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ESSENTIAL NORMALITY AND THE DECOMPOSABILITY OF HOMOGENEOUS SUBMODULES

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ABSTRACT. We establish the essential normality of a large new class of homogeneous submodules of the finite rank d-shift Hilbert module. The main idea is a notion of essential decomposability that determines when a submodule can be decomposed into the algebraic sum of essentially normal submodules. We prove that every essentially decomposable submodule is essentially normal, and introduce methods for establishing that a submodule is essentially decomposable. It turns out that many submodules have this property. We prove that many of the submodules considered by other authors are essentially decomposable, and in addition establish the essential decomposability of a large new class of homogeneous submodules. Our results support Arveson's conjecture that every homogeneous submodule of the finite rank d-shift Hilbert module is essentially normal.

1. INTRODUCTION

In this paper we establish new results in higher-dimensional operator theory that support Arveson's conjecture of a corresponence between algebraic varieties and C*-algebras of essentially normal operators. Specifically, we prove the essential normality of a large new class of homogeneous submodules of the finite rank d-shift Hilbert module introduced by Arveson in [Arv98]. Our work provides a new perspective on Arveson's conjecture that every homogeneous submodule is essentially normal.

For fixed $d \geq 1$, let $\mathbb{C}[z] = \mathbb{C}[z_1, \ldots, z_d]$ denote the algebra of complex polynomials in d variables. With the introduction of an appropriate inner product, $\mathbb{C}[z]$ can be completed to a space of analytic functions on the complex unit ball called the Drury-Arveson space, which we denote by H^2_d. The coordinate multiplication operators $M_{z_1}, \ldots, M_{z_d}$, defined on $\mathbb{C}[z]$ by

$$(M_{z_i} p)(z_1, \ldots, z_d) = z_i p(z_1, \ldots, z_d), \quad p \in \mathbb{C}[z],\ 1 \leq i \leq d,$$

extend to bounded linear operators on H^2_d, and $(M_{z_1}, \ldots, M_{z_d})$ forms a contractive d-tuple of operators called the d-shift. The space H^2_d can be naturally viewed as a module over $\mathbb{C}[z]$, with the module action given by

$$pf = p(M_{z_1}, \ldots, M_{z_d}) f, \quad p \in \mathbb{C}[z],\ f \in H^2_d.$$

Endowed with this module action, H^2_d is called the d-shift Hilbert module.

The d-shift and the d-shift Hilbert module H^2_d are of fundamental importance in multivariable operator theory, and they have received a great deal of attention in recent years (see for example [Arv98], [Arv00], [Arv02], [Arv05], [Arv07], [DRS11], [DRS12], [Dou06], [GW08], [Esc11], [Sha11]).

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Let M be a submodule of H^2_d, so that M is an invariant subspace for each coordinate operator $M_{z_1}, \ldots, M_{z_d}$. If we identify the quotient space H^2_d/M with the orthogonal complement $M^\perp$, then we obtain a d-tuple of quotient operators $(T_1, \ldots, T_d)$ by compressing the d-shift $(M_{z_1}, \ldots, M_{z_d})$ to H^2_d/M. Since the operators $M_{z_1}, \ldots, M_{z_d}$ commute, the operators $T_1, \ldots, T_d$ also commute, and in fact, every commuting contractive d-tuple of operators can be realized as a quotient of the d-shift in precisely this way, provided that one is willing to increase the multiplicity and consider vector-valued functions (see for example [Arv98]).

The quotient module H^2_d/M is said to be *p-essentially normal* if the self-commutators

$$T_i^* T_j - T_j T_i^*, \quad 1 \leq i, j \leq d$$

belong to the Schatten p-class $\mathcal{L}^p$ for $1 \leq p \leq \infty$ (where $\mathcal{L}^\infty$ denotes the ideal of compact operators $\mathcal{K}$). We also obtain a d-tuple of operators $(S_1, \ldots, S_d)$ by restricting the elements in the d-shift $(M_{z_1}, \ldots, M_{z_d})$ to M, and the module M is similarly said to be p-essentially normal if the self-commutators

$$S_i^* S_j - S_j S_i^*, \quad 1 \leq i, j \leq d$$

belong to $\mathcal{L}^p$ for $1 \leq p \leq \infty$. In fact, it turns out that these notions of essential normality are equivalent for $p > d$, since the submodule M is p-essentially normal if and only if the quotient module H^2_d/M is p-essentially normal (see for example [Arv05]).

The purpose of this paper is to consider the essential normality of submodules of the d-shift Hilbert module H^2_d, and more generally, the essential normality of submodules of the finite rank d-shift Hilbert module $H^2_d \otimes \mathbb{C}^r$, obtained by tensoring H^2_d with $\mathbb{C}^r$, for a positive integer $r \geq 1$. Note that elements in $H^2_d \otimes \mathbb{C}^r$ can be viewed as analytic vector-valued functions on the complex unit ball.

Arveson observed in [Arv02] that $H^2_d \otimes \mathbb{C}^r$ is itself p-essentially normal for every $p > d$, and motivated by applications to multivariable Fredholm Theory, he asked whether every homogeneous submodule of the finite rank d-shift Hilbert module, i.e. every submodule generated by homogeneous polynomials, is p-essentially normal for every $p > d$.

In [Arv05], Arveson conjectured that this question should have an affirmative answer, and established the truth of his conjecture for submodules of $H^2_d \otimes \mathbb{C}^r$ generated by monomials, i.e. polynomials of the form $z_1^{\alpha_1} \cdots z_d^{\alpha_d} \otimes \xi$ for $\alpha = (\alpha_1, \ldots, \alpha_d)$ in $\mathbb{N}_0^d$ and ξ in $\mathbb{C}$. More recently, in [GW08], Guo and Wang proved that every submodule of $H^2_d \otimes \mathbb{C}^r$ generated by a single homogeneous polynomial is p-essentially normal for every $p > d$. Additionally, they proved that for $d \leq 3$, every homogeneous submodule of $H^2_d \otimes \mathbb{C}^r$ is p-essentially normal for every $p > d$. However, none of these results apply to homogeneous submodules of $H^2_d \otimes \mathbb{C}^r$ when $d \geq 4$ and the submodule is generated by two or more non-monomials.

In this paper, we establish Arveson's conjecture for a large new class of homogeneous submodules of the finite-rank d-shift Hilbert module. This class includes homogeneous submodules of $H^2_d \otimes \mathbb{C}^r$, with d arbitrarily large, that are generated by an arbitrary number of non-monomials. For example, we obtain the following result.

Theorem 1.1. *Let $F_1, \ldots, F_n$ be sets of homogeneous polynomials in $\mathbb{C}[z_1, \ldots, z_d]$. Suppose that there are disjoint subsets $Z_1, \ldots, Z_n$ of $\{z_1, \ldots, z_d\}$, each of size at* *most 2, such that*

$$F_i \subseteq \mathbb{C}[Z_i], \quad 1 \le i \le n.$$

Let $X_1, \ldots, X_n$ be arbitrary sets of vectors in $\mathbb{C}^r$. Then the $H^2_d \otimes \mathbb{C}^r$ submodule generated by the set of vector-valued polynomials

$$\{p \otimes \xi \mid p \in F_i,\ \xi \in X_i,\ 1 \le i \le n\}$$

is p-essentially normal for every $p > d$.

We obtain Theorem 1.1 as a special case of the following more broadly applicable result.

Theorem 1.2. *Let $F_1, \ldots, F_n$ be sets of polynomials in $\mathbb{C}[z_1, \ldots, z_d]$ that each generate p-essentially normal submodules of H^2_d. Suppose that there are disjoint subsets $Z_1, \ldots, Z_n$ of $\{z_1, \ldots, z_d\}$, such that*

$$F_i \subseteq \mathbb{C}[Z_i], \quad 1 \le i \le n.$$

Let $X_1, \ldots, X_n$ be arbitrary sets of vectors in $\mathbb{C}^r$. Then the $H^2_d \otimes \mathbb{C}^r$ submodule generated by the set of vector-valued polynomials

$$\{p \otimes \xi \mid p \in F_i,\ \xi \in X_i,\ 1 \le i \le n\}$$

is p-essentially normal for every $p > d$.

More generally, we obtain Theorem 1.2 as an application of a new method for establishing the essential normality of a submodule of $H^2_d \otimes \mathbb{C}^r$. First, we observe that if $M_1, \ldots, M_n$ are p-essentially normal submodules of $H^2_d \otimes \mathbb{C}^r$ with the property that the algebraic sum $M_1 + \ldots + M_n$ is closed, then the $H^2_d \otimes \mathbb{C}^r$ submodule generated by $M_1, \ldots, M_n$ is also p-essentially normal. Reversing this argument tells us that if we want to prove the p-essential normality of an $H^2_d \otimes \mathbb{C}^r$ submodule M, then we should try to obtain a decomposition of M as $M = M_1 + \ldots + M_n$, where $M_1, \ldots, M_n$ are p-essentially normal submodules of $H^2_d \otimes \mathbb{C}^r$.

Note that by Guo and Wang's result on the essential normality of submodules generated by a single homogeneous polynomial, every homogeneous submodule of $H^2_d \otimes \mathbb{C}^r$ can be written as a closed sum of essentially normal submodules. Therefore, the main difficulty is understanding when, if ever, the algebraic sum $M_1 + \ldots + M_n$ is closed. While this problem seems quite difficult in general, we prove below that this sum is closed in many interesting cases.

Our results also imply the essential normality of submodules that have been considered by other authors. For example, our results imply the essential decomposability of submodules of $H^2_d \otimes \mathbb{C}^r$ generated by monomials, and so we obtain a new proof of Arveson's main result in [Arv05].

In addition to this introduction, this paper has five other sections. In Section 2, we provide a brief review of the basic background material. In Section 3, we introduce the notion of essential decomposability, and relate it to Shalit's stable division property from [Sha11]. In Section 4, we introduce a notion of perpendicularity for a family of submodules that implies essential decomposability. In Section 5, we establish the main results on essential normality.

2. Preliminaries

2.1. **The d-shift Hilbert module.** For fixed $d \geq 1$, let $\mathbb{C}[z] = \mathbb{C}[z_1, \ldots, z_d]$ denote the algebra of complex polynomials in d variables. For monomials in $\mathbb{C}[z]$, it is convenient to use the multi-index notation

$$z^\alpha = z_1^{\alpha_1} \cdots z_d^{\alpha_d}, \quad \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d.$$

The *Drury-Arveson space* H_d^2 is the completion of $\mathbb{C}[z]$ with respect to the inner product $\langle \cdot, \cdot \rangle$, defined on monomials by

$$\langle z^\alpha, z^\beta \rangle = \delta_{\alpha\beta} \frac{\alpha!}{|\alpha|!}, \quad \alpha, \beta \in \mathbb{N}_0^d,$$

where we have written $\alpha! = \alpha_1! \cdots \alpha_d!$ and $|\alpha| = \alpha_1 + \ldots + \alpha_d$ for $\alpha = (\alpha_1, \ldots, \alpha_d)$ in $\mathbb{N}_0^d$.

Let $M_{z_1}, \ldots, M_{z_d}$ denote the coordinate multiplication operators on $\mathbb{C}[z]$ corresponding to the variables $z_1, \ldots, z_d$ respectively,

$$M_{z_i} p = z_i p, \quad p \in \mathbb{C}[z], \ 1 \leq i \leq d.$$

Then these operators extend to bounded linear operators on H_d^2, and the d-tuple $(M_{z_1}, \ldots, M_{z_d})$ is called the *d-shift*.

The elements in H_d^2 can be identified with analytic functions on the complex unit ball, and H_d^2 can be viewed as a Hilbert module over the algebra of polynomials $\mathbb{C}[z]$, with the module action defined by

$$pf = p(M_{z_1}, \ldots, M_{z_d})f, \quad p \in \mathbb{C}[z], \ f \in H_d^2.$$

Endowed with this module action, H_d^2 is called the *d-shift Hilbert module*. The importance of this construction in multivariable operator theory was recognized by Arveson in his comprehensive treatment [Arv98].

Let N denote the number operator, the unbounded self-adjoint operator defined on monomials in $\mathbb{C}[z]$ by

$$N z^\alpha = |\alpha| z^\alpha, \quad \alpha \in \mathbb{N}_0^d,$$

and extended to polynomials in $\mathbb{C}[z]$ by linearity. Then the operator $(N+1)^{-1}$ extends to a bounded operator on H_d^2. Let $\partial_1, \ldots, \partial_d$ denote the operators that act on $\mathbb{C}[z]$ by partial differentiation with respect to the variables $z_1, \ldots, z_d$ respectively. Then restricted to $\mathbb{C}[z]$, we can write

$$M_{z_i}^* = (N+1)^{-1} \partial_i, \quad 1 \leq i \leq d.$$

The fact that the adjoint operators take this form is one reason for the importance of the d-shift (see [Arv98] for details).

More generally, for a polynomial p in $\mathbb{C}[z]$, let M_p denote the operator on H_d^2 corresponding to multiplication by p,

$$M_p f = pf, \quad f \in H_d^2.$$

2.2. **The finite rank d-shift Hilbert module.** We will need to consider a higher multiplicity version of H_d^2. For fixed $r \geq 1$, the *d-shift Hilbert module of rank r*, $H_d^2 \otimes \mathbb{C}^r$, is the Hilbert space tensor product of H_d^2 with the r-dimensional Hilbert space $\mathbb{C}^r$. Note that we could also realize $H_d^2 \otimes \mathbb{C}^r$ as the completion of the algebraic tensor product $\mathbb{C}[z] \otimes \mathbb{C}^r$. We will follow [Arv07] and write $r\mathbb{C}[z]$ and rH_d^2 for $\mathbb{C}[z] \otimes \mathbb{C}^r$ and $H_d^2 \otimes \mathbb{C}^r$ respectively.

Since the meaning will always be clear from the context, it will be convenient to also let $M_{z_1}, \ldots, M_{z_d}$ denote the coordinate multiplication operators on rH^2_d,

$$M_{z_i}(f \otimes \xi) = M_{z_i} f \otimes \xi, \quad f \in H^2_d,\ \xi \in \mathbb{C}^r,\ 1 \le i \le d.$$

Note that cordinate multiplication operators on rH^2_d can also be realized as the tensor product of the coordinate multiplication operators on H^2_d with the identity operator on $\mathbb{C}^r$. The d-tuple $(M_{z_1}, \ldots, M_{z_d})$ is called the *d-shift of rank r*.

The elements in rH^2_d can be viewed as vector-valued analytic functions on the complex unit ball, and rH^2_d can also be viewed as a Hilbert module over the algebra of polynomials $\mathbb{C}[z]$. In this case, the module action is defined on the elementary tensors in rH^2_d by

$$p(f \otimes \xi) = p(M_{z_1}, \ldots, M_{z_d})(f \otimes \xi), \quad p \in \mathbb{C}[z],\ f \in H^2_d,\ \xi \in \mathbb{C}^r,$$

and extended to all of rH^2_d by linearity.

2.3. **Essential normality.** Let N be a submodule of rH^2_d. Then N is invariant for the coordinate multiplication operators $M_{z_1}, \ldots, M_{z_d}$ on rH^2_d, so we can consider the corresponding restrictions $S_1, \ldots, S_d$ to N. The submodule N is said to be *p-essentially normal* if the self-commutators

$$S_i^* S_j - S_j S_i^*, \quad 1 \le i, j \le d,$$

belong to the Schatten p-class $\mathcal{L}^p$ for $1 \le p \le \infty$. If $p = \infty$, then we will say that N is *essentially normal*.

If we identify the quotient space rH^2_d/N with the orthogonal complement $N^\perp$, then we can also consider the compressions $T_1, \ldots, T_d$ of $M_{z_1}, \ldots, M_{z_d}$ respectively to rH^2_d/N. The quotient module rH^2_d/N is similarly said to be *p-essentially normal* if the self-commutators

$$T_i^* T_j - T_j T_i^*, \quad 1 \le i, j \le d,$$

belong to the Schatten class $\mathcal{L}^p$ for $1 \le p \le \infty$.

For a submodule N, let P_N denote the projection onto N. We will require the following result of Arveson, which is Theorem 4.3 of [Arv05].

Theorem 2.1 (Arveson)**.** *Let N be a submodule of rH^2_d. Then for every $p > d$, the following are equivalent:*

1. *N is p-essentially normal,*
2. *rH^2_d/N is p-essentially normal,*
3. *for $1 \le i \le d$, the commutators $M_{z_i} P_N - P_N M_{z_i}$ belong to the Schatten p-class $\mathcal{L}^{2p}$.*

In our work, we will mostly use condition (3) of Theorem 2.1.

3. Essential decomposability

3.1. **Essential decomposability.** In this section, we will consider a notion of decomposability for a submodule that implies essential normality. Let $N_1, \ldots, N_n$ be submodules of rH^2_d. Then we will write $N_1 + \ldots + N_n$ for the (not necessarily closed) algebraic sum

$$N_1 + \ldots + N_n = \{x_1 + \ldots + x_n \mid x_i \in N_i \text{ for } 1 \le i \le n\}.$$

Definition 3.1. Let N be a submodule of rH^2_d. Then N is said to be *p-essentially decomposable* if there are p-essentially normal rH^2_d submodules $N_1, \ldots, N_n$ such that

$$N = N_1 + \ldots + N_n.$$

If $p = \infty$, then N is said to be *essentially decomposable.*

Remark 3.2. Note that if N is a p-essentially normal submodule of rH^2_d, then it is trivially p-essentially decomposable.

Theorem 3.3. *Every p-essentially decomposable submodule of rH^2_d is p-essentially normal for $p > d$.*

Proof. Let N be a p-essentially decomposable submodule of rH^2_d with decomposition $N = N_1 + \ldots + N_n$, where $N_1, \ldots, N_n$ are p-essentially normal rH^2_d submodules. Let $M = N_1 \oplus \ldots \oplus N_n$. Then M is a submodule of $nrH^2_d = (rH^2_d)^n$, and the p-essential normality of $N_1, \ldots, N_n$ implies the p-essential normality of M. Define $L : (rH^2_d)^n \to rH^2_d$ by

$$L(x_1, \ldots, x_n) = x_1 + \ldots + x_n, \quad (x_1, \ldots, x_n) \in (rH^2_d)^n.$$

Then $L(M^{(n)}_{z_i})^* = M^*_{z_i} L$ for each i, where $M^{(n)}_{z_i}$ denotes the coordinate multiplication operator on $(rH^2_d)^n$. In particular, the restriction of L to M is a $2p$-morphism from $(rH^2_d)^n$ to rH^2_d, in the sense of Definition 4.3 of [Arv07]. Furthermore, $L(M) = N$ is closed. Since M is p-essentially normal, it follows from Theorem 4.4 of [Arv07] that N is also p-essentially normal. □

Remark 3.4. A direct proof of Theorem 3.3 that avoids the notion of a p-morphism can be given by emulating the first part of the proof of Theorem 4.4 of [Arv07].

We will also require the following lemma, which can be proved by a simple modification of the proof of Corollary 3 of [FW71].

Lemma 3.5. *Let $N_1, \ldots, N_n$ be submodules of rH^2_d. Then the algebraic sum $N_1 + \ldots + N_n$ is closed if and only if the range of the operator $P_{N_1} + \ldots + P_{N_n}$ is closed.*

Remark 3.6. A classical result of Friedrichs [Fri37] implies that the algebraic sum of two subspaces N_1 and N_2 is closed if and only if the (Friedrichs) angle between them is positive. Recently in [BGM10], Badea, Grivaux and Müller established an analogue of Friedrichs' result for an arbitrary number of subspaces $N_1, \ldots, N_n$, by considering a generalized notion of angle. Although we do not require their results in the present paper, we believe that similar ideas may eventually prove useful in resolving Arveson's conjecture.

3.2. **Stable division.** The stable division property was introduced by Shalit in [Sha11], in connection with Arveson's conjecture. However, the notion of stable division is also of independent interest, since it concerns the numerical stability of multivariable polynomial division.

Definition 3.7. An rH^2_d submodule N is said to have the *stable division property* if there is a family of homogeneous polynomials $\{p_1, \ldots, p_n\}$ generating N and a constant $C \geq 0$ such that, for any polynomial p in N, there are polynomials $q_1, \ldots, q_n$ in $\mathbb{C}[z]$ satisfying $p = q_1 p_1 + \ldots + q_n p_n$ and

$$\|q_1 p_1\| + \ldots + \|q_n p_n\| \leq C\|p\|.$$

The family $\{p_1, \ldots, p_n\}$ is said to be a *stable generating set* for N.

The next result was discovered at the suggestion of Shalit. It establishes a connection between the ideas in this paper and his work on the stable division property in [Sha11].

Theorem 3.8. *Let $p_1, \dots, p_n$ be homogeneous polynomials in $r\mathbb{C}[z]$, and let N_1, $\dots, N_n$ denote the corresponding rH^2_d submodules they generate. Then $\{p_1, \dots, p_n\}$ is a stable generating set if and only if the algebraic sum $N_1 + \dots + N_n$ is closed.*

Proof. Let $N = \overline{N_1 + \dots + N_n}$. Suppose first that $N_1 + \dots + N_n$ is closed. Then the operator $T : N_1 \oplus \dots \oplus N_n \to N$, defined by

$$T(x_1, \dots, x_n) = x_1 + \dots + x_n, \quad (x_1, \dots, x_n) \in N_1 \oplus \dots \oplus N_n,$$

has closed range N. Hence there is a constant $C \geq 0$ such that for any f in N, there are f_i in N_i satisfying $f = f_1 + \dots + f_n$ and

$$\|f_1\| + \dots + \|f_n\| \leq C\|f\|.$$

If $f = p$ is a homogeneous polynomial, then we can replace each f_i with the homogeneous polynomial q_i' obtained by projecting f_i onto the homogeneous component of rH^2_d containing p. Since N_i is generated by the homogeneous polynomial p_i, it is left invariant by this projection. Hence q_i' still belongs to N_i, and we can write $q_i' = q_i p_i$ for some polynomial q_i in $\mathbb{C}[z]$. Since an arbitrary polynomial can be written as an orthogonal sum of homogeneous polynomials of different degrees, it follows that $\{p_1, \dots, p_n\}$ is a stable generating set for N.

Conversely, suppose $\{p_1, \dots, p_n\}$ is a stable generating set for N. Fix f in N, and let $(s_k)_{k=1}^\infty$ be a sequence of polynomials in N converging to f. By the stable division property, there is a constant $C \geq 0$ such that, for each $k \geq 1$, there are polynomials $q_{k,i}$ in $\mathbb{C}[z]$ satisfying $s_k = q_{k,1}p_1 + \dots + q_{k,n}p_n$ and

$$\|q_{k,1}p_1\| + \dots + \|q_{k,n}p_n\| \leq C\|s_k\|.$$

For each i, the sequence $(q_{k,i}p_i)_{k=1}^\infty$ is clearly bounded, and by passing to a subsequence we can assume that it is weakly convergent to some f_i in N_i. Then for all g in rH^2_d,

$$\langle f - (f_1 + \dots + f_n), g\rangle = \lim_{k\to\infty} \langle s_k - (q_{k,1}p_1 + \dots + q_{k,n}p_n), g\rangle = 0,$$

and it follows that $f = f_1 + \dots + f_k$. Hence f belongs to $N_1 + \dots + N_n$. Since f was arbitrary, we conclude that $N = N_1 + \dots + N_n$. ☐

Shalit proved in Theorem 2.3 of [Sha11] that many families of homogeneous submodules of H^2_2 have the stable division property. However, consideration of Shalit's proof reveals that it actually implies the following stronger result. (For background material on Groebner bases, see, for example, [CLS92].)

Theorem 3.9 (Shalit). *Let $p_1, \dots, p_n$ be homogeneous polynomials in $\mathbb{C}[z]$ such that $\{p_1, \dots, p_n\}$ is a Groebner basis. Suppose there is a subset Z of $\{z_1, \dots, z_d\}$, of size at most 2, such that $p_1, \dots, p_n \in \mathbb{C}[Z]$. Then the family $\{p_1, \dots, p_n\}$ is a stable generating set.*

Applying Theorem 3.8 to Theorem 3.9, we obtain the following result.

Proposition 3.10. *Let $N_1, \dots, N_n$ be submodules of H^2_d generated by homogeneous polynomials $p_1, \dots, p_n$ respectively in $\mathbb{C}[z]$. Suppose that $\{p_1, \dots, p_n\}$ is a Groebner basis, and suppose that there is a subset Z of $\{z_1, \dots, z_d\}$, of size at most 2, such that $p_1, \dots, p_n \in \mathbb{C}[Z]$. Then the algebraic sum $N_1 + \dots + N_n$ is closed.*

Applying Theorem 3.8 to an example from [Sha11] provides an example of two submodules of H_d^2 with non-closed algebraic sum, and demonstrates that there is no straightforward generalization of Proposition 3.10 to polynomials in three or more variables.

Example 3.11. Let N_1 and N_2 denote the H_3^2 submodules generated by the polynomials p_1 and p_2 respectively, where

$$p_1(z_1, z_2, z_3) = z_1^2 + z_2 z_3$$
$$p_2(z_1, z_2, z_3) = z_2^2,$$

and let $N = \overline{N_1 + N_2}$. In Example 2.6 of [Sha11] it was shown that the family $\{p_1, p_2\}$ generates N, but is not a stable generating set. Hence by Theorem 3.8, the algebraic sum $N_1 + N_2$ is not closed. Since $\{p_1, p_2\}$ is a Groebner basis with respect to the lexicographical monomial ordering, this also shows that Theorem 3.8 does not generalize to polynomials in three or more variables.

However, N is essentially decomposable. Indeed, let K_1, K_2, K_3, K_4 denote the submodules of H_3^2 generated by the polynomials q_1, q_2, q_3, q_4 respectively, where

$$q_1(z_1, z_2, z_3) = z_1^4$$
$$q_2(z_1, z_2, z_3) = z_1^2 z_2$$
$$q_3(z_1, z_2, z_3) = z_1^2 + z_2 z_3$$
$$q_4(z_1, z_2, z_3) = z_2^2.$$

Then the family $\{q_1, q_2, q_3, q_4\}$ is a stable generating set for N. Hence by Theorem 3.8, $N = K_1 + K_2 + K_3 + K_4$.

We also briefly mention Eschmeier's recent paper [Esc11], which introduces a related property of a family of polynomials, in connection with Arveson's essential normality conjecture. In fact, as pointed out in [Sha11], Eschmeier's property is implied by the stable division property.

4. Perpendicular submodules

4.1. **Perpendicularity.** In this section we consider a property of a family of submodules of rH_d^2 that implies the algebraic sum of the submodules is closed.

Definition 4.1. Let $N_1, \ldots, N_n$ be submodules of rH_d^2. The family $\{N_1, \ldots, N_n\}$ is *perpendicular* if

$$N_i \cap (N_i \cap N_j)^\perp \perp N_j \cap (N_i \cap N_j)^\perp, \quad 1 \le i < j \le n. \tag{4.1}$$

Proposition 4.2. *Let $N_1, \ldots, N_n$ be submodules of rH_d^2. Then the family $\{N_1, \ldots, N_n\}$ is perpendicular if and only if the projections $P_{N_1}, \ldots, P_{N_n}$ commute.*

Proof. We recall the simple fact that if P and Q are projections with range $\operatorname{ran}(P)$ and $\operatorname{ran}(Q)$ respectively, then P and Q commute if and only if

$$\operatorname{ran}(P) \cap (\operatorname{ran}(P) \cap \operatorname{ran}(Q))^\perp \perp \operatorname{ran}(Q) \cap (\operatorname{ran}(P) \cap \operatorname{ran}(Q))^\perp .$$

Therefore, the result follows immediately from Definition 4.1. □

Ken Davidson pointed out that the following lemma is a well-known result from the theory of CSL (commutative subspace lattice) algebras.

Lemma 4.3. *Let $\{N_1, \ldots, N_n\}$ be a perpendicular family of submodules of rH^2_d, and let $K_1, \ldots, K_m$ be subspaces contained in the subspace lattice generated by $N_1, \ldots, N_n$. Then $\{K_1, \ldots, K_m\}$ is also a perpendicular family of submodules of rH^2_d.*

Proof. The projections $P_{K_1}, \ldots, P_{K_n}$ are contained in the von Neumann algebra generated by the projections $P_{N_1}, \ldots, P_{N_n}$, and by Proposition 4.2, this von Neumann algebra is commutative. In particular, the projections $P_{K_1}, \ldots, P_{K_n}$ commute, and another application of Proposition 4.2 implies that the family $\{K_1, \ldots, K_m\}$ is perpendicular. □

4.2. **Criteria for perpendicularity.** In this section, we will consider criteria for a family of submodules to be perpendicular.

Lemma 4.4. *Let $N_1, \ldots, N_n$ be submodules of H^2_d, and for $1 \leq i \leq n$, let $p_{i1}, \ldots, p_{im_i}$ be polynomials that generate N_i. If the operators*

$$M_{p_{i1}} M^*_{p_{i1}} + \ldots + M_{p_{im_i}} M^*_{p_{im_i}}, \quad 1 \leq i \leq n,$$

commute, then the family $\{N_1, \ldots, N_n\}$ is perpendicular.

Proof. For $1 \leq i, j \leq n$, P_{N_i} and P_{N_j} are the range projections of the operators $M_{p_{i1}} M^*_{p_{i1}} + \ldots + M_{p_{im_i}} M^*_{p_{im_i}}$ and $M_{p_{j1}} M^*_{p_{j1}} + \ldots + M_{p_{jm_j}} M^*_{p_{jm_j}}$ respectively. In particular, the projections $P_{N_i}$and P_{N_j} are contained in the von Neumann algebra generated by $M_{p_{i1}} M^*_{p_{i1}} + \ldots + M_{p_{im_i}} M^*_{p_{im_i}}$ and $M_{p_{j1}} M^*_{p_{j1}} + \ldots + M_{p_{jm_j}} M^*_{p_{jm_j}}$. Since these latter operators are self-adjoint, and since they commute, this von Neumann algebra is commutative, meaning in particular that the projections P_{N_i} and P_{N_j} commute. Since i and j were arbitrary, Proposition 4.2 implies that the family $\{N_1, \ldots, N_n\}$ is perpendicular. □

To apply Lemma 4.4, we will require an identity of Guo and Wang from [GW08]. Before presenting the identity, it will be convenient to introduce some special notation for operators that are related to the number operator N defined in Section 2.1. For a function $f : \mathbb{Z} \to \mathbb{Z}$, let $[f(N)]$ denote the (potentially unbounded) self-adjoint operator defined on monomials in $\mathbb{C}[z]$ by

$$[f(N)]\, z^\alpha = f(|\alpha|)\, z^\alpha, \quad \alpha \in \mathbb{N}^d_0,$$

and extended by linearity to polynomials in $\mathbb{C}[z]$. Then, for example, restricted to $\mathbb{C}[z]$, we can write the adjoints of the coordinate multiplication operators $M^*_{z_1}, \ldots, M^*_{z_d}$ on H^2_d as

$$M^*_{z_i} = \left[\frac{1}{N+1}\right] \partial_i, \quad 1 \leq i \leq d,$$

where $\partial_1, \ldots, \partial_d$ denote the operators that act on $\mathbb{C}[z]$ by partial differentiation in the variable $z_1, \ldots, z_d$ respectively. If the operator $[f(N)]$ happens to extend to a bounded operator on H^2_d, then we will also write $[f(N)]$ for this extension. Recall that for a polynomial p in $\mathbb{C}[z]$, we write M_p to denote the operator on H^2_d corresponding to multiplication by p. If p is homogeneous of degree n, then it is easy to check that, restricted to $\mathbb{C}[z]$, we can write

$$[f(N)]\, M_p = M_p\, [f(N+n)]\,.$$

These facts, combined with the general Leibniz rule

$$\partial^{\alpha}(pq) = \sum_{\substack{\beta\in\mathbb{N}_0^d \\ \beta\le\alpha}} \binom{\alpha}{\beta} \left(\partial^{\alpha-\beta}p\right)\left(\partial^{\beta}q\right), \quad \alpha\in\mathbb{N}_0^d,\ p,q\in\mathbb{C}[z],$$

where $\partial^{\alpha} = \partial_{\alpha_1}\cdots\partial_{\alpha_d}$, lead to the following identity of Guo and Wang from [GW08].

Proposition 4.5 (Guo-Wang identity). *Let p and q be homogeneous polynomials in $\mathbb{C}[z]$ of degree m and n respectively. Then*

$$M_p^* M_q = \sum_{\alpha\in\mathbb{N}_0^d} \frac{1}{\alpha!}\left[\frac{N!\,(N+m-n)!}{(N+m)!\,(N-n+|\alpha|)!}\right] M_{\partial^{\alpha}q} M_{\partial^{\alpha}p}^*.$$

We will apply Proposition 4.5 to determine when the hypotheses of Lemma 4.4 hold.

Lemma 4.6. *Let p and q be homogeneous polynomials in $\mathbb{C}[z]$ of degree m and n respectively. Then*

$$\begin{aligned} &M_pM_p^*M_qM_q^* - M_qM_q^*M_pM_p^* \\ &\quad= \sum_{\alpha\in\mathbb{N}_0^d\setminus\{0\}} \frac{1}{\alpha!}\left[\tfrac{(N-m)!(N-n)!}{N!(N-m-n+|\alpha|)!}\right]\left(M_pM_{\partial^{\alpha}q}M_q^*M_{\partial^{\alpha}p}^* - M_{\partial^{\alpha}p}M_qM_{\partial^{\alpha}q}^*M_p^*\right).\end{aligned}$$

Proof. The Guo-Wang identity from Proposition 4.5 gives

$$\begin{aligned} M_pM_p^*M_qM_q^* &= \sum_{\alpha\in\mathbb{N}_0^d} M_p\frac{1}{\alpha!}\left[\tfrac{N!(N+m-n)!}{(N+m)!(N-n+|\alpha|)!}\right] M_{\partial^{\alpha}q}M_{\partial^{\alpha}p}^*M_q^* \\ &= \sum_{\alpha\in\mathbb{N}_0^d} \frac{1}{\alpha!}\left[\tfrac{(N-m)!(N-n)!}{N!(N-m-n+|\alpha|)!}\right] M_pM_{\partial^{\alpha}q}M_q^*M_{\partial^{\alpha}p}^*,\end{aligned}$$

and by symmetry this implies

$$M_qM_q^*M_pM_p^* = \sum_{\alpha\in\mathbb{N}_0^d} \frac{1}{\alpha!}\left[\tfrac{(N-m)!(N-n)!}{N!(N-m-n+|\alpha|)!}\right] M_{\partial^{\alpha}p}M_qM_{\partial^{\alpha}q}^*M_p^*.$$

The result now follows by taking the difference of these identities. □

Lemma 4.7. *Let p and q be homogeneous polynomials in $\mathbb{C}[z]$ of degree m and n respectively. Then $M_pM_p^*$ and $M_qM_q^*$ commute if and only if the operator*

$$\sum_{\alpha\in\mathbb{N}_0^d\setminus\{0\}} \frac{1}{\alpha!}\left[\frac{(N-m)!\,(N-n)!}{N!\,(N-m-n-|\alpha|)!}\right] M_pM_{\partial^{\alpha}q}M_q^*M_{\partial^{\alpha}p}^*$$

is self-adjoint.

Proof. This follows immediately from Lemma 4.6, using the observation that for α in $\mathbb{N}_0^d$,

$$\begin{aligned} &M_pM_{\partial^{\alpha}q}M_q^*M_{\partial^{\alpha}p}^* - M_{\partial^{\alpha}p}M_qM_{\partial^{\alpha}q}^*M_p^* \\ &\qquad= M_pM_{\partial^{\alpha}q}M_q^*M_{\partial^{\alpha}p}^* - \left(M_pM_{\partial^{\alpha}q}M_q^*M_{\partial^{\alpha}p}^*\right)^*.\end{aligned}$$

□

Lemma 4.8. *Let p and q be homogeneous polynomials in $\mathbb{C}[z]$ of degree m and n respectively. Then $M_pM_p^*$ and $M_qM_q^*$ commute if each operator*

$$M_pM_{\partial^\alpha q}M_q^*M_{\partial^\alpha p}^*, \quad \alpha \in \mathbb{N}_0^d \setminus \{0\}$$

is self-adjoint.

Proof. This follows immediately from Lemma 4.7. □

Lemma 4.9. *Let p and q be linear polynomials in $\mathbb{C}[z]$. Then the operators $M_pM_P^*$ and $M_qM_q^*$ commute if either $p = q$ or $p \perp q$.*

Proof. Write the polynomials p and q as

$$p(z_1, \ldots, z_d) = a_1z_1 + \ldots + a_dz_d$$
$$q(z_1, \ldots, z_d) = b_1z_1 + \ldots + b_dz_d.$$

Then

$$\sum_{\alpha\in\mathbb{N}_0^d\setminus\{0\}} \frac{1}{\alpha!}\left[\tfrac{(N-m)!(N-n)!}{N!(N-m-n+|\alpha|)!}\right] M_pM_{\partial^\alpha q}M_q^*M_{\partial^\alpha p}^* \tag{4.2}$$

$$= \sum_{i=1}^d \left[\tfrac{(N-1)!(N-1)!}{N!(N+1)!}\right] a_i\overline{b_i}M_pM_q^*$$
$$= \langle p, q\rangle \left[\tfrac{(N-1)!(N-1)!}{N!(N+1)!}\right] M_pM_q^*.$$

Hence the operator (4.2) is self-adjoint if either $p = q$ or $p \perp q$, and the result follows by Lemma 4.7. □

Lemma 4.10. *Let z^λ and z^μ be monomials in $\mathbb{C}[z]$ for λ and μ in $\mathbb{N}_0^d$. Then the operators $M_{z^\lambda}M_{z^\lambda}^*$ and $M_{z^\mu}M_{z^\mu}^*$ commute.*

Proof. For α in $\mathbb{N}_0^d$,

$$z^\lambda(\partial^\alpha z^\mu) = c_\mu z^{\lambda_1+\mu_1-\alpha_1}\cdots z^{\lambda_d+\mu_d-\alpha_d},$$

where

$$c_\mu = \begin{cases}\prod_{i=1}^d \mu_i(\mu_i-1)\cdots(\mu_i-\alpha_i+1) & \text{if } \alpha_i \le \mu_i \text{ for } 1 \le i \le d,\\ 0 & \text{otherwise.}\end{cases}$$

Similarly,

$$(\partial^\alpha z^\lambda)z^\mu = c_\lambda z^{\lambda_1+\mu_1-\alpha_1}\cdots z^{\lambda_d+\mu_d-\alpha_d},$$

where

$$c_\lambda = \begin{cases}\prod_{i=1}^d \lambda_i(\lambda_i-1)\cdots(\lambda_i-\alpha_i+1) & \text{if } \alpha_i \le \lambda_i \text{ for } 1 \le i \le d,\\ 0 & \text{otherwise.}\end{cases}$$

Let $\nu = (\lambda_1+\mu_1-\alpha_1, \ldots, \lambda_d+\mu_d-\alpha_d)$. Then

$$M_{z^\lambda}M_{\partial^\alpha z^\mu}M_{z^\mu}^*M_{\partial^\alpha z^\lambda}^* = \begin{cases} c_\lambda c_\mu M_{z^\nu}M_{z^\nu}^* & \text{if } \nu \in \mathbb{N}_0^d,\\ 0 & \text{otherwise.}\end{cases}$$

In particular, this operator is self-adjoint. Therefore, by Lemma 4.8, the operators $M_{z^\lambda}M_{z^\lambda}^*$ and $M_{z^\mu}M_{z^\mu}^*$ commute. □

Lemma 4.11. *Let p and q be homogeneous in $\mathbb{C}[z]$ in distinct variables. Then the operators $M_pM_p^*$ and $M_qM_q^*$ commute.*

Proof. Since p and q are polynomials in disjoint variables, for every $\alpha \in \mathbb{N}_0^d \backslash \{0\}$, at least one of $\partial^\alpha p$ and $\partial^\alpha q$ must be zero, and hence at least one of $M_{\partial^\alpha p}$ and $M_{\partial^\alpha q}$ must be zero. In particular, this implies that $M_p M_{\partial^\alpha q} M_q^* M_{\partial^\alpha p}^* = 0$, and it follows from Lemma 4.8 that $M_pM_p^*$ and $M_qM_q^*$ commute. □

4.3. **Perpendicular submodules.** In this section, we will establish the perpendicularity of many families of submodules of H_d^2 using the criteria from Section 4.2.

Proposition 4.12. *Let $N_1, \ldots, N_n$ be submodules of H_d^2 that are generated by mutually orthogonal sets of linear polynomials $F_1, \ldots, F_n$ respectively. Then the family $\{N_1, \ldots, N_n\}$ is perpendicular.*

Proof. This follows immediately from Lemma 4.4 and Proposition 4.2. □

Proposition 4.13. *Let $N_1, \ldots, N_n$ be submodules of H_d^2 each generated by monomials. Then* the family *$\{N_1, \ldots, N_n\}$ is perpendicular.*

Proof. This follows immediately from Lemma 4.4 and Lemma 4.10. □

Proposition 4.14. *Let $N_1, \ldots, N_n$ be submodules of H_d^2 generated by sets of homogeneous polynomials $F_1, \ldots, F_n$ respectively. Suppose that there are disjoint subsets $Z_1, \ldots, Z_n$ of $\{z_1, \ldots, z_d\}$ such that*

$$F_i \subseteq \mathbb{C}\left[Z_i\right], \quad 1 \leq i \leq n.$$

Then the family $\{N_1, \ldots, N_n\}$ is perpendicular.

Proof. This follows immediately from Lemma 4.4 and Lemma 4.11. □

We can strengthen Proposition 4.14 using results of Carlini and Reznick. The following result is Lemma 3.1 in [Rez93].

Proposition 4.15 (Rez93)**.** *Let $p_1, \ldots, p_n$ be homogeneous polynomials in $\mathbb{C}[z]$. If the sets*

$$\left\{\partial_{z^\alpha} p_i \mid |\alpha| = \deg\left(p_i\right) - 1,\ \alpha \in \mathbb{N}_0^d\right\}, \quad 1 \leq i \leq n,$$

are mutually orthogonal, then there is a unitary change of variables such that the polynomials $p_1, \ldots, p_n$ are polynomials in disjoint variables.

The following result is Proposition 1 in [Car06].

Proposition 4.16 (Car06)**.** *Let $p_1, \ldots, p_n$ be homogeneous polynomials in $\mathbb{C}[z]$. If the sets*

$$\left\{\nabla p_i\left(z\right) \mid z \in \mathbb{C}^d\right\}, \quad 1 \leq i \leq n,$$

are mutually orthogonal, where ∇p denotes the gradient of p, then there is a unitary change of variables such that the polynomials $p_1, \ldots, p_n$ are polynomials in disjoint variables.

We immediately obtain the following two results.

Proposition 4.17. *Let $N_1, \ldots, N_n$ be submodules of H_d^2 generated by sets of homogeneous polynomials $F_1, \ldots, F_n$ respectively. If the sets*

$$\left\{\partial^\alpha\left(p\right) \mid |\alpha| = \deg\left(p\right) - 1,\ \alpha \in \mathbb{N}_0^d,\ p \in F_i\right\}, \quad 1 \leq i \leq n,$$

are mutually orthogonal, then the family $\{N_1, \ldots, N_n\}$ is perpendicular.

Proposition 4.18. *Let $N_1, \ldots, N_n$ be submodules of H_d^2 generated by sets of homogeneous polynomials $F_1, \ldots, F_n$ respectively. If the sets*

$$\left\{ (\nabla p)(\lambda) \mid \lambda \in \mathbb{C}^d, \; p \in F_i \right\}, \quad 1 \leq i \leq n,$$

are mutually orthogonal, then the family $\{N_1, \ldots, N_n\}$ is perpendicular.

4.4. **Perpendicularity and tensor products.** The results obtained in Section 4.2 and Section 4.3 only apply to submodules of H_d^2. Because we also need to consider higher-rank submodules of rH_d^2, in this section we consider tensor products of perpendicular submodules.

Theorem 4.19. *Let $\{N_1, \ldots, N_n\}$ be a perpendicular family of submodules of H_d^2, and let $V_1, \ldots V_n$ be arbitrary subspaces of $\mathbb{C}^r$. Let $M_1, \ldots, M_n$ denote the rH_d^2 submodules*

$$M_i = N_i \otimes V_i, \quad 1 \leq i \leq n.$$

Then the algebraic sum $M_1 + \ldots + M_n$ is closed.

Proof. Let $E_1, \ldots, E_n$ denote the projections onto $V_1, \ldots, V_n$ respectively. Then it's clear that

$$P_{M_i} = P_{N_i} \otimes E_i, \quad 1 \leq i \leq n.$$

We will prove that the operator

$$P_{M_1} + \ldots + P_{M_n}$$

has closed range. By Lemma 3.5, this will imply the desired result.

We proceed by induction on n, the size of the family $\{N_1, \ldots, N_n\}$. For $n = 1$, the result is trivially true. Therefore, suppose that $n \geq 2$ with the induction hypothesis that the result is true for a perpendicular family of submodules of H_d^2 if the size of the family is at most $n - 1$.

Let $Q_0, \ldots, Q_n$ denote the projections onto rH_d^2 defined by

$$\begin{aligned}
Q_0 &= P_{N_1} \cdots P_{N_n} \otimes I \\
Q_1 &= P_{N_1}^{\perp} \otimes I \\
Q_2 &= P_{N_1} P_{N_2}^{\perp} \otimes I \\
&\vdots \\
Q_n &= P_{N_1} \cdots P_{N_{n-1}} P_{N_n}^{\perp} \otimes I.
\end{aligned}$$

Since the family $\{N_1, \ldots, N_n\}$ is perpendicular, Proposition 4.2 implies that the projections $P_{N_1}, \ldots, P_{N_n}$ commute, and hence that the projections $Q_0, \ldots, Q_n$ also commute. It's also clear that

$$Q_i P_{M_j} = P_{M_j} Q_i, \quad 1 \leq i, j \leq n. \tag{4.3}$$

Furthermore, by construction the projections $Q_0, \ldots, Q_n$ are orthogonal,

$$Q_i Q_j = 0, \quad 0 \leq i < j \leq n, \tag{4.4}$$

and we can decompose the identity operator on rH_d^2 as

$$I = Q_0 + Q_1 + Q_2 + \ldots + Q_n. \tag{4.5}$$

Therefore, by (4.3), (4.4) and (4.5), we can write

$$P_{M_1} + \ldots + P_{M_n} = \left(\sum_{i=0}^{n} Q_i\right)\left(\sum_{j=1}^{n} P_{M_j}\right)\left(\sum_{i=0}^{m} Q_i\right) = \sum_{i=0}^{n}\sum_{j=0}^{n} Q_i P_{M_j}. \tag{4.6}$$

Now, for $1 \le j \le n$,

$$\begin{aligned} Q_0 P_{M_j} &= (P_{N_1} \cdots P_{N_n} \otimes I)\left(P_{N_j} \otimes E_j\right) \\ &= P_{N_1} \cdots P_{N_n} \otimes E_j \\ &= Q_0 (I \otimes E_j), \end{aligned}$$

and this gives

$$Q_0 \left(\sum_{j=1}^{n} P_{M_j}\right) = Q_0 \sum_{j=1}^{n} (I \otimes E_j). \tag{4.7}$$

By a similar calculation, for $1 \le i \le n$,

$$\begin{aligned} Q_i P_{M_i} &= \left(P_{N_1} \cdots P_{N_{i-1}} P_{N_i}^{\perp} \otimes I\right)(P_{N_i} \otimes E_i) \\ &= P_{N_1} \cdots P_{N_{i-1}} P_{N_i}^{\perp} P_{N_i} \otimes E_i \\ &= 0, \end{aligned}$$

and this gives

$$Q_i \sum_{j=1}^{n} P_{M_j} = Q_i \sum_{\substack{j=1 \\ j \ne i}}^{n} P_{M_j}. \tag{4.8}$$

Let X_0 denote the operator on rH_d^2 defined by

$$X_0 = I \otimes \sum_{j=1}^{n} E_j,$$

and let $X_1, \ldots, X_n$ denote the operators on $H_d^2 \otimes \mathbb{C}^r$ defined by

$$X_i = \sum_{\substack{j=1 \\ j \ne i}}^{n} P_{M_j}, \quad 1 \le i \le n.$$

Then by the induction hypothesis and Lemma 3.5, the operators $X_0, \ldots, X_n$ each have closed range, and by (4.6), (4.7) and (4.8), we can write

$$P_{M_1} + \ldots + P_{M_n} = Q_0 X_0 + \ldots + Q_n X_n.$$

Since the operators $X_0, \ldots, X_n$ commute with the projections $Q_0, \ldots, Q_n$, it follows that the operators $Q_0 X_0, \ldots, Q_n X_n$ each have closed range. Therefore, since the projections $Q_0, \ldots, Q_n$ are orthogonal, this means that we have written $P_{M_1} + \ldots + P_{M_n}$ as the direct sum of $n+1$ operators that each have closed range. It follows that the range of this operator is also closed. □

5. Essential normality

5.1. **Essential normality and perpendicularity.**

Lemma 5.1. *Let N be a p-essentially normal submodule of H^2_d for $p > d$, and let V be an arbitrary subspace of $\mathbb{C}^r$. Then the rH^2_d submodule $N \otimes V$ is also p-essentially normal.*

Proof. Let E denote the projection onto V, and let $M = N \otimes V$. Then it's clear that

$$P_M = P_N \otimes E.$$

By Theorem 2.1, the p-essential normality of N implies that the projection P_N $2p$-essentially commutes with the coordinate multiplication operators $M_{z_1}, \ldots, M_{z_d}$ on H^2_d, i.e.

$$M_{z_i} P_N - P_N M_{z_i} \in \mathcal{L}^{2p}, \quad 1 \leq i \leq d,$$

where $\mathcal{L}^{2p}$ denotes the set of Schatten $2p$-class operators on H^2_d. Recall from Section 2.2 that we can write the coordinate multiplication operators on rH^2_d as $M_{z_1} \otimes I, \ldots, M_{z_d} \otimes I$. Hence for $1 \leq i \leq d$,

$$\begin{aligned}(M_{z_i} \otimes I) P_M - P_M (M_{z_i} \otimes I) &= (M_{z_i} \otimes I)(P_N \otimes E) - (P_N \otimes E)(M_{z_i} \otimes I) \\ &= (M_{z_i} P_N - P_N M_{z_i}) \otimes E \in \mathcal{L}^{2p},\end{aligned}$$

since E is a finite rank projection. Therefore, by Theorem 2.1, M is $2p$-essentially normal. ☐

Theorem 5.2. *Let $\{N_1, \ldots, N_n\}$ be a perpendicular family of p-essentially decomposable submodules of H^2_d for $p > d$, and let $V_1, \ldots, V_n$ be arbitrary subspaces of $\mathbb{C}^r$. Let $M_1, \ldots, M_n$ denote the rH^2_d submodules*

$$M_i = N_i \otimes V_i, \quad 1 \leq i \leq n.$$

Then the rH^2_d submodule $\overline{M_1 + \ldots + M_n}$ is also p-essentially normal.

Proof. By Lemma 5.1, the submodules $M_1, \ldots, M_n$ are p-essentially normal. Let M denote the rH^2_d submodule $M = \overline{M_1 + \ldots + M_n}$. By Theorem 4.19, we can write $M = M_1 + \ldots + M_n$. It follows that M is p-essentially decomposable, and hence by Theorem 3.3 that M is p-essentially normal. ☐

5.2. **Essential normality.** In this section, we establish our main results on the essential normality of homogeneous submodules of rH^2_d. We will require Guo and Wang's result, Theorem 2.2 from [GW08], about the essential normality of singly generated homogeneous submodules.

Theorem 5.3 (Guo-Wang)**.** *Every submodule of rH^2_d generated by a single homogeneous polynomial is p-essentially normal for every $p > d$.*

The next result is well known. It was proved, for example, by Shalit in [Sha11], using his results on stable division. The methods introduced here provide a new proof.

Theorem 5.4. *Every submodule of H^2_d generated by linear polynomials is p-essentially normal for every $p > d$.*

Proof. Let N be a submodule of H^2_d generated by linear polynomials $p_1, \ldots, p_n$ in $\mathbb{C}[z]$. By applying the Gram-Schmidt process if necessary, we can assume that the set $\{p_1, \ldots, p_n\}$ is orthogonal in H^2_d. Let $N_1, \ldots, N_n$ denote the H^2_d submodules generated by $p_1, \ldots, p_n$ respectively. Then Theorem 5.3 implies that these submodules are each p-essentially normal for every $p > d$, and Proposition 4.12 implies that the family $\{N_1, \ldots, N_n\}$ is perpendicular. Note that $N = \overline{N_1 + \ldots + N_n}$. Hence by Theorem 5.2, N is also p-essentially normal for every $p > d$. □

The next result is new. It establishes the essential normality of submodules of rH^2_d that are generated by certain linear polynomials. We note Arveson's result from [Arv07] that the problem of the essential normality of homogeneous submodules of rH^2_d is equivalent to the problem of the essential normality of submodules of rH^2_d generated by arbitrary linear polynomials.

Theorem 5.5. *Let $F_1, \ldots, F_n$ be mutually orthogonal sets of linear polynomials, and let $X_1, \ldots, X_n$ be arbitrary sets of vectors in $\mathbb{C}^r$. Then the rH^2_d submodule generated by the set of vector-valued polynomials*

$$\{p \otimes \xi \mid p \in F_i,\ \xi \in X_i,\ 1 \leq i \leq n\} \tag{5.1}$$

is p-essentially normal for every $p > d$.

Proof. Let $N_1, \ldots, N_n$ denote the H^2_d submodules generated by $F_1, \ldots, F_n$ respectively, and let $V_1, \ldots, V_n$ denote the $\mathbb{C}^r$ subspaces spanned by $X_1, \ldots, X_n$ respectively. Let M denote the rH^2_d submodule generated by the set (5.1), and let $M_1, \ldots, M_n$ denote the rH^2_d submodules

$$M_i = N_i \otimes V_i, \quad 1 \leq i \leq n.$$

Then Theorem 5.4 implies that each of the submodules $N_1, \ldots, N_n$ is p-essentially normal for every $p > d$, and Proposition 4.12 implies that the family $\{N_1, \ldots, N_n\}$ is perpendicular. Note that $M = \overline{M_1 + \ldots + M_n}$. Hence by Theorem 5.2, M is also p-essentially normal for every $p > d$. □

The next theorem is Arveson's main result from [Arv05]. Shalit also gave a proof of this result in [Sha11] using his results on stable division. The methods introduced here provide a new and simple proof. Recall that a monomial of rH^2_d is an element in $r\mathbb{C}[z]$ of the form $z^\alpha \otimes \xi$ for some α in $\mathbb{N}^d_0$ and ξ in $\mathbb{C}^r$.

Theorem 5.6 (Arveson)**.** *Every submodule of rH^2_d generated by monomials is p-essentially normal for every $p > d$.*

Proof. Let N be a submodule of rH^2_d generated by monomials, say $z^{\alpha_1} \otimes \xi_1, \ldots, z^{\alpha_n} \otimes \xi_n$ in $r\mathbb{C}[z]$ for $\alpha_1, \ldots, \alpha_n$ in $\mathbb{N}^d_0$. Let $N_1, \ldots, N_n$ denote the rH^2_d submodules generated by $z^{\alpha_1}, \ldots, z^{\alpha_n}$ respectively, and let $V_1, \ldots, V_n$ denote the one-dimensional subspaces of $\mathbb{C}^r$ spanned by $\xi_1, \ldots, \xi_n$ respectively. Then Theorem 5.3 implies that the submodules $N_1, \ldots, N_n$ are p-essentially normal for every $p > d$, and Proposition 4.13 implies that the family $\{N_1, \ldots, N_n\}$ is perpendicular. Note that $N = \overline{N_1 + \ldots + N_n}$. Hence by Theorem 5.2, N is also p-essentially normal for every $p > d$. □

Recall Shalit's result from [Sha11] that a submodule generated by polynomials in two variables has the stable division property. Shalit used this result to prove the next theorem that these submodules are essentially normal. However, starting from Proposition 3.10, we can also view Shalit's proof of the stable division property for these submodules as a method for establishing essential decomposability. In this case, the methods introduced here provide a new proof.

Theorem 5.7 (Shalit). *Let F be a set of homogeneous polynomials. Suppose that there is a subset Z of $\{z_1, \ldots, z_d\}$, of size at most 2, such that $F \subseteq \mathbb{C}[Z]$. Then the H^2_d submodule generated by F is p-essentially normal for every $p > d$.*

Proof. Let N denote the H^2_d submodule generated by F, and let $\{p_1, \ldots, p_n\}$ be a Groebner basis consisting of homogeneous polynomials that generates N. Let $N_1, \ldots, N_n$ denote the H^2_d submodules generated by $p_1, \ldots, p_n$ respectively. Then by Theorem 5.3, $N_1, \ldots, N_n$ are p-essentially normal for every $p > d$. Note that the polynomials $p_1, \ldots, p_n$ belong to $\mathbb{C}[Z]$. Hence by Proposition 3.10, $N = N_1 + \ldots + N_n$, and N is p-essentially decomposable for every $p > d$. It follows from Theorem 3.3 that M is p-essentially normal for every $p > d$. ☐

The next result is new. It implies the essential normality of a large new class of submodules of rH^2_d.

Theorem 5.8. *Let $F_1, \ldots, F_n$ be sets of homogeneous polynomials that each generate p-essentially normal submodules of H^2_d for $p > d$. Suppose that there are disjoint subsets $Z_1, \ldots, Z_n$ of $\{z_1, \ldots, z_d\}$ such that*

$$F_i \subseteq \mathbb{C}[Z_i], \quad 1 \leq i \leq n.$$

Let $X_1, \ldots, X_n$ be arbitrary sets of vectors in $\mathbb{C}^r$. Then the rH^2_d submodule generated by the set of vector-valued polynomials

$$\{p \otimes \xi \mid p \in F_i,\ \xi \in X_i,\ 1 \leq i \leq n\} \tag{5.2}$$

is p-essentially normal.

Proof. Let $N_1, \ldots, N_n$ denote the H^2_d submodules generated by $F_1, \ldots, F_n$ respectively, and let $V_1, \ldots, V_n$ denote the $\mathbb{C}^r$ subspaces spanned by $X_1, \ldots, X_n$ respectively. Let M denote the rH^2_d submodule generated by the set (5.2), and let $M_1, \ldots, M_n$ denote the rH^2_d submodules

$$M_i = N_i \otimes V_i, \quad 1 \leq i \leq n.$$

The submodules $N_1, \ldots, N_n$ are p-essentially normal by assumption, and Proposition 4.14 implies that the family $\{N_1, \ldots, N_n\}$ is perpendicular. Note that $M = \overline{M_1 + \ldots + M_n}$. Hence by Theorem 5.2, M is also p-essentially normal. ☐

Replacing the use of Proposition 4.14 in the proof of Theorem 5.8 with Proposition 4.17 and Proposition 4.18 respectively, we immediately obtain the following strengthened results.

Theorem 5.9. *Let $F_1, \ldots, F_n$ be sets of homogeneous polynomials that each generate p-essentially normal submodules of H^2_d for $p > d$. Suppose that the sets*

$$\left\{\partial^\alpha(p) \mid |\alpha| = \deg(p) - 1,\ \alpha \in \mathbb{N}_0^d,\ p \in F_i\right\}, \quad 1 \leq i \leq n,$$

are mutually orthogonal. Let $X_1, \ldots, X_n$ be arbitrary sets of vectors in $\mathbb{C}^r$. Then the rH^2_d submodule generated by the set of vector-valued polynomials

$$\{p \otimes \xi \mid p \in F_i,\ \xi \in X_i,\ 1 \leq i \leq n\}$$

is p-essentially normal.

Theorem 5.10. *Let $F_1, \ldots, F_n$ be sets of homogeneous polynomials that each generate p-essentially normal submodules of H^2_d for $p > d$. Suppose that the sets*

$$\left\{ (\nabla p)(\lambda) \mid \lambda \in \mathbb{C}^d,\ p \in F_i \right\}, \quad 1 \leq i \leq n,$$

are mutually orthogonal. Let $X_1, \ldots, X_n$ be arbitrary sets of vectors in $\mathbb{C}^r$. Then the rH^2_d submodule generated by the set of vector-valued polynomials

$$\{ p \otimes \xi \mid p \in F_i,\ \xi \in X_i,\ 1 \leq i \leq n \}$$

is p-essentially normal.

The next theorem follows from a combination of the results above.

Theorem 5.11. *Let $F_1, \ldots, F_n$ be sets of homogeneous polynomials in $\mathbb{C}[z]$. Suppose that there are disjoint subsets $Z_1, \ldots, Z_n$ of $\{z_1, \ldots, z_d\}$, each of size at most 2, such that*

$$F_i \subseteq \mathbb{C}[Z_i], \quad 1 \leq i \leq n.$$

Let $X_1, \ldots, X_n$ be arbitrary sets of vectors in $\mathbb{C}^r$. Then the $H^2_d \otimes \mathbb{C}^r$ submodule generated by the set of vector-valued polynomials.

$$\{ p \otimes \xi \mid p \in F_i,\ \xi \in X_i,\ 1 \leq i \leq n \}$$

is p-essentially normal for every $p > d$.

Proof. This follows immediately from Theorem 5.7 and Theorem 5.8. ☐

Example 5.12. For every even $d \geq 1$ and $n \geq 1$, let N denote the H^2_d submodule generated by the set of polynomials $p_1, \ldots, p_{d/2}$, where

$$\begin{aligned} p_1(z_1, \ldots, z_d) &= z_1^n + z_1^{n-1} z_2 + \ldots + z_1 z_2^{n-1} + z_2^n \\ &\vdots \\ p_{d/2}(z_1, \ldots, z_d) &= z_{d-1}^n + z_{d-1}^{n-1} z_d + \ldots + z_{d-1} z_d^{n-1} + z_d^n \\ &. \end{aligned}$$

Then Theorem 5.11 implies that N is p-essentially normal for every $p > d$.

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