

ON THE CLASSIFICATION OF FIBRATIONS

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ABSTRACT. We identify the homotopy type of the moduli of maps with a given homotopy type of the base and the homotopy fiber. A new model for the space of weak equivalences and its classifying space is given.

1. INTRODUCTION

Classification questions are often about understanding components of a category. It is not unusual however that with a category one can associate a unique homotopy type of a simplicial set whose set of components coincides with the set of components of the category. Such a space carries more information about the category than just the set of its components. For example:

Definition 1.1. Let X and F be spaces. Let $\text{Fib}(X, F)$ be the category whose objects are maps $f: A \rightarrow B$, where B is weakly equivalent to X and the homotopy fiber of f , over any base point in B , is weakly equivalent to F . The set of morphisms in $\text{Fib}(X, F)$ between $f: A \rightarrow B$ and $f': A' \rightarrow B'$ consists of pairs of weak equivalences $\phi: A \rightarrow A'$ and $\psi: B \rightarrow B'$ for which $f'\phi = \psi f$. The composition of morphisms is induced by the usual composition of maps.

A classical result states that the components of $\text{Fib}(X, F)$ are in bijection with homotopy classes of maps $[X, Bwe(F, F)]$, where $Bwe(F, F)$ is the classifying space of the topological monoid of weak equivalences of F . This is a classical theorem proved by Stasheff in [19] and re-proved and generalized by May in [15].

Instead of looking at the set of components, it is more desirable to study the entire moduli space of fibrations. One would like to understand the homotopy type of the category $\text{Fib}(X, F)$ and not just the set of its components. Naively one can try to form the nerve of $\text{Fib}(X, F)$ and then identify its homotopy type. However, since $\text{Fib}(X, F)$ is not equivalent to a small category, this cannot be done so directly. Instead we are going to show that $\text{Fib}(X, F)$ has what we call a **core** (see Definition 5.2), which is a small category whose nerve approximates the homotopy type of $\text{Fib}(X, F)$. Our classification statement can then be formulated as follows:

Theorem A. *The category $\text{Fib}(X, F)$ has a core whose nerve admits a map to $Bwe(X, X)$ whose homotopy fiber is weakly equivalent to the mapping space $\text{map}(X, Bwe(F, F))$. Furthermore this map has a section.*

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It turns out that the above theorem is a particular case of a much more general statement that holds in an arbitrary model category. The purpose of such a generalization is more than showing that analogous classification statements hold in a much broader context. Statements that hold in an arbitrary model category often have more conceptual proofs in which one does not need to use the nature of objects considered but rather basic fundamental facts from homotopy theory. In this way arguments are becoming more transparent. It was Dwyer and Kan who first realized and proved that such general classification statements are true. In their sequence of papers including [5–9] they develop a strategy and techniques for dealing with classification questions. An important part of their program was the investigation of the notion of continuity in model categories. They showed that model categories have mapping spaces whose homotopy type is unique. They also gave a particular model for them using so-called hammocks.

In this paper we follow, in principle, the plan of Dwyer and Kan. Our realization of their strategy is different, however. For example, homotopical smallness is an essential ingredient in our work. Another important difference is our use of a model for mapping spaces developed in [4]. The general statement is about the homotopy type of the category of weak equivalences \mathcal{M}_{we} of a model category \mathcal{M} . Its objects are the objects of \mathcal{M} and morphisms are all the weak equivalences in \mathcal{M} . To understand its homotopy type, we study the components of \mathcal{M}_{we} . For an object X in \mathcal{M} , we denote by X_{we} the full subcategory of \mathcal{M}_{we} that consists of all the objects in \mathcal{M} which are weakly equivalent to X . This subcategory is also called a **component** of \mathcal{M}_{we} . Our key result states (see Theorem 17.1):

Theorem B. *Let I be a small category and X be an object in a model category \mathcal{M} . The category of functors $\text{Fun}(I, X_{\text{we}})$ has a core which is weakly equivalent to the mapping space $\text{map}(N(I), \text{Bwe}(X, X))$.*

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2. CATEGORICAL CONSTRUCTIONS AND NOTATION

To describe sets we use Zermelo-Fraenkel set theory with the axiom of choice.

2.1. The term category is used as defined in [14, Section 7]. The category \mathcal{A}^{op} is the opposite category of \mathcal{A} . A natural transformation between functors $f, g: \mathcal{B} \rightarrow \mathcal{A}$ is denoted by $\phi: g \rightarrow f$. It consists of morphisms $\phi_b: g(b) \rightarrow f(b)$ in \mathcal{A} for any object b in \mathcal{B} such that $f(\beta)\phi_{b_1} = \phi_{b_0}g(\beta)$ for any morphism $\beta: b_1 \rightarrow b_0$ in \mathcal{B} .

The category of sets is denoted by Sets . A category is small if it has a set of objects. The symbol Cat denotes the category of small categories and Δ its full subcategory whose objects are posets $[n] := \{0 < \dots < n\}$ for $n \geq 0$.

The symbol $B \subset A$ denotes the fact that B is a subset or a subcategory or a subspace, depending on whether A is a set or a category or a space.

2.2. The symbol \mathcal{M} always denotes a model category which, in addition to the standard axioms **MC1-MC5** (see e.g. [3, 10, 12, 17]), we require to be closed under arbitrary colimits and limits, to have a functorial fibrant replacement, and that any commutative square on the left below can be extended **functorially** to a commutative diagram on the right with the indicated morphisms being cofibrations and

acyclic fibrations:

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 \alpha_1 \downarrow & & \downarrow \alpha_2 \\
 X' & \xrightarrow{f'} & Y'
 \end{array}
 \qquad
 \begin{array}{ccc}
 X & \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{P(f)} \simeq \end{array} & Y \\
 \alpha_1 \downarrow & \begin{array}{c} P(\alpha_1, \alpha_2) \\ \downarrow \end{array} & \downarrow \alpha_2 \\
 X' & \begin{array}{c} \xrightarrow{f'} \\ \xrightarrow{P(f')} \simeq \end{array} & Y'
 \end{array}$$

The restriction of P to the full subcategory of arrows in \mathcal{M} of the form $\emptyset \rightarrow X$ is a functorial cofibrant replacement in \mathcal{M} and will be depicted by $X \hookrightarrow P(X)$.

Simplicial sets are also called spaces, and their category with the standard model structure (see for example [11]) is denoted by Spaces. The full subcategory of Spaces whose objects are the standard simplices $\Delta[n]$ is isomorphic to Δ .

2.3. A system \mathcal{F} of categories indexed by a category \mathcal{C} consists of a category \mathcal{F}_c for any object c in \mathcal{C} and a functor $\mathcal{F}_\alpha: \mathcal{F}_{c_0} \rightarrow \mathcal{F}_{c_1}$ for any morphism $\alpha: c_1 \rightarrow c_0$ in \mathcal{C} (note contravariance). These functors are required to satisfy: $\mathcal{F}_{\text{id}_c} = \text{id}$ for any object c ; and $\mathcal{F}_{\alpha\alpha'} = \mathcal{F}_{\alpha'}\mathcal{F}_\alpha$ for any morphisms $\alpha': c_2 \rightarrow c_1$ and $\alpha: c_1 \rightarrow c_0$. Note that when all \mathcal{F}_c are small, the system of categories \mathcal{F} is simply a functor $\mathcal{C}^{\text{op}} \rightarrow \text{Cat}$.

A subsystem $\mathcal{G} \subset \mathcal{F}$ consists of a subcategory $\mathcal{G}_c \subset \mathcal{F}_c$ for any object c in \mathcal{C} such that, for any morphism $\alpha: c_1 \rightarrow c_0$, \mathcal{F}_α takes \mathcal{G}_{c_0} to \mathcal{G}_{c_1} .

2.4. Let \mathcal{F} be a system of categories indexed by \mathcal{C} . Its Grothendieck construction, denoted by $\text{Gr}_{\mathcal{C}}\mathcal{F}$, is the category whose objects are pairs (c, x) where c is an object in \mathcal{C} and x in \mathcal{F}_c . The set of morphisms between (c_1, x_1) and (c_0, x_0) is the set of pairs $(\alpha: c_1 \rightarrow c_0, \beta: x_1 \rightarrow \mathcal{F}_\alpha(x_0))$ where α is a morphism in \mathcal{C} and β is a morphism in \mathcal{F}_{c_1} . The composition of $(\alpha': c_2 \rightarrow c_1, \beta': x_2 \rightarrow \mathcal{F}_{\alpha'}(x_1))$ and $(\alpha: c_1 \rightarrow c_0, \beta: x_1 \rightarrow \mathcal{F}_\alpha(x_0))$ is defined to be the pair

$$(c_2 \xrightarrow{\alpha'} c_1 \xrightarrow{\alpha} c_0, x_2 \xrightarrow{\beta'} \mathcal{F}_{\alpha'}(x_1) \xrightarrow{\mathcal{F}_{\alpha'}(\beta)} \mathcal{F}_{\alpha'}(\mathcal{F}_\alpha(x_0)) = \mathcal{F}_{\alpha\alpha'}(x_0)).$$

The **projection** $\pi: \text{Gr}_{\mathcal{C}}\mathcal{F} \rightarrow \mathcal{C}$ is the functor that assigns to an object (c, x) (resp. morphism (α, β)) in $\text{Gr}_{\mathcal{C}}\mathcal{F}$ the object c (resp. morphism α) in \mathcal{C} . For any object c in \mathcal{C} the functor $\mathcal{F}_c \rightarrow \text{Gr}_{\mathcal{C}}\mathcal{F}$ which assigns to an object x the pair (c, x) and to a morphism $\beta: x \rightarrow y$ the pair (id_c, β) is called the **standard inclusion**.

2.5. Let $f: \mathcal{B} \rightarrow \mathcal{A}$ be a functor and a be an object in \mathcal{A} . The objects of the **under category** $a \uparrow f$ are pairs (b, α) where b is an object in \mathcal{B} and $\alpha: a \rightarrow f(b)$ is a morphism in \mathcal{A} . The set of morphisms between (b_1, α_1) and (b_0, α_0) in $a \uparrow f$ is the set of morphisms $\beta: b_1 \rightarrow b_0$ in \mathcal{B} for which $f(\beta)\alpha_1 = \alpha_0$. The category $a \uparrow \text{id}_{\mathcal{A}}$ is also denoted by $a \uparrow \mathcal{A}$. By forgetting the second component we obtain a functor $(a \uparrow f) \rightarrow \mathcal{B}$ called **forgetful**.

The objects of the **over category** $f \downarrow a$ are pairs (b, α) where b is an object in \mathcal{B} and $\alpha: f(b) \rightarrow a$ is a morphism in \mathcal{A} . The set of morphisms between (b_1, α_1) and (b_0, α_0) is the set of morphisms $\beta: b_1 \rightarrow b_0$ in \mathcal{B} for which $\alpha_0 f(\beta) = \alpha_1$.

Consider the following commutative diagram of functors:

$$\begin{array}{ccc}
 \mathcal{D} & \xrightarrow{g} & \mathcal{C} \\
 e \downarrow & & \downarrow h \\
 \mathcal{B} & \xrightarrow{f} & \mathcal{A}
 \end{array}$$

Let c be an object in \mathcal{C} . The symbol $(e, h): c \uparrow g \rightarrow h(c) \uparrow f$ denotes the functor that maps an object (d, α) to $(e(d), h(\alpha))$ and a morphism $\alpha: d_1 \rightarrow d_0$ to $e(\alpha)$.

Let $\gamma: a_1 \rightarrow a_0$ be a morphism in \mathcal{A} . The functor $\gamma \uparrow f: a_0 \uparrow f \rightarrow a_1 \uparrow f$ assigns to (b, α) the object $(b, \alpha\gamma)$ and to a morphism β the same β . The assignment $a \mapsto (a \uparrow f)$ and $\lambda \mapsto (\gamma \uparrow f)$ is a system of categories indexed by \mathcal{A} denoted by $-\uparrow f$. Its Grothendieck construction $\text{Gr}_{\mathcal{A}}(-\uparrow f)$ is isomorphic to a category whose objects are pairs $(b, \alpha: a \rightarrow f(b))$ of an object b in \mathcal{B} and a morphism α in \mathcal{A} . The set of morphisms between $(b_1, \alpha_1: a_1 \rightarrow f(b_1))$ and $(b_0, \alpha_0: a_0 \rightarrow f(b_0))$ consists of pairs $(\beta: b_1 \rightarrow b_0, \gamma: a_1 \rightarrow a_0)$ where γ is a morphism in \mathcal{A} and β is a morphism in \mathcal{B} making the following square commutative:

$$\begin{array}{ccc} a_1 & \xrightarrow{\gamma} & a_0 \\ \alpha_1 \downarrow & & \downarrow \alpha_0 \\ f(b_1) & \xrightarrow{f(\beta)} & f(b_0) \end{array}$$

The functor $\hat{\pi}: \text{Gr}_{\mathcal{A}}(-\uparrow f) \rightarrow \mathcal{B}$ assigns to an object (b, α) in $\text{Gr}_{\mathcal{A}}(-\uparrow f)$ the object b in \mathcal{B} and to a morphism (β, γ) in $\text{Gr}_{\mathcal{A}}(-\uparrow f)$ the morphism β in \mathcal{B} . The functor $\hat{f}: \mathcal{B} \rightarrow \text{Gr}_{\mathcal{A}}(-\uparrow f)$ assigns to an object b in \mathcal{B} the object $(b, \text{id}_{f(b)})$ in $\text{Gr}_{\mathcal{A}}(-\uparrow f)$ and to a morphism β in \mathcal{B} the morphism $(\beta, f(\beta))$ in $\text{Gr}_{\mathcal{A}}(-\uparrow f)$. These functors fit into the following commutative diagram:

$$\begin{array}{ccc} & \text{id} & \\ & \curvearrowright & \\ \mathcal{B} & \xrightarrow{\hat{f}} & \text{Gr}_{\mathcal{A}}(-\uparrow f) & \xrightarrow{\hat{\pi}} & \mathcal{B} \\ & \searrow f & \downarrow \pi & & \\ & & \mathcal{A} & & \end{array}$$

Note that there is a natural transformation between the functors $\hat{f}\hat{\pi}$ and $\text{id}_{\text{Gr}_{\mathcal{A}}(-\uparrow f)}$ which for an object $(b, \alpha: a \rightarrow f(b))$ is given by the morphism (id_b, α) .

2.6. $\text{Fun}(I, \mathcal{C})$ is the category of functors indexed by a **small** category I with values in a category \mathcal{C} and natural transformations as morphisms. The set of natural transformations between $F: I \rightarrow \mathcal{C}$ and $G: I \rightarrow \mathcal{C}$ is denoted by $\text{Nat}(F, G)$.

Let $f: I \rightarrow J$ be a functor of small categories. The composition with f functor is denoted by $f^*: \text{Fun}(J, \mathcal{C}) \rightarrow \text{Fun}(I, \mathcal{C})$. It assigns to a natural transformation $\{\psi_j\}_{j \in J}$ the natural transformation $\{\psi_{f(i)}\}_{i \in I}$.

A natural transformation $\phi: F \rightarrow G$ in $\text{Fun}(I, \mathcal{M})$ is called a weak equivalence if $\phi_i: F(i) \rightarrow G(i)$ is a weak equivalence for any i in I . $\text{Ho}(\text{Fun}(I, \mathcal{M}))$ denotes the localization of $\text{Fun}(I, \mathcal{M})$ with respect to weak equivalences which exists by [3, Theorem 11.3].

2.7. A connected component of a category \mathcal{C} containing an object c is the class of all objects d for which there is a finite sequence of morphisms $d = x_0 \rightarrow x_1 \leftarrow \dots \leftarrow x_n = c$ in \mathcal{C} . The symbol $\pi_0(\mathcal{C})$ denotes the discrete category (identities are the only morphisms) whose objects are connected components of \mathcal{C} and $\pi_0: \mathcal{C} \rightarrow \pi_0(\mathcal{C})$ denotes the unique functor mapping an object to its component.

Part I. Categories and homotopy. In this part we review two ways of doing homotopy theory on categories. In one the homotopy relation is induced by natural transformations. This notion however is too strong for us. We need weak equivalences. A standard way of introducing them is to transport weak notions

from simplicial sets using the nerve. This works for small categories. Our aim is to extend the weak notions to categories such as $\text{Fib}(X, F)$ that can be approximated by small categories. We call such categories essentially small. The nerve of such a category is what Dwyer and Kan called a homotopically small simplicial class (as opposed to a simplicial set; see [6]). In this part we develop basic properties of essentially small categories.

3. CATEGORICAL HOMOTOPY

Here is a standard dictionary of homotopy notions on arbitrary categories:

- Functors $f, g: \mathcal{B} \rightarrow \mathcal{A}$ are **homotopic** if there are functors $\{h_k: \mathcal{B} \rightarrow \mathcal{A}\}_{0 \leq k \leq n}$ and natural transformations $f = h_0 \rightarrow h_1 \leftarrow \cdots \leftarrow h_n = g$.
- $f: \mathcal{B} \rightarrow \mathcal{A}$ is a **homotopy equivalence** if it has a homotopy inverse, i.e., a functor $g: \mathcal{A} \rightarrow \mathcal{B}$ for which fg and gf are homotopic to $\text{id}_{\mathcal{A}}$ and $\text{id}_{\mathcal{B}}$.
- $f: \mathcal{B} \rightarrow \mathcal{A}$ is a **strong fibration** if, for any morphism $\gamma: a_1 \rightarrow a_0$ in \mathcal{A} , $\gamma \uparrow f: a_0 \uparrow f \rightarrow a_1 \uparrow f$ (see Section 2.5) is a homotopy equivalence.
- A **strong homotopy pull-back** is a commutative square of functors:

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{g} & \mathcal{C} \\ e \downarrow & & \downarrow h \\ \mathcal{B} & \xrightarrow{f} & \mathcal{A} \end{array}$$

where f is a strong fibration and $(e, h): c \uparrow g \rightarrow h(c) \uparrow f$ (see Section 2.5) is a homotopy equivalence for any object c in \mathcal{C} .

If $f: \mathcal{A} \rightarrow \mathcal{A}$ is either left or right adjoint, then f is a homotopy equivalence whose homotopy inverse is given by the adjoint. Another example of a homotopy equivalence is given by $\hat{f}: \mathcal{B} \rightarrow \text{Gr}_{\mathcal{A}}(- \uparrow f)$ (see Section 2.5).

Note that in the definition of a strong fibration, the under categories $a \uparrow f$ are allowed to be empty, in which case, for any b connected to a by a zig-zag of morphisms, $b \uparrow f$ would also be empty.

Note also the lack of symmetry in our definition of a strong homotopy pull-back. This never becomes an issue as we orient all the considered squares in the right way.

4. SMALL CATEGORIES

The nerve is used to translate weak homotopy notions from Spaces into Cat. It is a functor $N: \text{Cat} \rightarrow \text{Spaces}$ that assigns to a small category I a simplicial set $N(I)$ whose set of n -dimensional simplices is, for $n > 0$, the set of n -composable morphisms in I and, if $n = 0$, is the set of objects in I .

Here is a basic dictionary (compare with notions recalled in Section 3):

- A functor $f: J \rightarrow I$ of small categories is called a **weak equivalence** if $N(f): N(J) \rightarrow N(I)$ is a weak equivalence of spaces.
- A functor $f: J \rightarrow I$ of small categories is called a **quasifibration** if $\alpha \uparrow f: i_0 \uparrow f \rightarrow i_1 \uparrow f$ is a weak equivalence for any $\alpha: i_1 \rightarrow i_0$ in I .
- A commutative square of small categories is called a **homotopy pull-back** if after applying the nerve we obtain a homotopy pull-back of spaces.

It is well known that if functors of small categories are homotopic as functors, then their nerves are homotopic as maps. Consequently, a homotopy equivalence,

resp. a strong fibration, of small categories is a weak equivalence, resp. a quasifibration. To prove that for small categories strong homotopy pull-backs are homotopy pull-backs, we need Thomason’s and Puppe’s theorems:

Proposition 4.1. *Let I be a small category and $F, G: I^{\text{op}} \rightarrow \text{Cat}$ be functors.*

- (1) $N(\text{Gr}_I F)$ is weakly equivalent to $\text{hocolim}_{I^{\text{op}}} N(F)$.
- (2) Let $f: F \rightarrow G$ be a natural transformation. Assume f_i is a weak equivalence for any object i in I . Then $\text{Gr}_I f$ is a weak equivalence.
- (3) Assume $F(\alpha): F(i_0) \rightarrow F(i_1)$ is a weak equivalence for any $\alpha: i_1 \rightarrow i_0$. Then, for any object i in I , the following is a homotopy pull-back square:

$$\begin{array}{ccc} F_i & \longrightarrow & \text{Gr}_I F \\ \downarrow & & \downarrow \pi \\ [0] & \xrightarrow{i} & I \end{array}$$

where π is the projection, $F(i) \rightarrow \text{Gr}_I F$ is the standard inclusion (see Section 2.4), and $i: [0] \rightarrow I$ is the functor that sends the object 0 to i .

- (4) Let $f: J \rightarrow I$ be a quasifibration of small categories. Then, for any object i in I , the following is a homotopy pull-back square:

$$\begin{array}{ccc} i \uparrow f & \xrightarrow{\text{forget}} & J \\ \downarrow & & \downarrow f \\ i \uparrow I & \xrightarrow{\text{forget}} & I \end{array}$$

Proof. Statement (1) is Thomason’s theorem [20]; see also [3]. Statement (2) is a consequence of (1). Statement (3) follows from (1) and Puppe’s theorem [16]; see also [2]. Statement (4) is Quillen’s Theorem A and follows from (3). □

Here is a method to verify that a square of small categories is a homotopy pull-back. It should be compared with [13] where the term homotopy pull-back is used to describe squares that are not homotopy pull-backs as defined in this paper.

Proposition 4.2. *Let the following be a commutative diagram of small categories:*

$$\begin{array}{ccc} L & \xrightarrow{g} & K \\ e \downarrow & & \downarrow h \\ J & \xrightarrow{f} & I \end{array}$$

Assume $f: J \rightarrow I$ is a quasifibration and $(e, h): k \uparrow g \rightarrow h(k) \uparrow f$ is a weak equivalence for any object k in K . Then the above square is a homotopy pull-back.

Proof. The assumptions imply that g is also a quasi fibration. Thus by Proposition 4.1(4) the induced map on the homotopy fibers of $N(g)$ and $N(f)$ is a weak equivalence. □

5. ESSENTIALLY SMALL CATEGORIES

The aim of this section is to explain how certain “large” categories can be approximated by small categories. It is based on the following well-known fact:

Lemma 5.1. *If $I_0 \subset I_1 \subset \dots$ is a sequence of small categories where each inclusion is a weak equivalence, then $I_0 \subset \text{colim } I_n = \bigcup_{n \geq 0} I_n$ is also a weak equivalence.*

Here is our key definition:

Definition 5.2. A **core** of a category \mathcal{C} is a small subcategory $I \subset \mathcal{C}$ such that, for any small subcategory $J \subset \mathcal{C}$ with $I \subset J$, there is a small subcategory $K \subset \mathcal{C}$ for which $J \subset K$ and the inclusion $I \subset K$ is a weak equivalence. A category is said to be **essentially small** if it has a core.

For example, if \mathcal{C} has a small skeleton, then this skeleton is its core.

Proposition 5.3. *Let \mathcal{C} be a category.*

- (1) *If $I \subset \mathcal{C}$ and $J \subset \mathcal{C}$ are cores, then I and J are weakly equivalent.*
- (2) *A discrete essentially small category is small.*
- (3) *If \mathcal{C} is essentially small, then the components of \mathcal{C} form a set. If $I \subset \mathcal{C}$ is a core, then this inclusion induces a bijection between $\pi_0(I)$ and $\pi_0(\mathcal{C})$.*

Lemma 5.4. (1) *Let $I \subset \mathcal{C}$ be a core and $I' \subset \mathcal{C}$ a small subcategory containing I . Then $I' \subset \mathcal{C}$ is a core if and only if $I \subset I'$ is a weak equivalence.*

- (2) *Let $J \subset \mathcal{C}$ be a small subcategory and $I \subset \mathcal{C}$ be a core. Then there is a full subcategory $K \subset \mathcal{C}$ such that $J \subset K \supset I$ and $I \subset K$ is a weak equivalence.*

Proof. (1) Assume $I \subset I'$ is a weak equivalence. Let $J \subset \mathcal{C}$ be a small subcategory such that $I' \subset J$. Since $I \subset \mathcal{C}$ is a core, there is a small subcategory $K \subset \mathcal{C}$ for which $J \subset K$ and $I \subset K$ is a weak equivalence. By the “2 out of 3” property the inclusion $I' \subset K$ is also a weak equivalence. This shows that $I' \subset \mathcal{C}$ is a core.

Assume $I' \subset \mathcal{C}$ is a core. We define inductively a sequence of small subcategories $I_0 \subset I'_0 \subset I_1 \subset I'_1 \subset \dots \subset \mathcal{C}$. Set $I_0 = I$ and $I'_0 = I'$. Assume $n > 0$. Let $I_n \subset \mathcal{C}$ be a small subcategory containing I'_{n-1} for which $I_0 \subset I_n$ is a weak equivalence. It exists since I_0 is a core in \mathcal{C} . Similarly, let $I'_n \subset \mathcal{C}$ be a small subcategory containing I_n for which $I'_0 \subset I'_n$ is a weak equivalence. Note that $\bigcup_{n \geq 0} I_n = \bigcup_{n \geq 0} I'_n$. Moreover, according to Lemma 5.1, the inclusions $I_0 \subset \bigcup_{n \geq 0} I_n = \bigcup_{n \geq 0} I'_n \supset I'_0$ are weak equivalences. It follows that $I_0 \subset I'_0$ is a weak equivalence as well.

(2) Define a sequence of small subcategories $I_0 \subset K_0 \subset I_1 \subset K_1 \subset \dots \subset \mathcal{C}$. Set $I_0 = I$ and K_0 to be the full subcategory of \mathcal{C} on objects in I_0 and J . Let $n > 0$. Define $I_n \subset \mathcal{C}$ to be a small subcategory such that $K_{n-1} \subset I_n$ and $I_0 \subset I_n$ is a weak equivalence. Define K_n to be the full subcategory of \mathcal{C} on the set of objects in I_n . Set $K := \bigcup_{n \geq 0} K_n$. Note K is full in \mathcal{C} since the K_n 's are. As $K = \bigcup_{n \geq 0} I_n$ and $I = I_0 \subset I_n$ is a weak equivalence, $I = I_0 \subset K$ is also a weak equivalence. \square

Proof of Proposition 5.3. (1) According to Lemma 5.4(2) there is a small subcategory $K \subset \mathcal{C}$ such that $I \subset K \supset J$ and $I \subset K$ is a weak equivalence. Since $I \subset \mathcal{C}$ is a core we can use Lemma 5.4(1) to conclude that $K \subset \mathcal{C}$ is also a core. By assumption $J \subset \mathcal{C}$ is a core. The inclusion $J \subset K$ is thus a weak equivalence.

- (2) Just note that weak equivalences of discrete categories are isomorphisms.
- (3) This follows from the fact that $\pi_0 : \mathcal{C} \rightarrow \pi_0(\mathcal{C})$ maps a core to a core. \square

By Proposition 5.3(1) the homotopy type of a core is a well-defined invariant.

We can thus use it to introduce various homotopy notions on essentially small categories. For example, we can define homotopy groups of an essentially small category as the homotopy groups of the nerve of its core. Usefulness of such invariants

depends on their functorial properties. For that we need to extend Definition 5.2 to:

Definition 5.5. Let \mathcal{F} be a system of categories indexed by a small category I (see Section 2.3). A **core** of \mathcal{F} is a subsystem $F \subset \mathcal{F}$ such that $F_i \subset \mathcal{F}_i$ is a core for any object i in I . A system \mathcal{F} is called **essentially small** if it has a core.

Proposition 5.6. Let \mathcal{F} be a system of categories indexed by a small category I .

- (1) \mathcal{F} is essentially small if and only if, for any i , \mathcal{F}_i is essentially small.
- (2) Assume that $F \subset \mathcal{F} \supset F'$ are cores. Then there is a core $H \subset \mathcal{F}$ such that $F \subset H \supset F'$ and $H_i \subset \mathcal{F}_i$ is a full subcategory for any object i in I .
- (3) If $F \subset \mathcal{F}$ is a core, then $\text{Gr}_I F \subset \text{Gr}_I \mathcal{F}$ is a core.

Lemma 5.7. Assume \mathcal{F} is a system of essentially small categories. If $G_i \subset \mathcal{F}_i$ are small subcategories, then there is a core $H \subset \mathcal{F}$ such that $G_i \subset H_i$ and $H_i \subset \mathcal{F}_i$ is a full subcategory for any i .

Proof. We construct inductively a sequence of small subcategories, for any object i in I , $(G_i)_0 \subset (H_i)_1 \subset (G_i)_1 \subset (H_i)_2 \subset (G_i)_2 \subset \dots \subset \mathcal{F}_i$. Set $(G_i)_0 := G_i$. Assume $n > 0$. Let $(H_i)_n \subset \mathcal{F}_i$ be a core such that $(G_i)_{n-1} \subset (H_i)_n$. It exists by Lemma 5.4(2). Define $(G_i)_n$ to be the full subcategory of \mathcal{F}_i on the set of objects $\bigcup_{\alpha: i \rightarrow j} \mathcal{F}_\alpha((H_j)_n)$ where the index $\alpha: i \rightarrow j$ runs over all possible morphisms in I with domain i . The purpose of this definition is to ensure that, for any $n > 0$,

- a. $(H_i)_n \subset \mathcal{F}_i$ is a core for any object i in I ;
- b. $(G_i)_n \subset \mathcal{F}_i$ is a full subcategory for any object i in I ;
- c. $\mathcal{F}_\alpha: \mathcal{F}_j \rightarrow \mathcal{F}_i$ takes $(H_j)_n$ to $(G_i)_n$ for any morphism $\alpha: i \rightarrow j$ in I .

Define $H_i := \bigcup_{n>0} (H_i)_n$. The above properties and Lemma 5.4(1) imply:

- a. $H_i \subset \mathcal{F}_i$ is a core.
- b. $H_i \subset \mathcal{F}_i$ is a full subcategory.
- c. $\mathcal{F}_\alpha: \mathcal{F}_j \rightarrow \mathcal{F}_i$ takes H_j to H_i for any morphism $\alpha: i \rightarrow j$ in I . □

Proof of Proposition 5.6. (1) and (2) are direct consequences of Lemma 5.7. To prove (3) choose a small subcategory $J \subset \text{Gr}_I \mathcal{F}$ containing $\text{Gr}_I F$. For any object i in I , let $G_i \subset \mathcal{F}_i$ be the full subcategory on the set of all objects x in \mathcal{F}_i for which $(i, x) \in J$. According to Lemma 5.7 there is a core $H \subset \mathcal{F}$ such that $G_i \subset H_i$ for any i . Consider the inclusions $\text{Gr}_I F \subset J \subset \text{Gr}_I H \subset \text{Gr}_I \mathcal{F}$. Since $F_i \subset H_i$ is a weak equivalence (see Lemma 5.4), then so is $\text{Gr}_I F \subset \text{Gr}_I H$ (see Proposition 4.1(2)). □

Proposition 5.8. Let $f: \mathcal{B} \rightarrow \mathcal{A}$ and $r: \mathcal{A} \rightarrow \mathcal{B}$ be functors. Assume \mathcal{A} is essentially small and $rf: \mathcal{B} \rightarrow \mathcal{B}$ is homotopic to $\text{id}_{\mathcal{B}}$. Then \mathcal{B} is essentially small and there are cores $A \subset \mathcal{A}$ and $B \subset \mathcal{B}$ for which the following diagram commutes:

$$\begin{array}{ccccc} \mathcal{B} & \xrightarrow{f} & \mathcal{A} & \xrightarrow{r} & \mathcal{B} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{B} & \xrightarrow{f} & \mathcal{A} & \xrightarrow{r} & \mathcal{B} \end{array}$$

Proof. Choose a sequence of natural transformations $rf = h_0 \rightarrow \dots \leftarrow h_m = \text{id}_{\mathcal{B}}$. For small subcategories $I \subset \mathcal{A}$ and $K \subset \mathcal{B}$, define inductively a sequence of small subcategories of \mathcal{A} and \mathcal{B} that fit into the following commutative diagram:

$$\begin{array}{ccccccc} I := I_0 & \subset & I_1 & \subset & I_2 & \subset & \dots \subset \mathcal{A} \\ & & r \downarrow & & r \downarrow & & \downarrow r \\ K := K_0 & \subset & K_1 & \subset & K_2 & \subset & \dots \subset \mathcal{B} \end{array}$$

Let $n > 0$. Using Lemma 5.4(2), define $I_n \subset \mathcal{A}$ to be a core which is a full subcategory and contains the set of objects that either belong to I_{n-1} or are of the form $f(b)$ where b is in K_{n-1} . Set K_n to be the full subcategory of \mathcal{B} on the set of objects that either belong to K_{n-1} , are of the form $r(a)$ where a is in I_n , or are of the form $h_k(b)$ where b is in K_{n-1} . The purpose of this definition is to ensure:

- a. $f: \mathcal{B} \rightarrow \mathcal{A}$ takes K_{n-1} to I_n and $r: \mathcal{A} \rightarrow \mathcal{B}$ takes I_n to K_n ;
- b. $I_n \subset \mathcal{A}$ is a core for any $n > 0$;
- c. $h_k: \mathcal{B} \rightarrow \mathcal{B}$ takes K_{n-1} to K_n for any $0 \leq k \leq m$;
- d. $K_n \subset \mathcal{B}$ is a full subcategory.

Define $I_\infty := \bigcup_{n \geq 0} I_n$ and $K_\infty := \bigcup_{n \geq 0} K_n$. The above requirements imply:

- a. there is a commutative diagram

$$\begin{array}{ccccc} K_\infty & \xrightarrow{f} & I_\infty & \xrightarrow{r} & K_\infty \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{B} & \xrightarrow{f} & \mathcal{A} & \xrightarrow{r} & \mathcal{B} \end{array}$$
- b. $I_\infty \subset \mathcal{A}$ is a core;
- c. $h_k: \mathcal{B} \rightarrow \mathcal{B}$ takes K_∞ to K_∞ for any $0 \leq k \leq m$;
- d. $rf: K_\infty \rightarrow K_\infty$ is homotopic to the identity functor. The appropriate “zig-zag” is given by restricting the natural transformations between the h_k ’s to the full subcategory K_∞ .

We need to prove that $K_\infty \subset \mathcal{B}$ is a core. Let $J \subset \mathcal{B}$ be a small subcategory containing K_∞ . The above construction applied to $I_\infty \subset \mathcal{A}$ and $J \subset \mathcal{B}$ yields

$$\begin{array}{ccccc} K_\infty & \xrightarrow{f} & I_\infty & \xrightarrow{r} & K_\infty \\ \downarrow & & \downarrow & & \downarrow \\ J_\infty & \xrightarrow{f} & (I_\infty)_\infty & \xrightarrow{r} & J_\infty \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{B} & \xrightarrow{f} & \mathcal{A} & \xrightarrow{r} & \mathcal{B} \end{array}$$

Since $(I_\infty)_\infty \subset \mathcal{A}$ is a core, $I_\infty \subset (I_\infty)_\infty$ is a weak equivalence (see Lemma 5.4(1)). As $rf: J_\infty \rightarrow J_\infty$ and $rf: K_\infty \rightarrow K_\infty$ are homotopic to the appropriate identity functors, then $K_\infty \subset J_\infty$, as a homotopy retract of a weak equivalence, is a weak equivalence. □

Corollary 5.9. *Let $f: \mathcal{B} \rightarrow \mathcal{A}$ be a homotopy equivalence. Then \mathcal{A} is essentially small if and only if \mathcal{B} is essentially small.*

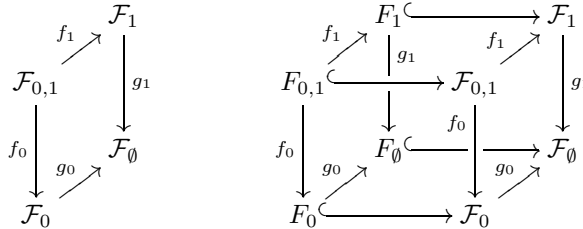
6. WEAK HOMOTOPY NOTIONS FOR ESSENTIALLY SMALL CATEGORIES

Our aim is to extend the dictionary from Section 4 to essentially small categories. A functor $f: \mathcal{F}_1 \rightarrow \mathcal{F}_0$ is a system of categories indexed by the poset [1]. Thus its core consists of cores $F_1 \subset \mathcal{F}_1$ and $F_0 \subset \mathcal{F}_0$ such that f takes F_1 to F_0 . The restricted functor $f: F_1 \rightarrow F_0$ fits into a commutative diagram:

$$\begin{array}{ccc} F_1 \hookrightarrow \mathcal{F}_1 & & \\ f \downarrow & & \downarrow f \\ F_0 \hookrightarrow \mathcal{F}_0 & & \end{array}$$

By the “2 out of 3” property of weak equivalences and Proposition 5.6(2), if $f: F_1 \rightarrow F_0$ and $f: F'_1 \rightarrow F'_0$ are cores of $f: \mathcal{F}_1 \rightarrow \mathcal{F}_0$, then $f: F_1 \rightarrow F_0$ is a weak equivalence if and only if $f: F'_1 \rightarrow F'_0$ is so.

Similarly, the commutative square on the left below is a system of categories indexed by the poset of all the subsets of $\{0, 1\}$. Its core consists of cores $F_\emptyset \subset \mathcal{F}_\emptyset$, $F_0 \subset \mathcal{F}_0$, $F_1 \subset \mathcal{F}_1$, and $F_{0,1} \subset \mathcal{F}_{0,1}$, making the right cube commutative:



Again, by the “2 out of 3” property and Proposition 5.6(2), if one core of a commutative square is a homotopy pull-back then so is any other. This justifies:

Definition 6.1.

- A functor is a **weak equivalence** if it has a core which is a weak equivalence.
- A functor $f: \mathcal{B} \rightarrow \mathcal{A}$ is a **quasifibration** if $\alpha \uparrow f: a_0 \uparrow f \rightarrow a_1 \uparrow f$ is a weak equivalence for any morphism $\alpha: a_1 \rightarrow a_0$ in \mathcal{A} .
- A commutative square of functors is a **homotopy pull-back** if it has a core which is a homotopy pull-back.

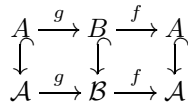
Note that a weak equivalence is only defined between essentially small categories. Therefore, one implicitly assumes in the definition of a quasifibration that the under categories $a \uparrow f$ are essentially small for all a .

Proposition 6.2. *Let $f: \mathcal{B} \rightarrow \mathcal{A}$ be a homotopy equivalence of essentially small categories. Then f is a weak equivalence.*

Proof. Let g be a homotopy inverse to f . Consider a system of categories indexed by the free category on the graph on the left given by the diagram on the right:

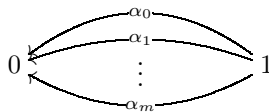


This system is essentially small (see Proposition 5.6(1)) and hence has a core. It consists of small subcategories $A \subset \mathcal{A}$ and $B \subset \mathcal{B}$ that fit into a commutative diagram:



To show that $f: A \rightarrow B$ is a weak equivalence, it is enough to prove that the compositions $fg: A \rightarrow A$ and $gf: B \rightarrow B$ are weak equivalences.

Choose a sequence of natural transformations $fg = h_0 \rightarrow \dots \leftarrow h_m = \text{id}_A$. The functors $\{h_k: \mathcal{A} \rightarrow \mathcal{A}\}_{0 \leq k \leq m}$ form a system of categories indexed by:



It has a core given by full subcategories $A_0 \subset \mathcal{A}$ and $A_1 \subset \mathcal{A}$ containing A (see Lemma 5.7). Take the restrictions $h_k: A_0 \rightarrow A_1$. Since A_0 and A_1 are cores of

\mathcal{A} , $h_m: A_0 \rightarrow A_1$ is a weak equivalence as it is the restriction of the identity. Use fullness of A_1 in \mathcal{A} to get a sequence of natural transformations $h_0 \rightarrow \cdots \leftarrow h_m$ between these restrictions. Thus $h_0: A_0 \rightarrow A_1$ is a weak equivalence too and hence, by the “2 out of 3” property, so is $fg: A \rightarrow A$. By symmetry gf is also a weak equivalence. \square

If \mathcal{F} is a system of essentially small categories indexed by a small category I , then according to Proposition 5.6(3), $\text{Gr}_I \mathcal{F}$ is essentially small. This is probably not true in general if we just assume that I is essentially small.

Lemma 6.3. *Let \mathcal{F} be a system of essentially small categories indexed by an essentially small category \mathcal{A} . If, for any morphism $\alpha: a_1 \rightarrow a_0$ in \mathcal{A} , the functor $\mathcal{F}_\alpha: \mathcal{F}_{a_0} \rightarrow \mathcal{F}_{a_1}$ is a weak equivalence, then $\text{Gr}_{\mathcal{A}} \mathcal{F}$ is essentially small.*

Proof. Choose a core $A \subset \mathcal{A}$. Let F be a core of the restriction of \mathcal{F} to A (see Proposition 5.6(1)). We claim $\text{Gr}_A F \subset \text{Gr}_{\mathcal{A}} \mathcal{F}$ is a core. Let $J \subset \text{Gr}_{\mathcal{A}} \mathcal{F}$ be a small subcategory containing $\text{Gr}_A F$ and $A' \subset \mathcal{A}$ be a core containing the full subcategory on all the objects of the form $\pi(x)$ where x is in J and $\pi: \text{Gr}_{\mathcal{A}} \mathcal{F} \rightarrow \mathcal{A}$ is the projection (see Section 2.4). Let F' be a core of the restriction of \mathcal{F} to A' such that $\text{Gr}_A F \subset J \subset \text{Gr}_{A'} F'$ (see Lemma 5.7). According to Proposition 4.1(3), the homotopy fibers of the nerves of the projections $\pi: \text{Gr}_A F \rightarrow A$ and $\pi: \text{Gr}_{A'} F' \rightarrow A'$ over a vertex given by an object a in A are weakly equivalent to the nerves of F_a and F'_a respectively. As these categories are cores of \mathcal{F}_a , they are weakly equivalent, and consequently the following square is a homotopy pull-back:

$$\begin{CD} \text{Gr}_A F @>>> \text{Gr}_{A'} F' \\ @V \pi VV @VV \pi V \\ A @>>> A' \end{CD}$$

Now, since $A \subset A'$ is a weak equivalence, so too is $\text{Gr}_A F \subset \text{Gr}_{A'} F'$. \square

Corollary 6.4. *Let $f: \mathcal{B} \rightarrow \mathcal{A}$ be a quasifibration. If \mathcal{A} is essentially small, then \mathcal{B} is essentially small.*

Proof. Consider the system $- \uparrow f$ indexed by \mathcal{A}^{op} (see Section 2.4). Recall that we have a homotopy equivalence $\hat{f}: \mathcal{B} \rightarrow \text{Gr}_{\mathcal{A}}(- \uparrow f)$. Thus according to Corollary 5.9, \mathcal{B} is essentially small if and only if $\text{Gr}_{\mathcal{A}}(- \uparrow f)$ is so. We can now apply Lemma 6.3. \square

Proposition 6.5. *Let $f: \mathcal{B} \rightarrow \mathcal{A}$ be a strong fibration between essentially small categories. Then f is a quasifibration.*

Proof. We claim: *for small subcategories $A' \subset \mathcal{A}$ and $B' \subset \mathcal{B}$, there is a core $f: B \rightarrow A$ of f which is a quasifibration and such that $A' \subset A$ and $B' \subset B$.*

Assume the claim. To prove the proposition we need to show $a \uparrow f$ is essentially small for any a in \mathcal{A} . The under category of the restriction of f to $C \subset \mathcal{B}$ is denoted by $a \uparrow C$. Use the claim to get a quasifibration core $f: B \rightarrow A$ of $f: \mathcal{B} \rightarrow \mathcal{A}$ such that a is in A . We will show that $a \uparrow B \subset a \uparrow f$ is a core. Let $J \subset a \uparrow f$ be a small subcategory containing $a \uparrow B$ and B' be the full subcategory of \mathcal{B} on the set of objects b for which there is $\alpha: a \rightarrow f(b)$ with (b, α) in J . Note $J \subset a \uparrow B'$. Use the claim again to get a quasifibration core $f: \hat{B} \rightarrow \hat{A}$ of $f: \mathcal{A} \rightarrow \mathcal{B}$ such that $A \subset \hat{A}$

and $B' \subset \hat{B}$. All this fits into a commutative diagram:

$$\begin{array}{ccccc} B & \subset & B' & \subset & \hat{B} & \subset & \mathcal{B} \\ & \searrow f & & & \downarrow f & & \downarrow f \\ & & A & \subset & \hat{A} & \subset & \mathcal{A} \end{array}$$

The inclusions $A \subset \hat{A}$ and $B \subset \hat{B}$ are weak equivalences (see Lemma 5.4(1)). Since $f: B \rightarrow A$ and $f: \hat{B} \rightarrow \hat{A}$ are quasifibrations, the homotopy fibers of their nerves over the components containing a are given by $N(a \uparrow B)$ and $N(a \uparrow \hat{B})$ (see Proposition 4.1(4)). These two observations imply that $a \uparrow B \subset a \uparrow \hat{B}$ is a weak equivalence.

It remains to show the claim. Since $f: \mathcal{B} \rightarrow \mathcal{A}$ is a strong fibration, for any $\alpha: a_1 \rightarrow a_0$ in \mathcal{A} there is $\phi_\alpha: a_1 \uparrow f \rightarrow a_0 \uparrow f$ for which $\phi_\alpha(\alpha \uparrow f)$ and $(\alpha \uparrow f)\phi_\alpha$ are homotopic to the identity functors. Choose $\{h_{\alpha,k}: a_0 \uparrow f \rightarrow a_0 \uparrow f\}_{0 \leq k \leq m}$ and $\{g_{\alpha,k}: a_1 \uparrow f \rightarrow a_1 \uparrow f\}_{0 \leq k \leq l}$ and natural transformations $\phi_\alpha(\alpha \uparrow f) = h_{\alpha,0} \rightarrow \cdots \leftarrow h_{\alpha,m} = \text{id}$ and $(\alpha \uparrow f)\phi_\alpha = g_{\alpha,0} \rightarrow \cdots \leftarrow g_{\alpha,l} = \text{id}$. By induction define a sequence of subsystems of $f: \mathcal{B} \rightarrow \mathcal{A}$:

$$\begin{array}{ccccccccccc} B_0 & \subset & D_1 & \subset & B_1 & \subset & D_2 & \subset & B_2 & \subset & \cdots & \subset & \mathcal{B} \\ f \downarrow & & f \downarrow & & f \downarrow & & f \downarrow & & f \downarrow & & & & \downarrow f \\ A_0 & \subset & C_1 & \subset & A_1 & \subset & C_2 & \subset & A_2 & \subset & \cdots & \subset & \mathcal{A} \end{array}$$

Let $f: B_0 \rightarrow A_0$ be a core of $f: \mathcal{B} \rightarrow \mathcal{A}$ such that $A' \subset A_0$ and $B' \subset B_0$. It exists by Lemma 5.7. Assume $n > 0$ and that the sequence above is defined for indices smaller than n . Let D_n be the full subcategory of \mathcal{B} whose set of objects b satisfies:

- b either belongs to B_{n-1} or
- there is an object b' in B_{n-1} and morphisms $\alpha: a_1 \rightarrow a_0$ in A_{n-1} and $\beta': a_1 \rightarrow f(b')$ and $\beta: a_0 \rightarrow f(b)$ in \mathcal{A} such that $\phi_\alpha(b', \beta') = (b, \beta)$, or
- there is b' in B_{n-1} and $\alpha: a_1 \rightarrow a_0$ in A_{n-1} and $\beta': a_0 \rightarrow f(b')$ and $\beta: a_0 \rightarrow f(b)$ in \mathcal{A} such that $h_{\alpha,k}(b', \beta') = (b, \beta)$ for some $0 \leq k \leq m$, or
- there is b' in B_{n-1} and $\alpha: a_1 \rightarrow a_0$ in A_{n-1} and $\beta': a_1 \rightarrow f(b')$ and $\beta: a_1 \rightarrow f(b)$ in \mathcal{A} such that $g_{\alpha,k}(b', \beta') = (b, \beta)$ for some $0 \leq k \leq l$.

Define C_n to be the full subcategory of \mathcal{A} on the set of objects that belong either to A_{n-1} or are of the form $f(b)$ where b is in D_n . The purpose is to ensure that for any $\alpha: a_1 \rightarrow a_0$ in A_{n-1} :

- a. $\phi_\alpha: a_1 \uparrow f \rightarrow a_0 \uparrow f$ takes $a_1 \uparrow B_{n-1}$ to $a_0 \uparrow D_n$,
- b. $h_{\alpha,k}: a_0 \uparrow f \rightarrow a_0 \uparrow f$ takes $a_0 \uparrow B_{n-1}$ to $a_0 \uparrow D_n$ for any $0 \leq k \leq m$,
- c. $g_{\alpha,k}: a_1 \uparrow f \rightarrow a_1 \uparrow f$ takes $a_1 \uparrow B_{n-1}$ to $a_1 \uparrow D_n$ for any $0 \leq k \leq l$.

Let $f: B_n \rightarrow A_n$ be a core of $f: \mathcal{B} \rightarrow \mathcal{A}$ such that $C_n \subset A_n$ and $D_n \subset B_n$. Define $A := \bigcup_{n \geq 0} A_n$ and $B := \bigcup_{n \geq 0} B_n$. According to Lemmas 5.1 and 5.4(1) $f: \mathcal{B} \rightarrow \mathcal{A}$ is also a core of $f: B \rightarrow A$. We are going to show that $f: B \rightarrow A$ is a strong fibration. The requirements above imply that, for any $\alpha: a_1 \rightarrow a_0$ in A :

- a. $\phi_\alpha: a_1 \uparrow f \rightarrow a_0 \uparrow f$ takes $a_1 \uparrow B$ to $a_0 \uparrow B$;
- b. $h_{\alpha,k}: a_0 \uparrow f \rightarrow a_0 \uparrow f$ takes $a_0 \uparrow B$ to $a_0 \uparrow B$ for any $0 \leq k \leq m$;
- c. $g_{\alpha,k}: a_1 \uparrow f \rightarrow a_1 \uparrow f$ takes $a_1 \uparrow B$ to $a_1 \uparrow B$ for any $0 \leq k \leq l$.

Since B is a full subcategory in \mathcal{B} , for any $\alpha: a_1 \rightarrow a_0$ in A , by restricting to $a_0 \uparrow B$ and $a_1 \uparrow B$ we have two sequences of natural transformations $\phi_\alpha(\alpha \uparrow f) = h_{\alpha,0} \rightarrow \cdots \leftarrow h_{\alpha,m} = \text{id}$ and $(\alpha \uparrow f)\phi_\alpha = g_{\alpha,0} \rightarrow \cdots \leftarrow g_{\alpha,l} = \text{id}$, showing that $\phi_\alpha(\alpha \uparrow f): a_0 \uparrow B \rightarrow a_0 \uparrow B$ and $(\alpha \uparrow f)\phi_\alpha: a_1 \uparrow B \rightarrow a_1 \uparrow B$ are both homotopic

to the identity functors. The functor $\alpha \uparrow B: a_0 \uparrow B \rightarrow a_1 \uparrow B$ is therefore a homotopy equivalence and hence a weak equivalence, which shows the claim. \square

Corollary 6.6. *Let the following be a strong homotopy pull-back square:*

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{g} & \mathcal{C} \\ e \downarrow & & \downarrow h \\ \mathcal{B} & \xrightarrow{f} & \mathcal{A} \end{array}$$

Assume that \mathcal{A} , \mathcal{B} , and \mathcal{C} are essentially small categories. Then \mathcal{D} is also essentially small and the above square is a homotopy pull-back.

Proof. The strong homotopy pull-back assumption implies $(e, h): c \uparrow g \rightarrow h(c) \uparrow f$ is a homotopy equivalence for any c in \mathcal{C} , and $f: \mathcal{B} \rightarrow \mathcal{A}$ and $g: \mathcal{D} \rightarrow \mathcal{C}$ are strong fibrations. Since \mathcal{A} and \mathcal{B} are essentially small, by Proposition 6.5, $f: \mathcal{B} \rightarrow \mathcal{A}$ is a quasifibration and thus $a \uparrow f$ is essentially small for any a in \mathcal{A} . By Corollary 5.9, $c \uparrow g$ is then also essentially small for any c in \mathcal{C} . The functor $g: \mathcal{D} \rightarrow \mathcal{C}$ is therefore a quasifibration. As \mathcal{C} is essentially small, Corollary 6.4 implies that so is \mathcal{D} .

By the claim in the proof of Proposition 6.5, there is a core $g: D \rightarrow C$ of g which is a quasifibration. Let $B' \subset \mathcal{B}$ be the full subcategory on the set of all objects $e(d)$ where d is in D and let $A' \subset \mathcal{A}$ be the full subcategory on the set of all objects $h(c)$ where c is in C . The same claim yields a core $f: B \rightarrow A$ of f which is a quasifibration and such that $A' \subset A$ and $B' \subset B$. This leads to a commutative diagram of categories:

$$\begin{array}{ccccc} & & C^c & \xrightarrow{\quad} & C \\ & g \nearrow & \downarrow h & & \downarrow h \\ D & \xrightarrow{\quad} & D & \xrightarrow{g} & C \\ & \downarrow e & \downarrow e & & \downarrow h \\ & & A^c & \xrightarrow{\quad} & A \\ e \downarrow & f \nearrow & \downarrow e & & \downarrow h \\ B & \xrightarrow{\quad} & B & \xrightarrow{f} & A \end{array}$$

In the proof of Proposition 6.5 it was also shown that $a \uparrow B \subset a \uparrow f$ and $c \uparrow D \subset c \uparrow g$ are cores for any a in A and c in C . By the “2 out of 3” property $(e, h): c \uparrow D \rightarrow h(c) \uparrow B$ is a weak equivalence. The following square is therefore a homotopy-pull-back:

$$\begin{array}{ccc} D & \xrightarrow{g} & C \\ e \downarrow & & \downarrow h \\ B & \xrightarrow{f} & A \end{array} \quad \square$$

Part II. The aim of this part is to present a construction of the spaces of weak equivalences and their deloopings in an arbitrary model category based on [4].

7. SIMPLEX CATEGORIES

Let A be a simplicial set. Its simplex category (see [3, Section 6]), denoted by the same symbol A , is a category whose objects are simplices of A i.e., maps of the form $\sigma: \Delta[n] \rightarrow A$. The set of morphisms in A between $\tau: \Delta[m] \rightarrow A$ and $\sigma: \Delta[n] \rightarrow A$ consists of the maps $\alpha: \Delta[m] \rightarrow \Delta[n]$ for which $\tau = \sigma\alpha$. A map of spaces $f: A \rightarrow B$ induces a functor $f: A \rightarrow B$. It assigns to $\sigma: \Delta[n] \rightarrow A$ the

composition $f\sigma: \Delta[n] \rightarrow B$. This defines a functor from Spaces to Cat called the simplex category. Its composition with the nerve is called the **subdivision**:

$$\text{Spaces} \begin{array}{c} \xrightarrow{\text{simplex category}} \\ \searrow \text{subdivision} \end{array} \text{Cat} \xrightarrow{N} \text{Spaces}$$

The subdivision has the following properties:

- (1) A space A is contractible if and only if $N(A)$ is contractible.
- (2) A map $f: A \rightarrow B$ is a weak equivalence if and only if $N(f)$ is.
- (3) For any $f: A \rightarrow B$, the map $N(f)$ is **reduced**, i.e., it maps non-degenerate simplices to non-degenerate simplices (see [3, Definition 12.9]).
- (4) The subdivision is a left adjoint and hence it commutes with colimits. In particular, if the left square below is a push-out, then so is the right square:

$$\begin{array}{ccc} A \xrightarrow{f} B & & N(A) \xrightarrow{N(f)} N(B) \\ g \downarrow & \searrow & \downarrow N(g) \\ C \xrightarrow{k} D & & N(C) \xrightarrow{N(k)} N(D) \end{array} \quad \begin{array}{ccc} & & \downarrow N(h) \\ & & N(D) \end{array}$$

The symbols $\text{Fun}(-, \mathcal{C})$ and $\text{Fun}(N(-), \mathcal{C})$ denote systems of categories indexed by Spaces (see Section 2.3) given by the assignments $A \mapsto \text{Fun}(A, \mathcal{C})$, $(f: A \rightarrow B) \mapsto f^*$, $A \mapsto \text{Fun}(N(A), \mathcal{C})$, and $(f: A \rightarrow B) \mapsto N(f)^*$.

7.1. Clutching construction. To construct and analyze functors indexed by simplex categories one can use the geometry of the underlying spaces. For example, the clutching construction can be described as follows. An **initial data** consists of a push-out square of spaces where the indicated maps are inclusions,

$$\begin{array}{ccc} A & \xhookrightarrow{i} & C \\ f \downarrow & & \downarrow g \\ B & \xhookrightarrow{j} & D \end{array}$$

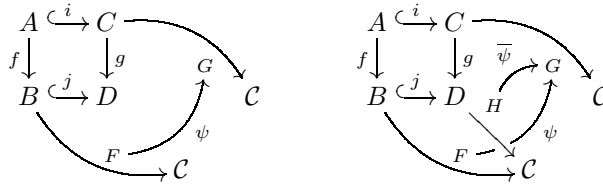
two functors $F: B \rightarrow \mathcal{C}$ and $G: C \rightarrow \mathcal{C}$, and a natural transformation $\psi: f^*F \rightarrow i^*G$ in $\text{Fun}(A, \mathcal{C})$. Out of this data we are going to construct a functor $H: D \rightarrow \mathcal{C}$ and a natural transformation $\bar{\psi}: g^*H \rightarrow G$ in $\text{Fun}(C, \mathcal{C})$. This functor is called the **clutching of F and G along ψ** and is denoted by $H(\psi, F, G)$. The functor H and the natural transformation $\bar{\psi}$ are supposed to satisfy the following properties:

- (1) $j^*H = F$;
- (2) the following diagram commutes:

$$\begin{array}{ccc} f^*F & \xrightarrow{\psi} & i^*G \\ \parallel & & \uparrow i^*\bar{\psi} \\ f^*j^*H = (jf)^*H = (gi)^*H = i^*g^*H & & \end{array}$$

- (3) the morphism $\bar{\psi}_\sigma: Hg(\sigma) \rightarrow G(\sigma)$ is an isomorphism for any simplex σ in C which is not in the image of i .

We use the following diagrams to depict an initial data and its clutching:



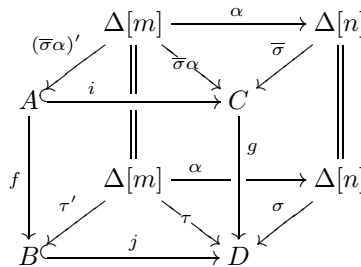
By the push-out assumption in the initial data, there is a bijective correspondence between the set D_n and the disjoint union $j(B_n) \coprod g(C_n \setminus i(A_n))$. Furthermore, $j: B_n \rightarrow j(B_n)$, $g: C_n \setminus i(A_n) \rightarrow g(C_n \setminus i(A_n))$, and $i: A_n \rightarrow i(A_n)$ are bijections. This justifies the use of the following notation. If $\sigma: \Delta[n] \rightarrow C$ belongs to $i(A)$, then $\sigma': \Delta[n] \rightarrow A$ denotes the unique simplex for which $i\sigma' = \sigma$. If $\sigma: \Delta[n] \rightarrow D$ belongs to $j(B)$, then $\sigma': \Delta[n] \rightarrow B$ denotes the unique simplex for which $j\sigma' = \sigma$. If $\sigma: \Delta[n] \rightarrow D$ does not belong to $j(B)$, then $\bar{\sigma}: \Delta[n] \rightarrow C$ denotes the unique simplex in C for which $g\bar{\sigma} = \sigma$. We can now define:

$$H(\sigma) := \begin{cases} F(\sigma') & \text{if } \sigma \in j(B), \\ G(\bar{\sigma}) & \text{if } \sigma \notin j(B). \end{cases}$$

Let $\alpha: \Delta[m] \rightarrow \Delta[n]$ be a morphism in D between $\tau: \Delta[m] \rightarrow D$ and $\sigma: \Delta[n] \rightarrow D$. We are going to define $H(\alpha): H(\tau) \rightarrow H(\sigma)$. Note that if σ belongs to $j(B)$, then so does τ . Thus there are no morphisms in D between any simplex that does not belong to $j(B)$ and a simplex that belongs to $j(B)$. Three possibilities remain

<p>• If $\tau \in j(B)$ and $\sigma \in j(B)$, then</p> $\begin{array}{ccc} H(\tau) & \xrightarrow{H(\alpha)} & H(\sigma) \\ \parallel & & \parallel \\ F(\tau') & \xrightarrow{F(\alpha)} & F(\sigma') \end{array}$	<p>• If $\tau \notin j(B)$ and $\sigma \notin j(B)$, then</p> $\begin{array}{ccc} H(\tau) & \xrightarrow{H(\alpha)} & H(\sigma) \\ \parallel & & \parallel \\ G(\bar{\tau}) & \xrightarrow{G(\alpha)} & G(\bar{\sigma}) \end{array}$
--	--

• If $\tau \in j(B)$ and $\sigma \notin j(B)$, then we have a commutative diagram of spaces:



Define $H(\alpha): H(\tau) \rightarrow H(\sigma)$ as the following composition:

$$\begin{array}{ccccc} H(\tau) & \xrightarrow{\quad} & H(\alpha) & \xrightarrow{\quad} & H(\sigma) \\ \parallel & & \parallel & & \parallel \\ F(\tau') & \xrightarrow{\quad} & F(f(\bar{\sigma}\alpha)') & \xrightarrow{\psi(\bar{\sigma}\alpha)'} & G(i(\bar{\sigma}\alpha)') & \xrightarrow{\quad} & G(\bar{\sigma}\alpha) & \xrightarrow{G(\alpha)} & G(\bar{\sigma}) \end{array}$$

This procedure indeed defines a functor $H: D \rightarrow \mathcal{M}$ such that $Hj = F$, which is the requirement (1). It remains to construct $\bar{\psi}: Hg \rightarrow G$.

- If $\sigma: \Delta[n] \rightarrow C$ belongs to $i(A)$, then $\overline{\psi}_\sigma: Hg(\sigma) \rightarrow G(\sigma)$ is defined as:

$$\begin{array}{ccc} Hg(\sigma) & \xrightarrow{\overline{\psi}_\sigma} & G(\sigma) \\ \parallel & & \parallel \\ Hgi(\sigma') & \xlongequal{=} Hjf(\sigma') \xlongequal{=} Ff(\sigma') \xrightarrow{\psi_{\sigma'}} & Gi(\sigma') \end{array}$$

- If $\sigma: \Delta[n] \rightarrow C$ does not belong to $i(A)$, then $\overline{\psi}_\sigma$ is given by the identity $\text{id}: Hg(\sigma) = G(\sigma) \rightarrow G(\sigma)$.

The morphisms $\{\overline{\psi}_\sigma\}_{\sigma \in C}$ form a natural transformation between g^*H and G that fulfills the requirements (2) and (3).

Assume now that we have a push-out square of spaces, where the indicated maps are inclusions,

$$\begin{array}{ccc} A & \xhookrightarrow{i} & C \\ f \downarrow & & \downarrow g \\ B & \xhookrightarrow{j} & D \end{array}$$

two functors $F: B \rightarrow C$ and $G: D \rightarrow C$, and a natural transformation $\psi: F \rightarrow j^*G$ in $\text{Fun}(B, C)$. This data induces an initial data for the clutching that consists of the above push-out, functors $F: A \rightarrow C$ and $g^*G: C \rightarrow C$, and a natural transformation $f^*\psi: f^*F \rightarrow f^*j^*G = i^*g^*G$. Its clutching is a functor $H(f^*\psi, F, g^*G): D \rightarrow C$ and a natural transformation $\overline{f^*\psi}: g^*H(f^*\psi, F, g^*G) \rightarrow g^*G$.

Proposition 7.2. *There is a unique natural transformation $\widehat{\psi}: H(f^*\psi, F, g^*G) \rightarrow G$ such that $j^*\widehat{\psi} = \psi$ and $g^*\widehat{\psi} = \overline{f^*\psi}$.*

Proof. Let σ be a simplex in D . If it belongs to B , define $\widehat{\psi}_\sigma$ to be ψ_σ . If it does not, define $\widehat{\psi}_\sigma$ to be the identity morphism $\text{id}: H(f^*\psi, F, g^*G)(\sigma) = g^*G(\overline{\sigma}) = G(\sigma)$. These morphisms define the desired natural transformation $\widehat{\psi}$. \square

8. BOUNDED FUNCTORS

In a simplex category A the face and degeneracy morphisms

$$\begin{array}{ccc} \Delta[n] & \xrightarrow{d_i} & \Delta[n+1] \\ & \searrow d_i\sigma & \swarrow \sigma \\ & & A \end{array} \qquad \begin{array}{ccc} \Delta[n+1] & \xrightarrow{s_i} & \Delta[n] \\ & \searrow s_i\sigma & \swarrow \sigma \\ & & A \end{array}$$

are subject to the usual cosimplicial identities, and they generate all the morphisms in A . A functor indexed by A is called **bounded** [3, Definition 10.1] if it assigns an isomorphism to any degeneracy morphism s_i in A . If S denotes the set of all these degeneracy morphisms in A , then a bounded functor is a functor indexed by the localized category $A[S^{-1}]$. The symbol $\text{Fun}^b(A, C)$ denotes the category of bounded functors indexed by A with values in C . The sets of morphisms in $\text{Fun}^b(A, C)$ consist of all natural transformations. For example, let I be a small category and $\epsilon: N(I) \rightarrow I$ be the functor defined as follows (see also [3, Definition 6.6]). For a simplex $\sigma = (i_n \xrightarrow{\alpha_n} \dots \xrightarrow{\alpha_1} i_0)$ in $N(I)$, let

$$\epsilon(\sigma) := i_0, \quad \epsilon(s_k: s_k\sigma \rightarrow \sigma) := \text{id}_{i_0}, \quad \epsilon(d_k: d_k\sigma \rightarrow \sigma) := \begin{cases} \text{id}_{i_0} & \text{if } k > 0, \\ \alpha_1 & \text{if } k = 0. \end{cases}$$

Since ϵ maps all the degeneracy morphisms to identities, it is a bounded functor. Consequently $\epsilon^*: \text{Fun}(I, C) \rightarrow \text{Fun}(N(I), C)$ has bounded values. The induced functor is denoted by the same symbol, $\epsilon^*: \text{Fun}(I, C) \rightarrow \text{Fun}^b(N(I), C)$.

If $F: B \rightarrow \mathcal{C}$ is bounded, then, for any map $f: A \rightarrow B$, so is $Ff: A \rightarrow \mathcal{C}$. Thus the subcategories $\text{Fun}^b(A, \mathcal{C}) \subset \text{Fun}(A, \mathcal{C})$ form a subsystem of $\text{Fun}(-, \mathcal{C})$. The same is true for the subcategories $\text{Fun}^b(N(A), \mathcal{C}) \subset \text{Fun}(N(A), \mathcal{C})$. We denote these subsystems by $\text{Fun}^b(-, \mathcal{C})$ and $\text{Fun}^b(N(-), \mathcal{C})$ respectively.

If \mathcal{C} is closed under colimits, then the left Kan extension $f^k: \text{Fun}(A, \mathcal{C}) \rightarrow \text{Fun}(B, \mathcal{C})$ also preserves the property of being bounded ([3, Theorem 10.6]). Thus in this case we have a pair of adjoint functors $f^k: \text{Fun}^b(A, \mathcal{C}) \rightleftarrows \text{Fun}^b(B, \mathcal{C}) : f^*$.

Proposition 8.1. *Notation as in Section 7.1. If $F: B \rightarrow \mathcal{C}$ and $G: C \rightarrow \mathcal{C}$ are bounded, then so is their clutching $H(\psi, F, G)$.*

Proof. This is a consequence of properties (1) and (3) given in Section 7.1. □

Let \mathcal{M} be a model category and A a simplicial set. A natural transformation $\phi: F \rightarrow G$ in $\text{Fun}^b(A, \mathcal{M})$ is called a **weak equivalence (fibration)** if $\phi_\sigma: F(\sigma) \rightarrow G(\sigma)$ is a weak equivalence (fibration) in \mathcal{M} for any σ in A . It is called a **cofibration** if, for any **non-degenerate** $\sigma: \Delta[n] \rightarrow A$, the morphism

$$\text{colim}(\text{colim}_{\partial\Delta[n]}G \xleftarrow{\text{colim}_{\partial\Delta[n]}\phi} \text{colim}_{\partial\Delta[n]}F \rightarrow F(\sigma)) \longrightarrow G(\sigma)$$

induced by the commutativity of the following square is a cofibration:

$$\begin{array}{ccc} \text{colim}_{\partial\Delta[n]}F & \longrightarrow & \text{colim}_{\Delta[n]}F = F(\sigma) \\ \text{colim}_{\partial\Delta[n]}\phi \downarrow & & \text{colim}_{\Delta[n]}\phi \downarrow \phi_\sigma \\ \text{colim}_{\partial\Delta[n]}G & \longrightarrow & \text{colim}_{\Delta[n]}G = G(\sigma) \end{array}$$

Theorem 8.2 ([3, Theorem 21.1]). *The above choice of weak equivalences, fibrations, and cofibrations equips $\text{Fun}^b(A, \mathcal{M})$ with a model category structure.*

Let X be an object in \mathcal{M} . A bounded functor $F: A \rightarrow X_{\text{we}}$ is called cofibrant if its composition with $X_{\text{we}} \subset \mathcal{M}$ is cofibrant in $\text{Fun}^b(A, \mathcal{M})$. The symbol $\text{Cof}(A, X_{\text{we}})$ denotes the full subcategory of $\text{Fun}^b(A, X_{\text{we}})$ of cofibrant functors.

A map of simplicial sets $f: A \rightarrow B$ can send a non-degenerate simplex in A to a degenerate simplex in B . This is the reason why the subcategories $\text{Cof}(A, X_{\text{we}}) \subset \text{Fun}^b(A, X_{\text{we}})$ do not form a subsystem of $\text{Fun}^b(-, X_{\text{we}})$.

Proposition 8.3. *Let $f: A \rightarrow B$ be a map of spaces.*

- (1) *If ϕ is a weak equivalence in $\text{Fun}^b(A, \mathcal{M})$ between cofibrant objects, then $f^k\phi$ is a weak equivalence in $\text{Fun}^b(B, \mathcal{M})$.*
- (2) *If ϕ is an (acyclic) cofibration in $\text{Fun}^b(A, \mathcal{M})$, then $f^k\phi$ is an (acyclic) cofibration in $\text{Fun}^b(B, \mathcal{M})$.*
- (3) *Assume f is reduced (see Section 7). If ϕ is an (acyclic) cofibration in $\text{Fun}^b(B, \mathcal{M})$, then so is $f^*\phi$ in $\text{Fun}^b(A, \mathcal{M})$.*
- (4) *If ϕ is an (acyclic) cofibration in $\text{Fun}^b(N(B), \mathcal{M})$, then so is the natural transformation $N(f)^*\phi$ in $\text{Fun}^b(N(A), \mathcal{M})$.*

Proof. Statement (1) is [3, Proposition 13.3(2)]. Statement (2) is [3, Theorem 11.2.]. Statement (3) follows from the definition and (4) is a particular case. □

Since, for any map $f: A \rightarrow B$ of spaces, $N(f)$ is reduced, the subcategories $\text{Cof}(N(A), X_{\text{we}}) \subset \text{Fun}^b(N(A), X_{\text{we}})$ form a subsystem of $\text{Fun}^b(N(-), X_{\text{we}})$.

9. HOMOTOPY COLIMITS AND DERIVED LEFT KAN EXTENSIONS

The following proposition is part of [3, Theorem 11.3(1)] which states that $\epsilon^k: \text{Fun}^b(N(I), \mathcal{M}) \rightleftarrows \text{Fun}(I; \mathcal{M}) : \epsilon^*$ is a left model approximation ([3, 5.1]):

Proposition 9.1. *Let I be a small category.*

- (1) *Assume that $\phi: F \rightarrow G$ is a weak equivalence between cofibrant objects in $\text{Fun}^b(N(I), \mathcal{M})$. Then $\epsilon^k\phi$ is a weak equivalence in $\text{Fun}(I, \mathcal{M})$.*
- (2) *Let $F: I \rightarrow \mathcal{M}$ be a functor and $\psi: G \rightarrow \epsilon^*F$ be a weak equivalence in $\text{Fun}^b(N(I), \mathcal{M})$. If $G: N(I) \rightarrow \mathcal{M}$ is cofibrant, then the morphism $\epsilon^kG \rightarrow F$ which is adjoint to ψ is a weak equivalence.*

Let P be a functorial cofibrant replacement in $\text{Fun}^b(N(I), \mathcal{M})$ (see Section 2.2). We use the same symbol P to denote the following composition:

$$\text{Fun}(I, \mathcal{M}) \xrightarrow{\epsilon^*} \text{Fun}^b(N(I), \mathcal{M}) \xrightarrow{P} \text{Fun}^b(N(I), \mathcal{M}) \xrightarrow{\epsilon^k} \text{Fun}(I, \mathcal{M}) .$$

Let $PF = \epsilon^kP\epsilon^*F \rightarrow F$ be the natural transformation adjoint to the cofibrant replacement $P\epsilon^*F \rightarrow \epsilon^*F$. According to [3, Theorem 11.3(1)] it is a weak equivalence. The functor $P: \text{Fun}(I, \mathcal{M}) \rightarrow \text{Fun}^b(N(I), \mathcal{M})$ with this natural transformation is called a **cofibrant replacement** in $\text{Fun}(I, \mathcal{M})$ and $\text{colim}_I P(-): \text{Fun}(I, \mathcal{M}) \rightarrow \mathcal{M}$ is called the **homotopy colimit**. There is a natural transformation $\text{colim}_I P(-) \rightarrow \text{colim}_I (-)$ given by the colimit of the cofibrant replacement. According to [3, Theorem 11.3(2)], this natural transformation is the total left derived functor of the colimit, which justifies the name homotopy colimit.

Analogous statements hold for arbitrary left Kan extensions. Let $f: I \rightarrow J$ be a functor between small categories. The functor $f^kPF: J \rightarrow \mathcal{M}$ is called the **derived left Kan extension** of $F: I \rightarrow \mathcal{M}$ along f . Let $f^kPF \rightarrow f^kF$ be the natural transformation given by the left Kan extension of the cofibrant replacement. In this way we obtain a functor $f^kP: \text{Fun}(I, \mathcal{M}) \rightarrow \text{Fun}(J, \mathcal{M})$ and a natural transformation $f^kP \rightarrow f^k$. According to [3, Theorem 11.3(3)], this natural transformation is the total left derived functor of the left Kan extension.

Let j be an object in J and $g: f \downarrow j \rightarrow I$ be the forgetful functor. The value of the derived left Kan extension $f^kPF(j) = \text{colim}_{f \downarrow j} (PF)g$ is weakly equivalent to the homotopy colimit of the composition $Fg: f \downarrow j \rightarrow \mathcal{M}$.

10. HOMOTOPY CONSTANT FUNCTORS AND THEIR HOMOTOPY COLIMITS

A functor $F: I \rightarrow \mathcal{M}$, indexed by a small category I , is called **homotopy constant** if it is isomorphic in $\text{Ho}(\text{Fun}(I, \mathcal{M}))$ to a constant functor. To be homotopy constant it is necessary for a functor to have values in a component X_{we} for some X in \mathcal{M} . In general however, this is not enough. Similar to the fact that fibrations of spaces over a contractible base are weakly equivalent to product fibrations, in the context of functors according to [3, Corollary 29.2], we have:

Proposition 10.1. *Let I be a small contractible category and P a cofibrant replacement in $\text{Fun}(I, \mathcal{M})$. Then, for any $F: I \rightarrow X_{\text{we}}$, the morphism $PF(i) \rightarrow \text{colim}_I PF$, induced by the inclusion of an object i in I , is a weak equivalence and F is homotopy constant.*

If I is contractible, then the homotopy colimit $\operatorname{colim}_I P(-) : \operatorname{Fun}(I, \mathcal{M}) \rightarrow \mathcal{M}$ maps the subcategory $\operatorname{Fun}(I, X_{\text{we}}) \subset \operatorname{Fun}(I, \mathcal{M})$ into $X_{\text{we}} \subset \mathcal{M}$ (see Proposition 10.1) inducing a functor $\operatorname{colim}_I P(-) : \operatorname{Fun}(I, X_{\text{we}}) \rightarrow X_{\text{we}}$. More generally let $f : I \rightarrow J$ be a functor of small categories such that $f \downarrow j$ is contractible for any j in J . Then the derived left Kan extension $f^k P(-) : \operatorname{Fun}(I, \mathcal{M}) \rightarrow \operatorname{Fun}(J, \mathcal{M})$ maps $\operatorname{Fun}(I, X_{\text{we}})$ into $\operatorname{Fun}(J, X_{\text{we}})$ inducing a functor $f^k P(-) : \operatorname{Fun}(I, X_{\text{we}}) \rightarrow \operatorname{Fun}(J, X_{\text{we}})$ (see Proposition 10.1). In particular, let $f : A \rightarrow B$ be a map of simplicial sets. For any $\sigma : \Delta[n] \rightarrow B$ the category $f \downarrow \sigma$ is the simplex category of the pull-back $df(\sigma) = \lim(\Delta[n] \xrightarrow{-\sigma} B \xleftarrow{f} A)$. Thus, if $df(\sigma)$ is contractible for any σ , then the derived left Kan extension induces a functor $f^k P : \operatorname{Fun}(A, X_{\text{we}}) \rightarrow \operatorname{Fun}(B, X_{\text{we}})$.

If a bounded functor $F : A \rightarrow \mathcal{M}$ is weakly equivalent to a constant functor in $\operatorname{Fun}(A, \mathcal{M})$, it does not necessarily have to be weakly equivalent to a constant functor in $\operatorname{Fun}^b(A, \mathcal{M})$. Furthermore, even if A is contractible and $F : A \rightarrow X_{\text{we}}$ is bounded and cofibrant, then the morphism $F(\sigma) \rightarrow \operatorname{colim}_A F$, induced by the inclusion of a simplex σ into A , may not be a weak equivalence. This is not true even if we assume that F is weakly equivalent to a constant functor in $\operatorname{Fun}^b(A, \mathcal{M})$. An additional assumption that A is the nerve of a small category is needed. The following is a consequence of Proposition 10.1:

Proposition 10.2. *Let $F : N(I) \rightarrow \mathcal{M}$ be a bounded and cofibrant functor weakly equivalent in $\operatorname{Fun}^b(N(I), \mathcal{M})$ to a constant functor and σ a simplex in $N(I)$. If I is contractible, then the morphism $F(\sigma) \rightarrow \operatorname{colim}_{N(I)} F$ is a weak equivalence.*

11. MAPPING SPACES IN MODEL CATEGORIES

The homotopy colimits of constant functors are homotopy invariant with respect to the indexing categories and thus the functor $\operatorname{Spaces} \ni A \mapsto \operatorname{hocolim}_A X \in \operatorname{Ho}(\mathcal{M})$ is a composition of the localization $\operatorname{Spaces} \rightarrow \operatorname{Ho}(\operatorname{Spaces})$ and a functor denoted by $X \otimes_l - : \operatorname{Ho}(\operatorname{Spaces}) \rightarrow \operatorname{Ho}(\mathcal{M})$ ([4, Proposition 7.1]). A key result in [4] states that $X \otimes_l -$ has a right adjoint $\operatorname{map}(X, -) : \operatorname{Ho}(\mathcal{M}) \rightarrow \operatorname{Ho}(\operatorname{Spaces})$. Its value $\operatorname{map}(X, Y)$ is what we take to be the homotopy type of the mapping space between X and Y in \mathcal{M} . If \mathcal{M} is a simplicial model category, then the mapping space between a cofibrant and a fibrant object given by the simplicial structure on \mathcal{M} is weakly equivalent to the value of this right adjoint.

Notation 11.1. The full subcategory in $\operatorname{Fun}^b(N(\Delta[0]), \mathcal{M})$ of cofibrant and fibrant objects which are weakly equivalent in $\operatorname{Fun}^b(N(\Delta[0]), \mathcal{M})$ to constant functors is denoted by $\operatorname{Cons}(N(\Delta[0]), \mathcal{M})$.

The unique map $p : \Delta[n] \rightarrow \Delta[0]$ induces a functor $N(p)^* : \operatorname{Fun}^b(N(\Delta[0]), \mathcal{M}) \rightarrow \operatorname{Fun}^b(N(\Delta[n]), \mathcal{M})$. To make formulas more readable, the effect of $N(p)^*$ is denoted by adding the symbol $[n]$. Thus, if $f : F \rightarrow G$ is a natural transformation in $\operatorname{Fun}^b(N(\Delta[0]), \mathcal{M})$, then $f[n] : F[n] \rightarrow G[n]$ denotes the natural transformation $N(p)^* f : N(p)^* F \rightarrow N(p)^* G$ in $\operatorname{Fun}^b(N(\Delta[n]), \mathcal{M})$. In particular, $F = F[0]$ for any F in $\operatorname{Fun}^b(N(\Delta[0]), \mathcal{M})$. Note that $F[n] = N(\alpha)^* F[m]$ for any map $\alpha : \Delta[n] \rightarrow \Delta[m]$ and any functor F in $\operatorname{Fun}^b(N(\Delta[0]), \mathcal{M})$.

We now recall the construction of mapping spaces in a model category from [4].

Step 1. Let $\alpha: \Delta[n] \rightarrow \Delta[m]$ be a map. For objects F and G in $\text{Cons}(N(\Delta[0]), \mathcal{M})$, define a function of sets $\text{map}(F, G)_\alpha: \text{map}(F, G)_m \rightarrow \text{map}(F, G)_n$ by the formula

$$\begin{array}{ccc} \text{map}(F, G)_m & \xrightarrow{\text{map}(F, G)_\alpha} & \text{map}(F, G)_n \\ \text{Nat}(F[m], G[m]) & \xrightarrow{N(\alpha)^*} & \text{Nat}(F[n], G[n]). \end{array}$$

In this way we obtain a simplicial set $\text{map}(F, G)$.

Step 2. Let $f: E \rightarrow F$ and $g: G \rightarrow H$ be morphisms in $\text{Cons}(N(\Delta[0]), \mathcal{M})$. Define a map of simplicial sets $\text{map}(f, g): \text{map}(F, G) \rightarrow \text{map}(E, H)$ by the formula

$$\begin{array}{ccc} \text{map}(F, G)_n & \xrightarrow{\text{map}(f, g)_n} & \text{map}(E, H)_n \\ \parallel & \xrightarrow{\text{Nat}(f[n], g[n])} & \parallel \\ \text{Nat}(F[n], G[n]) & \xrightarrow{\text{Nat}(f[n], g[n])} & \text{Nat}_{\mathcal{M}}(E[n], H[n]). \end{array}$$

In this way we have constructed a functor:

$$\text{map}(-, -): \text{Cons}(N(\Delta[0]), \mathcal{M})^{\text{op}} \times \text{Cons}(N(\Delta[0]), \mathcal{M}) \rightarrow \text{Spaces}.$$

Step 3. For any F, G , and H in $\text{Cons}(N(\Delta[0]), \mathcal{M})$, define the composition $\circ: \text{map}(F, G) \times \text{map}(G, H) \rightarrow \text{map}(F, H)$ to be in each degree the composition of natural transformations:

$$\begin{array}{ccc} \text{map}(F, G)_n \times \text{map}(G, H)_n & = \text{Nat}(F[n], G[n]) \times \text{Nat}(G[n], H[n]) \ni (\phi, \psi) \\ \downarrow \circ_n & & \downarrow \\ \text{map}(F, H)_n & = \text{Nat}(F[n], H[n]) \ni \psi\phi. \end{array}$$

Step 4. For any F in $\text{Cons}(N(\Delta[0]), \mathcal{M})$, define $e_F: \Delta[0] \rightarrow \text{map}(F, F)$ to be given by the 0-dimensional simplex $\text{id}_{F[0]}$ in $\text{map}(F, F)_0 = \text{Nat}(F[0], F[0])$.

Proposition 11.2. *Let E, F, G , and H be objects in $\text{Cons}(N(\Delta[0]), \mathcal{M})$.*

(1) *The following diagrams commute (\circ is associative and has a unit):*

$$\begin{array}{ccc} \text{map}(E, F) \times \text{map}(F, G) \times \text{map}(G, H) & \xrightarrow{\circ \times \text{id}} & \text{map}(E, G) \times \text{map}(G, H) \\ \text{id} \times \circ \downarrow & & \downarrow \circ \\ \text{map}(E, F) \times \text{map}(F, H) & \xrightarrow{\circ} & \text{map}(E, H) \end{array}$$

$$\begin{array}{ccc} & \xrightarrow{\text{pr}} & \\ \text{map}(E, F) \times \Delta[0] & \xrightarrow{\text{id} \times e_F} \text{map}(E, F) \times \text{map}(F, F) & \xrightarrow{\circ} \text{map}(E, F) \end{array}$$

$$\begin{array}{ccc} \Delta[0] \times \text{map}(E, F) & \xrightarrow{e_E \times \text{id}} \text{map}(E, E) \times \text{map}(E, F) & \xrightarrow{\circ} \text{map}(E, F) \\ & \xrightarrow{\text{pr}} & \end{array}$$

- (2) $\text{map}(E, F)$ is Kan.
- (3) If $f: G \rightarrow E$ and $g: F \rightarrow H$ are weak equivalences, then the map of spaces $\text{map}(f, g): \text{map}(E, F) \rightarrow \text{map}(G, H)$ is a weak equivalence.
- (4) Let σ be a simplex in $N(\Delta[0])$. The assignment mapping a 0-dimensional simplex $f: F = F[0] \rightarrow G[0] = G$ in $\text{map}(F, G)$ to the morphism in $\text{Ho}(\mathcal{M})$ represented by $f_\sigma: F(\sigma) \rightarrow G(\sigma)$ is a bijection between $\pi_0 \text{map}(F, G)$ and $[F(\sigma), G(\sigma)]$. This bijection is natural in F and G and takes the composition \circ to the composition of morphisms in $\text{Ho}(\mathcal{M})$.

Proof. (1) follows from analogous properties of the composition of natural transformations. (2) is a particular case of [4, Proposition 8.5] and (3) of [4, Corollary 8.4]. Finally, (4) is a consequence of [4, Corollary 8.6 and Proposition 9.2(2)]. \square

Let us choose a functorial factorization P in $\text{Fun}^b(N(\Delta[0]), \mathcal{M})$ as defined in Section 2.2 and a functorial fibrant replacement R in \mathcal{M} . By applying R object-wise we obtain a functorial fibrant replacement in $\text{Fun}^b(N(\Delta[0]), \mathcal{M})$. For any bounded functor $F: N(\Delta[0]) \rightarrow \mathcal{M}$ consider the factorization of $\emptyset \rightarrow RF$ given by P :

$$\begin{array}{ccc} \emptyset & \hookrightarrow & PRF \\ \downarrow & & \downarrow \simeq \\ F & \xrightarrow{\simeq} & RF \end{array}$$

Define QF to be PRF . It is a functorial cofibrant-fibrant replacement in the category $\text{Fun}^b(N(\Delta[0]), \mathcal{M})$. The natural weak equivalences $F \xrightarrow{\simeq} RF \xleftarrow{\simeq} PRF = QF$ are called the **standard comparison morphisms** between QF and F .

For an object X in \mathcal{M} , the same symbol $X: N(\Delta[0]) \rightarrow \mathcal{M}$ is used to denote the constant functor with value X . Note that QX is an object in $\text{Cons}(N(\Delta[0]), \mathcal{M})$. In this way we obtain a functor $Q: \mathcal{M} \rightarrow \text{Cons}(N(\Delta[0]), \mathcal{M})$. Its composition with the mapping space defined above is denoted by $\text{map}(-, -): \mathcal{M}^{\text{op}} \times \mathcal{M} \rightarrow \text{Spaces}$.

12. THE SPACES OF WEAK EQUIVALENCES

In Section 11 we described a construction of mapping spaces. Here we discuss the spaces of weak equivalences. Let F and G be objects in $\text{Cons}(N(\Delta[0]), \mathcal{M})$ and σ be a simplex in $N(\Delta[0])$. The set of components $\pi_0 \text{map}(F, G)$ is in bijection with the set of morphisms $[F(\sigma), G(\sigma)]$ in $\text{Ho}(\mathcal{M})$. The component of $\text{map}(F, G)$ corresponding to α in $[F(\sigma), G(\sigma)]$ is denoted by $\text{map}(F, G)_\alpha$. Define

$$\text{we}(F, G) := \coprod_{\alpha \in [F(\sigma), G(\sigma)] \text{ is an isomorphism}} \text{map}(F, G)_\alpha.$$

Since $N(\Delta[0])$ is connected and F and G are weakly equivalent to constant functors, $\text{we}(F, G)$ does not depend on the choice of the simplex σ .

The spaces of weak equivalences $\text{we}(F, G)$ do not form a subfactor in $\text{map}(-, -)$, since in general $\text{map}(f, \text{id}): \text{map}(F, G) \rightarrow \text{map}(E, G)$ does not take the component corresponding to an isomorphism in $[F(\sigma), G(\sigma)]$ to a component corresponding to an isomorphism in $[E(\sigma), G(\sigma)]$. However, if both $f: E \rightarrow F$ and $g: G \rightarrow H$ are weak equivalences, then $\text{map}(f, g)$ takes $\text{we}(F, G)$ into $\text{we}(E, H)$, and in this case we use the symbol $\text{we}(f, g): \text{we}(F, G) \rightarrow \text{we}(E, H)$ to denote the induced map. Thus $\text{we}(f, g)$ is defined only if f and g are weak equivalences.

Let F and G be objects in $\text{Cons}(N(\Delta[0]), \mathcal{M})$. Define $\text{Natwe}(F[n], G[n])$ to be the subset of $\text{map}(F, G)_n = \text{Nat}(F[n], G[n])$ which consists of the natural transformations $f: F[n] \rightarrow G[n]$ in $\text{Fun}^b(N(\Delta[n]), \mathcal{M})$ which are weak equivalences.

Proposition 12.1. $\text{we}(F, G)_n = \text{Natwe}(F[n], G[n])$.

Proof. Let f be an element in $\text{map}(F, G)_n = \text{Nat}(F[n], G[n])$. As $F[n]$ and $G[n]$ are weakly equivalent to constant functors, f is a weak equivalence if and only if for some τ in $N(\Delta[n])$, $f_\tau: F[n](\tau) \rightarrow G[n](\tau)$ is a weak equivalence. Choose a map $v: \Delta[0] \rightarrow \Delta[n]$ and a simplex σ in $N(\Delta[0])$. Let $\tau = N(v)(\sigma)$. Note that $F[n](\tau) = F(\sigma)$ and $G[n](\tau) = G(\sigma)$. According to Proposition 11.2(4) f is an n -dimensional simplex in $\text{map}(F, G)$ that belongs to the component determined

by the morphism in $[F(\sigma), G(\sigma)]$ represented by $f_\tau: F[n](\tau) = F(\sigma) \rightarrow G(\sigma) = G[n](\tau)$. This morphism is an isomorphism in $\text{Ho}(\mathcal{M})$ if and only if f_τ is a weak equivalence. \square

For example, since $\text{id}: F[0] \rightarrow F[0]$ belongs to $\text{Natwe}(F[0], F[0])$, the map $e_F: \Delta[0] \rightarrow \text{map}(F, F)$ factors as a composition of a map which we denote by the same symbol $e_F: \Delta[0] \rightarrow \text{we}(F, F)$ and the inclusion $\text{we}(F, F) \subset \text{map}(F, F)$.

Compositions of weak equivalences are weak equivalences, and hence the restriction of the composition operation \circ fits into a commutative diagram:

$$\begin{array}{ccc} \text{we}(F, G) \times \text{we}(G, H) & \xrightarrow{\circ} & \text{we}(F, H) \\ \downarrow & & \downarrow \\ \text{map}(F, G) \times \text{map}(G, H) & \xrightarrow{\circ} & \text{map}(F, H) \end{array}$$

This together with Proposition 11.2 gives:

Corollary 12.2. *Let $E, F, G,$ and H be objects in $\text{Cons}(N(\Delta[0]), \mathcal{M})$.*

(1) *The following diagrams commute (\circ is associative and has a unit):*

$$\begin{array}{ccc} \text{we}(E, F) \times \text{we}(F, G) \times \text{we}(G, H) & \xrightarrow{\circ \times \text{id}} & \text{we}(E, G) \times \text{we}(G, H) \\ \text{id} \times \circ \downarrow & & \downarrow \circ \\ \text{we}(E, F) \times \text{we}(F, H) & \xrightarrow{\circ} & \text{we}(E, H) \end{array}$$

$$\begin{array}{ccc} & \xrightarrow{\text{pr}} & \\ \text{we}(E, F) \times \Delta[0] & \xrightarrow[\text{id} \times e_F]{} & \text{we}(E, F) \times \text{we}(F, F) \xrightarrow{\circ} \text{we}(E, F) \\ & \xrightarrow{\text{pr}} & \end{array}$$

$$\begin{array}{ccc} \Delta[0] \times \text{we}(E, F) & \xrightarrow{e_E \times \text{id}} & \text{we}(E, E) \times \text{we}(E, F) \xrightarrow{\circ} \text{we}(E, F) \\ & \xrightarrow{\text{pr}} & \end{array}$$

(2) *$\text{we}(F, G)$ is Kan.*

(3) *If $f: G \rightarrow E$ and $g: F \rightarrow H$ are weak equivalences, then so is the map of spaces $\text{we}(f, g): \text{we}(E, F) \rightarrow \text{we}(G, H)$.*

We finish this section with:

Proposition 12.3. *Let $F, G,$ and H be objects in $\text{Cons}(N(\Delta[0]), \mathcal{M})$. Then the following squares are homotopy pull-backs:*

$$\begin{array}{ccc} \text{we}(F, G) \times \text{we}(G, H) & \xrightarrow{\circ} & \text{we}(F, H) \\ \text{pr} \downarrow & & \downarrow \\ \text{we}(F, G) & \longrightarrow & \Delta[0] \end{array} \qquad \begin{array}{ccc} \text{we}(F, G) \times \text{we}(G, H) & \xrightarrow{\circ} & \text{we}(F, H) \\ \text{pr} \downarrow & & \downarrow \\ \text{we}(G, H) & \longrightarrow & \Delta[0] \end{array}$$

Proof. As the arguments are analogous we present only the proof that the left square is a homotopy pull-back. The case when one of the spaces $\text{we}(F, G)$, or $\text{we}(G, H)$, or $\text{we}(F, H)$ is empty is clear. Assume then that all these spaces are not empty. We need to show that \circ induces a weak equivalence between the homotopy fibers of pr and $\text{we}(F, H)$. Choose a vertex in $\text{we}(F, G)$ represented by a weak equivalence $f: F[0] \rightarrow G[0]$. The homotopy fiber of the map pr over the component of f is given by $\text{we}(G, H)$. The restriction of the composition \circ to this homotopy fiber is given by $\text{we}(f, \text{id})$, which is a weak equivalence by Corollary 12.2(3). \square

Let X and Y be objects in \mathcal{M} . Define $\text{we}(X, Y)$ as $\text{we}(QX, QY)$, where Q is the functorial cofibrant-fibrant replacement in $\text{Fun}^b(N(\Delta[0]), \mathcal{M})$ (see Section 11).

13. DELOOPINGS OF THE SPACES OF WEAK EQUIVALENCES

In this section we describe the standard delooping of the spaces of weak equivalences with the monoid structure given by the composition. The proof of the main result and needed techniques are placed in the appendix (Section 19).

Notation 13.1. Let X be an object in \mathcal{M} . The symbol $\text{Cons}(N(\Delta[0]), X_{\text{we}})$ denotes the full subcategory of $\text{Fun}^b(N(\Delta[0]), X_{\text{we}})$ whose objects are functors whose composition with the inclusion $X_{\text{we}} \subset \mathcal{M}$ is cofibrant, fibrant, and weakly equivalent to the constant functor X in $\text{Fun}^b(N(\Delta[0]), \mathcal{M})$ (compare with Notation 11.1). The set of morphisms between two such functors F and G is given by $\text{Natwe}(F, G)$.

Let \mathcal{S} be a collection of objects in $\text{Cons}(N(\Delta[0]), X_{\text{we}})$. Define \mathcal{S}_n to be the full subcategory of $\text{Fun}^b(N(\Delta[n]), X_{\text{we}})$ of objects of the form $F[n]$ where F is in \mathcal{S} (see Notation 11.1). The set of morphisms between $F[n]$ and $G[n]$ in \mathcal{S}_n is given by $\text{Natwe}(F[n], G[n])$. For any $\alpha: \Delta[n] \rightarrow \Delta[m]$, $F[n] = N(\alpha)^*F[m]$. The restriction of $N(\alpha)^*: \text{Fun}^b(N(\Delta[m]), X_{\text{we}}) \rightarrow \text{Fun}^b(N(\Delta[n]), X_{\text{we}})$ therefore induces a functor $N(\alpha)^*: \mathcal{S}_m \rightarrow \mathcal{S}_n$ which we denote by $\mathcal{S}_\alpha: \mathcal{S}_m \rightarrow \mathcal{S}_n$. In this way we obtain a system of categories \mathcal{S}_- indexed by Δ .

Assume \mathcal{S} is a **set** of objects in $\text{Cons}(N(\Delta[0]), X_{\text{we}})$. For $n \geq 0$, the category \mathcal{S}_n is then small. By taking the nerves we obtain a functor $N(\mathcal{S}_-): \Delta^{\text{op}} \rightarrow \text{Spaces}$.

Definition 13.2. $Bwe(\mathcal{S})$ is defined to be the diagonal of $N(\mathcal{S}_-)$.

By definition, the set of 0-dimensional simplices $Bwe(\mathcal{S})_0$ is given by the set of objects in \mathcal{S}_0 which is the set \mathcal{S} . For $n > 0$, the set of n -dimensional simplices $Bwe(\mathcal{S})_n$ is the set of n -composable morphisms in \mathcal{S}_n :

$$Bwe(\mathcal{S})_n = \coprod_{(X_n, \dots, X_0) \in \mathcal{S}^{n+1}} \prod_{k=n}^{k=1} \text{Natwe}(X_k[n], X_{k-1}[n]).$$

If $\mathcal{S}' \subset \mathcal{S}$, then \mathcal{S}'_n is a full subcategory of \mathcal{S}_n . We call the induced map $N(\mathcal{S}'_-) \subset N(\mathcal{S}_-)$ the standard inclusion. The following is a consequence of Proposition 19.2:

Proposition 13.3. *Let $\mathcal{S}' \subset \mathcal{S}$ be non-empty sets of objects in $\text{Cons}(N(\Delta[0]), X_{\text{we}})$.*

- (1) $Bwe(\mathcal{S})$ is a connected space.
- (2) The loop space $\Omega Bwe(\mathcal{S})$ is weakly equivalent to $\text{we}(X, X)$.
- (3) The map $Bwe(\mathcal{S}') \rightarrow Bwe(\mathcal{S})$, induced by the standard inclusion, is a weak equivalence.

Part III. In this part we show that $\text{Fun}(I, X_{\text{we}})$ is essentially small and the nerve of its core is weakly equivalent to $\text{map}(N(I), Bwe(X, X))$. This will be applied to prove Theorem A.

14. THE CATEGORY OF WEAK EQUIVALENCES

Theorem 14.1. *Let X be an object in \mathcal{M} . The category X_{we} is essentially small and the nerve of its core is weakly equivalent to $Bwe(X, X)$.*

In [7] Dwyer and Kan used the name “the special classification complex of X ” (denoted by scX) for the nerve of a core of X_{we} . The above theorem is a version of their [7, Proposition 2.3] in the case \mathcal{M} is simplicial.

Proof. Let \mathcal{T} be the collection of all the objects in $\text{Cons}(N(\Delta[0]), X_{\text{we}})$. Consider the system \mathcal{T}_- indexed by Δ (see Notation 13.1). The objects in its Grothendieck construction $\text{Gr}_\Delta \mathcal{T}_-$ are given by functors $F[n]$ where $n \geq 0$ and F is an object in $\text{Cons}(N(\Delta[0]), X_{\text{we}})$. A morphism in $\text{Gr}_\Delta \mathcal{T}_-$ between two such functors $F[n]$ and $G[m]$ is a pair $(\alpha: \Delta[n] \rightarrow \Delta[m], \phi: F[n] \rightarrow G[n])$ where α is a map and ϕ is a natural weak equivalence. Define a functor $\Psi: \text{Gr}_\Delta \mathcal{T}_- \rightarrow \mathcal{M}$ as follows:

$$\begin{array}{ccc} \Psi(F[n]) & \xrightarrow{\Psi(\alpha, \phi)} & \Psi(G[m]) \\ \text{ii} & & \text{ii} \\ \text{colim}_{N(\Delta[n])} F[n] & \xrightarrow{\text{colim}(\phi)} \text{colim}_{N(\Delta[n])} G[n] & \xrightarrow{\text{colim}_{N(\alpha)}} \text{colim}_{N(\Delta[m])} G[m] \end{array}$$

Since $\Delta[n]$ is contractible and $F[n]$ is cofibrant, by Proposition 10.2, Ψ has values in X_{we} . We claim the induced functor $\Psi: \text{Gr}_\Delta \mathcal{T}_- \rightarrow X_{\text{we}}$ is a homotopy equivalence. To prove the claim choose a functorial factorization P in $\text{Fun}^b(N(\Delta[0]), \mathcal{M})$ as defined in Section 2.2 and a functorial fibrant replacement R in \mathcal{M} . By applying P to morphisms of the form $\emptyset \rightarrow F$ we obtain a functorial cofibrant replacement in $\text{Fun}^b(N(\Delta[0]), \mathcal{M})$ (see Section 2.2). By applying R object-wise we obtain a functorial fibrant replacement in $\text{Fun}^b(N(\Delta[0]), \mathcal{M})$. Set $QF = PRF$ (see the end of Section 11). Let us denote by the same symbol $Q: X_{\text{we}} \rightarrow \text{Cons}(N(\Delta[0]), X_{\text{we}})$ the restriction of Q to the constant functors. Define $\Phi: X_{\text{we}} \rightarrow \text{Gr}_\Delta \mathcal{T}_-$ to be the following composition:

$$X_{\text{we}} \xrightarrow{Q} \text{Cons}(N(\Delta[0]), X_{\text{we}}) = \mathcal{T}_0 \xrightarrow{\text{standard inclusion}} \text{Gr}_\Delta \mathcal{T}_-$$

Φ

We will show that $\Psi\Phi$ and $\Phi\Psi$ are homotopic to the identity functors. The composition $\Psi\Phi: X_{\text{we}} \rightarrow X_{\text{we}}$ assigns to Y the object $\text{colim}_{N(\Delta[0])} QY$. Think about the simplex $\sigma: \Delta[0] \rightarrow \Delta[0]$ as a vertex in $N(\Delta[0])$. According to Proposition 10.2 the morphism $QY(\sigma) \rightarrow \text{colim}_{N(\Delta[0])} QY$, induced by the inclusion of σ in $N(\Delta[0])$, is a weak equivalence. Consequently the following morphisms form a “zig-zag” of natural weak equivalences between $\text{id}_{X_{\text{we}}}$ and $\Psi\Phi$:

$$\underbrace{Y \rightarrow RY \leftarrow QY(\sigma)}_{\text{standard comparison}} \rightarrow \text{colim}_{N(\Delta[0])} QY = \Psi\Phi(Y).$$

The composition $\Phi\Psi: \text{Gr}_\Delta \mathcal{T}_- \rightarrow \text{Gr}_\Delta \mathcal{T}_-$ assigns to $F[n]: N(\Delta[n]) \rightarrow X_{\text{we}}$ the functor $Q(\text{colim}_{N(\Delta[n])} F[n]): N(\Delta[0]) \rightarrow X_{\text{we}}$. Consider the left Kan extension $N(p)^k(F[n]): N(\Delta[0]) \rightarrow \mathcal{M}$. Denote by $N(p)^k(F[n]) \rightarrow \text{colim}_{N(\Delta[n])} F[n]$ the canonical natural transformation into the constant functor $\text{colim}_{N(\Delta[n])} F[n]$. We denote the composition of this natural transformation with the fibrant replacement $\text{colim}_{N(\Delta[n])} F[n] \rightarrow R(\text{colim}_{N(\Delta[n])} F[n])$ by π . Finally, take the following factorizations in $\text{Fun}^b(N(\Delta[0]), \mathcal{M})$ given in Section 2.2:

$$\begin{array}{ccccc} \emptyset & \hookrightarrow & PR(\text{colim}_{N(\Delta[n])} F[n]) & \xrightarrow{\simeq} & R(\text{colim}_{N(\Delta[n])} F[n]) \\ \downarrow & & \downarrow \simeq & & \parallel \\ N(p)^k(F[n]) & \hookrightarrow & P(\pi) & \xrightarrow{\simeq} & R(\text{colim}_{N(\Delta[n])} F[n]) \\ & & \searrow \pi & & \end{array}$$

Observe that $N(p)^k(F[n])$ is cofibrant in $\text{Fun}^b(N(\Delta[0]), \mathcal{M})$ (see Proposition 8.3(2)). Therefore, the functors $P(\pi)$ and $PR(\text{colim}_{N(\Delta[n])} F[n])$ are objects in

$\text{Cons}(N(\Delta[0]), X_{\text{we}})$. Next consider the following sequence of morphisms in $\text{Gr}_\Delta \mathcal{T}_-$:

$$F[n] \xrightarrow{(p, \phi)} P(\pi) \leftarrow PR(\text{colim}_{N(\Delta[n])} F[n]) = Q(\text{colim}_{N(\Delta[n])} F[n]) = \Phi\Psi(F[n]),$$

where $p: \Delta[n] \rightarrow \Delta[0]$ is the unique map, $\phi: F[n] \rightarrow P(\pi)[n]$ is adjoint to $N(p)^k(F[n]) \hookrightarrow P(\pi)$, and $P(\pi) \leftarrow PR(\text{colim}_{N(\Delta[n])} F[n])$ is the vertical morphism in the above commutative diagram. These morphisms give a “zig-zag” of natural weak equivalences between $\text{id}_{\text{Gr}_\Delta \mathcal{T}_-}$ and $\Phi\Psi$.

Since X_{we} and $\text{Gr}_\Delta \mathcal{T}_-$ are homotopy equivalent, one of them is essentially small if and only if the other one is (see Corollary 5.9). Consider the one element set $\{QX\}$. We claim that $\text{Gr}_\Delta\{QX\}_- \subset \text{Gr}_\Delta \mathcal{T}_-$ is a core. Let $J \subset \text{Gr}_\Delta \mathcal{T}_-$ be a small subcategory containing $\text{Gr}_\Delta\{QX\}_-$ and \mathcal{S} be the set of all F in $\text{Cons}(N(\Delta[0]), X_{\text{we}})$ such that, for some $n \geq 0$, $F[n]$ is in J . We have a sequence of inclusions $\text{Gr}_\Delta\{QX\}_- \subset J \subset \text{Gr}_\Delta \mathcal{S}$. Its composition is a weak equivalence by Proposition 13.3(3), which shows the claim.

By Thomason’s theorem as shown in Proposition 4.1(1) and the fact that a homotopy colimit of a simplicial space is weakly equivalent to its diagonal, we get that the following spaces are weakly equivalent to each other: the nerve $N(\text{Gr}_\Delta\{QX\}_-)$, the homotopy colimit $\text{hocolim}_{\Delta^{\text{op}}} N(\{QX\}_-)$ and the diagonal $B\text{we}(X, X)$ of $N(\{QX\}_-)$ (see Definition 13.2). \square

15. COFINALITY

Proposition 15.1. *Let $f: I \rightarrow J$ be a functor between small categories. Assume that, for any object j in J , the over category $f \downarrow j$ is contractible. Then the functor $f^*: \text{Fun}(J, X_{\text{we}}) \rightarrow \text{Fun}(I, X_{\text{we}})$ is a homotopy equivalence.*

Proof. Let P be a cofibrant replacement in $\text{Fun}(I, \mathcal{M})$ (see Section 9). Recall that the derived left Kan extension $f^k P: \text{Fun}(I, \mathcal{M}) \rightarrow \text{Fun}(J, \mathcal{M})$ maps the subcategory $\text{Fun}(I, X_{\text{we}}) \subset \text{Fun}(I, \mathcal{M})$ into $\text{Fun}(J, X_{\text{we}})$ (see Section 10). We claim that the functor $f^k P: \text{Fun}(I, X_{\text{we}}) \rightarrow \text{Fun}(J, X_{\text{we}})$ is a homotopy inverse to f^* .

Let $F: J \rightarrow X_{\text{we}}$ be a functor, $Pf^*F \rightarrow f^*F$ the cofibrant replacement, and $\phi_F: f^k Pf^*F \rightarrow F$ its adjoint. Choose an object $a = (i, \alpha: f(i) \rightarrow j)$ in $f \downarrow j$ and consider the following commutative diagram:

$$\begin{array}{ccc} (Pf^*F)(i) & \xrightarrow{\cong} & F(f(i)) \xrightarrow{F(\alpha)} F(j) \\ \downarrow & & \nearrow (\phi_F)_j \\ \text{colim}_{f \downarrow j} Pf^*F & = & (f^k Pf^*F)(j) \end{array}$$

where the left vertical morphism is induced by the inclusion of the object a in $f \downarrow j$. According to Proposition 10.1 this morphism is a weak equivalence and hence so is $(\phi_F)_j$. The morphisms $\{\phi_F: f^k Pf^*F \rightarrow F\}_F$ therefore form a natural transformation between $f^k Pf^*: \text{Fun}(J, X_{\text{we}}) \rightarrow \text{Fun}(J, X_{\text{we}})$ and the identity functor.

By the same argument the morphism $\psi_F: PF \rightarrow f^* f^k PF$, which is adjoint to $\text{id}_{f^k PF}$, is also a natural weak equivalence. Thus the natural transformations $F \leftarrow PF$ and ψ_F form a “zig-zag” connecting $f^* f^k P$ with $\text{id}_{\text{Fun}(I, X_{\text{we}})}$. \square

Using the above proposition and Corollary 5.9 we can extend Theorem 14.1 to:

Corollary 15.2. *If I is a small contractible category, then $\text{Fun}(I, X_{\text{we}})$ is essentially small and the nerve of its core is weakly equivalent to $B\text{we}(X, X)$.*

The cofinality result of Proposition 15.1 can also be used to translate between categories of functors indexed by arbitrary small categories and simplex categories:

Corollary 15.3. *Let I be a small category and $\epsilon: N(I) \rightarrow I$ be the functor defined in Section 8. Then both functors $\epsilon^*: \text{Fun}(I, X_{\text{we}}) \rightarrow \text{Fun}(N(I), X_{\text{we}})$ and $\epsilon^*: \text{Fun}(I, X_{\text{we}}) \rightarrow \text{Fun}^b(N(I), X_{\text{we}})$ are homotopy equivalences.*

Proof. The statement basically follows from Proposition 15.1 and the fact that the categories $\epsilon \downarrow i$ are contractible. According to the proof of Proposition 15.1, the homotopy inverse to the first functor in the statement of the corollary is given by $\epsilon^k P: \text{Fun}(N(I), X_{\text{we}}) \rightarrow \text{Fun}(I, X_{\text{we}})$. Its restriction to $\text{Fun}^b(N(I), X_{\text{we}})$ is a homotopy inverse to the second functor. \square

The following special cases of Proposition 15.1 are of particular interest to us:

- Corollary 15.4.**
- (1) *If $df(\sigma) = \lim(\Delta[n] \xrightarrow{\sigma} B \xleftarrow{f} A)$ is contractible for any σ in B , then $f^*: \text{Fun}(B, X_{\text{we}}) \rightarrow \text{Fun}(A, X_{\text{we}})$ is a homotopy equivalence.*
 - (2) *Let K be a contractible simplicial set and $\text{pr}: B \times K \rightarrow B$ be the projection. Then $\text{pr}^*: \text{Fun}(B, X_{\text{we}}) \rightarrow \text{Fun}(B \times K, X_{\text{we}})$ is a homotopy equivalence.*
 - (3) *Let $A_0 \subset A_1 \subset A_2 \subset \dots \subset A$ be an increasing sequence of subspaces of A such that $A = \bigcup_{i \geq 0} A_i$. Define $f: \text{Tel} \rightarrow A$ to be the following map:*

$$\begin{array}{ccc} \text{Tel} & := & \text{colim} \left(\coprod_{i \geq 0} A_i \times \Delta[1] \xleftarrow{g_0 \amalg g_1} (\coprod_{i \geq 0} A_i) \amalg (\coprod_{i \geq 0} A_i) \xrightarrow{\text{id} \amalg \text{id}} \coprod_{i \geq 0} A_i \right) \\ f \downarrow & & \amalg_{i \geq 0} \text{pr} \downarrow \qquad \qquad \qquad \parallel \qquad \qquad \qquad \parallel \\ A & = & \text{colim} \left(\coprod_{i \geq 0} A_i \longleftarrow (\coprod_{i \geq 0} A_i) \amalg (\coprod_{i \geq 0} A_i) \xrightarrow{\text{id} \amalg \text{id}} \coprod_{i \geq 0} A_i \right) \end{array}$$

where on each component:

- g_0 is the inclusion $A_i = A_i \times \Delta[0] \xrightarrow{\text{id} \times d_0} A_i \times \Delta[1]$;
- g_1 is the inclusion $A_i \subset A_{i+1} = A_{i+1} \times \Delta[0] \xrightarrow{\text{id} \times d_1} A_{i+1} \times \Delta[1]$.

Then $f^*: \text{Fun}(A, X_{\text{we}}) \rightarrow \text{Fun}(\text{Tel}, X_{\text{we}})$ is a homotopy equivalence.

16. CLUTCHING

Recall that $\text{Cof}(A, X_{\text{we}})$ denotes the full subcategory of $\text{Fun}^b(A, X_{\text{we}})$ whose objects are bounded functors $F: A \rightarrow X_{\text{we}}$ whose composition with $X_{\text{we}} \subset \mathcal{M}$ is cofibrant in $\text{Fun}^b(A, \mathcal{M})$ (see Section 8).

Proposition 16.1. $\text{Cof}(A, X_{\text{we}}) \subset \text{Fun}^b(A, X_{\text{we}})$ is a homotopy equivalence.

Proof. A homotopy inverse is given by a cofibrant replacement. \square

Recall that if $f: A \rightarrow B$ is reduced (see Section 7), then $f^*: \text{Fun}^b(B, X_{\text{we}}) \rightarrow \text{Fun}^b(A, X_{\text{we}})$ maps the subcategory $\text{Cof}(B, X_{\text{we}})$ into $\text{Cof}(A, X_{\text{we}})$.

Proposition 16.2. *If $f: A \rightarrow B$ is reduced, then $f^*: \text{Cof}(B, X_{\text{we}}) \rightarrow \text{Cof}(A, X_{\text{we}})$ is a strong fibration (see Section 3).*

Proof. We need to show that $\psi \uparrow f^*: F_0 \uparrow f^* \rightarrow F_1 \uparrow f^*$ is a homotopy equivalence for any $\psi: F_1 \rightarrow F_0$ in $\text{Cof}(A, X_{\text{we}})$. Let (G, α_1) be an object in $F_1 \uparrow f^*$. It consists of a functor $G: B \rightarrow X_{\text{we}}$ in $\text{Cof}(B, X_{\text{we}})$ and a natural weak equivalence

$\alpha_1: F_1 \rightarrow f^*G$. Consider the following commutative diagram in $\text{Fun}^b(B, \mathcal{M})$:

$$\begin{array}{ccc}
 f^k F_1 & \xrightarrow{\quad} & P(\widetilde{\alpha}_1) \xrightarrow{\simeq} G \\
 f^k \psi \downarrow \simeq & & \downarrow \simeq \\
 f^k F_0 & \xrightarrow{\widetilde{\alpha}_0} & \Phi(G, \alpha_1)
 \end{array}$$

$\xrightarrow{\widetilde{\alpha}_1}$ (top arrow) $\xrightarrow{\simeq}$ (right arrow) $\xrightarrow{\simeq}$ (bottom arrow) $\xrightarrow{\widetilde{\alpha}_0}$ (left arrow)

where $\widetilde{\alpha}_1: f^k F_1 \rightarrow G$ is adjoint to α_1 , $f^k F_1 \hookrightarrow P(\widetilde{\alpha}_1) \xrightarrow{\simeq} G$. Since ψ is a weak equivalence between cofibrant objects, its left Kan extension $f^k \psi$ is also a weak equivalence between cofibrant objects (see Proposition 8.3). Thus the natural transformation $P(\widetilde{\alpha}_1) \rightarrow \Phi(G, \alpha_1)$ is a weak equivalence and $\widetilde{\alpha}_0$ is a cofibration. The functor $\Phi(G, \alpha_1)$ is therefore cofibrant and has values in X_{we} . Let $\alpha_0: F_0 \rightarrow f^* \Phi(G, \alpha_1)$ be adjoint to $\widetilde{\alpha}_0$. In this way, out of an object (G, α_1) in $F_1 \uparrow f^*$ we have constructed an object $(\Phi(G, \alpha_1), \alpha_0)$ in $F_0 \uparrow f^*$. This whole procedure is functorial. The induced functor is denoted by $\Phi: F_1 \uparrow f^* \rightarrow F_0 \uparrow f^*$. We claim that Φ is a homotopy inverse to $\psi \uparrow f^*: F_0 \uparrow f^* \rightarrow F_1 \uparrow f^*$. The weak equivalences $\Phi(G, \alpha_1) \xleftarrow{\simeq} P(\widetilde{\alpha}_1) \xrightarrow{\simeq} G$ give a “zig-zag” of natural transformations between the composition $(\psi \uparrow f^*)\Phi$ and $\text{id}_{F_1 \uparrow f^*}$. Let $(G: B \rightarrow X_{\text{we}}, \lambda: F_0 \rightarrow f^*G)$ be an object in $F_0 \uparrow f^*$ and consider the following commutative diagram in $\text{Fun}^b(C, \mathcal{M})$:

$$\begin{array}{ccc}
 f^k F_1 & \xrightarrow{\quad} & P(\widetilde{\lambda} f^k \psi) \xrightarrow{\simeq} G \\
 f^k \psi \downarrow \simeq & & \downarrow \simeq \\
 f^k F_0 & \xrightarrow{\quad} & \Phi(G, \widetilde{\lambda} f^k \psi) \xrightarrow{\simeq} G
 \end{array}$$

$\xrightarrow{\widetilde{\lambda} f^k \psi}$ (top arrow) $\xrightarrow{\simeq}$ (right arrow) $\xrightarrow{\simeq}$ (bottom arrow) $\xrightarrow{\widetilde{\lambda}}$ (left arrow)

where $\Phi(G, \widetilde{\lambda} f^k \psi) \xrightarrow{\simeq} G$ is given by the universal property of a push-out. These weak equivalences form a natural transformation between $\Phi(\psi \uparrow f^*)$ and $\text{id}_{F_0 \uparrow f^*}$. \square

Proposition 16.3. *Assume that the square on the left below is a push-out with i an inclusion and f reduced (see Section 7). Then g is also reduced and the square on the right is a strong homotopy pull-back (see Section 3):*

$$\begin{array}{ccc}
 A \xrightarrow{i} C & & \text{Cof}(D, X_{\text{we}}) \xrightarrow{j^*} \text{Cof}(B, X_{\text{we}}) \\
 f \downarrow \quad \downarrow g & & g^* \downarrow \quad \downarrow f^* \\
 B \xrightarrow{j} D & & \text{Cof}(C, X_{\text{we}}) \xrightarrow{i^*} \text{Cof}(A, X_{\text{we}})
 \end{array}$$

Proof. The proof of the fact that g is reduced is left for the reader. By definition, the claimed square is a strong homotopy pull-back if two requirements are satisfied: i^* is a strong fibration and, for any object F in $\text{Cof}(B, X_{\text{we}})$, the functor $(g^*, f^*): F \uparrow j^* \rightarrow f^* F \uparrow i^*$ is a homotopy equivalence (see Section 3). The first requirement is the content of Proposition 16.2. It remains to prove the second one. A short argument for (g^*, f^*) being a homotopy equivalence is that the clutching construction is its homotopy inverse. Here is a more detailed explanation of why this is so. An object (G, ψ) in $f^* F \uparrow i^*$ consists of a functor G in $\text{Cof}(C, X_{\text{we}})$ and a natural weak equivalence $\psi: f^* F \rightarrow i^* G$. This is an example of a clutching data (see Section 7.1). Its clutching is a functor $H(\psi, F, G): D \rightarrow \mathcal{M}$ and a natural transformation $\overline{\psi}: g^* H(\psi, F, G) \rightarrow G$. The functor $H(\psi, F, G)$ is bounded (see Proposition 8.1) and $\overline{\psi}$ is a weak equivalence. However, $H(\psi, F, G)$ may not be cofibrant in $\text{Fun}^b(D, \mathcal{M})$. Let $\alpha: j^k F \rightarrow H(\psi, F, G)$ be the adjoint to

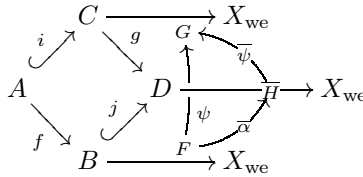
the identity functor $F = j^*H(\psi, F, G)$. Define $\overline{H}(\psi, F, G)$ to be the functor that fits into the following functorial factorization in $\text{Fun}^b(D, \mathcal{M})$ (see Section 2.2):

$$j^k F \begin{array}{c} \hookrightarrow \overline{H}(\psi, F, G) \xrightarrow{\simeq} H(\psi, F, G) \\ \searrow \alpha \nearrow \end{array}$$

By applying g^* to the acyclic fibration on the right and composing it with $\overline{\psi}$, we get a natural transformation denoted by the same symbol $\overline{\psi}: g^*\overline{H}(\psi, F, G) \rightarrow G$:

$$g^*\overline{H}(\psi, F, G) \begin{array}{c} \xrightarrow{\simeq} g^*H(\psi, F, G) \xrightarrow{\overline{\psi}} G \\ \searrow \overline{\psi} \nearrow \end{array}$$

Let $\overline{\alpha}: F \rightarrow j^*\overline{H}(\psi, F, G)$ be the adjoint to $j^k F \hookrightarrow \overline{H}(\psi, F, G)$. Since F is cofibrant, then so are $j^k F$ and $\overline{H}(\psi, F, G)$. Thus $\overline{H}(\psi, F, G)$ is an object in $\text{Cof}(D, X_{\text{we}})$. This data can be arranged into a commutative diagram:



Out of an object (G, ψ) in $f^*F \uparrow i^*$, we have constructed an object $(\overline{H}(\psi, F, G), \overline{\alpha})$ in $F \uparrow j^*$. All the steps in this construction are functorial. The obtained functor is denoted by $\Phi: f^*F \uparrow i^* \rightarrow F \uparrow j^*$. We claim that $(g^*, f^*)\Phi$ and $\Phi(g^*, f^*)$ are homotopic to the identity functors. The morphisms $\overline{\psi}: g^*\overline{H}(\Psi, F, G) \rightarrow G$ in $f^*F \uparrow i^*$ form a natural transformation between $(g^*, f^*)\Phi$ and $\text{id}_{f^*F \uparrow i^*}$.

Let (G, ψ) be an object in $F \uparrow j^*$ consisting of a functor G in $\text{Cof}(D, X_{\text{we}})$ and a weak equivalence $\psi: F \rightarrow j^*G$. The composition of $\widehat{\psi}: H(f^*\psi, F, g^*G) \rightarrow G$ given in Proposition 7.2 with the cofibrant replacement $\overline{H}(f^*\psi, F, g^*G) \rightarrow H(f^*\psi, F, g^*G)$ is a natural transformation between $\Phi(g^*, f^*)$ and $\text{id}_{F \uparrow j^*}$. \square

17. THE CATEGORY OF FUNCTORS WITH VALUES IN X_{we}

Theorem 17.1. *Let I be a small category. Then $\text{Fun}(I, X_{\text{we}})$ is essentially small and the nerve of its core is weakly equivalent to $\text{map}(N(I), B\text{we}(X, X))$.*

Our strategy to prove Theorem 17.1 is to show that the nerve of a core of $\text{Fun}(I, X_{\text{we}})$ is weakly equivalent to the homotopy limit of the constant functor indexed by I with value $B\text{we}(X, X)$. For this strategy to work we need to choose these cores in a certain functorial way with respect to I . We set this functoriality first.

Let $\phi: I \rightarrow \text{Spaces}$ be a functor. For any morphism $\lambda: i \rightarrow j$ in I and any commutative diagram on the left below we have the following commutative diagram of functor categories on the right:

$$\begin{array}{ccc} \Delta[n] \xrightarrow{\alpha} \Delta[m] & & \text{Fun}(\Delta[n], X_{\text{we}}) \xleftarrow{\alpha^*} \text{Fun}(\Delta[m], X_{\text{we}}) \\ \sigma \downarrow & & \sigma^* \uparrow \\ \phi(i) \xrightarrow{\phi(\lambda)} \phi(j) & & \text{Fun}(\phi(i), X_{\text{we}}) \xleftarrow{\phi(\lambda)^*} \text{Fun}(\phi(j), X_{\text{we}}) \\ p \downarrow & & p^* \uparrow \\ \Delta[0] \xlongequal{\quad} \Delta[0] & & \text{Fun}(\Delta[0], X_{\text{we}}) \xlongequal{\quad} \text{Fun}(\Delta[0], X_{\text{we}}) \end{array}$$

The symbol $\mathcal{F}(\phi, X)$ denotes the system of categories given by all the functors in the right diagram above for all the morphisms λ in I . To describe this system we use the following notation. Let i be an object in I and $\sigma: \Delta[n] \rightarrow \phi(i)$ a simplex:

$$\begin{array}{ccccc}
 & & \mathcal{F}(\phi, X)_p & & \\
 & \swarrow & & \searrow & \\
 \mathcal{F}(\phi, X)_{e_{\phi(i)}} & \xrightarrow{\mathcal{F}(\phi, X)_p} & \mathcal{F}(\phi, X)_{t_{\phi(i)}} & \xrightarrow{\mathcal{F}(\phi, X)_\pi} & \mathcal{F}(\phi, X)_\sigma \\
 \parallel & & \parallel & & \parallel \\
 \text{Fun}(\Delta[0], X_{\text{we}}) & \xrightarrow{p^*} & \text{Fun}(\phi(i), X_{\text{we}}) & \xrightarrow{\sigma^*} & \text{Fun}(\Delta[n], X_{\text{we}}) \\
 & \searrow & & \swarrow & \\
 & & p^* & &
 \end{array}$$

For example let A be a space and $A: [0] \rightarrow \text{Spaces}$ be the constant functor with value A . The corresponding system of categories $\mathcal{F}(A, X)$ is given by the following functors indexed by morphisms $\alpha: \sigma \rightarrow \tau$ in A :

$$\begin{array}{ccccc}
 & & \mathcal{F}_p & & \\
 & \swarrow & & \searrow & \\
 \mathcal{F}(A, X)_{e_A} & \xrightarrow{\mathcal{F}_p} & \mathcal{F}(A, X)_{t_A} & \xrightarrow{\mathcal{F}_\pi} & \mathcal{F}(A, X)_\tau & \xrightarrow{\mathcal{F}_\alpha} & \mathcal{F}(A, X)_\sigma \\
 \parallel & & \parallel & & \parallel & & \parallel \\
 \text{Fun}(\Delta[0], X_{\text{we}}) & \xrightarrow{p^*} & \text{Fun}(A, X_{\text{we}}) & \xrightarrow{\tau^*} & \text{Fun}(\Delta[m], X_{\text{we}}) & \xrightarrow{\alpha^*} & \text{Fun}(\Delta[n], X_{\text{we}}) \\
 & \searrow & & \swarrow & & \searrow & \\
 & & p^* & & & \sigma^* &
 \end{array}$$

Proposition 17.2. *The system $\mathcal{F}(A, X)$ is essentially small and, for any of its cores $F \subset \mathcal{F}(A, X)$, the maps $N(F_{t_A}) \rightarrow \text{holim}_{\sigma \in A} N(F_\sigma) \leftarrow \text{holim}_{\sigma \in A} N(F_{e_A})$, induced by $F_\pi: F_{t_A} \rightarrow F_\sigma$ and $F_\sigma \leftarrow F_{e_A}: F_p$, are weak equivalences.*

Proof. If K is a contractible space, then $\text{Fun}(K, X_{\text{we}})$ is essentially small (see Corollary 15.2). Thus to prove that $\mathcal{F}(A, X)$ is an essentially small system, we need to show that $\text{Fun}(A, X_{\text{we}})$ is an essentially small category (see Proposition 5.6(1)).

Let \mathcal{T} be the collection of all spaces for which the proposition is true. To show that any space belongs to \mathcal{T} , we prove that \mathcal{T} satisfies the following properties:

- (1) If in the following push square A, B , and C belong to \mathcal{T} , then so does D :

$$\begin{array}{ccc}
 A & \xrightarrow{i} & C \\
 f \downarrow & & \downarrow g \\
 B & \xrightarrow{j} & D
 \end{array}$$

- (2) Let $f: A \rightarrow B$ be a map such that $df(\sigma) = \lim(\Delta[n] \xrightarrow{\sigma} B \xleftarrow{f} A)$ is contractible for any σ in B . Then A belongs to \mathcal{T} if and only if B does.
- (3) If A is contractible, then A belongs to \mathcal{T} .
- (4) Let S be a set. If $A_s \in \mathcal{T}$ for any $s \in S$, then $\coprod_{s \in S} A_s \in \mathcal{T}$.
- (5) Let $A_0 \subset A_1 \subset \dots \subset A$ be a filtration of A such that $A = \bigcup_{i \geq 0} A_i$. If $A_i \in \mathcal{T}$ for any i , then $A \in \mathcal{T}$.

In each of the following steps we prove the corresponding property.

Step (1). By Corollary 15.3 and Proposition 16.1 the functors $\text{Cof}(N(I), X_{\text{we}}) \subset \text{Fun}^b(N(I), X_{\text{we}})$ and $\epsilon^*: \text{Fun}(I, X_{\text{we}}) \rightarrow \text{Fun}^b(N(I), X_{\text{we}})$ are homotopy equivalences. Thus if one of those categories is essentially small, then so is any other (see Corollary 5.9) in which case these functors are weak equivalences (see Proposition 6.2).

Since the following square on the left is a push-out with all the maps being reduced, according to Proposition 16.3 we have a strong homotopy pull-back on the right:

$$\begin{array}{ccc}
 N(A) \xrightarrow{N(i)} N(C) & & \text{Cof}(N(D), X_{\text{we}}) \xrightarrow{N(j)^*} \text{Cof}(N(B), X_{\text{we}}) \\
 N(f) \downarrow & & N(g)^* \downarrow \\
 N(B) \xrightarrow{N(j)} N(D) & & \text{Cof}(N(C), X_{\text{we}}) \xrightarrow{N(i)^*} \text{Cof}(N(A), X_{\text{we}}) \\
 & & \downarrow N(f)^*
 \end{array}$$

By the assumption $\text{Fun}(A, X_{\text{we}})$, $\text{Fun}(B, X_{\text{we}})$, and $\text{Fun}(C, X_{\text{we}})$ are essentially small, and hence so are $\text{Cof}(N(A), X_{\text{we}})$, $\text{Cof}(N(B), X_{\text{we}})$, and $\text{Cof}(N(C), X_{\text{we}})$. We can use Corollary 6.6 to conclude that $\text{Cof}(N(D), X_{\text{we}})$ is also essentially small. The right square above is thus a homotopy pull-back and consequently so is

$$\begin{array}{ccc}
 \text{Fun}(D, X_{\text{we}}) & \xrightarrow{j^*} & \text{Fun}(B, X_{\text{we}}) \\
 g^* \downarrow & & \downarrow f^* \\
 \text{Fun}(C, X_{\text{we}}) & \xrightarrow{i^*} & \text{Fun}(A, X_{\text{we}})
 \end{array}$$

Let ϕ be the functor indexed by the poset category of all subsets of $\{0, 1\}$ given by the commutative square in statement (1). The system $\mathcal{F}(\phi, X_{\text{we}})$ is essentially small as its values are so (see Proposition 5.6(1)). Let F be its core. If K is among $\{A, B, C\}$, then the maps $N(F_{t_K}) \rightarrow \text{holim}_{\sigma \in K} N(F_\sigma) \leftarrow \text{holim}_{\sigma \in K} N(F_{e_K})$ are weak equivalences. These maps fit into a commutative diagram:

$$\begin{array}{ccccc}
 N(F_{t_D}) & \rightarrow & \text{holim}_{\sigma \in D} N(F_\sigma) & \longleftarrow & \text{holim}_{\sigma \in D} N(F_{e_D}) \\
 \downarrow & \searrow & \downarrow & \searrow & \downarrow \\
 N(F_{t_B}) & \longrightarrow & \text{holim}_{\sigma \in B} N(F_\sigma) & \longleftarrow & \text{holim}_{\sigma \in B} N(F_{e_B}) \\
 \downarrow & \searrow & \downarrow & \searrow & \downarrow \\
 N(F_{t_C}) & \longrightarrow & \text{holim}_{\sigma \in C} N(F_\sigma) & \longleftarrow & \text{holim}_{\sigma \in C} N(F_{e_C}) \\
 \downarrow & \searrow & \downarrow & \searrow & \downarrow \\
 N(F_{t_A}) & \longrightarrow & \text{holim}_{\sigma \in A} N(F_\sigma) & \longleftarrow & \text{holim}_{\sigma \in A} N(F_{e_A})
 \end{array}$$

The right side, the left side, and the middle squares are homotopy pull-backs. The first one is because the functors involved are constant, the second because of the previous discussion, and the middle because the right horizontal maps are weak equivalences. By the inductive assumption the horizontal left bottom and front maps are weak equivalences. It follows that so is the fourth one, proving Step (1).

Step (2). Let $\phi: [1] \rightarrow \text{Spaces}$ be a functor given by the map $f: A \rightarrow B$. Consider the system $\mathcal{F}(\phi, X)$. By Corollary 15.4(1), $f^*: \text{Fun}(B, X_{\text{we}}) \rightarrow \text{Fun}(A, X_{\text{we}})$ is a homotopy equivalence. Thus if one of these categories is essentially small, then $\mathcal{F}(\phi, X)$ is essentially small and it has a core F . Consider the following commutative diagram induced by the functors in such a core:

$$\begin{array}{ccccc}
 N(F_{t_B}) & \longrightarrow & \text{holim}_{\sigma \in B} N(F_\sigma) & \longleftarrow & \text{holim}_{\sigma \in B} N(F_{e_B}) \\
 \downarrow & & \downarrow & & \downarrow \\
 N(F_{t_A}) & \longrightarrow & \text{holim}_{\sigma \in A} N(F_\sigma) & \longleftarrow & \text{holim}_{\sigma \in A} N(F_{e_A})
 \end{array}$$

The right vertical map is a weak equivalence since the functors involved are constant. The right horizontal maps are weak equivalences as the homotopy limit preserves weak equivalences. It follows that so is the middle vertical map. The left vertical map is a weak equivalence by Corollary 15.4(1). We can then conclude

that if one of the left horizontal maps is a weak equivalence, then so is the other. This means that A belongs to \mathcal{T} if and only if B does, which proves Step (2).

Step (3). According to Corollary 15.2 the category $\text{Fun}(A, X_{\text{we}})$ is essentially small, and consequently $\mathcal{F}(A, X)$ is an essentially small system (see Proposition 5.6(1)). Let $F \subset \mathcal{F}(A, X)$ be its core and consider the following commutative diagram:

$$\begin{array}{ccc} N(F_{e_A}) & \xrightarrow{a} & \text{holim}_{\sigma \in A} N(F_{e_A}) \\ N(F_p) \downarrow & & \downarrow \\ N(F_{t_A}) & \longrightarrow & \text{holim}_{\sigma \in A} N(F_{\sigma}) \end{array}$$

where a is the diagonal map from $N(F_{e_A})$ to the homotopy limit of the constant functor $N(F_{e_A})$ and the rest of the maps are induced by functors in the system F . Since A is contractible, the vertical maps and a are weak equivalences (see Corollary 15.2). Thus so is the bottom map, and A belongs to \mathcal{T} .

Step (4). A consequence of the fact that products preserve weak equivalences.

Step (5). The map $f: \text{Tel} \rightarrow A$ (see Corollary 15.4(3)) satisfies the requirements in Step (2). Thus it suffices to prove that Tel belongs to \mathcal{T} . Again by Step (2) the products $A_i \times \Delta[1]$ belong to \mathcal{T} , and so by Step (4) the components of the push-out describing Tel also belong to \mathcal{T} . We can now use Step (1). □

Proof of Theorem 17.1. Apply Proposition 17.2 to $\mathcal{F}(N(I), X)$ to see that the nerve of a core of $\mathcal{F}(N(I), X)_{t_{N(I)}} = \text{Fun}(N(I), X_{\text{we}})$ is weakly equivalent to the homotopy limit of the constant functor $\text{holim}_{\sigma \in N(I)} N(F_{e_{N(I)}}) \simeq \text{map}(N(I), N(F_{e_{N(I)}}))$. By Corollary 15.2 the nerve of $F_{e_{N(I)}}$, which is a core of

$$\mathcal{F}(N(I), X)_{e_{N(I)}} = \text{Fun}(\Delta[0], X_{\text{we}}),$$

is weakly equivalent to $B\text{we}(X, X)$. To finish the proof recall that according to Corollary 15.3, the functor $\epsilon^*: \text{Fun}(I, X_{\text{we}}) \rightarrow \text{Fun}(N(I), X_{\text{we}})$ is a homotopy equivalence. □

As a direct consequence of Proposition 17.2 we also get:

Corollary 17.3. *If $f: I \rightarrow J$ is a weak equivalence of small categories, then $f^*: \text{Fun}(J, X_{\text{we}}) \rightarrow \text{Fun}(I, X_{\text{we}})$ is a weak equivalence.*

18. THEOREM A

The aim of this final section is to prove Theorem A from the introduction.

Definition 18.1. Let X and F be spaces. $\text{Ext}(X, F)$ is a subcategory of $\text{Fib}(X, F)$ whose objects are maps $f: E \rightarrow X$ with a fixed codomain X . The set of morphisms in $\text{Ext}(X, F)$ between $f: E \rightarrow X$ and $f': E' \rightarrow X$ consists of weak equivalences $g: E \rightarrow E'$ for which $f'g = f$.

To prove Theorem A we are going to use the following constructions.

18.2. Fibrant replacement. In addition to the factorization given in Section 2.2, the category Spaces has the following factorization: any commutative square on the

left below can be extended **functorially** to a commutative diagram on the right with the indicated morphisms being acyclic cofibrations and fibrations:

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 \alpha_1 \downarrow & & \downarrow \alpha_2 \\
 X' & \xrightarrow{f'} & Y'
 \end{array}
 \qquad
 \begin{array}{ccc}
 X & \begin{array}{c} \xrightarrow{\gamma_f} \\ \simeq \\ \xrightarrow{\gamma_{f'}} \end{array} & R(f) \xrightarrow{\mu_f} Y \\
 \alpha_1 \downarrow & & \downarrow R(\alpha_1, \alpha_2) \\
 X' & \begin{array}{c} \xrightarrow{\gamma_{f'}} \\ \simeq \\ \xrightarrow{\gamma_{f'}} \end{array} & R(f') \xrightarrow{\mu_{f'}} Y'
 \end{array}$$

The commutativity of the right diagram means that the morphisms $\gamma_f: X \rightarrow R(f)$ and $\mu_f: R(f) \rightarrow Y$ form natural transformations. Define $\mu: \text{Ext}(X, F) \rightarrow \text{Ext}(X, F)$ to be the functor assigning to an object $f: E \rightarrow X$ the fibration $\mu_f: R(f) \rightarrow X$ and to a morphism $g: E \rightarrow E'$, between $f: E \rightarrow X$ and $f': E' \rightarrow X$ the map $R(f', \text{id}_X): R(f) \rightarrow R(f')$.

18.3. Decomposition ([3, Section 9]). Let $f: E \rightarrow X$ be a map. Define $df: X \rightarrow \text{Spaces}$ to be the functor that assigns to a morphism $\alpha: \Delta[n] \rightarrow \Delta[m]$ in X between $\sigma: \Delta[n] \rightarrow X$ and $\tau: \Delta[m] \rightarrow X$ the map

$$\begin{array}{ccc}
 df(\sigma) & := & \lim (\Delta[n] \xrightarrow{\sigma} X \xleftarrow{f} E) \\
 df(\alpha) \downarrow & & \alpha \downarrow \quad \parallel \quad \parallel \\
 df(\tau) & := & \lim (\Delta[m] \xrightarrow{\tau} X \xleftarrow{f} E)
 \end{array}$$

$d(\text{id}_X): X \rightarrow \text{Spaces}$ is also denoted by Δ_X . Its value on $\sigma: \Delta[n] \rightarrow X$ is the space $\Delta[n]$. For any such σ , define $df_\sigma: df(\sigma) \rightarrow \Delta_X(\sigma)$ to be the map that fits into the pull-back square on the left below. These maps form a natural transformation that we denote by $df: df \rightarrow \Delta_X$. The horizontal maps in this square satisfy the universal property of the colimit inducing **canonical** isomorphisms on the right:

$$\begin{array}{ccc}
 df(\sigma) & \longrightarrow & E \\
 df_\sigma \downarrow & & \downarrow f \\
 \Delta_X(\sigma) & \xlongequal{\quad} & \Delta[n] \xrightarrow{\sigma} X
 \end{array}
 \qquad
 \begin{array}{ccc}
 \text{colim}_X df & \xrightarrow{\cong} & E \\
 \text{colim}_{\sigma \in X} df_\sigma \downarrow & & \downarrow f \\
 \text{colim}_X \Delta_X & \xrightarrow{\cong} & X
 \end{array}$$

For any morphism $\psi: f \rightarrow f'$ in $\text{Spaces} \downarrow X$, define $d\psi: df \rightarrow df'$ as

$$\begin{array}{ccc}
 df(\sigma) & := & \lim (\Delta[n] \xrightarrow{\sigma} X \xleftarrow{f} E) \\
 d\psi_\sigma \downarrow & & \parallel \quad \parallel \quad \downarrow \psi \\
 df'(\sigma) & := & \lim (\Delta[n] \xrightarrow{\sigma} X \xleftarrow{f'} E')
 \end{array}$$

In this way we obtain a functor $d: \text{Spaces} \downarrow X \rightarrow \text{Fun}(X, \text{Spaces}) \downarrow \Delta_X$ which we call the **decomposition**. Its composition with the fibrant replacement is called the **derived decomposition** and is denoted by ∂ :

$$\text{Ext}(X, F) \xrightarrow{\mu} \text{Ext}(X, F) \xrightarrow{d} \text{Fun}(X, F_{\text{we}}) \downarrow \Delta_X$$

$\xrightarrow{\quad \partial \quad}$

18.4. Assembly. Let $\text{colim}: \text{Fun}(X, \text{Spaces}) \downarrow \Delta_X \rightarrow \text{Spaces} \downarrow X$ be the functor whose value on an object $g: G \rightarrow \Delta_X$ is the composition of the colimit $\text{colim}_X g$ with the canonical isomorphism $\text{colim}_X \Delta_X \cong X$. It maps a morphism $\psi: g \rightarrow g'$ to $\text{colim}_X \psi$. Let us choose a cofibrant replacement P in $\text{Fun}(X, \text{Spaces})$. For an

object $g: G \rightarrow \Delta_X$ in $\text{Fun}(X, F) \downarrow \Delta_X$, take the cofibrant replacement $PG \rightarrow G$ and define $\int g$ to be the composition

$$\text{hocolim}_X G \cong \text{colim}_X PG \longrightarrow \text{colim}_X G \xrightarrow{\text{colim}_X g} \text{colim}_X \Delta_X \xrightarrow{\cong} X$$

$\xrightarrow{\int g}$

Note that if g belongs to $\text{Fun}(X, F_{\text{we}}) \downarrow \Delta_X$, then by Proposition 4.1, the homotopy fiber of $\int g$ is weakly equivalent to F , and hence $\int g$ is an object in $\text{Ext}(X, F)$. We call this composition of P and the colimit the **assembly functor** and denote it by $\int: \text{Fun}(X, F_{\text{we}}) \downarrow \Delta_X \rightarrow \text{Ext}(X, F)$.

Proposition 18.5. $\partial: \text{Ext}(X, F) \rightarrow \text{Fun}(X, F_{\text{we}}) \downarrow \Delta_X$ is a homotopy equivalence.

Proof. We are going to show that the assembly functor is a homotopy inverse to ∂ . For an object $f: E \rightarrow X$ in $\text{Ext}(X, F)$ form a commutative diagram:

$$\begin{array}{ccccc} \text{colim}_X P\partial f & \longrightarrow & \text{colim}_X \partial f = \text{colim}_X d\mu_f & \xrightarrow{\cong} & R(f) & \xleftarrow{\gamma_f} & E \\ \downarrow & & \searrow \int \partial f & & \mu_f \downarrow & \swarrow f & \\ \text{colim}_X P\Delta_X & \longrightarrow & \text{colim}_X \Delta_X & \xrightarrow{\cong} & X & & \end{array}$$

The top horizontal maps form a “zig-zag” connecting $\int \partial$ with the identity functor.

Similarly, for an object $g: G \rightarrow \Delta_X$ in $\text{Fun}(X, F_{\text{we}}) \downarrow \Delta_X$ and a simplex $\sigma: \Delta[n] \rightarrow X$, consider the following commutative diagram where the maps $G(\sigma) \rightarrow \text{colim}_X G$ and $PG(\sigma) \rightarrow \text{colim}_X PG$ are induced by the inclusion of σ into X , and $PG(\sigma) \rightarrow (d\mu_{\int g})(\sigma)$ is induced by the universal property of a pull-back:

$$\begin{array}{ccccc} & & G(\sigma) & \longrightarrow & \text{colim}_X G \\ & & \uparrow & & \downarrow \text{colim}_X g \\ & PG(\sigma) & \longrightarrow & \text{colim}_X PG & \downarrow f g \\ & \uparrow & & \downarrow \mu_{\int g} & \swarrow \cong \\ (d\mu_{\int g})(\sigma) & \longrightarrow & R(\int g) & \longrightarrow & X \\ & \downarrow & \downarrow & & \downarrow \sigma \\ & \Delta[n] & \xrightarrow{\sigma} & X & \end{array}$$

$\xrightarrow{\int g}$

The maps $(d\mu_{\int g})(\sigma) \leftarrow PG(\sigma) \rightarrow G(\sigma)$ form natural transformations $d\mu_{\int g} \leftarrow PG \rightarrow G$ connecting $\partial \int$ and the identity functor. □

We can now prove our first classification theorem for fibrations of spaces:

Theorem 18.6. *The category $\text{Ext}(X, F)$ is essentially small and the nerve of its core is weakly equivalent to $\text{map}(X, \text{Bwe}(F, F))$.*

Proof. According to Proposition 18.5 and Theorem 17.1, we need to show that $\text{Fun}(X, F_{\text{we}}) \downarrow \Delta_X$ is homotopy equivalent to $\text{Fun}(X, F_{\text{we}})$. Let $\Phi: \text{Fun}(X, F_{\text{we}}) \downarrow \Delta_X \rightarrow \text{Fun}(X, F_{\text{we}})$ be the functor that forgets the augmentation. Its right adjoint $-\times \Delta_X: \text{Fun}(X, F_{\text{we}}) \rightarrow \text{Fun}(X, F_{\text{we}}) \downarrow \Delta_X$ assigns to $G: X \rightarrow F_{\text{we}}$ the projection $\text{pr}: G \times \Delta_X \rightarrow \Delta_X$. This adjointness implies that $-\times \Delta_X$ is a homotopy inverse of Φ . □

To understand $\text{Fib}(X, F)$ we will use the range functor $\mathcal{R}: \text{Fib}(X, F) \rightarrow X_{\text{we}}$ that assigns to a morphism (ϕ, ψ) in $\text{Fib}(X, F)$ (see Definition 1.1) the map ψ .

Proposition 18.7. $\mathcal{R}: \text{Fib}(X, F) \rightarrow X_{\text{we}}$ is a quasifibration (see Definition 6.1).

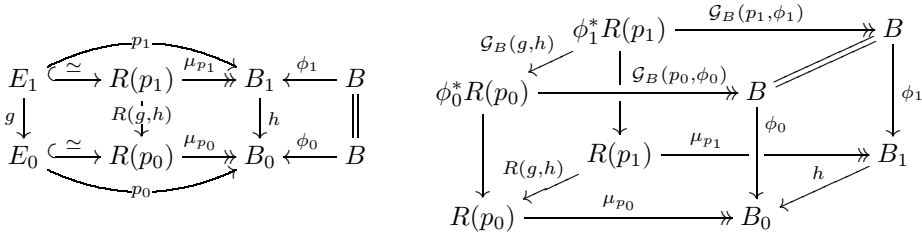
Proof. Let B be in X_{we} . Our first goal is to show that $B \uparrow \mathcal{R}$ is homotopy equivalent to $\text{Ext}(B, F)$, which would prove for example that $B \uparrow \mathcal{R}$ is essentially small (see Corollary 5.9 and Theorem 18.6). An object in $B \uparrow \mathcal{R}$ is a pair of maps $p: E \rightarrow B'$ and $B' \leftarrow B: \phi$ where ϕ is a weak equivalence and p is in $\text{Fib}(X, F)$. A morphism between such objects is a pair of weak equivalences (g, h) making the following diagram commutative:

$$(*) \quad \begin{array}{ccccc} E_1 & \xrightarrow{p_1} & B_1 & \xleftarrow{\phi_1} & B \\ g \downarrow & & \downarrow h & & \parallel \\ E_0 & \xrightarrow{p_0} & B_0 & \xleftarrow{\phi_0} & B \end{array}$$

Define $\mathcal{F}_B: \text{Ext}(B, F) \rightarrow B \uparrow \mathcal{R}$ to be the functor given by the assignment

$$\begin{array}{ccc} E_1 \xrightarrow{p_1} B & & E_1 \xrightarrow{p_1} B = B \\ g \downarrow \quad \parallel & \xrightarrow{\mathcal{F}_B} & g \downarrow \quad \parallel \quad \parallel \\ E_0 \xrightarrow{p_0} B & & E_0 \xrightarrow{p_0} B = B \end{array}$$

Let $\mathcal{G}_B: B \uparrow \mathcal{R} \rightarrow \text{Ext}(B, F)$ be the functor that maps the morphism $(*)$ in $B \uparrow \mathcal{R}$ to the morphism in $\text{Ext}(B, F)$ described as follows. Use the fibrant replacement R to extend $(*)$ to a diagram on the left below (see Section 18.2) and set $\mathcal{G}_B(g, h): \mathcal{G}_B(p_1, \phi_1) \rightarrow \mathcal{G}_B(p_0, \phi_0)$ to be the unique morphism in $\text{Ext}(B, F)$ that fits into the following commutative cube on the right where the front and back squares are pull-backs:



For any object $E \xrightarrow{p} B' \xleftarrow{\phi} B$ in $B \uparrow \mathcal{R}$ the following morphisms give a “zig-zag” of natural transformations between $\mathcal{F}_B \mathcal{G}_B$ and the identity functor:

$$\begin{array}{ccc} \mathcal{F}_B \mathcal{G}_B(p, \phi) & \phi^* R(p) \xrightarrow{\mathcal{G}_B(p, \phi)} B = B & \\ \downarrow & \downarrow & \downarrow \phi \parallel \\ (\mu_p, \phi) & R(p) \xrightarrow{\mu_p} B' \xleftarrow{\phi} B & \\ \uparrow & \simeq \uparrow & \parallel \\ (p, \phi) & E_0 \xrightarrow{p} B' \xleftarrow{\phi} B & \parallel \end{array}$$

Instead of showing directly that the other composition $\mathcal{G}_B \mathcal{F}_B$ is homotopic to the identity, we prove slightly more. Let $f: C \rightarrow B$ be a morphism in X_{we} and $f_*: \text{Ext}(C, F) \rightarrow \text{Ext}(B, F)$ be the composition with f functor:

$$\begin{array}{ccc} E_1 \xrightarrow{p_1} C & & E_1 \xrightarrow{fp_1} B \\ g \downarrow \quad \parallel & \xrightarrow{f_*} & g \downarrow \quad \parallel \\ E_0 \xrightarrow{p_0} C & & E_0 \xrightarrow{fp_0} B \end{array}$$

We claim that $\text{Ext}(B, F) \xrightarrow{\mathcal{F}_B} B \uparrow \mathcal{R} \xrightarrow{f \uparrow \mathcal{R}} C \uparrow \mathcal{R} \xrightarrow{\mathcal{G}_C} \text{Ext}(C, F)$ is a homotopy inverse to f_* . The effect of $f_* \mathcal{G}_C(f \uparrow \mathcal{R}) \mathcal{F}_B$ on an object $p: E \rightarrow B$ in $\text{Ext}(B, F)$ can be understood through the following commutative diagram

$$\begin{array}{ccccc}
 E & \xrightarrow{\cong} & R(p) & \xleftarrow{\quad} & f^* R(p) \\
 & \searrow p & \downarrow \mu_p & \swarrow f_* \mathcal{G}_C(f \uparrow \mathcal{R}) \mathcal{F}_B(p) & \downarrow \mathcal{G}_C(p, f) = \mathcal{G}_C(f \uparrow \mathcal{R}) \mathcal{F}_B(p) \\
 & & B & \xleftarrow{f} & C
 \end{array}$$

The maps $E \hookrightarrow R(p) \leftarrow f^* R(p)$ give a “zig-zag” of natural transformations between $f_* \mathcal{G}_C(f \uparrow \mathcal{R}) \mathcal{F}_B$ and $\text{id}_{\text{Ext}(B, F)}$. Conversely, by applying $\mathcal{G}_C(f \uparrow \mathcal{R}) \mathcal{F}_B f_*$ to an object $q: E \rightarrow C$ in $\text{Ext}(C, F)$, we get a commutative diagram:

$$\begin{array}{ccccc}
 & & & \xrightarrow{\quad} & \\
 & & & \searrow & \\
 & & & E & \xrightarrow{\cong} & R(fq) \\
 & & & \downarrow q & \searrow fq & \downarrow \mu_{fq} \\
 & & & C & \xrightarrow{f} & B \\
 & & & \swarrow \mathcal{G}_C(f \uparrow \mathcal{R}) \mathcal{F}_B f_*(q) = \mathcal{G}_C(fq, f) & &
 \end{array}$$

where $E \rightarrow f^* R(fq)$ is the unique map into the pull-back given by the commutativity of the inner square. These maps give a natural transformation between $\mathcal{G}_C(f \uparrow \mathcal{R}) \mathcal{F}_B f_*$ and $\text{id}_{\text{Ext}(C, F)}$. For example if $f = \text{id}_B$, then we see that $\mathcal{G}_B \mathcal{F}_B$ is homotopic to $\text{id}_{\text{Ext}(B, F)}$. Thus according to Theorem 18.6 and Proposition 6.2, $f \uparrow \mathcal{R}: B \uparrow \mathcal{R} \rightarrow C \uparrow \mathcal{R}$ is a weak equivalence of essentially small categories, which shows the proposition. \square

With Theorem 18.6 and Proposition 18.7 we have the tools necessary to prove:

Proof of Theorem A. Consider the range functor $\mathcal{R}: \text{Fib}(X, F) \rightarrow X_{\text{we}}$. The categories $\text{Fib}(X, F)$ and $\text{Gr}_{X_{\text{we}}}(- \uparrow \mathcal{R})$ are homotopy equivalent (see Section 3). By Theorem 14.1 X_{we} is essentially small and by Proposition 18.7 the system $- \uparrow \mathcal{R}$ satisfies the requirement of Lemma 6.3. Thus $\text{Gr}_{X_{\text{we}}}(- \uparrow \mathcal{R})$ is essentially small and there is a core Ξ of X_{we} and a core Ψ of the restriction of $- \uparrow \mathcal{R}$ to Ξ for which $\text{Gr}_{\Xi} \Psi$ is a core of $\text{Gr}_{X_{\text{we}}}(- \uparrow \mathcal{R})$. If B is an object in Ξ , by Thomason’s theorem as shown in Proposition 4.1(3), the homotopy fiber of the nerve of the projection $\text{Gr}_{\Xi} \Psi \rightarrow \Xi$ over the component containing B is weakly equivalent to the nerve of $\Psi(B)$. This homotopy fiber is then weakly equivalent to the mapping space $\text{map}(B, B_{\text{we}}(F, F))$ (see the proof of Proposition 18.7 and Theorem 18.6).

The projection $\text{Gr}_{\Xi} \Psi \rightarrow \Xi$ clearly has a section. The theorem now follows, since the nerve of Ξ is weakly equivalent to $B_{\text{we}}(B, B) \cong B_{\text{we}}(X, X)$ (see Theorem 14.1). \square

By applying the long exact sequence of homotopy groups to the map in Theorem A and using the fact that this map has a section, we get the classical classification of fibrations of Stasheff [19]:

Corollary 18.8. *There is a bijection between components of $\text{Fib}(X, F)$ and the set of homotopy classes $[X, B_{\text{we}}(F, F)]$.*

19. APPENDIX: DELOOPING OF HOMOTOPY GROUPOIDS

In this appendix we recall the standard delooping machinery and prove Proposition 13.3.

Definition 19.1. Let \mathcal{S} be a set. A homotopy groupoid indexed by \mathcal{S} consists of:

- fibrant-cofibrant spaces $G(r, t)$ indexed by pairs (r, t) of elements in \mathcal{S} ;
- maps $\circ: G(r, s) \times G(s, t) \rightarrow G(r, t)$ indexed by triples (r, s, t) in \mathcal{S} ;
- maps $e_r: \Delta[0] \rightarrow G(r, r)$ indexed by elements r in \mathcal{S} .

These sequences are required to satisfy the following properties:

- (1) (associativity) for any r, s, t, v in \mathcal{S} the following diagram commutes:

$$\begin{array}{ccc} G(r, s) \times G(s, t) \times G(t, v) & \xrightarrow{\circ \times \text{id}} & G(r, t) \times G(t, v) \\ \text{id} \times \circ \downarrow & & \downarrow \circ \\ G(r, s) \times G(s, v) & \xrightarrow{\circ} & G(r, v) \end{array}$$

- (2) (identity) for any r, s in \mathcal{S} the following diagrams commute:

$$\begin{array}{ccc} \Delta[0] \times G(r, s) & \xrightarrow{e_r \times \text{id}} & G(r, r) \times G(r, s) & G(r, s) \times G(s, s) & \xleftarrow{\text{id} \times e_s} & G(r, s) \times \Delta[0] \\ & \searrow \text{pr} & \downarrow \circ & \downarrow \circ & \swarrow \text{pr} & \\ & & G(r, s) & G(r, s) & & \end{array}$$

- (3) for any r, s, t in \mathcal{S} the following squares are homotopy pull-backs:

$$\begin{array}{ccc} G(r, s) \times G(s, t) & \xrightarrow{\circ} & G(r, t) & G(r, s) \times G(s, t) & \xrightarrow{\circ} & G(r, t) \\ \text{pr} \downarrow & & \downarrow & \text{pr} \downarrow & & \downarrow \\ G(r, s) & \longrightarrow & \Delta[0] & G(s, t) & \longrightarrow & \Delta[0] \end{array}$$

A homotopy groupoid is simply a small category enriched over Spaces with an additional assumption given by requirement (3). A homotopy groupoid G indexed by \mathcal{S} is also denoted by $G_{\mathcal{S}}$. If all the spaces $G(s, t)$ are non-empty, G is called **connected**. In this case requirement (3) of Definition 19.1 implies that all these spaces are weakly equivalent to each other. This unique homotopy type is called the homotopy type of the connected groupoid G . For example, let \mathcal{S} be a set of objects in $\text{Cons}(N(\Delta[0]), \mathcal{M})$ (see Notation 11.1). The spaces $\text{we}(F, G)$ with the composition operations \circ as defined in Section 12 and the identities given by the maps $e_F: \Delta[0] \rightarrow \text{we}(F, F)$ form a homotopy groupoid denoted by $\text{we}(\mathcal{S})$. Requirements (1) and (2) are the content of Corollary 12.2(1), and (3) of Proposition 12.3. If all objects in \mathcal{S} are weakly equivalent to each other, $\text{we}(\mathcal{S})$ is connected.

The definition of a homotopy groupoid $G_{\mathcal{S}}$ is designed so we can form a so-called bar construction. It is a map of simplicial spaces $\pi: \mathcal{E}G \rightarrow \mathcal{B}G$, and here is how to construct it. Let $t_{-1} \in \mathcal{S}$ be an arbitrary but fixed element and $n \geq 0$. For any tuple $(t_n, \dots, t_0) \in \mathcal{S}^{n+1}$ set

$$\begin{array}{ccc} \mathcal{E}G_{t_n, \dots, t_0} & \xrightarrow{\pi_{t_n, \dots, t_0}} & \mathcal{B}G_{t_n, \dots, t_0} \\ \parallel & & \parallel \\ \prod_{k=n}^{k=0} G(t_k, t_{k-1}) & \xrightarrow{\text{projection}} & \prod_{k=n}^{k=1} G(t_k, t_{k-1}) \end{array}$$

Thus $\mathcal{E}G_{t_n, \dots, t_0} = \mathcal{B}G_{t_n, \dots, t_0} \times G(t_0, t_{-1})$ and π_{t_n, \dots, t_0} is the projection. For example, in the case $n = 0$, since the product of an empty set of spaces is $\Delta[0]$, $\pi_t: \mathcal{E}G_t = G(t, t_{-1}) \rightarrow \Delta[0] = \mathcal{B}G_t$ is the unique map. By assembling all these spaces we

define

$$\begin{array}{ccc}
 \mathcal{E}G_n & \xrightarrow{\pi_n} & \mathcal{B}G_n \\
 \parallel & & \parallel \\
 \prod_{(t_n, \dots, t_0) \in \mathcal{S}^{n+1}} \mathcal{E}G_{t_n, \dots, t_0} & \xrightarrow{\prod_{(t_n, \dots, t_0) \in \mathcal{S}^{n+1}} \pi_{t_n, \dots, t_0}} & \prod_{(t_n, \dots, t_0) \in \mathcal{S}^{n+1}} \mathcal{B}G_{t_n, \dots, t_0}
 \end{array}$$

Let $\phi: [m] \rightarrow [n]$ be a morphism in Δ . Set $\phi(-1) = -1$. Define

$$\begin{array}{ccc}
 \prod_{k=n}^{k=1} G(t_k, t_{k-1}) = \mathcal{B}G_{t_n, \dots, t_0} & \xrightarrow{\mathcal{B}G_{t_n, \dots, t_0, \phi}} & \mathcal{B}G_{t_{\phi(m)}, \dots, t_{\phi(0)}} = \prod_{k=m}^{k=1} G(t_{\phi(k)}, t_{\phi(k-1)}) \\
 \prod_{k=n}^{k=0} G(t_k, t_{k-1}) = \mathcal{E}G_{t_n, \dots, t_0} & \xrightarrow{\mathcal{E}G_{t_n, \dots, t_0, \phi}} & \mathcal{E}G_{t_{\phi(m)}, \dots, t_{\phi(0)}} = \prod_{k=m}^{k=0} G(t_{\phi(k)}, t_{\phi(k-1)})
 \end{array}$$

to be the maps whose projections onto $G(t_{\phi(i)}, t_{\phi(i-1)})$ are given by the following compositions in the case $\phi(i) = \phi(i - 1)$:

$$\mathcal{B}G_{t_n, \dots, t_0} \rightarrow \Delta[0] \xrightarrow{e_{\phi(i)}} G(t_{\phi(i)}, t_{\phi(i-1)}), \quad \mathcal{E}G_{t_n, \dots, t_0} \rightarrow \Delta[0] \xrightarrow{e_{\phi(i)}} G(t_{\phi(i)}, t_{\phi(i-1)}),$$

and the following compositions in the case $\phi(i) > \phi(i - 1)$:

$$\begin{array}{ccc}
 \mathcal{B}G_{t_n, \dots, t_0} & \xrightarrow{\text{projection}} & \prod_{k=\phi(i)}^{k=\phi(i-1)+1} G(t_k, t_{k-1}) & \xrightarrow{\text{composition}} & G(t_{\phi(i)}, t_{\phi(i-1)}) \\
 \mathcal{E}G_{t_n, \dots, t_0} & \xrightarrow{\text{projection}} & \prod_{k=\phi(i)}^{k=\phi(i-1)+1} G(t_k, t_{k-1}) & \xrightarrow{\text{composition}} & G(t_{\phi(i)}, t_{\phi(i-1)})
 \end{array}$$

For any $\phi: [m] \rightarrow [n]$ in Δ , define $\mathcal{B}G_\phi: \mathcal{B}G_n \rightarrow \mathcal{B}G_m$ and $\mathcal{E}G_\phi: \mathcal{E}G_n \rightarrow \mathcal{E}G_m$ to be the maps on which components are given by $\mathcal{B}G_{t_n, \dots, t_0, \phi}$ and $\mathcal{E}G_{t_n, \dots, t_0, \phi}$. The requirements (1) and (2) of Definition 19.1 are exactly what is needed for $\mathcal{B}G$ and $\mathcal{E}G$ to be simplicial spaces (see for example [15, 18]). From the above definition it is also clear that $\pi: \mathcal{E}G \rightarrow \mathcal{B}G$ is a map of simplicial spaces. Note further that if $S' \subset S$, then $\mathcal{E}G_{S'}$ and $\mathcal{B}G_{S'}$ are simplicial subspaces of $\mathcal{E}G_S$ and $\mathcal{B}G_S$.

Proposition 19.2. *Let $S' \subset S$ be non-empty sets and G_S be a connected homotopy groupoid indexed by S . Let $G_{S'}$ be the restriction of G_S to S' . Then:*

- (1) $\text{hocolim}_{\Delta\text{-op}} \mathcal{B}G_S$ is connected and $\text{hocolim}_{\Delta\text{-op}} \mathcal{E}G_S$ is contractible.
- (2) G_S has the homotopy type of the loop space $\Omega \text{hocolim}_{\Delta\text{-op}} \mathcal{B}G_S$.
- (3) The map $\text{hocolim}_{\Delta\text{-op}} \mathcal{B}G_{S'} \rightarrow \text{hocolim}_{\Delta\text{-op}} \mathcal{B}G_S$, induced by the inclusion of simplicial spaces $\mathcal{B}G_{S'} \subset \mathcal{B}G_S$, is a weak equivalence.

In the rest of this section we prove Proposition 19.2. We start with two classical lemmas whose proofs can be found for example in [1], [2] or also see [11, page 190]. The first lemma is about contractibility of the realization of a simplicial space that admits a so called extra degeneracy. The second lemma is often referred to as Quillen’s Theorem B.

Lemma 19.3. *Let X be a simplicial space. Set $d_0: X_0 \rightarrow \Delta[0] =: X_{-1}$ to be the unique map. Assume that there are maps $s: X_n \rightarrow X_{n+1}$ for $n \geq -1$ such that $d_0 s = \text{id}$ and $d_i s = s d_{i-1}$ for $i > 0$. Then $\text{hocolim}_{\Delta\text{-op}} X$ is contractible.*

Lemma 19.4. *Let $\psi: F \rightarrow G$ be a natural transformation in $\text{Fun}(I, \text{Spaces})$. Assume the commutative diagram on the left below is a homotopy pull-back for any $\alpha: i \rightarrow j$ in I . Then, for any k in I , the diagram on the right is a homotopy pull-back, where the horizontal maps are induced by the inclusion of k in I :*

$$\begin{array}{ccc} F(i) \xrightarrow{F(\alpha)} F(j) & & F(k) \longrightarrow \text{hocolim}_I F \\ \psi_i \downarrow & & \psi_k \downarrow \\ G(i) \xrightarrow{G(\alpha)} G(j) & & G(k) \longrightarrow \text{hocolim}_I G \end{array} \qquad \begin{array}{ccc} & & \downarrow \text{hocolim}_I \psi \\ & & \text{hocolim}_I F \\ & & \downarrow \text{hocolim}_I \psi \\ & & \text{hocolim}_I G \end{array}$$

Proof of Proposition 19.2. (1) $\text{hocolim}_{\Delta\text{op}} \mathcal{B}G_{\mathcal{S}}$ is weakly equivalent to the diagonal of $\mathcal{B}G_{\mathcal{S}}$. For two vertices $s, t \in \coprod_{t_0 \in \mathcal{S}} \Delta[0]_0 = \text{diag}(\mathcal{B}G_{\mathcal{S}})_0$, pick an element f in $G(t, s)_0$ and set $v = s_0 f \in \text{diag}(\mathcal{B}G_{\mathcal{S}})_1$. Note that $d_0 v = s$ and $d_1 v = t$. The space $\text{diag}(\mathcal{B}G_{\mathcal{S}})$ is therefore connected and hence so is $\text{hocolim}_{\Delta\text{op}} \mathcal{B}G_{\mathcal{S}}$.

Let t_n, \dots, t_0 be in \mathcal{S} . Define $s_{t_n, \dots, t_0}: \mathcal{E}G_{t_n, \dots, t_0} \rightarrow \mathcal{E}G_{t_n, \dots, t_0, t_{-1}}$ as

$$\begin{array}{ccc} \mathcal{E}G_{t_n, \dots, t_0} \equiv \prod_{k=n}^{k=0} G(t_k, t_{k-1}) & \equiv & \prod_{k=n}^{k=0} G(t_k, t_{k-1}) \times \Delta[0] \\ & \searrow^{s_{t_n, \dots, t_0}} & \downarrow \text{id} \times e_{t_{-1}} \\ & & \mathcal{E}G_{t_n, \dots, t_0, t_{-1}} \equiv \prod_{k=n}^{k=0} G(t_k, t_{k-1}) \times G(t_{-1}, t_{-1}) \end{array}$$

Let $s: \mathcal{E}G_n \rightarrow \mathcal{E}G_{n+1}$ be given on components by s_{t_n, \dots, t_0} . Set $\mathcal{E}G_{-1} = \Delta[0]$ and $s: \mathcal{E}G_{-1} = \Delta[0] \rightarrow \mathcal{E}G_0$ to be the composition of $e_{t_{-1}}: \Delta[0] \rightarrow G(t_{-1}, t_{-1})$ and the inclusion $G(t_{-1}, t_{-1}) \subset \coprod_{t \in \mathcal{S}} G(t, t_{-1}) = \mathcal{E}G_0$. Since these maps satisfy the assumptions of Lemma 19.3, we can conclude $\text{hocolim}_{\Delta\text{op}} \mathcal{E}G_{\mathcal{S}}$ is contractible.

(2) Requirement (3) in Definition 19.1 implies that the square on the left below is a homotopy pull-back. Since being a homotopy pull-back can be checked component-wise, the square on the right is also a homotopy pull-back:

$$\begin{array}{ccc} \mathcal{E}G_{t_n, \dots, t_0} \xrightarrow{\mathcal{E}G_{t_n, \dots, t_0, \phi}} \mathcal{E}G_{t_{\phi(m)}, \dots, t_{\phi(0)}} & & \mathcal{E}G_m \xrightarrow{\mathcal{E}G_{\phi}} \mathcal{E}G_n \\ \pi_{t_n, \dots, t_0} \downarrow & & \pi_m \downarrow \\ \mathcal{B}G_{t_n, \dots, t_0} \xrightarrow{\mathcal{B}G_{t_n, \dots, t_0, \phi}} \mathcal{B}G_{t_{\phi(m)}, \dots, t_{\phi(0)}} & & \mathcal{B}G_m \xrightarrow{\mathcal{B}G_{\phi}} \mathcal{B}G_n \end{array} \qquad \begin{array}{ccc} & & \downarrow \pi_n \\ & & \mathcal{B}G_{\phi} \end{array}$$

We can then use Lemma 19.4 to conclude that, for any t in \mathcal{S} , the square

$$\begin{array}{ccc} G(t, t_{-1}) \equiv \mathcal{E}G_t \longrightarrow \text{hocolim}_{\Delta\text{op}} \mathcal{E}G & & \\ \pi_t \downarrow & & \downarrow \text{hocolim}_{\Delta\text{op}} \pi \\ \Delta[0] \equiv \mathcal{B}G_t \longrightarrow \text{hocolim}_{\Delta\text{op}} \mathcal{B}G & & \end{array}$$

is a homotopy pull-back which proves statement (2).

(3) Let t be an object in \mathcal{S}' . Consider the following commutative diagram:

$$\begin{array}{ccccc} G(t, t_{-1}) \equiv \mathcal{E}G_t & \longrightarrow & \text{hocolim}_{\Delta\text{op}} \mathcal{E}G_{\mathcal{S}'} & \longrightarrow & \text{hocolim}_{\Delta\text{op}} \mathcal{E}G_{\mathcal{S}} \\ \pi_t \downarrow & & \text{hocolim}_{\Delta\text{op}} \pi \downarrow & & \text{hocolim}_{\Delta\text{op}} \pi \downarrow \\ \Delta[0] \equiv \mathcal{B}G_t & \longrightarrow & \text{hocolim}_{\Delta\text{op}} \mathcal{B}G_{\mathcal{S}'} & \longrightarrow & \text{hocolim}_{\Delta\text{op}} \mathcal{B}G_{\mathcal{S}} \end{array}$$

The squares in this diagram are homotopy pull-backs, the spaces $\text{hocolim}_{\Delta\text{op}} \mathcal{B}G_{\mathcal{S}'}$ and $\text{hocolim}_{\Delta\text{op}} \mathcal{B}G_{\mathcal{S}}$ are connected, and $\text{hocolim}_{\Delta\text{op}} \mathcal{E}G_{\mathcal{S}'}$ and $\text{hocolim}_{\Delta\text{op}} \mathcal{E}G_{\mathcal{S}}$ are contractible. Thus $\text{hocolim}_{\Delta\text{op}} \mathcal{B}G_{\mathcal{S}'} \rightarrow \text{hocolim}_{\Delta\text{op}} \mathcal{B}G_{\mathcal{S}}$ is a weak equivalence. \square

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