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ADDING A LOT OF COHEN REALS BY ADDING A FEW. I

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ABSTRACT. In this paper we produce models $V_1 \subseteq V_2$ of set theory such that adding κ-many Cohen reals to V_2 adds λ-many Cohen reals to V_1, for some $\lambda > \kappa$. We deal mainly with the case when V_1 and V_2 have the same cardinals.

1. Introduction

A basic fact about Cohen reals is that adding λ-many Cohen reals cannot produce more than λ-many Cohen reals.[1] More precisely, if $\langle s_\alpha : \alpha < \lambda \rangle$ are λ-many Cohen reals over V, then in $V[\langle s_\alpha : \alpha < \lambda \rangle]$ there are no λ^+-many Cohen reals over V. But if instead of dealing with one universe V we consider two; then the above may no longer be true.

The purpose of this paper is to produce models $V_1 \subseteq V_2$ such that adding κ-many Cohen reals to V_2 adds λ-many Cohen reals to V_1, for some $\lambda > \kappa$. We deal mainly with the case when V_1 and V_2 have the same cardinals.

2. Models with the same reals

In this section we produce models $V_1 \subseteq V_2$ as above with the same reals. We first state a general result.

Theorem 2.1. *Let V_1 be an extension of V. Suppose that in V_1:*

(a) $\kappa < \lambda$ are infinite cardinals,

(b) λ is regular,

(c) there exists an increasing sequence $\langle \kappa_n : n < \omega \rangle$ of regular cardinals cofinal in κ; in particular, $cf(\kappa) = \omega$,

(d) there exists an increasing (mod finite) sequence $\langle f_\alpha : \alpha < \lambda \rangle$ of functions in $\prod_{n<\omega}(\kappa_{n+1} \setminus \kappa_n)$,[2] and

(e) there exists a club $C \subseteq \lambda$ which avoids points of countable V-cofinality.

Then adding κ-many Cohen reals over V_1 produces λ-many Cohen reals over V.

Proof. We consider two cases.

Case $\lambda = \kappa^+$. Force to add κ-many Cohen reals over V_1. Split them into two sequences of length κ denoted by $\langle r_\imath : \imath < \kappa \rangle$ and $\langle r'_\imath : \imath < \kappa \rangle$. Also let

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[1]By "λ-many Cohen reals" we mean "a generic object $\langle s_\alpha : \alpha < \lambda \rangle$ for the poset $\mathbb{C}(\lambda)$ of finite partial functions from $\lambda \times \omega$ to 2".

[2]Note that condition (d) holds automotically for $\lambda = \kappa^+$; given any collection $\mathcal{F}$ of κ-many elements of $\prod_{n<\omega}(\kappa_{n+1} \setminus \kappa_n)$ there exists f such that for each $g \in \mathcal{F}$, $f(n) > g(n)$ for all large n. Thus we can define, by induction on $\alpha < \kappa^+$, an increasing (mod finite) sequence $\langle f_\alpha : \alpha < \kappa^+ \rangle$ in $\prod_{n<\omega}(\kappa_{n+1} \setminus \kappa_n)$.

$\langle f_\alpha : \alpha < \kappa^+\rangle \in V_1$ be an increasing (mod finite) sequence in $\prod_{n<\omega}(\kappa_{n+1} \setminus \kappa_n)$. Let $\alpha < \kappa^+$. We define a real s_α as follows:

Case 1. $\alpha \in C$. Then

$$\forall n < \omega,\ s_\alpha(n) = r_{f_\alpha(n)}(0).$$

Case 2. $\alpha \notin C$. Let α^* and α^{**} be two successive points of C so that $\alpha^* < \alpha < \alpha^{**}$. Let $\langle \alpha_\imath : \imath < \kappa\rangle$ be some fixed enumeration of the interval (α^*, α^{**}). Then for some $\imath < \kappa$, $\alpha = \alpha_\imath$. Let $k(\imath) = \min\{k < \omega : r'_\imath(k) = 1\}$. Set

$$\forall n < \omega,\ s_\alpha(n) = r_{f_\alpha(k(\imath)+n)}(0).$$

The following lemma completes the proof.

Lemma 2.2. *$\langle s_\alpha : \alpha < \kappa^+\rangle$ is a sequence of κ^+-many Cohen reals over V.*

Notation 2.3. For each set I, let $\mathbb{C}(I)$ be the Cohen forcing notion for adding I-many Cohen reals. Thus $\mathbb{C}(I)=\{p : p$ is a finite partial function from $I\times\omega$ into $2\}$, ordered by reverse inclusion.

Proof. First note that $\langle\langle r_\imath : \imath < \kappa\rangle, \langle r'_\imath : \imath < \kappa\rangle\rangle$ is $\mathbb{C}(\kappa)\times\mathbb{C}(\kappa)$-generic over V_1. By the c.c.c. of $\mathbb{C}(\kappa^+)$ it suffices to show that for any countable set $I \subseteq \kappa^+$, $I \in V$, the sequence $\langle s_\alpha : \alpha \in I\rangle$ is $\mathbb{C}(I)$-generic over V. Thus it suffices to prove the following:

$(*)$ for every $(p, q) \in \mathbb{C}(\kappa)\times\mathbb{C}(\kappa)$ and every open dense subset $D \in V$ of $\mathbb{C}(I)$, there is $(\overline{p}, \overline{q}) \leq (p, q)$ such that $(\overline{p}, \overline{q}) \Vdash$ " $\langle \underset{\sim}{s}_\alpha : \alpha \in I\rangle$ extends some element of D".

Let (p, q) and D be as above. For simplicity suppose that $p = q = \emptyset$. By (e) there are only finitely many $\alpha^* \in C$ such that $I \cap [\alpha^*, \alpha^{**}) \neq \emptyset$, where $\alpha^{**} = \min(C\setminus(\alpha^*+1))$. For simplicity suppose that there are two $\alpha_1^* < \alpha_2^*$ in C with this property. Let $n^* < \omega$ be such that for all $n \geq n^*$, $f_{\alpha_1^*}(n) < f_{\alpha_2^*}(n)$. Let $p \in \mathbb{C}(\kappa)$ be such that

$$\mathrm{dom}(p) = \{\langle \beta, 0\rangle : \exists n < n^*(\beta = f_{\alpha_1^*}(n) \text{ or } \beta = f_{\alpha_2^*}(n))\}.$$

Then for $n < n^*$ and $j \in \{1, 2\}$,

$$(p, \emptyset)\Vdash \underset{\sim}{s}_{\alpha_j^*}(n) = \underset{\sim}{r}_{f_{\alpha_j^*}(n)}(0) = p(f_{\alpha_j^*(n)}, 0).$$

Thus $(p, \emptyset)$ decides $s_{\alpha_1^*} \restriction n^*$ and $s_{\alpha_2^*} \restriction n^*$. Let $b \in D$ be such that $\langle b(\alpha_1^*), b(\alpha_2^*)\rangle$ extends $\langle s_{\alpha_1^*} \restriction n^*, s_{\alpha_2^*} \restriction n^*\rangle$, where $b(\alpha)$ is defined by $b(\alpha) : \{n : (\alpha, n) \in \mathrm{dom}(b)\} \longrightarrow 2$ and $b(\alpha)(n) = b(\alpha, n)$. Let

$$p' = p \cup \bigcup_{j\in\{1,2\}} \{\langle f_{\alpha_j^*}(n), 0, b(\alpha_j^*, n)\rangle : n \geq n^*, (\alpha_j^*, n) \in \mathrm{dom}(b)\}.$$

Then $p' \in \mathbb{C}(\kappa)$[3] and

$$(p', \emptyset)\Vdash \text{``}\langle \underset{\sim}{s}_{\alpha_1^*}, \underset{\sim}{s}_{\alpha_2^*}\rangle \quad \text{extends} \quad \langle b(\alpha_1^*), b(\alpha_2^*)\rangle\text{''}.$$

[3]This is because for $n \geq n^*$, $f_{\alpha_1^*}(n) \neq f_{\alpha_2^*}(n)$, and for $j \in \{1, 2\}$, $f_{\alpha_j^*}(n) \notin \{f_{\alpha_j^*}(m) : m < n\}$; thus there are no collisions.

For $j \in \{1,2\}$, let $\{\alpha_{j_0},\dots,\alpha_{jk_j-1}\}$ be an increasing enumeration of components of b in the interval $(\alpha_j^*,\alpha_j^{**})$ (i.e. those $\alpha \in (\alpha_j^*,\alpha_j^{**})$ such that $(\alpha,n)\in \mathrm{dom}(b)$ for some n). For $j\in\{1,2\}$ and $l<k_j$ let $\alpha_{jl}=\alpha_{\imath_{jl}}$, where $\imath_{jl}<\kappa$ is the index of α_{jl} in the enumeration of the interval $(\alpha_j^*,\alpha_j^{**})$ considered in Case 2 above. Let $m^*<\omega$ be such that for all $n\ge m^*$, $j\in\{1,2\}$ and $l_j<l_j'<k_j$ we have

$$f_{\alpha_1^*}(n)<f_{\alpha_{1\ell_1}}(n)<f_{\alpha_{1\ell_1'}}(n)<f_{\alpha_2^*}(n)<f_{\alpha_{2\ell_2}}(n)<f_{\alpha_{2\ell_2'}}(n).$$

Let

$$\overline{q}=\{\langle \imath_{jl},n,0\rangle : j\in\{1,2\}, l<k_j, n<m^*\}\cup\{\langle \imath_{jl},m^*,1\rangle : j\in\{1,2\}, l<k_j\}.$$

Then $\overline{q}\in\mathbb{C}(\kappa)$ and for $j\in\{1,2\}$ and $n<m^*$, $(\emptyset,\overline{q})\Vdash$"$r'_{\imath_{jl}}(n)=0$ and $r'_{\imath_{jl}}(m^*)=1$"; thus $(\emptyset,\overline{q})\Vdash k(j,l)=\min\{k<\omega : r'_{\imath_{jl}}(k)=1\}=m^*$. Let

$$\overline{p}=p'\cup\bigcup_{j\in\{1,2\}}\{\langle f_{\alpha_{jl}}(m^*+n),0,b(\alpha_{jl},n)\rangle : l<k_j, (\alpha_{jl},n)\in\mathrm{dom}(b)\}.$$

It is easily seen that $\overline{p}\in\mathbb{C}(\kappa)$ is well defined, and for $j\in\{1,2\}$ and $l<k_j$,

$$(\overline{p},\overline{q})\Vdash \text{“}\underset{\sim}{s}_{\alpha_{jl}} \text{ extends } b(\alpha_{jl})\text{”}.$$

Thus

$$(\overline{p},\overline{q})\Vdash \text{“}\langle \underset{\sim}{s}_{\alpha} : \alpha\in I\rangle \text{ extends } b\text{”}.$$

$(*)$ follows and we are done. □

Case $\lambda>\kappa^+$. Force to add κ-many Cohen reals over V_1. We now construct λ-many Cohen reals over V as in the above case using C and $\langle f_\alpha : \alpha<\lambda\rangle$. Case 2 of the definition of $\langle s_\alpha : \alpha<\lambda\rangle$ is now problematic since the cardinality of an interval (α^*,α^{**}) (using the above notation) may now be above κ and we have only κ-many Cohen reals to play with. Let us proceed as follows in order to overcome this.

Let us rearrange the Cohen reals as $\langle r_{n,\alpha} : n<\omega, \alpha<\kappa\rangle$ and $\langle r_\eta : \eta\in[\kappa]^{<\omega}\rangle$. We define by induction on levels a tree $T\subseteq[\lambda]^{<\omega}$, its projection $\pi(T)\subseteq[\kappa]^{<\omega}$, and for each $n<\omega$ and $\alpha\in Lev_n(T)$, a real s_α. The union of the levels of T will be λ, so $\langle s_\alpha : \alpha<\lambda\rangle$ will be defined.

For $n=0$, let $Lev_0(T)=\langle\rangle=Lev_0(\pi(T))$.

For $n=1$, let $Lev_1(T)=C, Lev_1(\pi(T))=\{0\}$, i.e. $\pi(\langle\alpha\rangle)=\langle 0\rangle$ for every $\alpha\in C$. For $\alpha\in C$ we define a real s_α by

$$\forall m<\omega,\ s_\alpha(m)=r_{1,f_\alpha(m)}(0).$$

Suppose now that $n>1$ and $T\restriction n$ and $\pi(T)\restriction n$ are defined. We define $Lev_n(T)$, $Lev_n(\pi(T))$ and reals s_α for $\alpha\in Lev_n(T)$. Let $\eta\in T\restriction(n-1)$, $\alpha^*,\alpha^{**}\in Suc_T(\eta)$ and $\alpha^{**}=\min(Suc_T(\eta)\setminus(\alpha^*+1))$. We then define $Suc_T(\eta^\frown\langle\alpha^{**}\rangle)$ if it is not yet defined.[4]

Case A. $|\alpha^{**}\setminus\alpha^*|\le\kappa$.

Fix some enumeration $\langle\alpha_\imath : \imath<\rho\le\kappa\rangle$ of $\alpha^{**}\setminus\alpha^*$. Let

- $Suc_T(\eta^\frown\langle\alpha^{**}\rangle)=\alpha^{**}\setminus\alpha^*$,
- $Suc_T(\eta^\frown\langle\alpha^{**}\rangle^\frown\langle\alpha\rangle)=\langle\rangle$ for $\alpha\in\alpha^{**}\setminus\alpha^*$,

[4]Then $Lev_n(T)$ will be the union of such $Suc_T(\eta^\frown\langle\alpha^{**}\rangle)$'s.

- $Suc_{\pi(T)}(\pi(\eta^\frown\langle\alpha^{**}\rangle)) = \rho = |\alpha^{**} \setminus \alpha^*|$,
- $Suc_{\pi(T)}(\pi(\eta^\frown\langle\alpha^{**}\rangle)^\frown\langle\imath\rangle) = \langle\rangle$ for $\imath < \rho$.

Now we define s_α for $\alpha \in \alpha^{**} \setminus \alpha^*$. Let $\imath$ be such that $\alpha = \alpha_\imath$. Let $k = \min\{m < \omega : r_{\pi(\eta^\frown\langle\alpha^{**}\rangle)^\frown\langle\imath\rangle}(m) = 1\}$, Finally let

$$\forall m < \omega, s_\alpha(m) = r_{n,f_\alpha(k+m)}(0).$$

Case B. $|\alpha^{**} \setminus \alpha^*| > \kappa$ and $cf(\alpha^{**}) < \kappa$.

Let $\rho = cf\alpha^{**}$ and let $\langle\alpha^{**}_\nu : \nu < \rho\rangle$ be a normal sequence cofinal in α^{**} with $\alpha^{**}_0 > \alpha^*$. Let

- $Suc_T(\eta^\frown\langle\alpha^{**}\rangle) = \{\alpha^{**}_\nu : \nu < \rho\}$,
- $Suc_{\pi(T)}(\pi(\eta^\frown\langle\alpha^{**}\rangle)) = \rho$.

Now we define $s_{\alpha^{**}_\nu}$ for $\nu < \rho$. Let $k = \min\{m < \omega : r_{\pi(\eta^\frown\langle\alpha^{**}\rangle)^\frown\langle\nu\rangle}(m) = 1\}$ and let

$$\forall m < \omega, s_{\alpha^{**}_\nu}(m) = r_{n,f_{\alpha^{**}_\nu}(k+m)}(0).$$

Case C. $cf(\alpha^{**}) > \kappa$.

Let ρ and $\langle\alpha^{**}_\nu : \nu < \rho\rangle$ be as in Case B. Let

- $Suc_T(\eta^\frown\langle\alpha^{**}\rangle) = \{\alpha^{**}_\nu : \nu < \rho\}$,
- $Suc_{\pi(T)}(\pi(\eta^\frown\langle\alpha^{**}\rangle)) = \langle 0\rangle$.

We define $s_{\alpha^{**}_\nu}$ for $\nu < \rho$. Let $k = \min\{m < \omega : r_{\pi(\eta^\frown\langle\alpha^{**}\rangle)^\frown\langle 0\rangle}(m) = 1\}$ and let

$$\forall m < \omega, s_{\alpha^{**}_\nu}(m) = r_{n,f_{\alpha^{**}_\nu}(k+m)}(0).$$

By the definition, T is a well-founded tree and $\bigcup_{n<\omega} Lev_n(T) = \lambda$. The following lemma completes our proof.

Lemma 2.4. *$\langle s_\alpha : \alpha < \lambda\rangle$ is a sequence of λ-many Cohen reals over V.*

Proof. First note that $\langle\langle r_{n,\alpha} : n < \omega, \alpha < \kappa\rangle, \langle r_\eta : \eta \in [\kappa]^{<\omega}\rangle\rangle$ is $\mathbb{C}(\omega \times \kappa) \times \mathbb{C}([\kappa]^{<\omega})$-generic over V_1. By the c.c.c. of $\mathbb{C}(\lambda)$ it suffices to show that for any countable set $I \subseteq \lambda$, $I \in V$, the sequence $\langle s_\alpha : \alpha \in I\rangle$ is $\mathbb{C}(I)$-generic over V. Thus it suffices to prove the following:

(∗) For every $(p, q) \in \mathbb{C}(\omega \times \kappa) \times \mathbb{C}([\kappa]^{<\omega})$ and every open dense subset $D \in V$ of $\mathbb{C}(I)$, there is $(\overline{p}, \overline{q}) \leq (p, q)$ such that $(\overline{p}, \overline{q}) \Vdash$ "$\langle \underset{\sim}{s}_\alpha : \alpha \in I\rangle$ extends some element of D".

Let (p, q) and D be as above. For simplicity suppose that $p = q = \emptyset$. For each $n < \omega$ let $I_n = I \cap Lev_n(T)$. Then $I_0 = \emptyset$ and $I_1 = I \cap C$ is finite. For simplicity let $I_1 = \{\alpha^*_1, \alpha^*_2\}$, where $\alpha^*_1 < \alpha^*_2$. Pick $n^* < \omega$ such that for all $n \geq n^*$, $f_{\alpha^*_1}(n) < f_{\alpha^*_2}(n)$. Let $p_0 \in \mathbb{C}(\omega \times \kappa)$ be such that

$$\mathrm{dom}(p_0) = \{\langle 1, \beta, 0\rangle : \exists n < n^*(\beta = f_{\alpha^*_1}(n) \text{ or } \beta = f_{\alpha^*_2}(n))\}.$$

Then for $n < n^*$ and $j \in \{1, 2\}$,

$$(p_0, \emptyset) \Vdash \underset{\sim}{s}_{\alpha^*_j}(n) = \underset{\sim}{r}_{1,f_{\alpha^*_j}(n)}(0) = p_0(1, f_{\alpha^*_j}(n), 0).$$

Thus $(p_0, \emptyset)$ decides $s_{\alpha_1^*} \restriction n^*$ and $s_{\alpha_2^*} \restriction n^*$. Let $b \in D$ be such that $\langle b(\alpha_1^*), b(\alpha_2^*)\rangle$ extends $\langle s_{\alpha_1^*} \restriction n^*, s_{\alpha_2^*} \restriction n^*\rangle$. Let

$$p_1 = p_0 \cup \bigcup_{j\in\{1,2\}} \{\langle 1, f_{\alpha_j^*}(n), 0, b(\alpha_j^*, n)\rangle : n \geq n^*, (\alpha_j^*, n) \in \mathrm{dom}(b)\}.$$

Then $p_1 \in \mathbb{C}(\omega \times \kappa)$ is well defined, and letting $q_1 = \emptyset$, we have

$$(p_1, q_1) \Vdash \text{“}\langle \underset{\sim}{s}_{\alpha_1^*}, \underset{\sim}{s}_{\alpha_2^*}\rangle \text{ extends } \langle b(\alpha_1^*), b(\alpha_2^*)\rangle\text{”}.$$

For each $n < \omega$ let J_n be the set of all components of b which are in I_n, i.e. $J_n = \{\alpha \in I_n : \exists n, (\alpha, n) \in \mathrm{dom}(b)\}$. We note that $J_0 = \emptyset$ and $J_1 = I_1 = \{\alpha_1^*, \alpha_2^*\}$. Also note that for all but finitely many $n < \omega, J_n = \emptyset$. Thus let us suppose $t < \omega$ is such that for all $n > t$, $J_n = \emptyset$. Let us consider J_2. For each $\alpha \in J_2$ there are three cases to be considered:[5]

Case 1. There are $\alpha^* < \alpha^{**}$ in $Lev_1(T) = C$, $\alpha^{**} = \min(C \setminus (\alpha^* + 1))$ such that $|\alpha^{**} \setminus \alpha^*| \leq \kappa$ and $\alpha \in Suc_T(\langle\alpha^{**}\rangle) = \alpha^{**} \setminus \alpha^*$. Let $\imath_\alpha$ be the index of α in the enumeration of $\alpha^{**} \setminus \alpha^*$ considered in Case A above, and let $k_\alpha = \min\{m < \omega : r_{\pi(\langle\alpha^{**}\rangle)^\frown\langle\imath_\alpha\rangle}(m) = 1\}$. Then

$$\forall m < \omega,\ s_\alpha(m) = r_{2, f_\alpha(k_\alpha + m)}(0).$$

Case 2. There are $\alpha^* < \alpha^{**}$ as above such that $|\alpha^{**} \setminus \alpha^*| > \kappa$ and $\rho = cf\alpha^{**} < \kappa$. Let $\langle \alpha_\nu^{**} : \nu < \rho\rangle$ be as in Case B. Then $\alpha = \alpha_{\nu_\alpha}^{**}$ for some $\nu_\alpha < \rho$ and if $k_\alpha = \min\{m < \omega : r_{\pi(\langle\alpha^{**}\rangle)^\frown\langle\nu_\alpha\rangle}(m) = 1\}$. Then

$$\forall m < \omega,\ s_\alpha(m) = r_{2, f_\alpha(k_\alpha + m)}(0).$$

Case 3. There are $\alpha^* < \alpha^{**}$ as above such that $\rho = cf\alpha^{**} > \kappa$. Let $\langle \alpha_\nu^{**} : \nu < \rho\rangle$ be as in Case C. Then $\alpha = \alpha_{\nu_\alpha}^{**}$ for some $\nu_\alpha < \rho$, and if $k_\alpha = \min\{m < \omega : r_{\pi(\langle\alpha^{**}\rangle)^\frown\langle 0\rangle}(m) = 1\}$, then

$$\forall m < \omega,\ s_\alpha(m) = r_{2, f_\alpha(k_\alpha + m)}(0).$$

Let $m^* < \omega$ be such that for all $n \geq m^*$ and $\alpha < \alpha'$ in $J_1 \cup J_2$, $f_\alpha(n) < f_{\alpha'}(n)$. Let

$$\begin{aligned} q_2 = \{\langle \eta, n, 0\rangle : n < m^*, \exists \alpha \in J_2 (\eta = \pi(\langle\alpha^{**}\rangle)^\frown\langle i_\alpha\rangle &\text{ or} \\ \eta = \pi(\langle\alpha^{**}\rangle)^\frown\langle \nu_\alpha\rangle &\text{ or} \\ \eta = \pi(\langle\alpha^{**}\rangle)^\frown\langle 0\rangle)\} & \\ \cup\ \{\langle \eta, m^*, 1\rangle : \exists \alpha \in J_2 (\eta = \pi(\langle\alpha^{**}\rangle)^\frown\langle i_\alpha\rangle &\text{ or} \\ \eta = \pi(\langle\alpha^{**}\rangle)^\frown\langle \nu_\alpha\rangle &\text{ or} \\ \eta = \pi(\langle\alpha^{**}\rangle)^\frown\langle 0\rangle)\}. & \end{aligned}$$

Then $q_2 \in \mathbb{C}([\kappa]^{<\omega})$ is well defined, and for each $\alpha \in J_2$, $(\phi, q_2) \Vdash k_\alpha = m^*$. Let

$$p_2 = p_1 \cup \{\langle 2, f_\alpha(m^* + m), 0, b(\alpha, m)\rangle : \alpha \in J_2, (\alpha, m) \in \mathrm{dom}(b)\}.$$

Then $p_2 \in \mathbb{C}(\omega \times \kappa)$ is well defined, $(p_2, q_2) \leq (p_1, q_1)$, and for $\alpha \in J_2$ and $m < \omega$ with $(\alpha, m) \in \mathrm{dom}(b)$,

$$(p_2, q_2) \Vdash \underset{\sim}{s}_\alpha(m) = \underset{\sim}{r}_{2, f_\alpha(k_\alpha + m)}(0) = p_2(2, f_\alpha(k_\alpha + m), 0) = b(\alpha, m) = b(\alpha)(m).$$

[5]Note that all the action in Cases 1-3 below is happening in the generic extension; in particular, we did not yet determine the value of k_α.

Thus $(p_2, q_2) \Vdash$"$\underset{\sim}{s}_\alpha$ extend $b(\alpha)$", and hence

$$(p_2, q_2) \Vdash \text{“}\langle \underset{\sim}{s}_\alpha : \alpha \in J_1 \cup J_2 \rangle \text{ extends } \langle b(\alpha) : \alpha \in J_1 \cup J_2 \rangle\text{”}.$$

By induction suppose that we have defined $(p_1, q_1) \geq (p_2, q_2) \geq \ldots \geq (p_j, q_j)$ for $j < t$, where for $1 \leq i \leq j$,

$$(p_i, q_i) \Vdash \text{“}\langle \underset{\sim}{s}_\alpha : \alpha \in J_1 \cup \ldots \cup J_i \rangle \text{ extends } \langle b(\alpha) : \alpha \in J_1 \cup \ldots \cup J_i \rangle\text{”}.$$

We define $(p_{j+1}, q_{j+1}) \leq (p_j, q_j)$ such that for each $\alpha \in J_{j+1}$,

$$(p_{j+1}, q_{j+1}) \Vdash \text{“}\underset{\sim}{s}_\alpha \text{ extends } b(\alpha)\text{”}.$$

Let $\alpha \in J_{j+1}$. Then we can find $\eta \in T \restriction j$ and $\alpha^* < \alpha^{**}$ such that $\alpha^*, \alpha^{**} \in Suc_T(\eta)$, $\alpha^{**} = \min(Suc_T(\eta) \setminus (\alpha^* + 1))$ and $\alpha \in Suc_T(\eta^\frown \langle \alpha^{**} \rangle)$. As before there are three cases to be considered:[6]

Case 1. $|\alpha^{**} \setminus \alpha^*| \leq \kappa$. Then let i_α be the index of α in the enumeration of $\alpha^{**} \setminus \alpha^*$ considered in Case A and let $k_\alpha = \min\{m < \omega : r_{\pi(\eta^\frown \langle \alpha^{**} \rangle)^\frown \langle i_\alpha \rangle}(m) = 1\}$. Then

$$\forall m < \omega, s_\alpha(m) = r_{j+1, f_\alpha(k_\alpha + m)}(0).$$

Case 2. $|\alpha^{**} \setminus \alpha^*| > \kappa$ and $\rho = cf\alpha^{**} < \kappa$. Let $\langle \alpha^{**}_\nu : \nu < \rho \rangle$ be as in Case B and let $\nu_\alpha < \rho$ be such that $\alpha = \alpha^{**}_{\nu_\alpha}$. Let $k_\alpha = \min\{m < \omega : r_{\pi(\eta^\frown \langle \alpha^{**} \rangle)^\frown \langle \nu_\alpha \rangle}(m) = 1\}$. Then

$$\forall m < \omega, s_\alpha(m) = r_{j+1, f_\alpha(k_\alpha + m)}(0).$$

Case 3. $\rho = cf\alpha^{**} > \kappa$. Let $\langle \alpha^{**}_\nu : \nu < \rho \rangle$ be as in Case C. Let $\nu_\alpha < \rho$ be such that $\alpha = \alpha^{**}_{\nu_\alpha}$, and let $k_\alpha = \min\{m < \omega : r_{\pi(\eta^\frown \langle \alpha^{**} \rangle)^\frown \langle 0 \rangle}(m) = 1\}$. Then

$$\forall m < \omega, s_\alpha(m) = r_{j+1, f_\alpha(k_\alpha + m)}(0).$$

Let $m^* < \omega$ be such that for all $n \geq m^*$ and $\alpha < \alpha'$ in $J_1 \cup \ldots \cup J_{j+1}$, $f_\alpha(n) < f_{\alpha'}(n)$. Let

$$\begin{aligned} q_{j+1} = q_j \cup \{\langle \overline{\eta}, n, 0 \rangle : n < m^*, \exists \alpha \in J_{j+1} &\text{ (for some unique } \eta \in T \restriction j, \\ \alpha^{**} \in Suc_T(\eta), &\text{ we have } \alpha \in Suc_T(\eta^\frown \langle \alpha^{**} \rangle) \\ &\text{and } (\overline{\eta} = \pi(\eta^\frown \langle \alpha^{**} \rangle)^\frown \langle i_\alpha \rangle \\ &\text{or } \overline{\eta} = \pi(\eta^\frown \langle \alpha^{**} \rangle)^\frown \langle \nu_\alpha \rangle \\ &\text{or } \overline{\eta} = (\pi(\eta^\frown \langle \alpha^{**} \rangle)^\frown \langle 0 \rangle))\} \\ \cup \{\langle \overline{\eta}, m^*, 1 \rangle : \exists \alpha \in J_{j+1} &\text{ (for some unique } \eta \in T \restriction j, \\ \alpha^{**} \in Suc_T(\eta), &\text{ we have } \alpha \in Suc_T(\eta^\frown \langle \alpha^{**} \rangle) \\ &\text{and } (\overline{\eta} = \pi(\eta^\frown \langle \alpha^{**} \rangle)^\frown \langle i_\alpha \rangle \\ &\text{or } \overline{\eta} = \pi(\eta^\frown \langle \alpha^{**} \rangle)^\frown \langle \nu_\alpha \rangle \\ &\text{or } \overline{\eta} = (\pi(\eta^\frown \langle \alpha^{**} \rangle)^\frown \langle 0 \rangle))\}. \end{aligned}$$

It is easily seen that $q_{j+1} \in \mathbb{C}([\kappa]^{<\omega})$, and for each $\alpha \in J_{j+1}$, $(\phi, q_{j+1}) \Vdash k_\alpha = m^*$. Let

$$p_{j+1} = p_j \cup \{\langle j+1, f_\alpha(m^* + m), 0, b(\alpha, m) \rangle : \alpha \in J_{j+1}, (\alpha, m) \in \mathrm{dom}(b)\}.$$

[6]Again note that all the action in Cases 1-3 below is happening in the generic extension.

Then $p_{j+1} \in \mathbb{C}(\omega \times \kappa)$ is well defined and $(p_{j+1}, q_{j+1}) \leq (p_j, q_j)$, and for $\alpha \in J_{j+1}$ we have

$$(p_{j+1}, q_{j+1}) \Vdash \underset{\sim}{s}_\alpha(m) = \underset{\sim}{r}_{j+1, f_\alpha(k_\alpha+m)}(0) = p_{j+1}(j+1, f_\alpha(k_\alpha + m), 0) \\ = b(\alpha, m) = b(\alpha)(m).$$

Thus $(p_{j+1}, q_{j+1}) \Vdash$ "$\underset{\sim}{s}_\alpha$ extends $b(\alpha)$". Finally let $(\overline{p}, \overline{q}) = (p_t, q_t)$. Then for each component α of b,

$$(\overline{p}, \overline{q}) \Vdash \text{“}\underset{\sim}{s}_\alpha \text{ extends } b(\alpha)\text{”}.$$

Hence

$$(\overline{p}, \overline{q}) \Vdash \text{“}\langle \underset{\sim}{s}_\alpha : \alpha \in I \rangle \text{ extends } b\text{”}.$$

$(*)$ follows and we are done. □

Theorem 2.1 follows. □

We now give several applications of the above theorem.

Theorem 2.5. *Suppose that V satisfies GCH, $\kappa = \bigcup_{n<\omega} \kappa_n$ and $\bigcup_{n<\omega} o(\kappa_n) = \kappa$ (where $o(\kappa_n)$ is the Mitchell order of κ_n). Then there exists a cardinal preserving generic extension V_1 of V satisfying GCH and having the same reals as V does, so that adding κ-many Cohen reals over V_1 produces κ^+-many Cohen reals over V.*

Proof. Rearranging the sequence $\langle \kappa_n : n < \omega \rangle$ we may assume that $o(\kappa_{n+1}) > \kappa_n$ for each $n < \omega$. Let $0 < n < \omega$. By [Mag1], there exists a forcing notion $\mathbb{P}_n$ such that:

- Each condition in $\mathbb{P}_n$ is of the form (g, G), where g is an increasing function from a finite subset of κ_n^+ into κ_{n+1} and G is a function from $\kappa_n^+ \setminus \mathrm{dom}(g)$ into $\mathcal{P}(\kappa_{n+1})$ such that for each $\alpha \in \mathrm{dom}(G), G(\alpha)$ belongs to a suitable normal measure.[7] We may also assume that conditions have no parts below or at κ_n, and sets of measure one are like this as well.
- Forcing with $\mathbb{P}_n$ preserves cardinals and the GCH, and adds no new subsets to κ_n.
- If G_n is $\mathbb{P}_n$-generic over V, then in $V[G_n]$ there is a normal function $g_n^* : \kappa_n^+ \longrightarrow \kappa_{n+1}$ such that $ran(g_n^*)$ is a club subset of κ_{n+1} consisting of measurable cardinals of V such that $V[G_n] = V[g_n^*]$.

Let $\mathbb{P}^* = \prod_{n<\omega} \mathbb{P}_n$, and let

$$\mathbb{P} = \{\langle \langle g_n, G_n \rangle : n < \omega \rangle \in \mathbb{P}^* : g_n = \emptyset, \text{ for all but finitely many } n\}.$$

Then $\mathbb{P}$ satisfies the $\kappa^+ - c.c.$,[8] and using simple modification of arguments from [Mag1], [Mag2] we can show that forcing with $\mathbb{P}$ preserves cardinals and the GCH. Let G be $\mathbb{P}$-generic over V, and let $g_n^* : \kappa_n^+ \longrightarrow \kappa_{n+1}$ be the generic function added by the part of the forcing corresponding to $\mathbb{P}_n$, for $0 < n < \omega$. Let $X = \bigcup_{0<n<\omega}((ran(g_n^*) \setminus \kappa_n^+) \cup \{\kappa_{n+1}\})$ and let $g^* : \kappa \longrightarrow \kappa$ be an enumeration of X in increasing order. Then $X = ran(g^*)$ is club in κ and consists entirely of measurable cardinals of V. Also, $V[G] = V[g^*]$.

[7] In fact, if $\alpha > \max(\mathrm{dom}(g))$, then $G(\alpha)$ belongs to a normal measure on κ_{n+1}, and if $\alpha < \max(\mathrm{dom}(g))$, then $G(\alpha)$ belongs to a normal measure on $g(\beta)$ where $\beta = \min(\mathrm{dom}(g) \setminus \alpha)$.

[8] This is because any two conditions $\langle \langle g_n, G_n \rangle : n < \omega \rangle$ and $\langle \langle g_n, H_n \rangle : n < \omega \rangle$ in $\mathbb{P}$ are compatible.

Working in $V[G]$, let $\mathbb{Q}$ be the usual forcing notion for adding a club subset of κ^+ which avoids points of countable V-cofinality. Thus $\mathbb{Q} = \{p : p$ is a closed bounded subset of κ^+ and avoids points of countable V-cofinality$\}$, ordered by end extension. Let H be $\mathbb{Q}$-generic over $V[G]$ and $C = \bigcup\{p : p \in H\}$.

Lemma 2.6. *(a) $(\mathbb{Q}, \leq)$ satisfies the κ^{++}-c.c.,*
(b) $(\mathbb{Q}, \leq)$ is $< \kappa^+$-distributive,
(c) C is a club subset of κ^+ which avoids points of countable V-cofinality.

(a) and (c) of the above lemma are trivial. For use later we prove a more general version of (b).

Lemma 2.7. *Let $V \subseteq W$, let ν be regular in W and suppose that:*
(a) W is a ν-c.c. extension of V.
(b) For every $\lambda < \nu$ which is regular in W, there is $\tau < \nu$ so that $cf^W(\tau) = \lambda$ and τ has a club subset in W which avoids points of countable V-cofinality.
In W let $\mathbb{Q} = \{p \subseteq \nu : p$ is closed and bounded in ν and avoids points of countable V-cofinality$\}$. Then in W, $\mathbb{Q}$ is $< \nu$-distributive.

Proof. This lemma first appeared in [G-N-S]. We prove it for completeness. Suppose that $W = V[G]$, where G is $\mathbb{P}$-generic over V for a ν-c.c. forcing notion $\mathbb{P}$. Let $\lambda < \nu$ be regular, $q \in \mathbb{Q}$, $\underset{\sim}{f} \in W^{\mathbb{Q}}$ and

$$q \Vdash \underset{\sim}{f} : \lambda \longrightarrow ON.$$

We find an extension of q which decides $\underset{\sim}{f}$. By (b) we can find $\tau < \nu$ and $g : \lambda \longrightarrow \tau$ such that $cf^W(\tau) = \lambda$, g is normal and $C = ran(g)$ is a club of τ which avoids points of countable V-cofinality.

In W, let $\theta > \nu$ be large enough regular. Working in V, let $\overline{H} \prec V_\theta$ and $R : \tau \longrightarrow ON$ be such that

- $Card(\overline{H}) < \nu$,
- $\overline{H}$ has $\lambda, \tau, \nu, \mathbb{P}$ and $\mathbb{P}$-names for $p, \mathbb{Q}, \underset{\sim}{f}, g$ and C as elements,
- $ran(R)$ is cofinal in $\sup(\overline{H} \cap \nu)$,
- $R \restriction \beta \in \overline{H}$ for each $\beta < \tau$.

Let $H = \overline{H}[G]$. Then $\sup(H \cap \nu) = \sup(\overline{H} \cap \nu)$, since $\mathbb{P}$ is ν-c.c., $H \prec V_\theta^W$, and if $\gamma = sup(H \cap \nu)$, then $cf^W(\gamma) = cf^W(\tau) = \lambda$. For $\alpha < \lambda$ let $\gamma_\alpha = R(g(\alpha))$. Then

- $\langle \gamma_\alpha : \alpha < \lambda \rangle \in W$ is a normal sequence cofinal in γ,
- $\langle \gamma_\alpha : \alpha < \beta \rangle \in H$ for each $\beta < \lambda$, since $R \restriction g(\beta) \in \overline{H}$,
- $cf^V(\gamma_\alpha) = cf^V(g(\alpha)) \neq \omega$ for each $\alpha < \lambda$, since R is normal and $g(\alpha) \in C$.

Let $D = \{\gamma_\alpha : \alpha < \lambda\}$. We define by induction a sequence $\langle q_\eta : \eta < \lambda \rangle$ of conditions in $\mathbb{Q}$ such that for each $\eta < \lambda$,

- $q_0 = q$,
- $q_\eta \in H$,
- $q_{\eta+1} \leq q_\eta$,
- $q_{\eta+1}$ decides $\underset{\sim}{f}(\eta)$,
- $D \cap (\max(q_\eta), \max(q_{\eta+1})) \neq \emptyset$,
- $q_\eta = \bigcup_{\rho<\eta} q_\rho \cup \{\delta_\eta\}$, where $\delta_\eta = \sup_{\rho<\eta}(\max(q_\rho))$, if η is a limit ordinal.

We may further suppose that

- q_η's are chosen in a uniform way (say via a well-ordering which is built in to $\overline{H}$).

We can define such a sequence using the fact that H contains all initial segments of D and that $\delta_\eta \in D$ for every limit ordinal $\eta < \lambda$ (and hence $cf^V(\delta_\eta) \neq \omega$).

Finally let $q_\lambda = \bigcup_{\eta<\lambda} q_\eta \cup \{\delta_\lambda\}$, where $\delta_\lambda = \sup_{\eta<\lambda}(\max(q_\eta))$. Then $\delta_\lambda \in D \cup \{\gamma\}$, hence $cf^V(\delta_\lambda) \neq \omega$. It follows that $q_\lambda \in \mathbb{Q}$ is well defined. Trivially $q_\lambda \leq q$ and q_λ decides $\underset{\sim}{f}$. The lemma follows. □

Let $V_1 = V[G * H]$. The following is obvious:

Lemma 2.8. (a) *V and V_1 have the same cardinals and reals,*
(b) *$V_1 \models GCH$.*

It follows from Theorem 2.1 that adding κ-many Cohen reals over V_1 adds κ^+-many Cohen reals over V. This concludes the proof of Theorem 2.5. □

Let us show that some large cardinals are needed for the previous result.

Theorem 2.9. *Assume that $V_1 \supseteq V$ and V_1 and V have the same cardinals and reals. Suppose that for some uncountable cardinal κ of V_1, adding κ-many Cohen reals to V_1 produces κ^+-many Cohen reals to V. Then in V_1 there is an inner model with a measurable cardinal.*

Proof. Suppose on the contrary that in V_1 there is no inner model with a measurable cardinal. Thus by the Dodd-Jensen covering lemma (see [D-J1], [D-J2]) $(\mathcal{K}(V_1), V_1)$ satisfies the covering lemma, where $\mathcal{K}(V_1)$ is the Dodd-Jensen core model as computed in V_1.

Claim 2.10. $\mathcal{K}(V) = \mathcal{K}(V_1)$.

Proof. The claim is well known and follows from the fact that V and V_1 have the same cardinals. We present a proof for completeness.[9] Suppose not. Clearly $\mathcal{K}(V) \subseteq \mathcal{K}(V_1)$, so let $A \subseteq \alpha, A \in \mathcal{K}(V_1), A \notin \mathcal{K}(V)$. Then there is a mouse of $\mathcal{K}(V_1)$ to which A belongs, hence there is such a mouse of $\mathcal{K}(V_1)$-power α. It then follows that for every limit cardinal $\lambda > \alpha$ of V_1 there is a mouse with critical point λ to which A belongs, and the filter is generated by end segments of

$$\{\chi : \chi < \lambda, \chi \text{ a cardinal in } V_1\}.$$

As V and V_1 have the same cardinals, this mouse is in V, and hence in $\mathcal{K}(V)$. □

Let us denote this common core model by $\mathcal{K}$. Then $\mathcal{K} \subseteq V$, and hence (V, V_1) satisfies the covering lemma. It follows that $([\kappa^+]^{\leq\omega_1})^V$ is unbounded in $([\kappa^+]^{\leq\omega})^{V_1}$, and since $\omega_1^V = \omega_1^{V_1}$, we can easily show that $([\kappa^+]^{\leq\omega})^V$ is unbounded in $([\kappa^+]^{\leq\omega})^{V_1}$. Since V_1 and V have the same reals, $([\kappa^+]^{\leq\omega})^V = ([\kappa^+]^{\leq\omega})^{V_1}$, and we get a contradiction. □

If we relax our assumptions, and allow some cardinals to collapse, then no large cardinal assumptions are needed.

Theorem 2.11. (a) *Suppose V is a model of GCH. Then there is a generic extension V_1 of V satisfying GCH so that the only cardinal of V which is collapsed in V_1 is $\aleph_1$ and such that adding $\aleph_\omega$-many Cohen reals to V_1 produces $\aleph_{\omega+1}$-many of them over V.*

(b) *Suppose V satisfies GCH. Then there is a generic extension V_1 of V satisfying GCH and having the same reals as V does, so that the only cardinals of V*

[9] Our proof is the same as in the proof of [Sh2, Theorem VII. 4.2(1)].

which are collapsed in V_1 are $\aleph_2$ and $\aleph_3$ and such that adding $\aleph_\omega$-many Cohen reals to V_1 produces $\aleph_{\omega+1}$-many of them over V.

Proof. (a) Working in V, let $\mathbb{P} = Col(\aleph_0, \aleph_1)$ and let G be $\mathbb{P}$-generic over V. Also let $S = \{\alpha < \omega_2 : cf^V(\alpha) = \omega_1\}$. Then S remains stationary in $V[G]$. Working in $V[G]$, let $\mathbb{Q}$ be the standard forcing notion for adding a club subset of S with countable conditions, and let H be $\mathbb{Q}$-generic over $V[G]$. Let $C = \bigcup H$. Then C is a club subset of $\omega_1^{V[G]} = \omega_2^V$ such that $C \subseteq S$, and in particular C avoids points of countable V-cofinality. Working in $V[G * H]$, let

$$\mathbb{R} = \langle\langle \mathbb{P}_\nu : \aleph_2 \leq \nu \leq \aleph_{\omega+2}, \nu \text{ regular}\rangle, \langle \underset{\sim}{\mathbb{Q}}_\nu : \aleph_2 \leq \nu \leq \aleph_{\omega+1}, \nu \text{ regular}\rangle\rangle$$

be the Easton support iteration by letting $\underset{\sim}{\mathbb{Q}}_\nu$ name the poset $\{p \subset \nu : p \text{ is closed and bounded in } \nu \text{ and avoids points of countable } V\text{-cofinality}\}$ as defined in $V[G*H]^{\mathbb{P}_\nu}$. Let

$$K = \langle\langle G_\nu : \aleph_2 \leq \nu \leq \aleph_{\omega+2}, \nu \text{ regular}\rangle, \langle H_\nu : \aleph_2 \leq \nu \leq \aleph_{\omega+1}, \nu \text{ regular}\rangle\rangle$$

be $\mathbb{R}$-generic over $V[G * H]$ (i.e. G_ν is $\mathbb{P}_\nu$-generic over $V[G * H]$ and H_ν is $\mathbb{Q}_\nu = \underset{\sim}{\mathbb{Q}}_\nu[G_\nu]$-generic over $V[G * H * G_\nu]$). Then

Lemma 2.12. (a) *$\mathbb{P}_\nu$ adds a club disjoint from $\{\alpha < \lambda : cf^V(\alpha) = \omega\}$ for each regular $\lambda \in (\aleph_1, \nu)$.*

(b) *(By Lemma 2.7) $V[G * H * G_\nu] \models$ "$\mathbb{Q}_\nu$ is $< \nu$-distributive".*

(c) *$V[G * H]$ and $V[G * H * K]$ have the same cardinals and reals, and satisfy GCH.*

(d) *In $V[G * H * K]$ there is a club subset C of $\aleph_{\omega+1}$ which avoids points of countable V-cofinality.*

Let $V_1 = V[G * H * K]$. By the above results, V_1 satisfies GCH and the only cardinal of V which is collapsed in V_1 is $\aleph_1$. The proof of the fact that adding $\aleph_\omega$-many Cohen reals over V_1 produces $\aleph_{\omega+1}$-many of them over V follows from Theorem 2.1.

(b) Working in V, let $\mathbb{P}$ be the following version of Namba forcing:

$$\mathbb{P} = \{T \subseteq \omega_2^{<\omega} : T \text{ is a tree, and for every } s \in T, \text{ the set } \{t \in T : t \supset s\} \text{ has size } \aleph_2\}$$

ordered by inclusion. Let G be $\mathbb{P}$-generic over V. It is well known that forcing with $\mathbb{P}$ adds no new reals, preserves cardinals $\geq \aleph_4$ and that $|\aleph_2^V|^{V[G]} = |\aleph_3^V|^{V[G]} = \aleph_1^{V[G]} = \aleph_1^V$ (see [Sh1]). Let $S = \{\alpha < \omega_3 : cf^V(\alpha) = \omega_2\}$.

Lemma 2.13. *S remains stationary in $V[G]$.*

Proof. See [Ve-W, Lemma 3]. □

Now the rest of the proof is exactly as in (a).

The theorem follows. □

By the same line but using stronger initial assumptions, adding κ-many Cohen reals may produce λ-many of them for λ much larger than κ^+.

Theorem 2.14. *Suppose that κ is a strong cardinal, $\lambda \geq \kappa$ is regular and GCH holds. Then there exists a cardinal preserving generic extension V_1 of V having the same reals as V does, so that adding κ-many Cohen reals over V_1 produces λ-many of them over V.*

Proof. Working in V, build for each δ a measure sequence $\vec{u}_\delta$ from a j witnessing "κ is δ-strong" out to the first weak repeat point. Find $\vec{u}$ such that $\vec{u} = \vec{u}_\delta$ for unboundedly many δ. Let $\mathbb{R}_{\vec{u}}$ be the corresponding Radin forcing notion and let G be $\mathbb{R}_{\vec{u}}$-generic over V. Then

Lemma 2.15. (a) *Forcing with $\mathbb{R}_{\vec{u}}$ preserves cardinals and the GCH and adds no new reals.*

(b) *In $V[G]$, there is a club $C_\kappa \subseteq \kappa$ consisting of inaccessible cardinals of V and $V[G] = V[C_\kappa]$.*

(c) *κ remains strong in $V[G]$.*

Proof. See [Git2] and [Cu]. □

Working in $V[G]$, let

$$E = \langle \langle U_\alpha : \alpha < \lambda \rangle, \langle \pi_{\alpha\beta} : \alpha \leq_E \beta \rangle \rangle$$

be a nice system satisfying conditions (0)-(9) in [Git2, page 37]. Also let

$$\mathbb{R} = \langle \langle \mathbb{P}_\nu : \kappa^+ \leq \nu \leq \lambda^+, \nu \text{ regular} \rangle, \langle \underset{\sim}{\mathbb{Q}}_\nu : \kappa^+ \leq \nu \leq \lambda, \nu \text{ regular} \rangle \rangle$$

be the Easton support iteration by letting $\underset{\sim}{\mathbb{Q}}_\nu$ name the poset $\{p \subseteq \nu : p$ is closed and bounded in ν and avoids points of countable V-cofinality$\}$ as defined in $V[G]^{\mathbb{P}_\nu}$. Let

$$K = \langle \langle G_\nu : \kappa^+ \leq \nu \leq \lambda^+, \nu \text{ regular} \rangle, \langle H_\nu : \kappa^+ \leq \nu \leq \lambda, \nu \text{ regular} \rangle \rangle$$

be $\mathbb{R}$-generic over $V[G]$. Then

Lemma 2.16. (a) *$\mathbb{P}_\nu$ adds a club disjoint form $\{\alpha < \delta : cf^V(\alpha) = \omega\}$ for each regular $\delta \in (\kappa, \nu)$.*

(b) *(By Lemma 2.7) $V[G * G_\nu] \models$ "$\mathbb{Q}_\nu = \underset{\sim}{\mathbb{Q}}_\nu[G_\nu]$ is $< \nu$-distributive".*

(c) *$V[G]$ and $V[G * K]$ have the same cardinals, and satisfy GCH.*

(d) *$\mathbb{R}$ is $\leq \kappa$-distributive, hence forcing with $\mathbb{R}$ adds no new κ-sequences.*

(e) *In $V[G * K]$, for each regular cardinal $\kappa \leq \nu \leq \lambda$ there is a club $C_\nu \subseteq \nu$ such that C_ν avoids points of countable V-cofinality.*

By Lemma 2.16(d), E remains a nice system in $V[G * K]$, except that the condition (0) is replaced by $(\lambda, \leq_E)$ is κ^+-directed closed. Hence working in $V[G * K]$, by results of [Git-Mag1], [Git-Mag2] and [Mer], we can find a forcing notion S such that if L is S-generic over $V[G * K]$, then

- $V[G * K]$ and $V[G * K * L]$ have the same cardinals and reals.
- In $V[G * K * L]$, $2^\kappa = \lambda$, $cf(\kappa) = \aleph_0$ and there is an increasing sequence $\langle \kappa_n : n < \omega \rangle$ of regular cardinals cofinal in κ and an increasing (mod finite) sequence $\langle f_\alpha : \alpha < \lambda \rangle$ in $\prod_{n<\omega}(\kappa_{n+1} \setminus \kappa_n)$.

Let $V_1 = V[G * K * L]$. Then V_1 and V have the same cardinals and reals. The fact that adding κ-many Cohen reals over V_1 produces λ-many Cohen reals over V follows from Theorem 2.1. □

If we allow many cardinals between V and V_1 to collapse, then using [Git-Mag1, Sec. 2] one can obtain the following:

Theorem 2.17. *Suppose that there is a strong cardinal and GCH holds. Let $\alpha < \omega_1$. Then there is a model $V_1 \supset V$ having the same reals as V and satisfying GCH below $\aleph_\omega^{V_1}$ such that adding $\aleph_\omega^{V_1}$-many Cohen reals to V_1 produces $\aleph_{\alpha+1}^{V_1}$-many of them over V.*

Proof. Proceed as in Theorem 2.14 to produce the model $V[G * K]$. Then working in $V[G*K]$, we can find a forcing notion S such that if L is S-generic over $V[G*K]$, then

- $V[G * K]$ and $V[G * K * L]$ have the same reals.
- In $V[G * K * L]$, cardinals $\geq \kappa$ are preserved, $\kappa = \aleph_\omega, GCH$ holds below $\aleph_\omega, 2^\kappa = \aleph_{\alpha+1}$ and there is an increasing (mod finite) sequence $\langle f_\beta : \beta < \aleph_{\alpha+1}\rangle$ in $\prod_{n<\omega}(\aleph_{n+1} \setminus \aleph_n)$.

Let $V_1 = V[G * K * L]$. Then V_1 and V have the same reals. The fact that adding $\aleph_\omega^{V_1}$-many Cohen reals over V_1 produces $\aleph_{\alpha+1}^{V_1}$-many Cohen reals over V follows from Theorem 2.1. □

3. Models with the same cofinality function but different reals

This section is completely devoted to the proof of the following theorem.

Theorem 3.1. *Suppose that V satisfies GCH. Then there is a cofinality preserving generic extension V_1 of V satisfying GCH so that adding a Cohen real over V_1 produces $\aleph_1$-many Cohen reals over V.*

The basic idea of the proof will be to split ω_1 into ω sets such that none of them will contain an infinite set of V. Then something like in section 2 will be used for producing Cohen reals. It turned out however that just not containing an infinite set of V is not enough. We will use a stronger property. As a result the forcing turns out to be more complicated. We are now going to define the forcing sufficient for proving the theorem. Fix a nonprincipal ultrafilter U over ω.

Definition 3.2. Let $(\mathbb{P}_U, \leq, \leq^*)$ be the Prikry (or in this context Mathias) forcing with U, i.e.

- $\mathbb{P}_U = \{\langle s, A\rangle \in [\omega]^{<\omega} \times U : \max(s) < \min(A)\}$,
- $\langle t, B\rangle \leq \langle s, A\rangle \iff t$ end extends s and $(t \setminus s) \cup B \subseteq A$,
- $\langle t, B\rangle \leq^* \langle s, A\rangle \iff t = s$ and $B \subseteq A$.

We call $\leq^*$ a direct or $*$-extension. The following are the basic facts on this forcing that will be used further.

Lemma 3.3. (*a*) *The generic object of $\mathbb{P}_U$ is generated by a real.*

(*b*) *$(\mathbb{P}_U, \leq)$ satisfies the c.c.c.*

(*c*) *If $\langle s, A\rangle \in \mathbb{P}_U$ and $b \subseteq \omega \setminus (\max(s) + 1)$ is finite, then there is a $*$-extension of $\langle s, A\rangle$, forcing the generic real to be disjoint to b.*

Proof. (*a*) If G is $\mathbb{P}_U$-generic over V, then let $r = \bigcup\{s : \exists A, \langle s, A\rangle \in G\}$. r is a real and $G = \{\langle s, A\rangle \in \mathbb{P}_U : r$ end extends s and $r \setminus s \subseteq A\}$.

(*b*) Trivial using the fact that for $\langle s, A\rangle, \langle t, B\rangle \in \mathbb{P}_U$, if $s = t$, then $\langle s, A\rangle$ and $\langle t, B\rangle$ are compatible.

(*c*) Consider $\langle s, A \setminus (\max(b) + 1)\rangle$. □

We now define our main forcing notion.

Definition 3.4. $p \in \mathbb{P}$ iff $p = \langle p_0, \underset{\sim}{p}_1\rangle$ where
(1) $p_0 \in \mathbb{P}_U$,
(2) $\underset{\sim}{p}_1$ is a $\mathbb{P}_U$-name such that for some $\alpha < \omega_1$, $p_0 \Vdash \underset{\sim}{p}_1 : \alpha \longrightarrow \omega$ and such that the following hold

(2a) For every $\beta < \alpha$, $\underset{\sim}{p}_1(\beta) \subseteq \mathbb{P}_U \times \omega$ is a $\mathbb{P}_U$−name for a natural number such that

- $\underset{\sim}{p}_1(\beta)$ is partial function from $\mathbb{P}_U$ into ω,
- for some fixed $l < \omega$, dom $\underset{\sim}{p}_1(\beta) \subseteq \{\langle s, \omega \setminus \max(s)+1\rangle : s \in [\omega]^l\}$,
- for all $\beta_1 \neq \beta_2 < \alpha, \mathrm{range}(\underset{\sim}{p}_1(\beta_1)) \cap \mathrm{range}(\underset{\sim}{p}_1(\beta_2))$ is finite.[10]

(2b) for every $I \subseteq \alpha$, $I \in V$, $p'_0 \leq p_0$ and finite $J \subseteq \omega$ there is a finite set $a \subseteq \alpha$ such that for every finite set $b \subseteq I \setminus a$ there is $p''_0 \leq^* p'_0$ such that $p''_0 \Vdash (\forall \beta \in b, \forall k \in J, \underset{\sim}{p}_1(\beta) \neq k) \& (\forall \beta_1 \neq \beta_2 \in b, \underset{\sim}{p}_1(\beta_1) \neq \underset{\sim}{p}_1(\beta_2))$.

Notation 3.5. (1) Call α the length of p (or $\underset{\sim}{p}_1$) and denote it by $\mathrm{lh}(p)$ (or $\mathrm{lh}(\underset{\sim}{p}_1)$).
(2) For $n < \omega$ let $\underset{\sim}{I}_{p,n}$ be a $\mathbb{P}_U$-name such that $p_0 \Vdash \underset{\sim}{I}_{p,n} = \{\beta < \alpha : \underset{\sim}{p}_1(\beta) = n\}$. Then we can identify $\underset{\sim}{p}_1$ with $\langle \underset{\sim}{I}_{p,n} : n < \omega\rangle$.

Remark 3.6. (2a) will guarantee that for $\beta < \alpha$, $p_0 \Vdash \underset{\sim}{p}_1(\beta) \in \omega$. The last condition in (2a) is a technical fact that will be used in several parts of the argument. The condition (2b) appears technical but will be crucial for producing numerous Cohen reals.

Definition 3.7. For $p = \langle p_0, \underset{\sim}{p}_1\rangle, q = \langle q_0, \underset{\sim}{q}_1\rangle \in \mathbb{P}$, define
(1) $p \leq q$ iff

- $p_0 \leq_{\mathbb{P}_U} q_0$,
- $\mathrm{lh}(q) \leq \mathrm{lh}(p)$,
- $p_0 \Vdash \forall n < \omega, \underset{\sim}{I}_{q,n} = \underset{\sim}{I}_{p,n} \cap \mathrm{lh}(q)$.

(2) $p \leq^* q$ iff

- $p_0 \leq^*_{\mathbb{P}_U} q_0$,
- $p \leq q$.

We call $\leq^*$ a direct or $*$-extension.

Remark 3.8. In the definition of $p \leq q$, we can replace the last condition by $p_0 \Vdash \underset{\sim}{q}_1 = \underset{\sim}{p}_1 \restriction \mathrm{lh}(q)$.

Lemma 3.9. *Let $\langle p_0, \underset{\sim}{p}_1\rangle \Vdash$ "α is an ordinal". Then there are $\mathbb{P}_U$-names $\underset{\sim}{\beta}$ and $\underset{\sim}{q}_1$ such that $\langle p_0, \underset{\sim}{q}_1\rangle \leq^* \langle p_0, \underset{\sim}{p}_1\rangle$ and $\langle p_0, \underset{\sim}{q}_1\rangle \Vdash \underset{\sim}{\alpha} = \underset{\sim}{\beta}$.*

Proof. Suppose for simplicity that $\langle p_0, \underset{\sim}{p}_1\rangle = \langle\langle\langle\,\rangle, \omega\rangle, \phi\rangle$. Let θ be large enough regular and let $\langle N_n : n < \omega\rangle$ be an increasing sequence of countable elementary submodels of H_θ such that $\mathbb{P}$, $\underset{\sim}{\alpha} \in N_0$ and $N_n \in N_{n+1}$ for each $n < \omega$. Let $N = \bigcup_{n<\omega} N_n$, $\delta_n = N_n \cap \omega_1$ for $n < \omega$ and $\delta = \bigcup_{n<\omega} \delta_n = N \cap \omega_1$. Let $\langle J_n : n < \omega\rangle \in N_0$ be a sequence of infinite subsets of $\omega \setminus \{0\}$ such that $\bigcup_{n<\omega} J_n = \omega \setminus \{0\}$, $J_n \subseteq J_{n+1}$, and $J_{n+1} \setminus J_n$ is infinite for each $n < \omega$. Also let $\langle \alpha_i : 0 < i < \omega\rangle$ be an enumeration of δ such that for every $n < \omega$, $\{\alpha_i : i \in J_n\} \in N_{n+1}$ is an enumeration of δ_n and $\{\alpha_i : i \in J_{n+1}\} \cap \delta_n = \{\alpha_i : i \in J_n\}$.

[10] Thus if G and r are as in the proof of Lemma 3.3 with $p_0 \in G$, then $p_o \Vdash$ "$\underset{\sim}{p}_1(\beta)$ is the l-th element of r".

We define by induction on the length of s, a sequence $\langle p^s : s \in [\omega]^{<\omega}\rangle$ of conditions such that

- $p^s = \langle p^s_0, \underset{\sim}{p}^s_1\rangle = \langle\langle s, A_s\rangle, \underset{\sim}{p}^s_1\rangle$,
- $p^s \in N_{s(\mathrm{lh}(s)-1)+1}$,
- $\mathrm{lh}(p^s) = \delta_{s(\mathrm{lh}(s)-1)+1}$,
- if t does not contradict p^s_0 (i.e. if the t end extends s and $t \setminus s \subseteq A_S$), then $p^t \leq p^s$.

For $s = \langle\,\rangle$, let $p^{\langle\,\rangle} = \langle\langle\langle\,\rangle, \omega\rangle, \phi\rangle$. Suppose that $\langle\,\rangle \neq s \in [\omega]^{<\omega}$ and $p^{s\restriction \mathrm{lh}(s)-1}$ is defined. We define p^s. First we define $t^{s\restriction \mathrm{lh}(s)-1} \leq^* p^{s\restriction \mathrm{lh}(s)-1}$ as follows: If there is no $*$-extension of $p^{s\restriction \mathrm{lh}(s)-1}$ deciding $\underset{\sim}{\alpha}$, then let $t^{s\restriction \mathrm{lh}(s)-1} = p^{s\restriction \mathrm{lh}(s)-1}$. Otherwise let $t^{s\restriction \mathrm{lh}(s)-1} \in N_{s(\mathrm{lh}(s)-2)+1}$ be such an extension. Note that $\mathrm{lh}(t^{s\restriction \mathrm{lh}(s)-1}) \leq \delta_{s(\mathrm{lh}(s)-2)+1}$.

Let $t^{s\restriction \mathrm{lh}(s)-1} = \langle t_0, \underset{\sim}{t}_1\rangle, t_0 = \langle s \restriction \mathrm{lh}(s)-1, A\rangle$. Let $C \subseteq \omega$ be an infinite set almost disjoint to $\langle \mathrm{range}(\underset{\sim}{t}_1(\beta)) : \beta < \mathrm{lh}(\underset{\sim}{t}_1)\rangle$. Split C into ω infinite disjoint sets C_i, $i < \omega$. Let $\langle c_{ij} : j < \omega\rangle$ be an increasing enumeration of C_i, $i < \omega$. We may suppose that all of these is done in $N_{s(\mathrm{lh}(s)-1)+1}$. Let $p^s = \langle p^s_0, \underset{\sim}{p}^s_1\rangle$, where

- $p^s_0 = \langle s, A \setminus (\max(s)+1)\rangle$,
- for $\beta < \mathrm{lh}(\underset{\sim}{t}_1)$, $\underset{\sim}{p}^s_1(\beta) = \underset{\sim}{t}_1(\beta)$,
- for $i \in J_{s(\mathrm{lh}(s)-1)}$ such that $\alpha_i \in \delta_{s(\mathrm{lh}(s)-1)} \setminus \mathrm{lh}(\underset{\sim}{t}_1)$,

$$\underset{\sim}{p}^s_1(\alpha_i) = \left\{\langle\langle s^\frown\langle r_1, \ldots, r_i\rangle, \omega \setminus (r_i+1)\rangle, c_{ir_i}\rangle : r_1 > \max(s), \langle r_1, \ldots, r_i\rangle \in [\omega]^i\right\}.$$

Trivially $p^s \in N_{s(\mathrm{lh}(s)-1)+1}$, $\mathrm{lh}(p^s) = \delta_{s(\mathrm{lh}(s)-1)}$, and if $s(\mathrm{lh}(s)-1) \in A$, then $p^s \leq t^{s\restriction \mathrm{lh}(s)-1}$.

Claim 3.10. $p^s \in \mathbb{P}$.

Proof. We check the conditions in Definition 3.4.

(1) i.e. $p^s_0 \in \mathbb{P}_U$ is trivial.

(2) It is clear that $p^s_0 \Vdash \underset{\sim}{p}^s_1 : \delta_{s(\mathrm{lh}(s)-1)} \longrightarrow \omega$ and that (2a) holds. Let us prove (2b). Thus suppose that $I \subseteq \delta_{s(\mathrm{lh}(s)-1)}$, $I \in V$, $p \leq p^s_0$ and $J \subseteq \omega$ is finite. First we apply (2b) to $\langle p, \underset{\sim}{t}_1\rangle, I \cap \mathrm{lh}(\underset{\sim}{t}_1)$, p and J to find a finite set $a' \subseteq \mathrm{lh}(\underset{\sim}{t}_1)$ such that

$(*)$ For every finite set $b \subseteq I \cap \mathrm{lh}(\underset{\sim}{t}_1) \setminus a'$ there is $p' \leq^* p$ such that $p' \Vdash (\forall\beta \in b, \forall k \in J, \underset{\sim}{t}_1(\beta) \neq k)$ & $(\forall\beta_1 \neq \beta_2 \in b, \underset{\sim}{t}_1(\beta_1) \neq \underset{\sim}{t}_1(\beta_2))$.

Let $p = \langle s^\frown\langle r_1, \ldots, r_m\rangle, B\rangle$. Suppose that $\delta_{s(\mathrm{lh}(s)-1)} \setminus \mathrm{lh}(\underset{\sim}{t}_1) = \{\alpha_{J_1}, \ldots, \alpha_{J_i}, \ldots\}$, where $J_1 < J_2 < \ldots$ are in $J_{s(\mathrm{lh}(s)-1)}$. Let

$$a = a' \cup \{\alpha_{J_1}, \ldots, \alpha_{J_m}\}.$$

We show that a is as required. Thus suppose that $b \subseteq I \setminus a$ is finite. Apply $(*)$ to $b \cap \mathrm{lh}(\underset{\sim}{t}_1)$ to find $p' = \langle s^\frown\langle r_1, \ldots, r_m\rangle, B'\rangle \leq^* p$ such that

$$p' \Vdash (\forall\beta \in b \cap \mathrm{lh}(\underset{\sim}{t}_1), \forall k \in J, \underset{\sim}{t}_1(\beta) \neq k) \\ \&(\forall\beta_1 \neq \beta_2 \in b \cap \mathrm{lh}(\underset{\sim}{t}_1), \underset{\sim}{t}_1(\beta_1) \neq \underset{\sim}{t}_1(\beta_2)).$$

Also note that

$$p' \Vdash \forall\beta \in b \cap \mathrm{lh}(\underset{\sim}{t}_1), \underset{\sim}{p}^s_1(\beta) = \underset{\sim}{t}_1(\beta).$$

Pick $k < \omega$ such that

$$\forall \beta \in b \cap \mathrm{lh}(\underset{\sim}{t}_1), \forall \alpha_i \in b \setminus \mathrm{lh}(\underset{\sim}{t}_1), \mathrm{range}(\underset{\sim}{p}^s_1(\beta_1)) \cap (\mathrm{range}(\underset{\sim}{p}^s_1(\alpha_i)) \setminus k) = \phi.$$

Let $q = \langle s^\frown\langle r_1, \ldots, r_m\rangle, B\rangle = \langle s^\frown\langle r_1, \ldots, r_m\rangle, B' \setminus (\max(J) + k + 1)\rangle$. Then $q \leq^* p' \leq^* p$. We show that q is as required. We need to show that

(1) $q \Vdash \forall \beta \in b \setminus \mathrm{lh}(\underset{\sim}{t}_1), \forall k \in J, \underset{\sim}{p}^s_1(\beta) \neq k$,
(2) $q \Vdash \forall \beta_1 \neq \beta_2 \in b \setminus \mathrm{lh}(\underset{\sim}{t}_1), \underset{\sim}{p}^s_1(\beta_1) \neq \underset{\sim}{p}^s_1(\beta_2)$,
(3) $q \Vdash \forall \beta_1 \in b \cap \mathrm{lh}(\underset{\sim}{t}_1), \forall \beta_2 \in b \setminus \mathrm{lh}(\underset{\sim}{t}_1), \underset{\sim}{p}^s_1(\beta_1) \neq \underset{\sim}{p}^s_1(\beta_2)$.

Now (1) follows from the fact that $q \Vdash$ "$\underset{\sim}{p}^s_1(\alpha_i) \geq (i - m)$-th element of $B > \max(J)$". (2) follows from the fact that for $i \neq j < \omega$, $C_i \cap C_j = \emptyset$, and $\mathrm{range}(\underset{\sim}{p}^s_1(\alpha_i)) \subseteq C_i$. (3) follows from the choice of k. The claim follows. $\square$

This completes our definition of the sequence $\langle p^s : s \in [\omega]^{<\omega}\rangle$. Let

$$\underset{\sim}{q}_1 = \{\langle p^s_0, \langle \beta, \underset{\sim}{p}^s_1(\beta)\rangle\rangle : s \in [\omega]^{<\omega}, \beta < \mathrm{lh}(p^s)\}.$$

Then $\underset{\sim}{q}_1$ is a $\mathbb{P}_U$-name, and for $s \in [\omega]^{<\omega}$, $p^s_0 \Vdash \underset{\sim}{p}^s_1 = \underset{\sim}{q}_1 \restriction \mathrm{lh}(\underset{\sim}{p}^s_1)$.

Claim 3.11. $\langle\langle\langle\,\rangle, \omega\rangle, \underset{\sim}{q}_1\rangle \in \mathbb{P}$.

Proof. We check conditions in Definition 3.4.

(1) i.e. $\langle\langle\,\rangle, \omega\rangle \in \mathbb{P}_U$ is trivial.

(2) It is clear from our definition that

$$\langle\langle\,\rangle, \omega\rangle \Vdash \text{“}\underset{\sim}{q}_1 \text{ is a well-defined function into } \omega\text{”}.$$

Let us show that $\mathrm{lh}(\underset{\sim}{q}_1) = \delta$. By the construction it is trivial that $\mathrm{lh}(\underset{\sim}{q}_1) \leq \delta$. We show that $\mathrm{lh}(\underset{\sim}{q}_1) \geq \delta$. It suffices to prove the following:

$(*)$ For every $\tau < \delta$ and $p \in \mathbb{P}_U$ there is $q \leq p$ such that $q \Vdash$ "$\underset{\sim}{q}_1(\tau)$ is defined".

Fix $\tau < \delta$ and $p = \langle s, A\rangle \in \mathbb{P}_U$ as in $(*)$. Let t be an end extension of s such that $t \setminus s \subseteq A$ and $\delta_{t(\mathrm{lh}(t)-1)} > \tau$. Then p^t_0 and p are compatible and $p^t_0 \Vdash$ "$\underset{\sim}{q}_1(\tau) = \underset{\sim}{p}^t_1(\tau)$ is defined". Let $q \leq p^t_0, p$. Then $q \Vdash$ "$\underset{\sim}{q}_1(\tau)$ is defined" and $(*)$ follows. Thus $\mathrm{lh}(\underset{\sim}{q}_1) = \delta$.

(2a) is trivial. Let us prove (2b). Thus suppose that $I \subseteq \delta$, $I \in V$, $p \leq \langle\langle\,\rangle, \omega\rangle$ and $J \subseteq \omega$ is finite. Let $p = \langle s, A\rangle$.

First we consider the case where $s = \langle\,\rangle$. Let $a = \emptyset$. We show that a is as required. Thus let $b \subseteq I$ be finite. Let $n \in A$ be such that $n > \max(J) + 1$ and $b \subseteq \delta_n$. Let $t = s^\frown\langle n\rangle$. Note that

$$\forall \beta_1 \neq \beta_2 \in b, \mathrm{range}(\underset{\sim}{p}^t_1(\beta_1)) \cap \mathrm{range}(\underset{\sim}{p}^t_1(\beta_2)) = \emptyset.$$

Let $q = \langle\langle\,\rangle, B\rangle = \langle\langle\,\rangle, A \setminus (\max(J) + 1)\rangle$. Then $q \leq^* p$ and q is compatible with p^t_0. We show that q is as required. We need to show that

(1) $q \Vdash \forall \beta \in b, \forall k \in J, \underset{\sim}{q}_1(\beta) \neq k$,
(2) $q \Vdash \forall \beta_1 \neq \beta_2 \in b, \underset{\sim}{q}_1(\beta_1) \neq \underset{\sim}{q}_1(\beta_2)$.

For (1), if it fails, then we can find $\langle r, D\rangle \leq q, p^t_0$, $\beta \in b$ and $k \in J$ such that $\langle r, D\rangle \leq^* p^r_0$ and $\langle r, D\rangle \Vdash \underset{\sim}{q}_1(\beta) = k$. But $\langle r, D\rangle \Vdash \underset{\sim}{q}_1(\beta) = \underset{\sim}{p}^r_1(\beta) = \underset{\sim}{p}^t_1(\beta)$, and hence $\langle r, D\rangle \Vdash \underset{\sim}{p}^t_1(\beta) = k$. This is impossible since $\min(D) \geq \min(B) > \max(J)$. For (2), if it fails, then we can find $\langle r, D\rangle \leq q, p^t_0$ and $\beta_1 \neq \beta_2 \in b$ such that $\langle r, D\rangle \leq^* p^r_0$ and $\langle r, D\rangle \Vdash \underset{\sim}{q}_1(\beta_1) = \underset{\sim}{q}_1(\beta_2)$. As above it follows that

$\langle r, D\rangle \Vdash \underset{\sim}{p}{}_1^t(\beta_1) = \underset{\sim}{p}{}_1^t(\beta_2)$. This is impossible since for $\beta_1 \neq \beta_2 \in b$, $\mathrm{range}(\underset{\sim}{p}{}_1^t(\beta_1)) \cap \mathrm{range}(\underset{\sim}{p}{}_1^t(\beta_2)) = \emptyset$. Hence q is as required and we are done.

Now consider the case $s \neq \langle\,\rangle$. First we apply (2b) to t^s, $I \cap \mathrm{lh}(t^s)$, p and J to find a finite set $a' \subseteq \mathrm{lh}(t^s)$ such that

(∗∗) For every finite set $b \subseteq I \cap \mathrm{lh}(t^s) \setminus a'$ there is $p' \leq^* p$ such that $p' \Vdash (\forall \beta \in b, \forall k \in J,\ \underset{\sim}{p}{}_1^s(\beta) \neq k) \& (\forall \beta_1 \neq \beta_2 \in b, \underset{\sim}{p}{}_1^s(\beta_1) \neq \underset{\sim}{p}{}_1^s(\beta_2))$.

Let $t^s = \langle t_0, \underset{\sim}{t}{}_1\rangle, \delta_{s(\mathrm{lh}(s)-1)+1} \setminus \delta_{s(\mathrm{lh}(s)-1)} = \{\alpha_{J_1}, \alpha_{J_2}, \ldots\}$, where $J_1 < J_2 < \ldots$ are in $J_{s(\mathrm{lh}(s)-1)+1}$. Define

$$a = a' \cup \{\alpha_1, \alpha_2, \ldots, \alpha_{J_{\mathrm{lh}(s)+1}}\}.$$

We show that a is as required. First apply (∗∗) to $b \cap \mathrm{lh}(t^s)$ to find $p' = \langle s, A'\rangle \leq^* p$ such that

$$p' \Vdash (\forall \beta \in b \cap \mathrm{lh}(t^s), \forall k \in J, \underset{\sim}{t}{}_1(\beta) \neq k) \& (\forall \beta_1 \neq \beta_2 \in b \cap \mathrm{lh}(t^s), \underset{\sim}{t}{}_1(\beta_1) \neq \underset{\sim}{t}{}_1(\beta_2)).$$

Pick $n \in A'$ such that $n > \max(J) + 1$ and $b \subseteq \delta_n$, and let $r = s^\frown\langle n\rangle$. Then

$$\forall \beta_1 \neq \beta_2 \in b \setminus \mathrm{lh}(t^s), \mathrm{range}(\underset{\sim}{p}{}_1^r(\beta_1)) \cap \mathrm{range}(\underset{\sim}{p}{}_1^r(\beta_2)) = \emptyset.$$

Pick $k < \omega$ such that $k > n$ and

$$\forall \beta_1 \in b \cap \mathrm{lh}(t^s), \forall \beta_2 \in b \setminus \mathrm{lh}(t^s), \mathrm{range}(\underset{\sim}{p}{}_1^r(\beta_1)) \cap (\mathrm{range}(\underset{\sim}{p}{}_1^r(\beta_2)) \setminus k) = \emptyset.$$

Let $q = \langle s, B\rangle = \langle s, A' \setminus (\max(J) + k + 1) \cup \{n\}\rangle$. Then $q \leq^* p' \leq^* p$ and q is compatible with p_0^r (since $n \in B$). We show that q is as required. We need to prove the following:

(1) $q \Vdash \forall \beta \in b, \forall k \in J, \underset{\sim}{q}{}_1(\beta) \neq k$,
(2) $q \Vdash \forall \beta_1 \neq \beta_2 \in b \setminus \mathrm{lh}(t^s), \underset{\sim}{q}{}_1(\beta_1) \neq \underset{\sim}{q}{}_1(\beta_2)$,
(3) $q \Vdash \forall \beta_1 \in b \cap \mathrm{lh}(t^s), \forall \beta_2 \in b \setminus \mathrm{lh}(t^s), \underset{\sim}{q}{}_1(\beta_1) \neq \underset{\sim}{q}{}_1(\beta_2)$.

The proofs of (1) and (2) are as in the case $s = \langle\,\rangle$. Let us prove (3). Suppose that (3) fails. Thus we can find $\langle u, D\rangle \leq q, p_0^r$, $\beta_1 \in b \cap \mathrm{lh}(t^s)$ and $\beta_2 \in b \setminus \mathrm{lh}(t^s)$ such that $\langle u, D\rangle \leq^* p_0^u$ and $\langle u, D\rangle \Vdash \underset{\sim}{q}{}_1(\beta_1) = \underset{\sim}{q}{}_1(\beta_2)$. But for $\beta \in b$, $\langle u, D\rangle \Vdash \underset{\sim}{q}{}_1(\beta) = \underset{\sim}{p}{}_1^u(\beta) = \underset{\sim}{p}{}_1^r(\beta)$, and hence $\langle u, D\rangle \Vdash \underset{\sim}{p}{}_1^r(\beta_1) = \underset{\sim}{p}{}_1^r(\beta_2)$. Now note that $\beta_2 = \alpha_i$ for some $i > \mathrm{lh}(s) + 1$, $\min(D) \geq n$ and $\min(D \setminus \{n\}) > k$. Hence by the construction of p^r,

$$\langle u, D\rangle \Vdash \text{“}\underset{\sim}{p}{}_1^r(\beta_2) \geq (i - \mathrm{lh}(s))\text{-th element of } D > k\text{”}.$$

By our choice of k, $\mathrm{range}(\underset{\sim}{p}{}_1^r(\beta_1)) \cap (\mathrm{range}(\underset{\sim}{p}{}_1^r(\beta_2)) \setminus k) = \emptyset$ and we get a contradiction. (3) follows. Thus q is as required, and the claim follows. □

Let

$$\underset{\sim}{\beta} = \{\langle p_0^s, \delta\rangle : s \in [\omega]^{<\omega}, \exists \gamma(\delta < \gamma, p^s \Vdash \underset{\sim}{\alpha} = \gamma)\}.$$

Then $\underset{\sim}{\beta}$ is a $\mathbb{P}_U$-name of an ordinal.

Claim 3.12. $\langle\langle\langle\,\rangle, \omega\rangle, \underset{\sim}{q}{}_1\rangle \Vdash \underset{\sim}{\alpha} = \underset{\sim}{\beta}$.

Proof. Suppose not. There are two cases to be considered.

Case 1. There are $\langle r_0, \underset{\sim}{r}_1\rangle \le \langle\langle\langle\,\rangle,\omega\rangle, \underset{\sim}{q}_1\rangle$ and δ such that $\langle r_0, \underset{\sim}{r}_1\rangle \Vdash$"$\delta \in \underset{\sim}{\alpha}$ and $\delta \notin \underset{\sim}{\beta}$". We may suppose that for some ordinal α, $\langle r_0, \underset{\sim}{r}_1\rangle \Vdash \underset{\sim}{\alpha} = \alpha$. Then $\delta < \alpha$. Let $r_0 = \langle s, A\rangle$. Consider $p^s = \langle p^s_0, \underset{\sim}{p}^s_1\rangle$. Then p^s_0 is compatible with r_0 and there is a $*$-extension of p^s deciding $\underset{\sim}{\alpha}$. Let $t \in N_{s(\mathrm{lh}(s)-1)+1}$ be the $*-$extension of p^s deciding $\underset{\sim}{\alpha}$ chosen in the proof of Lemma 3.9. Let $t = \langle t_0, \underset{\sim}{t}_1\rangle, t_0 = \langle s, B\rangle$, and let γ be such that $\langle t_0, \underset{\sim}{t}_1\rangle \Vdash \underset{\sim}{\alpha} = \gamma$. Let $n \in A \cap B$. Then

- $p_0^{s^\frown\langle n\rangle}, t_0$ and p^s_0 are compatible and $\langle s^\frown\langle n\rangle, A\cap B\cap A_{s^\frown\langle n\rangle}\rangle$ extends them,
- $p^{s^\frown\langle n\rangle} \le t$.

Thus $p^{s^\frown\langle n\rangle} \Vdash \underset{\sim}{\alpha} = \gamma$. Let $u = \langle s^\frown\langle n\rangle, A\cap B\cap A_{s^\frown\langle n\rangle} \setminus (n+1)\rangle$.

Then $u \le p_0^{s^\frown\langle n\rangle}$ and $u \Vdash$"$\underset{\sim}{r}_1$ extends $\underset{\sim}{p}_1^{s^\frown\langle n\rangle}$ which extends $\underset{\sim}{t}_1$". Thus $\langle u, \underset{\sim}{r}_1\rangle \le t, \langle r_0, \underset{\sim}{r}_1\rangle, p^{s^\frown\langle n\rangle}$. It follows that $\alpha = \gamma$. Now $\delta < \gamma$ and $p^{s^\frown\langle n\rangle} \Vdash \underset{\sim}{\alpha} = \gamma$. Hence $\langle p_0^{s^\frown\langle n\rangle}, \delta\rangle \in \underset{\sim}{\beta}$ and $p^{s^\frown\langle n\rangle} \Vdash \delta \in \underset{\sim}{\beta}$. This is impossible since $\langle r_0, \underset{\sim}{r}_1\rangle \Vdash \delta \notin \underset{\sim}{\beta}$.

Case 2. There are $\langle r_0, \underset{\sim}{r}_1\rangle \le \langle\langle\langle\,\rangle,\omega\rangle, \underset{\sim}{q}_1\rangle$ and δ such that $\langle r_0, \underset{\sim}{r}_1\rangle \Vdash$"$\delta \in \underset{\sim}{\beta}$ and $\delta \notin \underset{\sim}{\alpha}$". We may further suppose that for some ordinal α, $\langle r_0, \underset{\sim}{r}_1\rangle \Vdash \underset{\sim}{\alpha} = \alpha$. Thus $\delta \ge \alpha$. Let $r = \langle s, A\rangle$. Then as above p^s_0 is compatible with r and there is a $*$-extension of p^s deciding $\underset{\sim}{\alpha}$. Choose t as in Case 1, $t = \langle t_0, \underset{\sim}{t}_1\rangle, t_0 = \langle s, B\rangle$ and let γ be such that $\langle t_0, \underset{\sim}{t}_1\rangle \Vdash \underset{\sim}{\alpha} = \gamma$. Let $n \in A\cap B$. Then as in Case 1, $\alpha = \gamma$ and $p^{s^\frown\langle n\rangle} \Vdash \underset{\sim}{\alpha} = \gamma$. On the other hand, since $\langle r_0, \underset{\sim}{r}_1\rangle \Vdash \delta \in \underset{\sim}{\beta}$, we can find $\overline{s}$ such that $\overline{s}$ does not contradict $p_0^{s^\frown\langle n\rangle}$, $\langle p^{\overline{s}}_0, p^{\overline{s}}_1\rangle \Vdash \underset{\sim}{\alpha} = \overline{\gamma}$, for some $\overline{\gamma} > \delta$ and $\langle p^{\overline{s}}_0, \delta\rangle \in \underset{\sim}{\beta}$. Now $\overline{\gamma} = \gamma = \alpha > \delta$, which is in contradiction with $\delta \ge \alpha$. The claim follows. □

This completes the proof of Lemma 3.9. □

Lemma 3.13. *Let $\langle p_0, \underset{\sim}{p}_1\rangle \Vdash \underset{\sim}{f} : \omega \longrightarrow ON$. Then there are $\mathbb{P}_U$-names $\underset{\sim}{g}$ and $\underset{\sim}{q}_1$ such that $\langle p_0, \underset{\sim}{q}_1\rangle \le^* \langle p_0, \underset{\sim}{p}_1\rangle$ and $\langle p_0, \underset{\sim}{q}_1\rangle \Vdash \underset{\sim}{f} = \underset{\sim}{g}$.*

Proof. For simplicity suppose that $\langle p_0, \underset{\sim}{p}_1\rangle = \langle\langle\langle\,\rangle,\omega\rangle,\emptyset\rangle$. Let θ be large enough regular and let $\langle N_n : n<\omega\rangle$ be an increasing sequence of countable elementary submodels of H_θ such that $\mathbb{P}, \underset{\sim}{f} \in N_0$ and $N_n \in N_{n+1}$ for every $n<\omega$. Let $N = \bigcup_{n<\omega} N_n$, $\delta_n = N_n \cap \omega_1$ for $n<\omega$ and $\delta = \bigcup_{n<\omega}\delta_n = N\cap\omega_1$. Let $\langle J_n : n<\omega\rangle \in N_0$ and $\langle \alpha_i : 0<i<\omega\rangle$ be as in Lemma 3.9.

We define by induction a sequence $\langle p^s : s \in [\omega]^{<\omega}\rangle$ of conditions and a sequence $\langle \underset{\sim}{\beta}_s : s\in[\omega]^{<\omega}\rangle$ of $\mathbb{P}_U$-names for ordinals such that

- $p^s = \langle p^s_0, \underset{\sim}{p}^s_1\rangle = \langle\langle s, \omega\setminus(\max(s)+1)\rangle, \underset{\sim}{p}^s_1\rangle$,
- $p^s \in N_{s(\mathrm{lh}(s)-1)+1}$,
- $\mathrm{lh}(p^s) \ge \delta_{s(\mathrm{lh}(s)-1)}$,
- $p^s \Vdash$"$\underset{\sim}{f}(\mathrm{lh}(s)-1) = \underset{\sim}{\beta}_s$",
- if t end extends s, then $p^t \le p^s$.

For $s = \langle\,\rangle$, let $p^{\langle\,\rangle} = \langle\langle\langle\,\rangle,\omega\rangle,\emptyset\rangle$. Now suppose that $s \ne \langle\,\rangle$ and $p^{s\restriction\mathrm{lh}(s)-1}$ is defined. We define p^s. Let $C_{s\restriction\mathrm{lh}(s)-1}$ be an infinite subset of ω almost disjoint to $\langle \mathrm{range}(\underset{\sim}{p}_1^{s\restriction\mathrm{lh}(s)-1}(\beta)) : \beta < \mathrm{lh}(p^{s\restriction\mathrm{lh}(s)-1})\rangle$. Split $C_{s\restriction\mathrm{lh}(s)-1}$ into ω infinite disjoint sets $\langle C_{s\restriction\mathrm{lh}(s)-1,t} : t\in[\omega]^{<\omega}$ and the t end extends $s\restriction\mathrm{lh}(s)-1\rangle$. Again split $C_{s\restriction\mathrm{lh}(s)-1,s}$ into ω infinite disjoint sets $\langle C_i : i<\omega\rangle$. Let $\langle c_{ij} : j<\omega\rangle$ be an increasing enumeration of C_i, $i<\omega$. We may suppose that all of these are done in $N_{s(\mathrm{lh}(s)-1)+1}$. Let $q^s = \langle q^s_0, \underset{\sim}{q}^s_1\rangle$, where

- $q^s_0 = \langle s, \omega\setminus(\max(s)+1)\rangle$,
- for $\beta < \mathrm{lh}(p^{s\restriction\mathrm{lh}(s)-1})$, $\underset{\sim}{q}^s_1(\beta) = \underset{\sim}{p}_1^{s\restriction\mathrm{lh}(s)-1}(\beta)$,

- for $i \in J_{s(\mathrm{lh}(s)-1)}$ such that $\alpha_i \in \delta_{s(\mathrm{lh}(s)-1)} \setminus \mathrm{lh}(p^{s\restriction \mathrm{lh}(s)-1})$,
$$\underset{\sim}{q}{}^s_1(\alpha_i) = \{\langle\langle s^\frown\langle r_1, \ldots, r_i\rangle, \omega \setminus (r_i+1)\rangle, c_{ir_i}\rangle : r_1 > \max(s), \langle r_1, \ldots, r_i\rangle \in [\omega]^i\}.$$

Then $q^s \in N_{s(\mathrm{lh}(s)-1)+1}$ and as in the proof of claim 3.10, $q^s \in \mathbb{P}$. By Lemma 3.9, applied inside $N_{s(\mathrm{lh}(s)-1)+1}$, we can find $\mathbb{P}_U$-names $\underset{\sim}{\beta}_s$ and $\underset{\sim}{p}{}^s_1$ such that $\langle q^s_0, \underset{\sim}{p}{}^s_1\rangle \leq \langle q^s_0, \underset{\sim}{q}{}^s_1\rangle$ and $\langle q^s_0, \underset{\sim}{p}{}^s_1\rangle \Vdash \underset{\sim}{f}(\mathrm{lh}(s)-1) = \underset{\sim}{\beta}_s$. Let $p^s = \langle p^s_0, \underset{\sim}{p}{}^s_1\rangle = \langle q^s_0, \underset{\sim}{p}{}^s_1\rangle$. Then $p^s \leq p^{s\restriction \mathrm{lh}(s)-1}$ and $p^s \Vdash \underset{\sim}{f} \restriction \mathrm{lh}(s) = \{\langle i, \underset{\sim}{\beta}_{s\restriction i+1}\rangle : i < \mathrm{lh}(s)\}$.

This completes our definition of the sequences $\langle p^s : s \in [\omega]^{<\omega}\rangle$ and $\langle \underset{\sim}{\beta}_s : s \in [\omega]^{<\omega}\rangle$. Let

$$\begin{aligned} \underset{\sim}{q}{}_1 &= \{\langle p^s_0, \langle \beta, \underset{\sim}{p}{}^s_1(\beta)\rangle\rangle : s \in [\omega]^{<\omega}, \beta < \mathrm{lh}(p^s)\}, \\ \underset{\sim}{g} &= \{\langle p^s_0, \langle i, \underset{\sim}{\beta}_{s\restriction i+1}\rangle\rangle : s \in [\omega]^{<\omega}, i < \mathrm{lh}(s)\}. \end{aligned}$$

Then $\underset{\sim}{q}{}_1$ and $\underset{\sim}{g}$ are $\mathbb{P}_U$-names.

Claim 3.14. $\langle\langle\langle\,\rangle, \omega\rangle, \underset{\sim}{q}{}_1\rangle \in \mathbb{P}$.

Proof. We check conditions in Definition 3.4.

(1) i.e. $\langle\langle\,\rangle, \omega\rangle \in \mathbb{P}_U$ is trivial.

(2) It is clear by our construction that

$$\langle\langle\,\rangle, \omega\rangle \Vdash \text{“}\underset{\sim}{q}{}_1 \text{ is a well-defined function”}$$

and as in the proof of Claim 3.11, we can show that $\mathrm{lh}(\underset{\sim}{q}{}_1) = \delta$. (2a) is trivial. Let us prove (2b). Thus suppose that $I \subseteq \delta$, $I \in V$, $p \leq \langle\langle\,\rangle, \omega\rangle$ and $J \subseteq \omega$ is finite. Let $p = \langle s, A\rangle$. If $s = \langle\,\rangle$, then as in the proof of Claim 3.11, we can show that $a = \emptyset$ is a required. Thus suppose that $s \neq \langle\,\rangle$. First we apply (2b) to p^s, $I \cap \mathrm{lh}(p^s)$, p and J to find $a' \subseteq \mathrm{lh}(p^s)$ such that

$(*)$ For every finite $b \subseteq I \cap \mathrm{lh}(p^s) \setminus a'$ there is $p' \leq^* p$ such that $p' \Vdash (\forall\beta \in b, \forall k \in J, \underset{\sim}{p}{}^s_1(\beta) \neq k) \& (\forall \beta_1 \neq \beta_2 \in b, \underset{\sim}{p}{}^s_1(\beta_1) \neq \underset{\sim}{p}{}^s_1(\beta_2))$.

Let $\delta_{s(\mathrm{lh}(s)-1)+1} \setminus \delta_{s(\mathrm{lh}(s)-1)} = \{\alpha_{J_1}, \ldots, \alpha_{J_i}, \ldots\}$ where $J_1 < J_2 < \ldots$ are in $J_{s(\mathrm{lh}(s)-1)+1}$. Let

$$a = a' \cup \{\alpha_1, \alpha_2, \ldots, \alpha_{J_{\mathrm{lh}(s)}}\}.$$

We show that a is as required. Let $b \subseteq I \setminus a$ be finite. First we apply $(*)$ to $b \cap \mathrm{lh}(p^s)$ to find $p' = \langle s, A'\rangle \leq^* p$ such that

$$p' \Vdash (\forall \beta \in b \cap \mathrm{lh}(p^s), \forall k \in J, \underset{\sim}{p}{}^s_1(\beta) \neq k) \& (\forall \beta_1 \neq \beta_2 \in b \cap \mathrm{lh}(p^s), \underset{\sim}{p}{}^s_1(\beta_1) \neq \underset{\sim}{p}{}^s_1(\beta_2)).$$

Also note that for $\beta \in b \cap \mathrm{lh}(p^s)$, $p' \Vdash \underset{\sim}{q}{}_1(\beta) = \underset{\sim}{p}{}^s_1(\beta)$. Pick m such that $\max(s) + \max(J) + 1 < m < \omega$, and if t end extends s and $m < \max(t)$, then $C_{s,t}$ is disjoint to J and to $\mathrm{range}(\underset{\sim}{p}{}^s_1(\beta))$ for $\beta \in b \cap \mathrm{lh}(p^s)$. Then pick $n > m, n \in A'$ such that $b \subseteq \delta_n$, and let $t = s^\frown\langle n\rangle$. Then

- $\forall \beta_1 \neq \beta_2 \in b \setminus \mathrm{lh}(p^s)$, $\mathrm{range}(\underset{\sim}{p}{}^t_1(\beta_1)) \cap \mathrm{range}(\underset{\sim}{p}{}^t_1(\beta_2)) = \emptyset$,
- $\forall \beta_1 \in b \cap \mathrm{lh}(p^s), \forall \beta_2 \in b \setminus \mathrm{lh}(p^s)$, $\mathrm{range}(\underset{\sim}{p}{}^t_1(\beta_1)) \cap \mathrm{range}(\underset{\sim}{p}{}^t_1(\beta_2)) = \emptyset$,
- $\forall \beta \in b \setminus \mathrm{lh}(p^s), \mathrm{range}(\underset{\sim}{p}{}^t_1(\beta)) \cap J = \emptyset$.

Let $q = \langle s, B\rangle = \langle s, A' \setminus (n+1)\rangle$. Then $q \leq^* p' \leq^* p$ and using the above facts we can show that

$$q \Vdash (\forall\beta \in b, \forall k \in J, \underset{\sim}{q}_1(\beta) = \underset{\sim}{p}^t_1(\beta) \neq k) \& (\forall\beta_1 \neq \beta_2 \in b, \underset{\sim}{q}_1(\beta_1) = \underset{\sim}{p}^t_1(\beta_1) \neq \underset{\sim}{p}^t_1(\beta_2) = \underset{\sim}{q}_1(\beta_2)).$$

Thus q is as required and the claim follows. □

Claim 3.15. $\langle\langle\langle\,\rangle, \omega\rangle, \underset{\sim}{q}_1\rangle \Vdash \underset{\sim}{f} = \underset{\sim}{g}$.

Proof. Suppose not. Then we can find $\langle r_0, \underset{\sim}{r}_1\rangle \leq \langle\langle\langle\,\rangle, \omega\rangle, \underset{\sim}{q}_1\rangle$ and $i < \omega$ such that $\langle r_0, \underset{\sim}{r}_1\rangle \Vdash \underset{\sim}{f}(i) \neq \underset{\sim}{g}(i)$. Let $r_0 = \langle s, A\rangle$. Then r_0 is compatible with p^s_0 and $r_0 \Vdash$ "$\underset{\sim}{r}_1$ extends $\underset{\sim}{p}^s_1$". Hence $\langle r_0, \underset{\sim}{r}_1\rangle \leq \langle p^s_0, \underset{\sim}{p}^s_1\rangle = p^s$. Now $p^s \Vdash \underset{\sim}{g}(i) = \underset{\sim}{\beta}_{s\restriction i+1} = \underset{\sim}{f}(i)$, and we get a contradiction. The claim follows. □

This completes the proof of Lemma 3.13. □

The following is now immediate.

Lemma 3.16. *The forcing* $(\mathbb{P}, \leq)$ *preserves cofinalities.*

Proof. By Lemma 3.13, $\mathbb{P}$ preserves cofinalities $\leq \omega_1$. On the other hand, by a Δ-system argument, $\mathbb{P}$ satisfies the ω_2-c.c. and hence it preserves cofinalities $\geq \omega_2$. □

Lemma 3.17. *Let* G *be* $(\mathbb{P}, \leq)$*-generic over* V*. Then* $V[G] \models GCH$*.*

Proof. By Lemma 3.13, $V[G] \models CH$. Now let $\kappa \geq \omega_1$. Then

$$(2^\kappa)^{V[G]} \leq ((|\mathbb{P}|^{\omega_1})^\kappa)^V \leq (2^\kappa)^V = \kappa^+.$$

The result follows. □

Now we return to the proof of Theorem 3.1. Suppose that G is $(\mathbb{P}, \leq)$-generic over V, and let $V_1 = V[G]$. Then V_1 is a cofinality and GCH preserving generic extension of V. We show that adding a Cohen real over V_1 produces $\aleph_1$-many Cohen reals over V. Thus force to add a Cohen real over V_1. Split it into ω Cohen reals over V_1. Denote them by $\langle r_{n,m} : n, m < \omega\rangle$. Also let $\langle f_i : i < \omega_1\rangle \in V$ be a sequence of almost disjoint functions from ω into ω. First we define a sequence $\langle s_{n,i} : i < \omega_1\rangle$ of reals by

$$\forall k < \omega,\ s_{n,i}(k) = r_{n, f_i(k)}(0).$$

Let $\langle I_n : n < \omega\rangle$ be the partition of ω_1 produced by G. For $\alpha < \omega_1$ let

- $n(\alpha) =$ that $n < \omega$ such that $\alpha \in I_n$,
- $i(\alpha) =$ that $i < \omega_1$ such that α is the i-th element of $I_{n(\alpha)}$.

We define a sequence $\langle t_\alpha : \alpha < \omega_1\rangle$ of reals by $t_\alpha = s_{n(\alpha), i(\alpha)}$. The following lemma completes the proof of Theorem 3.1.

Lemma 3.18. $\langle t_\alpha : \alpha < \omega_1\rangle$ *is a sequence of* $\aleph_1$*-many Cohen reals over* V*.*

Proof. First note that $\langle r_{n,m} : n, m < \omega\rangle$ is $\mathbb{C}(\omega \times \omega)$-generic over V_1. By the c.c.c. of $\mathbb{C}(\omega_1)$ it suffices to show that for every countable $I \subseteq \omega_1$, $I \in V$, $\langle t_\alpha : \alpha \in I\rangle$ is $\mathbb{C}(I)$-generic over V. Thus it suffices to prove the following:

(∗) For every $\langle\langle p_0, \underset{\sim}{p}_1\rangle, q\rangle \in \mathbb{P} * \mathbb{C}(\omega \times \omega)$ and every open dense subset $D \in V$ of $\mathbb{C}(I)$, there is $\langle\langle q_0, \underset{\sim}{q}_1\rangle, r\rangle \leq \langle\langle p_0, \underset{\sim}{p}_1\rangle, q\rangle$ such that $\langle\langle q_0, \underset{\sim}{q}_1\rangle, r\rangle \Vdash$ "$\langle \underset{\sim}{t}_\nu : \nu \in I\rangle$ extends some element of D".

Let $\langle\langle p_0, \underset{\sim}{p}_1\rangle, q\rangle$ and D be as above. Let $\alpha = \sup(I)$. We may suppose that $\mathrm{lh}(\underset{\sim}{p}_1) \geq \alpha$. Let $J = \{n : \exists m, k, \langle n, m, k\rangle \in \mathrm{dom}(q)\}$. We apply (2b) to $\langle p_0, \underset{\sim}{p}_1\rangle$, I, p_0 and J to find a finite set $a \subseteq I$ such that:

(∗∗) For every finite $b \subseteq I \setminus a$ there is $p_0' \leq^* p_0$ such that $p_0' \Vdash (\forall \beta \in b, \forall k \in J, \underset{\sim}{p}_1(\beta) \neq k) \& (\forall \beta_1 \neq \beta_2 \in b, \underset{\sim}{p}_1(\beta_1) \neq \underset{\sim}{p}_1(\beta_2))$.

Let

$$S = \{\langle \nu, k, j\rangle : \nu \in a, k < \omega, j < 2, \langle n(\nu), f_{i(\nu)}(k), 0, j\rangle \in q\}.$$

Then $S \in \mathbb{C}(\omega_1)$. Pick $k_0 < \omega$ such that for all $\nu_1 \neq \nu_2 \in a$, and $k \geq k_0$, $f_{i(\nu_1)}(k) \neq f_{i(\nu_2)}(k)$. Let

$$S^* = S \cup \{\langle \nu, k, 0\rangle : \nu \in a, k < \kappa_0,\ \langle \nu, k, 1\rangle \notin S\}.$$

The reason for defining S^* is to avoid possible collisions. Then $S^* \in \mathbb{C}(\omega_1)$. Pick $S^{**} \in D$ such that $S^{**} \leq S^*$. Let $b = \{\nu : \exists k, j, \langle \nu, k, j\rangle \in S^{**}\} \setminus q$. By (∗∗) there is $p_0' \leq^* p_0$ such that

$$p_0' \Vdash (\forall \nu \in b, \forall k \in J, \underset{\sim}{p}_1(\nu) \neq k) \& (\forall \nu_1 \neq \nu_2 \in b, \underset{\sim}{p}_1(\nu_1) \neq \underset{\sim}{p}_1(\nu_2)).$$

Let $p_0'' \leq p_0'$ be such that $\langle p_0'', \underset{\sim}{p}_1\rangle$ decides all the colors of elements of $a \cup b$. Let

$$q^* = q \cup \{\langle n(\nu), f_{i(\nu)}(k), 0, S^{**}(\nu, k)\rangle : \langle \nu, k\rangle \in \mathrm{dom}(S^{**})\}.$$

Then q^* is well defined and $q^* \in \mathbb{C}(\omega \times \omega)$. Now $q^* \leq q$, $\langle\langle p_0'', \underset{\sim}{p}_1\rangle, q^*\rangle \leq \langle\langle p_0, \underset{\sim}{p}_1\rangle, q\rangle$, and for $\langle \nu, k\rangle \in \mathrm{dom}(S^{**})$,

$$\langle\langle p_0'', \underset{\sim}{p}_1\rangle, q^*\rangle \Vdash S^{**}(\nu, k) = q^*(n(\nu), f_{i(\nu)}(k), 0) = \underset{\sim}{r}_{n(\nu), f_{i(\nu)}(k)}(0) = \underset{\sim}{t}_\nu(k).$$

It follows that

$$\langle\langle p_0'', \underset{\sim}{p}_1\rangle, q^*\rangle \Vdash \text{“}\langle \underset{\sim}{t}_\nu : \nu \in I\rangle \text{ extends } S^{**}\text{”}.$$

(∗) and hence Lemma 3.18 follow. □

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