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ERRATUM TO "A CONNES-AMENABLE, DUAL BANACH ALGEBRA NEED NOT HAVE A NORMAL, VIRTUAL DIAGONAL"

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Abstract. In Trans. Amer. Math. Soc., vol. 358 (2006), pp. 391–402, we claimed that, for an amenable, non-compact [SIN]-group G, the dual Banach algebra $\mathcal{WAP}(G)^*$ is Connes-amenable, but lacks a normal virtual diagonal. The proof presented contains a gap. In this erratum, we indicate how the faulty proof can be repaired.

Introduction

In [5], the claim was made that, for a non-compact [SIN]-group G, the dual Banach algebra $\mathcal{WAP}(G)^*$ lacked a normal, virtual diagonal ([5, Theorem 3.5]). It is straightforward that $\mathcal{WAP}(G)^*$ is Connes-amenable for every amenable, locally compact group G. The proof for the non-existence of a normal, virtual diagonal for $\mathcal{WAP}(G)^*$ if G is a non-compact [SIN]-group, however, is more subtle.

In [5], we proceeded through the following steps:

(1) the existence of a normal, virtual diagonal for $\mathcal{WAP}(G)^*$ forces the closed ideal $\mathcal{C}_0(G)^\perp \cong M(G_{\mathcal{WAP}} \setminus G)$ of $\mathcal{WAP}(G)^*$ to have an identity (here, $G_{\mathcal{WAP}}$ denotes the weakly almost periodic compactification of G);
(2) this identity has norm one (this is at the heart of the proof of [5, Corollary 2.3]);
(3) this forces the semigroup $G_{\mathcal{WAP}} \setminus G$ to have an identity ([5, Proposition 2.1]);
(4) if G is a non-compact [SIN]-group, this is impossible due to work by S. Ferri and D. Strauss ([2]).

As it turns out, step (2) contains a gap: the argument to show that an identity of $\mathcal{C}_0(G)^\perp$ necessarily has norm one is not valid. In this erratum, we indicate how this difficulty can be circumvented, so that the main result of [5] remains valid. In the process, we extend a result by L. J. Lardy that characterizes those abelian, locally compact, semitopological semigroups whose measure algebras have an identity ([4, Theorem 3.3]) to the not necessarily abelian situation.

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1. Left identities in measure algebras of locally compact semitopological semigroups

A semitopological semigroup is a semigroup equipped with a Hausdorff topology such that the multiplication is separately continuous; if the underlying topological space is locally compact, we speak of a locally compact, semitopological semigroup. If S is a locally compact, semitopological semigroup, then the Banach space $M(S) \cong \mathcal{C}_0(S)^*$ of all regular complex measures on S becomes a Banach algebra through convolution, i.e.,

$$\langle f, \mu * \nu \rangle := \int_S \int_S f(st) \, d\mu(s) \, d\nu(t) \qquad (\mu, \nu \in M(S), \ f \in \mathcal{C}_0(S)).$$

Obviously, if S has an identity, say e, then $M(S)$ has an identity, namely the point mass δ_e. On the other hand, there are finite, abelian semigroups S without an identity such that $\ell^1(S)$ has an identity ([3, 11.1.6]).

In [3], those abelian semigroups S for which $\ell^1(S)$ has an identity were characterized. This result was later extended to the general locally compact (but still abelian) situation by L. J. Lardy ([4]). We prove a "left version" of Lardy's result for not necessarily abelian, locally compact, semitopological semigroups.

Definition 1.1. Let S be a semigroup. A set $U \subset S$ is called a *set of local left units* for S if, for each $s \in S$, there is $u \in U$ with $us = s$. A set of local left units for S is called *minimal* if none of its proper subsets is a set of local left units for S.

Theorem 1.2. *The following are equivalent for a locally compact, semitopological semigroup:*
(i) *$M(S)$ has a left identity;*
(ii) *S contains a minimal, finite set of local left units.*

We follow Lardy's proof and indicate where adjustments have to be made.

Lemma 1.3. *Let S be a semigroup, and let $U \subset S$ be a set of local left units for S. Then the following are equivalent:*
(i) *U is minimal;*
(ii) *if $u, v \in U$ and $vu = u$, then $v = u$.*

Proof. (i) $\Longrightarrow$ (ii): Same as (i) $\Longrightarrow$ (ii) in the proof of [4, Proposition 2.1]

(ii) $\Longrightarrow$ (i): Assume that U is not minimal, i.e., there is $u \in U$ such that $U \setminus \{u\}$ is a set of local left units for S. Then there is $v \in U \setminus \{u\}$ such that $vu = u$. By (ii), this means that $v = u$, which is impossible. (Compare the proof of (iii) $\Longrightarrow$ (i) of [4, Proposition 2.1].) □

Corollary 1.4. *Let S be a semigroup, and let $U \subset S$ be a minimal set of local left units for S. Then U consists of idempotents.*

Lemma 1.5. *Let S be a locally compact, semitopological semigroup, and let $K \subset S$ be a compact set of local left units for S. Then K contains a minimal set of local left units for S.*

Proof. Just as the proof of [4, Proposition 2.4]. □

Lemma 1.6. *Let S be a locally compact, semitopological semigroup such that $M(S)$ has a left identity λ. Then $\lambda(\{t \in S : ts = s\}) = 1$ holds for each $s \in S$.*

Proof. Just as the proof of [4, Lemma 3.1]. □

Proof of Theorem 1.2. (i) $\Longrightarrow$ (ii): Suppose that $M(S)$ has a left identity λ. As the variation $|\lambda|$ of λ is regular, there is a compact set $K \subset S$ such that $|\lambda|(G \setminus K) < 1$. For $s \in S$, set $X_s := \{t \in S : ts = s\}$, and note that

$$1 \leq |\lambda|(X_s) = |\lambda|(X_s \cap K) + |\lambda|(X_s \setminus K)$$

by Lemma 1.6. As $|\lambda|(X_s \setminus K) \leq |\lambda|(G \setminus K) < 1$, it follows that $|\lambda|(X_s \cap K) > 0$, so that, in particular, $X_s \cap K \neq \varnothing$. This means that K is a set of local left units for S. By Lemma 1.5, K contains a minimal set U of local left units for S. Let $u, v \in U$ be such that $X_u \cap X_v \neq \varnothing$, i.e., there is $t \in S$ such that $tu = u$ and $tv = v$. As U is a set of local left units for S, there is $w \in U$ such that $wt = t$; it follows that $wu = wtu = tu = u$ and, similarly, $wv = v$. Since U is minimal, Lemma 1.3 yields that $u = w = v$. This means that, for $u, v \in U$ such that $u \neq v$, we have $X_u \cap X_v = \varnothing$. As $\lambda(X_s) = 1$ for $s \in S$ by Lemma 1.6, this is possible only if U is finite.

(ii) $\Longrightarrow$ (i): Let $\{u_1, \ldots, u_N\}$ be a finite, minimal set of local left units for S. Then it is routine to verify that

$$\sum_{n=1}^{N} (-1)^{n+1} \sum_{1 \leq j_1 < \cdots < j_n \leq N} \delta_{u_{j_1} \cdots u_{j_n}}$$

is a left identity for $M(S)$. □

Remark. Already in [3], it was observed that a "left version" of [3, Theorems 7.3 and 7.5], i.e., Theorem 1.2 in the discrete case, holds ([3, 7.4]).

2. Non-existence of a normal, virtual diagonal for $\mathcal{WAP}(G)^*$ if G is a non-compact [SIN]-group

We shall now use Theorem 1.2 to give a valid proof of [5, Theorem 3.5]:

Theorem 2.1. *Let G be a non-compact* [SIN]*-group. Then $\mathcal{WAP}(G)^*$ does not have a normal, virtual diagonal.*

As we already pointed out in the introduction, the existence of a normal, virtual diagonal for $\mathcal{WAP}(G)^*$ forces $M(G_{\mathcal{WAP}} \setminus G)$ to have a (left) identity. From there, we shall arrive at a contradiction.

The following theorem is essentially [1, Corollary 4]:

Theorem 2.2. *Let G be a non-compact* [SIN]*-group, and let Ω denote the interior of the set*

$$\{x \in G_{\mathcal{WAP}} \setminus G : ux \neq x \text{ for all } u \in G_{\mathcal{WAP}} \setminus \{e\}\}.$$

Then Ω is dense in $G_{\mathcal{WAP}} \setminus G$, and $\Omega G \subset \Omega$ holds.

Proof. The first part of the theorem follows immediately from [1, Corollary 4], and the second part is clear because $G_{\mathcal{WAP}} \setminus G$ is an ideal of $G_{\mathcal{WAP}}$ and G operates on $G_{\mathcal{WAP}}$ from the right as bijections. □

Linking Theorems 1.2 and 2.2, we obtain:

Proposition 2.3. *Let G be a non-compact* [SIN]*-group. Then $M(G_{\mathcal{WAP}} \setminus G)$ does not have a left identity.*

Proof. Assume towards a contradiction that $M(G_{\mathcal{WAP}} \setminus G)$ has a left identity. Then Theorem 1.2 asserts that there is a finite set U of local left units for $G_{\mathcal{WAP}} \setminus G$. Let Ω be the set specified in Theorem 2.2. As $\bigcup_{u \in U} uG$ is dense in $G_{\mathcal{WAP}} \setminus G$, it follows that $\Omega \cap \bigcup_{u \in U} uG \neq \varnothing$, so that $\Omega \cap u_0 G \neq \varnothing$ for some $u_0 \in U$. Since $\Omega G \subset \Omega$, this means that $u_0 \in \Omega$. By Corollary 1.4, $u_0^2 = u_0$ holds, which contradicts the definition of Ω. □

Remark. For the proof of Proposition 2.3, it was only needed that the set U consists of idempotents, but not that it is finite.

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References

[1] M. Filali, *On the actions of a locally compact group on some of its semigroup compactifications*, Math. Proc. Cambridge Philos. Soc. **143** (2007), no. 1, 25–39, DOI 10.1017/S0305004107000242. MR2340973 (2009d:22004)

[2] S. Ferri and D. Strauss, *A note on the $\mathcal{WAP}$-compactification and the $\mathcal{LUC}$-compactification of a topological group*, Semigroup Forum **69** (2004), no. 1, 87–101, DOI 10.1007/s00233-003-0026-8. MR2063981 (2005b:22002)

[3] Edwin Hewitt and Herbert S. Zuckerman, *The l_1-algebra of a commutative semigroup*, Trans. Amer. Math. Soc. **83** (1956), 70–97. MR0081908 (18,465b)

[4] L. J. Lardy, *On the identity in a measure algebra*, Proc. Amer. Math. Soc. **19** (1968), 807–810. MR0227782 (37 #3366)

[5] Volker Runde, *A Connes-amenable, dual Banach algebra need not have a normal, virtual diagonal*, Trans. Amer. Math. Soc. **358** (2006), no. 1, 391–402 (electronic), DOI 10.1090/S0002-9947-05-03827-4. MR2171239 (2006j:46059)

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