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AMITSUR'S CONJECTURE FOR POLYNOMIAL H-IDENTITIES OF H-MODULE LIE ALGEBRAS

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ABSTRACT. Consider a finite dimensional H-module Lie algebra L over a field of characteristic 0 where H is a Hopf algebra. We prove the analog of Amitsur's conjecture on asymptotic behaviour for codimensions of polynomial H-identities of L under some assumptions on H. In particular, the conjecture holds when H is finite dimensional semisimple. As a consequence, we obtain the analog of Amitsur's conjecture for graded codimensions of any finite dimensional Lie algebra graded by an arbitrary group and for G-codimensions of any finite dimensional Lie algebra with a rational action of a reductive affine algebraic group G by automorphisms and anti-automorphisms.

1. INTRODUCTION

In the 1980s, a conjecture about the asymptotic behaviour of codimensions of ordinary polynomial identities was made by S. A. Amitsur. Amitsur's conjecture was proved in 1999 by A. Giambruno and M. V. Zaicev [16, Theorem 6.5.2] for associative algebras, in 2002 by M. V. Zaicev [30] for finite dimensional Lie algebras, and in 2011 by A. Giambruno, I. P. Shestakov, M. V. Zaicev for finite dimensional Jordan and alternative algebras [15]. In 2011 the author proved its analog for polynomial identities of finite dimensional representations of Lie algebras [17].

Alongside ordinary polynomial identities of algebras, graded polynomial identities and G- and H-identities are important too [5–10, 22]. Usually, finding such identities is easier than finding the ordinary ones. Furthermore, each of these types of identities completely determines the ordinary polynomial identities. Therefore the question arises as to whether the conjecture holds for graded codimensions and G- and H-codimensions. The analog of Amitsur's conjecture for codimensions of graded identities was proved in 2010–2011 by E. Aljadeff, A. Giambruno, and D. La Mattina [2, 3, 13] for all associative PI-algebras graded by a finite group. As a consequence, they proved the analog of the conjecture for G-codimensions for any associative PI-algebra with an action of a finite Abelian group G by automorphisms. The case when $G = \mathbb{Z}_2$ acts on a finite dimensional associative algebra by automorphisms and anti-automorphisms (i.e. polynomial identities with involution) was

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considered by A. Giambruno and M. V. Zaicev [16, Theorem 10.8.4] in 1999. In 2012 the author [19] proved the analog of Amitsur's conjecture for polynomial H-identities of finite dimensional associative algebras with a generalized H-action. As a consequence, the analog of Amitsur's conjecture was proved for G-codimensions of finite dimensional associative algebras with an action of an arbitrary finite group G by automorphisms and anti-automorphisms, and for H-codimensions of finite dimensional H-module associative algebras for a finite dimensional semisimple Hopf algebra H.

In 2011 the author [18] proved the analog of Amitsur's conjecture for graded polynomial identities of finite dimensional Lie algebras graded by a finite Abelian group and for G-identities of finite dimensional Lie algebras with an action of any finite group (not necessarily Abelian).

This article is concerned with the analog of Amitsur's conjecture for graded codimensions of Lie algebras graded by an arbitrary group (Subsection 1.1), G-codimensions of Lie algebras with a rational action of a reductive affine algebraic group G by automorphisms and anti-automorphisms (Subsection 1.2), and H-codimensions of H-module Lie algebras (Subsections 1.3–1.8). The results of Subsections 1.1 and 1.2 (including the results of [18] as a special case) are derived in Section 7 from the results of Subsection 1.7.

In Subsection 1.8 we provide an explicit formula for the Hopf PI-exponent that is a natural generalization of the formula for the ordinary PI-exponent [30, Definition 2]. Of course, this formula can be used for graded codimensions and G-codimensions as well. The formula has immediate applications. In particular, in Section 8 we apply it to calculate the Hopf PI-exponent for several classes of algebras and to provide a criterion of H-simplicity.

The results obtained provide a useful tool to study polynomial identities in Lie algebras and hence to study Lie algebras themselves.

1.1. **Graded polynomial identities and their codimensions.** Let G be a group. Denote by $F\{X^{\mathrm{gr}}\}$ the absolutely free nonassociative algebra on the countable set $X^{\mathrm{gr}} = \bigcup_{g\in G} X^{(g)}$, $X^{(g)} = \{x_1^{(g)}, x_2^{(g)}, \ldots\}$, over a field F, i.e. the algebra of nonassociative noncommutative polynomials in variables from X^{gr}.

The algebra $F\{X^{\mathrm{gr}}\}$ has the following natural G-grading. The indeterminates from $X^{(g)}$ are said to be homogeneous of degree g. The G-degree of a monomial $x_{i_1}^{(g_1)} \ldots x_{i_t}^{(g_t)} \in F\{X^{\mathrm{gr}}\}$ (arbitrary arrangement of brackets) is defined to be $g_1 g_2 \ldots g_t$, as opposed to its total degree, which is defined to be t. Denote by $F\{X^{\mathrm{gr}}\}^{(g)}$ the subspace of the algebra $F\{X^{\mathrm{gr}}\}$ spanned by all the monomials having G-degree g. Notice that $F\{X^{\mathrm{gr}}\}^{(g)} F\{X^{\mathrm{gr}}\}^{(h)} \subseteq F\{X^{\mathrm{gr}}\}^{(gh)}$, for every $g, h \in G$. It follows that
$$F\{X^{\mathrm{gr}}\} = \bigoplus_{g\in G} F\{X^{\mathrm{gr}}\}^{(g)}$$
is a G-grading.

Consider the intersection I of all graded ideals of $F\{X^{\mathrm{gr}}\}$ containing the set

$$(1) \qquad \big\{u(vw) + v(wu) + w(uv) \mid u, v, w \in F\{X^{\mathrm{gr}}\}\big\} \cup \big\{u^2 \mid u \in F\{X^{\mathrm{gr}}\}\big\}.$$

Then $L(X^{\mathrm{gr}}) := F\{X^{\mathrm{gr}}\}/I$ is *the free G-graded Lie algebra* on X^{gr}, i.e. for any G-graded Lie algebra $L = \bigoplus_{g\in G} L^{(g)}$ and a map $\psi\colon X^{\mathrm{gr}} \to L$ such that $\psi(X^{(g)}) \subseteq L^{(g)}$, there exists a unique homomorphism $\bar\psi\colon L(X^{\mathrm{gr}}) \to L$ of graded algebras such that $\bar\psi\big|_{X^{\mathrm{gr}}} = \psi$.

We use the commutator notation for the multiplication in $L(X^{\mathrm{gr}})$ and other Lie algebras. In this notation, $L = \bigoplus_{g\in G} L^{(g)}$ is *graded* if $[L^{(g)}, L^{(h)}] \subseteq L^{(gh)}$.

Remark. If G is Abelian, then $L(X^{\mathrm{gr}})$ is the ordinary free Lie algebra with free generators from X^{gr} since the ordinary ideal of $F\{X^{\mathrm{gr}}\}$ generated by (1) is already graded. However, if $gh \neq hg$ for some $g, h \in G$, then $[x_i^{(g)}, x_j^{(h)}] = 0$ in $L(X^{\mathrm{gr}})$ for all $i, j \in \mathbb{N}$.

Let $f = f(x_{i_1}^{(g_1)}, \dots, x_{i_t}^{(g_t)}) \in L(X^{\mathrm{gr}})$. We say that f is a *graded polynomial identity* of a G-graded Lie algebra $L = \bigoplus_{g\in G} L^{(g)}$ and write $f \equiv 0$ if $f(a_{i_1}^{(g_1)}, \dots, a_{i_t}^{(g_t)}) = 0$ for all $a_{i_j}^{(g_j)} \in L^{(g_j)}$, $1 \leqslant j \leqslant t$. In other words, f is a graded polynomial identity if for any homomorphism $\psi\colon L(X^{\mathrm{gr}}) \to L$ of graded algebras we have $\psi(f) = 0$. The set $\mathrm{Id}^{\mathrm{gr}}(L)$ of graded polynomial identities of L is a graded ideal of $L(X^{\mathrm{gr}})$. The case of ordinary polynomial identities is included for the trivial group $G = \{e\}$.

Example 1. Let $G = \mathbb{Z}_2 = \{\bar 0, \bar 1\}$, $\mathfrak{gl}_2(F) = \mathfrak{gl}_2(F)^{(\bar 0)} \oplus \mathfrak{gl}_2(F)^{(\bar 1)}$, where $\mathfrak{gl}_2(F)^{(\bar 0)} = \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}$ and $\mathfrak{gl}_2(F)^{(\bar 1)} = \begin{pmatrix} 0 & F \\ F & 0 \end{pmatrix}$. Then $[x^{(\bar 0)}, y^{(\bar 0)}] \in \mathrm{Id}^{\mathrm{gr}}(\mathfrak{gl}_2(F))$.

Example 2 ([26])**.** Consider $L = \left\{ \begin{pmatrix} \mathfrak{gl}_2(F) & 0 \\ 0 & \mathfrak{gl}_2(F) \end{pmatrix} \right\} \subseteq \mathfrak{gl}_4(F)$ with the following grading by the third symmetric group S_3:

$$L^{(e)} = \left\{ \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix} \right\}, \quad L^{\left((12)\right)} = \left\{ \begin{pmatrix} 0 & \beta & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\},$$

$$L^{\left((23)\right)} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda \\ 0 & 0 & \gamma & 0 \end{pmatrix} \right\},$$

the other components are zero, $\alpha, \beta, \gamma, \lambda \in F$. Then $\left[x^{\left((12)\right)}, y^{\left((23)\right)}\right] \in \mathrm{Id}^{\mathrm{gr}}(L)$ and $[x^{(e)}, y^{(e)}] \in \mathrm{Id}^{\mathrm{gr}}(L)$. In fact, $\left[x^{\left((12)\right)}, y^{\left((23)\right)}\right] = 0$ in $L(X^{\mathrm{gr}})$.

Let S_n be the nth symmetric group, $n \in \mathbb{N}$, and

$$V_n^{\mathrm{gr}} := \langle [x_{\sigma(1)}^{(g_1)}, x_{\sigma(2)}^{(g_2)}, \dots, x_{\sigma(n)}^{(g_n)}] \mid g_i \in G, \sigma \in S_n \rangle_F.$$

(All long commutators in the article are left-normed, although this is not important in this particular case by virtue of the Jacobi identity.) The nonnegative integer

$$c_n^{\mathrm{gr}}(L) := \dim \left(\frac{V_n^{\mathrm{gr}}}{V_n^{\mathrm{gr}} \cap \mathrm{Id}^{\mathrm{gr}}(L)} \right)$$

is called the nth *codimension of graded polynomial identities* or the nth *graded codimension* of L.

Remark. Let $\tilde G \supseteq G$ be another group. Denote by $L(X^{\widetilde{\mathrm{gr}}})$, $\mathrm{Id}^{\widetilde{\mathrm{gr}}}(L)$, $V_n^{\widetilde{\mathrm{gr}}}$, $c_n^{\widetilde{\mathrm{gr}}}(L)$ the objects corresponding to the $\tilde G$-grading. Let I be the ideal of $L(X^{\widetilde{\mathrm{gr}}})$ generated by

$x_j^{(g)}$, $j \in \mathbb{N}$, $g \notin G$. We can identify $L(X^{\mathrm{gr}})$ with the subalgebra in $L(X^{\widetilde{\mathrm{gr}}})$. Then $L(X^{\widetilde{\mathrm{gr}}}) = L(X^{\mathrm{gr}}) \oplus I$, $\mathrm{Id}^{\widetilde{\mathrm{gr}}}(L) = \mathrm{Id}^{\mathrm{gr}}(L) \oplus I$, $V_n^{\widetilde{\mathrm{gr}}} = V_n^{\mathrm{gr}} \oplus (V_n^{\widetilde{\mathrm{gr}}} \cap I)$,

$$V_n^{\widetilde{\mathrm{gr}}} \cap \mathrm{Id}^{\widetilde{\mathrm{gr}}}(L) = (V_n^{\mathrm{gr}} \cap \mathrm{Id}^{\mathrm{gr}}(L)) \oplus (V_n^{\widetilde{\mathrm{gr}}} \cap I)$$

(direct sums of subspaces). Hence $c_n^{\widetilde{\mathrm{gr}}}(L) = c_n^{\mathrm{gr}}(L)$ for all $n \in \mathbb{N}$. In particular, we can always replace the grading group with the subgroup generated by the elements corresponding to the nonzero components.

The analog of Amitsur's conjecture for graded codimensions can be formulated as follows.

Conjecture. *There exists* $\mathrm{PIexp}^{\mathrm{gr}}(L) := \lim\limits_{n\to\infty} \sqrt[n]{c_n^{\mathrm{gr}}(L)}$ *which is a nonnegative integer.*

Remark. I. B. Volichenko [29] gave an example of an infinite dimensional Lie algebra L with a nontrivial polynomial identity for which the growth of codimensions $c_n(L)$ of ordinary polynomial identities is overexponential. M. V. Zaicev and S. P. Mishchenko [24, 31] gave an example of an infinite dimensional Lie algebra L with a nontrivial polynomial identity such that there exists a fractional $\mathrm{PIexp}(L) := \lim\limits_{n\to\infty} \sqrt[n]{c_n(L)}$.

Theorem 1. *Let L be a finite dimensional non-nilpotent Lie algebra over a field F of characteristic* 0*, graded by an arbitrary group G. Then there exist constants $C_1, C_2 > 0$, $r_1, r_2 \in \mathbb{R}$, $d \in \mathbb{N}$ such that $C_1 n^{r_1} d^n \leqslant c_n^{\mathrm{gr}}(L) \leqslant C_2 n^{r_2} d^n$ for all $n \in \mathbb{N}$.*

Corollary. *The above analog of Amitsur's conjecture holds for such codimensions.*

Remark. If L is nilpotent, i.e. $[x_1, \dots, x_p] \equiv 0$ for some $p \in \mathbb{N}$, then, by the Jacobi identity, $V_n^{\mathrm{gr}} \subseteq \mathrm{Id}^{\mathrm{gr}}(L)$ and $c_n^{\mathrm{gr}}(L) = 0$ for all $n \geqslant p$.

Theorem 2. *Let $L = L_1 \oplus \ldots \oplus L_s$ (direct sum of graded ideals) be a finite dimensional Lie algebra over a field F of characteristic* 0*, graded by an arbitrary group G. Then* $\mathrm{PIexp}^{\mathrm{gr}}(L) = \max_{1 \leqslant i \leqslant s} \mathrm{PIexp}^{\mathrm{gr}}(L_i)$.

Theorems 1 and 2 will be obtained as consequences of Theorems 9 and 10 below in Subsection 7.1.

1.2. **Polynomial G-identities and their codimensions.** Let L be a Lie algebra over a field F. Recall that $\psi \in \mathrm{GL}(L)$ is an *automorphism* of L if $\psi([a,b]) = [\psi(a), \psi(b)]$ for all $a, b \in L$ and an *anti-automorphism* of L if $\psi([a,b]) = [\psi(b), \psi(a)]$ for all $a, b \in L$. Obviously, ψ is an anti-automorphism if and only if $(-\psi)$ is an automorphism. Automorphisms of L form the group denoted by $\mathrm{Aut}(L)$. Automorphisms and anti-automorphisms of A form the group denoted by $\mathrm{Aut}^*(L)$. Note that $\mathrm{Aut}(L)$ is a normal subgroup of $\mathrm{Aut}^*(L)$ of index 2.

Let G be a group with a fixed (normal) subgroup G_0 of index $\leqslant 2$. We say that a Lie algebra L is an *algebra with G-action* or a *G-algebra* if L is endowed with a homomorphism $\zeta\colon G \to \mathrm{Aut}^*(L)$ such that $\zeta^{-1}(\mathrm{Aut}(L)) = G_0$. Note that if G is an affine algebraic group acting rationally on L, then G_0 is closed.

We use the exponential notation for the action of a group and its group algebra, and write $a^g := \zeta(g)(a)$ for $g \in G$ and $a \in L$.

Denote by $L(X|G)$ the free Lie algebra over F with free formal generators x_j^g, $j \in \mathbb{N}$, $g \in G$. Here $X := \{x_1, x_2, x_3, \ldots\}$, $x_j := x_j^1$. Define

$$[x_{i_1}^{g_1}, x_{i_2}^{g_2}, \ldots, x_{i_{n-1}}^{g_{n-1}}, x_{i_n}^{g_n}]^g := [x_{i_1}^{gg_1}, x_{i_2}^{gg_2}, \ldots, x_{i_{n-1}}^{gg_{n-1}}, x_{i_n}^{gg_n}] \text{ for } g \in G_0,$$
$$[x_{i_1}^{g_1}, x_{i_2}^{g_2}, \ldots, x_{i_{n-1}}^{g_{n-1}}, x_{i_n}^{g_n}]^g := [x_{i_n}^{gg_n}, [x_{i_{n-1}}^{gg_{n-1}}, \ldots, [x_{i_2}^{gg_2}, x_{i_1}^{gg_1}] \ldots] \text{ for } g \in G \backslash G_0.$$

Then $L(X|G)$ becomes *the free Lie G-algebra* with free generators x_j, $j \in \mathbb{N}$. We call its elements Lie G-polynomials.

Let L be a Lie G-algebra over F. A G-polynomial $f(x_1, \ldots, x_n) \in L(X|G)$ is a *G-identity* of L if $f(a_1, \ldots, a_n) = 0$ for all $a_i \in L$. In this case we write $f \equiv 0$. The set $\mathrm{Id}^G(L)$ of all G-identities of L is an ideal in $L(X|G)$ invariant under G-action.

Example 3. Consider $\psi \in \mathrm{Aut}(\mathfrak{gl}_2(F))$ defined by the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\psi} := \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.$$

Then $[x + x^{\psi}, y + y^{\psi}] \in \mathrm{Id}^G(\mathfrak{gl}_2(F))$, where $G = \langle \psi \rangle \cong \mathbb{Z}_2$.

Denote by V_n^G the space of all multilinear Lie G-polynomials in $x_1, \ldots, x_n$, i.e.

$$V_n^G = \langle [x_{\sigma(1)}^{g_1}, x_{\sigma(2)}^{g_2}, \ldots, x_{\sigma(n)}^{g_n}] \mid g_i \in G, \sigma \in S_n \rangle_F.$$

Then the number $c_n^G(L) := \dim \left(\frac{V_n^G}{V_n^G \cap \mathrm{Id}^G(L)} \right)$ is called the nth *codimension of polynomial G-identities* or the nth *G-codimension* of L.

The analog of Amitsur's conjecture for G-codimensions can be formulated as follows.

Conjecture. *There exists* $\mathrm{PIexp}^G(L) := \lim\limits_{n \to \infty} \sqrt[n]{c_n^G(L)} \in \mathbb{Z}_+$.

We claim that the following theorem holds:

Theorem 3. *Let L be a finite dimensional nonnilpotent Lie algebra over an algebraically closed field F of characteristic* 0. *Suppose a reductive affine algebraic group G acts on L rationally by automorphisms and anti-automorphisms. Then there exist constants $C_1, C_2 > 0$, $r_1, r_2 \in \mathbb{R}$, $d \in \mathbb{N}$ such that $C_1 n^{r_1} d^n \leqslant c_n^G(L) \leqslant C_2 n^{r_2} d^n$ for all $n \in \mathbb{N}$.*

In addition, we derive the following theorem (which one could regard as a special case of Theorem 3).

Theorem 4 ([18, Theorem 2])**.** *Let L be a finite dimensional nonnilpotent Lie algebra over a field F of characteristic* 0. *Suppose a finite group G acts on L by automorphisms and anti-automorphisms. Then there exist constants $C_1, C_2 > 0$, $r_1, r_2 \in \mathbb{R}$, $d \in \mathbb{N}$ such that $C_1 n^{r_1} d^n \leqslant c_n^G(L) \leqslant C_2 n^{r_2} d^n$ for all $n \in \mathbb{N}$.*

Corollary. *The above analog of Amitsur's conjecture holds for such codimensions.*

Remark. If L is nilpotent, i.e. $[x_1, \ldots, x_p] \equiv 0$ for some $p \in \mathbb{N}$, then $V_n^G \subseteq \mathrm{Id}^G(L)$ and $c_n^G(L) = 0$ for all $n \geqslant p$.

Theorem 5. *Let $L = L_1 \oplus \ldots \oplus L_s$ (direct sum of ideals) be a finite dimensional Lie algebra over an algebraically closed field F of characteristic* 0. *Suppose a reductive affine algebraic group G acts on L rationally by automorphisms and anti-automorphisms and the ideals L_i are G-invariant. Then* $\mathrm{PIexp}^G(L) = \max_{1 \leqslant i \leqslant s} \mathrm{PIexp}^G(L_i)$.

Theorem 6. *Let $L = L_1 \oplus \ldots \oplus L_s$ (direct sum of ideals) be a finite dimensional Lie algebra over a field F of characteristic* 0. *Suppose a finite group G acts on L by automorphisms and anti-automorphisms, and the ideals L_i are G-invariant. Then*

$$\mathrm{PIexp}^G(L) = \max_{1\leqslant i\leqslant s} \mathrm{PIexp}^G(L_i).$$

Theorems 3, 4, 5, and 6 will be obtained as consequences of Theorems 9 and 10 below in Subsection 7.2.

1.3. **Polynomial H-identities and their codimensions.** Analogously, one can consider polynomial H-identities for H-module Lie algebras. The connection between graded, G- and H-identities will become clear in Section 7.

Let H be a Hopf algebra over a field F. An algebra A is an *H-module algebra* if A is endowed with a left H-action $h\otimes a \mapsto ha$ or, equivalently, with a homomorphism $H \to \mathrm{End}_F(A)$, such that $h(ab) = (h_{(1)}a)(h_{(2)}b)$ for all $h \in H$, $a, b \in A$. Here we use Sweedler's notation $\Delta h = h_{(1)} \otimes h_{(2)}$, where Δ is the comultiplication in H. We refer the reader to [1, 11, 25, 28] for an account of Hopf algebras and algebras with Hopf algebra actions.

In particular, a Lie algebra L is an *H-module Lie algebra* if

$$h\,[a,b] = [h_{(1)}a, h_{(2)}b] \text{ for all } h \in H,\ a, b \in L. \tag{2}$$

Let $F\{X\}$ be the absolutely free nonassociative algebra on the set $X := \{x_1, x_2, x_3, \ldots\}$. Then $F\{X\} = \bigoplus_{n=1}^{\infty} F\{X\}^{(n)}$, where $F\{X\}^{(n)}$ is the linear span of all monomials of total degree n. Let H be a Hopf algebra over a field F. Consider the algebra

$$F\{X|H\} := \bigoplus_{n=1}^{\infty} H^{\otimes n} \otimes F\{X\}^{(n)}$$

with the multiplication $(u_1 \otimes w_1)(u_2 \otimes w_2) := (u_1 \otimes u_2) \otimes w_1 w_2$ for all $u_1 \in H^{\otimes j}$, $u_2 \in H^{\otimes k}$, $w_1 \in F\{X\}^{(j)}$, $w_2 \in F\{X\}^{(k)}$. We use the notation

$$x^{h_1}_{i_1} x^{h_2}_{i_2} \ldots x^{h_n}_{i_n} := (h_1 \otimes h_2 \otimes \ldots \otimes h_n) \otimes x_{i_1} x_{i_2} \ldots x_{i_n}$$

(the arrangements of brackets on x_{i_j} and on $x^{h_j}_{i_j}$ are the same). Here $h_1 \otimes h_2 \otimes \ldots \otimes h_n \in H^{\otimes n}$, $x_{i_1} x_{i_2} \ldots x_{i_n} \in F\{X\}^{(n)}$.

Note that if $(\gamma_\beta)_{\beta\in\Lambda}$ is a basis in H, then $F\{X|H\}$ is isomorphic to the absolutely free nonassociative algebra over F with free formal generators $x^{\gamma_\beta}_i$, $\beta \in \Lambda$, $i \in \mathbb{N}$.

Define on $F\{X|H\}$ the structure of a left H-module by

$$h\,(x^{h_1}_{i_1} x^{h_2}_{i_2} \ldots x^{h_n}_{i_n}) = x^{h_{(1)}h_1}_{i_1} x^{h_{(2)}h_2}_{i_2} \ldots x^{h_{(n)}h_n}_{i_n},$$

where $h_{(1)} \otimes h_{(2)} \otimes \ldots \otimes h_{(n)}$ is the image of h under the comultiplication Δ applied $(n-1)$ times, $h \in H$. Then $F\{X|H\}$ is *the absolutely free H-module nonassociative algebra* on X; i.e. for each map $\psi\colon X \to A$ where A is an H-module algebra, there exists a unique homomorphism $\bar\psi\colon F\{X|H\} \to A$ of algebras and H-modules, such that $\bar\psi\big|_X = \psi$. Here we identify X with the set $\{x^1_j \mid j \in \mathbb{N}\} \subset F\{X|H\}$.

Consider the H-invariant ideal I in $F\{X|H\}$ generated by the set

$$\big\{u(vw) + v(wu) + w(uv) \mid u, v, w \in F\{X|H\}\big\} \cup \big\{u^2 \mid u \in F\{X|H\}\big\}. \tag{3}$$

Then $L(X|H) := F\{X|H\}/I$ is *the free H-module Lie algebra* on X, i.e. for any H-module Lie algebra L and a map $\psi\colon X \to L$, there exists a unique homomorphism

$\bar\psi\colon L(X|H) \to L$ of algebras and H-modules such that $\bar\psi\big|_X = \psi$. We refer to the elements of $L(X|H)$ as *Lie H-polynomials.*

Remark. If H is cocommutative and $\operatorname{char} F \ne 2$, then $L(X|H)$ is the ordinary free Lie algebra with free generators $x_i^{\gamma_\beta}$, $\beta \in \Lambda$, $i \in \mathbb{N}$, where $(\gamma_\beta)_{\beta\in\Lambda}$ is a basis in H, since the ordinary ideal of $F\{X|H\}$ generated by (3) is already H-invariant. However, if $h_{(1)} \otimes h_{(2)} \ne h_{(2)} \otimes h_{(1)}$ for some $h \in H$, we still have

$$[x_i^{h_{(1)}}, x_j^{h_{(2)}}] = h[x_i, x_j] = -h[x_j, x_i] = -[x_j^{h_{(1)}}, x_i^{h_{(2)}}] = [x_i^{h_{(2)}}, x_j^{h_{(1)}}]$$

in $L(X|H)$ for all $i, j \in \mathbb{N}$; i.e. in the case $h_{(1)} \otimes h_{(2)} \ne h_{(2)} \otimes h_{(1)}$ the algebra $L(X|H)$ is not free as an ordinary Lie algebra.

Let L be an H-module Lie algebra for some Hopf algebra H over a field F. An H-polynomial $f \in L(X|H)$ is an *H-identity* of L if $\psi(f) = 0$ for all homomorphisms $\psi\colon L(X|H) \to L$ of algebras and H-modules. In other words, $f(x_1, x_2, \dots, x_n)$ is a polynomial H-identity of L if and only if $f(a_1, a_2, \dots, a_n) = 0$ for any $a_i \in L$. In this case we write $f \equiv 0$. The set $\operatorname{Id}^H(L)$ of all polynomial H-identities of L is an H-invariant ideal of $L(X|H)$.

Example 4. Consider $e_0, e_1 \in \operatorname{End}_F(\mathfrak{gl}_2(F))$ defined by the formulas

$$e_0 \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

and

$$e_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}.$$

Then $H := Fe_0 \oplus Fe_1$ (direct sum of ideals) is a Hopf algebra with the counit ε, where $\varepsilon(e_0) := 1$, $\varepsilon(e_1) := 0$, the comultiplication Δ where

$$\Delta(e_0) := e_0 \otimes e_0 + e_1 \otimes e_1,$$

$$\Delta(e_1) := e_0 \otimes e_1 + e_1 \otimes e_0,$$

and the antipode $S := \mathrm{id}$. Moreover, $\mathfrak{gl}_2(F)$ is an H-module Lie algebra and $[x^{e_0}, y^{e_0}] \in \operatorname{Id}^H(\mathfrak{gl}_2(F))$. As we shall see in Subsection 7.1, this H-action and this H-identity corresponds to the $\mathbb{Z}_2$-grading and the graded identity from Example 1.

Denote by V_n^H the space of all multilinear Lie H-polynomials in $x_1, \dots, x_n$, $n \in \mathbb{N}$, i.e.

$$V_n^H = \langle [x_{\sigma(1)}^{h_1}, x_{\sigma(2)}^{h_2}, \dots, x_{\sigma(n)}^{h_n}] \mid h_i \in H, \sigma \in S_n \rangle_F \subset L(X|H).$$

Then the number $c_n^H(L) := \dim\left(\frac{V_n^H}{V_n^H \cap \operatorname{Id}^H(L)}\right)$ is called the nth *codimension of polynomial H-identities* or the nth *H-codimension* of L.

Remark. One can treat polynomial H-identities of L as identities of a nonassociative algebra (i.e. use $F\{X|H\}$ instead of $L(X|H)$) and define their codimensions. However, these codimensions will coincide with $c_n^H(L)$ since the nth H-codimension equals the dimension of the subspace in $\operatorname{Hom}_F(L^{\otimes n}; L)$ that consists of those n-linear functions that can be represented by H-polynomials. (See also the proof of Lemma 1 below.)

1.4. **Some bounds for H-codimensions.** In this subsection we prove several inequalities for H-codimensions and ordinary codimensions. They show that H-codimensions have an asymptotic behaviour that is in some sense similar to the behaviour of the ordinary codimensions. Hence we indeed may expect from H-codimensions to satisfy the analog of Amitsur's conjecture (see Subsection 1.5).

As in the case of ordinary codimensions, we have the following upper bound:

Lemma 1. *Let L be an H-module Lie algebra for some Hopf algebra H over an arbitrary field F. Then $c_n^H(L) \leqslant (\dim L)^{n+1}$ for all $n \in \mathbb{N}$.*

Proof. Consider H-polynomials as n-linear maps from L to L. Then we have a natural map $V_n^H \to \mathrm{Hom}_F(L^{\otimes n}; L)$ with the kernel $V_n^H \cap \mathrm{Id}^H(L)$ that leads to the embedding

$$\frac{V_n^H}{V_n^H \cap \mathrm{Id}^H(L)} \hookrightarrow \mathrm{Hom}_F(L^{\otimes n}; L).$$

Thus

$$c_n^H(L) = \dim\left(\frac{V_n^H}{V_n^H \cap \mathrm{Id}^H(L)}\right) \leqslant \dim \mathrm{Hom}_F(L^{\otimes n}; L) = (\dim L)^{n+1}.$$

□

Denote by V_n the space of ordinary multilinear Lie polynomials in the noncommuting variables $x_1, \dots, x_n$ and by $\mathrm{Id}(L)$ the set of ordinary polynomial identities of L. In other words, $V_n = V_n^{\mathrm{gr}}$ and $\mathrm{Id}(L) = \mathrm{Id}^{\mathrm{gr}}(L)$ for the trivial grading on L by the trivial group $\{e\}$. Then $c_n(L) := \dim \frac{V_n}{V_n \cap \mathrm{Id}(L)}$.

The next lemma is an analog of [16, Lemmas 10.1.2 and 10.1.3].

Lemma 2. *Let L be an H-module Lie algebra for some Hopf algebra H over a field F and let $\zeta \colon H \to \mathrm{End}_F(L)$ be the homomorphism corresponding to the H-action. Then*

$$c_n(L) \leqslant c_n^H(L) \leqslant (\dim \zeta(H))^n c_n(L) \text{ for all } n \in \mathbb{N}.$$

Proof. As in Lemma 1, we consider polynomials as n-linear maps from L to L and identify $\frac{V_n}{V_n \cap \mathrm{Id}(L)}$ and $\frac{V_n^H}{V_n^H \cap \mathrm{Id}^H(L)}$ with the corresponding subspaces in $\mathrm{Hom}_F(L^{\otimes n}; L)$. Then

$$\frac{V_n}{V_n \cap \mathrm{Id}(L)} \subseteq \frac{V_n^H}{V_n^H \cap \mathrm{Id}^H(L)} \subseteq \mathrm{Hom}_F(L^{\otimes n}; L)$$

and the lower bound follows.

Choose such $f_1, \dots, f_t \in V_n$ so that their images form a basis in $\frac{V_n}{V_n \cap \mathrm{Id}(L)}$. Then for any monomial $[x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}]$, $\sigma \in S_n$, there exist $\alpha_{i,\sigma} \in F$ such that

$$[x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}] - \sum_{i=1}^{t} \alpha_{i,\sigma} f_i(x_1, \dots, x_n) \in \mathrm{Id}(L). \tag{4}$$

Let $\left(\zeta(\gamma_j)\right)_{j=1}^m$, $\gamma_j \in H$, be a basis in $\zeta(H)$. Then for every $h \in H$ there exist such $\alpha_j \in F$ so that $\zeta(h) = \sum_{j=1}^m \alpha_j \zeta(\gamma_j)$ and

$$x^h - \sum_{j=1}^{m} \alpha_j x^{\gamma_j} \in \mathrm{Id}^H(L). \tag{5}$$

Thus the linear span of $[x^{\gamma_{i_1}}_{\sigma(1)}, x^{\gamma_{i_2}}_{\sigma(2)}, \ldots, x^{\gamma_{i_n}}_{\sigma(n)}]$, $\sigma \in S_n$, $1 \leqslant i_j \leqslant m$, coincides with V_n^H modulo $V_n^H \cap \mathrm{Id}^H(L)$. Note that (4) implies

$$[x^{\gamma_{i_1}}_{\sigma(1)}, x^{\gamma_{i_2}}_{\sigma(2)}, \ldots, x^{\gamma_{i_n}}_{\sigma(n)}] - \sum_{i=1}^{t} \alpha_{i,\sigma} f_i(x_1^{\gamma_{i_1}}, \ldots, x_n^{\gamma_{i_n}}) \in \mathrm{Id}^H(L).$$

Hence any H-polynomial from V_n^H can be expressed modulo $\mathrm{Id}^H(L)$ as a linear combination of H-polynomials $f_i(x_1^{\gamma_{i_1}}, \ldots, x_n^{\gamma_{i_n}})$. The number of such polynomials equals $m^n t = \left(\dim \zeta(H)\right)^n c_n(L)$ that finishes the proof. □

1.5. **The analog of Amitsur's conjecture for H-codimensions.** Let L be an H-module Lie algebra. The analog of Amitsur's conjecture for H-codimensions of L can be formulated as follows.

Conjecture. *There exists* $\mathrm{PIexp}^H(L) := \lim_{n\to\infty} \sqrt[n]{c_n^H(L)} \in \mathbb{Z}_+$.

We call $\mathrm{PIexp}^H(L)$ the *Hopf PI-exponent* of L.

Theorem 7. *Let L be a finite dimensional nonnilpotent H-module Lie algebra over a field F of characteristic* 0 *where H is a finite dimensional semisimple Hopf algebra. Then there exist constants $C_1, C_2 > 0$, $r_1, r_2 \in \mathbb{R}$, $d \in \mathbb{N}$ such that $C_1 n^{r_1} d^n \leqslant c_n^H(L) \leqslant C_2 n^{r_2} d^n$ for all $n \in \mathbb{N}$.*

Remark. If L is nilpotent, i.e. $[x_1, \ldots, x_p] \equiv 0$ for some $p \in \mathbb{N}$, then $V_n^H \subseteq \mathrm{Id}^H(L)$ and $c_n^H(L) = 0$ for all $n \geqslant p$.

Corollary. *The above analog of Amitsur's conjecture holds for such codimensions.*

Theorem 8. *Let $L = L_1 \oplus \ldots \oplus L_s$ (direct sum of H-invariant ideals) be a finite dimensional H-module Lie algebra over a field F of characteristic* 0 *where H is a finite dimensional semisimple Hopf algebra. Then* $\mathrm{PIexp}^H(L) = \max_{1\leqslant i\leqslant s} \mathrm{PIexp}^H(L_i)$.

Theorems 7 and 8 are obtained as consequences of more general results in Subsection 1.7.

1.6. **(H, L)-modules.** In this subsection we give the definitions needed to formulate the more general result and to present the explicit formula for the Hopf PI-exponent.

Let L be an H-module Lie algebra for some Hopf algebra H over a field F.

We say that M is an *(H, L)-module* if M is both a left H- and L-module and

(6) $$h(av) = (h_{(1)}a)(h_{(2)}v) \text{ for all } a \in L,\ v \in M,\ h \in H.$$

By (2), each H-invariant ideal of L can be regarded as a left (H, L)-module under the adjoint representation of L.

An (H, L)-module M is *irreducible* if for any H- and L-invariant subspace $M_1 \subseteq M$ we have either $M_1 = 0$ or $M_1 = M$. We say that M is *completely reducible* if M is the direct sum of irreducible (H, L)-submodules.

If M is an H-module, then $\mathrm{End}_F(M)$ has the natural structure of an associative H-module algebra:

(7) $$h\psi := \zeta(h_{(1)})\psi\zeta(Sh_{(2)}), \text{ where } h \in H \text{ and } \psi \in \mathrm{End}_F(M).$$

Here $\zeta\colon H \to \mathrm{End}_F(M)$ is the homomorphism corresponding to the H-action. The Lie algebra $\mathfrak{gl}(M)$ inherits the structure of an H-module from $\mathrm{End}_F(M)$. However, if H is not cocommutative, we cannot claim that $\mathfrak{gl}(M)$ satisfies (2).

If M is an (H,L)-module where L is an H-module Lie algebra and $\varphi\colon L \to \mathfrak{gl}(M)$ is the corresponding homomorphism, then φ is a homomorphism of H-modules. Moreover, (6) is equivalent to

$$\zeta(h)\varphi(a) = \varphi(h_{(1)}a)\zeta(h_{(2)}),\ \text{where } h \in H,\ a \in L. \tag{8}$$

We say that an (H,L)-module M is *faithful* if it is faithful as an L-module, i.e. if $\ker\varphi = 0$.

1.7. ***H*-nice Lie algebras.** Let L be a finite dimensional H-module Lie algebra where H is a Hopf algebra over an algebraically closed field F of characteristic 0. We say that L is *H-nice* if either L is semisimple or the following conditions hold:

(1) the nilpotent radical N and the solvable radical R of L are H-invariant;
(2) *(Levi decomposition)* there exists an H-invariant maximal semisimple subalgebra $B \subseteq L$ such that $L = B \oplus R$ (direct sum of H-modules);
(3) *(Wedderburn — Mal'cev decompositions)* for any H-submodule $W \subseteq L$ and associative H-module subalgebra $A_1 \subseteq \mathrm{End}_F(W)$, the Jacobson radical $J(A_1)$ is H-invariant and there exists an H-invariant maximal semisimple associative subalgebra $\tilde{A}_1 \subseteq A_1$ such that $A_1 = \tilde{A}_1 \oplus J(A_1)$ (direct sum of H-submodules);
(4) for any H-invariant Lie subalgebra $L_0 \subseteq \mathfrak{gl}(L)$ such that L_0 is an H-module algebra and L is a completely reducible L_0-module disregarding H-action, L is a completely reducible (H,L_0)-module.

Example 5. Suppose L is a finite dimensional H-module Lie algebra where H is a finite dimensional semisimple Hopf algebra over F. Then L is H-nice.

Proof. By [20, Theorem 1], the solvable radical R and the nilpotent radical N of L are H-invariant. Moreover, by [20, Theorem 3], $L = B \oplus R$ (direct sum of H-submodules), where B is some maximal semisimple subalgebra of L. By [23] and [27, Corollary 2.7], we have an H-invariant Wedderburn — Malcev decomposition. By [20, Theorem 10], condition (4) holds. Hence L is H-nice. ☐

Let G be any group. Denote by FG the group algebra of G. Then FG is a Hopf algebra with the comultiplication $\Delta(g) = g \otimes g$, the counit $\varepsilon(g) = 1$, and the antipode $S(g) = g^{-1}$, $g \in G$.

Example 6. Let L be a finite dimensional Lie algebra over F. Suppose a reductive affine algebraic group G acts on L rationally by automorphisms. Then L is an FG-nice algebra where the action of the Hopf algebra FG on L is the extension of the G-action by linearity.

Proof. First, the solvable radical R and the nilpotent radical N are G-invariant since they are invariant under all automorphisms. Hence condition (1) holds. By [20, Theorem 5], we have a G-invariant Levi decomposition.

Suppose $W \subseteq L$ is an FG-submodule. Consider the FG-action (7) on $\mathrm{End}_F(W)$. It corresponds to the natural rational G-action on $\mathrm{End}_F(W)$: $\psi^g w = (\psi w^{g^{-1}})^g$ for $w \in W$, $\psi \in \mathrm{End}_F(W)$, $g \in G$. Hence, by [27, Corollary 2.10], for any FG-module subalgebra $A_1 \subseteq \mathrm{End}_F(W)$ we have an FG-invariant Wedderburn — Mal'cev decomposition.

Analogously, for any FG-module Lie subalgebra $L_0 \subseteq \mathfrak{gl}(L)$ the space L is a (G, L_0)-*module*, i.e. $(\psi a)^g = \psi^g a^g$ for all $\psi \in L_0$, $a \in L$, $g \in G$. By [20, Theorem 11], if L is a completely reducible L_0-module disregarding G-action, it is a completely reducible (FG, L_0)-module.

Therefore, L is an FG-nice algebra. □

Another important example of an H-nice algebra is Example 9 in Subsection 7.1 below. (In fact, one can regard Example 9 as a special case of Example 6.)

Theorem 9 is the main result of the article.

Theorem 9. *Let L be a nonnilpotent H-nice Lie algebra over an algebraically closed field F of characteristic* 0*. Then there exist constants $C_1, C_2 > 0$, $r_1, r_2 \in \mathbb{R}$, $d \in \mathbb{N}$ such that $C_1 n^{r_1} d^n \leqslant c_n^H(L) \leqslant C_2 n^{r_2} d^n$ for all $n \in \mathbb{N}$.*

Theorem 10. *Let $L = L_1 \oplus \ldots \oplus L_s$ (direct sum of H-invariant ideals) be an H-module Lie algebra over an algebraically closed field F of characteristic* 0 *where H is a Hopf algebra. Suppose L_i are H-nice algebras. Then there exists* $\operatorname{PIexp}^H(L) = \max_{1 \leqslant i \leqslant s} \operatorname{PIexp}^H(L_i)$.

Theorems 9 and 10 are proved at the end of Section 6. A formula for $d = \operatorname{PIexp}^H(L)$ is given in Subsection 1.8.

Proof of Theorems 7 *and* 8. Suppose L is a finite dimensional H-module Lie algebra where H is a finite dimensional semisimple Hopf algebra over a field of characteristic 0.

H-codimensions do not change upon an extension of the base field. The proof is analogous to the cases of ordinary codimensions of associative [16, Theorem 4.1.9] and Lie algebras [30, Section 2]. Thus without loss of generality we may assume F to be algebraically closed.

Using Example 5, we derive Theorem 7 from Theorem 9. Analogously, Theorem 8 is obtained from Theorem 10. □

1.8. **Formula for the Hopf PI-exponent.** Suppose L is an H-nice Lie algebra.

Consider H-invariant ideals $I_1, I_2, \ldots, I_r$, $J_1, J_2, \ldots, J_r$, $r \in \mathbb{Z}_+$, of the algebra L such that $J_k \subseteq I_k$, satisfying the conditions

1. (1) I_k/J_k is an irreducible (H, L)-module;
2. (2) for any H-invariant B-submodules T_k such that $I_k = J_k \oplus T_k$, there exist numbers $q_i \geqslant 0$ such that

$$\big[[T_1, \underbrace{L, \ldots, L}_{q_1}], [T_2, \underbrace{L, \ldots, L}_{q_2}], \ldots, [T_r, \underbrace{L, \ldots, L}_{q_r}]\big] \neq 0.$$

Let M be an L-module. Denote by $\operatorname{Ann} M$ its annihilator in L. Let

$$d(L) := \max \left(\dim \frac{L}{\operatorname{Ann}(I_1/J_1) \cap \cdots \cap \operatorname{Ann}(I_r/J_r)} \right),$$

where the maximum is found among all $r \in \mathbb{Z}_+$ and all $I_1, \ldots, I_r$, $J_1, \ldots, J_r$ satisfying conditions (1)–(2). We claim that $\operatorname{PIexp}^H(L) = d(L)$ and prove Theorem 9 for $d = d(L)$.

The following example will be used in the proof of Theorems 9 and 10 in the case of semisimple L (when we do not require conditions (3) and (4) from Subsection 1.7).

Example 7. If L is a semisimple H-module Lie algebra, then by [20, Theorem 6],

$$L = B = B_1 \oplus B_2 \oplus \ldots \oplus B_q$$

(direct sum of H-invariant ideals), where B_i are H-simple Lie algebras. In this case

$$d(L) = \max_{1 \leqslant k \leqslant q} \dim B_k.$$

Proof. Note that if I is an H-simple ideal of L, then $[I, L] \neq 0$ and hence $[I, B_i] \neq 0$ for some $1 \leqslant i \leqslant q$. However, $[I, B_i] \subseteq B_i \cap I$ is an H-invariant ideal. Thus $I = B_i$. And if I is an H-invariant ideal of L, then it is semisimple and each of its simple components coincides with one of B_i. Thus if $I \subseteq J$ are H-invariant ideals of L and I/J is an irreducible (H, L)-module, then $I = B_i \oplus J$ for some $1 \leqslant i \leqslant q$ and $\dim(L/\operatorname{Ann}(I/J)) = \dim B_i$.

Suppose $I_1, \ldots, I_r$, $J_1, \ldots, J_r$ satisfy conditions (1)–(2). Let $I_k = B_{i_k} \oplus J_k$, $1 \leqslant k \leqslant r$. Then

$$[[B_{i_1}, L, \ldots, L], [B_{i_2}, L, \ldots, L], \ldots, [B_{i_r}, L, \ldots, L]] \neq 0$$

for some number of copies of L. Hence $i_1 = \ldots = i_r$ and

$$\dim \frac{L}{\operatorname{Ann}(I_1/J_1) \cap \cdots \cap \operatorname{Ann}(I_r/J_r)} = \dim B_{i_1}.$$

Therefore $d(L) = \max_{1 \leqslant k \leqslant q} \dim B_k$. □

1.9. **S_n-cocharacters.** One of the main tools in the investigation of polynomial identities is provided by the representation theory of symmetric groups. The symmetric group S_n acts on the space $\frac{V_n^H}{V_n^H \cap \operatorname{Id}^H(L)}$ by permuting the variables. Irreducible FS_n-modules are described by partitions $\lambda = (\lambda_1, \ldots, \lambda_s) \vdash n$ and their Young diagrams D_λ. The character $\chi_n^H(L)$ of the FS_n-module $\frac{V_n^H}{V_n^H \cap \operatorname{Id}^H(L)}$ is called the nth *cocharacter* of polynomial H-identities of L. We can rewrite it as a sum

$$\chi_n^H(L) = \sum_{\lambda \vdash n} m(L, H, \lambda) \chi(\lambda)$$

of irreducible characters $\chi(\lambda)$. Let $e_{T_\lambda} = a_{T_\lambda} b_{T_\lambda}$ and $e^*_{T_\lambda} = b_{T_\lambda} a_{T_\lambda}$, where $a_{T_\lambda} = \sum_{\pi \in R_{T_\lambda}} \pi$ and $b_{T_\lambda} = \sum_{\sigma \in C_{T_\lambda}} (\operatorname{sign} \sigma)\sigma$, be Young symmetrizers corresponding to a Young tableau T_λ. Then $M(\lambda) = FSe_{T_\lambda} \cong FSe^*_{T_\lambda}$ is an irreducible FS_n-module corresponding to a partition $\lambda \vdash n$. We refer the reader to [4, 12, 16] for an account of S_n-representations and their applications to polynomial identities.

In Sections 2 and 4 we discuss (H, L)-modules and their annihilators.

In Section 5 we prove that if $m(L, H, \lambda) \neq 0$, then the corresponding Young diagram D_λ has at most d long rows. This implies the upper bound.

In Section 3 we consider faithful irreducible (H, L_0)-modules where L_0 is a reductive H-module Lie algebra. For an arbitrary $k \in \mathbb{N}$, we construct an associative H-polynomial that is alternating in $2k$ sets, each consisting of $\dim L_0$ variables. This polynomial is not an identity of the corresponding representation of L_0. In Section 6 we choose reductive algebras and faithful irreducible modules, and glue the corresponding alternating polynomials. This enables us to find $\lambda \vdash n$ with $m(L, H, \lambda) \neq 0$ such that $\dim M(\lambda)$ has the desired asymptotic behavior and the lower bound is proved.

2. Some properties of (H, L)-modules

Lemma 3. *Let M be an (H, L)-module for some Hopf algebra H and H-module Lie algebra L over a field F. Then $\operatorname{Ann} M$ is an H-invariant ideal of L.*

Proof. Note that $[a, b]v = a(bv) - b(av) = 0$ for all $v \in M$, $a \in \operatorname{Ann} M$, $b \in L$. Hence $\operatorname{Ann} M$ is an ideal. Moreover,

$$(ha)v = (h_{(1)}\varepsilon(h_{(2)})a)v = (h_{(1)}a)((\varepsilon(h_{(2)})1)v) = (h_{(1)}a)(h_{(2)}(Sh_{(3)})v)$$
$$= (h_{(1)(1)}a)(h_{(1)(2)}(Sh_{(2)})v) = h_{(1)}(a(Sh_{(2)})v) = 0$$

for all $v \in M$, $a \in \operatorname{Ann} M$, $h \in H$. Thus $\operatorname{Ann} M$ is an H-submodule. □

In Lemmas 4, 5, 6, M is a finite dimensional (H, L)-module where H is a Hopf algebra over an algebraically closed field F of characteristic 0 and L is an H-module Lie algebra with the solvable radical R which we require to be H-invariant. Recall that we denote by $\zeta\colon H \to \operatorname{End}_F(M)$ and $\varphi\colon L \to \mathfrak{gl}(M)$ the homomorphisms corresponding to the (H, L)-module structure. Denote by A the associative subalgebra of $\operatorname{End}_F(M)$ generated by the operators from $\varphi(L)$ and $\zeta(H)$.

Lemma 4. *$\varphi([L, R]) \subseteq J(A)$, where $J(A)$ is the Jacobson radical of A.*

Proof. Let $M = W_0 \supseteq W_1 \supseteq W_2 \supseteq \ldots \supseteq W_t = \{0\}$ be a composition series in M of L-submodules not necessarily H-invariant. Then each W_i/W_{i+1} is an irreducible L-module. Denote the corresponding homomorphism by $\varphi_i\colon L \to \mathfrak{gl}(W_i/W_{i+1})$. Then by E. Cartan's theorem [21, Proposition 1.4.11], $\varphi_i(L)$ is semisimple or the direct sum of a semisimple ideal and the center of $\mathfrak{gl}(W_i/W_{i+1})$. Thus $\varphi_i([L, L])$ is semisimple and $\varphi_i([L, L] \cap R) = 0$. Since $[L, R] \subseteq [L, L] \cap R$, we have $\varphi_i([L, R]) = 0$ and $[L, R]W_i \subseteq W_{i+1}$.

Denote by I the associative ideal of A generated by $\varphi([L, R])$. Then I^t is the associative ideal generated by elements of the form

$$a_1\bigl(\zeta(h_{10})b_{11}\zeta(h_{11})b_{12} \ldots \zeta(h_{1,s_1-1})b_{1,s_1}\zeta(h_{1,s_1})\bigr)$$
$$\cdot a_2\bigl(\zeta(h_{20})b_{21}\zeta(h_{21})b_{22} \ldots \zeta(h_{2,s_2-1})b_{2,s_2}\zeta(h_{2,s_2})\bigr)$$
$$\ldots a_{t-1}\bigl(\zeta(h_{t-1,0})b_{t-1,1}\zeta(h_{t-1,1})b_{t-1,2} \ldots \zeta(h_{t-1,s_{t-1}-1})b_{t-1,s_{t-1}}\zeta(h_{t-1,s_{t-1}})\bigr)a_t,$$

where $a_i \in \varphi([L, R])$, $b_{ij} \in \varphi(L)$, $h_{ij} \in H$. Using (8), we move all $\zeta(h_{ij})$ to the right and obtain that I^t is generated by elements

$$b = a_1\bigl((h'_{11}b_{11}) \ldots (h'_{1,s_1}b_{1,s_1})\bigr)\ (h_2a_2)\ \bigl((h'_{21}b_{21}) \ldots (h'_{2,s_2}b_{2,s_2})\bigr)$$
$$\ldots (h_{t-1}a_{t-1})\bigl((h'_{t-1,1}b_{t-1,1}) \ldots (h'_{t-1,s_{t-1}}b_{t-1,s_{t-1}})\bigr)(h_ta_t)\zeta(h_{t+1}),$$

where $h_i, h'_{ij} \in H$. However, each $h_ia_i \in \varphi([L, R])$ since R is H-invariant. Hence $(h_ia_i)W_{k-1} \subseteq W_k$ for all $1 \leqslant k \leqslant t$. Thus $b = 0$, $I^t = 0$, and $\varphi([L, R]) \subseteq J(A)$. □

Lemma 5. *Suppose for every H-invariant associative subalgebra $A_1 \subseteq \operatorname{End}_F(M)$ there exists an H-invariant Wedderburn — Mal'cev decomposition. Let $W = \langle a_1, \ldots, a_t\rangle_F$ be a subspace in R such that $\varphi(W)$ is an H-submodule in $\varphi(R)$. Then $\varphi(a_i) = c_i + d_i$, $1 \leqslant i \leqslant t$, where c_i and d_i are polynomials in $\varphi(a_j)$, $1 \leqslant j \leqslant t$, without a constant term, c_i are commuting diagonalizable operators on M, and $d_i \in J(A)$. Moreover, $\langle c_1, \ldots, c_t\rangle_F$ and $\langle d_1, \ldots, d_t\rangle_F$ are H-submodules in $\operatorname{End}_F(M)$.*

Proof. Consider the H-invariant Lie algebra $\varphi(R) + J(A) \subseteq \mathfrak{gl}(M)$. Note that $\varphi(R) + J(A)$ is solvable since $J(A)$ is nilpotent and

$$(\varphi(R) + J(A))/J(A) \cong \varphi(R)/(\varphi(R) \cap J(A))$$

is a homomorphic image of the solvable algebra R. By Lie's Theorem, there exists a basis in M in which all the operators from $\varphi(R) + J(A)$ have upper triangular matrices. Denote the corresponding embedding $A \hookrightarrow M_s(F)$ by ψ. Here $s := \dim M$.

Let A_1 be the associative subalgebra of $\mathrm{End}_F(M)$ generated by $\varphi(a_i)$, $1 \leqslant i \leqslant t$. This algebra is H-invariant since $\varphi(W)$ is an H-submodule in $\varphi(R)$. Hence A_1 is an associative H-module algebra. By our assumptions, the Jacobson radical $J(A_1)$ is H-invariant and we have an H-invariant Wedderburn — Mal'cev decomposition $A_1 = \tilde{A}_1 \oplus J(A_1)$ (direct sum of H-submodules), where $\tilde{A}_1$ is an H-invariant semisimple subalgebra of A_1. Since $\psi(\varphi(R)) \subseteq \mathfrak{t}_s(F)$, we have $\psi(A_1) \subseteq UT_s(F)$. Here $UT_s(F)$ is the associative algebra of upper triangular $s \times s$ matrices. There is a decomposition

$$UT_s(F) = Fe_{11} \oplus Fe_{22} \oplus \dots \oplus Fe_{ss} \oplus \tilde{N},$$

where

$$\tilde{N} := \langle e_{ij} \mid 1 \leqslant i < j \leqslant s \rangle_F$$

is a nilpotent ideal. Thus there is no subalgebra in A_1 isomorphic to $M_2(F)$ and $\tilde{A}_1 = Fe_1 \oplus \dots \oplus Fe_q$ for some idempotents $e_i \in A_1$. For every a_j, denote its component in $J(A_1)$ by d_j and its component in $Fe_1 \oplus \dots \oplus Fe_q$ by c_j. Note that e_i are commuting diagonalizable operators. Thus they have a common basis of eigenvectors in M, and c_i are commuting diagonalizable operators too. Moreover,

$$hc_j + hd_j = h\varphi(a_j) \in \langle \varphi(a_i) \mid 1 \leqslant i \leqslant t \rangle_F \subseteq \langle c_i \mid 1 \leqslant i \leqslant t \rangle_F \oplus \langle d_i \mid 1 \leqslant i \leqslant t \rangle_F \subseteq \tilde{A}_1 \oplus J(A_1)$$

for all $h \in H$. However, $\tilde{A}_1$ and $J(A_1)$ are H-invariant and $hc_j \in \tilde{A}_1$, $hd_j \in J(A_1)$. Thus $hc_j \in \langle c_1, \dots, c_t \rangle_F$ and $hd_j \in \langle d_1, \dots, d_t \rangle_F$. Therefore, $\langle c_1, \dots, c_t \rangle_F$ and $\langle d_1, \dots, d_t \rangle_F$ are H-submodules in $\mathrm{End}_F(M)$.

We claim that the space $J(A_1) + J(A)$ generates a nilpotent H-invariant ideal I in A. First, $\psi(J(A_1))$ and $\psi(J(A))$ are contained in $UT_s(F)$ and consist of nilpotent elements. Thus the corresponding matrices have zero diagonal elements and $\psi(J(A_1)), \psi(J(A)) \subseteq \tilde{N}$. Denote

$$\tilde{N}_k := \langle e_{ij} \mid i + k \leqslant j \rangle_F \subseteq \tilde{N}.$$

Then

$$\tilde{N} = \tilde{N}_1 \supsetneqq \tilde{N}_2 \supsetneqq \dots \supsetneqq \tilde{N}_{m-1} \supsetneqq \tilde{N}_s = \{0\}.$$

Let $\mathrm{ht}_{\tilde{N}}\, a := k$ if $\psi(a) \in \tilde{N}_k$, $\psi(a) \notin \tilde{N}_{k+1}$.

Since $J(A)$ is nilpotent, $(J(A))^p = 0$ for some $p \in \mathbb{N}$. We claim that $I^{s+p} = 0$. Using (8), we move all $\zeta(h)$, $h \in H$, to the right, and obtain that the space I^{s+p} is the span of $b_1 j_1 b_2 j_2 \dots j_{s+p} b_{s+p+1} \zeta(h)$, where $j_k \in J(A_1) \cup J(A)$, $b_k \in A_2 \cup \{1\}$, $h \in H$. Here A_2 is the subalgebra of $\mathrm{End}_F(M)$ generated by the operators from $\varphi(L)$ only. If at least p elements j_k belong to $J(A)$, then the product equals 0. Thus we may assume that at least s elements j_k belong to $J(A_1)$.

Let $j_i \in J(A_1)$, $b_i \in A_2 \cup \{1\}$. We prove by induction on ℓ that $j_1 b_1 j_2 b_2 \dots b_{\ell-1} j_\ell$ can be expressed as a sum of $\tilde{j}_1 \tilde{j}_2 \dots \tilde{j}_\alpha j'_1 j'_2 \dots j'_\beta a$, where $\tilde{j}_i \in J(A_1)$, $j'_i \in J(A)$,

$a \in A_2 \cup \{1\}$, and $\alpha + \sum_{i=1}^{\beta} \mathrm{ht}_{\tilde N}\, j'_i \geqslant \ell$. Indeed, suppose that $j_1 b_1 j_2 b_2 \dots b_{\ell-2} j_{\ell-1}$ can be expressed as a sum of $\tilde j_1 \tilde j_2 \dots \tilde j_\gamma j'_1 j'_2 \dots j'_\varkappa a$, where $\tilde j_i \in J(A_1)$, $j'_i \in J(A)$, $a \in A_2 \cup \{1\}$, and $\gamma + \sum_{i=1}^{\varkappa} \mathrm{ht}_{\tilde N}\, j'_i \geqslant \ell - 1$. Then $j_1 b_1 j_2 b_2 \dots j_{\ell-1} b_{\ell-1} j_\ell$ is a sum of

$$\tilde j_1 \tilde j_2 \dots \tilde j_\gamma j'_1 j'_2 \dots j'_\varkappa a b_{\ell-1} j_\ell = \tilde j_1 \tilde j_2 \dots \tilde j_\gamma j'_1 j'_2 \dots j'_\varkappa [a b_{\ell-1}, j_\ell] + \tilde j_1 \tilde j_2 \dots \tilde j_\gamma j'_1 j'_2 \dots j'_\varkappa j_\ell (a b_{\ell-1}).$$

Note that, by virtue of the Jacobi identity and Lemma 4, $[ab_{\ell-1}, j_\ell] \in J(A)$. Thus it is sufficient to consider only the second term. However,

$$\tilde j_1 \tilde j_2 \dots \tilde j_\gamma j'_1 j'_2 \dots j'_\varkappa j_\ell (a b_{\ell-1}) = \tilde j_1 \tilde j_2 \dots \tilde j_\gamma j_\ell j'_1 j'_2 \dots j'_\varkappa (a b_{\ell-1}) + \sum_{i=1}^{\varkappa} \tilde j_1 \tilde j_2 \dots \tilde j_\gamma j'_1 j'_2 \dots j'_{i-1} [j'_i, j_\ell] j'_{i+1} \dots j'_\varkappa (a b_{\ell-1}).$$

Since $[j'_i, j_\ell] \in J(A)$ and $\mathrm{ht}_{\tilde N} [j'_i, j_\ell] \geqslant 1 + \mathrm{ht}_{\tilde N}\, j'_i$, all the terms have the desired form. Therefore,

$$j_1 b_1 j_2 b_2 \dots j_{s-1} b_{s-1} j_s \in \psi^{-1}(\tilde N_s) = \{0\},$$

$I^{s+p} = 0$, and

$$J(A) \subseteq J(A_1) + J(A) \subseteq I \subseteq J(A).$$

In particular, $d_i \in J(A_1) \subseteq J(A)$. □

Lemma 6. *Let M be a finite dimensional irreducible (H, L)-module. Suppose for every H-invariant associative subalgebra $A_1 \subseteq \mathrm{End}_F(M)$ there exists an H-invariant Wedderburn — Mal'cev decomposition. Then*

(1) *$M = M_1 \oplus \dots \oplus M_q$ for some L-submodules M_i, $1 \leqslant i \leqslant q$;*
(2) *elements of R act on each M_i by scalar operators.*

Proof. Let $\varphi(r_1), \dots, \varphi(r_t)$ be a basis in $\varphi(R)$. By Lemma 5, $\varphi(r_i) = r'_i + r''_i$, where r'_i are commuting diagonalizable operators on M and $r''_i \in J(A)$. Note that by the Density Theorem, $A = \mathrm{End}_F(M)$. Hence $J(A) = 0$ and $\varphi(r_i) = r'_i$. Therefore, $\varphi(r_i)$ have a common basis of eigenvectors and we can choose subspaces M_i, $1 \leqslant i \leqslant q$, $q \in \mathbb{N}$, such that

$$M = M_1 \oplus \dots \oplus M_q,$$

and each M_i is the intersection of eigenspaces of $\varphi(r_i)$. Note that by Lemma 4,

$$[\varphi(r_i), \varphi(a)] \in J(a) = 0 \text{ for all } a \in L.$$

Thus M_i are L-submodules and the lemma is proved. □

3. Polynomial H-identities of representations and alternating H-polynomials

In this section we prove auxiliary propositions needed to obtain the lower bound.

Let H be a Hopf algebra over a field F. Analogously to the absolutely free nonassociative H-module algebra $F\{X|H\}$, we define the free associative H-module algebra $F\langle X|H\rangle$ on the set $X := \{x_1, x_2, x_3, \dots\}$. Here we follow [6]. First, consider the free associative algebra $F\langle X\rangle = \bigoplus_{n=1}^{\infty} F\langle X\rangle^{(n)}$, where $F\langle X\rangle^{(n)}$ is the linear span of all associative monomials in variables from X of total degree n. Consider the algebra

$$F\langle X|H\rangle := \bigoplus_{n=1}^{\infty} H^{\otimes n} \otimes F\langle X\rangle^{(n)}$$

with the multiplication $(u_1 \otimes w_1)(u_2 \otimes w_2) := (u_1 \otimes u_2) \otimes w_1 w_2$ for all $u_1 \in H^{\otimes j}$, $u_2 \in H^{\otimes k}$, $w_1 \in F\langle X \rangle^{(j)}$, $w_2 \in F\langle X \rangle^{(k)}$. We use the notation

$$x^{h_1}_{i_1} x^{h_2}_{i_2} \dots x^{h_n}_{i_n} := (h_1 \otimes h_2 \otimes \dots \otimes h_n) \otimes x_{i_1} x_{i_2} \dots x_{i_n}.$$

Here $h_1 \otimes h_2 \otimes \dots \otimes h_n \in H^{\otimes n}$, $x_{i_1} x_{i_2} \dots x_{i_n} \in F\langle X \rangle^{(n)}$.

Note that if $(\gamma_\alpha)_{\alpha \in \Lambda}$ is a basis in H, then $F\langle X|H \rangle$ is isomorphic to the free associative algebra over F with free formal generators $x_i^{\gamma_\alpha}$, $\alpha \in \Lambda$, $i \in \mathbb{N}$.

Define on $F\langle X|H \rangle$ the structure of a left H-module by

$$h\,(x^{h_1}_{i_1} x^{h_2}_{i_2} \dots x^{h_n}_{i_n}) = x^{h_{(1)}h_1}_{i_1} x^{h_{(2)}h_2}_{i_2} \dots x^{h_{(n)}h_n}_{i_n},$$

where $h \in H$. Then $F\langle X|H \rangle$ is *the free H-module associative algebra* on X, i.e. for each map $\psi\colon X \to A$, where A is an associative H-module algebra, there exists a unique homomorphism $\bar\psi\colon F\langle X|H \rangle \to A$ of algebras and H-modules, such that $\bar\psi\big|_X = \psi$. Here we identify X with the set $\{x^1_j \mid j \in \mathbb{N}\} \subset F\langle X|H \rangle$.

Let L be an H-module Lie algebra, let M be an (H,L)-module, and let $\varphi\colon L \to \mathfrak{gl}(M)$ be the corresponding representation. A polynomial $f(x_1,\dots,x_n) \in F\langle X|H \rangle$ is an *H-identity* of φ if $f(\varphi(a_1),\dots,\varphi(a_n)) = 0$ for all $a_i \in L$. In other words, f is an H-identity of φ if for every homomorphism $\psi\colon F\langle X|H \rangle \to \mathrm{End}_F(M)$ of algebras and H-modules such that $\psi(X) \subseteq \varphi(L)$, we have $\psi(f) = 0$. The set $\mathrm{Id}^H(\varphi)$ of all H-identities of φ is an H-invariant two-sided ideal in $F\langle X|H \rangle$.

Denote by P^H_n, $n \in \mathbb{N}$, the subspace of associative multilinear H-polynomials in variables $x_1,\dots,x_n$. In other words,

$$P^H_n = \left\langle x^{h_1}_{\sigma(1)} x^{h_2}_{\sigma(2)} \dots x^{h_n}_{\sigma(n)} \;\middle|\; \sigma \in S_n,\, h_1,\dots,h_n \in H \right\rangle_F \subset F\langle X|H \rangle.$$

Lemma 7 is an analog of [15, Lemma 1]. Recall that if M is an (H,L)-module, we denote by $\varphi\colon L \to \mathfrak{gl}(M)$ and $\zeta\colon H \to \mathrm{End}_F(M)$ the corresponding homomorphisms.

Lemma 7. *Let L be an H-module Lie algebra where H is a Hopf algebra over an algebraically closed field F of characteristic* 0, *and let M be a faithful finite dimensional irreducible (H,L)-module. Let $(\zeta(\gamma_j))_{j=1}^m$ be a basis of $\zeta(H)$. Then for some $n \in \mathbb{N}$ there exist polynomials $f_j \in P^H_n$, $1 \leqslant j \leqslant m$, alternating in $\{x_1,\dots,x_\ell\}$ and in $\{y_1,\dots,y_\ell\} \subseteq \{x_{\ell+1},\dots,x_n\}$ where $\ell := \dim L$, such that*

$$\sum_{j=1}^m f_j(\varphi(\bar x_1),\dots,\varphi(\bar x_n))\,\zeta(\gamma_j) = \mathrm{id}_M$$

for some $\bar x_i \in L$. In particular, there exists a polynomial $f \in P^H_n \setminus \mathrm{Id}^H(\varphi)$ alternating in $\{x_1,\dots,x_\ell\}$ and in $\{y_1,\dots,y_\ell\} \subseteq \{x_{\ell+1},\dots,x_n\}$.

Proof. Since M is irreducible, by the Density Theorem, $\mathrm{End}_F(M) \cong M_q(F)$ is generated by operators from $\zeta(H)$ and $\varphi(L)$. Here $q := \dim M$. Consider Regev's polynomial

$$\hat f(x_1,\dots,x_{q^2}; y_1,\dots,y_{q^2}) := \sum_{\substack{\sigma \in S_q,\\ \tau \in S_q}} (\mathrm{sign}(\sigma\tau)) x_{\sigma(1)}\, y_{\tau(1)}\, x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)}\, y_{\tau(2)} y_{\tau(3)} y_{\tau(4)} \dots x_{\sigma(q^2-2q+2)} \dots x_{\sigma(q^2)}\, y_{\tau(q^2-2q+2)} \dots y_{\tau(q^2)}.$$

This is a central polynomial [16, Theorem 5.7.4] for $M_q(F)$; i.e. $\hat f$ is not a polynomial identity for $M_q(F)$ and its values belong to the center of $M_q(F)$. Since

$\hat{f}$ is alternating, it becomes a nonzero scalar operator under a substitution of the elements of any basis for $\{x_1, \dots, x_\ell\}$ and $\{y_1, \dots, y_\ell\}$.

Let $a_1, \dots, a_\ell$ be a basis of L. Recall that if we have the product of elements of $\varphi(L)$ and $\zeta(H)$, we can always move the elements from $\zeta(H)$ to the right, using inequality (8). Then $\varphi(a_1), \dots, \varphi(a_\ell)$, $\left(\varphi\left(a_{i_{11}}\right)\dots\varphi\left(a_{i_{1,m_1}}\right)\right)\zeta(h_1)$, $\dots$, $\left(\varphi\left(a_{i_{r,1}}\right)\dots\varphi\left(a_{i_{r,m_r}}\right)\right)\zeta(h_r)$ is a basis of $\operatorname{End}_F(M)$ for appropriate $i_{jk}\in\{1,2,\dots,\ell\}$, $h_j \in H$, since $\operatorname{End}_F(M)$ is generated by operators from $\zeta(H)$ and $\varphi(L)$. We replace $x_{\ell+j}$ with $z_{j1}z_{j2}\dots z_{j,m_j}\zeta(h_j)$ and $y_{\ell+j}$ with $z'_{j1}z'_{j2}\dots z'_{j,m_j}\zeta(h_j)$ in $\hat{f}$ and denote the expression obtained by $\tilde{f}$. Using (8) again, we can move all $\zeta(h)$, $h \in H$, in $\tilde{f}$ to the right and rewrite $\tilde{f}$ as $\sum_{j=1}^{m} f_j\,\zeta(\gamma_j)$, where each $f_j \in P^H_{2\ell+2\sum_{i=1}^{r} m_i}$ is a polynomial alternating in $x_1, \dots, x_\ell$ and in $y_1, \dots, y_\ell$. Note that $\tilde{f}$ becomes a nonzero scalar operator on M under the substitution $x_i = y_i = \varphi(a_i)$ for $1 \leqslant i \leqslant \ell$ and $z_{jk} = z'_{jk} = \varphi(a_{i_{jk}})$ for $1 \leqslant j \leqslant r$, $1 \leqslant k \leqslant m_j$. Dividing all f_j by this scalar, we obtain the required polynomials. In particular, $f_j \notin \operatorname{Id}^H(\varphi)$ for some $1 \leqslant j \leqslant m$ and we can take $f = f_j$. □

Lemma 8. *Let $\alpha_1, \alpha_2, \dots, \alpha_q$, $\beta_1, \dots, \beta_q \in F$, where F is an infinite field, $1 \leqslant k \leqslant q$, $\alpha_i \neq 0$ for $1 \leqslant i < k$, $\alpha_k = 0$, and $\beta_k \neq 0$. Then there exists such $\gamma \in F$ that $\alpha_i + \gamma\beta_i \neq 0$ for all $1 \leqslant i \leqslant k$.*

Proof. It is sufficient to choose $\gamma \notin \left\{-\frac{\alpha_1}{\beta_1}, \dots, -\frac{\alpha_{k-1}}{\beta_{k-1}}, 0\right\}$. It is possible to do since F is infinite. □

Lemma 9. *Let $L = B \oplus Z(L)$ be a finite dimensional reductive H-module Lie algebra where H is a Hopf algebra over an algebraically closed field F of characteristic 0, B is an H-invariant maximal semisimple subalgebra, and $Z(L)$ is the center of L with a basis r_1, r_2, $\dots$, r_t. Let M be a faithful finite dimensional irreducible (H, L)-module. Suppose that for all H-invariant associative subalgebras in $\operatorname{End}_F(M)$ there exists an H-invariant Wedderburn — Mal'cev decomposition. Then there exists such alternating in $x_1, x_2, \dots, x_t$ polynomial $f \in P^H_t$ that $f(\varphi(r_1), \dots, \varphi(r_t))$ is a nondegenerate operator on M.*

Proof. By Lemma 3, $Z(L)$ is H-invariant. By Lemma 6, $M = M_1 \oplus \dots \oplus M_q$, where M_j are L-submodules and r_i acts on each M_j as a scalar operator. Note that it is sufficient to prove that for each j there exists such alternating in $x_1, x_2, \dots, x_t$ polynomial $f_j \in P^H_t$ that $f_j(\varphi(r_1), \dots, \varphi(r_t))$ multiplies each element of M_j by a nonzero scalar. Indeed, in this case Lemma 8 implies the existence of such $f = \lambda_1 f_1 + \dots + \lambda_q f_q$, $\lambda_i \in F$, that $f(\varphi(r_1), \dots, \varphi(r_t))$ acts on each M_i as a nonzero scalar.

By Lemma 7, there exist $n \in \mathbb{N}$ and polynomials $\hat{f}_k \in P^H_n$ alternating in $\{x_1, x_2, \dots, x_\ell\}$, $\ell := \dim L$, such that

$$\text{(9)}\qquad \sum_{k=1}^{m} \hat{f}_k(\varphi(r_1), \varphi(r_2), \dots, \varphi(r_t), \varphi(\bar{x}_{t+1}), \varphi(\bar{x}_{t+2}), \dots, \varphi(\bar{x}_n))\,\zeta(\gamma_k) = \mathrm{id}_M$$

for some $\bar{x}_i \in L$. (We can substitute the elements of any basis of L for $\{x_1, x_2, \dots, x_\ell\}$ since each $\hat{f}_k$ is alternating in the first ℓ variables.) Note that $[\varphi(r_i), \varphi(a)] = 0$ for all $a \in L$ and $1 \leqslant i \leqslant t$. Hence we can move all $\varphi(r_i)$ to the left and rewrite (9) as

$$\sum_k \tilde{f}_k(\varphi(r_1), \varphi(r_2), \dots, \varphi(r_t))b_k = \mathrm{id}_M,$$

where $\tilde{f}_k \in P_t^H$ are polynomials alternating in $\{x_1, x_2, \ldots, x_t\}$, and $b_k \in \mathrm{End}_F(M)$. Hence for every $1 \leqslant j \leqslant q$ there exists k such that $\tilde{f}_k(\varphi(r_1), \varphi(r_2), \ldots, \varphi(r_t))\big|_{M_j} \neq 0$. Since $\tilde{f}_j(\varphi(r_1), \varphi(r_2), \ldots, \varphi(r_t))$ is a scalar operator on M_j, we can take $f_j := \tilde{f}_k$. ☐

Let $k\ell \leqslant n$, where $k, \ell, n \in \mathbb{N}$ are some numbers. Denote by $Q^H_{\ell,k,n} \subseteq P^H_n$ the subspace spanned by all polynomials that are alternating in k disjoint subsets of variables $\{x_1^i, \ldots, x_\ell^i\} \subseteq \{x_1, x_2, \ldots, x_n\}$, $1 \leqslant i \leqslant k$.

Theorem 11 is an analog of [15, Theorem 1].

Theorem 11. *Let $L = B \oplus Z(L)$ be a reductive H-module Lie algebra where H is a Hopf algebra over an algebraically closed field F of characteristic 0, B is an H-invariant maximal semisimple subalgebra, and $\ell := \dim L$. Let M be a faithful finite dimensional irreducible (H, L)-module. Denote the corresponding representation $L \to \mathfrak{gl}(M)$ by φ. Suppose that either $Z(L) = 0$ or for any H-invariant associative subalgebra in $\mathrm{End}_F(M)$ there exists an H-invariant Wedderburn — Mal'cev decomposition. Then there exists $T \in \mathbb{Z}_+$ such that for any $k \in \mathbb{N}$ there exists $f \in Q^H_{\ell,2k,2k\ell+T} \backslash \mathrm{Id}^H(\varphi)$.*

Proof. Let $f_1 = f_1(x_1, \ldots, x_\ell,\ y_1, \ldots, y_\ell, z_1, \ldots, z_T)$ be the polynomial f from Lemma 7 alternating in $x_1, \ldots, x_\ell$ and in $y_1, \ldots, y_\ell$. Since $f_1 \in Q^H_{\ell,2,2\ell+T} \backslash \mathrm{Id}^H(\varphi)$, we may assume that $k > 1$. Note that

$$f_1^{(1)}(u_1, v_1, x_1, \ldots, x_\ell,\ y_1, \ldots, y_\ell, z_1, \ldots, z_T)$$

$$:= \sum_{i=1}^{\ell} f_1(x_1, \ldots, [u_1, [v_1, x_i]], \ldots, x_\ell,\ y_1, \ldots, y_\ell, z_1, \ldots, z_T)$$

is alternating in $x_1, \ldots, x_\ell$ and in $y_1, \ldots, y_\ell$, and

$$f_1^{(1)}(\bar{u}_1, \bar{v}_1, \bar{x}_1, \ldots, \bar{x}_\ell,\ \bar{y}_1, \ldots, \bar{y}_\ell, \bar{z}_1, \ldots, \bar{z}_T)$$

$$= \mathrm{tr}(\mathrm{ad}_{\varphi(L)}\, \bar{u}_1\, \mathrm{ad}_{\varphi(L)}\, \bar{v}_1) f_1(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_\ell,\ \bar{y}_1, \ldots, \bar{y}_\ell, \bar{z}_1, \ldots, \bar{z}_T)$$

for any substitution of elements from $\varphi(L)$ since we may assume $\bar{x}_1, \ldots, \bar{x}_\ell$ to be different basis elements. Here $(\mathrm{ad}\, a)b = [a, b]$.

Let

$$f_1^{(j)}(u_1, \ldots, u_j, v_1, \ldots, v_j, x_1, \ldots, x_\ell,\ y_1, \ldots, y_\ell, z_1, \ldots, z_T)$$

$$:= \sum_{i=1}^{\ell} f_1^{(j-1)}(u_1, \ldots, u_{j-1}, v_1, \ldots, v_{j-1}, x_1, \ldots, [u_j, [v_j, x_i]], \ldots, x_\ell,\ y_1, \ldots, y_\ell, z_1, \ldots, z_T),$$

$2 \leqslant j \leqslant s$, $s := \dim B$. Note that if we substitute an element from $\varphi(Z(L))$ for u_i or v_i, then $f_1^{(j)}$ vanish since $Z(L)$ is the center of L. Again,

$$f_1^{(j)}(\bar{u}_1, \ldots, \bar{u}_j, \bar{v}_1, \ldots, \bar{v}_j, \bar{x}_1, \ldots, \bar{x}_\ell,\ \bar{y}_1, \ldots, \bar{y}_\ell, \bar{z}_1, \ldots, \bar{z}_T)$$

$$= \mathrm{tr}(\mathrm{ad}_{\varphi(L)}\, \bar{u}_1\, \mathrm{ad}_{\varphi(L)}\, \bar{v}_1)\, \mathrm{tr}(\mathrm{ad}_{\varphi(L)}\, \bar{u}_2\, \mathrm{ad}_{\varphi(L)}\, \bar{v}_2) \ldots \mathrm{tr}(\mathrm{ad}_{\varphi(L)}\, \bar{u}_j\, \mathrm{ad}_{\varphi(L)}\, \bar{v}_j)$$

$$\cdot f_1(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_\ell,\ \bar{y}_1, \ldots, \bar{y}_\ell, \bar{z}_1, \ldots, \bar{z}_T). \tag{10}$$

Let η be the polynomial f from Lemma 9. We define

$$f_2(u_1,\dots,u_\ell,v_1,\dots,v_\ell,x_1,\dots,x_\ell,y_1,\dots,y_\ell,z_1,\dots,z_T)$$
$$:=\sum_{\sigma,\tau\in S_\ell}\operatorname{sign}(\sigma\tau)f_1^{(s)}(u_{\sigma(1)},\dots,u_{\sigma(s)},v_{\tau(1)},\dots,v_{\tau(s)},x_1,\dots,x_\ell,\ y_1,\dots,y_\ell,z_1,\dots,z_T)$$
$$\cdot\eta(u_{\sigma(s+1)},\dots,u_{\sigma(\ell)})\eta(v_{\tau(s+1)},\dots,v_{\tau(\ell)}).$$

Then $f_2\in Q^H_{\ell,4,4\ell+T}$. Suppose $a_1,\dots,a_s\in\varphi(B)$ and $a_{s+1},\dots,a_\ell\in\varphi(Z(L))$ form a basis of $\varphi(L)$. Consider a substitution $x_i=y_i=u_i=v_i=a_i$, $1\leqslant i\leqslant\ell$. Suppose that the values $z_j=\bar z_j$, $1\leqslant j\leqslant T$, are chosen in such a way that $f_1(a_1,\dots,a_\ell,a_1,\dots,a_\ell,\bar z_1,\dots,\bar z_T)\neq 0$. We claim that f_2 does not vanish either. Indeed,

$$f_2(a_1,\dots,a_\ell,a_1,\dots,a_\ell,a_1,\dots,a_\ell,a_1,\dots,a_\ell,\bar z_1,\dots,\bar z_T)$$
$$=\sum_{\sigma,\tau\in S_\ell}\operatorname{sign}(\sigma\tau)f_1^{(s)}(a_{\sigma(1)},\dots,a_{\sigma(s)},a_{\tau(1)},\dots,a_{\tau(s)},a_1,\dots,a_\ell,\ a_1,\dots,a_\ell,\bar z_1,\dots,\bar z_T)$$
$$\cdot\eta(a_{\sigma(s+1)},\dots,a_{\sigma(\ell)})\eta(a_{\tau(s+1)},\dots,a_{\tau(\ell)})$$
$$=\left(\sum_{\sigma,\tau\in S_s}\operatorname{sign}(\sigma\tau)f_1^{(s)}(a_{\sigma(1)},\dots,a_{\sigma(s)},a_{\tau(1)},\dots,a_{\tau(s)},a_1,\dots,a_\ell,\ a_1,\dots,a_\ell,\bar z_1,\dots,\bar z_T)\right)$$
$$\cdot\left(\sum_{\pi,\omega\in S\{s+1,\dots,\ell\}}\operatorname{sign}(\pi\omega)\eta(a_{\pi(s+1)},\dots,a_{\pi(\ell)})\eta(a_{\omega(s+1)},\dots,a_{\omega(\ell)})\right).$$

since a_j, $s<j\leqslant\ell$, belong to the center of $\varphi(L)$ and $f_j^{(s)}$ vanishes if we substitute such a_i for u_i or v_i. Here $S\{s+1,\dots,\ell\}$ is the symmetric group on $\{s+1,\dots,\ell\}$. Note that η is alternating. Using (10), we obtain

$$f_2(a_1,\dots,a_\ell,a_1,\dots,a_\ell,a_1,\dots,a_\ell,a_1,\dots,a_\ell,\bar z_1,\dots,\bar z_T)$$
$$=\left(\sum_{\sigma,\tau\in S_s}\operatorname{sign}(\sigma\tau)\operatorname{tr}(\operatorname{ad}_{\varphi(L)}a_{\sigma(1)}\operatorname{ad}_{\varphi(L)}a_{\tau(1)})\dots\operatorname{tr}(\operatorname{ad}_{\varphi(L)}a_{\sigma(s)}\operatorname{ad}_{\varphi(L)}a_{\tau(s)})\right)$$
$$\cdot f_1(a_1,\dots,a_\ell,\ a_1,\dots,a_\ell,\bar z_1,\dots,\bar z_T)((\ell-s)!)^2\left(\eta(a_{s+1},\dots,a_\ell)\right)^2.$$

Note that

$$\sum_{\sigma,\tau\in S_s}\operatorname{sign}(\sigma\tau)\operatorname{tr}(\operatorname{ad}_{\varphi(L)}a_{\sigma(1)}\operatorname{ad}_{\varphi(L)}a_{\tau(1)})\dots\operatorname{tr}(\operatorname{ad}_{\varphi(L)}a_{\sigma(s)}\operatorname{ad}_{\varphi(L)}a_{\tau(s)})$$
$$=\sum_{\sigma,\tau\in S_s}\operatorname{sign}(\sigma\tau)\operatorname{tr}(\operatorname{ad}_{\varphi(L)}a_1\operatorname{ad}_{\varphi(L)}a_{\tau\sigma^{-1}(1)})\dots\operatorname{tr}(\operatorname{ad}_{\varphi(L)}a_s\operatorname{ad}_{\varphi(L)}a_{\tau\sigma^{-1}(s)})$$
$$\overset{(\tau'=\tau\sigma^{-1})}{=}\sum_{\sigma,\tau'\in S_s}\operatorname{sign}(\tau')\operatorname{tr}(\operatorname{ad}_{\varphi(L)}a_1\operatorname{ad}_{\varphi(L)}a_{\tau'(1)})\dots\operatorname{tr}(\operatorname{ad}_{\varphi(L)}a_s\operatorname{ad}_{\varphi(L)}a_{\tau'(s)})$$
$$=s!\det(\operatorname{tr}(\operatorname{ad}_{\varphi(L)}a_i\operatorname{ad}_{\varphi(L)}a_j))_{i,j=1}^s=s!\det(\operatorname{tr}(\operatorname{ad}_{\varphi(B)}a_i\operatorname{ad}_{\varphi(B)}a_j))_{i,j=1}^s\neq 0$$

since the Killing form $\operatorname{tr}(\operatorname{ad}x\operatorname{ad}y)$ of the semisimple Lie algebra $\varphi(B)$ is nondegenerate. Thus

$$f_2(a_1,\dots,a_\ell,a_1,\dots,a_\ell,a_1,\dots,a_\ell,a_1,\dots,a_\ell,\bar z_1,\dots,\bar z_T)\neq 0.$$

Note that if f_1 is alternating in some of $z_1,\dots,z_T$, the polynomial f_2 is alternating in those variables too. Thus if we apply the same procedure to f_2 instead of f_1, we obtain $f_3 \in Q^H_{\ell,6,6\ell+T}$. Analogously, we define f_4 using f_3, f_5 using f_4, etc. Eventually, we obtain $f := f_k \in Q^H_{\ell,2k,2k\ell+T} \setminus \mathrm{Id}^H(\varphi)$. $\square$

4. On factors of the adjoint representation of L

In Sections 4–6 we consider an H-nice Lie algebra L (see Subsection 1.7).

Lemma 10. *Consider the adjoint action of B on L. Then L is a completely reducible (H, B)-module. Moreover, there exists an H-submodule $S \subseteq R$ such that $L = B \oplus S \oplus N$ (direct sum of H-submodules) and $[B, S] = 0$.*

Proof. If L is semisimple, i.e. $L = B$, then by [20, Theorem 6], L is a direct sum of H-simple ideals, i.e. L is a completely reducible (H, B)-module.

Suppose $L \ne B$. Note that $(\mathrm{ad}\, B) \subseteq \mathfrak{gl}(L)$ is a finite dimensional semisimple Lie algebra. Hence, by the Weyl theorem, L is a completely reducible $(\mathrm{ad}\, B)$-module. Thus, by condition (4) of Subsection 1.7, L is a completely reducible $(H, \mathrm{ad}\, B)$- and (H, B)-module.

Since the nilpotent radical N of L is an H-submodule, there exists an H-invariant B-submodule $S \subseteq R$ such that $R = S \oplus N$ (direct sum of H-submodules). Therefore $[B, S] \subseteq S$. However, by [21, Proposition 2.1.7], $[B, S] \subseteq [L, R] \subseteq N$. Hence $[B, S] = 0$. $\square$

Lemma 11 is an H-invariant analog of [30, Lemma 4].

Lemma 11. *Let $J \subseteq I \subseteq L$ be H-invariant ideals such that I/J is an irreducible (H, L)-module. Then*

1. $\mathrm{Ann}(I/J) \cap B$ *and* $\mathrm{Ann}(I/J) \cap S$ *are H-submodules of L;*
2. $\mathrm{Ann}(I/J) = (\mathrm{Ann}(I/J) \cap B) \oplus (\mathrm{Ann}(I/J) \cap S) \oplus N$.

Proof. By Lemma 3, $\mathrm{Ann}(I/J)$, $\mathrm{Ann}(I/J) \cap B$, and $\mathrm{Ann}(I/J) \cap S$ are H-submodules.

Note that N is a nilpotent ideal. Hence $\underbrace{[N, [N, \ldots, [N}_{p}, I] \ldots] = 0$ for some $p \in \mathbb{N}$. Thus $N^p(I/J) = 0$. However, I/J is an irreducible (H, L)-module and either $N(I/J) = 0$ or $N(I/J) = I/J$. Hence $N(I/J) = 0$ and $N \subseteq \mathrm{Ann}(I/J)$. In order to prove the lemma, it is sufficient to show that if $b + s \in \mathrm{Ann}(I/J)$, $b \in B$, $s \in S$, then $b, s \in \mathrm{Ann}(I/J)$. Denote by $\varphi \colon L \to \mathfrak{gl}(I/J)$ the homomorphism corresponding to L-module structure on I/J. Then $\varphi(b) + \varphi(s) = 0$ and

$$[\varphi(b), \varphi(B)] = [-\varphi(s), \varphi(B)] = 0$$

since $[B, S] = 0$. Hence $\varphi(b)$ belongs to the center of $\varphi(B)$ and $\varphi(b) = \varphi(s) = 0$ since $\varphi(B)$ is semisimple. Thus $b, s \in \mathrm{Ann}(I/J)$ and the lemma is proved. $\square$

5. Upper bound

Fix a composition chain of H-invariant ideals

$$L = L_0 \supsetneqq L_1 \supsetneqq L_2 \supsetneqq \ldots \supsetneqq N \supsetneqq \ldots \supsetneqq L_{\theta-1} \supsetneqq L_\theta = \{0\}.$$

Let $\mathrm{ht}\, a := \max_{a \in L_k} k$ for $a \in L$.

Remark. If $d = d(L) = 0$, then $L = \mathrm{Ann}(L_{i-1}/L_i)$ for all $1 \leqslant i \leqslant \theta$ and $[a_1, a_2, \ldots, a_n] = 0$ for all $a_i \in L$ and $n \geqslant \theta + 1$. Thus $c_n^H(L) = 0$ for all $n \geqslant \theta + 1$. Therefore we assume $d > 0$.

Let $Y := \{y_{11}, y_{12}, \dots, y_{1j_1}; y_{21}, y_{22}, \dots, y_{2j_2}; \dots; y_{m1}, y_{m2}, \dots, y_{mj_m}\}$, $Y_1, \dots, Y_q$, and $\{z_1, \dots, z_m\}$ be subsets of $\{x_1, x_2, \dots, x_n\}$ such that $Y_i \subseteq Y$, $|Y_i| = d+1$, $Y_i \cap Y_j = \varnothing$ for $i \neq j$, $Y \cap \{z_1, \dots, z_m\} = \varnothing$, $j_i \geqslant 0$. Denote

$$f_{m,q} := \mathrm{Alt}_1 \dots \mathrm{Alt}_q \Big[[z_1^{h_1}, y_{11}^{h_{11}}, y_{12}^{h_{12}}, \dots, y_{1j_1}^{h_{1j_1}}], [z_2^{h_2}, y_{21}^{h_{21}}, y_{22}^{h_{22}}, \dots, y_{2j_2}^{h_{2j_2}}], \dots, [z_m^{h_m}, y_{m1}^{h_{m1}}, y_{m2}^{h_{m2}}, \dots, y_{mj_m}^{h_{mj_m}}] \Big],$$

where Alt_i is the operator of alternation on the variables of Y_i, $h_i, h_{ij} \in H$.

Let $\xi\colon L(X|H) \to L$ be the homomorphism of H-module algebras induced by some substitution $\{x_1, x_2, \dots, x_n, \dots\} \to L$. We say that ξ is *proper* for $f_{m,q}$ if $\xi(z_1) \in B \cup S \cup N$, $\xi(z_i) \in N$ for $2 \leqslant i \leqslant m$, and $\xi(y_{ik}) \in B \cup S$ for $1 \leqslant i \leqslant m$, $1 \leqslant k \leqslant j_i$.

Lemma 12. *Let ξ be a proper homomorphism for $f_{m,q}$. Then $\xi(f_{m,q})$ can be rewritten as a sum of $\psi(f_{m+1,q'})$, where ψ is a proper homomorphism for $f_{m+1,q'}$, $q' \geqslant q - (\dim L)m - 2$. ($Y'$, Y'_i, $z'_1, \dots, z'_{m+1}$ may be different for different terms.)*

Proof. Let $\alpha_i := \mathrm{ht}\,\xi(z_i)$. We will use induction on $\sum_{i=1}^m \alpha_i$. (The sum will grow.) Note that $\alpha_i \leqslant \theta \leqslant \dim L$ and the induction will eventually stop. Denote $I_i := L_{\alpha_i}$, $J_i := L_{\alpha_{i+1}}$.

First, consider the case when $I_1, \dots, I_m$, $J_1, \dots, J_m$ do not satisfy conditions (1)–(2). In this case we can choose H-invariant B-submodules T_i, $I_i = T_i \oplus J_i$, such that

$$[[T_1, \underbrace{L, \dots, L}_{q_1}], [T_2, \underbrace{L, \dots, L}_{q_2}], \dots, [T_m, \underbrace{L, \dots, L}_{q_m}]] = 0 \qquad (11)$$

for all $q_i \geqslant 0$. Rewrite $\xi(z_i) = a'_i + a''_i$, $a'_i \in T_i$, $a''_i \in J_i$. Note that $\mathrm{ht}\, a''_i > \mathrm{ht}\,\xi(z_i)$. Since $f_{m,q}$ is multilinear, we can rewrite $\xi(f_{m,q})$ as a sum of similar terms $\tilde\xi(f_{m,q})$ where $\tilde\xi(z_i)$ equals either a'_i or a''_i. By (11), the term where all $\tilde\xi(z_i) = a'_i \in T_i$ equals 0. For the other terms $\tilde\xi(f_{m,q})$ we have $\sum_{i=1}^m \mathrm{ht}\,\tilde\xi(z_i) > \sum_{i=1}^m \mathrm{ht}\,\xi(z_i)$.

Thus without lost of generality we may assume that $I_1, \dots, I_m$, $J_1, \dots, J_m$ satisfy conditions (1)–(2). In this case, $\dim(\mathrm{Ann}(I_1/J_1) \cap \ldots \cap \mathrm{Ann}(I_m/J_m)) \geqslant \dim(L) - d$. By virtue of Lemma 11,

$$\mathrm{Ann}(I_1/J_1) \cap \ldots \cap \mathrm{Ann}(I_m/J_m) = (B \cap \mathrm{Ann}(I_1/J_1) \cap \ldots \cap \mathrm{Ann}(I_m/J_m)) \oplus (S \cap \mathrm{Ann}(I_1/J_1) \cap \ldots \cap \mathrm{Ann}(I_m/J_m)) \ \oplus\ N.$$

Choose a basis in B that includes a basis of $B \cap \mathrm{Ann}(I_1/J_1) \cap \ldots \cap \mathrm{Ann}(I_m/J_m)$, and a basis in S that includes the basis of $S \cap \mathrm{Ann}(I_1/J_1) \cap \ldots \cap \mathrm{Ann}(I_m/J_m)$. Since $f_{m,q}$ is multilinear, we may assume that only basis elements are substituted for $y_{k\ell}$. Note that $f_{m,q}$ is alternating in Y_i. Hence, if $\xi(f_{m,q}) \neq 0$, then for every $1 \leqslant i \leqslant q$ there exists $y_{kj} \in Y_i$ such that either

$$\xi(y_{kj}) \in B \cap \mathrm{Ann}(I_1/J_1) \cap \ldots \cap \mathrm{Ann}(I_m/J_m)$$

or

$$\xi(y_{kj}) \in S \cap \mathrm{Ann}(I_1/J_1) \cap \ldots \cap \mathrm{Ann}(I_m/J_m).$$

Consider the case when $\xi(y_{kj}) \in B \cap \mathrm{Ann}(I_1/J_1) \cap \ldots \cap \mathrm{Ann}(I_m/J_m)$ for some y_{kj}. Since L is a completely reducible (H,B)-module (Lemma 10), we can choose H-invariant B-submodules T_k such that $I_k = T_k \oplus J_k$. We may assume that $\xi(z_k) \in T_k$ since elements of J_k have greater heights. Therefore $[\xi(z_k^{h_k}), a] \in T_k \cap J_k$ for all $a \in B \cap \mathrm{Ann}(I_1/J_1) \cap \ldots \cap \mathrm{Ann}(I_m/J_m)$. Hence $[\xi(z_k^{h_k}), a] = 0$. Moreover, $B \cap \mathrm{Ann}(I_1/J_1) \cap \ldots \cap \mathrm{Ann}(I_m/J_m)$ is an H-invariant ideal of B and $[B, S] = 0$. Thus, applying Jacobi's identity several times, we obtain

$$\xi([z_k^{h_k}, y_{k1}^{h_{k1}}, \ldots, y_{kj_k}^{h_{kj_k}}]) = 0.$$

Expanding the alternations, we get $\xi(f_{m,q}) = 0$.

Consider the case when $\xi(y_{k\ell}) \in S \cap \mathrm{Ann}(I_1/J_1) \cap \ldots \cap \mathrm{Ann}(I_m/J_m)$ for some $y_{k\ell} \in Y_q$. Expand the alternation Alt_q in $f_{m,q}$ and rewrite $f_{m,q}$ as a sum of

$$\tilde{f}_{m,q-1} := \mathrm{Alt}_1 \ldots \mathrm{Alt}_{q-1}\big[[z_1^{h_1}, y_{11}^{h_{11}}, y_{12}^{h_{12}}, \ldots, y_{1j_1}^{h_{1j_1}}], [z_2^{h_2}, y_{21}^{h_{21}}, y_{22}^{h_{22}}, \ldots, y_{2j_2}^{h_{2j_2}}], \ldots, [z_m^{h_m}, y_{m1}^{h_{m1}}, y_{m2}^{h_{m2}}, \ldots, y_{mj_m}^{h_{mj_m}}]\big].$$

The operator Alt_q may change indices, however we keep the notation $y_{k\ell}$ for the variable with the property $\xi(y_{k\ell}) \in S \cap \mathrm{Ann}(I_1/J_1) \cap \ldots \cap \mathrm{Ann}(I_m/J_m)$. Now the alternation does not affect $y_{k\ell}$. Note that

$$[z_k^{h_k}, y_{k1}^{h_{k1}}, \ldots, y_{k\ell}^{h_{k\ell}}, \ldots, y_{kj_k}^{h_{kj_k}}] = [z_k^{h_k}, y_{k\ell}^{h_{k\ell}}, y_{k1}^{h_{k1}}, \ldots, y_{kj_k}^{h_{kj_k}}]$$
$$+ \sum_{\beta=1}^{\ell-1} [z_k^{h_k}, y_{k1}^{h_{k1}}, \ldots, y_{k,\beta-1}^{h_{k,\beta-1}}, [y_{k\beta}^{h_{k\beta}}, y_{k\ell}^{h_{k\ell}}], y_{k,\beta+1}^{h_{k,\beta+1}}, \ldots, y_{k,\ell-1}^{h_{k,\ell-1}}, y_{k,\ell+1}^{h_{k,\ell+1}}, \ldots, y_{kj_k}^{h_{kj_k}}].$$

In the first term we replace $[z_k^{h_k}, y_{k\ell}^{h_{k\ell}}]$ with z'_k and define $\xi'(z'_k) := \xi([z_k^{h_k}, y_{k\ell}^{h_{k\ell}}])$, $\xi'(x) := \xi(x)$ for other variables x. Then $\mathrm{ht}\,\xi'(z'_k) > \mathrm{ht}\,\xi(z_k)$ and we can use the inductive assumption. If $y_{k\beta} \in Y_j$ for some j, then we expand the alternation Alt_j in this term in $\tilde{f}_{m,q-1}$. If $\xi(y_{k\beta}) \in B$, then the term is zero. If $\xi(y_{k\beta}) \in S$, then $\xi([y_{k\beta}^{h_{k\beta}}, y_{k\ell}^{h_{k\ell}}]) \in N$. We replace $[y_{k\beta}^{h_{k\beta}}, y_{k\ell}^{h_{k\ell}}]$ with an additional variable z'_{m+1} and define $\psi(z'_{m+1}) := \xi([y_{k\beta}^{h_{k\beta}}, y_{k\ell}^{h_{k\ell}}])$, $\psi(x) := \xi(x)$ for other variables x. Applying Jacobi's identity several times, we obtain the polynomial of the desired form. In each inductive step we reduce q by no more than 1, and the maximal number of inductive steps equals $(\dim L)m$. This finishes the proof. □

Since N is a nilpotent ideal, $N^p = 0$ for some $p \in \mathbb{N}$.

Lemma 13. *If $\lambda = (\lambda_1, \ldots, \lambda_s) \vdash n$ and $\lambda_{d+1} \geqslant p((\dim L)p + 3)$ or $\lambda_{\dim L+1} > 0$, then $m(L, H, \lambda) = 0$.*

Proof. It is sufficient to prove that $e^*_{T_\lambda} f \in \mathrm{Id}^H(L)$ for every $f \in V_n^H$ and a Young tableau T_λ, $\lambda \vdash n$, with $\lambda_{d+1} \geqslant p((\dim L)p + 3)$ or $\lambda_{\dim L+1} > 0$.

Fix some basis of L that is a union of bases of B, S, and N. Since polynomials are multilinear, it is sufficient to substitute only basis elements. Note that $e^*_{T_\lambda} = b_{T_\lambda} a_{T_\lambda}$ and b_{T_λ} alternates the variables of each column of T_λ. Hence if we make a substitution and $e^*_{T_\lambda} f$ does not vanish, then this implies that different basis elements are substituted for the variables of each column. But if $\lambda_{\dim L+1} > 0$, then the length of the first column is greater than $(\dim L)$. Therefore, $e^*_{T_\lambda} f \in \mathrm{Id}^H(L)$.

Consider the case $\lambda_{d+1} \geqslant p((\dim L)p + 3)$. Let ξ be a substitution of basis elements for the variables $x_1, \ldots, x_n$. Then $e^*_{T_\lambda} f$ can be rewritten as a sum of polynomials $f_{m,q}$ where $1 \leqslant m \leqslant p$, $q \geqslant p((\dim L)p + 2)$, and z_i, $2 \leqslant i \leqslant m$, are replaced with elements of N. (For different terms $f_{m,q}$, numbers m and q, variables z_i, y_{ij}, and sets Y_i can be different.) Indeed, we expand symmetrization on all variables and alternation on the variables replaced with elements from N. If we have no variables replaced with elements from N, then we take $m = 1$, rewrite the polynomial f as a sum of long commutators, in each long commutator expand the alternation on the set that includes one of the variables in the inner commutator, and denote that variable by z_1. Suppose we have variables replaced with elements from N. We denote them by z_k. Then, using Jacobi's identity, we can put one of such variables inside a long commutator and group all the variables, replaced with elements from $B \cup S$, around z_k such that each z_k is inside the corresponding long commutator.

Applying Lemma 12 many times, we increase m. The ideal N is nilpotent and $\xi(f_{p+1,q}) = 0$ for every q and a proper homomorphism ξ. Reducing q by no more than $p((\dim L)p + 2)$, we obtain $\xi(e^*_{T_\lambda} f) = 0$. □

We also need the upper bound for the multiplicities.

Theorem 12. *Let L be a finite dimensional H-module Lie algebra over a field F of characteristic 0 where H is a Hopf algebra. Then there exist constants $C_3 > 0$, $r_3 \in \mathbb{N}$ such that*

$$\sum_{\lambda \vdash n} m(L, H, \lambda) \leqslant C_3 n^{r_3} \text{ for all } n \in \mathbb{N}.$$

Remark. Cocharacters do not change upon an extension of the base field F (the proof is completely analogous to [16, Theorem 4.1.9]), so we may assume F to be algebraically closed.

Proof. Consider ordinary polynomial identities and cocharacters of L. In fact, we may define them as H-identities and H-cocharacters for $H = F$: $V_n := V_n^F$, $\chi_n(L) := \chi_n^F(L)$, $m(L, \lambda) := m(L, F, \lambda)$, $\mathrm{Id}(L) := \mathrm{Id}^F(L)$. By [14, Theorem 3.1],

$$\sum_{\lambda \vdash n} m(L, \lambda) \leqslant C_4 n^{r_4} \tag{12}$$

for some $C_4 > 0$ and $r_4 \in \mathbb{N}$.

Let $G_1 \subseteq G_2$ be finite groups and let W be an FG_2-module. Denote by $W \downarrow G_1$ the module W with the G_2-action restricted to G_1.

Let $\zeta \colon H \to \mathrm{End}_F(L)$ be the homomorphism corresponding to the H-action, and let $\left(\zeta(\gamma_j)\right)_{j=1}^m$, $\gamma_j \in H$, be a basis in $\zeta(H)$.

Consider the diagonal embedding $\varphi \colon S_n \to S_{mn}$,

$$\varphi(\sigma) := \left(\begin{array}{cccc|cccc|c} 1 & 2 & \dots & n & n+1 & n+2 & \dots & 2n & \dots \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) & n+\sigma(1) & n+\sigma(2) & \dots & n+\sigma(n) & \dots \end{array} \right)$$

and the S_n-homomorphism $\pi \colon (V_{mn} \downarrow \varphi(S_n)) \to V_n^H$ defined by $\pi(x_{n(i-1)+t}) = x_t^{\gamma_i}$, $1 \leqslant i \leqslant m$, $1 \leqslant t \leqslant n$. Note that $\pi(V_{mn} \cap \mathrm{Id}(L)) \subseteq V_n^H \cap \mathrm{Id}^H(L)$. By (5), the FS_n-module $\frac{V_n^H}{V_n^H \cap \mathrm{Id}^H(L)}$ is a homomorphic image of the FS_n-module $\left(\frac{V_{mn}}{V_{mn} \cap \mathrm{Id}(L)}\right) \downarrow \varphi(S_n)$. Denote by $\mathrm{length}(M)$ the number of irreducible components of a module M. Then

$$\sum_{\lambda \vdash n} m(L, H, \lambda) = \mathrm{length}\left(\frac{V_n^H}{V_n^H \cap \mathrm{Id}^H(L)}\right) \leqslant \mathrm{length}\left(\left(\frac{V_{mn}}{V_{mn} \cap \mathrm{Id}(L)}\right) \downarrow \varphi(S_n)\right).$$

Therefore, it is sufficient to prove that $\mathrm{length}\left(\left(\frac{V_{mn}}{V_{mn} \cap \mathrm{Id}(L)}\right) \downarrow \varphi(S_n)\right)$ is polynomially bounded. Replacing $|G|$ with m in [18, Lemma 10 and 12] (or, alternatively, using the proof of [10, Theorem 13 (b)]), we derive this from (12). □

Now we can prove

Theorem 13. *If L is an H-nice algebra and $d > 0$, then there exist constants $C_2 > 0$, $r_2 \in \mathbb{R}$ such that $c_n^H(L) \leqslant C_2 n^{r_2} d^n$ for all $n \in \mathbb{N}$. In the case $d = 0$, the algebra L is nilpotent.*

Proof. Lemma 13 and [16, Lemmas 6.2.4, 6.2.5] imply

$$\sum_{m(L,H,\lambda)\neq 0} \dim M(\lambda) \leqslant C_5 n^{r_5} d^n$$

for some constants $C_5, r_5 > 0$. Together with Theorem 12 this inequality yields the upper bound. □

Remark. In fact, we prove Theorem 13 for all H-module Lie algebras L such that Lemma 10 holds for L.

6. Lower bound

By the definition of $d = d(L)$ (see Subsection 1.8), there exist H-invariant ideals $I_1, I_2, \dots, I_r, J_1, J_2, \dots, J_r$, $r \in \mathbb{Z}_+$, of the algebra L, satisfying conditions (1)–(2), $J_k \subseteq I_k$, such that

$$d = \dim \frac{L}{\operatorname{Ann}(I_1/J_1) \cap \dots \cap \operatorname{Ann}(I_r/J_r)}.$$

We consider the case $d > 0$.

Without loss of generality we may assume that

$$\bigcap_{k=1}^{r} \operatorname{Ann}(I_k/J_k) \neq \bigcap_{\substack{k=1,\\ k\neq \ell}}^{r} \operatorname{Ann}(I_k/J_k)$$

for all $1 \leqslant \ell \leqslant r$. In particular, L has nonzero action on each I_k/J_k.

Remark. If L is semisimple, then, by Example 7, we may assume that $r = 1$, $I_1 = B_1$ for some H-simple ideal B_1 of $L = B$, and $J_1 = 0$. In this case, we skip Lemmas 14–18, and define $T_1 = \tilde{T}_1 = B_1$ (which is a faithful irreducible (H, B_1)-module), $j_1 = 1$, and $q_1 = 0$. In Lemmas 14–18 we assume that L satisfy conditions (1)–(4) of Subsection 1.7.

Our aim is to present a partition $\lambda \vdash n$ with $m(L, H, \lambda) \neq 0$ such that $\dim M(\lambda)$ has the desired asymptotic behavior. We will glue alternating polynomials constructed in Theorem 11 for faithful irreducible modules over reductive algebras. In order to do this, we have to choose the reductive algebras.

Lemma 14. *There exist H-invariant ideals $B_1, \dots, B_r$ in B and H-submodules $\tilde{R}_1, \dots, \tilde{R}_r \subseteq S$ (some of $\tilde{R}_i$ and B_j may be zero) such that*

1. $B_1 + \ldots + B_r = B_1 \oplus \ldots \oplus B_r$;
2. $\tilde{R}_1 + \ldots + \tilde{R}_r = \tilde{R}_1 \oplus \ldots \oplus \tilde{R}_r$;
3. $\sum_{k=1}^{r} \dim(B_k \oplus \tilde{R}_k) = d$;
4. I_k/J_k *is a faithful* $(B_k \oplus \tilde{R}_k \oplus N)/N$*-module;*
5. I_k/J_k *is an irreducible* $\left(H, \left(\sum_{i=1}^{r} (B_i \oplus \tilde{R}_i) \oplus N\right)/N\right)$*-module;*
6. $B_i I_k/J_k = \tilde{R}_i I_k/J_k = 0$ *for* $i > k$.

Proof. Consider $N_\ell := \bigcap_{k=1}^{\ell} \mathrm{Ann}(I_k/J_k)$, $1 \leqslant \ell \leqslant r$, $N_0 := L$. Note that N_ℓ are H-invariant. Since B is semisimple, by [20, Theorem 6], we can choose such H-invariant ideals B_ℓ that $N_{\ell-1} \cap B = B_\ell \oplus (N_\ell \cap B)$. Also, applying condition (4) of Subsection 1.7 to zero algebra, we obtain that L is a completely reducible H-module. Hence we can choose such H-submodules $\tilde{R}_\ell$ that $N_{\ell-1} \cap S = \tilde{R}_\ell \oplus (N_\ell \cap S)$. Therefore properties (1), (2), (6) hold.

By Lemma 11, $N_k = (N_k \cap B) \oplus (N_k \cap S) \oplus N$. Thus property (4) holds. Furthermore,

$$N_{\ell-1} = B_\ell \oplus (N_\ell \cap B) \oplus \tilde{R}_\ell \oplus (N_\ell \cap S) \oplus N = (B_\ell \oplus \tilde{R}_\ell) \oplus N_\ell$$

(direct sum of subspaces). Hence $L = \left(\bigoplus_{i=1}^{r} (B_i \oplus \tilde{R}_i)\right) \oplus N_r$, and properties (3) and (5) hold too. □

Denote by $\zeta \colon H \to \mathrm{End}_F(L)$ the homomorphism corresponding to the H-action. Let A be the associative subalgebra in $\mathrm{End}_F(L)$ generated by operators from $(\mathrm{ad}\, L)$ and $\zeta(H)$. Let $a_{\ell 1}, \dots, a_{\ell, k_\ell}$ be a basis of $\tilde{R}_\ell$.

Lemma 15. *There exist decompositions* $\mathrm{ad}\, a_{ij} = c_{ij} + d_{ij}$, $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant k_i$, *such that $c_{ij} \in A$ acts as a diagonalizable operator on L, $d_{ij} \in J(A)$, elements c_{ij} commute with each other, and c_{ij} and d_{ij} are polynomials in* $\mathrm{ad}\, a_{ij}$. *Moreover, $R_\ell := \langle c_{\ell 1}, \dots, c_{\ell, k_\ell} \rangle_F$ are H-submodules in A.*

Proof. We apply Lemma 5 for $W = \tilde{R}_\ell$ and $\varphi = \mathrm{ad}$. □

Let

$$\tilde{B} := \left(\bigoplus_{i=1}^{r} \mathrm{ad}\, B_i\right) \oplus \langle c_{ij} \mid 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant k_i \rangle_F,$$

$$\tilde{B}_0 := (\mathrm{ad}\, B) \oplus \langle c_{ij} \mid 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant k_i \rangle_F \subseteq A.$$

By virtue of the definition of S (see Section 4), Lemma 15, and the Jacobi identity,

$$[c_{ij}, \mathrm{ad}\, B] = 0. \tag{13}$$

Hence (2) holds for both $\tilde{B}$ and $\tilde{B}_0$. Thus $\tilde{B}$ and $\tilde{B}_0$ are H-module Lie algebras.

Lemma 16. *The space L is a completely reducible $(H, \tilde{B}_0)$-module. Moreover, L is a completely reducible $(H, (\mathrm{ad}\, B_k) \oplus R_k)$-module for any $1 \leqslant k \leqslant r$.*

Proof. By condition (4) of Subsection 1.7, it is sufficient to show that L is a completely reducible $\tilde{B}_0$-module and a completely reducible $(\mathrm{ad}\, B_k) \oplus R_k$-module disregarding the H-action. The elements c_{ij} are diagonalizable on L and commute. Therefore, an eigenspace of any c_{ij} is invariant under the action of other $c_{k\ell}$. Using induction, we split $L = \bigoplus_{i=1}^{\alpha} W_i$, where W_i are intersections of eigenspaces of $c_{k\ell}$ and elements $c_{k\ell}$ act as scalar operators on W_i. By (13), the spaces W_i are B-submodules and L is a completely reducible $\tilde{B}_0$-module and $(\mathrm{ad}\, B_k) \oplus R_k$-module since B and B_k are semisimple. □

Lemma 17. *There exist complementary subspaces $I_k = \tilde{T}_k \oplus J_k$ such that*

1. *$\tilde{T}_k$ is an H-invariant B-submodule and an irreducible $(H, \tilde{B})$-submodule;*
2. *$\tilde{T}_k$ is a completely reducible faithful $(H, (\mathrm{ad}\, B_k) \oplus R_k)$-module;*
3. *$\sum_{k=1}^{r} \dim((\mathrm{ad}\, B_k) \oplus R_k) = d$;*
4. *$B_i \tilde{T}_k = R_i \tilde{T}_k = 0$ for $i > k$.*

Proof. By Lemma 16, L is a completely reducible $(H, \tilde{B}_0)$-module. Therefore, for every J_k we can choose a complementary H-invariant $\tilde{B}_0$-submodule $\tilde{T}_k$ in I_k. Then $\tilde{T}_k$ is both a B- and a $\tilde{B}$-submodule.

Note that

$$(\operatorname{ad} a_{ij})w = c_{ij}w \quad \text{for all } w \in I_k/J_k$$

since I_k/J_k is an irreducible A-module and $J(A)\, I_k/J_k = 0$. Hence, by Lemma 14, I_k/J_k is a faithful $(\operatorname{ad} B_k) \oplus R_k$-module, $R_i\, I_k/J_k = 0$ for $i > k$ and the elements c_{ij} are linearly independent. Moreover, by property (5) of Lemma 14, I_k/J_k is an irreducible $\left(H, \left(\sum_{i=1}^r (B_i \oplus \tilde{R}_i) \oplus N\right)/N\right)$-module. However $\left(\sum_{i=1}^r (B_i \oplus \tilde{R}_i) \oplus N\right)$ $/N$ acts on I_k/J_k by the same operators as $\tilde{B}$. Thus $\tilde{T}_k \cong I_k/J_k$ is an irreducible $(H, \tilde{B})$-module. Property (1) is proved. By Lemma 16, L is a completely reducible $(H, (\operatorname{ad} B_k) \oplus R_k)$-module for any $1 \leqslant k \leqslant r$. Using the isomorphism $\tilde{T}_k \cong I_k/J_k$, we obtain properties (2) and (4) from the remarks above. Property (3) is a consequence of property (3) of Lemma 14. □

Lemma 18. *For all* $1 \leqslant k \leqslant r$ *we have*

$$\tilde{T}_k = T_{k1} \oplus T_{k2} \oplus \cdots \oplus T_{km},$$

where T_{kj} *are faithful irreducible* $(H, (\operatorname{ad} B_k) \oplus R_k)$*-submodules,* $m \in \mathbb{N}$*,* $1 \leqslant j \leqslant m$.

Proof. By Lemma 17, property (2),

$$\tilde{T}_k = T_{k1} \oplus T_{k2} \oplus \cdots \oplus T_{km}$$

for some irreducible $(H, (\operatorname{ad} B_k) \oplus R_k)$-submodules. Suppose T_{kj} is not faithful for some $1 \leqslant j \leqslant m$. Hence $bT_{kj} = 0$ for some $b \in (\operatorname{ad} B_k) \oplus R_k$, $b \neq 0$. Note that $\tilde{B} = ((\operatorname{ad} B_k) \oplus R_k) \oplus \tilde{B}_k$, where

$$\tilde{B}_k := \bigoplus_{i \neq k} (\operatorname{ad} B_i) \oplus \bigoplus_{i \neq k} R_i$$

and $[(\operatorname{ad} B_k) \oplus R_k, \tilde{B}_k] = 0$. Denote by $\widehat{B}_k$ the associative subalgebra of $\operatorname{End}_F(\tilde{T}_k)$ with 1 generated by operators from $\tilde{B}_k$. This subalgebra is H-invariant. Then

$$[(\operatorname{ad} B_k) \oplus R_k, \widehat{B}_k] = 0$$

and $\sum_{a \in \widehat{B}_k} aT_{kj} \supseteq T_{kj}$ is an H-invariant $\tilde{B}$-submodule of $\tilde{T}_k$ since

$$h\left(\sum_{a \in \widehat{B}_k} aT_{kj}\right) = \sum_{a \in \widehat{B}_k} (h_{(1)}a)(h_{(2)}T_{kj}) \subseteq \sum_{a \in \widehat{B}_k} aT_{kj}$$

for all $h \in H$. Thus $\tilde{T}_k = \sum_{a \in \widehat{B}_k} aT_{kj}$ and

$$b\tilde{T}_k = \sum_{a \in \widehat{B}_k} baT_{kj} = \sum_{a \in \widehat{B}_k} a(bT_{kj}) = 0.$$

We get a contradiction with faithfulness of $\tilde{T}_k$. □

By condition (2) of the definition of d (see Subsection 1.8), there exist numbers $q_1, \ldots, q_r \in \mathbb{Z}_+$ such that

$$[[\tilde{T}_1, \underbrace{L, \ldots, L}_{q_1}], [\tilde{T}_2, \underbrace{L, \ldots, L}_{q_2}] \ldots, [\tilde{T}_r, \underbrace{L, \ldots, L}_{q_r}]] \neq 0.$$

Choose $n_i \in \mathbb{Z}_+$ with the maximal $\sum_{i=1}^r n_i$ such that

$$[[\left(\prod_{k=1}^{n_1} j_{1k}\right)\tilde{T}_1, \underbrace{L,\dots,L}_{q_1}], [\left(\prod_{k=1}^{n_2} j_{2k}\right)\tilde{T}_2, \underbrace{L,\dots,L}_{q_2}] \dots, [\left(\prod_{k=1}^{n_r} j_{rk}\right)\tilde{T}_r, \underbrace{L,\dots,L}_{q_r}]] \neq 0$$

for some $j_{ik} \in J(A)$. Let $j_i := \prod_{k=1}^{n_i} j_{ik}$. Then $j_i \in J(A) \cup \{1\}$ and

$$[[j_1\tilde{T}_1, \underbrace{L,\dots,L}_{q_1}], [j_2\tilde{T}_2, \underbrace{L,\dots,L}_{q_2}], \dots, [j_r\tilde{T}_r, \underbrace{L,\dots,L}_{q_r}]] \neq 0,$$

but

$$[[j_1\tilde{T}_1, \underbrace{L,\dots,L}_{q_1}], \dots, [j_k(j\tilde{T}_k), \underbrace{L,\dots,L}_{q_k}], \dots, [j_r\tilde{T}_r, \underbrace{L,\dots,L}_{q_r}]] = 0 \tag{14}$$

for all $j \in J(A)$ and $1 \leqslant k \leqslant r$.

By virtue of Lemma 18, for every k we can choose a faithful irreducible $(H, (\operatorname{ad} B_k) \oplus R_k)$-submodule $T_k \subseteq \tilde{T}_k$ such that

$$[[j_1 T_1, \underbrace{L,\dots,L}_{q_1}], [j_2 T_2, \underbrace{L,\dots,L}_{q_2}] \dots, [j_r T_r, \underbrace{L,\dots,L}_{q_r}]] \neq 0. \tag{15}$$

Lemma 19. *Let* $\psi\colon \bigoplus_{i=1}^r (B_i \oplus \tilde{R}_i) \to \bigoplus_{i=1}^r ((\operatorname{ad} B_i) \oplus R_i)$ *be the linear isomorphism defined by formulas* $\psi(b) = \operatorname{ad} b$ *for all* $b \in B_i$ *and* $\psi(a_{i\ell}) = c_{i\ell}$, $1 \leqslant \ell \leqslant k_\ell$. *Let* f_i *be multilinear associative* H*-polynomials, and* $\bar{x}_1^{(i)}, \dots, \bar{x}_{n_i}^{(i)} \in \bigoplus_{i=1}^r B_i \oplus \tilde{R}_i$, $\bar{t}_i \in \tilde{T}_i$, $\bar{u}_{ik} \in L$, *be some elements. Then*

$$\begin{aligned}
&[[j_1 f_1(\operatorname{ad} \bar{x}_1^{(1)}, \dots, \operatorname{ad} \bar{x}_{n_1}^{(1)})\bar{t}_1, \bar{u}_{11}, \dots, \bar{u}_{1q_1}], \dots,\\
&\qquad [j_r f_r(\operatorname{ad} \bar{x}_1^{(r)}, \dots, \operatorname{ad} \bar{x}_{n_r}^{(r)})\bar{t}_r, \bar{u}_{r1}, \dots, \bar{u}_{rq_r}]]\\
&= [[j_1 f_1(\psi(\bar{x}_1^{(1)}), \dots, \psi(\bar{x}_{n_1}^{(1)}))\bar{t}_1, \bar{u}_{11}, \dots, \bar{u}_{1q_1}], \dots,\\
&\qquad [j_r f_r(\psi(\bar{x}_1^{(r)}), \dots, \psi(\bar{x}_{n_r}^{(r)}))\bar{t}_r, \bar{u}_{r1}, \dots, \bar{u}_{rq_r}]].
\end{aligned}$$

In other words, we can replace $\operatorname{ad} a_{i\ell}$ *with* $c_{i\ell}$ *and the result does not change.*

Proof. We rewrite $\operatorname{ad} a_{i\ell} = c_{i\ell} + d_{i\ell} = \psi(a_i) + d_{i\ell}$ and use the multilinearity of f_i. By (14), terms with $d_{i\ell}$ vanish. □

Lemma 20. *If* $d \neq 0$, *then there exists a number* $n_0 \in \mathbb{N}$ *such that for every* $n \geqslant n_0$ *there exist disjoint subsets* $X_1, \dots, X_{2k} \subseteq \{x_1, \dots, x_n\}$, $k := \left[\frac{n-n_0}{2d}\right]$, $|X_1| = \dots = |X_{2k}| = d$ *and a polynomial* $f \in V_n^H \setminus \operatorname{Id}^H(L)$ *alternating in the variables of each set* X_j.

Proof. Denote by $\varphi_i\colon (\operatorname{ad} B_i) \oplus R_i \to \mathfrak{gl}(T_i)$ the representation corresponding to the action of $(\operatorname{ad} B_i) \oplus R_i$ on T_i. By virtue of Theorem 11, there exist constants $m_i \in \mathbb{Z}_+$ such that for any k there exist multilinear polynomials $f_i \in Q^H_{d_i, 2k, 2kd_i+m_i} \setminus \operatorname{Id}^H(\varphi_i)$, $d_i := \dim((\operatorname{ad} B_i) \oplus R_i)$, alternating in the variables from disjoint sets $X_\ell^{(i)}$, $1 \leqslant \ell \leqslant 2k$, $|X_\ell^{(i)}| = d_i$.

By virtue of (15),

$$[[j_1\bar{t}_1, \bar{u}_{11}, \dots, \bar{u}_{1,q_1}], [j_2\bar{t}_2, \bar{u}_{21}, \dots, \bar{u}_{2,q_2}], \dots, [j_r\bar{t}_r, \bar{u}_{r1}, \dots, \bar{u}_{r,q_r}]] \neq 0,$$

for some $\bar{u}_{i\ell} \in L$ and $\bar{t}_i \in T_i$. All $j_i \in J(A) \cup \{1\}$ are polynomials in elements from $\zeta(H)$ and $\operatorname{ad} L$. Denote by $\tilde{m}$ the maximal degree of them.

Recall that each T_i is a faithful irreducible $(H, (\operatorname{ad} B_i) \oplus R_i)$-module. Therefore by the Density Theorem, $\operatorname{End}_F(T_i)$ is generated by operators from $\zeta(H)$ and $(\operatorname{ad} B_i) \oplus R_i$. Note that $\operatorname{End}_F(T_i) \cong M_{\dim T_i}(F)$. Thus every matrix unit $e^{(i)}_{j\ell} \in M_{\dim T_i}(F)$ can be represented as a polynomial in operators from $\zeta(H)$ and $(\operatorname{ad} B_i) \oplus R_i$. Choose such polynomials for all i and all matrix units. Denote by m_0 the maximal degree of those polynomials.

Let $n_0 := r(2m_0 + \tilde m + 1) + \sum_{i=1}^r (m_i + q_i)$. Now we choose f_i for $k = \left[\frac{n-n_0}{2d}\right]$. In addition, we choose $\tilde f_1$ for $\tilde k = \left[\frac{n-2kd-m_1}{2d_1}\right] + 1$ and φ_1. The polynomials f_i will deliver us the required alternations. However, the total degree of the product may be less than n. We will use $\tilde f_1$ to increase the number of variables and obtain a polynomial of degree n.

Since $f_i \notin \operatorname{Id}^H(\varphi_i)$ and $\tilde f_1 \notin \operatorname{Id}^H(\varphi_1)$, there exist $\bar x_{i1}, \ldots, \bar x_{i,2kd_i+m_i} \in (\operatorname{ad} B_i) \oplus R_i$ such that $f_i(\bar x_{i1}, \ldots, \bar x_{i,2kd_i+m_i}) \neq 0$, and $\bar x_1, \ldots, \bar x_{2\tilde k d_1+m_1} \in (\operatorname{ad} B_1) \oplus R_1$ such that $\tilde f_1(\bar x_1, \ldots, \bar x_{2\tilde k d_1+m_1}) \neq 0$. Hence

$$e^{(i)}_{\ell_i\ell_i} f_i(\bar x_{i1}, \ldots, \bar x_{i,2kd_i+m_i}) e^{(i)}_{s_i s_i} \neq 0$$

and

$$e^{(1)}_{\tilde\ell\tilde\ell} \tilde f_1(\bar x_1, \ldots, \bar x_{2\tilde k d_1+m_1}) e^{(1)}_{\tilde s\tilde s} \neq 0$$

for some matrix units $e^{(i)}_{\ell_i\ell_i}, e^{(i)}_{s_i s_i} \in \operatorname{End}_F(T_i)$, $1 \leqslant \ell_i, s_i \leqslant \dim T_i$, $e^{(1)}_{\tilde\ell\tilde\ell}, e^{(1)}_{\tilde s\tilde s} \in \operatorname{End}_F(T_1)$, $1 \leqslant \tilde\ell, \tilde s \leqslant \dim T_1$. Thus

$$\sum_{\ell=1}^{\dim_{T_i}} e^{(i)}_{\ell\ell_i} f_i(\bar x_{i1}, \ldots, \bar x_{i,2kd_i+m_i}) e^{(i)}_{s_i\ell}$$

is a nonzero scalar operator in $\operatorname{End}_F(T_i)$.

Hence

$$[[j_1\left(\sum_{\ell=1}^{\dim T_1} e^{(1)}_{\ell\ell_1} f_1(\bar x_{11}, \ldots, \bar x_{1,2kd_1+m_1}) e^{(1)}_{s_1\tilde\ell} \tilde f_1(\bar x_1, \ldots, \bar x_{2\tilde k d_1+m_1}) e^{(1)}_{\tilde s\ell}\right) \bar t_1, \bar u_{11}, \ldots, \bar u_{1q_1}],$$
$$[j_2\left(\sum_{\ell=1}^{\dim T_2} e^{(2)}_{\ell\ell_2} f_2(\bar x_{21}, \ldots, \bar x_{2,2kd_2+m_2}) e^{(2)}_{s_2\ell}\right) \bar t_2, \bar u_{21}, \ldots, \bar u_{2q_2}], \ldots,$$
$$[j_r\left(\sum_{\ell=1}^{\dim T_r} e^{(r)}_{\ell\ell_r} f_r(\bar x_{r1}, \ldots, \bar x_{r,2kd_r+m_r}) e^{(r)}_{s_r\ell}\right) \bar t_r, \bar u_{r1}, \ldots, \bar u_{rq_r}]] \neq 0.$$

We assume that each f_i is a polynomial in $x_{i1}, \ldots, x_{i,2kd_i+m_i}$ and $\tilde f_1$ is a polynomial in $x_1, \ldots, x_{2\tilde k d_1+m_1}$. Denote $X_\ell := \bigcup_{i=1}^r X^{(i)}_\ell$, where f_i is alternating in the variables of each $X^{(i)}_\ell$. Let Alt_ℓ be the operator of alternation in the variables from X_ℓ. Consider

$$\tilde f(x_1, \ldots, x_{2\tilde k d_1+m_1}; x_{11}, \ldots, x_{1,2kd_1+m_1}; \ldots;\ x_{r1}, \ldots, x_{r,2kd_r+m_r})$$
$$:= \operatorname{Alt}_1 \operatorname{Alt}_2 \ldots \operatorname{Alt}_1 \operatorname{Alt}_2 \ldots \operatorname{Alt}_{2k} [[j_1\left(\sum_{\ell=1}^{\dim T_1} e^{(1)}_{\ell\ell_1} f_1(x_{11}, \ldots, x_{1,2kd_1+m_1}) e^{(1)}_{s_1\tilde\ell}\right.$$
$$\left.\cdot \tilde f_1(x_1, \ldots, x_{2\tilde k d_1+m_1}) e^{(1)}_{\tilde s\ell}\right) \bar t_1, \bar u_{11}, \ldots, \bar u_{1q_1}],$$

$$[j_2\left(\sum_{\ell=1}^{\dim T_2} e^{(2)}_{\ell\ell_2} f_2(x_{21},\dots,x_{2,2kd_2+m_2})e^{(2)}_{s_2\ell}\right)\bar t_2,\bar u_{21},\dots,\bar u_{2q_2}],\dots,$$

$$[j_r\left(\sum_{\ell=1}^{\dim T_r} e^{(r)}_{\ell\ell_r} f_r(x_{r1},\dots,x_{r,2kd_r+m_r})e^{(r)}_{s_r\ell}\right)\bar t_r,\bar u_{r1},\dots,\bar u_{rq_r}]].$$

Then

$$\tilde f(\bar x_1,\dots,\bar x_{2\tilde k d_1+m_1};\bar x_{11},\dots,\bar x_{1,2kd_1+m_1};\dots;\ \bar x_{r1},\dots,\bar x_{r,2kd_r+m_r})$$

$$= (d_1!)^{2k}\dots(d_r!)^{2k}[[j_1\left(\sum_{\ell=1}^{\dim T_1} e^{(1)}_{\ell\ell_1} f_1(\bar x_{11},\dots,\bar x_{1,2kd_1+m_1})e^{(1)}_{s_1\tilde\ell}\right.$$

$$\left.\cdot\,\tilde f_1(\bar x_1,\dots,\bar x_{2\tilde k d_1+m_1})e^{(1)}_{\tilde s\ell}\right)\bar t_1,\bar u_{11},\dots,\bar u_{1q_1}],\dots,$$

$$[j_r\left(\sum_{\ell=1}^{\dim T_r} e^{(r)}_{\ell\ell_r} f_r(\bar x_{r1},\dots,\bar x_{r,2kd_r+m_r})e^{(r)}_{s_r\ell}\right)\bar t_r,\bar u_{r1},\dots,\bar u_{rq_r}]]\neq 0$$

since f_i are alternating in each $X^{(i)}_\ell$ and, by Lemma 17, $((\operatorname{ad}B_i)\oplus R_i)\tilde T_\ell = 0$ for $i>\ell$. Now we rewrite $e^{(i)}_{\ell j}$ as polynomials in elements of $(\operatorname{ad}B_i)\oplus R_i$ and $\zeta(H)$. Using linearity of $\tilde f$ in $e^{(i)}_{\ell j}$, we can replace $e^{(i)}_{\ell j}$ with the products of elements from $(\operatorname{ad}B_i)\oplus R_i$ and $\zeta(H)$, and the expression will not vanish for some choice of the products. Using (8), we can move all $\zeta(h)$ to the right. By Lemma 19, we can replace all elements from $(\operatorname{ad}B_i)\oplus R_i$ with elements from $B_i\oplus\tilde R_i$ and the expression will be still nonzero. Denote by $\psi\colon \bigoplus_{i=1}^r(B_i\oplus\tilde R_i)\to\bigoplus_{i=1}^r((\operatorname{ad}B_i)\oplus R_i)$ the corresponding linear isomorphism. Now we rewrite j_i as polynomials in elements $\operatorname{ad}L$ and $\zeta(H)$. Since $\tilde f$ is linear in j_i, we can replace j_i with one of the monomials, i.e. with the product of elements from $\operatorname{ad}L$ and $\zeta(H)$. Using (8), we again move all $\zeta(h)$ to the right. Then we replace the elements from $\operatorname{ad}L$ with new variables, and

$$\hat f := \operatorname{Alt}_1\operatorname{Alt}_2\dots\operatorname{Alt}_{2k}\Bigl[\Bigl[\Bigl[y_{11},[y_{12},\dots[y_{1\alpha_1},\Bigl[z_{11},[z_{12},\dots,[z_{1\beta_1},$$

$$(f_1(\operatorname{ad}x_{11},\dots,\operatorname{ad}x_{1,2kd_1+m_1}))^{h_1}[w_{11},[w_{12},\dots,[w_{1\theta_1},$$

$$(\tilde f_1(\operatorname{ad}x_1,\dots,\operatorname{ad}x_{2\tilde kd_1+m_1}))^{\tilde h}[w_1,[w_2,\dots,[w_{\tilde\theta},t_1^{h'_1}]\dots\Bigr],u_{11},\dots,u_{1q_1}\Bigr],$$

$$\Bigl[\Bigl[y_{21},[y_{22},\dots[y_{2\alpha_2},\Bigl[z_{21},[z_{22},\dots,[z_{2\beta_2},$$

$$(f_2(\operatorname{ad}x_{21},\dots,\operatorname{ad}x_{2,2kd_2+m_2}))^{h_2}[w_{21},[w_{22},\dots,[w_{2\theta_2},t_2^{h'_2}]\dots\Bigr],u_{21},\dots,u_{2q_2}\Bigr],\dots,$$

$$\Bigl[\Bigl[y_{r1},[y_{r2},\dots,[y_{r\alpha_r},\Bigl[z_{r1},[z_{r2},\dots,[z_{r\beta_r},$$

$$(f_r(\operatorname{ad}x_{r1},\dots,\operatorname{ad}x_{r,2kd_r+m_r}))^{h_r}[w_{r1},[w_{r2},\dots,[w_{r\theta_r},t_r^{h'_r}]\dots\Bigr],u_{r1},\dots,u_{rq_r}\Bigr]\Bigr]$$

for some $0\leqslant\alpha_i\leqslant\tilde m$, $\quad 0\leqslant\beta_i,\theta_i,\tilde\theta\leqslant m_0$, $\quad h_i,h'_i,\tilde h\in H$, $\quad \bar y_{i\ell},\bar z_{i\ell},\bar w_{i\ell},\bar w_i\in L$ does not vanish under the substitution $t_i=\bar t_i$, $u_{i\ell}=\bar u_{i\ell}$, $x_{i\ell}=\psi^{-1}(\bar x_{i\ell})$, $x_i=\psi^{-1}(\bar x_i)$, $y_{i\ell}=\bar y_{i\ell}$, $z_{i\ell}=\bar z_{i\ell}$, $w_{i\ell}=\bar w_{i\ell}$, $w_i=\bar w_i$.

Hence

$$f_0 := \mathrm{Alt}_1 \mathrm{Alt}_2 \dots \mathrm{Alt}_{2k} \Bigl[\Bigl[\Bigl[y_{11}, [y_{12}, \dots [y_{1\alpha_1}, \Bigl[z_{11}, [z_{12}, \dots, [z_{1\beta_1},$$

$$(f_1(\operatorname{ad} x_{11}, \dots, \operatorname{ad} x_{1,2kd_1+m_1}))^{h_1}[w_{11}, [w_{12}, \dots, [w_{1\theta_1}, t_1] \dots \Bigr], u_{11}, \dots, u_{1q_1}\Bigr],$$

$$\Bigl[\Bigl[y_{21}, [y_{22}, \dots [y_{2\alpha_2}, \Bigl[z_{21}, [z_{22}, \dots, [z_{2\beta_2},$$

$$(f_2(\operatorname{ad} x_{21}, \dots, \operatorname{ad} x_{2,2kd_2+m_2}))^{h_2}[w_{21}, [w_{22}, \dots, [w_{2\theta_2}, t_2^{h'_2}] \dots \Bigr], u_{21}, \dots, u_{2q_2}\Bigr], \dots,$$

$$\Bigl[\Bigl[y_{r1}, [y_{r2}, \dots, [y_{r\alpha_r}, \Bigl[z_{r1}, [z_{r2}, \dots, [z_{r\beta_r},$$

$$(f_r(\operatorname{ad} x_{r1}, \dots, \operatorname{ad} x_{r,2kd_r+m_r}))^{h_r}[w_{r1}, [w_{r2}, \dots, [w_{r\theta_r}, t_r^{h'_r}] \dots \Bigr], u_{r1}, \dots, u_{rq_r}\Bigr]\Bigr]$$

does not vanish under the substitution

$$t_1 = (\tilde{f}_1(\operatorname{ad} \bar{x}_1, \dots, \operatorname{ad} \bar{x}_{2\tilde{k}d_1+m_1}))^{\tilde{h}}[\bar{w}_1, [\bar{w}_2, \dots, [\bar{w}_{\tilde{\theta}}, h'_1 \bar{t}_1] \dots],$$

$t_i = \bar{t}_i$ for $2 \leqslant i \leqslant r$; $u_{i\ell} = \bar{u}_{i\ell}$, $x_{i\ell} = \psi^{-1}(\bar{x}_{i\ell})$, $y_{i\ell} = \bar{y}_{i\ell}$, $z_{i\ell} = \bar{z}_{i\ell}$, $w_{i\ell} = \bar{w}_{i\ell}$.

Note that $f_0 \in V_{\tilde{n}}^H$, $\tilde{n} := 2kd + r + \sum_{i=1}^r (m_i + q_i + \alpha_i + \beta_i + \theta_i) \leqslant n$. If $n = \tilde{n}$, then we take $f := f_0$. Suppose $n > \tilde{n}$. Note that $(\tilde{f}_1(\operatorname{ad} \bar{x}_1, \dots, \operatorname{ad} \bar{x}_{2\tilde{k}d_1+m_1}))^{\tilde{h}}$ $\cdot[\bar{w}_1, [\bar{w}_2, \dots, [\bar{w}_{\tilde{\theta}}, h'_1 \bar{t}_1] \dots]$ is a linear combination of long commutators. Each of these commutators contains at least $2\tilde{k}d_1 + m_1 + 1 > n - \tilde{n} + 1$ elements of L. Hence f_0 does not vanish under a substitution $t_1 = [\bar{v}_1, [\bar{v}_2, [\dots, [\bar{v}_q, h'_1 \bar{t}_1] \dots]$ for some $q \geqslant n - \tilde{n}$, $\bar{v}_i \in L$; $t_i = \bar{t}_i$ for $2 \leqslant i \leqslant r$; $u_{i\ell} = \bar{u}_{i\ell}$, $x_{i\ell} = \psi^{-1}(\bar{x}_{i\ell})$, $y_{i\ell} = \bar{y}_{i\ell}$, $z_{i\ell} = \bar{z}_{i\ell}$, $w_{i\ell} = \bar{w}_{i\ell}$. Therefore,

$$f := \mathrm{Alt}_1 \mathrm{Alt}_2 \dots \mathrm{Alt}_{2k} \Bigl[\Bigl[\Bigl[y_{11}, [y_{12}, \dots [y_{1\alpha_1}, \Bigl[z_{11}, [z_{12}, \dots, [z_{1\beta_1},$$

$$(f_1(\operatorname{ad} x_{11}, \dots, \operatorname{ad} x_{1,2kd_1+m_1}))^{h_1}[w_{11}, [w_{12}, \dots, [w_{1\theta_1},$$

$$\Bigl[v_1, [v_2, [\dots, [v_{n-\tilde{n}}, t_1] \dots] \dots \Bigr], u_{11}, \dots, u_{1q_1}\Bigr],$$

$$\Bigl[\Bigl[y_{21}, [y_{22}, \dots [y_{2\alpha_2}, \Bigl[z_{21}, [z_{22}, \dots, [z_{2\beta_2},$$

$$(f_2(\operatorname{ad} x_{21}, \dots, \operatorname{ad} x_{2,2kd_2+m_2}))^{h_2}[w_{21}, [w_{22}, \dots, [w_{2\theta_2}, t_2^{h'_2}] \dots \Bigr], u_{21}, \dots, u_{2q_2}\Bigr],$$

$$\dots, \Bigl[\Bigl[y_{r1}, [y_{r2}, \dots, [y_{r\alpha_r}, \Bigl[z_{r1}, [z_{r2}, \dots, [z_{r\beta_r},$$

$$(f_r(\operatorname{ad} x_{r1}, \dots, \operatorname{ad} x_{r,2kd_r+m_r}))^{h_r}[w_{r1}, [w_{r2}, \dots, [w_{r\theta_r}, t_r^{h'_r}] \dots \Bigr], u_{r1}, \dots, u_{rq_r}\Bigr]\Bigr]$$

does not vanish under the substitution $v_\ell = \bar{v}_\ell$, $1 \leqslant \ell \leqslant n - \tilde{n}$,

$$t_1 = [\bar{v}_{n-\tilde{n}+1}, [\bar{v}_{n-\tilde{n}+2}, [\dots, [\bar{v}_q, h'_1 \bar{t}_1] \dots];$$

$t_i = \bar{t}_i$ for $2 \leqslant i \leqslant r$; $u_{i\ell} = \bar{u}_{i\ell}$, $x_{i\ell} = \psi^{-1}(\bar{x}_{i\ell})$, $y_{i\ell} = \bar{y}_{i\ell}$, $z_{i\ell} = \bar{z}_{i\ell}$, $w_{i\ell} = \bar{w}_{i\ell}$. Note that $f \in V_n^H$ and satisfies all the conditions of the lemma. □

Lemma 21. *Let k, n_0 be the numbers from Lemma 20. Then for every $n \geqslant n_0$ there exists a partition $\lambda = (\lambda_1, \dots, \lambda_s) \vdash n$, $\lambda_i \geqslant 2k - C$ for every $1 \leqslant i \leqslant d$, with $m(L, H, \lambda) \neq 0$. Here $C := p((\dim L)p + 3)((\dim L) - d)$, where $p \in \mathbb{N}$ is such a number that $N^p = 0$.*

Proof. Consider the polynomial f from Lemma 20. It is sufficient to prove that $e^*_{T_\lambda} f \notin \mathrm{Id}^H(L)$ for some tableau T_λ of the desired shape λ. It is known that $FS_n = \bigoplus_{\lambda, T_\lambda} FS_n e^*_{T_\lambda}$, where the summation runs over the set of all standard tableax T_λ, $\lambda \vdash n$. Thus $FS_n f = \sum_{\lambda, T_\lambda} FS_n e^*_{T_\lambda} f \not\subseteq \mathrm{Id}^H(L)$ and $e^*_{T_\lambda} f \notin \mathrm{Id}^H(L)$ for some $\lambda \vdash n$. We claim that λ is of the desired shape. It is sufficient to prove that $\lambda_d \geqslant 2k - C$, since $\lambda_i \geqslant \lambda_d$ for every $1 \leqslant i \leqslant d$. Each row of T_λ includes numbers of no more than one variable from each X_i, since $e^*_{T_\lambda} = b_{T_\lambda} a_{T_\lambda}$ and a_{T_λ} is symmetrizing the variables of each row. Thus $\sum_{i=1}^{d-1} \lambda_i \leqslant 2k(d-1) + (n - 2kd) = n - 2k$. By virtue of Lemma 13, $\sum_{i=1}^{d} \lambda_i \geqslant n - C$. Therefore $\lambda_d \geqslant 2k - C$. □

Proof of Theorem 9. The Young diagram D_λ from Lemma 21 contains the rectangular subdiagram D_μ, $\mu = (\underbrace{2k - C, \ldots, 2k - C}_{d})$. The branching rule for S_n implies that if we consider the restriction of S_n-action on $M(\lambda)$ to S_{n-1}, then $M(\lambda)$ becomes the direct sum of all nonisomorphic FS_{n-1}-modules $M(\nu)$, $\nu \vdash (n-1)$, where each D_ν is obtained from D_λ by deleting one box. In particular, $\dim M(\nu) \leqslant \dim M(\lambda)$. Applying the rule $(n - d(2k - C))$ times, we obtain $\dim M(\mu) \leqslant \dim M(\lambda)$. By the hook formula,

$$\dim M(\mu) = \frac{(d(2k - C))!}{\prod_{i,j} h_{ij}},$$

where h_{ij} is the length of the hook with edge in (i, j). By Stirling's formula,

$$c_n^H(L) \geqslant \dim M(\lambda) \geqslant \dim M(\mu) \geqslant \frac{(d(2k - C))!}{((2k - C + d)!)^d}$$

$$\sim \frac{\sqrt{2\pi d(2k - C)} \left(\frac{d(2k-C)}{e}\right)^{d(2k-C)}}{\left(\sqrt{2\pi(2k - C + d)} \left(\frac{2k-C+d}{e}\right)^{2k-C+d}\right)^d} \sim C_6 k^{r_6} d^{2kd}$$

for some constants $C_6 > 0$, $r_6 \in \mathbb{Q}$, as $k \to \infty$. Since $k = \left[\frac{n-n_0}{2d}\right]$, this gives the lower bound. The upper bound has been proved in Theorem 13. □

Proof of Theorem 10. Suppose $L = L_1 \oplus \ldots \oplus L_q$, where L_i are H-nice ideals. First, $c_n^H(L) \geqslant c_n^H(L_i)$ for all $n \in \mathbb{N}$ and $1 \leqslant i \leqslant q$ since L_i are H-invariant subalgebras of L. Hence

$$\max_{1 \leqslant i \leqslant q} \mathrm{PIexp}^H(L_i) \leqslant \varliminf_{n \to \infty} \sqrt[n]{c_n^H(L)}.$$

Suppose $f_0 \in V_n^H$, $n \in \mathbb{N}$. In order to prove that $f_0 \in \mathrm{Id}^H(L)$, it is sufficient to substitute only basis elements. Choose a basis in L that is the union of bases in L_i. Then if we substitute elements from different L_i, the polynomial f_0 vanishes. Hence it is sufficient to prove that $f_0 \in \mathrm{Id}^H(L_i)$ for all $1 \leqslant i \leqslant s$. Let $d := \max_{1 \leqslant i \leqslant q} d(L_i) = \max_{1 \leqslant i \leqslant q} \mathrm{PIexp}^H(L_i)$. Then Lemma 13 implies that if $\lambda = (\lambda_1, \ldots, \lambda_s) \vdash n$ and $\lambda_{d+1} \geqslant p((\dim L)p + 3)$ or $\lambda_{\dim L + 1} > 0$, then $m(L, H, \lambda) = 0$. Repeating the arguments of Theorem 13, we apply Theorem 12 and obtain that there exist constants $C_2 > 0$, $r_2 \in \mathbb{R}$ such that $c_n^H(L) \leqslant C_2 n^{r_2} d^n$ for all $n \in \mathbb{N}$. Hence

$$\varlimsup_{n \to \infty} \sqrt[n]{c_n^H(L)} \leqslant d = \max_{1 \leqslant i \leqslant q} \mathrm{PIexp}^H(L_i).$$

The lower bound has already been obtained. □

7. Applications

In this section we derive Theorems 1, 3, and 4 from Theorem 9 and Theorems 2, 5, and 6 from Theorem 10.

7.1. **Applications to graded codimensions.**

7.1.1. *Gradings by finite groups.* In the case of an infinite group we will use a trick (see Lemma 26 below) to pass from an arbitrary group to a finitely generated Abelian one. Unfortunately, this trick makes it impossible to use the explicit formula for the Hopf PI-exponent. However, in the case when the group is finite, we can avoid this trick and prove Lemma 22 that enables us to derive properties of graded codimensions from properties of H-codimensions directly and, in particular, to calculate the graded PI-exponent using Subsection 1.8.

Let L be a Lie algebra over a field F. Suppose L is a right H-comodule for some Hopf algebra H. Denote by $\rho\colon L \to L \otimes H$ the corresponding comodule map. We say that L is an *H-comodule Lie algebra* if $\rho([a,b]) = [a_{(0)}, b_{(0)}] \otimes a_{(1)}b_{(1)}$ for all $a, b \in L$. Here we use Sweedler's notation $\rho(a) = a_{(0)} \otimes a_{(1)}$.

Example 8. If $L = \bigoplus_{g\in G} L^{(g)}$ is a Lie algebra over a field F graded by a group G, then L is an FG-comodule algebra where $\rho(a^{(g)}) = a^{(g)} \otimes g$ for all $g \in G$ and $a^{(g)} \in L^{(g)}$. Conversely, each FG-comodule Lie algebra L has the G-grading $L = \bigoplus_{g\in G} L^{(g)}$, where

$$L^{(g)} = \{a \in L \mid \rho(a) = a \otimes g\}.$$

If H is finite dimensional, then every H-comodule Lie algebra becomes an H^*-module Lie algebra where $H^* := \mathrm{Hom}_F(H,F)$ is the Hopf algebra dual to H and $h^*a = h^*(a_{(1)})a_{(0)}$, $h^* \in H^*$, $a \in L$. In particular, if G is finite, a G-graded Lie algebra L is an $(FG)^*$-module Lie algebra. Conversely, each $(FG)^*$-module Lie algebra L has the G-grading $L = \bigoplus_{g\in G} L^{(g)}$, where

$$L^{(g)} = \{a \in L \mid ha = h(g)a \text{ for all } h \in (FG)^*\}.$$

Furthermore, any homomorphism of graded algebras is a homomorphism of $(FG)^*$-module algebras and vice versa.

Let $(h_g)_{g\in G}$ be the basis in $(FG)^*$ dual to the basis $(g)_{g\in G}$ of FG, i.e.

$$h_{g_1}(g_2) = \left\{ \begin{array}{ll} 1, & g_1 = g_2, \\ 0, & g_1 \ne g_2. \end{array} \right. \tag{16}$$

Note that

$$h_{g_1}h_{g_2} = \left\{ \begin{array}{ll} h_{g_1}, & g_1 = g_2, \\ 0, & g_1 \ne g_2, \end{array} \right.$$

i.e. $(FG)^*$ is isomorphic as an algebra to the direct sum of copies of F.

We treat $L(X|(FG)^*)$ and $L(X^{\mathrm{gr}})$ as both graded and $(FG)^*$-module algebras. The homomorphism $\varphi\colon L(X|(FG)^*) \to L(X^{\mathrm{gr}})$ of $(FG)^*$-module algebras defined by $\varphi(x_j) = \sum_{g\in G} x_j^{(g)}$, $h \in (FG)^*$, $j \in \mathbb{N}$, is an isomorphism since the homomorphism $\xi\colon L(X^{\mathrm{gr}}) \to L(X|(FG)^*)$ of graded algebras defined by $\xi(x_j^{(g)}) = x_j^{h_g}$, $g \in G$, $j \in \mathbb{N}$, is the inverse of φ.

Indeed,

$$\varphi(\xi(x_j^{(g)})) = \varphi(x_j^{h_g}) = h_g\varphi(x_j) = \sum_{g_0\in G} h_g(g_0)x_j^{(g_0)} = x_j^{(g)}$$

and
$$\xi(\varphi(x_j)) = \sum_{g\in G} \xi(x_j^{(g)}) = \sum_{g\in G} x_j^{h_g} = x_j^{\sum_{g\in G} h_g} = x_j.$$

Lemma 22. *Let L be a G-graded Lie algebra where G is a finite group. Consider the corresponding $(FG)^*$-action on L. Then*

(1) $\varphi\left(\mathrm{Id}^{(FG)^*}(L)\right) = \mathrm{Id}^{\mathrm{gr}}(L)$;

(2) $c_n^{(FG)^*}(L) = c_n^{\mathrm{gr}}(L)$.

Proof. Let $f \in \mathrm{Id}^{(FG)^*}(L)$. Suppose $\psi\colon L(X^{\mathrm{gr}}) \to L$ is a homomorphism of graded algebras. Then $\psi\varphi\colon L(X|(FG)^*) \to L$ is a homomorphism of $(FG)^*$-module algebras. Hence $\psi(\varphi(f)) = 0$ and $\varphi(f) \in \mathrm{Id}^{\mathrm{gr}}(L)$.

Conversely, let $f \in \mathrm{Id}^{\mathrm{gr}}(L)$. Suppose $\psi\colon L(X|(FG)^*) \to L$ is a homomorphism of $(FG)^*$-module algebras. Then $\psi\varphi^{-1}\colon L(X^{\mathrm{gr}}) \to L$ is a homomorphism of graded algebras. Therefore $\psi(\varphi^{-1}(f)) = 0$ and $\varphi^{-1}(f) \in \mathrm{Id}^{(FG)^*}(L)$. The first assertion is proved.

The second assertion follows from the first one and the equality $\varphi\left(V_n^{(FG)^*}\right) = V_n^{\mathrm{gr}}$. ☐

Remark. The H-action and the polynomial H-identity from Example 4 correspond to the grading and the graded polynomial identity from Example 1.

Now we can easily derive Theorem 1, in the case when G is finite, from Theorem 7, Lemma 22, and the fact that $(FG)^*$ is isomoriphic as an algebra to the direct sum of fields.

7.1.2. *Gradings by arbitrary groups.* The case when G is infinite is treated using a similar duality. Although Lemma 23 is known, we sketch the proof for the reader's convenience.

Lemma 23. *Let F be an algebraically closed field of characteristic 0, let G be a finitely generated Abelian group, and let $\hat{G} = \mathrm{Hom}(G, F^\times)$ be the group of homomorphisms from G into the multiplicative group $F^\times$ of the field F. Consider the elements of $F\hat{G}$ as functions on G. Then for any pairwise distinct $\gamma_1, \dots, \gamma_m \in G$ there exist $h_1, \dots, h_m \in F\hat{G}$ that $h_i(\gamma_j) = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$*

Proof. First consider the case when G is finite. Then, applying the orthogonality relations, we obtain that the space of functions on G is a linear span of $\hat{G}$, and we can find such h_i.

Now consider the case when $G = \langle g \rangle$ is an infinite cyclic group. Then we can take $\chi \in \hat{G}$, $\chi(g^k) = \lambda^k$, where $\lambda \in F^\times$ is a fixed element of an infinite order. Using the Vandermonde argument, we obtain that 1, χ, χ^2, …, χ^{m-1} are linearly independent as functions on $\gamma_1, \dots, \gamma_m$, and we can find the required $h_1, \dots, h_m$.

In the general case $G = G_1 \times \mathbb{Z}^s$, where G_1 is a finite group. Hence $\hat{G} = \hat{G}_1 \times (F^\times)^s$, where $\hat{G}_1$ is the group of characters of G_1. Now we choose the elements for each component, consider their products, and obtain the required $h_1, \dots, h_m$. ☐

If G is a finitely generated Abelian group and $\hat{G} = \mathrm{Hom}(G, F^\times)$, then each G-graded space $V = \bigoplus_{g\in G} V^{(g)}$ becomes an $F\hat{G}$-module: $\chi v^{(g)} = \chi(g) v^{(g)}$ for all $\chi \in \hat{G}$ and $v^{(g)} \in V^{(g)}$.

We have the following important property:

Lemma 24. *Let L be a finite dimensional Lie algebra over an algebraically closed field F of characteristic* 0*, graded by a finitely generated Abelian group G. Consider the $F\hat{G}$-action on L defined above. Then $c_n^{\mathrm{gr}}(L) = c^{F\hat{G}}(L)$ for all $n \in \mathbb{N}$.*

Proof. Let $\{\gamma_1, \ldots, \gamma_m\} := \{g \in G \mid L^{(g)} \neq 0\}$. This set is finite since L is finite dimensional.

Define the homomorphism $\xi\colon L(X|F\hat{G}) \to L(X^{\mathrm{gr}})$ of algebras and $F\hat{G}$-modules by the formula $\xi(x_i) = \sum_{i=1}^m x_i^{(\gamma_i)}$. Note that $\xi(\mathrm{Id}^{F\hat{G}}(L)) \subseteq \mathrm{Id}^{\mathrm{gr}}(L)$ since for any homomorphism $\psi\colon L(X^{\mathrm{gr}}) \to L$ of G-graded algebras, ψ is a homomorphism of $F\hat{G}$-modules, and if $f \in \mathrm{Id}^{F\hat{G}}(L)$, then $\psi(\xi(f)) = 0$. Hence we can define $\tilde{\xi}\colon L(X|F\hat{G})/\,\mathrm{Id}^{F\hat{G}}(L) \to L(X^{\mathrm{gr}})/\,\mathrm{Id}^{\mathrm{gr}}(L)$.

Let $h_1, \ldots, h_m \in F\hat{G}$ be the elements from Lemma 23 corresponding to $\gamma_1, \ldots, \gamma_m$. Then $a^{h_i} \in L^{(\gamma_i)}$ is the γ_i-component of a for all $a \in L$ and $1 \leqslant i \leqslant m$. In particular,

$$x^h - \sum_{i=1}^m h(\gamma_i) x^{h_i} \in \mathrm{Id}^{F\hat{G}}(L) \text{ for all } h \in F\hat{G}. \tag{17}$$

We define the homomorphism of algebras $\eta\colon L(X^{\mathrm{gr}}) \to L(X|F\hat{G})$ by the formula $\eta(x_j^{(\gamma_i)}) = x_j^{h_i}$ for all $1 \leqslant i \leqslant m$, $j \in \mathbb{N}$, and $\eta(x_j^{(g)}) = 0$ for $g \notin \{\gamma_1, \ldots, \gamma_m\}$. Note that $\eta(\mathrm{Id}^{\mathrm{gr}}(L)) \subseteq \mathrm{Id}^{F\hat{G}}(L)$. Indeed, if $\psi\colon L(X|F\hat{G}) \to L$ is a homomorphism of algebras and $F\hat{G}$-modules, then $\psi(\eta(x_j^{(\gamma_i)})) = \psi(x_j)^{h_i} \in L^{(\gamma_i)}$ for any choice $\psi(x_j) \in L$. Hence $\psi\eta\colon L(X^{\mathrm{gr}}) \to L$ is a graded homomorphism, $\psi(\eta(\mathrm{Id}^{\mathrm{gr}}(L))) = 0$, and $\eta(\mathrm{Id}^{\mathrm{gr}}(L)) \subseteq \mathrm{Id}^{F\hat{G}}(L)$. Thus we can define $\tilde{\eta}\colon L(X^{\mathrm{gr}})/\,\mathrm{Id}^{\mathrm{gr}}(L) \to L(X|F\hat{G})/\,\mathrm{Id}^{F\hat{G}}(L)$.

Denote by $\bar{f}$ the image of a polynomial f in a factor space. Then

$$\tilde{\xi}\tilde{\eta}(\bar{x}_j^{(\gamma_i)}) = \left(\sum_{k=1}^m \bar{x}_j^{(\gamma_k)}\right)^{h_i} = \bar{x}_j^{(\gamma_i)} \text{ for all } 1 \leqslant i \leqslant m,\ j \in \mathbb{N}.$$

Since $x_j^{(g)} \in \mathrm{Id}^{\mathrm{gr}}(L)$ for all $g \notin \{\gamma_1, \ldots, \gamma_m\}$, the map $\tilde{\xi}\tilde{\eta}$ coincides with the identity map on the generators. Hence $\tilde{\xi}\tilde{\eta} = \mathrm{id}_{L(X^{\mathrm{gr}})/\,\mathrm{Id}^{\mathrm{gr}}(L)}$. By (17),

$$\tilde{\eta}\tilde{\xi}(\bar{x}_j^h) = \tilde{\eta}\left(\sum_{i=1}^m h(\gamma_i)\bar{x}_j^{(\gamma_i)}\right) = \sum_{i=1}^m h(\gamma_i)\bar{x}_j^{h_i} = \bar{x}_j^h \text{ for all } j \in \mathbb{N},\ h \in F\hat{G}.$$

Similarly, we have $\tilde{\eta}\tilde{\xi} = \mathrm{id}_{L(X|F\hat{G})/\,\mathrm{Id}^{F\hat{G}}}$. Hence

$$L(X|F\hat{G})/\,\mathrm{Id}^{F\hat{G}}(L) \cong L(X^{\mathrm{gr}})/\,\mathrm{Id}^{\mathrm{gr}}(L).$$

In particular, $\frac{V_n^{F\hat{G}}}{V_n^{F\hat{G}} \cap \mathrm{Id}^{F\hat{G}}(L)} \cong \frac{V_n^{\mathrm{gr}}}{V_n^{\mathrm{gr}} \cap \mathrm{Id}^{\mathrm{gr}}(L)}$ and $c_n^{\mathrm{gr}}(L) = c^{F\hat{G}}(L)$ for all $n \in \mathbb{N}$. $\square$

Furthermore, G-graded algebras deliver us another example of an H-nice algebra:

Example 9. Let L be a finite dimensional Lie algebra over an algebraically closed field F of characteristic 0, graded by a finitely generated Abelian group G. Then L is an $F\hat{G}$-nice algebra.

Proof. First, the nilpotent and the solvable radicals are invariant under all automorphisms. Hence they are $\hat{G}$- and $F\hat{G}$-invariant, and by Lemma 23, G-graded. By [20, Theorem 4], we have an $F\hat{G}$-invariant Levi decomposition. If $V = \bigoplus_{g \in G} V^{(g)}$

is a finite dimensional G-graded vector space, we have a natural G-grading on $\mathrm{End}_F(V)$: $\mathrm{Hom}_F(V^{(g_1)}, V^{(g_2)}) \subseteq \mathrm{End}_F(V)^{(g_2 g_1^{-1})}$, $g_1, g_2 \in G$. (The action of the maps from $\mathrm{Hom}_F(V^{(g_1)}, V^{(g_2)})$ on the other components is zero.) This G-grading corresponds to the natural $F\hat{G}$-action on $\mathrm{End}_F(V)$: $(h\psi)(v) = h_{(1)}\psi((Sh_{(2)})v)$ for $h \in F\hat{G}$, $\psi \in \mathrm{End}_F(V)$, $v \in V$. Hence the graded Wedderburn — Mal'cev theorem [27, Corollary 2.8] implies the $F\hat{G}$-invariant one. (The Jacobson radical is invariant under all automorphisms.) Analogously, condition (4) is a consequence of [20, Theorem 9]. Hence L is $F\hat{G}$-nice. $\square$

Now we can prove a particular case of Theorem 1:

Theorem 14. *Let L be a finite dimensional non-nilpotent Lie algebra over a field F of characteristic 0, graded by a finitely generated Abelian group G. Then there exist constants $C_1, C_2 > 0$, $r_1, r_2 \in \mathbb{R}$, $d \in \mathbb{N}$ such that $C_1 n^{r_1} d^n \leqslant c_n^{\mathrm{gr}}(L) \leqslant C_2 n^{r_2} d^n$ for all $n \in \mathbb{N}$.*

Proof. Graded codimensions do not change upon an extension of the base field. The proof is analogous to the cases of ordinary codimensions of associative [16, Theorem 4.1.9] and Lie algebras [30, Section 2]. Thus without loss of generality we may assume F to be algebraically closed.

By Example 9, L is $F\hat{G}$-nice, and Theorem 14 is a consequence of Theorem 9 and Lemma 24. $\square$

Now we show that the case of an arbitrary group G can be reduced to the case of a finitely generated Abelian group.

We need the following known result (see e.g. [26, Lemma 2.1]):

Lemma 25. *Let L be a Lie algebra graded by a group G. Suppose $[L^{(g_1)}, \ldots, L^{(g_k)}] \neq 0$ for some $g_1, \ldots, g_k \in G$. Then $g_i g_j = g_j g_i$ for all $1 \leqslant i, j \leqslant k$.*

Let G_0 be a subgroup of G. Denote $L_{G_0} := \bigoplus_{g \in G_0} L^{(g)}$.

Lemma 26. *Let L be a finite dimensional Lie algebra over a field F of characteristic 0 graded by an arbitrary group G. Then there exist finitely generated Abelian subgroups $G_1, \ldots, G_r \subseteq G$ such that*

$$c_n^{\mathrm{gr}}(L) = \sum_{i=1}^{r} c_n^{\mathrm{gr}}(L_{G_i}) - \sum_{i,j=1}^{r} c_n^{\mathrm{gr}}(L_{G_i \cap G_j}) + \sum_{i,j,k=1}^{r} c_n^{\mathrm{gr}}(L_{G_i \cap G_j \cap G_k}) - \ldots + (-1)^{r-1} c_n^{\mathrm{gr}}(L_{G_1 \cap G_2 \cap \ldots \cap G_r}). \tag{18}$$

Proof. First, we notice that $V_n^{\mathrm{gr}} = \bigoplus_{g_1, \ldots, g_n \in G} V_{g_1, \ldots, g_n}$ where

$$V_{g_1, \ldots, g_n} := \left\langle \left[x_{\sigma(1)}^{(g_{\sigma(1)})}, x_{\sigma(2)}^{(g_{\sigma(2)})}, \ldots, x_{\sigma(n)}^{(g_{\sigma(n)})} \right] \;\middle|\; \sigma \in S_n \right\rangle_F.$$

Using the process of linearization (see e.g. [4, Theorem 4.2.3]), we get

$$\frac{V_n^{\mathrm{gr}}}{V_n^{\mathrm{gr}} \cap \mathrm{Id}^{\mathrm{gr}}(L)} \cong \bigoplus_{g_1, \ldots, g_n \in G} \frac{V_{g_1, \ldots, g_n}}{V_{g_1, \ldots, g_n} \cap \mathrm{Id}^{\mathrm{gr}}(L)}. \tag{19}$$

Let $\{\gamma_1, \ldots, \gamma_m\} := \{g \in G \mid L^{(g)} \neq 0\}$. This set is finite since L is finite dimensional. Denote $V_{n_1, \ldots, n_m} := V_{\bar{g}}$, where

$$\bar{g} := (\underbrace{\gamma_1, \ldots, \gamma_1}_{n_1}, \underbrace{\gamma_2, \ldots, \gamma_2}_{n_2}, \ldots, \underbrace{\gamma_m, \ldots, \gamma_m}_{n_m}), \ n_i \in \mathbb{Z}_+.$$

Then (19) implies

$$c_n^{\mathrm{gr}}(L) = \sum_{n_1+\ldots+n_m=n} \binom{n}{n_1,\ldots,n_m} c_{n_1,\ldots,n_m}(L), \tag{20}$$

where $c_{n_1,\ldots,n_m}(L) := \dim \frac{V_{n_1,\ldots,n_m}}{V_{n_1,\ldots,n_m}\cap \mathrm{Id}^{\mathrm{gr}}(L)}$.

Let G_0 be a subgroup of G. If $n_i = 0$ for all $\gamma_i \notin G_0$, we have $c_{n_1,\ldots,n_m}(L) = c_{n_1,\ldots,n_m}(L_{G_0})$. (By a remark in Subsection 1.1, it is not important whether we consider G_0- or G-gradings on L_{G_0}.) Fix $n \in \mathbb{N}$ and consider the sets $\Theta(G_0)$ of all m-tuples $(n_1,\ldots,n_m)$ such that $n_i \geqslant 0$, $n_1+\ldots+n_m = n$, and $n_i = 0$ for all $\gamma_i \notin G_0$.

Now we introduce a discrete measure μ on $\Theta(G)$ by the formula

$$\mu\big((n_1,\ldots,n_m)\big) = \binom{n}{n_1,\ldots,n_m} c_{n_1,\ldots,n_m}(L).$$

Then (20) implies $c_n^{\mathrm{gr}}(L_{G_0}) = \mu(\Theta(G_0))$. By Lemma 25, $c_{n_1,\ldots,n_m}(L) = 0$ if $n_i, n_j \neq 0$ for some $\gamma_i\gamma_j \neq \gamma_j\gamma_i$. Denote the set of such m-tuples by Θ_0. Then $\mu(\Theta_0) = 0$. Hence each nonzero $c_{n_1,\ldots,n_m}(L)$ equals $c_{n_1,\ldots,n_m}(L_{G_0})$ for some finitely generated Abelian subgroup G_0 in G. Suppose $G_1,\ldots,G_r$ are all Abelian subgroups in G generated by subsets of $\{\gamma_1,\ldots,\gamma_m\}$. Then $\Theta(G) = \Theta_0 \cup \bigcup_{i=1}^r \Theta(G_i)$, and using the inclusion-exclusion principle, we get

$$c_n^{\mathrm{gr}}(L) = \mu(\Theta(G)) = \mu\left(\bigcup_{i=1}^r \Theta(G_i)\right)$$

$$= \sum_{i=1}^r \mu(\Theta(G_i)) - \sum_{i,j=1}^r \mu(\Theta(G_i)\cap\Theta(G_j)) + \ldots + (-1)^{r-1}\mu(\Theta(G_1)\cap\Theta(G_2)\cap\ldots\cap\Theta(G_r))$$

$$= \sum_{i=1}^r \mu(\Theta(G_i)) - \sum_{i,j=1}^r \mu(\Theta(G_i\cap G_j)) + \ldots + (-1)^{r-1}\mu(\Theta(G_1\cap G_2\cap\ldots\cap G_r))$$

$$= \sum_{i=1}^r c_n^{\mathrm{gr}}(L_{G_i}) - \sum_{i,j=1}^r c_n^{\mathrm{gr}}(L_{G_i\cap G_j}) + \ldots + (-1)^{r-1} c_n^{\mathrm{gr}}(L_{G_1\cap G_2\cap\ldots\cap G_r}).$$

□

Proof of Theorem 1. By Theorem 14, each summand of (18) has the desired asymptotic behaviour for its own d. Take the maximal d. Then there exist $C_2 > 0$ and $r_2 \in \mathbb{R}$ such that $c_n^{\mathrm{gr}}(L) \leqslant C_2 n^{r_2} d^n$ for all $n \in \mathbb{N}$. Our choice of d implies that $\mathrm{PIexp}^{\mathrm{gr}}(L_{G_0}) = d$ for some finitely generated Abelian subgroup $G_0 \subseteq G$. Hence there exist $C_1 > 0$ and $r_1 \in \mathbb{R}$ such that $C_1 n^{r_1} d^n \leqslant c_n^{\mathrm{gr}}(L_{G_0}) \leqslant c_n^{\mathrm{gr}}(L)$ for all $n \in \mathbb{N}$. Therefore, graded codimensions of L satisfy the analog of the Amitsur's conjecture. □

Proof of Theorem 2. Suppose

$$L = L_1 \oplus \ldots \oplus L_q,$$

where L_i are graded ideals. First, $\mathrm{PIexp}^{\mathrm{gr}}(L) \geqslant \mathrm{PIexp}^{\mathrm{gr}}(L_i)$ for all $1 \leqslant i \leqslant q$ since L_i are graded subalgebras of L. By Lemma 26, it is sufficient to show that

$$\mathrm{PIexp}^{\mathrm{gr}}(L_{G_0}) \leqslant \max_{1\leqslant i\leqslant q} \mathrm{PIexp}^{\mathrm{gr}}(L_i)$$

for all finitely generated Abelian subgroups $G_0 \subseteq G$. However, $L_{G_0} = (L_1)_{G_0} \oplus \ldots \oplus (L_q)_{G_0}$, and by Lemma 24, Example 9, and Theorem 10,

$$\begin{aligned}\operatorname{PIexp}^{\mathrm{gr}}(L_{G_0}) &= \operatorname{PIexp}^{F\hat{G}_0}(L_{G_0}) = \max_{1\leqslant i\leqslant q} \operatorname{PIexp}^{F\hat{G}_0}((L_i)_{G_0}) \\ &= \max_{1\leqslant i\leqslant q} \operatorname{PIexp}^{\mathrm{gr}}((L_i)_{G_0}) \leqslant \max_{1\leqslant i\leqslant q} \operatorname{PIexp}^{\mathrm{gr}}(L_i).\end{aligned}$$

□

Also we obtain the following upper bound:

Lemma 27. *Let L be a finite dimensional G-graded Lie algebra over a field F of characteristic 0 and let G be any group. If G is finitely generated Abelian, then*

$$c_n^{\mathrm{gr}}(L) \leqslant (\dim L)^{n+1} \text{ for all } n \in \mathbb{N}.$$

Otherwise, there exists a finitely generated Abelian subgroup G_0 and a constant $C \in \mathbb{N}$ such that

$$c_n^{\mathrm{gr}}(L) \leqslant C(\dim L_{G_0})^n \text{ for all } n \in \mathbb{N}.$$

Proof. Again, graded codimensions do not change upon an extension of the base field, and without loss of generality we may assume F to be algebraically closed.

If G is finitely generated Abelian, we apply Lemmas 1 and 24, and get the first assertion.

Using Lemma 26 and choosing the maximal $\dim L_{G_0}$, we obtain the second assertion from the first one. □

7.2. **Applications to G-codimensions.** First, we prove that in the case of Lie algebras we can restrict ourselves to the case when a group acts by automorphisms only.

Lemma 28. *Let G be a group with a fixed subgroup G_0 of index $\leqslant 2$. Let L be a Lie G-algebra. Denote by $\tilde{G}$ the group isomorphic to G and by $\tilde{g} \in \tilde{G}$ the element corresponding to $g \in G$ under the isomorphism $\tilde{G} \cong G$. Then L is a $\tilde{G}$-algebra where $\tilde{G}$ acts by automorphisms only and the $\tilde{G}$-action is defined by the formula*

$$a^{\tilde{g}} = \begin{cases} a^g & \text{if } g \in G_0, \\ -a^g & \text{if } g \in G\backslash G_0 \end{cases} \quad \text{for all } a \in L.$$

Conversely, each Lie $\tilde{G}$-algebra where the $\tilde{G}$ acts by automorphisms is a Lie G-algebra where G_0 acts by automorphisms and $G\backslash G_0$ acts by anti-automorphisms. Moreover, $c_n^{\tilde{G}}(L) = c_n^G(L)$ for all $n \in \mathbb{N}$. If G is an affine algebraic group acting on L rationally, then $\tilde{G}$ acts on L rationally too. (We assume that the structure of an affine algebraic variety on $\tilde{G}$ is the same as on G.)

Proof. Note that

$$[a,b]^{\tilde{g}} = [a^{\tilde{g}}, b^{\tilde{g}}] \text{ for all } g \in G.$$

Hence L is a $\tilde{G}$-algebra. The converse implication is evident. In this case we define

$$a^{g} = \begin{cases} a^{\tilde{g}} & \text{if } g \in G_0, \\ -a^{\tilde{g}} & \text{if } g \in G\backslash G_0 \end{cases} \quad \text{for all } a \in L.$$

Therefore, we have an isomorphism $\psi\colon L(X|G) \to L(X|\tilde{G})$ of G- and $\tilde{G}$-algebras such that $\psi(\operatorname{Id}^G(L)) = \operatorname{Id}^{\tilde{G}}(L)$, $\psi(V_n^G) = V_n^{\tilde{G}}$, and therefore $c_n^{\tilde{G}}(L) = c_n^G(L)$ for all $n \in \mathbb{N}$.

If G is an affine algebraic group, then G_0 is a closed subgroup of index $\leqslant 2$ and G is the disjoint union of closed subsets G_0 and $G \backslash G_0$. Denote by $\mathcal{O}(G)$ the algebra of polynomial functions on G. Let

$$I_1 = \{f \in \mathcal{O}(G) \mid f(g) = 0 \text{ for all } g \in G_0\}$$

and

$$I_2 = \{f \in \mathcal{O}(G) \mid f(g) = 0 \text{ for all } g \in G \backslash G_0\}.$$

Since $G_0 \cap (G \backslash G_0) = \varnothing$, by Hilbert's Nullstellensatz, $I_1 + I_2 = \mathcal{O}(G)$. Hence $1 = f_1 + f_2$, where $f_1 \in I_1$, $f_2 \in I_2$. Then $f_2(g) - f_1(g) = 1$ for all $g \in G_0$ and $f_2(g) - f_1(g) = -1$ for all $g \in G \backslash G_0$. Therefore, if we multiply all operators from $G \backslash G_0$ by (-1), the representation will still be rational. □

The following lemma enables us to derive properties of G-codimensions from properties of H-codimensions.

Lemma 29. *Let L be a Lie algebra with the action of a group G by automorphisms. Then L is an FG-module algebra where the action of the Hopf algebra FG on L is the extension of the G-action by linearity. Conversely, each Lie FG-module algebra is a Lie G-algebra. Moreover, $c_n^{FG}(L) = c_n^G(L)$ for all $n \in \mathbb{N}$.*

Proof. If we treat G as a subgroup in FG, we obtain

$$g[a, b] = [ga, gb] = [g_{(1)}a, g_{(2)}b] \text{ for all } g \in G.$$

Using the linearity, we get (2) for $H = FG$. Hence L is an FG-module algebra. The converse implication is evident.

Therefore, we have an isomorphism $\psi\colon L(X|G) \to L(X|FG)$ of FG-module and G-algebras such that $\psi(\mathrm{Id}^G(L)) = \mathrm{Id}^{FG}(L)$, $\psi(V_n^G) = V_n^{FG}$, and therefore $c_n^{FG}(L) = c_n^G(L)$ for all $n \in \mathbb{N}$. □

Proof of Theorem 3. We apply Lemmas 28, 29, Example 6, and Theorem 9. □

Proof of Theorem 4. We apply Lemmas 28, 29, and Theorem 7. □

Proof of Theorem 5. We apply Lemmas 28, 29, Example 6, and Theorem 10. □

Proof of Theorem 6. We apply Lemmas 28, 29, and Theorem 8. □

Also we obtain the following propositions.

Lemma 30. *Let L be a finite dimensional Lie algebra with G-action over any field F and let G be any group. Then*

$$c_n^G(L) \leqslant (\dim L)^{n+1} \text{ for all } n \in \mathbb{N}.$$

Proof. We apply Lemmas 1, 28, and 29. □

Lemma 31. *Let L be a Lie algebra with G-action over any field F and let G be any group. Then*

$$c_n(L) \leqslant c_n^G(L) \leqslant |G|^n c_n(L) \text{ for all } n \in \mathbb{N},$$

where $c_n(L)$ are ordinary codimensions.

Proof. We apply Lemmas 2, 28, and 29. □

8. Examples and criteria for simplicity

In this section we assume F to be an algebraically closed field of characteristic 0.

Example 10. Let B be a finite dimensional semisimple H-module Lie algebra where H is a Hopf algebra. If B is H-simple, then there exist $C > 0$, $r \in \mathbb{R}$ such that

$$Cn^r(\dim B)^n \leqslant c_n^H(B) \leqslant (\dim B)^{n+1} \text{ for all } n \in \mathbb{N}.$$

Proof. The upper bound follows from Lemma 1. The lower bound is a consequence of Theorem 9 and Example 7. □

By [20, Theorem 6], every finite dimensional semisimple H-module Lie algebra is the direct sum of H-simple Lie algebras. Here we calculate its Hopf PI-exponent.

Example 11. Let $L = B_1 \oplus B_2 \oplus \ldots \oplus B_q$ be a finite dimensional semisimple H-module Lie algebra where H is a Hopf algebra and B_i are H-simple Lie algebras. Let $d := \max_{1 \leqslant k \leqslant q} \dim B_k$. Then there exist $C_1, C_2 > 0$, $r_1, r_2 \in \mathbb{R}$ such that

$$C_1 n^{r_1} d^n \leqslant c_n^H(L) \leqslant C_2 n^{r_2} d^n \text{ for all } n \in \mathbb{N}.$$

Proof. This is a consequence of Theorem 9 and Example 7. □

Now we obtain a criterion for H-simplicity:

Theorem 15. *Let L be an H-nice Lie algebra where H is a Hopf algebra over F. Then* $\operatorname{PIexp}^H(L) = \dim L$ *if and only if L is semisimple and H-simple.*

Proof. If L is an H-simple semisimple algebra, then $\operatorname{PIexp}^H(L) = \dim L$ by Example 10.

Suppose $\operatorname{PIexp}^H(L) = \dim L$. Let N be the nilpotent radical of L. Let $I_1, \ldots, I_r$, $J_1, \ldots, J_r$ be H-invariant ideals of L satisfying conditions (1)–(2) (see Subsection 1.8). By Lemma 11, $N \subseteq \operatorname{Ann}(I_1/J_1) \cap \cdots \cap \operatorname{Ann}(I_r/J_r)$. Hence $\operatorname{PIexp}^H(L) \leqslant (\dim L) - (\dim N)$. Therefore $N = 0$, and by [21, Proposition 2.1.7], $[L, R] \subseteq N = 0$, where R is the solvable radical of L. Hence $R = Z(L) \subseteq N = 0$ and L is semisimple. Now we apply Example 11. □

In particular, Theorem 15 holds for finite dimensional H-module Lie algebras in the case when H is finite dimensional semisimple (see Example 5).

Applying Lemmas 28 and 29, we get

Theorem 16. *Let L be a finite dimensional Lie algebra over F. Suppose a finite group G acts on L by automorphisms and anti-automorphisms. Then* $\operatorname{PIexp}^G(L) = \dim L$ *if and only if L is a G-simple algebra.*

Applying Lemmas 28, 29 and Example 6, we get

Theorem 17. *Let L be a finite dimensional Lie algebra over F. Suppose a reductive affine algebraic group group G acts on L rationally by automorphisms and anti-automorphisms. Then* $\operatorname{PIexp}^G(L) = \dim L$ *if and only if L is a G-simple algebra.*

Moreover, we can apply the results above to graded Lie algebras.

Example 12. Let $B = \bigoplus_{g \in G} B^{(g)}$ be a finite dimensional graded simple Lie algebra where G is an arbitrary group. Then there exist $C > 0$, $r \in \mathbb{R}$ such that

$$Cn^r(\dim B)^n \leqslant c_n^{\mathrm{gr}}(B) \leqslant (\dim B)^{n+1} \text{ for all } n \in \mathbb{N}.$$

Proof. Suppose $g_1, g_2 \in G$, $g_1 g_2 \neq g_2 g_1$. Denote by I_j the (graded) ideal generated by $B^{(g_j)}$, $j = 1, 2$. Using Lemma 25 and the Jacobi identity, we get $[I_1, I_2] = 0$. Since B is graded simple, the finite set $\{g \in G \mid B^{(g)} \neq 0\}$ consists of commuting elements, and we may assume that G is finitely generated Abelian. Note that the solvable radical of L is $\hat{G}$-invariant and hence graded. Thus B is an $F\hat{G}$-simple semisimple Lie algebra, and we get the bounds from Lemma 24 and Example 10. $\square$

Note that if L is a finite dimensional semisimple graded Lie algebra, then by [20, Theorem 7], L is isomorphic to the direct sum of simple graded algebras.

Example 13. Let $L = B_1 \oplus B_2 \oplus \ldots \oplus B_q$ be a finite dimensional semisimple Lie algebra graded by any group, where B_i are graded simple algebras. Let $d := \max_{1 \leqslant k \leqslant q} \dim B_k$. Then there exist $C_1, C_2 > 0$, $r_1, r_2 \in \mathbb{R}$ such that $C_1 n^{r_1} d^n \leqslant c_n^{\mathrm{gr}}(L) \leqslant C_2 n^{r_2} d^n$ for all $n \in \mathbb{N}$.

Proof. This follows immediately from Theorems 1, 2 and Example 12. $\square$

Theorem 18. *Let L be a finite dimensional Lie algebra over F graded by an arbitrary group. Then* $\mathrm{PIexp}^{\mathrm{gr}}(L) = \dim L$ *if and only if L is a graded simple algebra.*

Proof. If L is a graded simple algebra, then $\mathrm{PIexp}^{\mathrm{gr}}(L) = \dim L$ by Example 10.

Suppose $\mathrm{PIexp}^{\mathrm{gr}}(L) = \dim L$. By Lemma 27, we may assume G to be finitely generated Abelian. Now we use Lemma 24, Example 9, and Theorem 15. $\square$

We conclude the section with an example of a nonsemisimple algebra graded by a non-Abelian group.

Example 14. Let $G = S_3$ and $L = \mathfrak{gl}_2(F) \oplus \mathfrak{gl}_2(F)$. Consider the following G-grading on L:

$$L^{(e)} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} \gamma & 0 \\ 0 & \mu \end{pmatrix} \right\},$$

$$L^{\left((12)\right)} = \left\{ \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \right\} \oplus 0, \qquad L^{\left((23)\right)} = 0 \oplus \left\{ \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \right\};$$

the other components are zero. Then there exist $C_1, C_2 > 0$, $r_1, r_2 \in \mathbb{R}$ such that

$$C_1 n^{r_1} 3^n \leqslant c_n^{\mathrm{gr}}(L) \leqslant C_2 n^{r_2} 3^n \text{ for all } n \in \mathbb{N}.$$

Proof. Note that $L = \mathfrak{sl}_2(F) \oplus \mathfrak{sl}_2(F) \oplus Z(L)$, where the center $Z(L)$ consists of scalar matrices of both copies of $\mathfrak{gl}_2(F)$. Hence $V_n^{\mathrm{gr}} \cap \mathrm{Id}^{\mathrm{gr}}(L) = V_n^{\mathrm{gr}} \cap \mathrm{Id}^{\mathrm{gr}}(\mathfrak{sl}_2(F) \oplus \mathfrak{sl}_2(F))$ for all $n \in \mathbb{N}$. Now we notice that both copies of $\mathfrak{sl}_2(F)$ are simple graded ideals of L and apply Example 13. $\square$

Many examples of Lie algebras with a G-grading and G-action are considered in [18, Subsection 1.5].

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