

AN L^p REGULARITY THEORY FOR HARMONIC MAPS

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ABSTRACT. Motivated by the harmonic map heat flow, we consider maps between Riemannian manifolds such that the tension field belongs to an L^p -space. Under an appropriate smallness condition, a certain degree of regularity follows. For suitable solutions of the harmonic map heat flow, we have a partial regularity result as a consequence.

1. INTRODUCTION

1.1. The problems. It is well-known from work by Hélein [21–23] that weakly harmonic maps on two-dimensional domains are necessarily smooth. In higher dimensions, the corresponding statement is not true, but under a stationarity condition, at least partial regularity results are available [3, 13]. There have been attempts to prove similar results for the harmonic map heat flow, i.e., the L^2 gradient flow belonging to the harmonic map problem. But previously this has been successful only in domains of dimension 4 or less [29]. (A paper by Liu [26] purports to give a proof in arbitrary dimensions, but it contains an error; see below for details.)

In this paper we study an equation that differs from the harmonic map equation by an L^p -term. We show that under suitable smallness conditions, we have the type of regularity that one would expect for a linear elliptic equation. The result then implies partial regularity for suitable solutions of the harmonic map heat flow as well.

Let M and N be smooth Riemannian manifolds. Suppose that N is compact and without boundary. By the Nash embedding theorem, we may assume that N is a submanifold of a Euclidean space \mathbb{R}^n . For a smooth map $u : M \rightarrow N$, let ∇u denote the gradient. Consider the Dirichlet functional

$$E(u) = \frac{1}{2} \int_M |\nabla u|^2 \, d\text{vol}_M.$$

Its critical points are called harmonic maps and satisfy the Euler-Lagrange equation

$$\Delta_M u + \text{trace } A(u)(\nabla u, \nabla u) = 0,$$

where Δ_M is the Laplace-Beltrami operator (with a sign convention that makes it negative semidefinite) and A is the second fundamental form of $N \subset \mathbb{R}^n$. The expression on the left-hand side of the equation is called the tension field of u .

The harmonic map heat flow is the L^2 gradient flow for the functional E . The corresponding equation is

$$\frac{\partial u}{\partial t} - \Delta_M u = \text{trace } A(u)(\nabla u, \nabla u).$$

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Smooth solutions satisfy an energy identity. This is not true for weak solutions in general, but if we study a suitable class of weak solutions, then we have an energy inequality of the form

$$E(u(t_2, \cdot)) + \int_{t_1}^{t_2} \int_M \left| \frac{\partial u}{\partial t} \right|^2 d\text{vol}_M dt \leq E(u(t_1, \cdot)).$$

If we solve the flow for initial data with finite energy, and if we have a type of solution that satisfies this inequality, then it follows that the restriction to almost any time slice satisfies

$$\Delta_M u + \text{trace } A(u)(\nabla u, \nabla u) \in L^2(M; \mathbb{R}^n).$$

Motivated by this observation, we study equations of the form

$$\Delta_M u + \text{trace } A(u)(\nabla u, \nabla u) = f$$

for $f \in L^p(M; \mathbb{R}^n)$ with $p > 1$. In particular, we study the question of regularity for weak solutions of such equations.

1.2. The main results. The geometry of M does not play a significant role in this context. For simplicity, we now replace M by the open unit ball B in \mathbb{R}^m . The results that we prove are known for two-dimensional domains, and thus we assume that $m \geq 3$ henceforth. We also write $B_r(x_0)$ for the open ball with centre $x_0 \in \mathbb{R}^m$ and radius $r > 0$, and we abbreviate $B_r = B_r(0)$. The harmonic map equation in B becomes

$$(1) \quad \Delta u + \text{trace } A(u)(\nabla u, \nabla u) = 0 \quad \text{in } B$$

for the Laplacian Δ in \mathbb{R}^m . For the harmonic map heat flow, consider the time interval $(-1, 1)$ for simplicity. The corresponding equation is

$$(2) \quad \frac{\partial u}{\partial t} - \Delta u = \text{trace } A(u)(\nabla u, \nabla u) \quad \text{in } (-1, 1) \times B.$$

For $p \in [1, \infty]$, define the Sobolev space

$$W^{1,p}(B; N) = \{u \in W^{1,p}(B; \mathbb{R}^n) : u(x) \in N \text{ for almost every } x \in B\}.$$

Consider weak solutions of (1) in $W^{1,2}(B; N)$. Without further assumptions, regularity of weakly harmonic maps cannot be expected. Indeed, Rivière [36] constructed an example of a weak solution that is discontinuous everywhere. On the other hand, it is known that weakly harmonic maps are smooth if they are small in an appropriate sense. The condition that we use in this paper is given in terms of certain Morrey spaces.

Let $p \in [1, \infty)$ and $\lambda \in (0, m]$. Let $D \subset \mathbb{R}^m$ be an open set. For a function $F \in L^p(D; V)$ with values in a normed vector space V , we write

$$\|F\|_{M^{p,\lambda}(D)} = \sup_{x_0 \in D} \sup_{r > 0} \left(r^{\lambda-m} \int_{D \cap B_r(x_0)} |F|^p dx \right)^{\frac{1}{p}}.$$

Furthermore,

$$M^{p,\lambda}(D; V) = \{F \in L^p(D; V) : \|F\|_{M^{p,\lambda}(D)} < \infty\}$$

and $M^{p,\lambda}(D) = M^{p,\lambda}(D; \mathbb{R})$. For all of our results, the condition

$$(3) \quad \|\nabla u\|_{M^{2,2}(B)} \leq \epsilon,$$

for a sufficiently small number $\epsilon > 0$, is important.

Theorem 1. *For every $p \in (1, \infty)$ there exists an $\epsilon > 0$ with the following property. Suppose that $u \in W^{1,2}(B; N)$ and $f \in L^p(B; \mathbb{R}^n)$ satisfy*

$$(4) \quad \Delta u + \text{trace } A(u)(\nabla u, \nabla u) = f \quad \text{in } B$$

weakly. If $\|\nabla u\|_{M^{2,2}(B)} \leq \epsilon$, then $u \in W_{\text{loc}}^{2,p}(B; \mathbb{R}^n) \cap W_{\text{loc}}^{1,2p}(B; N)$.

For harmonic maps that satisfy a stationarity condition, an inequality of the form (3) can be reduced to smallness of the energy of u with the help of a well-known monotonicity formula [35]. A conditional regularity result requiring (3) can then be turned into an unconditional, but partial regularity result with a covering argument. The conclusion is that a stationary weakly harmonic map is smooth away from a closed singular set of vanishing $(m-2)$ -dimensional Hausdorff measure [3].

For the perturbed harmonic map equation (4), there is no monotonicity formula in general. But there is a parabolic version of the monotonicity formula for the harmonic map heat flow, which was discovered by Struwe [46]. Weak solutions do not satisfy it in general, but it can be derived from a parabolic stationarity condition formulated by Feldman [15]. We do not state the formula here, because we merely use one of its consequences.

For $z_0 = (t_0, x_0) \in \mathbb{R} \times \mathbb{R}^m$ and $r > 0$, define the parabolic cylinder

$$B_r^*(z_0) = (t_0 - r^2, t_0 + r^2) \times B_r(x_0),$$

which we can also regard as a ball with respect to the parabolic metric

$$d^*((t_1, x_1), (t_2, x_2)) = \max \left\{ \sqrt{|t_1 - t_2|}, |x_1 - x_2| \right\}.$$

We write $B_r^* = B_r^*(0)$ and $B^* = B_1^*$.

Definition 2. *Let $u \in W^{1,2}(B^*; N)$. Suppose that there exists a constant $c_0 > 0$ such that for all $z_0 = (t_0, x_0) \in B^*$ and $r > 0$ with $B_r^*(z_0) \subset B^*$ and for all $z_1 = (t_1, x_1) \in B^*$ and $s > 0$ with $B_s^*(z_1) \subset B_{r/2}^*(z_0)$, the inequality*

$$s^{2-m} \left(\int_{B_s(x_1)} |\nabla u(t_1, x)|^2 dx + \int_{B_s^*(z_1)} \left| \frac{\partial u}{\partial t} \right|^2 dz \right) \leq \frac{c_0}{r^m} \int_{B_r^*(z_0)} |\nabla u|^2 dz$$

is satisfied. Then we say that u satisfies a monotonicity inequality.

Theorem 3. *Suppose that $u \in W^{1,2}(B^*; N)$ is a weak solution of (2) satisfying a monotonicity inequality. Then there exists a set $S \subset B^*$ such that*

- S is closed relative to B^* ,
- the m -dimensional Hausdorff measure of S with respect to the metric d^* vanishes, and
- u is smooth away from S .

Once Theorem 1 is established, this result follows with known methods. For completeness, we give a proof anyway in section 5.3.

1.3. Some background. The question of regularity of harmonic maps has quite a long history, going back to Morrey [28] for a two-dimensional domain and to Schoen and Uhlenbeck [40, 41] for higher dimensions. While these works concern energy minimising harmonic maps, the analysis of weak solutions of the Euler-Lagrange equation was pioneered by Hélein [21–23], who showed that for a two-dimensional domain, weakly harmonic maps are always smooth. Using similar ideas, Evans [13]

(for harmonic maps into spheres) and Bethuel [3] (for general target manifolds) proved partial regularity of stationary harmonic maps in higher dimensions.

More recently, Rivière [37] found a new approach for two-dimensional domains, and the method was extended to higher dimensions by Rivière and Struwe [38]. The work does not give a significant improvement of the results for harmonic maps, other than the fact that the assumptions on the regularity of N can be relaxed, but the method has other applications [25, 47]. The proof of Theorem 1 is also based on these ideas.

Some results on solutions of equation (4) for $f \in L^p(B; \mathbb{R}^n)$ with $p \geq \frac{m}{2}$ have been proved before. If $p > \frac{m}{2}$, then the equation, together with a condition of the form (3), implies continuity [29]. Once continuity is known, the conclusion of Theorem 1 follows with known methods. Thus the result is new only for $p \leq \frac{m}{2}$. A similar problem for $p > \frac{m}{2}$ was also studied by Sharp and Topping [43] (for two dimensions) and by Sharp [42] (for higher dimensions). The arguments used in these papers are based on Rivière's ideas [37] as well and thus related to the method that we use here. For $p = \frac{m}{2}$, an inequality in an Orlicz space involving exponential growth exists [32].

Partial regularity results for the harmonic map heat flow are known for $m \leq 4$. For $m = 2$, weak solutions with finitely many singular points have been constructed by Struwe [45] (on closed surfaces) and Chang [5] (for surfaces with boundary). A uniqueness result of Freire [16, 17] then implies better regularity than in Theorem 3 under weaker conditions (but still not for all weak solutions). The result has more recently been improved by Rupflin [39]. In higher dimensions, weak solutions with almost the regularity stated in Theorem 3 have been constructed by Chen and Struwe [8] and by Chen [7]. Partial regularity results for the special case $N = S^{n-1}$ have been obtained independently by Feldman [15] and by Chen, Li, and Lin [6]. For general target manifolds, but only for $m \leq 4$, similar results were proved by the author [29]. The problem in arbitrary dimensions was studied by Liu [26]. A partial regularity result is stated, but there is a gap in the proof. The problem occurs in Lemma 5.6, which states an L^∞ -inequality where only a BMO-estimate is justified. (More precisely, and more technically, on page 155, a uniform estimate for a fraction is stated, when there is no guarantee that the denominator is uniformly bounded away from 0.)

1.4. Strategy for the proof. The method that we use in this paper is based on a refinement of the ideas of Rivière and Struwe [38]. Indeed, the first steps in the proof of Theorem 1 are exactly the same.

The first important observation is that the harmonic map equation, and also equation (4), can be rewritten as follows. Let

$$1 = \sum_{\ell=1}^L \chi_\ell$$

be a smooth partition of unity such that for every $\ell = 1, \dots, L$, there exist smooth normal vector fields $\nu_{1\ell}, \dots, \nu_{K\ell}$ on N that form an orthonormal basis of the normal space at every point of $\text{supp } \chi_\ell$. Define $w_{k\ell} = \nu_{k\ell}(u)\chi_\ell(u)$. We write $\langle \cdot, \cdot \rangle$ for the inner product in \mathbb{R}^n and in $\mathbb{R}^m \otimes \mathbb{R}^n$, while we use a dot for the inner product in \mathbb{R}^m . Then we have $\langle \nabla u, w_{k\ell} \rangle = 0$. Hence

$$0 = \text{div } \langle \nabla u, w_{k\ell} \rangle = \langle \Delta u, w_{k\ell} \rangle + \langle \nabla u, \nabla w_{k\ell} \rangle.$$

Note that $-\text{trace } A(u)(\nabla u, \nabla u)$ is the normal part of Δu . If $u \in W^{1,2}(B; N)$ solves (4), then it follows that

$$\Delta u - f = \sum_{\ell=1}^L \sum_{k=1}^K \langle \Delta u, w_{k\ell} \rangle \nu_{k\ell}(u) = - \sum_{\ell=1}^L \sum_{k=1}^K \langle \nabla u, \nabla w_{k\ell} \rangle \nu_{k\ell}(u).$$

We write $u = (u^1, \dots, u^n)$ and use similar notation for the components of f , $\nu_{k\ell}$, and $w_{k\ell}$. Then for $i = 1, \dots, n$, we have

$$\Delta u^i - f^i = - \sum_{\ell=1}^L \sum_{k=1}^K \sum_{j=1}^n \nabla u^j \cdot (\nabla w_{k\ell}^j \nu_{k\ell}^i(u) - \nabla w_{k\ell}^i \nu_{k\ell}^j(u)).$$

Defining $\Omega = (\Omega^{ij})_{i,j=1,\dots,n}$ with

$$\Omega^{ij} = - \sum_{\ell=1}^L \sum_{k=1}^K (\nabla w_{k\ell}^j \nu_{k\ell}^i(u) - \nabla w_{k\ell}^i \nu_{k\ell}^j(u)),$$

we have the equation

$$(5) \quad \Delta u = \Omega \cdot \nabla u + f \quad \text{in } B.$$

The crucial observation for the analysis is the skew symmetry of Ω . Let $\mathfrak{so}(n)$ denote the Lie algebra comprising all skew symmetric $(n \times n)$ -matrices. If $|\nabla u| \in M^{2,2}(B)$, then Ω belongs to the space $M^{2,2}(B; \mathbb{R}^m \otimes \mathfrak{so}(n))$.

Now we give a brief outline of the strategy for the proof of Theorem 1. Again using an idea of Rivière and Struwe, we consider an $\text{SO}(n)$ -valued function $P \in W^{1,2}(B; \text{SO}(n))$ and calculate

$$\text{div}(P\nabla u) = (\nabla P P^{-1} + P\Omega P^{-1}) \cdot P\nabla u + P f.$$

The quantity $\tilde{\Omega} = \nabla P P^{-1} + P\Omega P^{-1}$ occurs naturally in the theory of gauge transformations. It is well-known that P can be chosen such that $\tilde{\Omega}$ is divergence free. Using this so-called Coulomb gauge, we can take advantage of a div-curl structure when we work with the expression $\tilde{\Omega} \cdot P\nabla u$. In particular, we can use compensated compactness results, such as the results of Coifman, Lions, Meyer, and Semmes [9], to estimate integrals involving this term. For solutions of (5) with $f = 0$, under an appropriate smallness condition, Rivière and Struwe were then able to estimate the decay rate of $\|\nabla u\|_{M^{1,1}(B_r(x_0))}$ for $x_0 \in B$ when r tends to 0. Using Morrey's lemma, they concluded that the solutions must be continuous, and higher regularity can then be obtained as well.

It is not obvious how to treat the additional term f directly using similar arguments. Therefore, we first solve a sequence of auxiliary equations. For technical reasons, we extend everything to \mathbb{R}^m with the help of a cut-off function, so we now assume that the equations are satisfied in the whole Euclidean space and $f \in L^p(\mathbb{R}^m)$. One may try to get rid of f by solving the equation

$$(6) \quad \text{div}(P\nabla v_0) = P f$$

and setting $w_0 = u - v_0$. This, however, introduces another term that we need to control. We now have

$$\text{div}(P\nabla w_0) = \tilde{\Omega} \cdot P\nabla w_0 + \tilde{\Omega} \cdot P\nabla v_0.$$

The idea for the proof of Theorem 1 is to repeat a similar process indefinitely: for $k = 0, 1, 2, \dots$, solve

$$(7) \quad \operatorname{div}(P\nabla v_{k+1}) = \tilde{\Omega} \cdot P\nabla v_k.$$

Set $w_{k+1} = w_k - v_{k+1}$. Then we have the equations

$$\operatorname{div}(P\nabla w_k) = \tilde{\Omega} \cdot P\nabla w_k + \tilde{\Omega} \cdot P\nabla v_k$$

for $k = 0, 1, 2, \dots$; moreover,

$$u = w_k + \sum_{\ell=1}^k v_\ell.$$

Equation (7) has a structure similar to the equation studied by Rivière and Struwe. With their methods, we can estimate $\|\nabla v_{k+1}\|_{M^{1,1}(\mathbb{R}^m)}$ in terms of $\|\nabla v_k\|_{M^{1,1}(\mathbb{R}^m)}$. We obtain a decay in k of the form

$$(8) \quad \|\nabla v_k\|_{M^{1,1}(\mathbb{R}^m)} \leq C\gamma^k,$$

and $\gamma > 0$ can be made arbitrarily small when ϵ is chosen sufficiently small. This inequality is then useful for the next step. We use a Gagliardo-Nirenberg type inequality involving the mean oscillation of a function. Recall that for $\phi \in L^1_{\text{loc}}(\mathbb{R}^m)$, using the shorthand notation

$$\bar{\phi}_{B_r(x_0)} = \int_{B_r(x_0)} \phi \, dx = \frac{1}{|B_r|} \int_{B_r(x_0)} \phi \, dx,$$

we have the BMO-seminorm

$$[\phi]_{\text{BMO}(\mathbb{R}^m)} = \sup_{x_0 \in \mathbb{R}^m} \sup_{r > 0} \int_{B_r(x_0)} |\phi - \bar{\phi}_{B_r(x_0)}| \, dx.$$

Given a number $p \in (1, \infty)$, there exists a constant $C_1 = C_1(m, p)$ such that

$$\|\phi\|_{L^{2p}(\mathbb{R}^m)}^2 \leq C_1 [\phi]_{\text{BMO}(\mathbb{R}^m)} \|\nabla^2 \phi\|_{L^p(\mathbb{R}^m)}.$$

The inequality is due to Adams and Frazier [2]. In combination with the Calderon-Zygmund inequality, we obtain a constant $C_2 = C_2(m, p)$ such that

$$(9) \quad \|\phi\|_{L^{2p}(\mathbb{R}^m)}^2 \leq C_2 [\phi]_{\text{BMO}(\mathbb{R}^m)} \|\Delta \phi\|_{L^p(\mathbb{R}^m)}.$$

We can prove a similar estimate for the differential operator in equation (7) instead of the Laplacian.

Set $p_0 = p$ and inductively define

$$p_{k+1} = \frac{2p_k}{p_k + 1}.$$

By the Poincaré inequality, we have a constant $C_3 = C_3(m, n)$ such that

$$[v_k]_{\text{BMO}(\mathbb{R}^m)} \leq C_3 \|\nabla v_k\|_{M^{1,1}(\mathbb{R}^m)}.$$

Using (6) and (9), it is then not too difficult to prove that $|\nabla v_0| \in L^{2p_0}(\mathbb{R}^m)$. Inductively, we can show that $|\nabla v_k| \in L^{2p_k}(\mathbb{R}^m)$ for solutions of (7). Note, however, that the constant C_2 will blow up when p tends to 1, and we have $p_k \rightarrow 1$ as $k \rightarrow \infty$. So we have to be careful here. Fortunately, the blow-up rate is not too bad; it is of order $(p-1)^{-2}$. When we use the exponents p_k , we obtain an exponential growth

in k . We can compensate for this with (8) if γ is small enough. As a consequence, we find convergence of the series

$$\sum_{k=1}^{\infty} v_k$$

in $W_{\text{loc}}^{1,2}(\mathbb{R}^m; \mathbb{R}^n)$, and it follows that the sequence $(w_k)_{k \in \mathbb{N}}$ converges in this space as well. The limit will be a solution of

$$\operatorname{div}(P\nabla w) = \tilde{\Omega} \cdot P\nabla w.$$

This is exactly the equation that corresponds to harmonic maps after the gauge transformation. The arguments of Rivière and Struwe apply, and as we work on \mathbb{R}^m , they imply that w is constant. Combining all the inequalities, we then obtain an estimate for $\|\nabla u\|_{L^2(B)}$ in terms of f (and some other terms coming from the use of a cut-off function earlier). In the next step, we use a Gehring lemma that implies higher integrability. From here on, we can use known results from the regularity theory of harmonic maps to derive the statement of the theorem.

2. TOOLS

In this section we collect a few known results that we need for the proof of Theorem 1.

2.1. Gauge transformations. For the method described in the introduction, the key observation is that a gauge transformation P exists that makes $\tilde{\Omega}$ divergence free. This follows from a theory developed for the study of Yang-Mills fields. The first results of this type were found by Uhlenbeck [50], and they were refined considerably by the independent works of Meyer and Rivière [27] and of Tao and Tian [49]. The version stated here is due to Rivière and Struwe [38, Lemma 3.1].

Theorem 4 (Rivière-Struwe). *There exist $\epsilon > 0$ and $C > 0$ with the following property. Suppose that $\Omega \in M^{2,2}(B; \mathbb{R}^m \otimes \mathfrak{so}(n))$ with $\|\Omega\|_{M^{2,2}(B)} \leq \epsilon$. Then there exists a $P \in W^{1,2}(B; \text{SO}(n))$ such that for*

$$\tilde{\Omega} = \nabla P P^{-1} + P \Omega P^{-1},$$

the conditions

$$(10) \quad \operatorname{div} \tilde{\Omega} = 0 \quad \text{in } B$$

and

$$(11) \quad x \cdot \tilde{\Omega} = 0 \quad \text{on } \partial B$$

are satisfied. Moreover,

$$\|\nabla P\|_{M^{2,2}(B)} + \|\tilde{\Omega}\|_{M^{2,2}(B)} \leq C \|\Omega\|_{M^{2,2}(B)}$$

and

$$(12) \quad \|\nabla P\|_{L^2(B)} + \|\tilde{\Omega}\|_{L^2(B)} \leq C \|\Omega\|_{L^2(B)}.$$

This is not exactly how Rivière and Struwe formulated the result. First, we have to replace P with P^{-1} and Ω with $-\Omega$ (and pass from vector fields to differential forms) to obtain their version. Second, some of the properties of P and $\tilde{\Omega}$ are not stated explicitly in their paper. They give a different boundary condition, but (11) follows from it when we consider the normal component on the boundary.

Furthermore, inequality (12), albeit not stated by Rivière and Struwe, is obtained by the arguments in the proof of their Lemma 4.2.

Note that (10) and (11) imply that the extension of $\tilde{\Omega}$ by 0 outside of B is divergence free in \mathbb{R}^m .

2.2. Compensated compactness. Estimates in the Hardy space $\mathcal{H}^1(\mathbb{R}^m)$ have long been used in the regularity theory of harmonic maps. (A definition of the space can be found, e.g., in a book by Stein [44].) The reason is the following duality between $\mathcal{H}^1(\mathbb{R}^m)$ and $\text{BMO}(\mathbb{R}^m)$ due to Fefferman and Stein [14].

Theorem 5 (Fefferman-Stein). *There exists a linear homeomorphism*

$$\Phi : \text{BMO}(\mathbb{R}^m) \rightarrow (\mathcal{H}^1(\mathbb{R}^m))^*$$

such that for every $f \in \text{BMO}(\mathbb{R}^m)$ and for every $g \in \mathcal{H}^1(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m)$ with compact support, we have

$$\Phi(f)(g) = \int_{\mathbb{R}^m} fg \, dx.$$

Note that the integral does not necessarily converge for all $f \in \text{BMO}(\mathbb{R}^m)$ and $g \in \mathcal{H}^1(\mathbb{R}^m)$ (hence the restriction). But as the set considered in the theorem is dense in $\mathcal{H}^1(\mathbb{R}^m)$ [44, Section III.2.2], we can typically still conclude that

$$\int_{\mathbb{R}^m} fg \, dx \leq C[f]_{\text{BMO}(\mathbb{R}^m)} \|g\|_{\mathcal{H}^1(\mathbb{R}^m)}$$

in cases where the integral does converge.

In order to make use of this inequality, we need to estimate certain functions in $\mathcal{H}^1(\mathbb{R}^m)$. To this end, we often take advantage of a div-curl structure and compensated compactness arguments. In particular, we can use the following result, due to Coifman, Lions, Meyer, and Semmes [9]. For its statement, we use the homogeneous Sobolev spaces $\dot{W}^{1,p}(\mathbb{R}^m; \mathbb{R}^n)$, obtained as the completion of $C_0^\infty(\mathbb{R}^m)$ with respect to the norm

$$\|\phi\|_{\dot{W}^{1,p}(\mathbb{R}^m)} = \|\nabla\phi\|_{L^p(\mathbb{R}^m)}.$$

Theorem 6 (Coifman-Lions-Meyer-Semmes). *Suppose that $p \in (1, \infty)$ and $p' = \frac{p}{p-1}$. Then there exists a constant $C > 0$ such that for all $f \in \dot{W}^{1,p}(\mathbb{R}^m)$ and for all $F \in L^{p'}(\mathbb{R}^m; \mathbb{R}^m)$ with $\text{div } F = 0$ in \mathbb{R}^m , the function $\nabla f \cdot F$ belongs to $\mathcal{H}^1(\mathbb{R}^m)$ and*

$$\|\nabla f \cdot F\|_{\mathcal{H}^1(\mathbb{R}^m)} \leq C \|\nabla f\|_{L^p(\mathbb{R}^m)} \|F\|_{L^{p'}(\mathbb{R}^m)}.$$

2.3. The Helmholtz-Hodge decomposition. The main inequality of this section is related to the duality between $\text{BMO}(\mathbb{R}^m)$ and $\mathcal{H}^1(\mathbb{R}^m)$ as well. It is a commutator estimate due to Coifman, Rochberg, and Weiss [10] for operators involving singular integrals. It is valid for all Calderon-Zygmund kernels, although we state only a special case of the theorem.

Suppose that $p \in (1, \infty)$ and $F \in L^p(\mathbb{R}^m; \mathbb{R}^m)$. Then it is well-known that there exists a unique decomposition of F into a gradient vector field and a divergence free vector field. That is, there exist $\phi \in \dot{W}^{1,p}(\mathbb{R}^m)$ and $\Phi \in L^p(\mathbb{R}^m; \mathbb{R}^m)$ with $\text{div } \Phi = 0$ in \mathbb{R}^m , such that

$$F = \nabla\phi + \Phi.$$

Both ϕ and Φ are unique. Thus the decomposition gives rise to two linear operators

$$h : L^p(\mathbb{R}^m; \mathbb{R}^m) \rightarrow \dot{W}^{1,p}(\mathbb{R}^m) \quad \text{and} \quad H : L^p(\mathbb{R}^m; \mathbb{R}^m) \rightarrow L^p(\mathbb{R}^m; \mathbb{R}^m)$$

such that for every $F \in L^p(\mathbb{R}^m; \mathbb{R}^m)$, we have $\operatorname{div} H(F) = 0$ and $F = \nabla h(F) + H(F)$. Furthermore, both of these operators are continuous.

We are interested in the commutator between H and another operator, given by multiplication with a fixed function in $\operatorname{BMO}(\mathbb{R}^m)$. Formally, this is not well-defined at first, as the second operator does not map $L^p(\mathbb{R}^m; \mathbb{R}^m)$ to itself in general. Nevertheless, it is a reasonable notion, owing to the following result.

Theorem 7 (Coifman-Rochberg-Weiss). *There is a constant $C > 0$ such that for all $f \in L^\infty(\mathbb{R}^m)$ and all $F \in L^p(\mathbb{R}^m; \mathbb{R}^m)$,*

$$\|H(fF) - fH(F)\|_{L^p(\mathbb{R}^m)} \leq C[f]_{\operatorname{BMO}(\mathbb{R}^m)} \|F\|_{L^p(\mathbb{R}^m)}.$$

It follows, in particular, that for all $g \in \dot{W}^{1,p}(\mathbb{R}^m)$ and $f \in L^\infty(\mathbb{R}^m)$, we have

$$\|H(f\nabla g)\|_{L^p(\mathbb{R}^m)} \leq C[f]_{\operatorname{BMO}(\mathbb{R}^m)} \|\nabla g\|_{L^p(\mathbb{R}^m)},$$

and this is the inequality that we will use. Another consequence is that the expression $H(fF) - fH(F)$ can be interpreted as an operator $L^p(\mathbb{R}^m; \mathbb{R}^m) \rightarrow L^p(\mathbb{R}^m; \mathbb{R}^m)$ for all $f \in \operatorname{BMO}(\mathbb{R}^m)$. But since we will apply the inequality only to functions that actually belong to $L^\infty(\mathbb{R}^m)$, this is irrelevant in the context of this paper.

2.4. Riesz potentials and Morrey spaces. It was shown by Adams [1] that singular integrals with Riesz potentials give rise to continuous maps between suitable Morrey spaces. In addition to the spaces $M^{p,\lambda}(\mathbb{R}^m)$ defined in the introduction, we also consider a weak version here. Let $p \in [1, \infty)$. For a measurable function $f : \mathbb{R}^m \rightarrow \mathbb{R}$, let

$$\|f\|_{M_*^{p,\lambda}(\mathbb{R}^m)} = \sup_{x_0 \in \mathbb{R}^m} \sup_{r>0} \sup_{t>0} (r^{\lambda-m} t^p |\{x \in B_r(x_0) : |f(x)| \geq t\}|)^{\frac{1}{p}}.$$

Define $M_*^{p,\lambda}(\mathbb{R}^m)$ to be the set of all measurable functions f on \mathbb{R}^m with $\|f\|_{M_*^{p,\lambda}(\mathbb{R}^m)} < \infty$. For every $a \in [\frac{1}{p}, 1)$, we then have

$$M^{p,\lambda}(\mathbb{R}^m) \subset M_*^{p,\lambda}(\mathbb{R}^m) \subset M^{ap,a\lambda}(\mathbb{R}^m).$$

For $\alpha \in (0, \frac{m}{p})$ and $f \in L^p(\mathbb{R}^m)$, let $I_\alpha f$ be the function with

$$I_\alpha f(x) = \int_{\mathbb{R}^m} |x-y|^{\alpha-m} f(y) dy.$$

Then we have the following estimates.

Theorem 8 (Adams). *Suppose that $p \in [1, \infty)$, $0 < \lambda \leq m$, and $0 < \alpha < \frac{\lambda}{p}$. Let $q = \frac{\lambda p}{\lambda - \alpha p}$. There exists a constant $C > 0$ such that for every $f \in L^p(\mathbb{R}^m) \cap M^{p,\lambda}(\mathbb{R}^m)$, the following hold:*

(i) *If $p > 1$, then $I_\alpha f \in M^{q,\lambda}(\mathbb{R}^m)$ with*

$$\|I_\alpha f\|_{M^{q,\lambda}(\mathbb{R}^m)} \leq C \|f\|_{M^{p,\lambda}(\mathbb{R}^m)}.$$

(ii) *If $p = 1$, then $I_\alpha f \in M_*^{q,\lambda}(\mathbb{R}^m)$ with*

$$\|I_\alpha f\|_{M_*^{q,\lambda}(\mathbb{R}^m)} \leq C \|f\|_{M^{1,\lambda}(\mathbb{R}^m)}.$$

We will need one more estimate of a similar type. This observation has been made before, but we give a proof for completeness.

Lemma 9. *There exists a constant C such that for every $g \in L^1(\mathbb{R}^m) \cap M^{1,1}(\mathbb{R}^m)$,*

$$[I_1 g]_{\text{BMO}(\mathbb{R}^m)} \leq C \|g\|_{M^{1,1}(\mathbb{R}^m)}.$$

Proof. Let $f = I_1 g$. As the inequality is invariant under rescaling of the domain, it suffices to show that

$$\int_B |f - \bar{f}_B| dx \leq C \|g\|_{M^{1,1}(\mathbb{R}^m)}.$$

Split f into $f_1 + f_2$ with

$$f_1(x) = \int_{B_2} |x - y|^{1-m} g(y) dy$$

and

$$f_2(x) = \int_{\mathbb{R}^m \setminus B_2} |x - y|^{1-m} g(y) dy.$$

Then there exists a constant $C_1 = C_1(m)$ such that

$$\|f_1\|_{L^1(B)} \leq C_1 \|g\|_{L^1(B_2)} \leq 2^{m-1} C_1 \|g\|_{M^{1,1}(\mathbb{R}^m)}.$$

Furthermore, for $x_1, x_2 \in B$ and $y \in \mathbb{R}^m \setminus B_2$, we have

$$\left| |x_1 - y|^{1-m} - |x_2 - y|^{1-m} \right| \leq C_2 |y|^{-m}$$

for a constant $C_2 = C_2(m)$. Since

$$\begin{aligned} \int_{\mathbb{R}^m \setminus B_2} |y|^{-m} |g(y)| dy &= \int_2^\infty r^{-m} \int_{\partial B_r} |g| d\sigma dr \\ &= -2^{-m} \int_{B_2} |g(y)| dy + m \int_2^\infty r^{-m-1} \int_{B_r} |g(y)| dy dr \\ &\leq m \|g\|_{M^{1,1}(\mathbb{R}^m)} \int_2^\infty \frac{dr}{r^2} \end{aligned}$$

by an integration by parts, we estimate

$$\text{osc}_B f_2 \leq \frac{C_2 m}{2} \|g\|_{M^{1,1}(\mathbb{R}^m)},$$

and the claim follows. \square

2.5. A Gagliardo-Nirenberg type inequality. Here we discuss an estimate similar to interpolation inequalities discovered independently by Gagliardo [18] and Nirenberg [33, 34]. In contrast to the classical versions, however, these inequalities involve the space $\text{BMO}(\mathbb{R}^m)$ in addition to $L^p(\mathbb{R}^m)$.

The inequality was stated by Adams and Frazier [2], but with only a sketch of the proof. For a special case, a different proof was given by Meyer and Rivière [27], and yet another method was used independently by Strzelecki [48] and the author [31].

Theorem 10 (Adams-Frazier). *Let $p_0 > 1$. There exists a constant C such that for all $p \in (1, p_0]$ and all $f \in W^{2,p}(\mathbb{R}^m) \cap \text{BMO}(\mathbb{R}^m)$, the gradient ∇f belongs to $L^{2p}(\mathbb{R}^m; \mathbb{R}^m)$ and*

$$\|\nabla f\|_{L^{2p}(\mathbb{R}^m)} \leq \frac{C}{p-1} [f]_{\text{BMO}(\mathbb{R}^m)} \|\nabla^2 f\|_{\text{BMO}(\mathbb{R}^m)}.$$

Again, there is one detail that is contained only implicitly in the papers cited. In order to see how the constant depends on p , we need to examine the proofs. The most transparent for this purpose is probably the proof given by Strzelecki. We see that there is only one instance of a constant that blows up when $p \rightarrow 1$, namely when the L^p -norm of a Hardy-Littlewood maximal function is estimated. The blow-up rate is of the order $(p-1)^{-1}$ [44, Section I.3.1].

We will combine this estimate with the Calderon-Zygmund inequality for the Laplacian, in order to replace the Hessian on the right-hand side by the Laplacian of f . This involves a constant that blows up for $p \rightarrow 1$ as well, and we need to know the blow-up rate. In most of the literature, this information is not given explicitly, but the most common proof uses the Marcinkiewicz interpolation theorem, and the corresponding constant is given explicitly, e.g., in a book by DiBenedetto [12, Theorem VIII.9.1]. An estimate of the constant is also given by Calderon and Zygmund [4].

Theorem 11 (Calderon-Zygmund). *Let $p_0 > 1$. There exists a constant $C > 0$ such that for $p \in (1, p_0]$ and for all $f \in W^{2,p}(\mathbb{R}^m)$,*

$$\|\nabla^2 f\|_{L^p(\mathbb{R}^m)} \leq \frac{C}{p-1} \|\Delta f\|_{L^p(\mathbb{R}^m)}.$$

The behaviour of the constants for $p \rightarrow \infty$ is irrelevant for our purpose, so we do not discuss it.

2.6. A Gehring type lemma. Finally, we will need the following result, which is an improvement of a lemma of Gehring [19] due to Giaquinta and Modica [20].

Theorem 12 (Giaquinta-Modica). *Suppose that $1 < p < q$ and $c > 0$. Then there exist $a > p$, $\theta > 0$, and $C > 0$ with the following property. Suppose that $f \in L^p(B)$ and $g \in L^q(B)$ are nonnegative functions such that for every ball $B_r(x_0) \subset B$,*

$$\left(\int_{B_{r/2}(x_0)} f^p dx \right)^{\frac{1}{p}} \leq \theta \left(\int_{B_r(x_0)} f^p dx \right)^{\frac{1}{p}} + c \int_{B_r(x_0)} f dx + \left(\int_{B_r(x_0)} g^p dx \right)^{\frac{1}{p}}.$$

Then $f \in L^a(B_{1/2})$ with

$$\left(\int_{B_{1/2}} f^a dx \right)^{\frac{1}{a}} \leq C \left(\int_B g^a dx \right)^{\frac{1}{a}} + C \left(\int_B f^p dx \right)^{\frac{1}{p}}.$$

3. ANALYSIS OF EQUATION (6)

In this section we derive a few results for solutions of equations of the type (6). For an $\text{SO}(n)$ -valued function P and an \mathbb{R}^n -valued function or distribution f on \mathbb{R}^m , consider the equation

$$(13) \quad \text{div}(P\nabla v) = f \quad \text{in } \mathbb{R}^m.$$

3.1. Existence. First we need to establish existence of solutions under reasonable conditions. To this end, we work in the homogeneous Sobolev spaces $\dot{W}^{1,p}(\mathbb{R}^m; \mathbb{R}^n)$. If $\frac{1}{p} + \frac{1}{p'} = 1$, then we write $\dot{W}^{-1,p'}(\mathbb{R}^m; \mathbb{R}^n)$ for the dual space of $\dot{W}^{1,p}(\mathbb{R}^m; \mathbb{R}^n)$. The following is the key estimate for the existence result.

Lemma 13. *Let $p \in (1, \infty)$. There exist $C > 0$ and $\epsilon > 0$ such that the following holds true. Suppose that $P \in \text{BMO}(\mathbb{R}^m; \text{SO}(n))$ and $f \in \dot{W}^{-1,p}(\mathbb{R}^m)$. If $[P]_{\text{BMO}(\mathbb{R}^m)} \leq \epsilon$, then any weak solution $v \in W^{1,p}(\mathbb{R}^m; \mathbb{R}^n)$ of (13) satisfies*

$$\|v\|_{\dot{W}^{1,p}(\mathbb{R}^m)} \leq C \|f\|_{\dot{W}^{-1,p}(\mathbb{R}^m)}.$$

In other words, despite the low degree of regularity of the coefficients of (13), we have the estimate expected from elliptic equations, provided that P is sufficiently close to a constant in the BMO sense.

Proof. For the Helmholtz-Hodge decomposition operators h and H defined in section 2.3, define $\phi = h(P\nabla v)$ and $\Phi = H(P\nabla v)$, so that

$$P\nabla v = \nabla\phi + \Phi.$$

By Theorem 7, there is a constant $C_1 = C_1(m, n, p)$ such that

$$\|\Phi\|_{L^p(\mathbb{R}^m)} \leq C_1 [P]_{\text{BMO}(\mathbb{R}^m)} \|\nabla v\|_{L^p(\mathbb{R}^m)}.$$

Furthermore, we have $\Delta\phi = f$ in \mathbb{R}^m , and thus by standard elliptic estimates, there is another constant $C_2 = C_2(m, n, p)$ with

$$\|\nabla\phi\|_{L^p(\mathbb{R}^m)} \leq C_2 \|f\|_{\dot{W}^{-1,p}(\mathbb{R}^m)}.$$

Since P takes values in $\text{SO}(n)$, we have

$$|\nabla v| = |P\nabla v| \leq |\nabla\phi| + |\Phi|$$

pointwise. As long as $C_1\epsilon \leq \frac{1}{2}$, the desired inequality follows. \square

Proposition 14. *Let $p \in (1, \infty)$. There exists a number $\epsilon > 0$ such that for all $P \in \text{BMO}(\mathbb{R}^m; \text{SO}(n))$ with $[P]_{\text{BMO}(\mathbb{R}^m)} \leq \epsilon$ and for all $f \in \dot{W}^{-1,p}(\mathbb{R}^m; \mathbb{R}^n)$, equation (13) has a unique solution in $\dot{W}^{1,p}(\mathbb{R}^m; \mathbb{R}^n)$.*

Proof. Choose ϵ so small that the preceding lemma applies. Uniqueness then follows immediately from the linearity and Lemma 13.

In order to prove existence, consider $P_t = tP + (1-t)I$ for $t \in \mathbb{R}$, where I is the identity ($n \times n$)-matrix. Define the operators

$$L_t : \dot{W}^{1,p}(\mathbb{R}^m; \mathbb{R}^n) \rightarrow \dot{W}^{-1,p}(\mathbb{R}^m; \mathbb{R}^n)$$

by

$$L_t v = \text{div}(P_t \nabla v).$$

Let Θ be the set of all $t \in [0, 1]$ such that L_t is invertible. Set $T = \sup \Theta$. Clearly, $0 \in \Theta$, so $T \geq 0$.

We claim that $T \in \Theta$. Injectivity of L_T follows from Lemma 13. To prove surjectivity, let $f \in \dot{W}^{-1,p}(\mathbb{R}^m; \mathbb{R}^n)$. Choose a sequence $(t_k)_{k \in \mathbb{N}}$ in Θ with $T = \lim_{k \rightarrow \infty} t_k$. For every $k \in \mathbb{N}$, let $v_k = L_{t_k}^{-1} f$. Then the sequence $(v_k)_{k \in \mathbb{N}}$ is bounded in $\dot{W}^{1,p}(\mathbb{R}^m; \mathbb{R}^n)$ by Lemma 13. Hence there exists a weakly convergent subsequence. The weak limit $v \in \dot{W}^{1,p}(\mathbb{R}^m; \mathbb{R}^n)$ satisfies $L_T v = f$.

Now consider the operator

$$\mathcal{L} : \dot{W}^{1,p}(\mathbb{R}^m; \mathbb{R}^n) \times \dot{W}^{-1,p}(\mathbb{R}^m; \mathbb{R}^n) \times \mathbb{R} \rightarrow \dot{W}^{-1,p}(\mathbb{R}^m; \mathbb{R}^n)$$

with

$$\mathcal{L}(v, f, t) = \text{div}(P_t \nabla v) - f.$$

It is continuously Fréchet differentiable with

$$D\mathcal{L}(v, f, t)(\omega, \phi, \tau) = \operatorname{div}(P_t \nabla \omega) - \phi + \tau \operatorname{div}((P - I) \nabla v).$$

In particular,

$$D\mathcal{L}(0, 0, T)(\omega, 0, 0) = L_T \omega.$$

We have seen that this operator is invertible. By the implicit function theorem, there exists a number $\rho > 0$ such that for all $t \in (T - \rho, T + \rho)$ and for all $f \in \dot{W}^{-1,p}(\mathbb{R}^m; \mathbb{R}^n)$ with $\|f\|_{\dot{W}^{-1,p}(\mathbb{R}^m)} < \delta$, there exists a $v \in \dot{W}^{1,p}(\mathbb{R}^m; \mathbb{R}^n)$ with

$$\mathcal{L}(v, f, t) = 0;$$

that is,

$$L_t v = f.$$

By the linearity, we can dispense with the smallness of $\|f\|_{\dot{W}^{-1,p}(\mathbb{R}^m)}$. In other words, the operator L_t is invertible for $t \in (T - \delta, T + \delta)$. It follows that $T = 1$, and the proof is complete. \square

3.2. An estimate in a Morrey space. The space $\dot{W}^{1,p}(\mathbb{R}^m; \mathbb{R}^n)$ is convenient when we want to prove existence of solutions. But once this is established, we need estimates in other spaces as well. In particular, we need to estimate the derivatives of solutions in certain Morrey spaces. To this end, we use the following lemma.

Lemma 15. *Let $p \in (1, 2)$. There exist $\epsilon > 0$, $\theta \in (0, 1)$, and $C > 0$ such that the following holds true. Suppose that $P \in \operatorname{BMO}(B; \operatorname{SO}(n))$ with $[P]_{\operatorname{BMO}(B)} \leq \epsilon$ and $f \in M^{1,2}(B; \mathbb{R}^n)$. If $v \in W^{1,p}(B; \mathbb{R}^n)$ is a solution of*

$$\operatorname{div}(P \nabla v) = f \quad \text{in } B,$$

then

$$\theta^{1-\frac{m}{p}} \|\nabla v\|_{L^p(B_\theta)} \leq \frac{1}{2} \|\nabla v\|_{L^p(B)} + C \|f\|_{M^{1,2}(B)}.$$

Proof. Choose $\eta \in C_0^\infty(B)$ with $\eta \equiv 1$ in $B_{1/2}$. Let $\tilde{v} = \eta(v - \bar{v}_B)$ and define $\phi = h(P \nabla \tilde{v})$ and $\Phi = H(P \nabla \tilde{v})$, so that

$$P \nabla \tilde{v} = \nabla \phi + \Phi.$$

We have

$$\|\Phi\|_{L^p(\mathbb{R}^m)} \leq C_1 [P]_{\operatorname{BMO}(B)} \|\nabla \tilde{v}\|_{L^p(\mathbb{R}^m)} \leq C_2 \epsilon \|\nabla v\|_{L^p(B)}$$

for certain constants $C_1 = C_1(m, n, p)$ and $C_2 = C_2(m, n, p, \eta)$. The first inequality follows from Theorem 7, and the second one from the Poincaré inequality. Now let G be the fundamental solution of the Laplace equation and define

$$\psi(x) = \int_B G(x - y) f(y) dy$$

for $x \in B$. Then by Theorem 8, we have a constant $C_3 = C_3(m, n, p)$ such that

$$\|\nabla \psi\|_{L^p(B)} \leq C_3 \|f\|_{M^{1,2}(B)}.$$

Let $\theta \in (0, \frac{1}{4}]$. We have $\Delta(\phi - \psi) = 0$ in $B_{1/2}$. Thus the mean value formula for harmonic functions implies

$$\|\nabla \phi - \nabla \psi\|_{L^p(B_\theta)} \leq C_4 \theta^{\frac{m}{p}} \|\nabla \phi - \nabla \psi\|_{L^p(B_{1/2})}$$

for a constant $C_4 = C_4(m, n)$.

Combining the inequalities, we find

$$\begin{aligned} \|\nabla v\|_{L^p(B_\theta)} &\leq \|\Phi\|_{L^p(B)} + \|\nabla\psi\|_{L^p(B)} + \|\nabla\phi - \nabla\psi\|_{L^p(B_\theta)} \\ &\leq (C_4 + 1) (\|\Phi\|_{L^p(B)} + \|\nabla\psi\|_{L^p(B)}) + C_4\theta^{\frac{m}{p}} \|\nabla v\|_{L^p(B)} \\ &\leq C_5 \left(\epsilon + \theta^{\frac{m}{p}} \right) \|\nabla v\|_{L^p(B)} + C_5 \|f\|_{M^{1,2}(B)}, \end{aligned}$$

where $C_5 = (C_4 + 1)(C_2 + C_3 + 1)$. We can choose ϵ and θ so small that

$$C_5 \left(\epsilon\theta^{1-\frac{m}{p}} + \theta \right) \leq \frac{1}{2},$$

and then the desired inequality follows. \square

Proposition 16. *Let $p \in (\frac{m}{m-1}, 2)$. Then there exist $\epsilon > 0$ and $C > 0$ with the following property. Suppose that $P \in \text{BMO}(\mathbb{R}^m; \text{SO}(n))$ with $[P]_{\text{BMO}(\mathbb{R}^m)} \leq \epsilon$. Let $f \in M^{1,2}(\mathbb{R}^m; \mathbb{R}^n)$ with compact support. Then there exists a unique solution $v \in \dot{W}^{1,p}(\mathbb{R}^m; \mathbb{R}^n)$ of (13), and it satisfies*

$$\|\nabla v\|_{M^{p,p}(\mathbb{R}^m)} \leq C \|f\|_{M^{1,2}(\mathbb{R}^m)}.$$

Proof. Choose $R > 0$ such that $\text{supp } f \subset B_R$. Let G be the fundamental solution of the Laplace equation again and define $\xi = G * f$, so that $\Delta\xi = f$ in \mathbb{R}^m . By Theorem 8, we have

$$\|\nabla\xi\|_{M_*^{2,2}(\mathbb{R}^m)} \leq C_1 \|f\|_{M^{1,2}(\mathbb{R}^m)}$$

for a constant $C_1 = C_1(m, n)$. Furthermore, there is a constant $C_2 = C_2(m, R)$ such that for all $x \in \mathbb{R}^m \setminus B_{2R}$,

$$|\nabla\xi(x)| \leq C_2 |x|^{1-m} \|f\|_{M^{1,2}(\mathbb{R}^m)}.$$

It follows that $\xi \in \dot{W}^{1,p}(\mathbb{R}^m; \mathbb{R}^n)$. Let $p' = \frac{p}{p-1}$ and consider a function $\phi \in \dot{W}^{1,p'}(\mathbb{R}^m; \mathbb{R}^n)$. Then

$$\int_{\mathbb{R}^m} f\phi \, dx = - \int_{\mathbb{R}^m} \nabla\xi \cdot \nabla\phi \, dx \leq \|\nabla\xi\|_{L^p(\mathbb{R}^m)} \|\nabla\phi\|_{L^{p'}(\mathbb{R}^m)}.$$

Thus $f \in \dot{W}^{-1,p}(\mathbb{R}^m; \mathbb{R}^n)$, and hence equation (13) has a unique solution $v \in \dot{W}^{1,p}(\mathbb{R}^m; \mathbb{R}^n)$ by Proposition 14.

We still have to prove the inequality. Choose a ball $B_r(x_0) \subset \mathbb{R}^m$. Consider the functions

$$\tilde{v}(x) = v(rx + x_0), \quad \tilde{P}(x) = P(rx + x_0), \quad \text{and} \quad \tilde{f}(x) = r^2 f(rx + x_0).$$

Then

$$\text{div}(\tilde{P}\nabla\tilde{v}) = \tilde{f} \quad \text{in } B.$$

Lemma 15 yields

$$\theta^{1-\frac{m}{p}} \|\nabla\tilde{v}\|_{L^p(B_\theta)} \leq \frac{1}{2} \|\nabla\tilde{v}\|_{L^p(B)} + C_3 \|\tilde{f}\|_{M^{1,2}(\mathbb{R}^m)}$$

for a constant $C_3 = C_3(m, n, p)$, provided that ϵ is sufficiently small. This inequality can be rewritten as

$$(\theta r)^{1-\frac{m}{p}} \|\nabla v\|_{L^p(B_{\theta r}(x_0))} \leq \frac{1}{2} r^{1-\frac{m}{p}} \|\nabla v\|_{L^p(B_r(x_0))} + C_3 \|f\|_{M^{1,2}(\mathbb{R}^m)}.$$

Since $|\nabla v| \in L^p(\mathbb{R}^m)$, we have

$$\lim_{r \rightarrow \infty} \left(r^{1-\frac{m}{p}} \|\nabla v\|_{L^p(B_r(x_0))} \right) = 0.$$

If x_0 is a Lebesgue point of $|\nabla v|^p$, then it follows that the quantity

$$\alpha(x_0) = \sup_{r>0} \left(r^{1-\frac{m}{p}} \|\nabla v\|_{L^p(B_r(x_0))} \right)$$

is finite. The above inequality then implies

$$\alpha(x_0) \leq 2C_3 \|f\|_{M^{1,2}(\mathbb{R}^m)}.$$

Since α is the supremum of continuous functions, it is lower semicontinuous. Hence the same inequality follows for every $x_0 \in \mathbb{R}^m$. \square

3.3. An interpolation inequality. We need one more inequality for solutions of (13).

Proposition 17. *Let $p_0 > 1$. There exist $\epsilon > 0$ and $C > 0$ with the following property. Suppose that $P \in \text{BMO}(\mathbb{R}^m; \text{SO}(n))$ with $[P]_{\text{BMO}(\mathbb{R}^m)} \leq \epsilon$. Let $p \in (1, p_0]$ and $f \in L^p(\mathbb{R}^m; \mathbb{R}^n)$. Suppose that $v \in \dot{W}^{1,2p}(\mathbb{R}^m; \mathbb{R}^n)$ solves (13). Then*

$$\|\nabla v\|_{L^{2p}(\mathbb{R}^m)}^2 \leq \frac{C}{(p-1)^2} \|\nabla v\|_{M^{1,1}(\mathbb{R}^m)} \|f\|_{L^p(\mathbb{R}^m)}.$$

Proof. Define $\phi = h(P\nabla v)$ and $\Phi = H(P\nabla v)$. Then $P\nabla v = \nabla\phi + \Phi$. According to Theorem 7, we have a constant $C_1 = C_1(m, n, p_0)$ such that

$$\|\Phi\|_{L^{2p}(\mathbb{R}^m)} \leq C_1 \epsilon \|\nabla v\|_{L^{2p}(\mathbb{R}^m)}.$$

Since $\Delta\phi = f$ in \mathbb{R}^m , a combination of Theorem 10 and Theorem 11 gives

$$\|\nabla\phi\|_{L^{2p}(\mathbb{R}^m)}^2 \leq \frac{C_2}{(p-1)^2} [\phi]_{\text{BMO}(\mathbb{R}^m)} \|f\|_{L^p(\mathbb{R}^m)}$$

for a constant $C_2 = C_2(m, n, p_0)$. Note also that the equation

$$\Delta\phi = \text{div}(P\nabla v)$$

implies

$$[\phi]_{\text{BMO}(\mathbb{R}^m)} \leq C_3 \|P\nabla v\|_{M^{1,1}(\mathbb{R}^m)} = C_3 \|\nabla v\|_{M^{1,1}(\mathbb{R}^m)}$$

for a constant $C_3 = C_3(m, n)$ by Lemma 9. Hence

$$\begin{aligned} \|\nabla v\|_{L^{2p}(\mathbb{R}^m)}^2 &\leq 2\|\Phi\|_{L^{2p}(\mathbb{R}^m)}^2 + 2\|\nabla\phi\|_{L^{2p}(\mathbb{R}^m)}^2 \\ &\leq 2C_1^2 \epsilon^2 \|\nabla v\|_{L^{2p}(\mathbb{R}^m)}^2 + \frac{2C_2 C_3}{(p-1)^2} \|\nabla v\|_{M^{1,1}(\mathbb{R}^m)} \|f\|_{L^p(\mathbb{R}^m)}. \end{aligned}$$

If $C_1 \epsilon \leq \frac{1}{2}$, the desired inequality follows. \square

4. ANALYSIS OF EQUATION (7)

We now study equations similar to (13), but with a more specific right-hand side. The aim of this section is to derive inequalities for solutions of equations of the form

$$\text{div}(P\nabla v) = \Omega \cdot P\nabla v_0 \quad \text{in } \mathbb{R}^m.$$

4.1. Another estimate in a Morrey space. The following is a variant of an inequality obtained by Rivière and Struwe [38]. Much of the proof is practically identical, but we give the details for completeness.

Proposition 18. *Let $\gamma > 0$ and $q \in (\frac{m}{m-1}, 1 + \frac{2}{m})$. There exists a number $\epsilon > 0$ such that the following holds true. Let $P, Q \in W_{\text{loc}}^{1,2}(\mathbb{R}^m; \text{SO}(n))$ with*

$$\|\nabla P\|_{M^{2,2}(\mathbb{R}^m)} + \|\nabla Q\|_{M^{2,2}(\mathbb{R}^m)} \leq \epsilon$$

and $\Omega \in M^{2,2}(\mathbb{R}^m; \mathbb{R}^m \otimes \mathfrak{so}(n))$ with compact support, satisfying $\text{div } \Omega = 0$ and $\|\Omega\|_{M^{2,2}(\mathbb{R}^m)} \leq \epsilon$. Suppose that $v_0 \in W_{\text{loc}}^{1,2}(\mathbb{R}^m; \mathbb{R}^n)$. Then there exists a unique $v \in \dot{W}^{1,q}(\mathbb{R}^m; \mathbb{R}^n)$ satisfying the equation

$$(14) \quad \text{div}(P\nabla v) = \Omega \cdot Q\nabla v_0.$$

Furthermore,

$$\|\nabla v\|_{M^{1,1}(\mathbb{R}^m)} \leq \gamma \|\nabla v_0\|_{M^{1,1}(\mathbb{R}^m)}.$$

Proof. As $|\Omega| \in M^{2,2}(\mathbb{R}^m)$ and $|Q\nabla v_0| \in L^2(\text{supp } \Omega)$, the Hölder inequality implies

$$\Omega \cdot Q\nabla v_0 \in M^{1,1+m/2}(\mathbb{R}^m, \mathbb{R}^n).$$

If G is the fundamental solution of the Laplace equation and $\mu = G * (\Omega \cdot Q\nabla v_0)$, it follows from Theorem 8 that

$$|\nabla \mu| \in M_*^{1+2/m, 1+m/2}(\mathbb{R}^m).$$

Since $\text{supp}(\Omega \cdot Q\nabla v_0)$ is compact, we conclude with the same arguments as in the proof of Proposition 16 that $\Omega \cdot Q\nabla v_0 \in \dot{W}^{-1,q}(\mathbb{R}^m; \mathbb{R}^n)$. Thus (14) has a unique solution $v \in \dot{W}^{1,q}(\mathbb{R}^m; \mathbb{R}^n)$ if ϵ is sufficiently small by Proposition 14. Next we want to verify the inequality.

We first assume that $v \in W_{\text{loc}}^{1,2}(\mathbb{R}^m; \mathbb{R}^n)$ and $\|\nabla v\|_{M^{1,1}(\mathbb{R}^m)}$ is finite. As everything is invariant under scaling, it then suffices to show that for some $\theta \in (0, 1)$,

$$(15) \quad \theta^{1-m} \int_{B_\theta} |\nabla v| dx \leq \frac{1}{2} \|\nabla v\|_{M^{1,1}(\mathbb{R}^m)} + \frac{\gamma}{2} \|\nabla v_0\|_{M^{1,1}(\mathbb{R}^m)}.$$

A similar inequality then follows for all balls in \mathbb{R}^m , and taking the supremum, we obtain the desired inequality.

Let $\eta \in C_0^\infty(B)$ with $\eta \equiv 1$ in $B_{1/2}$. Set $\tilde{v} = \eta(v - \bar{v}_B)$ and consider $\phi = h(P\nabla \tilde{v})$ and $\Phi = H(P\nabla \tilde{v})$. Then

$$\begin{aligned} \int_{\mathbb{R}^m} |\Phi|^2 dx &= \int_{\mathbb{R}^m} \langle \Phi, P\nabla \tilde{v} \rangle dx = - \int_{\mathbb{R}^m} \langle \Phi, \nabla P \tilde{v} \rangle dx \\ &\leq C_1 \|\Phi\|_{L^2(\mathbb{R}^m)} \|\nabla P\|_{L^2(B)} [v]_{\text{BMO}(B)} \end{aligned}$$

for a constant $C_1 = C_1(m, n, \eta)$ by Theorem 6 and Theorem 5. Hence

$$\int_B |\Phi| dx \leq |B|^{\frac{1}{2}} \|\Phi\|_{L^2(\mathbb{R}^m)} \leq C_1 |B|^{\frac{1}{2}} \epsilon [v]_{\text{BMO}(B)}.$$

Now let ψ be the solution of the Dirichlet problem

$$\begin{aligned} \Delta \psi &= 0 \quad \text{in } B_{1/2}, \\ \psi &= \phi \quad \text{on } \partial B_{1/2}. \end{aligned}$$

Then by the mean value formula for harmonic functions, there exists a constant $C_2 = C_2(m, n)$ such that

$$\int_{B_\theta} |\nabla\psi| dx \leq C_2\theta^m \int_{B_{1/2}} |\nabla\psi| dx.$$

We also note that

$$\Delta(\phi - \psi) = \Omega \cdot Q\nabla v_0 \quad \text{in } B_{1/2}.$$

Let ζ be the solution of the boundary value problem

$$\begin{aligned} -\Delta\zeta &= \operatorname{div} \left(\frac{\nabla\phi - \nabla\psi}{|\nabla\phi - \nabla\psi|} \right) \quad \text{in } B_{1/2}, \\ \zeta &= 0 \quad \text{on } \partial B_{1/2}, \end{aligned}$$

where $\frac{\nabla\phi - \nabla\psi}{|\nabla\phi - \nabla\psi|}$ is extended by 0 where $\nabla\phi = \nabla\psi$. Standard elliptic estimates imply that

$$\|\zeta\|_{L^\infty(B_{1/2})} + \|\nabla\zeta\|_{L^2(B_{1/2})} \leq C_3 = C_3(m, n).$$

We compute

$$\begin{aligned} \int_{B_{1/2}} |\nabla\phi - \nabla\psi| dx &= \int_{B_{1/2}} \langle \Delta\zeta, \phi - \psi \rangle dx = \int_{B_{1/2}} \langle \zeta, \Omega \cdot Q\nabla v_0 \rangle dx \\ &= \sum_{i,j,k=1}^n \int_{B_{1/2}} \nabla(\zeta_i Q_{jk}) \cdot \Omega^{ij} \left(v_0^k - \overline{(v_0^k)}_{B_{1/2}} \right) dx. \end{aligned}$$

Using Theorem 6 and Theorem 5 again, we conclude that there is a constant $C_4 = C_4(m, n)$ such that

$$\int_{B_{1/2}} |\nabla\phi - \nabla\psi| dx \leq C_4\epsilon[v_0]_{\operatorname{BMO}(B)},$$

provided that $\epsilon \leq 1$. Finally, we combine the inequalities and obtain

$$\begin{aligned} \int_{B_\theta} |\nabla v| dx &\leq \int_B |\Phi| dx + \int_{B_\theta} |\nabla\psi| dx + \int_{B_{1/2}} |\nabla\phi - \nabla\psi| dx \\ &\leq (1 + C_2) \left(\int_B |\Phi| dx + \int_{B_{1/2}} |\nabla\phi - \nabla\psi| dx \right) + C_2\theta^m \int_B |\nabla v| dx \\ &\leq C_5\epsilon ([v]_{\operatorname{BMO}(B)} + [v_0]_{\operatorname{BMO}(B)}) + C_2\theta^m \int_B |\nabla v| dx \end{aligned}$$

for $\theta \in (0, \frac{1}{2}]$, where $C_5 = (1 + C_2)(C_1|B|^{1/2} + C_4)$. Using the Poincaré inequality, we infer that there exists a constant $C_6 = C_6(m, n, \eta)$ such that

$$\theta^{1-m} \int_{B_\theta} |\nabla v| dx \leq C_6(\theta^{1-m}\epsilon + \theta)\|\nabla v\|_{M^{1,1}(\mathbb{R}^m)} + C_6\theta^{1-m}\epsilon\|\nabla v_0\|_{M^{1,1}(\mathbb{R}^m)}.$$

If θ and ϵ are sufficiently small, then we have (15).

It remains to show that we have, in fact,

$$\|\nabla v\|_{M^{1,1}(\mathbb{R}^m)} < \infty,$$

provided that v_0 has this property, and that we can dispense with the assumption that $v \in W_{\operatorname{loc}}^{1,2}(\mathbb{R}^m; \mathbb{R}^n)$. Let $\xi \in C_0^\infty(B)$ with $\xi \geq 0$ and

$$\int_{\mathbb{R}^m} \xi dx = 1.$$

For $\rho \in (0, 1]$, set $\xi_\rho(x) = \rho^{-m}\xi(x/\rho)$, and define $\Omega_\rho = \xi_\rho * \Omega$. Then Ω_ρ is still divergence free and has compact support, and for a constant $C_7 = C_7(m, n, \xi)$, we have $\|\Omega_\rho\|_{M^{2,2}(\mathbb{R}^m)} \leq C_7\epsilon$. Moreover, if ϵ is sufficiently small, then using a convolution with ξ_ρ and a projection to $\text{SO}(n)$, similarly to a method used by Schoen and Uhlenbeck [41, Section 4], we can construct a family of maps $P_\rho \in C^\infty(\mathbb{R}^m; \text{SO}(n))$ such that $P_\rho \rightarrow P$ in $L^a_{\text{loc}}(\mathbb{R}^m; \mathbb{R}^{n \times n})$ for every $a < \infty$, and there exists a constant $C_8 = C_8(m, n, \xi)$ such that $\|\nabla P_\rho\|_{M^{2,2}(\mathbb{R}^m)} \leq C_8\epsilon$.

We have $\Omega_\rho \cdot Q\nabla v_0 \in \dot{W}^{-1,q}(\mathbb{R}^m)$ and we have a uniform bound in this space. Hence by Proposition 14, there exists a bounded family of solutions $v_\rho \in \dot{W}^{1,q}(\mathbb{R}^m)$ of

$$\text{div}(P_\rho \nabla v_\rho) = \Omega_\rho \cdot Q\nabla v_0.$$

If $\|\nabla v_0\|_{M^{1,1}(\mathbb{R}^m)} < \infty$, then it follows from Proposition 16 that

$$\|\nabla v_\rho\|_{M^{1,1}(\mathbb{R}^m)} < \infty.$$

Furthermore, since

$$\Delta v_\rho = P_\rho^{-1}\Omega_\rho \cdot Q\nabla v_0 - P_\rho^{-1}\nabla P_\rho \cdot \nabla v_\rho,$$

we have $v_\rho \in W^{1,2}_{\text{loc}}(\mathbb{R}^m; \mathbb{R}^n)$ by standard elliptic estimates. With the previous arguments, we then prove

$$\|\nabla v_\rho\|_{M^{1,1}(\mathbb{R}^m)} \leq \gamma \|\nabla v_0\|_{M^{1,1}(\mathbb{R}^m)}$$

if ϵ is sufficiently small.

There exist $\rho_k \rightarrow 0$ such that $(v_{\rho_k})_{k \in \mathbb{N}}$ converges weakly in $\dot{W}^{1,q}(\mathbb{R}^m; \mathbb{R}^n)$. Let \hat{v} be the weak limit. Then it satisfies

$$\text{div}(P\nabla \hat{v}) = \Omega \cdot Q\nabla v_0.$$

This implies $\hat{v} = v$, for solutions to this equation are unique in $\dot{W}^{1,q}(\mathbb{R}^m; \mathbb{R}^n)$. It follows that

$$\|\nabla v\|_{M^{1,1}(\mathbb{R}^m)} \leq \gamma \|\nabla v_0\|_{M^{1,1}(\mathbb{R}^m)},$$

and this concludes the proof. \square

4.2. Another interpolation inequality. We also want a counterpart to Proposition 17 for solution of an equation of the type (7).

Proposition 19. *Let $p_0 > 1$. There exist $\epsilon > 0$ and $C > 0$ with the following property. Suppose that $P \in W^{1,2}_{\text{loc}}(\mathbb{R}^m; \text{SO}(n))$ with $\|\nabla P\|_{M^{2,2}(\mathbb{R}^m)} \leq \epsilon$ and such that $\text{supp}(P - I)$ is compact. Furthermore, suppose that $\Omega \in M^{2,2}(\mathbb{R}^m; \mathbb{R}^m \otimes \mathfrak{so}(n))$ has compact support and satisfies $\text{div } \Omega = 0$ and $\|\Omega\|_{M^{2,2}(\mathbb{R}^m)} \leq \epsilon$. Let $p \in (1, p_0]$ and $q \in (\frac{m}{m-1}, 1 + \frac{2}{m})$. Suppose that $v_0 \in \dot{W}^{1,2p}(\mathbb{R}^m; \mathbb{R}^n)$ with $\|\nabla v_0\|_{M^{1,1}(\mathbb{R}^m)} < \infty$, and let $v \in \dot{W}^{1,q}(\mathbb{R}^m; \mathbb{R}^n)$ be the solution of*

$$\text{div}(P\nabla v) = \Omega \cdot P\nabla v_0.$$

Then $\|\nabla v\|_{M^{1,1}(\mathbb{R}^m)} < \infty$ and $v \in \dot{W}^{1,4p/(p+1)}(\mathbb{R}^m; \mathbb{R}^n)$ with

$$(16) \quad \|\nabla v\|_{L^{4p/(p+1)}(\mathbb{R}^m)} \leq \frac{C}{(p-1)^2} \|\nabla v\|_{M^{1,1}(\mathbb{R}^m)} \|\Omega\|_{L^2(\mathbb{R}^m)} \|\nabla v_0\|_{L^{2p}(\mathbb{R}^m)}.$$

Proof. First assume that $v \in \dot{W}^{1,4p/(p+1)}(\mathbb{R}^m; \mathbb{R}^n)$. It follows from Proposition 18 that

$$\|\nabla v\|_{M^{1,1}(\mathbb{R}^m)} < \infty.$$

Inequality (16) follows from Proposition 17, as Hölder's inequality implies

$$\|\Omega \cdot P\nabla v_0\|_{L^{2p/(p+1)}(\mathbb{R}^m)} \leq \|\Omega\|_{L^2(\mathbb{R}^m)} \|\nabla v_0\|_{L^{2p}(\mathbb{R}^m)}.$$

Now we want to prove that $v \in \dot{W}^{1,4p/(p+1)}(\mathbb{R}^m; \mathbb{R}^n)$. Consider the same approximations Ω_ρ and P_ρ of Ω and P , respectively, as in the proof of Proposition 18. For every $\rho \in (0, 1]$, solve

$$\operatorname{div}(P_\rho \nabla v_\rho) = \Omega_\rho \cdot P\nabla v_0$$

in $\dot{W}^{1,q}(\mathbb{R}^m)$. The solutions are uniformly bounded in this space, since we have a uniform bound for the right-hand side in $\dot{W}^{-1,q}(\mathbb{R}^m)$.

We compute

$$(17) \quad \Delta v_\rho = P_\rho^{-1} \Omega_\rho \cdot P\nabla v_0 - P_\rho^{-1} \nabla P_\rho \cdot \nabla v_\rho.$$

Proposition 18 implies that

$$\sup_{\rho>0} \|\nabla v_\rho\|_{M^{1,1}(\mathbb{R}^m)} < \infty.$$

Using (17) and Theorem 10, we can improve the regularity of v_ρ step by step until we have $v_\rho \in \dot{W}^{1,4p/(p+1)}(\mathbb{R}^m)$. Proposition 17 then gives a constant $C_1 = C_1(m, n, p_0)$ such that

$$\|\nabla v_\rho\|_{L^{4p/(p+1)}(\mathbb{R}^m)} \leq \frac{C_1}{(p-1)^2} \|\nabla v_\rho\|_{M^{1,1}(\mathbb{R}^m)} \|\Omega_\rho\|_{L^2(\mathbb{R}^m)} \|\nabla v_0\|_{L^{2p}(\mathbb{R}^m)}$$

if ϵ is sufficiently small.

We infer the existence of a subsequence that converges weakly in

$$\dot{W}^{1,q}(\mathbb{R}^m; \mathbb{R}^n) \cap \dot{W}^{1,4p/(p+1)}(\mathbb{R}^m; \mathbb{R}^n).$$

The weak limit coincides with v by Proposition 14. In particular, we have $v \in \dot{W}^{1,4p/(p+1)}(\mathbb{R}^m; \mathbb{R}^n)$. \square

5. PROOFS OF THE MAIN RESULTS

In this section, we first establish an inequality on which the proof of Theorem 1 hinges. We then prove Theorem 1 and Theorem 3.

5.1. An estimate for the energy. Recall the equation

$$(18) \quad \Delta u = \Omega \cdot \nabla u + f \quad \text{in } B$$

derived in the introduction. We now use the preceding results to analyse its solutions.

Let $p \in (1, \frac{m}{m-2})$ with $p \leq 2$ and set

$$s = \frac{2mp}{m+2p}.$$

Note that $1 < s < 2$. Let $\epsilon, \delta > 0$. We consider $\Omega \subset M^{2,2}(B; \mathbb{R}^m \otimes \mathfrak{so}(n))$ and $f \in L^p(B; \mathbb{R}^n)$. Let $u \in W^{1,2}(B; \mathbb{R}^n)$ be a weak solution of (18). We assume that

$$\|\nabla u\|_{M^{2,2}(B)} + \|\Omega\|_{M^{2,2}(B)} \leq \epsilon.$$

We want to show that whenever ϵ is sufficiently small, then

(19)

$$\int_{B_{1/2}} |\nabla u|^2 dx \leq \delta \int_B |\Omega|^2 dx + C \left(\int_B (|\nabla u|^s + |\Omega|^s) dx \right)^{\frac{2}{s}} + C \left(\int_B |f|^p dx \right)^{\frac{1}{p}}.$$

This inequality is the key to Theorem 1. We divide its proof into several steps.

First step. We first want to turn (18) into an equation on \mathbb{R}^m rather than B . To this end, choose a cut-off function $\eta \in C_0^\infty(B_{3/4})$ with $0 \leq \eta \leq 1$ and $\eta \equiv 1$ in $B_{1/2}$. Define

$$\tilde{u} = \eta(u - \bar{u}_B).$$

Then we compute

$$\Delta \tilde{u} = \eta \Omega \cdot \nabla \tilde{u} + \eta f - \nabla \eta \cdot \Omega(u - \bar{u}_B) + 2\nabla \eta \cdot \nabla u + \Delta \eta(u - \bar{u}_B).$$

Define

$$\tilde{\Omega} = \eta \Omega, \quad \tilde{f}_1 = \eta f$$

and

$$\tilde{f}_2 = -\nabla \eta \cdot \Omega(u - \bar{u}_B) + 2\nabla \eta \cdot \nabla u + \Delta \eta(u - \bar{u}_B).$$

Furthermore, define $\tilde{f} = \tilde{f}_1 + \tilde{f}_2$, so that

$$\Delta \tilde{u} = \tilde{\Omega} \cdot \nabla \tilde{u} + \tilde{f}.$$

We claim that we still have

$$(20) \quad \|\nabla \tilde{u}\|_{M^{2,2}(\mathbb{R}^m)} + \|\tilde{\Omega}\|_{M^{2,2}(\mathbb{R}^m)} \leq C_1 \epsilon$$

for a constant $C_1 = C_1(m, n, \eta)$.

The estimate for $\tilde{\Omega}$ is clear. We have

$$\nabla \tilde{u} = \eta \nabla u + \nabla \eta(u - \bar{u}_B),$$

and the estimate for the first term on the right-hand side is clear as well. In order to estimate the other term, note that by the John-Nirenberg inequality [24], we have a constant $C_2 = C_2(m, n)$ such that

$$\int_B |u - \bar{u}_B|^m dx \leq C_2 \epsilon^m.$$

Let $x_0 \in B$ and $r > 0$. Then the Hölder inequality implies

$$\int_{B \cap B_r(x_0)} |\nabla \eta|^2 |u - \bar{u}_B|^2 dx \leq C_3 r^{m-2} \left(\int_{B \cap B_r(x_0)} |u - \bar{u}_B|^m dx \right)^{\frac{2}{m}}$$

for a constant $C_3 = C_3(m, \eta)$. Inequality (20) now follows.

We also have

$$\|\tilde{f}_1\|_{L^p(\mathbb{R}^m)} \leq \|f\|_{L^p(B)}$$

and there is a constant $C_4 = C_4(m, N, \eta)$ such that

$$\|\tilde{f}_2\|_{L^s(\mathbb{R}^m)} \leq C_4 \|\Omega\|_{L^s(B)} + C_4 \|\nabla u\|_{L^s(B)}.$$

Second step. We use the gauge transformation of Theorem 4. There exists a $P \in W^{1,2}(B; \text{SO}(n))$ such that for

$$\hat{\Omega} = \nabla P P^{-1} + P \tilde{\Omega} P^{-1},$$

extended by 0 outside of B , we have

$$\text{div } \hat{\Omega} = 0 \quad \text{in } \mathbb{R}^m.$$

Moreover, there exists a constant $C_5 = C_5(m, n, \eta)$ such that

$$\|\hat{\Omega}\|_{L^2(\mathbb{R}^m)} \leq C_5 \|\Omega\|_{L^2(B)}$$

and

$$\|\nabla P\|_{M^{2,2}(B)} + \|\hat{\Omega}\|_{M^{2,2}(\mathbb{R}^m)} \leq C_5 \epsilon.$$

It follows, in particular, that there is another constant $C_6 = C_6(m, n, \eta)$ such that

$$\text{dist}(\bar{P}_B, \text{SO}(n)) \leq \int_B |P - \bar{P}_B| dx \leq C_6 \epsilon.$$

Since for a constant $P_0 \in \text{SO}(n)$, replacing P by $P_0 P$ does not change the above properties, we may assume that

$$|\bar{P}_B - I| \leq C_6 \epsilon.$$

Choose another cut-off function $\xi \in C_0^\infty(B)$ with $\xi \equiv 1$ in $B_{3/4}$. Set

$$\tilde{P} = \xi P + (1 - \xi)I.$$

Then we have

$$\|\nabla \tilde{P}\|_{M^{2,2}(\mathbb{R}^m)} \leq C_7 \epsilon$$

for a constant $C_7 = C_7(m, n, \eta, \xi)$. Furthermore,

$$\text{div}(\tilde{P} \nabla \tilde{u}) = \hat{\Omega} \cdot \tilde{P} \nabla \tilde{u} + \tilde{P} \tilde{f} \quad \text{in } \mathbb{R}^m.$$

Third step. Fix $q \in (\frac{m}{m-1}, 1 + \frac{2}{m})$ with $q \leq \frac{mp}{m-p}$. We have $\tilde{P} \tilde{f}_1, \tilde{P} \tilde{f}_2 \in \dot{W}^{-1,q}(\mathbb{R}^m; \mathbb{R}^n)$, as both belong to $L^p(\mathbb{R}^m; \mathbb{R}^n)$ and have compact support. By Proposition 14, if ϵ is small enough, then there exist weak solutions $v_{01}, v_{02} \in \dot{W}^{1,q}(\mathbb{R}^m; \mathbb{R}^n)$ of

$$\text{div}(\tilde{P} \nabla v_{01}) = \tilde{P} \tilde{f}_1 \quad \text{in } \mathbb{R}^m$$

and

$$\text{div}(\tilde{P} \nabla v_{02}) = \tilde{P} \tilde{f}_2 \quad \text{in } \mathbb{R}^m.$$

Set $v_0 = v_{01} + v_{02}$ and $w_0 = \tilde{u} - v_0$. Then

$$\text{div}(\tilde{P} \nabla w_0) = \hat{\Omega} \cdot \tilde{P} \nabla \tilde{u},$$

which we can also write as

$$\text{div}(\tilde{P} \nabla w_0) = \hat{\Omega} \cdot \tilde{P} \nabla w_0 + \hat{\Omega} \cdot \tilde{P} \nabla v_0.$$

Since

$$\|\hat{\Omega} \cdot \tilde{P} \nabla \tilde{u}\|_{M^{1,2}(\mathbb{R}^m)} \leq C_1 C_5 \epsilon^2,$$

Proposition 16, together with the Hölder inequality, implies that there exists a constant $C_8 = C_8(m, n, q, \eta)$ such that

$$\|\nabla w_0\|_{M^{1,1}(\mathbb{R}^m)} \leq C_8 \epsilon^2,$$

provided that ϵ is sufficiently small. Assuming this, and assuming that $\epsilon \leq 1$, we then also obtain

$$\|\nabla v_0\|_{M^{1,1}(\mathbb{R}^m)} \leq C_9 \epsilon$$

for a constant $C_9 = C_9(m, n, q, \eta)$. Since $\|\tilde{f}_2\|_{M^{1,2}(\mathbb{R}^m)} \leq C_{10}\epsilon$ for a constant $C_{10} = C_{10}(m, N, \eta)$, it also follows from Proposition 16 that

$$\|\nabla v_{02}\|_{M^{1,1}(\mathbb{R}^m)} \leq C_{11}\epsilon$$

for some constant $C_{11} = C_{11}(m, N, q, \eta)$. Hence

$$\|\nabla v_{01}\|_{M^{1,1}(\mathbb{R}^m)} \leq (C_9 + C_{11})\epsilon.$$

If we can show that $v_{01} \in \dot{W}^{1,2p}(\mathbb{R}^m; \mathbb{R}^n)$, then Proposition 17 implies

$$\|\nabla v_{01}\|_{L^{2p}(\mathbb{R}^m)}^2 \leq C_{12}\epsilon\|\tilde{f}_1\|_{L^p(\mathbb{R}^m)}$$

for a constant $C_{12} = C_{12}(m, N, p, q, \eta)$, always assuming that ϵ is small enough. Furthermore, by the definition of s , we have $\tilde{f}_2 \in L^s(\mathbb{R}^m; \mathbb{R}^n) \subset \dot{W}^{-1,2p}(\mathbb{R}^m)$. If $v_{02} \in \dot{W}^{1,2p}(\mathbb{R}^m; \mathbb{R}^n)$, then by Lemma 13,

$$\|\nabla v_{02}\|_{L^{2p}(\mathbb{R}^m)}^2 \leq C_{13}\|\tilde{f}_2\|_{L^s(\mathbb{R}^m)}^2;$$

that is,

$$\|\nabla v_0\|_{L^{2p}(\mathbb{R}^m)}^2 \leq C_{14}^2 \left(\epsilon\|\tilde{f}_1\|_{L^p(\mathbb{R}^m)} + \|\tilde{f}_2\|_{L^s(\mathbb{R}^m)}^2 \right),$$

where $C_{14} = 2\sqrt{C_{12} + C_{13}}$.

We can prove that $v_{01}, v_{02} \in \dot{W}^{1,2p}(\mathbb{R}^m; \mathbb{R}^n)$ by approximating \tilde{P} with smooth functions $P_\rho \in C^\infty(\mathbb{R}^m; \text{SO}(n))$, similarly to the proofs of Proposition 18 and Proposition 19, such that $\text{supp}(P_\rho - I)$ is uniformly bounded and $\|\nabla P_\rho\|_{M^{2,2}(\mathbb{R}^m)}$ is uniformly small. For $\rho \in (0, 1)$, solve the equations

$$\text{div}(P_\rho \nabla v_{\rho 1}) = P_\rho \tilde{f}_1$$

and set $w_\rho = \tilde{u} - v_{\rho 1}$. Then we have

$$\text{div}(P_\rho \nabla w_\rho) = P_\rho \tilde{\Omega} \cdot \nabla \tilde{u} + P_\rho \tilde{f}_2 + \nabla P_\rho \cdot \nabla \tilde{u},$$

from which we obtain a uniform bound on $\|\nabla v_\rho\|_{M^{1,1}(\mathbb{R}^m)}$. Since

$$\Delta v_{\rho 1} = \tilde{f}_1 - P_\rho^{-1} \nabla P_\rho \cdot \nabla v_{\rho 1},$$

we show that $v_{\rho 1} \in \dot{W}^{1,2p}(\mathbb{R}^m, \mathbb{R}^n)$ with the help of Theorem 10. Proposition 17 then gives an estimate in this space that is uniform in ρ , and when we let $\rho \rightarrow 0$, we obtain $v_{01} \in \dot{W}^{1,2p}(\mathbb{R}^m; \mathbb{R}^n)$. The arguments are similar (but easier) for v_{02} .

Fourth step. Now solve recursively

$$(21) \quad \text{div}(\tilde{P} \nabla v_{k+1}) = \hat{\Omega} \cdot \tilde{P} \nabla v_k \quad \text{in } \mathbb{R}^m$$

and set $w_{k+1} = w_k - v_{k+1}$ for $k = 0, 1, 2, \dots$. Then we have

$$\text{div}(\tilde{P} \nabla w_k) = \hat{\Omega} \cdot \tilde{P} \nabla w_k + \hat{\Omega} \cdot \tilde{P} \nabla v_k \quad \text{in } \mathbb{R}^m.$$

We need to prove, however, that these equations have solutions in the appropriate spaces, satisfying good inequalities. To this end, define $p_0 = p$ and

$$p_{k+1} = \frac{2p_k}{p_k + 1}, \quad k = 0, 1, 2, \dots$$

Lemma 20. *Let $\gamma > 0$. If ϵ is small enough, then there exist functions v_1, v_2, \dots in $\dot{W}^{1,q}(\mathbb{R}^m; \mathbb{R}^n)$ such that (21) holds for $k = 0, 1, 2, \dots$. Furthermore, the inequalities*

$$(22) \quad \|\nabla v_k\|_{M^{1,1}(\mathbb{R}^m)} \leq C_9 \epsilon \gamma^k$$

and

$$(23) \quad \|\nabla v_k\|_{L^{2p_k}(\mathbb{R}^m)}^2 \leq \frac{C_{14}^2 \epsilon}{4^k} \|\hat{\Omega}\|_{L^2(\mathbb{R}^m)}^{2(1-2^{-k})} \left(\|\tilde{f}_1\|_{L^p(\mathbb{R}^m)}^{2^{-k}} + \frac{\|\tilde{f}_2\|_{L^s(\mathbb{R}^m)}^{2^{1-k}}}{\epsilon^{2^{-k}}} \right)$$

are satisfied.

Proof. We prove this by induction. We have already seen that v_0 satisfies both inequalities. Now suppose that (22) and (23) are true for v_k , where $k \in \mathbb{N} \cup \{0\}$. Then the conditions of Proposition 18 are satisfied (with $P = Q = \tilde{P}$). Hence a solution $v_{k+1} \in \dot{W}^{1,q}(\mathbb{R}^m; \mathbb{R}^n)$ of (21) exists. Moreover,

$$\|\nabla v_{k+1}\|_{M^{1,1}(\mathbb{R}^m)} \leq \gamma \|\nabla v_k\|_{M^{1,1}(\mathbb{R}^m)} \leq C_9 \epsilon \gamma^{k+1},$$

provided that ϵ is sufficiently small.

Note that

$$\begin{aligned} \|\hat{\Omega} \cdot \tilde{P} \nabla v_k\|_{L^{p_{k+1}}(\mathbb{R}^m)} &\leq \|\hat{\Omega}\|_{L^2(\mathbb{R}^m)} \|\nabla v_k\|_{L^{2p_k}(\mathbb{R}^m)} \\ &\leq 2^{-k} C_{14} \sqrt{\epsilon} \|\hat{\Omega}\|_{L^2(\mathbb{R}^m)}^{2(1-2^{-k-1})} \left(\|\tilde{f}_1\|_{L^p(\mathbb{R}^m)}^{2^{-k-1}} + \frac{\|\tilde{f}_2\|_{L^s(\mathbb{R}^m)}^{2^{-k}}}{\epsilon^{2^{-k-1}}} \right) \end{aligned}$$

by Hölder's inequality and the induction assumption. Apply Proposition 19 to v_{k+1} . This gives

$$\|\nabla v_{k+1}\|_{L^{2p_{k+1}}(\mathbb{R}^m)}^2 \leq \frac{C_{15} \gamma^{k+1} \epsilon^{3/2}}{2^k (p_{k+1} - 1)^2} \|\hat{\Omega}\|_{L^2(\mathbb{R}^m)}^{2(1-2^{-k-1})} \left(\|\tilde{f}_1\|_{L^p(\mathbb{R}^m)}^{2^{-k-1}} + \frac{\|\tilde{f}_2\|_{L^s(\mathbb{R}^m)}^{2^{1-k}}}{\epsilon^{2^{-k-1}}} \right)$$

for a constant $C_{15} = C_{15}(m, N, p, q, \eta)$. It is readily checked that

$$p_k - 1 \geq \frac{p-1}{3^k}.$$

Thus, if γ and ϵ are small enough, then

$$\frac{C_{15} \gamma^{k+1} \sqrt{\epsilon}}{2^k (p_{k+1} - 1)^2} \leq \frac{C_{14}^2}{4^{k+1}}.$$

Inequality (23) then follows. \square

Fifth step. Note that (23), together with Young's inequality, implies that

$$\|\nabla v_k\|_{L^{2p_k}(\mathbb{R}^m)}^2 \leq \frac{2C_{14}^2}{4^k} \left(\epsilon \|\hat{\Omega}\|_{L^2(\mathbb{R}^m)}^2 + \epsilon \|\tilde{f}_1\|_{L^p(\mathbb{R}^m)} + \|\tilde{f}_2\|_{L^s(\mathbb{R}^m)}^2 \right).$$

Thus

$$\sum_{k=0}^{\infty} \|\nabla v_k\|_{L^{2p_k}(\mathbb{R}^m)} \leq 4C_{14} \left(\sqrt{\epsilon} \|\hat{\Omega}\|_{L^2(\mathbb{R}^m)} + \sqrt{\epsilon} \|\tilde{f}_1\|_{L^p(\mathbb{R}^m)}^{1/2} + \|\tilde{f}_2\|_{L^s(\mathbb{R}^m)} \right).$$

Since

$$\tilde{u} = w_k + \sum_{\ell=0}^k v_\ell$$

for every $k \in \mathbb{N}$, the inequality implies that the sequence $(w_k)_{k \in \mathbb{N}}$ is convergent in $W_{\text{loc}}^{1,2}(\mathbb{R}^m; \mathbb{R}^n)$. Let $w \in W_{\text{loc}}^{1,2}(\mathbb{R}^m; \mathbb{R}^n)$ be its limit. Then we have

$$\operatorname{div}(\tilde{P}\nabla w) = \hat{\Omega} \cdot \tilde{P}\nabla w \quad \text{in } \mathbb{R}^m.$$

Using (22), we also see that $\|\nabla w\|_{M^{1,1}(\mathbb{R}^m)}$ is finite. Moreover, applying the arguments at the beginning of the proof of Proposition 18 to (21) and summing over k again, we conclude that $w \in \dot{W}^{1,q}(\mathbb{R}^m; \mathbb{R}^n)$. Thus by Proposition 18,

$$\|\nabla w\|_{M^{1,1}(\mathbb{R}^m)} = 0.$$

Hence $w = 0$, and we have

$$\tilde{u} = \sum_{k=0}^{\infty} v_k.$$

In particular, we have a constant $C_{16} = C_{16}(m, N, p)$ such that

$$\|\nabla \tilde{u}\|_{L^2(B)}^2 \leq C_{16} \left(\epsilon \|\hat{\Omega}\|_{L^2(\mathbb{R}^m)}^2 + \epsilon \|\tilde{f}_1\|_{L^p(\mathbb{R}^m)} + \|\tilde{f}_2\|_{L^s(\mathbb{R}^m)}^2 \right).$$

Recalling the inequalities derived for these quantities in the first and second steps, we finally obtain inequality (19).

5.2. Proof of Theorem 1. Let $p > 1$ and $f \in L^p(B)$. Suppose that $u \in W^{1,2}(B; N)$ is a weak solution of

$$(24) \quad \Delta u + \operatorname{trace} A(u)(\nabla u, \nabla u) = f \quad \text{in } B.$$

We first rewrite the equation in the form

$$\Delta u = \Omega \cdot \nabla u + f$$

as in the introduction. This construction has the property that

$$|\Omega| \leq C_1 |\nabla u|$$

pointwise for a constant $C_1 = C_1(m, N)$. If $\|\nabla u\|_{M^{2,2}(B)} \leq \epsilon$ (which we assume), then we also have

$$\|\Omega\|_{M^{2,2}(B)} \leq C_1 \epsilon.$$

Let $q = \min\{\frac{p+1}{2}, \frac{m-1}{m-2}\}$, so that we have $1 < q < p$ and $q < \frac{m}{m-2}$. Define $s = \frac{2mq}{m+2q}$ and fix $\delta > 0$. Apply inequality (19), for q instead of p , to rescaled versions of u . For any ball $B_r(x_0) \subset B$, we obtain the inequality

$$\begin{aligned} \left(\int_{B_{r/2}(x_0)} |\nabla u|^2 dx \right)^{\frac{1}{2}} &\leq \delta \left(\int_{B_r(x_0)} |\nabla u|^2 dx \right)^{\frac{1}{2}} + C_2 \left(\int_{B_r(x_0)} |\nabla u|^s dx \right)^{\frac{1}{s}} \\ &\quad + C_2 \left(\int_{B_r(x_0)} |f|^q dx \right)^{\frac{1}{2q}} \end{aligned}$$

for a constant $C_2 = C_2(m, N, \delta, p)$, provided that ϵ is sufficiently small. By Young's inequality, we have

$$\left(\int_{B_r(x_0)} |f|^q dx \right)^{\frac{1}{2q}} \leq \left(\frac{1}{q} \int_{B_r(x_0)} (|f|^q + q - 1) dx \right)^{\frac{1}{2}}.$$

Now we choose δ so small that Theorem 12 applies. We conclude that there are two numbers $a > 2$ and $C_3 > 0$, both of them dependent only on m , N , and p , such that

$$(25) \quad \left(\int_{B_{1/2}} |\nabla u|^a dx \right)^{\frac{1}{a}} \leq C_3 \left(\int_B |\nabla u|^2 dx \right)^{\frac{1}{2}} + C_3 \left(\int_B (|f|^p + 1) dx \right)^{\frac{a}{2p}}.$$

Once we have even such a small gain of regularity, the claim of Theorem 1 follows from known arguments. In particular, for $m = 4$ and $p = 2$, a similar proof is carried out in the paper [31]. For completeness, we give a short description of the arguments anyway. We first need the following lemma.

Lemma 21. *Suppose that $1 < p_0 < p_1$. Then there exist $\epsilon > 0$ and $C > 0$ such that the following holds true. Let $p \in [p_0, p_1]$ and $f \in L^p(B; \mathbb{R}^n)$. Suppose that $u \in W^{1,2p}(B; N)$ is a weak solution of (24). If $\|\nabla u\|_{M^{2,2}(B)} \leq \epsilon$, then*

$$\|\nabla u\|_{L^{2p}(B_{1/2})}^2 \leq C\epsilon \|f\|_{L^p(B)} + C\epsilon^2.$$

Proof. Let $\eta \in C_0^\infty(B)$ with $\eta \equiv 1$ in $B_{1/2}$. Define

$$v = \eta(u - \bar{u}_B).$$

Then we have

$$\Delta v = \eta f - \eta \operatorname{trace} A(u)(\nabla u, \nabla u) + 2\nabla \eta \cdot \nabla u + \Delta \eta (u - \bar{u}_B).$$

Let G be the fundamental solution of the Laplace equation and define

$$\begin{aligned} v_1 &= -G * (\eta \operatorname{trace} A(u)(\nabla u, \nabla u)), \\ v_2 &= G * (2\nabla \eta \cdot \nabla u + \Delta \eta (u - \bar{u}_B)), \end{aligned}$$

and

$$v_3 = v - v_1 - v_2.$$

Then $\Delta v_3 = \eta f$ in \mathbb{R}^m . Assume that $\epsilon \leq 1$. By Theorem 8 and Lemma 9, we have a constant $C_1 = C_1(m, N, \eta)$ such that

$$[v_1]_{\text{BMO}(\mathbb{R}^m)} + [v_2]_{\text{BMO}(\mathbb{R}^m)} \leq C_1 \epsilon.$$

It follows that we have another constant $C_2 = C_2(m, N, \eta)$ with

$$[v_3]_{\text{BMO}(\mathbb{R}^m)} \leq C_2 \epsilon.$$

Using Theorem 10 and Theorem 11, we now obtain

$$\|\nabla v_1\|_{L^{2p}(\mathbb{R}^m)}^2 \leq C_3 \epsilon \|\nabla u\|_{L^{2p}(B)}^2 \quad \text{and} \quad \|\nabla v_3\|_{L^{2p}(\mathbb{R}^m)}^2 \leq C_3 \epsilon \|f\|_{L^p(B)}$$

for a constant $C_3 = C_3(m, N, \eta, p_0, p_1)$. For v_2 , we use the Sobolev inequality instead. Let

$$s = \frac{2mp}{m + 2p}.$$

Then there is a constant $C_4 = C_4(m, n, p_0, p_1, \eta)$ such that

$$\|\nabla v_2\|_{L^{2p}(\mathbb{R}^m)}^2 \leq C_4 \|\nabla u\|_{L^s(B)}^2.$$

It follows that

$$\|\nabla u\|_{L^{2p}(B_{1/2})}^2 \leq C_5 \epsilon \|\nabla u\|_{L^{2p}(B)}^2 + C_5 \epsilon \|f\|_{L^p(B)} + C_5 \|\nabla u\|_{L^s(B)}^2$$

for another constant $C_5 = C_5(m, N, p_0, p_1)$. Apply this inequality to rescaled versions of u . This yields a constant $C_6 = C_6(m, N, p_0, p_1)$ such that

$$(26) \quad \int_{B_{r/2}(x_0)} |\nabla u|^{2p} dx \leq C_6 \epsilon^p \int_{B_r(x_0)} |\nabla u|^{2p} dx + C_6 \epsilon^p \int_{B_r(x_0)} |f|^p dx \\ + C_6 \left(\int_{B_r(x_0)} |\nabla u|^s dx \right)^{\frac{2p}{s}}$$

for every ball $B_r(x_0) \subset B$.

Let $\delta > 0$. By interpolation and Young's inequality, there exist $\alpha, \alpha' \in (1, \infty)$ with $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$, such that

$$\left(\int_{B_r(x_0)} |\nabla u|^s dx \right)^{\frac{2p}{s}} \leq \frac{\delta^{-\alpha'}}{\alpha'} \left(\int_{B_r(x_0)} |\nabla u|^2 dx \right)^p + \frac{\delta^\alpha}{\alpha} \int_{B_r(x_0)} |\nabla u|^{2p} dx.$$

Hence there exists a constant $C_7 = C_7(m, N, p_0, p_1, \delta)$ such that

$$\int_{B_{r/2}(x_0)} |\nabla u|^{2p} dx \leq \delta^\alpha \int_{B_r(x_0)} |\nabla u|^{2p} dx + C_7 \epsilon^p \int_{B_r(x_0)} |f|^p dx + C_7 \epsilon^{2p} r^{m-2p}$$

if ϵ is sufficiently small.

Now choose a collection of balls $\{B_{r_i}(x_i)\}_{i \in \mathbb{N}}$ such that

$$B_{1/2} = \bigcup_{i=1}^{\infty} B_{r_i/2}(x_i),$$

while every point of $B_{1/2}$ is contained in a bounded number of these balls, and such that for every $i \in \mathbb{N}$, we have $2r_i \leq \frac{1}{2} - |x_i| \leq 8r_i$. Define the function

$$\phi(x) = \left(\frac{1}{2} - |x| \right)^{2p} |\nabla u|^{2p}, \quad x \in B_{1/2}.$$

According to the above inequality,

$$\int_{B_{r_i/2}(x_i)} \phi dx \leq C_8 \int_{B_{r_i}(x_i)} (\delta^\alpha \phi + C_7 \epsilon^p |f|^p + C_7 \epsilon^{2p}) dx$$

for a constant $C_8 = C_8(m, p_0, p_1)$. Summing over i , we obtain another constant $C_9 = C_9(m, p_0, p_1)$ such that

$$\int_{B_{1/2}} \phi dx \leq C_9 \int_B (\delta^\alpha \phi + C_7 \epsilon^p |f|^p + C_7 \epsilon^{2p}) dx.$$

If δ is chosen sufficiently small, this implies the desired inequality. \square

Now let $p > 1$ and $f \in L^p(B; \mathbb{R}^n)$. Suppose that we have a weak solution $u \in W^{1,2}(B; N)$ of (24) such that $\|\nabla u\|_{M^{2,2}(B)}$ is small. Define

$$b = \sup \left\{ q \in [2, p] : |\nabla u| \in L_{\text{loc}}^{2q}(B) \right\}.$$

Then by (25) (and its counterpart for rescalings of u), we have $b > 2$. Using Lemma 21, we conclude that

$$|\nabla u| \in L_{\text{loc}}^{2b}(B).$$

Furthermore, if we had $b < p$, then we could combine inequality (26) with Theorem 12 to conclude that $|\nabla u| \in L^q_{\text{loc}}(B)$ for a number $q > b$, which obviously contradicts the definition of b . Hence $b = p$.

It follows that $u \in W^{1,2p}_{\text{loc}}(B; N)$. With standard elliptic estimates, we then see that $W^{2,p}_{\text{loc}}(B; \mathbb{R}^n)$ as well. This concludes the proof of Theorem 1.

5.3. Proof of Theorem 3. Now that we have Theorem 1, the partial regularity for the harmonic map heat flow follows with known arguments as well. We give an outline of the proof anyway. The first step is to estimate the decay of the energy in parabolic cylinders with shrinking radii.

Lemma 22. *Let $c_0 > 0$. There exist $\epsilon > 0$ and $\theta \in (0, 1)$ with the following property. Suppose that $u \in W^{1,2}(B^*; N)$ is a solution of (2) that satisfies a monotonicity inequality with constant c_0 . If $\|\nabla u\|_{L^2(B^*)} \leq \epsilon$, then*

$$\theta^{-m} \int_{B^*_\theta} |\nabla u|^2 dz \leq \frac{1}{2} \int_{B^*} |\nabla u|^2 dz.$$

Proof. We argue by contradiction. Suppose that the statement is false and fix $\theta \in (0, \frac{1}{8}]$. Then there exists a sequence of solutions $u_k \in W^{1,2}(B^*; N)$ of the harmonic map heat flow such that

$$\epsilon_k := \|\nabla u_k\|_{L^2(B^*)} \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

while

$$\theta^{-m} \int_{B^*_\theta} |\nabla u_k|^2 dz > \frac{\epsilon_k^2}{2}$$

for every k . By the monotonicity inequality,

$$\int_{B^*_{1/2}} \left| \frac{\partial u_k}{\partial t} \right|^2 dz \leq c_0 \epsilon_k^2,$$

and there exists a constant $C_1 = C_1(m, c_0)$ such that for every $t \in (-\frac{1}{4}, \frac{1}{4})$,

$$\|\nabla u_k(t, \cdot)\|_{M^{2,2}(B_{1/2})} \leq C_1 \epsilon_k.$$

If k is sufficiently large, then by Theorem 1, we have $|\nabla^2 u_k| \in L^2(B^*_{1/4})$. Using the inequality of Lemma 21 and standard elliptic estimates, we see that

$$\|\nabla^2 u_k\|_{L^2(B^*_{1/4})} \leq C_2 \epsilon_k$$

for a constant $C_2 = C_2(m, N, c_0)$.

Define $v_k = \epsilon_k^{-1} u_k$. Then this gives a bounded sequence in $W^{1,2}(B^*; \mathbb{R}^n)$ such that the quantities $\|\nabla^2 v_k\|_{L^2(B^*_{1/4})}$ are uniformly bounded as well. There is a subsequence that is weakly convergent in $W^{1,2}(B^*; \mathbb{R}^n)$. Discarding a subsequence, we may assume without loss of generality that $v_k \rightharpoonup v$ weakly for some $v \in W^{1,2}(B^*; \mathbb{R}^n)$. Using an interpolation inequality on the slices $\{t\} \times B_{1/4}$, we conclude that $\nabla v_k \rightarrow \nabla v$ strongly in $L^2(B^*_{1/8}; \mathbb{R}^m \otimes \mathbb{R}^n)$. Thus

$$\int_{B^*_\theta} |\nabla v|^2 dz \geq \frac{\theta^m}{2}.$$

It is readily checked that the limit v is a solution of the heat equation

$$\frac{\partial v}{\partial t} - \Delta v = 0 \quad \text{in } B^*.$$

By standard parabolic estimates, there exists a constant $C_2 = C_2(m, n)$ such that

$$\int_{B_\theta^*} |\nabla v|^2 dz \leq C_2 \theta^{m+2}.$$

We obtain a contradiction if θ is chosen sufficiently small. \square

Proposition 23. *Let $c_0 > 0$. There exists a number $\epsilon > 0$ with the following property. Suppose that $u \in W^{1,2}(B^*; N)$ is a weak solution of (2), satisfying a monotonicity inequality with constant c_0 . If $\|\nabla u\|_{L^2(B^*)} \leq \epsilon$, then $u \in C^\infty(B_{1/2}^*)$.*

Proof. We can apply Lemma 22 to rescaled versions of u . We conclude that there exist $C > 0$ and $\gamma > 0$ such that for every $z_0 \in B_{1/2}^*$ and for $0 < r \leq \frac{1}{2}$,

$$\int_{B_r^*(z_0)} |\nabla u|^2 dz \leq Cr^{m+\gamma}$$

and

$$\int_{B_r^*(z_0)} \left| \frac{\partial u}{\partial t} \right|^2 dz \leq Cr^{m+\gamma-2}.$$

(For the second inequality, we use the monotonicity inequality again.) These inequalities imply continuity [11, Teorema 3.I], and higher regularity then follows from results proved elsewhere [30, Theorem 5.2]. \square

Again, we can apply the proposition to rescaled versions of u . Thus for a solution of the harmonic map heat flow satisfying a monotonicity inequality, whenever we have a parabolic cylinder $B_r^*(z_0) \subset B^*$ such that

$$r^{-m} \int_{B_r^*(z_0)} |\nabla u|^2 dz$$

is sufficiently small, we obtain smoothness near z_0 . The statement of Theorem 3 now follows with a standard covering argument.

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