{B} is an abelian subvariety of \widehat{A}_Y , such that the composition of the natural morphisms

$$Q + \widehat{B} \hookrightarrow \widehat{A}_Y \rightarrow \widehat{K}$$

is surjective and then $(Q + \widehat{B}) \cap \widehat{\mu}(\widehat{A})$ is not empty.

If $\dim V^0(\omega_Y, t) = 0$, then

$$(1) \quad \dim(\widehat{\mu}(\widehat{A}) \cap V^0(\omega_Y)) = 0.$$

Hence

$$\dim((Q + \widehat{B}) \cap \widehat{\mu}(\widehat{A})) = 0,$$

and the projection $Q + \widehat{B} \rightarrow \widehat{K}$ is finite. Therefore, $\text{codim}_{\widehat{A}_Y}(Q + \widehat{B}) = \dim \widehat{A} \geq \dim Y - 1$.

On the other hand, we know by the proof of Theorem 3 in [EL] that $Q + \widehat{B}$ is also an irreducible component of $V^{\dim \widehat{A}}(\omega_Y)$. Hence $\dim \widehat{A} = \dim Y - 1$ and the

¹Hacon shows that $t_*\omega_X$ is a GV -sheaf and the twisted version follows in the same way by combining [H, Theorem 1.2] and [Kol4, Corollary 10.15].

natural morphism $A_Y \xrightarrow{(p,\mu)} B \times A$ is an isogeny. We then consider

$$\begin{array}{ccccc} Y & \xrightarrow{a_Y} & A_Y & \xrightarrow{\mu} & A \\ & \searrow p_B & \downarrow p & & \\ & & B & & \end{array}$$

By Proposition 2.6, the image $p_B(Y)$ is of dimension 1. We take the Stein factorization of p_B :

$$Y \xrightarrow{\gamma} Z \rightarrow B.$$

Since a general fiber F of γ is of general type and $(\mu \circ a_Y)|_F : F \rightarrow A$ is generically finite over its image, by [CH3, Theorem 1], there exists a positive dimensional component $T \subset V^0(\omega_F, (\mu \circ a_Y)|_F)$. Moreover, if $P \in T$ is a torsion line bundle, $\gamma_*(\omega_{Y/Z} \otimes a_Y^* \mu^* P)$ is a non-zero nef vector bundle on Z (see [V2, Corollary 3.6]²).

If Z is a smooth curve of genus ≥ 2 , then by Riemann-Roch, we deduce that

$$H^0(Y, \omega_Y \otimes a_Y^* \mu^* P) \neq 0,$$

for any torsion line bundle $P \in T$. Hence by semi-continuity,

$$\widehat{\mu}(T) \subset \widehat{\mu}(\widehat{A}) \cap V^0(\omega_Y) \hookrightarrow \widehat{A}_Y,$$

which contradicts (1).

If Z is an elliptic curve, then since Z generates B , Z is isogenous to B . Then $Q + \widehat{B}$ is an irreducible component of $V^0(\omega_Y)$ of dimension 1. This is impossible by Proposition 3.6 in [CDJ].

We then conclude the proof of Lemma 3.1. □

Proposition 3.2. *Let X be a smooth projective variety of general type. Assume that X is of Albanese fiber dimension 1. Then $\dim V^0(\omega_X) \geq 1$.*

Proof. We take the Stein factorization of a_X :

$$\begin{array}{ccc} X & & \\ g \downarrow & \searrow a_X & \\ W & \xrightarrow{f} & A_X. \end{array}$$

We denote $\dim X = n \geq 2$ and denote by $m \geq 2$ the genus of a general fiber of g .

Since X is of Albanese fiber dimension 1 and is of general type, $a_{X*}\omega_X$ is a non-zero coherent sheaf on A_X and hence is a non-zero GV -sheaf by [H, Corollary 4.2]. Hence by [H, Corollary 3.2], we have

$$V^0(a_{X*}\omega_X) \supset V^1(a_{X*}\omega_X) \supset \dots \supset V^n(a_{X*}\omega_X).$$

Therefore, $V^0(\omega_X) = V^0(a_{X*}\omega_X) \neq \emptyset$ by [CH3, Lemma 2.1].

We then argue by contradiction. Assume that $V^0(\omega_X) = V^0(a_{X*}\omega_X)$ is a union of finite number of points. Since $a_{X*}\omega_X$ is a GV -sheaf and $V^0(a_{X*}\omega_X)$ is a union of finite number of points, by [Mu, Example 3.2], $a_{X*}\omega_X$ is a homogeneous vector bundle on A_X . Hence $V^{\dim A_X}(a_{X*}\omega_X) \neq \emptyset$; therefore, $\dim A_X = \dim a_X(X) = n - 1$.

²This is stated there for P to be trivial, and the general case follows by the étale covering trick.

There are three steps to deduce a contradiction.

In the first step, we claim that a_X is a fibration. Hence f is an isomorphism and $R^1 a_{X*} \omega_X = \mathcal{O}_{A_X}$.

We take a general complete intersection curve C on A_X . By performing a base change via the morphism $C \hookrightarrow A_X$ to the above diagram, we have

$$\begin{array}{ccc} X_C & & \\ g_C \downarrow & \searrow h_C & \\ W_C & \xrightarrow{f_C} & C. \end{array}$$

By a Bertini type theorem (see [BS, Theorem 1.7.1]), both X_C and W_C are smooth projective varieties. Then $h_{C*} \omega_{X_C/C} = a_{X*} \omega_X|_C$.

If $\deg f > 1$, since a_X is the universal morphism to an abelian variety, f is ramified in codimension 1. Hence $f_C : W_C \rightarrow C$ is a ramified cover. We know that $g_{C*} \omega_{X_C/W_C}$ is a nef vector bundle on W_C (see [V2]). Then by Riemann-Roch, we conclude that

$$\deg h_{C*} \omega_{X_C/C} = \deg f_{C*} ((g_{C*} \omega_{X_C/W_C}) \otimes \omega_{W_C/C}) > 0$$

which contradicts the fact that $\deg(a_{X*} \omega_X|_C) = 0$. We finish the proof of the first step.

We then write $a_{X*} \omega_X = \bigoplus_{i=1}^s \mathcal{V}_{P_i}$, where \mathcal{V}_{P_i} is the tensor product of $P_i \in \text{Pic}^0(A_X)$ with a unipotent vector bundle on A_X and $\sum_{i=1}^s \text{rank}(\mathcal{V}_i) = m$. Notice that each P_i is a torsion line bundle (see [Si]).

In the second step, we claim that $P_i = \mathcal{O}_X$ for each i . In other words, each \mathcal{V}_i is a unipotent vector bundle.

We still argue by contradiction to prove this claim.

If there exists a non-trivial P_i , then $H^{n-1}(A_X, \mathcal{V}_{P_i} \otimes P_i^*) \neq 0$ and hence $H^{n-1}(X, \omega_X \otimes P_i^*) \neq 0$. Take the étale cover $\pi : \tilde{A}_X \rightarrow A_X$ induced by the group generated by all P_j . Let $\pi_X : \tilde{X} \rightarrow X$ be the induced étale cover. We notice that

$$q(\tilde{X}) = h^{n-1}(\tilde{X}, \omega_{\tilde{X}}) \geq h^{n-1}(X, \omega_X) + h^{n-1}(X, \omega_X \otimes P_i^*) > q(X).$$

We have the following commutative diagram:

$$\begin{array}{ccccc} & & \tilde{X} & \xrightarrow{\pi_X} & X \\ & \searrow^{a_{\tilde{X}}} & \downarrow \tilde{a}_X & & \downarrow a_X \\ & & \tilde{A}_X & \xrightarrow{\pi} & A_X \\ & \swarrow_{a_{\tilde{X}}(\tilde{X})} & \uparrow v & & \\ A_{\tilde{X}} & \xrightarrow{\mu} & \tilde{A}_X & & \end{array}$$

Since \tilde{a}_X is a fibration, either τ is generically finite and \tilde{X} is then of maximal Albanese dimension, or τ is a fibration and v is an isomorphism. The latter is impossible, since $a_{\tilde{X}}(\tilde{X})$ generates the whole abelian variety $A_{\tilde{X}}$. In the first case, \tilde{X} is of maximal Albanese dimension and \tilde{a}_X is surjective of relative dimension 1. We then apply Lemma 3.1 and deduce that $\dim V^0(\omega_{\tilde{X}}, \tilde{a}_X) \geq 1$. This is impossible because $\dim V^0(\omega_{\tilde{X}}, \tilde{a}_X) = \dim V^0(\omega_X) = 0$.

We then conclude the proof of the second step.

In the third step, we are going to deduce a contradiction. Since \mathcal{Y}_i is a unipotent vector bundle, $h^{n-1}(A_X, \mathcal{Y}_i) \geq 1$. We then have ([Kol2, Theorem 3.1])

$$\begin{aligned} n - 1 = h^{n-1}(X, \omega_X) &= h^{n-1}(A_X, a_{X*}\omega_X) + h^{n-2}(A_X, R^1 a_{X*}\omega_X) \\ &= \sum_{i=1}^s h^{n-1}(A_X, \mathcal{Y}_i) + h^{n-2}(A_X, \mathcal{O}_{A_X}) \\ &\geq s + n - 1, \end{aligned}$$

which is impossible.

We then conclude the proof of Proposition 3.2. □

We are now able to prove a slightly more general version of Theorem 1.1:

Theorem 3.3. *Let X be a smooth projective variety of general type and let $\alpha : X \rightarrow A$ be a morphism to an abelian variety such that $\dim \alpha(X) = \dim X - 1$. Then the translates through 0 of all irreducible components of $V^0(\omega_X, \alpha)$ generate \widehat{A} .*

Proof. By Theorem 2.1, we may write

$$V^0(\omega_X, \alpha) = \cup_i (Q_i + \widehat{B}_i),$$

where $Q_i \in \widehat{A}$ is a torsion point and \widehat{B}_i is an abelian subvariety of \widehat{A} for each i . Let $\widehat{T} \hookrightarrow \widehat{A}$ be the abelian subvariety generated by all \widehat{B}_i 's.

Assume that \widehat{T} is a proper subvariety of \widehat{A} . We consider the dual and get

$$\begin{array}{ccc} F & \xrightarrow{h} & K \\ \downarrow & & \downarrow j \\ X & \xrightarrow{\alpha(X) \subset} & A \\ & \searrow f & \downarrow p \\ & & f(X) \subset T \end{array}$$

Let F be an irreducible component of a general fiber of f , let K be the translate of the kernel of p containing $\alpha(F)$, and let $h : F \rightarrow K$ be the restriction of α to F . Notice that $\dim F - \dim h(F) = 1$. We now study the locus

$$V^0(\omega_F, h) := \{P \in \widehat{K} \mid H^0(F, \omega_F \otimes h^*P) \neq 0\}.$$

We know that for any $P \in \widehat{A}$, the sheaf $f_*(\omega_X \otimes P)$ is GV on T . Hence $V^0(\omega_X, \alpha)$ is mapped onto $V^0(\omega_F, h)$ via the morphism $\widehat{A} \xrightarrow{\widehat{j}} \widehat{K}$. Since \widehat{T} is generated by all \widehat{B}_i 's, for each irreducible component $Q_i + \widehat{B}_i$ of $V^0(\omega_X, \alpha)$, $\widehat{j}(Q_i + \widehat{B}_i)$ is a torsion point of \widehat{K} . Hence $V^0(\omega_F, h)$ is a union of finite number of points.

On the other hand, we consider

$$\begin{array}{ccc} & \xrightarrow{h} & \\ F & \xrightarrow{a_F} A_F & \xrightarrow{\pi} K. \end{array}$$

Since $\dim V^0(\omega_F, h) = 0$, then $\widehat{\pi}(\widehat{K}) \cap V^0(\omega_F)$ is a union of finite number of points in \widehat{A}_F and $\widehat{\pi} : \widehat{K} \rightarrow \widehat{A}_F$ is an isogeny onto its image. By Proposition 3.2, $\widehat{\pi}$ is not surjective. Since an irreducible component of a general fiber of h is a smooth projective curve of genus > 1 , as we have seen in the proof of Lemma 3.1, the composition of morphisms $V^0(\omega_F) \hookrightarrow \widehat{A}_F \rightarrow \widehat{A}_F/\widehat{\pi}(\widehat{K})$ is surjective. Hence there exists a positive dimensional irreducible component $P_0 + \widehat{B}$ of $V^0(\omega_F)$ of codimension equal to $\dim \widehat{\pi}(\widehat{K}) \geq \dim F - 1$.

If a_F is generically finite, we can simply apply Lemma 3.1 to get a contradiction. If F is of Albanese fiber dimension 1, we apply Proposition 2.6 to conclude that the image of the natural morphism $F \xrightarrow{a_F} A_F \rightarrow B$ is a point, which is impossible since a_F is the Albanese morphism of F .

Hence $\widehat{T} = \widehat{A}$ and we are done. \square

For surfaces X of Albanese fiber dimension 1 and of general type, since $\chi(X, \omega_X) > 0$ (see [B, Theorem X.4]), we always have $V^0(\omega_X, a_X) = \text{Pic}^0(X)$. In higher dimensions, we can construct varieties of Albanese fiber dimension 1 and of general type with $V^0(\omega_X, a_X)$ a proper subset of \widehat{A}_X in the same way as in the construction of Ein-Lazarsfeld threefold [EL, Example 1.13].

Example 3.4. Let $C_i \rightarrow E_i$ be ramified double covers over elliptic curves E_i for $i = 1, 2$, with associated involutions ι_i . Let $C_3 \rightarrow \mathbb{P}^1$ be a ramified double cover, with associated involution ι_3 . Let X be a desingularization of the quotient $(C_1 \times C_2 \times C_3)/(\iota_1, \iota_2, \iota_3)$. Hence X is of general type and the natural morphism $X \rightarrow E_1 \times E_2$ is the Albanese morphism of X , hence X is of Albanese fiber dimension 1. Moreover,

$$V^0(\omega_X, a_X) = (\{\mathcal{O}_{E_1}\} \times \widehat{E}_2) \cup (\widehat{E}_1 \times \{\mathcal{O}_{E_2}\}).$$

We can check that $\chi(X, \omega_X) = -(g(C_1) - 1)(g(C_2) - 1) < 0$, while the holomorphic Euler characteristic is always ≥ 0 for varieties of maximal Albanese dimension ([GL1]). Furthermore, we see that the intersection of the two irreducible components of $V^0(\omega_X)$ is 0, while we know that if Y is a variety of general type and of maximal Albanese dimension, then the intersection of any two maximal components of $V^0(\omega_Y)$ is of dimension ≥ 1 ([J2, Proposition 1.7]).

4. NON-GENERAL TYPE CASE

4.1. In this subsection, we assume that X is a smooth projective variety of Kodaira dimension $\kappa(X) \geq 0$ and $\alpha : X \rightarrow A$ is a morphism from X to an abelian variety. We denote by $I_X : X \rightarrow I(X)$ a model of the Iitaka fibration of X with $I(X)$ a smooth projective variety. Let F be a general fiber of I_X . Then F is a smooth projective variety of Kodaira dimension 0. By Kawamata's theorem ([Kaw, Theorem 1]), $\alpha(F)$ is a translate of an abelian subvariety K of A_X (independent of the choice of F).

We consider the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & A \\ I_X \downarrow & & \downarrow \text{pr} \\ I(X) & \xrightarrow{\alpha_I} & A/K. \end{array}$$

The following lemma is probably known to experts, but for lack of references, we give a proof here.

Lemma 4.1. *For any irreducible component $Q + \widehat{B}$ of $V^0(\omega_X, \alpha)$, where Q is a torsion point of \widehat{A} and \widehat{B} is an abelian subvariety of \widehat{A} , we have $\widehat{B} \subset \widehat{\text{pr}}(\widehat{A}/\widehat{K})$. In particular, if the translates through 0 of all irreducible components of $V^0(\omega_X, \alpha)$ generate \widehat{A} , then α factors through the Itaka fibration I_X of X .*

Proof. We argue by contradiction. We consider the exact sequence

$$0 \rightarrow \widehat{A}/\widehat{K} \xrightarrow{\widehat{\text{pr}}} \widehat{A} \xrightarrow{\lambda} \widehat{K} \rightarrow 0.$$

For a general fiber F of I_X , we notice that $\lambda(V^0(\omega_X, \alpha)) \subset V^0(\omega_F, \alpha|_F)$. If there exists an irreducible component $Q + \widehat{B}$ of $V^0(\omega_X, \alpha)$ such that $\widehat{B} \not\subset \widehat{\text{pr}}(\widehat{A}/\widehat{K})$, then $\lambda(\widehat{B})$ is a positive dimensional abelian of subvariety \widehat{K} . Hence $V^0(\omega_F, \alpha|_F) \supset \lambda(Q) + \lambda(\widehat{B})$ has positive dimension. Moreover, we know by Kawamata's theorem ([Kaw, Theorem 1]) that $\alpha|_F : F \rightarrow \alpha(F)$ is an algebraic fiber space. Hence the natural morphism

$$(\alpha|_F)^* : \text{Pic}^0(\alpha(F)) = \widehat{K} \rightarrow \text{Pic}^0(F)$$

is an embedding. Let $M \geq 1$ be the order of $\lambda(Q)$ in \widehat{K} . Then by repeating Clifford's lemma (see for instance [HP1, Lemma 2.14]), we have

$$P_{2M}(F) = \dim H^0(F, \omega_F^{2M}) \geq (2M - 1) \dim \lambda(\widehat{B}) + 1 > 1,$$

which contradicts the fact that $\kappa(F) = 0$. \square

4.2. Surfaces of Albanese fiber dimension 1 and of Kodaira dimension 0 are called hyperelliptic surfaces. They are completely classified: there are seven families of hyperelliptic surfaces (see [B, Chapter VI]). In higher dimensions, it was proved by Kawamata ([Kaw, Theorem 15]) that a variety of Albanese fiber dimension 1 and of Kodaira dimension 0 is birationally an étale fiber bundle over its Albanese variety. It is interesting to observe that the elementary method which we use to prove Proposition 3.2 gives an easy proof of this fact and it does not rely on the difficult addition theorem of Viehweg ([V1, Theorem I]).

Lemma 4.2. *Let F be a smooth projective variety of Albanese fiber dimension 1 and of Kodaira dimension 0. Then there exists an abelian variety B , and an elliptic curve E such that F is birational to $(B \times E)/H$, where H is a group of translations of B acting on E such that $E/H = \mathbb{P}^1$. See below the classification of H .*

Proof. Since F is a smooth projective variety of Albanese fiber dimension 1 and of Kodaira dimension 0, the Albanese morphism a_F of F is an algebraic fiber space (see [Kaw, Theorem 1]) and a general fiber of a_F is a curve of genus 1. We then consider the rank 1 sheaf $a_{F*}\omega_F$. Since $\dim V^0(a_{F*}\omega_F) = \dim V^0(\omega_F, a_F) = 0$ and $a_{F*}\omega_F$ is a GV -sheaf on A_F , we conclude that $a_{F*}\omega_F = P$ is a torsion line bundle on A_F .

Let $n = \dim F$, then $q(F) = \dim A_F = n - 1$. Moreover,

$$q(F) = h^{n-1}(A_F, a_{F*}\omega_F) + h^{n-2}(A_F, R^1 a_{F*}\omega_F) = h^{n-1}(A_F, a_{F*}\omega_F) + n - 1.$$

Hence $P \neq \mathcal{O}_F$. Let G be the subgroup of \widehat{A}_F generated by P . Then by considering the étale cover $\pi : \widetilde{F} \rightarrow F$ induced by G , we find that $q(\widetilde{F}) = n$. We now consider the Albanese morphism $a_{\widetilde{F}} : \widetilde{F} \rightarrow A_{\widetilde{F}}$, which is naturally G -equivariant. Since the Kodaira dimension of \widetilde{F} is also 0, by [Kaw, Theorem 1], we know that \widetilde{F} is

birational to its Albanese variety $A_{\tilde{F}}$. Therefore, G also acts faithfully on $A_{\tilde{F}}$. Considering the following commutative diagram

$$\begin{array}{ccc} \tilde{F} & \xrightarrow{\pi} & F \\ a_{\tilde{F}} \downarrow & & \downarrow \tau \\ A_{\tilde{F}} & \xrightarrow{\tilde{\pi}} & F' := A_{\tilde{F}}/G, \end{array}$$

since both π and $\tilde{\pi}$ are of degree $|G|$ and $a_{\tilde{F}}$ is birational, τ is also a birational morphism. Notice that $F' = A_{\tilde{F}}/G$ has, at worst, quotient singularities. Hence τ induces an isomorphism $\pi_1(F) \rightarrow \pi_1(F')$ (see [Kol3, Theorem 7.8]) and we have

$$\pi_1(F')/\tau_*\pi_*(\pi_1(\tilde{F})) \simeq \pi_1(F)/\pi_*(\pi_1(\tilde{F})) \simeq G.$$

Then the morphism $\tau \circ \pi : \tilde{F} \rightarrow F'$ factors through the étale cover

$$\tilde{\pi}' : \tilde{F}' \rightarrow F'$$

induced by the quotient $\pi_1(F')/\tau_*\pi_*(\pi_1(\tilde{F})) \simeq G$.

Hence by the universal property of the Stein factorization of $\tau \circ \pi$, we conclude that $A_{\tilde{F}} \simeq \tilde{F}'$ and $\tilde{\pi}$ is an étale morphism. Therefore, F' is a smooth projective variety and is a good minimal model of F .

Now we may and will assume that $\tilde{F} \simeq A_{\tilde{F}}$ and $F \simeq F'$. Since the composition of morphisms

$$\tilde{F} \xrightarrow{\pi} F \xrightarrow{a_F} A_F$$

is a quotient of abelian varieties and π is an étale morphism, a_F is a smooth morphism. Hence by [B, Proposition VI.8]³, there exists an étale cover $B \rightarrow A_F$ and an elliptic curve E such that $F \simeq (B \times E)/H$, where H is the Galois group of $B \rightarrow A_F$. Moreover, by the same proof of [B, Lemma VI.10], we may assume that H acts on $B \times E$ diagonally. If $q(F) = \dim F - 1 \geq 2$, the action of H on E is one of the following (by the same method of [B, List VI.20]):

- (1) $H = \mathbb{Z}/2\mathbb{Z}$ acting on E by symmetry;
- (2) $H = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ acting on E by $x \rightarrow -x$, $x \rightarrow x + \epsilon$ ($\epsilon \in E_2$, where E_2 is the group of points of E of order 2);
- (3) $H = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ acting on E by $x \rightarrow -x$, $x \rightarrow x + \epsilon_1$, $x \rightarrow x + \epsilon_2$ ($\epsilon_1, \epsilon_2 \in E_2$);
- (4) $H = \mathbb{Z}/4\mathbb{Z}$ acting on $E = E_i := \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}i)$ by $x \rightarrow ix$;
- (5) $H = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ acting on $E = E_i$ by $x \rightarrow ix$, $x \rightarrow x + \frac{1+i}{2}$;
- (6) $H = \mathbb{Z}/3\mathbb{Z}$ acting on $E = E_\rho := \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\rho)$ (where $\rho^3 = 1$) by $x \rightarrow \rho x$;
- (7) $H = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ acting on $E = E_\rho$ by $x \rightarrow \rho x$, $x \rightarrow x + \frac{1-\rho}{3}$;
- (8) $H = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ acting on $E = E_\rho$ by $x \rightarrow \rho x$, $x \rightarrow x + \frac{1-\rho}{3}$, $x \rightarrow x - \frac{1-\rho}{3}$;
- (9) $H = \mathbb{Z}/6\mathbb{Z}$ acting on $E = E_\rho$ by $x \rightarrow -\rho x$.

We notice that the cases (3) and (8) do not occur when $\dim F = 2$. □

³We notice that [B, Proposition VI.8] applies directly when A_F is an elliptic curve and the same proof works when A_F is an abelian variety.

5. PLURICANONICAL MAPS

In this section we deduce from Theorem 1.1 a quick proof of the birationality of the 4-canonical map of varieties of general type and of Albanese fiber dimension 1, which generalizes a result of Chen and Hacon [CH4, Corollary 4.3]. We use the strategy described in [PP2, Section 6], which is used by the authors of [J1] and [Ti] to study pluricanonical maps of varieties of maximal Albanese dimension.

Throughout this section we will always denote by the following commutative diagram

$$(2) \quad \begin{array}{ccc} X & & \\ \downarrow f & \searrow a_X & \\ Y & \xrightarrow{t} & A_X \end{array}$$

a modification of the Stein factorization of the Albanese morphism of X such that Y is a smooth projective variety. We denote by F a general fiber of f and, for a point $z \in Y$, we denote by F_z the fiber of f over z . We will say the Iitaka model of (X, K_X) dominates Y if there exists an integer $M > 0$ and an ample divisor A on Y such that $H^0(X, MK_X - f^*A) \neq 0$ (see [M, Proposition 1.14]).

Lemma 5.1. *Assume that X is a smooth projective variety and the Iitaka model of (X, K_X) dominates Y . Assume that K_F is semi-ample. Then the co-support of the multiplier ideal $\mathcal{J}(\|kK_X\|)$ is disjoint from a general fiber F for any $k \geq 1$.*

Proof. For simplicity we let $k = 1$. The argument works for all $k \geq 1$.

Since the Iitaka model of (X, K_X) dominates Y , we can take an ample divisor A on Y and an integer $M > 0$ such that there exists an effective divisor $D \in |MK_X - f^*A|$. By the subadditivity theorem (see [La, Corollary 11.2.4]), we have

$$\mathcal{J}(\|K_X\|)^m \supset \mathcal{J}(\|mK_X\|),$$

for any integer $m > 0$. Therefore we have for any $m > M$,

$$\begin{aligned} \mathcal{J}(\|K_X\|)^m \supset \mathcal{J}(\|mK_X\|) &= \mathcal{J}(\|(m - M)K_X + f^*A + D\|) \\ &\supset \mathcal{J}(\|(m - M)K_X + f^*A\|) \otimes \mathcal{O}_X(-D). \end{aligned}$$

On the other hand, Viehweg’s weak positivity theorem (see [V1, Theorem III]) implies that for $N \gg 0$, the restriction

$$H^0(X, \mathcal{O}_X(N(m - M)K_{X/Y} + Nf^*A)) \rightarrow H^0(F, \mathcal{O}_F(N(m - M)K_F))$$

is surjective. Since Y is of maximal Albanese dimension, K_Y is an effective divisor. Thus for all $N \gg 0$ the restriction

$$H^0(X, \mathcal{O}_X(N(m - M)K_X + Nf^*A)) \rightarrow H^0(F, \mathcal{O}_F(N(m - M)K_F))$$

is surjective.

Therefore we conclude by [La, Theorem 9.5.35] that for a general fiber F ,

$$\mathcal{J}(\|(m - M)K_X + f^*A\|)|_F = \mathcal{J}(\|(m - M)K_F\|) \simeq \mathcal{O}_F,$$

where the last isomorphism holds because K_F is semi-ample.

Combining all the isomorphisms, we get $\mathcal{J}(\|K_X\|)^m|_F \supset \mathcal{O}_F(-D)$ for all $m > M$. Hence $\mathcal{J}(\|K_X\|)|_F = \mathcal{O}_F$ and the co-support of the multiplier ideal $\mathcal{J}(\|K_X\|)$ is disjoint from a general fiber F . \square

We employ a trick of Tirabassi ([Ti]) to show that

Proposition 5.2. *Under the setting of diagram (2), we assume, moreover, that*

- 1) $2K_F$ is globally generated;
- 2) the translates through 0 of all irreducible components of $V^0(\omega_X)$ generate $\text{Pic}^0(X)$;

then there exists an open dense subset U of X such that for any $x \in U$, the sheaf

$$a_{X*}(\mathcal{I}_x \otimes \mathcal{O}_X(2K_X) \otimes \mathcal{J}(\|\omega_X\|))$$

is M -regular.

Proof. We notice that 1) implies that K_F is semi-ample and, by Lemma 4.1, 2) implies that the Iitaka model of (X, K_X) dominates Y . Hence we conclude by Lemma 5.1 that the co-support of $\mathcal{J}(\|\omega_X\|)$ does not dominate Y . Hence there exists an open dense subset Y_0 of Y such that $V := f^{-1}Y_0$ is disjoint from the co-support of $\mathcal{J}(\|\omega_X\|)$, $f : V \rightarrow Y_0$ is smooth, t is finite on Y_0 , and $2K_F$ is globally generated for any fiber F of f over Y_0 . Then for any $x \in V$, we have an exact sequence

$$\begin{aligned} 0 \rightarrow a_{X*}(\mathcal{I}_x \otimes \mathcal{O}_X(2K_X) \otimes \mathcal{J}(\|\omega_X\|)) &\rightarrow a_{X*}(\mathcal{O}_X(2K_X) \otimes \mathcal{J}(\|\omega_X\|)) \\ &\rightarrow \mathbb{C}_{a_X(x)} \rightarrow 0. \end{aligned}$$

By a variant of Nadel vanishing theorem (see [Kol4, 10.15] or [J3, Lemma 2.1]), we have $H^i(A_X, a_{X*}(\mathcal{O}_X(2K_X) \otimes \mathcal{J}(\|\omega_X\|)) \otimes P) = 0$, for any $i \geq 1$ and $P \in \text{Pic}^0(A_X)$. Hence, to conclude the proof of the proposition, all we need to prove is that

$$\text{codim}_{\widehat{A}_X}(V^1(a_{X*}(\mathcal{I}_x \otimes \mathcal{O}_X(2K_X) \otimes \mathcal{J}(\|\omega_X\|)))) \geq 2.$$

We observe from the above exact sequence that for $x \in V$,

$$V^1(a_{X*}(\mathcal{I}_x \otimes \mathcal{O}_X(2K_X) \otimes \mathcal{J}(\|\omega_X\|))) = \{P \in \widehat{A}_X \mid x \in \text{Bs}(|2K_X + P|)\}.$$

Now we write

$$V^0(\omega_X, a_X) = \bigcup_i (Q_i + \widehat{B}_i).$$

Let

$$V' = X - \bigcup_i \cap_{P \in Q_i + \widehat{B}_i} \text{Bs}(|K_X + P|).$$

Then for each $x \in V'$, we can choose an open dense subset \mathcal{U}_i of $Q_i + \widehat{B}_i$ for each i such that for any $P \in \bigcup_i \mathcal{U}_i$, x is not a base point of $|K_X + P|$. Then, x is not a base point of $|2K_X + Q|$, for any $Q \in \bigcup_i (\mathcal{U}_i + \mathcal{U}_i) = \bigcup_i (2Q_i + \widehat{B}_i)$.

Let $U = V \cap V'$. We then conclude that for any $x \in U$,

$$V^1(a_{X*}(\mathcal{I}_x \otimes \mathcal{O}_X(2K_X) \otimes \mathcal{J}(\|\omega_X\|))) \cap \bigcup_i (2Q_i + \widehat{B}_i) = \emptyset.$$

According to hypothesis 2), all the \widehat{B}_i 's generate the whole abelian variety \widehat{A}_X , hence we have (see, for instance, [Ti, Lemma 2])

$$\text{codim}_{\widehat{A}_X}(V^1(a_{X*}(\mathcal{I}_x \otimes \mathcal{O}_X(2K_X) \otimes \mathcal{J}(\|\omega_X\|)))) \geq 2. \quad \square$$

Now we are ready to prove the following theorem, which is slightly more general than Theorem 1.2.

Theorem 5.3. *Let X be a smooth projective variety of general type and of Albanese fiber dimension 1. Then, for any $P \in \text{Pic}^0(X)$, the map φ_{4K_X+P} induced by the linear system $|4K_X + P|$ is a birational map of X .*

Proof. We use the setting of diagram (2).

By Proposition 5.2, we can take an open dense subset U of Y and an open dense subset $V \subset f^{-1}U$ such that $f^{-1}U$ is disjoint from the co-support of $\mathcal{J}(|\omega_X|)$, $f : f^{-1}U \rightarrow U$ is smooth, t is finite on U , and for any $x \in V$, $a_{X*}(\mathcal{I}_x \otimes \omega_X^2 \otimes \mathcal{J}(|\omega_X|))$ is M -regular and thus is continuously globally generated on A_X by Theorem 2.5.

Any fiber F of f over U is a smooth projective curve of genus $g := g(F) \geq 2$. We notice that $|2K_F|$ induces an isomorphism of F if $g \geq 3$, and if $g = 2$, then F is a hyperelliptic curve with hyperelliptic involution τ_F and the linear system $|2K_F|$ induces the quotient morphism of F by τ_F .

We see from the choice of U that, for any $t \in U$, the natural map

$$f_*(\mathcal{O}_X(2K_X) \otimes \mathcal{J}(|\omega_X|)) \otimes \mathbb{C}_t \rightarrow H^0(F_t, \mathcal{O}_X(2K_X) \otimes \mathcal{J}(|\omega_X|)) \simeq H^0(F_t, \omega_{F_t}^2)$$

is an isomorphism.

For two different points $x, y \in V$, we denote respectively by $F_{f(x)}$ and $F_{f(y)}$ the fibers of f over $f(x)$ and $f(y)$.

By pushing forward via f the short exact sequence

$$0 \rightarrow \mathcal{I}_{x,y} \otimes \omega_X^2 \otimes \mathcal{J}(|\omega_X|) \rightarrow \mathcal{I}_x \otimes \omega_X^2 \otimes \mathcal{J}(|\omega_X|) \rightarrow \mathbb{C}_y \rightarrow 0,$$

we get the long exact sequence

$$(3) \quad 0 \rightarrow f_*(\mathcal{I}_{x,y} \otimes \omega_X^2 \otimes \mathcal{J}(|\omega_X|)) \rightarrow f_*(\mathcal{I}_x \otimes \omega_X^2 \otimes \mathcal{J}(|\omega_X|)) \xrightarrow{m_{x,y}} f_*\mathbb{C}_y \simeq \mathbb{C}_{f(y)} \rightarrow \dots$$

Step 1. For any two different points $x, y \in V$, $m_{x,y}$ is surjective, unless $g = 2$, $f(x) = f(y)$, and $y = \tau_{F_{f(x)}}(y)$.

The argument to prove the claim is contained in ([CH4, Step 2 of the proof of Theorem 4.2]). For the reader's convenience, we repeat it here.

If $f(x) \neq f(y)$, then

$$f_*(\mathcal{I}_x \otimes \omega_X^2 \otimes \mathcal{J}(|\omega_X|)) \otimes \mathbb{C}_{f(y)} \simeq H^0(F_{f(y)}, \omega_{F_{f(y)}}^2).$$

Since $\omega_{F_{f(y)}}^2$ is base point free, $m_{x,y}$ is surjective.

If $f(x) = f(y)$, we denote by $t = f(x)$ and notice that

$$\begin{aligned} f_*(\mathcal{I}_x \otimes \omega_X^2 \otimes \mathcal{J}(|\omega_X|)) \otimes \mathbb{C}_t \\ = \ker(f_*(\mathcal{O}_X(2K_X) \otimes \mathcal{J}(|\omega_X|)) \otimes \mathbb{C}_t \simeq H^0(F_t, \omega_{F_t}^2) \xrightarrow{ev_x} \mathbb{C}_x), \end{aligned}$$

and

$$\begin{aligned} f_*(\mathcal{I}_{x,y} \otimes \omega_X^2 \otimes \mathcal{J}(|\omega_X|)) \otimes \mathbb{C}_t \\ = \ker(f_*(\mathcal{O}_X(2K_X) \otimes \mathcal{J}(|\omega_X|)) \otimes \mathbb{C}_t \simeq H^0(F_t, \omega_{F_t}^2) \xrightarrow{(ev_x, ev_y)} \mathbb{C}_x \oplus \mathbb{C}_y). \end{aligned}$$

If $g \geq 3$ or $g = 2$ and $\tau_{F_t}(x) \neq y$, the linear system $|2K_{F_t}|$ separates x and y and hence (ev_x, ev_y) is surjective. Thus the inclusion

$$f_*(\mathcal{I}_{x,y} \otimes \omega_X^2 \otimes \mathcal{J}(|\omega_X|)) \otimes \mathbb{C}_t \hookrightarrow f_*(\mathcal{I}_x \otimes \omega_X^2 \otimes \mathcal{J}(|\omega_X|)) \otimes \mathbb{C}_t$$

is strict. Hence $m_{x,y}$ is again surjective.

Step 2. If $g \geq 3$, φ_{4K_X+P} is birational for any $P \in \text{Pic}^0(X)$.

Since $g \geq 3$, for any two different points $x, y \in V$, we have by Step 1 the following short exact sequence:

$$0 \rightarrow f_*(\mathcal{I}_{x,y} \otimes \omega_X^2 \otimes \mathcal{I}(\|\omega_X\|)) \rightarrow f_*(\mathcal{I}_x \otimes \omega_X^2 \otimes \mathcal{I}(\|\omega_X\|)) \xrightarrow{m_{x,y}} \mathbb{C}_{f(y)} \rightarrow 0.$$

Notice that t is finite on U and $a_X = t \circ f$. Hence we have the following short exact sequence on A_X :

$$(4) \quad 0 \rightarrow a_{X*}(\mathcal{I}_{x,y} \otimes \omega_X^2 \otimes \mathcal{I}(\|\omega_X\|)) \rightarrow a_{X*}(\mathcal{I}_x \otimes \omega_X^2 \otimes \mathcal{I}(\|\omega_X\|)) \xrightarrow{ev_{x,y}} \mathbb{C}_{a_X(y)} \rightarrow 0.$$

Because $a_{X*}(\mathcal{I}_x \otimes \omega_X^2 \otimes \mathcal{I}(\|\omega_X\|))$ is continuously globally generated, there exists an open dense subset $\mathcal{U}_{x,y}$ of $\text{Pic}^0(X)$ such that for any $Q \in \mathcal{U}_{x,y}$ the map

$$H^0(A_X, a_{X*}(\mathcal{I}_x \otimes \omega_X^2 \otimes \mathcal{I}(\|\omega_X\|)) \otimes Q) \xrightarrow{ev_{x,y}} \mathbb{C}_{a_X(y)}$$

is surjective. In other words, for any $Q \in \mathcal{U}_{x,y}$, there exists a global section

$$\begin{aligned} s_Q &\in H^0(A_X, a_{X*}(\mathcal{I}_x \otimes \omega_X^2 \otimes \mathcal{I}(\|\omega_X\|)) \otimes Q) \\ &\simeq H^0(X, \mathcal{I}_x \otimes \omega_X^2 \otimes \mathcal{I}(\|\omega_X\|) \otimes Q) \\ &\subset H^0(X, \mathcal{I}_x \otimes \omega_X^2 \otimes Q), \end{aligned}$$

which does not vanish at y , and hence the divisor $D_Q \in |2K_X + Q|$ defined by s_Q passes through x but does not pass through y . Therefore, for any $P \in \mathcal{U}_{x,y} + \mathcal{U}_{x,y} = \text{Pic}^0(X)$, the linear system $|4K_X + P|$ separating x and y . The map φ_{4K_X+P} is an injective morphism on V and we are done.

Step 3. If $g = 2$, φ_{4K_X+P} is also birational for any $P \in \text{Pic}^0(X)$.

Since $g = g(F) = 2$, the generic fiber of f is a hyperelliptic curve. Hence there exists a birational involution τ of X inducing τ_F in a general fiber F . After birational modifications of X (see for instance [Pr, 4.2]), we may assume that τ is a biregular involution of the smooth projective variety X . Then each connected component of the fixed loci of τ is smooth. After blowing up the fixed loci of τ , we may assume that each connected component of the fixed loci of τ is a smooth divisor of X and hence $Z := X/\langle \tau \rangle$ is also smooth. We choose U and V as before. Moreover, we may assume that $\tau(V) = V$. Now we have the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Z \\ & \searrow f & \downarrow g \\ & & Y, \end{array}$$

where φ is the quotient morphism induced by τ .

By Step 1, for two points $x, y \in V$ with $\tau(x) \neq y$, $m_{x,y}$ in the long exact sequence (3) is surjective. Hence the arguments in Step 2 show that φ_{4K_X+P} separates any two different points $x, y \in V$ with $\tau(x) \neq y$.

Therefore φ factors birationally through φ_{4K_X+P} and $\deg \varphi_{4K_X+P} \leq 2$. In order to finish the proof, we just need to prove that φ_{4K_X+P} does not factor birationally through φ .

Because τ induces the hyperelliptic involution on a general fiber of f , a general fiber L of g is isomorphic to \mathbb{P}^1 . Since φ is a finite morphism of degree 2 between smooth projective varieties, there exists a divisor D on Z such that

$$(5) \quad \begin{aligned} \varphi_* \mathcal{O}_X &= \mathcal{O}_Z \oplus \mathcal{O}_Z(-D), \\ \omega_X &= \varphi^*(\omega_Z \otimes \mathcal{O}_Z(D)). \end{aligned}$$

Moreover, $\deg(D|_L) = 3$.

By (5), we have

$$\varphi_* \mathcal{O}_X(4K_X) = \mathcal{O}_Z(4K_Z + 4D) \oplus \mathcal{O}_Z(4K_Z + 3D),$$

where $\mathcal{O}_Z(4K_Z + 4D)$ (resp. $\mathcal{O}_Z(4K_Z + 3D)$) is the τ -invariant part (resp. τ -anti-invariant part) of $\varphi_* \mathcal{O}_X(4K_X)$. Then there exist ideal sheaves \mathcal{I}_1 and \mathcal{I}_2 on Z such that

$$\begin{aligned} \varphi_*(\mathcal{O}_X(4K_X) \otimes \mathcal{I}(\|3K_X\|)) &= (\mathcal{O}_Z(4K_Z + 4D) \otimes \mathcal{I}_1) \\ &\oplus (\mathcal{O}_Z(4K_Z + 3D) \otimes \mathcal{I}_2). \end{aligned}$$

Because $\deg(\omega_Z|_L) = -2$, both $g_*(\mathcal{O}_Z(4K_Z + 4D))$ and $g_*(\mathcal{O}_Z(4K_Z + 3D))$ are non-zero torsion-free sheaves on Y . Since the co-support of $\mathcal{I}(\|3K_X\|)$ does not dominate Y , the sheaves $g_*(\mathcal{O}_Z(4K_Z + 4D) \otimes \mathcal{I}_1)$ and $g_*(\mathcal{O}_Z(4K_Z + 3D) \otimes \mathcal{I}_2)$ are also non-zero. Thus, since

$$t_* f_*(\mathcal{O}_X(4K_X) \otimes \mathcal{I}(\|3K_X\|))$$

is an IT^0 sheaf on A_X , we conclude that the sheaves

$$t_* g_*(\mathcal{O}_Z(4K_Z + 4D) \otimes \mathcal{I}_1) \quad \text{and} \quad t_* g_*(\mathcal{O}_Z(4K_Z + 3D) \otimes \mathcal{I}_2)$$

are non-zero IT^0 sheaves on A_X .

Therefore, for any $P \in \widehat{A}_X$, $H^0(Z, \mathcal{O}_Z(4K_Z + 3D) \otimes P) \neq 0$. Since $\mathcal{O}_Z(4K_Z + 3D) \otimes P$ is the τ -anti-invariant part of $\varphi_* \mathcal{O}_X(4K_X) \otimes P$, a non-zero global section of $\mathcal{O}_Z(4K_Z + 3D) \otimes P$ separates x and $\tau(x)$, for $x \in V$ general. Thus φ_{4K_X+P} does not factor birationally through φ and φ_{4K_X+P} is birational. \square

Remark 5.4. It is well known that the tricanonical map of a smooth projective curve of genus ≥ 2 is an isomorphism. Recently, the authors of [JLT] have shown that the tricanonical map is birational for a variety of maximal Albanese dimension and of general type. These lead the authors to believe that the tricanonical map of a variety of Albanese fiber dimension 1 and of general type is also birational.

By the same method in the proof of Theorem 5.3, it is quite easy to show that the tricanonical map is birational if $V^0(\omega_X) = \text{Pic}^0(X)$. Hence it leaves to analyze the case when $V^0(\omega_X) \neq \text{Pic}^0(X)$. One might want to mimic the argument in [JLT]. For each irreducible component $P + \widehat{B}$ of $V^0(\omega_X)$ of codimension k in $\text{Pic}^0(X)$, we do have a natural fibration by Proposition 2.6:

$$\begin{array}{ccc} X & & \\ \downarrow f & \searrow a_X & \\ Y & \xrightarrow{t} & A_X \\ \downarrow f_B & & \downarrow \\ X_B & \longrightarrow & B, \end{array}$$

where X_B is of dimension $\dim X - k - 1$. The problem here is that we don't have a good geometric explanation for the existence of $P + \widehat{B}$ unlike the irreducible component of V^0 of a variety of maximal Albanese dimension (see [CDJ, Theorem 3.1]). But the sheaf $R^k f_{B*}(f_*\omega_X \otimes P)$ does have certain positive properties over X_B and may be crucial to a proof of the birationality of the tricanonical map.

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MATHÉMATIQUES BÂTIMENT 425, UNIVERSITÉ PARIS-SUD, F-91405 ORSAY, FRANCE
E-mail address: Zhi.Jiang@math.u-psud.fr

DEPARTMENT OF MATHEMATICS, HUAZHONG NORMAL UNIVERSITY, WUHAN 430079, PEOPLE'S REPUBLIC OF CHINA
E-mail address: hsun@mail.ccnu.edu.cn

Current address: Department of Mathematics, Shanghai Normal University, Shanghai 200234, People's Republic of China