FRACTIONAL SOBOLEV EXTENSION AND IMBEDDING

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ABSTRACT. Let Ω be a domain of $\mathbb{R}^n$ with $n \geq 2$ and denote by $W^{s,p}(\Omega)$ the fractional Sobolev space for $s \in (0, 1)$ and $p \in (0, \infty)$. We prove that the following are equivalent:

- (i) there exists a constant $C_1 > 0$ such that for all $x \in \Omega$ and $r \in (0, 1]$,

$$|B(x, r) \cap \Omega| \geq C_1 r^n;$$
- (ii) Ω is a $W^{s,p}$ -extension domain for all $s \in (0, 1)$ and all $p \in (0, \infty)$;
- (iii) Ω is a $W^{s,p}$ -extension domain for some $s \in (0, 1)$ and some $p \in (0, \infty)$;
- (iv) Ω is a $W^{s,p}$ -imbedding domain for all $s \in (0, 1)$ and all $p \in (0, \infty)$;
- (v) Ω is a $W^{s,p}$ -imbedding domain for some $s \in (0, 1)$ and some $p \in (0, \infty)$.

1. INTRODUCTION

Let $n \geq 2$ and Ω be a domain (namely, connected open subset) of $\mathbb{R}^n$. For $s \in (0, 1)$ and $p \in (0, \infty)$, define the *fractional Sobolev space on the domain* Ω as

$$(1.1) \quad W^{s,p}(\Omega) \equiv \left\{ u \in L^p(\Omega) : \frac{|u(x) - u(y)|}{|x - y|^{n/p+s}} \in L^p(\Omega \times \Omega) \right\}$$

with the norm

$$(1.2) \quad \|u\|_{W^{s,p}(\Omega)} \equiv \left(\int_{\Omega} |u(x)|^p dx + \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dx dy \right)^{1/p},$$

which is also called a *Aronszajn, Gagliardo* or *Slobodeckij space* in the literature after the names of those who introduced them almost simultaneously; see [1, 6, 26]. The fractional Sobolev spaces are special cases of Besov and Triebel-Lizorkin spaces; for a comprehensive treatment and their applications in different subjects see [16, 17, 21, 22, 27–30] and the references therein.

Due to the applications, it attracts a lot of attention to extend fractional Sobolev (and also Besov and Triebel-Lizorkin) functions on a domain to the entire $\mathbb{R}^n$ continuously; see [4, 16, 17, 23–25, 27, 29, 30]. We say that $\Omega \subset \mathbb{R}^n$ is a *$W^{s,p}$ -extension domain* if every function $u \in W^{s,p}(\Omega)$ can be extended to a function $\tilde{u} \in W^{s,p}(\mathbb{R}^n)$ continuously, that is, $\tilde{u}(x) = u(x)$ for all $x \in \Omega$, and there exists a constant $C = C(n, p, s, \Omega)$ such that $\|\tilde{u}\|_{W^{s,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{s,p}(\Omega)}$. Jonsson and Wallin [17] (and also Shvartsman [24]) essentially proved that a regular domain must be a $W^{s,p}$ -extension domain for all $0 < s < 1$ and all $p \geq 1$. Recall that Ω is called a *regular domain* (also called a *plump domain*) if it satisfies the *measure*

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density condition: there exists a constant $C_1 > 0$ such that for all $x \in \Omega$ and all $r \in (0, 1]$,

$$(1.3) \quad |B(x, r) \cap \Omega| \geq C_1 r^n.$$

However, an arbitrary domain is not necessarily a $W^{s,p}$ -extension domain. For example, $\Omega \equiv \{x = (x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 1, 0 < x_2 < x_1^2\}$ is not a $W^{s,p}$ -extension domain for all $s \in (0, 1)$ and all $p \in (2/s, \infty)$, as we show by hand in Remark 1.5 below (without using Theorems 1.1 and 1.2). Such a fact was well understood earlier by mathematicians.

The main aim of this paper is to characterize the $W^{s,p}$ -extension domains for all $s \in (0, 1)$ and all $p \in (0, \infty)$ as below, and hence, give an answer to a question by Nezza, Palatucci and Valdinoci (see [21, Section 5]).

Theorem 1.1. *Let $n \geq 2$ and Ω be a domain of $\mathbb{R}^n$. Then the following are equivalent:*

- (i) Ω is a regular domain;
- (ii) Ω is a $W^{s,p}$ -extension domain for all $s \in (0, 1)$ and all $p \in (0, \infty)$;
- (iii) Ω is a $W^{s,p}$ -extension domain for some $s \in (0, 1)$ and some $p \in (0, \infty)$.

Extension properties play important roles in applications; in particular, they can be used to establish some imbedding properties. A domain $\Omega \in \mathbb{R}^n$ is said to be a $W^{s,p}$ -imbedding domain if the following holds:

- (a) when $sp < n$, there exists a constant $C > 0$ such that for all $u \in W^{s,p}(\Omega)$, we have $u \in L^{np/(n-sp)}(\Omega)$ and $\|u\|_{L^{np/(n-sp)}(\Omega)} \leq C\|u\|_{W^{s,p}(\Omega)}$;
- (b) when $sp = n$, there exist constants $C_3, C > 0$ such that for all $u \in W^{s,p}(\Omega)$ and all balls B ,

$$(1.4) \quad \inf_{c \in \mathbb{R}} \int_{B \cap \Omega} \exp \left(C_3 \frac{|u(x) - c|}{\|u\|_{W^{s,n/s}(\Omega)}} \right)^{n/(n-s)} dx \leq C|B|;$$

- (c) when $sp > n$, there exists a constant $C > 0$ such that for all $u \in W^{s,p}(\Omega)$ and every pair of $x, y \in \Omega$, we have $|u(x) - u(y)| \leq C\|u\|_{W^{s,p}(\Omega)}|x - y|^{s-n/p}$.

We also have the following results.

Theorem 1.2. *Let $n \geq 2$ and Ω be a domain of $\mathbb{R}^n$. Then the following are equivalent:*

- (i) Ω is a regular domain;
- (iv) Ω is a $W^{s,p}$ -imbedding domain for all $s \in (0, 1)$ and all $p \in (0, \infty)$;
- (v) Ω is a $W^{s,p}$ -imbedding domain for some $s \in (0, 1)$ and some $p \in (0, \infty)$.

The proofs of Theorems 1.1 and 1.2 will be given in Section 2. We borrow some ideas from [13, 14, 18, 24, 32].

Case (i) $\Rightarrow$ (ii). If $p \in [1, \infty)$, the proof has already been given by [16, 24] via constructing an extension operator with the mean value $u_X = \frac{1}{|X|} \int_X u(z) dz$. Recall that the procedure to construct an extension operator was essentially known after [18]. If $p \in (0, 1)$, the mean value u_X makes no sense since a function $u \in W^{s,p}(\Omega)$ may fail to be local integrable. So the extension operator in [14, 24] with the mean value does not work here. To overcome the possible non-integrability, we improve the extension operator in [14, 24] by replacing the mean value u_X with the median value $m_u(X)$ defined in (2.2) below; this is the main novelty of this paper. The

point is that the median value $m_u(X)$ is well defined for arbitrary measurable functions and enjoys the nice properties (2.5) and (2.6); see Lemma 2.2. The extension operator with the median value works for all $p \in (0, \infty)$ as shown in Section 2.

Case (ii) $\Rightarrow$ (iv) or (iii) $\Rightarrow$ (v). When $n = sp$, we need Lemma 2.4 below; when $sp < n$ and $p \in (0, 1)$, we need to use property (2.6) of the median value; the other cases are well known (see for example [21, Theorems 6.7 and 8.2]).

Case (v) $\Rightarrow$ (i). We first control the $W^{s,p}(\Omega)$ -norms of test functions by using the volume of the ball $B(x, r) \cap \Omega$ in Lemma 2.4 below. Then with a suitable slicing of the ball $B(x, r) \cap \Omega$ and iteration, we obtain a lower bound Cr^n for $|B(x, r) \cap \Omega|$ and hence give (1.3). We should point out that the idea to derive the measure density property from the imbedding was originally invented by Hajlasz, Koskela and Tuoninen [13] for Sobolev $W^{1,p}$ -extension domains with $p \in [1, \infty)$. Here we adapt their arguments to the setting of the fractional Sobolev $W^{s,p}$ -extension.

Finally, we make some remarks. The first remark says that the geometric characterizations of $W^{s,p}$ -extension/-imbedding domains have some jumps both when p goes from $p < \infty$ to $p = \infty$ for fixed $s \in (0, 1)$ and when s goes from $s < 1$ to $s = 1$ for fixed $p \in [1, \infty]$. In the second remark, we state some related results. The third remark focuses on a domain which is not a $W^{s,p}$ -extension domain for all $s \in (0, 1)$ and all $p > 2/s$.

Remark 1.3. At the endpoint case $s \in (0, 1)$ and $p = \infty$, and case $s = 1$ and $p \in [1, \infty]$, the geometric characterizations of $W^{s,p}$ -extension/-imbedding domains are quite different from Theorems 1.1 and 1.2. Precisely,

Case $s \in (0, 1)$ and $p = \infty$. We can define $W^{s,\infty}(\Omega)$ exactly by (1.1) with the norm

$$\|u\|_{W^{s,\infty}(\Omega)} \equiv \|u\|_{L^\infty(\Omega)} + \sup_{x,y \in \Omega, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^s}.$$

Then every domain is a $W^{s,\infty}$ -extension domain. Indeed, notice that $W^{s,\infty}(\Omega)$ is exactly the space of Hölder continuous functions of order s , and can be viewed as the Lipschitz space with respect to the distance $|\cdot - \cdot|^s$. By the McShane extension (see for example [15, Section 2.2]), every function $u \in W^{s,\infty}$ can be extended to a function $\bar{u}$, defined by

$$\bar{u}(x) = \sup_{z \in \Omega} [u(z) + L|x - z|^s]$$

for all $x \in \mathbb{R}^n$, where $L = \sup_{x,y \in \Omega, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^s}$. Set

$$\tilde{u} = \min\{\|u\|_{L^\infty(\Omega)}, \max\{-\|u\|_{L^\infty(\Omega)}, \bar{u}\}\}.$$

It is easy to check that $\tilde{u} = u$ on Ω , and, moreover, we obtain $\tilde{u} \in W^{s,\infty}(\mathbb{R}^n)$ with $\|\tilde{u}\|_{W^{s,\infty}(\mathbb{R}^n)} \leq \|u\|_{W^{s,\infty}(\Omega)}$ as desired. We omit the details.

Case $s = 1$ and $p \in [1, \infty]$. Define $W^{1,p}(\Omega)$ as the classical Sobolev space, that is, $W^{1,p} = \{u \in L^p(\mathbb{R}^n) : \nabla u \in L^p(\Omega)\}$ with the norm $\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}$, where ∇u denotes the distributional gradient of u . Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain. Then Gehring and Martio [8] proved that Ω is a $W^{1,\infty}$ -extension domain if and only if it is quasiconvex. Goldshtein, Latfullin and Vodop'yanov proved that Ω is a $W^{1,2}$ -extension domain if and only if it is a uniform domain; see [11, 12, 31] and also [18]. When $p \in (2, \infty)$, Buckley, Koskela and Shvartsman proved that Ω is a $W^{1,p}$ -extension/-imbedding domain if and only if it is a weak $(p-2)/(p-1)$ -cigar domain; see [3, 19, 25]. When $p \in [1, 2)$, Ω

is a $W^{1,p}$ -imbedding domain if and only if it is a John domain; see [2]. Higher dimensional analogies were also established therein.

Remark 1.4. The following results are closely relevant to our Theorems 1.1 and 1.2. In [13, 14], Hajlasz, Koskela and Tuominen first proved that the $W^{m,p}$ -extension/-imbedding domains satisfy the measure density property (1.3) for all $m \in \mathbb{N}$ and $p \in [1, \infty)$. With the aid of (1.3), they further give a characterization of Sobolev $W^{1,p}$ -extension domains for all $p \in [1, \infty)$. Rychkov [23] and Triebel [29] consider the extensions and restrictions of Besov and Triebel-Lizorkin spaces on Lipschitz domains. Moreover, Shvartsman [24] considered the extensions and restrictions of Besov and Triebel-Lizorkin spaces on regular domains. This, as well as the extension of Besov spaces on regular sets established by Jonsson and Wallin [17], also works for $W^{s,p}(\Omega)$ when $s \in (0, 1)$ and $p \in [1, \infty]$.

Remark 1.5. In this remark, we check by hand (without using Theorems 1.1 and 1.2) that the domain $\Omega \equiv \{x = (x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 1, 0 < x_2 < x_1^2\}$ is not a $W^{s,p}$ -extension domain for all $s \in (0, 1)$ and all $p > 2/s$. Such a fact was already well understood by mathematicians.

Let $s - 3/p < \alpha < s - 2/p$, and set $u(x) = |x|^\alpha$. Then $u \in W^{s,p}(\Omega)$. Indeed, by $|u(x) - u(y)| \sim |x|^{\alpha-1}|x - y|$, if $|x - y| \leq |x|/2$ and $|u(x) - u(y)| \leq |x - y|^\alpha$, we write

$$\begin{aligned} \|u\|_{W^{s,p}(\Omega)}^p &\lesssim \int_{\Omega} |x|^{\alpha p} dx + \int_{\Omega} \int_{B(x, |x|/2)} \frac{|x|^{p(\alpha-1)}}{|x - y|^{2-p+sp}} dy dx \\ &\quad + \int_{\Omega} \int_{\Omega \setminus B(x, |x|/2)} \frac{1}{|x - y|^{2+sp-\alpha p}} dy dx. \end{aligned}$$

Observing that when $x \neq 0$, $\int_{B(x, |x|/2)} \frac{1}{|x - y|^{2-p+sp}} dy \lesssim |x|^{p-sp}$ and

$$\int_{\mathbb{R}^n \setminus B(x, |x|/2)} \frac{1}{|x - y|^{2+sp-\alpha p}} dy \lesssim |x|^{\alpha p-sp},$$

by $2 + \alpha p - sp > -1$ (due to $s - 3/p < \alpha$), we have

$$\|u\|_{W^{s,p}(\Omega)}^p \lesssim 1 + \int_0^1 \int_0^{x_1^2} \frac{1}{|x|^{-\alpha p+sp}} dx_2 dx_1 \lesssim 1 + \int_0^1 x_1^{2+\alpha p-sp} dx_1 \lesssim 1.$$

Assume that u can be extended continuously as a function $\tilde{u} \in W^{s,p}(\mathbb{R}^2)$. Then by the imbedding of $W^{s,p}(\mathbb{R}^2)$ into the space of Hölder continuous functions of order $s - 2/p$, we know that

$$\sup_{x, y \in \Omega, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^{s-2/p}} \lesssim \|u\|_{W^{s,p}(\Omega)}.$$

However, it is easy to see that

$$\frac{|u(x) - u(x/2)|}{|x - x/2|^{s-2/p}} \sim |x|^{\alpha-s+2/p} \rightarrow \infty$$

as $x \rightarrow 0$, which is a contradiction. So Ω is not a $W^{s,p}$ -extension domain.

The notation used in what follows is standard. We denote by C a positive constant which is independent of the main parameters, but which may vary from line to line. Constants with subscripts, such as C_0 , do not change in different occurrences. The symbol $A \lesssim B$ or $B \gtrsim A$ means that $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$, we then write $A \sim B$. For any locally integrable function u and measurable

set X , we denote by $\bar{f}_X u$ the average of f on X , namely, $\bar{f}_X f \equiv \frac{1}{|X|} \int_X f \, dx$. For a set Ω and $x \in \mathbb{R}^n$, we use $d(x, \Omega)$ to denote $\inf_{z \in \Omega} |x - z|$, the distance from x to Ω .

2. PROOFS OF THEOREMS 1.1 AND 1.2

Obviously, it is easy to see that (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (v). It suffices to prove (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iv), (iii) $\Rightarrow$ (v), and (v) $\Rightarrow$ (i). Without loss of generality, we assume that $\text{diam } \Omega = \sup_{x, y \in \Omega} |x - y| \geq 2$.

(i) $\Rightarrow$ (ii). We first observe that $|\bar{\Omega} \setminus \Omega| = 0$; see [14, Lemma 9], but we also give the argument here. Indeed, for every $x \in \bar{\Omega} \setminus \Omega$ and $r \in (0, 1]$, take $x_j \in \Omega$ such that $x_j \rightarrow x$ as $j \rightarrow \infty$. We have

$$|B(x, r) \cap \Omega| = \lim_{j \rightarrow \infty} |B(x_j, r) \cap \Omega| \geq C_1 r^n,$$

and hence

$$\limsup_{r \rightarrow 0} \frac{|B(x, r) \cap (\bar{\Omega} \setminus \Omega)|}{|B(x, r)|} \leq 1 - C_1 C(n) < 1.$$

Thus x is not a Lebesgue point of $\chi_{\bar{\Omega} \setminus \Omega}$. By the Lebesgue differential theorem, the set of non-Lebesgue points, and hence $\bar{\Omega} \setminus \Omega$, has measure 0. Therefore, without loss of generality, we may assume that $\bar{\Omega} \neq \mathbb{R}^n$. Let $U \equiv \mathbb{R}^n \setminus \bar{\Omega}$. Then U is an open set and hence enjoys the following Whitney covering; see, for example, [14, Lemma 7].

Lemma 2.1. *There exist a family $\{B(x_i, r_i)\}_{i \in I}$ of countable balls and a constant $M \geq 1$ such that*

1. $r_i = d(x_i, \Omega)/10$ for all $i \in I$, and the family of balls $\{B(x_i, r_i/5)\}_{i \in I}$ is a maximal family of pairwise disjoint balls;
2. $U = \bigcup_{i \in I} B(x_i, r_i) = \bigcup_{i \in I} B(x_i, 5r_i)$;
3. if $x \in B(x_i, 5r_i)$ for some $i \in I$, then $5r_i < d(x, \Omega) < 15r_i$;
4. for each $i \in I$, there is $x_i^* \in \Omega$ such that $d(x_i, x_i^*) < 15r_i$;
5. $\sum_{i \in I} \chi_{B(x_i, 5r_i)}(x) \leq M$ for all $x \in U$.

Associated to this covering, there exists a partition of unity (see for example [14, Lemma 8]). That is, there exists a family of smooth functions $\{\varphi_i\}_{i \in I}$ such that

1. $\text{supp } \varphi_i \subset 2B(x_i, r_i)$ for all $i \in I$;
2. $\varphi_i(x) \geq 1/M$ for all $x \in B(x_i, r_i)$ and all $i \in I$;
3. there exists a constant $L > 0$ such that for all $i \in I$, $|\nabla \varphi_i| \leq L/r_i$;
4. $\sum_{i \in I} \varphi_i = \chi_\Omega$.

Let J be the collection of all $i \in I$ such that $r_i \leq 1$, and set $V \equiv \{x \in \mathbb{R}^n : d(x, \Omega) \leq 8\}$. For each $u \in W^{s,p}(\Omega)$, we define

$$(2.1) \quad \tilde{E}u(x) \equiv \begin{cases} u(x), & x \in \Omega, \\ 0, & x \in \bar{\Omega} \setminus \Omega, \\ \sum_{i \in J} \varphi_i(x) m_u(B(x_i^*, r_i) \cap \Omega), & x \in U, \end{cases}$$

where x_i^* is as in Lemma 2.1 and $m_u(X)$ denotes the *median value* of u on set X defined by

$$(2.2) \quad m_u(X) \equiv \max \left\{ a \in \mathbb{R}, |\{x \in X : u(x) < a\}| \leq \frac{|X|}{2} \right\}.$$

Moreover, let Ψ be a Lipschitz function on $\mathbb{R}^n$ such that $\Psi = 1$ on Ω , $\Psi = 0$ on $\mathbb{R}^n \setminus V$ and $0 \leq \Psi \leq 1$ on $V \setminus \Omega$. Set

$$(2.3) \quad Eu \equiv \Psi \tilde{E}u.$$

We point out that the extension operator $\tilde{E}$ in (2.1), and hence E in (2.3), is an improvement of the construction of [14, Proof of Theorem 6], where, instead of the median value $m_u(B(x_i^*, r_i) \cap \Omega)$, they use the mean value $\int_{B(x_i^*, r_i) \cap \Omega} u(z) dz$. But we do need the median value to handle the case $p \in (0, 1)$ since it has the important property (2.6).

We are going to show that Eu gives the desired extension of u into $W^{s,p}(\mathbb{R}^n)$. Obviously, $Eu = u$ on Ω . To see $\|Eu\|_{W^{s,p}(\mathbb{R}^n)} \lesssim \|u\|_{W^{s,p}(\Omega)}$, it suffices to prove that $\tilde{E}u \in W^{s,p}(V)$ and

$$(2.4) \quad \|\tilde{E}u\|_{W^{s,p}(V)} \lesssim \|u\|_{W^{s,p}(\Omega)}.$$

Indeed, assume that (2.4) holds for the moment. From this and $0 \leq \Psi \leq 1$, it follows that

$$\|Eu\|_{L^p(\mathbb{R}^n)} \leq \|\tilde{E}u\|_{L^p(V)} \lesssim \|u\|_{L^p(\Omega)}.$$

Moreover, by Fubini's theorem, we have

$$\begin{aligned} & \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|Eu(x) - Eu(y)|^p}{|x - y|^{n+sp}} dx dy \\ &= \int_V \int_V \frac{|\Psi(x)\tilde{E}u(x) - \Psi(y)\tilde{E}u(y)|^p}{|x - y|^{n+sp}} dx dy \\ &\lesssim \int_V \left(\int_{V \setminus B(x,1)} \frac{|\tilde{E}u(x)|^p + |\tilde{E}u(y)|^p}{|x - y|^{n+sp}} dy \right) dx \\ &\quad + \int_V \left(\int_{V \cap B(x,1)} \frac{|\Psi(x)\tilde{E}u(x) - \Psi(y)\tilde{E}u(y)|^p}{|x - y|^{n+sp}} dy \right) dx \\ &\equiv L_1 + L_2. \end{aligned}$$

By noticing that $\int_{\mathbb{R}^n \setminus B(x,1)} \frac{1}{|x - y|^{n+sp}} dx \lesssim 1$ and $\int_{\mathbb{R}^n \setminus B(y,1)} \frac{1}{|x - y|^{n+sp}} dy \lesssim 1$ and by Fubini's theorem, we have

$$\begin{aligned} L_1 &\leq \int_V |\tilde{E}u(x)|^p \left(\int_{V \setminus B(x,1)} \frac{1}{|x - y|^{n+sp}} dy \right) dx \\ &\quad + \int_V |\tilde{E}u(y)|^p \left(\int_{V \setminus B(y,1)} \frac{1}{|x - y|^{n+sp}} dx \right) dy \\ &\lesssim \|\tilde{E}u\|_{L^p(V)}^p. \end{aligned}$$

Since $\int_{B(x,1)} |x - y|^{-n-sp+p} dy \lesssim 1$, $0 \leq \Psi \leq 1$ and Ψ is a Lipschitz function, we also have

$$\begin{aligned} L_2 &\lesssim \int_V \left(\int_{V \cap B(x,1)} \frac{|\tilde{E}u(x) - \tilde{E}u(y)|^p}{|x - y|^{n+sp}} dy \right) dx \\ &\quad + \int_V \left(\int_{V \cap B(x,1)} \frac{|\Psi(x) - \Psi(y)|^p |\tilde{E}u(x)|^p}{|x - y|^{n+sp}} dy \right) dx \end{aligned}$$

$$\begin{aligned} &\lesssim \|\tilde{E}u\|_{W^{s,p}(V)}^p + \int_V |\tilde{E}u(x)|^p \left(\int_{B(x,1)} \frac{1}{|x-y|^{n+sp-p}} dy \right) dx \\ &\lesssim \|\tilde{E}u\|_{W^{s,p}(V)}^p. \end{aligned}$$

Thus $Eu \in W^{s,p}(\mathbb{R}^n)$ and $\|Eu\|_{W^{s,p}(\mathbb{R}^n)} \lesssim \|\tilde{E}u\|_{W^{s,p}(\Omega)}$. So if (2.4) holds, we will have $Eu \in W^{s,p}(\mathbb{R}^n)$ and $\|Eu\|_{W^{s,p}(\mathbb{R}^n)} \lesssim \|u\|_{W^{s,p}(\Omega)}$.

To prove (2.4), we need the following important properties of the median value, which were essentially proved in [5, Lemma 2.4] and [10, (2.4)] (see also [20, (5.9)]).

Lemma 2.2. *For every $\delta \in (0, 1]$ and $u \in L_{\text{loc}}^\delta(\Omega)$, we have*

$$(2.5) \quad u(x) = \lim_{r \rightarrow 0} m_u(B(x, r) \cap \Omega)$$

for almost all $x \in \Omega$. Moreover, for every ball B with its center in Ω and each $c \in \mathbb{R}$,

$$(2.6) \quad |m_u(B \cap \Omega) - c| \leq \left\{ 2 \int_{B \cap \Omega} |u(w) - c|^\delta dw \right\}^{1/\delta}.$$

Proof. Observe that for all $0 < r < \text{dist}(x, \partial\Omega)$, $B(x, r) \cap \Omega = B(x, r)$, and hence $m_u(B(x, r) \cap \Omega) = m_u(B(x, r))$. So the first conclusion (2.5) was exactly the one proved in [5, Lemma 2.2]. The second conclusion (2.6) was essentially proved in [10, (2.4)] (see also [20, (5.9)]). For convenience, we write the details here.

Let B be an arbitrary ball centered at Ω , and let $c \in \mathbb{R}$. We first claim that

$$m_u(B \cap \Omega) - c = m_{u-c}(B \cap \Omega) \text{ and } |m_u(B \cap \Omega)| \leq m_{|u|}(B \cap \Omega).$$

Indeed, observe that

$$|\{x \in B \cap \Omega, u(x) < m_u(B \cap \Omega)\}| \leq |B \cap \Omega|/2$$

implies that

$$|\{x \in B \cap \Omega, u(x) - c < m_u(B \cap \Omega) - c\}| \leq |B \cap \Omega|/2.$$

By the definition of $m_{u-c}(B \cap \Omega)$, we have $m_u(B \cap \Omega) - c \leq m_{u-c}(B \cap \Omega)$. Similarly,

$$|\{x \in B \cap \Omega, u(x) - c < m_{u-c}(B \cap \Omega)\}| \leq |B \cap \Omega|/2$$

implies that

$$|\{x \in B \cap \Omega, u(x) < m_{u-c}(B \cap \Omega) + c\}| \leq |B \cap \Omega|/2.$$

So by the definition of $m_u(B \cap \Omega)$, we have $m_{u-c}(B \cap \Omega) + c \leq m_u(B \cap \Omega)$. Therefore, we have $m_u(B \cap \Omega) - c = m_{u-c}(B \cap \Omega)$. Moreover, if $m_u(B \cap \Omega) \geq 0$, then

$$|\{x \in B \cap \Omega, |u(x)| < m_u(B \cap \Omega)\}| \leq |\{x \in B \cap \Omega, u(x) < m_u(B \cap \Omega)\}| \leq |B \cap \Omega|/2,$$

and hence, by the definition of $m_{|u|}(B \cap \Omega)$, implies that $m_u(B \cap \Omega) \leq m_{|u|}(B \cap \Omega)$.

If $m_u(B \cap \Omega) < 0$, for every $0 < a < |m_u(B \cap \Omega)|$, we have

$$\begin{aligned} |\{x \in B \cap \Omega, |u(x)| < a\}| &\leq |\{x \in B \cap \Omega, u(x) > -a\}| \\ &= |B \cap \Omega| - |\{x \in B \cap \Omega, u(x) \leq -a\}|. \end{aligned}$$

Observe that

$$|\{x \in B \cap \Omega, u(x) \leq -a\}| \geq |B \cap \Omega|/2;$$

otherwise, $-a \leq m_u(B \cap \Omega)$, which is a contradiction. We obtain

$$|\{x \in B \cap \Omega, |u(x)| < a\}| \leq |B \cap \Omega|/2,$$

which implies that $a \leq m_{|u|}(B \cap \Omega)$, and hence $|m_u(B \cap \Omega)| \leq m_{|u|}(B \cap \Omega)$.

The above claim leads to that

$$|m_u(B \cap \Omega) - c| = |m_{u-c}(B \cap \Omega)| \leq m_{|u-c|}(B \cap \Omega).$$

By this, (2.6) is reduced to

$$(2.7) \quad m_{|u-c|}(B \cap \Omega) \leq \left\{ 2 \int_{B \cap \Omega} |u(w) - c|^\delta d\mu(w) \right\}^{1/\delta}.$$

To see (2.7), set $\sigma \equiv \int_{B \cap \Omega} |u(w) - c|^\delta dw$. By Chebyshev's inequality, for every $a > 2$, we have

$$\begin{aligned} \left| \left\{ w \in B \cap \Omega : |u(w) - c| \geq (a\sigma)^{1/\delta} \right\} \right| &= \left| \left\{ w \in B \cap \Omega : |u(w) - c|^\delta \geq a\sigma \right\} \right| \\ &\leq (a\sigma)^{-1} \int_{B \cap \Omega} |u(w) - c|^\delta dw \\ &< \frac{|B \cap \Omega|}{2}. \end{aligned}$$

This yields that

$$\left| \left\{ w \in B \cap \Omega : |u(w) - c| < (a\sigma)^{1/\delta} \right\} \right| > \frac{|B \cap \Omega|}{2},$$

and hence, by the definition of $m_{|u-c|}(B \cap \Omega)$, we have $m_{|u-c|}(B \cap \Omega) \leq (a\sigma)^{1/\delta}$. Letting $a \rightarrow 2$, we obtain (2.7) and hence prove (2.6). This finishes the proof of Lemma 2.2. $\square$

We return to the proof of (2.4). First, we show that $\|\tilde{E}u\|_{L^p(V)} \lesssim \|u\|_{L^p(\Omega)}$. For $x \in V \setminus \overline{\Omega}$, denote by I_x the collection of $i \in I$ such that $x \in B(x_i, 2r_i)$. Then by Lemma 2.1,

$$(2.8) \quad \#I_x \leq M,$$

and for $i \in I_x$, by (1.3),

$$(2.9) \quad \begin{aligned} B(x_i^*, r_i) &\subset B(x, 5d(x, \Omega)), \\ |B(x_i^*, r_i) \cap \Omega| &\sim |B(x, 5d(x, \Omega)) \cap \Omega| \sim |B(x, 5d(x, \Omega))|. \end{aligned}$$

Notice that if $i \in I \setminus J$, then $r_i \geq 1$ and hence $d(z, \Omega) > 8r_i \geq 8$ for all $z \in B(x_i, 2r_i)$; that is, $B(x_i, 2r_i) \cap \Omega = \emptyset$. Thus $I_x \subset J$ and $\sum_{i \in I_x} \varphi_i(x) = 1$. Take $\delta \equiv \min\{1/2, p/2\}$. By Lemma 2.2, (2.8) and (2.9), we have

$$\begin{aligned} \tilde{E}u(x) &\leq \sum_{i \in I_x} \varphi_i(x) |m_u(B(x_i^*, r_i) \cap \Omega)| \\ &\lesssim \sum_{i \in I_x} \varphi_i(x) \left(\int_{B(x_i^*, r_i) \cap \Omega} |u(z)|^\delta dz \right)^{1/\delta} \\ &\lesssim \left(\int_{B(x, 5d(x, \Omega))} |u(z)|^\delta \chi_\Omega(z) dz \right)^{1/\delta} \\ &\lesssim \mathcal{M}_\delta(u\chi_\Omega)(x), \end{aligned}$$

where and in what follows,

$$\mathcal{M}_\delta(g)(x) \equiv \sup_{B(x,r)} \left(\int_{B(x,r)} |g(z)|^\delta dz \right)^{1/\delta} = [\mathcal{M}(|g|^\delta)]^{1/\delta},$$

and $\mathcal{M}$ is the Hardy-Littlewood operator. By the $L^{p/\delta}$ -boundedness of $\mathcal{M}$, we obtain $\|\tilde{E}u\|_{L^p(V)} \lesssim \|u\|_{L^p(\Omega)}$.

Moreover, we write

$$\begin{aligned} & \int_V \int_V \frac{|\tilde{E}u(x) - \tilde{E}u(y)|^p}{|x - y|^{n+sp}} dx dy \\ &= \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dx dy + 2 \int_{V \setminus \Omega} \int_\Omega \frac{|\tilde{E}u(x) - u(y)|^p}{|x - y|^{n+sp}} dy dx \\ & \quad + \int_{V \setminus \Omega} \int_{V \setminus \Omega} \frac{|\tilde{E}u(x) - \tilde{E}u(y)|^p}{|x - y|^{n+sp}} dx dy \\ &\equiv H_1 + H_2 + H_3. \end{aligned}$$

Obviously, $H_1 \leq \|u\|_{W^{s,p}(\Omega)}^p$. For $x \in V \setminus \Omega$ and $y \in \Omega$, by $\sum_{i \in I_x} \varphi_i(x) = 1$, Lemma 2.2, (2.8) and (2.9), we obtain

$$\begin{aligned} |\tilde{E}u(x) - u(y)| &\leq \sum_{i \in I_x} \varphi_i(x) |m_u(B(x_i^*, r_i) \cap \Omega) - u(y)| \\ &\leq \sum_{i \in I_x} \varphi_i(x) \left(\int_{B(x_i^*, r_i) \cap \Omega} |u(z) - u(y)|^\delta dz \right)^{1/\delta} \\ &\lesssim \left(\int_{B(x, 5d(x, \Omega)) \cap \Omega} |u(z) - u(y)|^\delta dz \right)^{1/\delta}. \end{aligned}$$

For $y \in \Omega$ and $z \in B(x, 5d(x, \Omega)) \cap \Omega$, since $|x - y| \geq d(x, \Omega)$, we always have

$$|z - y| \leq |z - x| + |x - y| \leq 5d(x, \Omega) + |x - y| \lesssim |x - y|.$$

Hence

$$\begin{aligned} \frac{|\tilde{E}u(x) - u(y)|}{|x - y|^{n/p+s}} &\lesssim \left(\int_{B(x, 5d(x, \Omega)) \cap \Omega} \frac{|u(z) - u(y)|^\delta}{|z - y|^{n\delta/p+s\delta}} dz \right)^{1/\delta} \\ &\lesssim \mathcal{M}_\delta \left(\frac{|u(\cdot) - u(y)|}{|\cdot - y|^{n/p+s}} \chi_\Omega(\cdot) \right) (x), \end{aligned}$$

which together with the $L^{p/\delta}$ -boundedness of Hardy-Littlewood maximal operator implies that

$$\begin{aligned} H_2 &\lesssim \int_{V \setminus \Omega} \int_\Omega \left[\mathcal{M}_\delta \left(\frac{|u(\cdot) - u(y)|}{|\cdot - y|^{n/p+s}} \chi_\Omega(\cdot) \right) (x) \right]^p dy dx \\ &\lesssim \int_\Omega \int_{\mathbb{R}^n} \left[\mathcal{M}_\delta \left(\frac{|u(\cdot) - u(y)|}{|\cdot - y|^{n/p+s}} \chi_\Omega(\cdot) \right) (x) \right]^p dx dy \\ &\lesssim \int_\Omega \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n/p+s}} \chi_\Omega(x) dx dy \\ &\lesssim \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dx dy \lesssim \|u\|_{W^{s,p}(\Omega)}^p. \end{aligned}$$

To estimate H_3 for the given x , we split $V \setminus \Omega$ into two parts:

$$X_1(x) \equiv \{y \in V \setminus \Omega : |x - y| \geq \frac{1}{2} \max\{d(x, \Omega), d(y, \Omega)\}\}$$

and

$$X_2(x) \equiv \{y \in V \setminus \Omega : |x - y| < \frac{1}{2} \max\{d(x, \Omega), d(y, \Omega)\}\}.$$

Write

$$\begin{aligned} H_3 &= \int_{V \setminus \Omega} \int_{X_1(x)} \frac{|\tilde{E}u(x) - \tilde{E}u(y)|^p}{|x - y|^{n+sp}} dy dx + \int_{V \setminus \Omega} \int_{X_2(x)} \frac{|\tilde{E}u(x) - \tilde{E}u(y)|^p}{|x - y|^{n+sp}} dy dx \\ &\equiv H_{3,1} + H_{3,2}. \end{aligned}$$

If $x \in V \setminus \Omega$ and $y \in X_1(x)$, by

$$\sum_{i \in I_x} \varphi_i(x) = 1 = \sum_{i \in I_y} \varphi_i(y)$$

we have

$$\tilde{E}u(x) - \tilde{E}u(y) = \sum_{i \in I_x} \sum_{j \in I_y} \varphi_i(x) \varphi_j(y) [m_u(B(x_i^*, r_i) \cap \Omega) - m_u(B(x_j^*, r_j) \cap \Omega)].$$

Applying Lemma 2.2 twice, by (2.9) we have

$$\begin{aligned} &|m_u(B(x_i^*, r_i) \cap \Omega) - m_u(B(x_j^*, r_j) \cap \Omega)| \\ &\lesssim \left(\int_{B(x_i^*, r_i) \cap \Omega} \int_{B(x_j^*, r_j) \cap \Omega} |u(z) - u(w)|^\delta dz dw \right)^{1/\delta} \\ &\lesssim \left(\int_{B(y, 5d(x, \Omega)) \cap \Omega} \int_{B(x, 5d(y, \Omega)) \cap \Omega} |u(z) - u(w)|^\delta dz dw \right)^{1/\delta}. \end{aligned}$$

Observe that for all $z \in B(x, 5d(x, \Omega)) \cap \Omega$ and $w \in B(y, 5d(y, \Omega)) \cap \Omega$, since $|x - y| \geq \frac{1}{2} \max\{d(x, \Omega), d(y, \Omega)\}$,

$$|z - w| \leq |x - y| + 5d(x, \Omega) + 5d(y, \Omega) \lesssim |x - y|.$$

This together with (2.8) leads to the fact that

$$\begin{aligned} \frac{|\tilde{E}u(x) - \tilde{E}u(y)|}{|x - y|^{n/p+s}} &\lesssim \left(\int_{B(x, 5d(x, \Omega)) \cap \Omega} \int_{B(y, 5d(y, \Omega)) \cap \Omega} \frac{|u(z) - u(w)|^\delta}{|z - w|^{n\delta/p+s\delta}} dz dw \right)^{1/\delta} \\ &\lesssim (\mathcal{M}_\delta \times \mathcal{M}_\delta)(F)(x, y), \end{aligned}$$

where

$$F(z, w) \equiv \frac{|u(z) - u(w)|}{|w - z|^{n/p+s}} \chi_\Omega(z) \chi_\Omega(w),$$

and $(\mathcal{M}_\delta \times \mathcal{M}_\delta)(F)(x, y)$ denotes the iterated Hardy-Littlewood maximal function of F . That is, first for a given w , taking the maximal function of $F(z, w)$ with respect to the variable z and evaluating at x , we get $\mathcal{M}_\delta(F(\cdot, w))(x)$. Then for a given x , taking maximal function of $\mathcal{M}_\delta(F(\cdot, w))(x)$ with respect to w and evaluating at y , we obtain $(\mathcal{M}_\delta \times \mathcal{M}_\delta)(F)(x, y)$.

By the $L^{p/\delta}$ -boundedness of Hardy-Littlewood operator, we obtain

$$\begin{aligned} H_{3,1} &\lesssim \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} [(\mathcal{M}_\delta \times \mathcal{M}_\delta)(F)(x, y)]^p dx dy \\ &\lesssim \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \chi_\Omega(x) \chi_\Omega(y) dx dy \\ &\lesssim \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dy dx \\ &\lesssim \|u\|_{W^{s,p}(\Omega)}^p. \end{aligned}$$

If $x \in V \setminus \Omega$ and $y \in X_2(x)$, noticing that $\sum_{i \in I_x \cup I_y} [\varphi_i(x) - \varphi_i(y)] = 0$, by Lemma 2.1, we arrive at

$$\begin{aligned} &|\tilde{E}u(x) - \tilde{E}u(y)| \\ &= \left| \sum_{i \in I_x \cup I_y} [\varphi_i(x) - \varphi_i(y)] m_u(B(x_i^*, r_i) \cap \Omega) \right| \\ &= \left| \sum_{i \in I_x \cup I_y} [\varphi_i(x) - \varphi_i(y)] [m_u(B(x_i^*, r_i) \cap \Omega) - m_u(B(x, 5d(x, \Omega)) \cap \Omega)] \right| \\ &\lesssim \sum_{i \in I_x \cup I_y} \frac{|x - y|}{r_i} |m_u(B(x_i^*, r_i) \cap \Omega) - m_u(B(x, 5d(x, \Omega)) \cap \Omega)|. \end{aligned}$$

Applying Lemma 2.2 twice with $\delta < p$ and the Hölder inequality, we have

$$\begin{aligned} &|m_u(B(x_i^*, r_i) \cap \Omega) - m_u(B(x, 5d(x, \Omega)) \cap \Omega)| \\ &\leq \left\{ \int_{B(x_i^*, r_i) \cap \Omega} \int_{B(x, 5d(x, \Omega)) \cap \Omega} |u(z) - u(w)|^\delta dw dz \right\}^{1/\delta} \\ &\leq \left\{ \int_{B(x_i^*, r_i) \cap \Omega} \left[\int_{B(x, 5d(x, \Omega)) \cap \Omega} |u(z) - u(w)|^p dw \right]^{\delta/p} dz \right\}^{1/\delta}. \end{aligned}$$

For all $i \in I_x \cup I_y$, we claim that $r_i \sim d(x, \Omega)$,

$$B(x_i^*, r_i) \subset B(x, 20d(x, \Omega)) \text{ and } |B(x_i^*, r_i)| \sim |B(x, 20d(x, \Omega)) \cap \Omega|.$$

If $i \in I_x$, this follows from (2.9). To see this for $i \in I_y$, observe that

$$(2.10) \quad \frac{1}{3}d(y, \Omega) \leq d(x, \Omega) \leq 3d(y, \Omega).$$

Indeed, taking $\bar{y} \in \bar{\Omega}$ so that $|y - \bar{y}| = d(y, \Omega)$, by $|x - y| \leq \frac{1}{2} \max\{d(x, \Omega), d(y, \Omega)\}$ we have

$$\begin{aligned} d(x, \Omega) &\leq |x - \bar{y}| \leq |x - y| + |y - \bar{y}| \\ &\leq \frac{1}{2}d(x, \Omega) + \frac{1}{2}d(y, \Omega) + d(y, \Omega) \\ &= \frac{1}{2}d(x, \Omega) + \frac{3}{2}d(y, \Omega), \end{aligned}$$

which implies that $d(x, \Omega) \leq 3d(y, \Omega)$. Similarly, we have $d(y, \Omega) \leq 3d(x, \Omega)$. Thus $B(y, 5d(y, \Omega)) \subset B(x, 20d(x, \Omega))$, and by (1.3),

$$|B(y, 5d(y, \Omega)) \cap \Omega| \sim |B(x, 20d(x, \Omega)) \cap \Omega|.$$

So, for all $i \in I_y$, by Lemma 2.1 and (1.3) we have

$$\begin{aligned} B(x_i^*, r_i) &\subset B(y, 5d(y, \Omega)) \subset B(x, 20d(x, \Omega)), \\ |B(x_i^*, r_i) \cap \Omega| &\sim |B(y, 5d(y, \Omega)) \cap \Omega| \sim |B(x, 20d(x, \Omega)) \cap \Omega|, \end{aligned}$$

as desired in the above claim.

Moreover, by (2.10), for all $z, w \in B(x, 20d(x, \Omega)) \cap \Omega$, we have $|z - w| \lesssim d(x, \Omega)$. This together with $\sharp I_x + \sharp I_y \lesssim 1$ (by Lemma 2.1) yields that

$$\begin{aligned} &|\tilde{E}u(x) - \tilde{E}u(y)| \\ &\leq \frac{|x - y|}{d(x, \Omega)} \left\{ \int_{B(x, 20d(x, \Omega)) \cap \Omega} \left[\int_{B(x, 5d(x, \Omega)) \cap \Omega} |u(z) - u(w)|^p dw \right]^{\delta/p} dz \right\}^{1/\delta} \\ &\leq \frac{|x - y|}{d(x, \Omega)^{1-s}} \left\{ \int_{B(x, 20d(x, \Omega)) \cap \Omega} \left[\int_{B(x, 5d(x, \Omega)) \cap \Omega} \frac{|u(z) - u(w)|^p}{|z - w|^{n+sp}} dw \right]^{\delta/p} dz \right\}^{1/\delta} \\ &\leq \frac{|x - y|}{d(x, \Omega)^{1-s}} \mathcal{M}_\delta \left(\left(\int_\Omega \frac{|u(\cdot) - u(w)|^p}{|\cdot - w|^{n+sp}} dw \right)^{1/p} \chi_\Omega(\cdot) \right) (x). \end{aligned}$$

This implies that

$$\begin{aligned} H_{3,2} &\lesssim \int_{V \setminus \Omega} \left(\int_{X_2(x)} \frac{|x - y|^{p-n-sp}}{d(x, \Omega)^{p-sp}} dy \right) \\ &\quad \times \left[\mathcal{M}_\delta \left(\left(\int_\Omega \frac{|u(\cdot) - u(w)|^p}{|\cdot - w|^{n+sp}} dw \right)^{1/p} \chi_\Omega(\cdot) \right) (x) \right]^p dx. \end{aligned}$$

Since (2.10) implies that $X_2(x) \subset B(x, 15d(x, \Omega))$, we have

$$\int_{X_2(x)} \frac{|x - y|^{p-n-sp}}{d(x, \Omega)^{p-sp}} dy \lesssim \int_{B(x, 15d(x, \Omega))} \frac{|x - y|^{p-n-sp}}{d(x, \Omega)^{p-sp}} dy \lesssim 1.$$

This together with the $L^p(\mathbb{R}^n)$ -boundedness of $\mathcal{M}_\delta$ yields that

$$\begin{aligned} H_{3,2} &\lesssim \int_{\mathbb{R}^n} \left[\mathcal{M}_\delta \left(\left(\int_\Omega \frac{|u(\cdot) - u(w)|^p}{|\cdot - w|^{n+sp}} dw \right)^{1/p} \chi_\Omega(\cdot) \right) (x) \right]^p dx \\ &\lesssim \int_{\mathbb{R}^n} \left[\left(\int_\Omega \frac{|u(x) - u(w)|^p}{|x - w|^{n+sp}} dw \right)^{1/p} \chi_\Omega(x) \right]^p dx \\ &\lesssim \int_\Omega \int_\Omega \frac{|u(x) - u(w)|^p}{|x - w|^{n+sp}} dw dx \\ &\lesssim \|u\|_{W^{s,p}(\Omega)}^p. \end{aligned}$$

Now we arrive at

$$H_3 \leq H_{3,1} + H_{3,2} \lesssim \|u\|_{W^{s,p}(\Omega)}^p.$$

Combining the estimates of H_1 , H_2 and H_3 yields (2.4). This gives (iii).

(ii) $\Rightarrow$ (iv). If $sp \neq n$ and $p \geq 1$, this is well known; see, for example, [21, Theorems 6.5 and 8.2].

Case $p \in (0, 1)$. For $u \in \dot{W}^{s,p}(\mathbb{R}^n)$, set

$$g(x) \equiv \sup_{j \in \mathbb{Z}} 2^{js} \left(\int_{B(x, 2^{-j})} |u(z) - m_u(B(x, 2^{-j}))|^p dz \right)^{1/p}.$$

For each j , by Lemma 2.2 and $|x - z| \leq 2^{-j}$ for all $z \in B(x, 2^{-j})$, we have

$$\begin{aligned} & \int_{B(x, 2^{-j})} |u(z) - m_u(B(x, 2^{-j}))|^p dz \\ & \lesssim \int_{B(x, 2^{-j})} |u(z) - u(x)|^p dz + |u(x) - m_u(B(x, 2^{-j}))|^p \\ & \lesssim \int_{B(x, 2^{-j})} |u(z) - u(x)|^p dz \\ & \lesssim 2^{-jsp} \int_{B(x, 2^{-j})} \frac{|u(z) - u(x)|^p}{2^{-j(n+sp)}} dz \\ & \lesssim 2^{-jsp} \int_{B(x, 2^{-j})} \frac{|u(z) - u(x)|^p}{|z - x|^{n+sp}} dz. \end{aligned}$$

Hence

$$g(x) \lesssim \sup_{j \in \mathbb{Z}} \left(\int_{B(x, 2^{-j})} \frac{|u(z) - u(x)|^p}{|x - z|^{n+sp}} dz \right)^{1/p} \lesssim \left(\int_{\mathbb{R}^n} \frac{|u(z) - u(x)|^p}{|x - z|^{n+sp}} dz \right)^{1/p},$$

which implies that $\|g\|_{L^p(\mathbb{R}^n)} \lesssim \|u\|_{W^{s,p}(\mathbb{R}^n)}$. For each ball $B(0, 2^{-k})$, we have

$$\begin{aligned} \|u\|_{L^{np/(n-sp)}(B(0, 2^{-k}))} & \leq \|u - m_u(B(\cdot, 2^{-k}))\|_{L^{np/(n-sp)}(B(0, 2^{-k}))} \\ & \quad + \sup_{x \in B(0, 2^{-k})} |m_u(B(x, 2^{-k}))| |B(0, 2^{-k})|^{(n-sp)/np}. \end{aligned}$$

Notice that by Lemma 2.2, for each $x \in B(0, 2^{-k})$,

$$\begin{aligned} & |m_u(B(x, 2^{-k}))| |B(0, 2^{-k})|^{(n-sp)/np} \\ & \leq |B(0, 2^{-k})|^{(n-sp)/np} \left(\int_{B(x, 2^{-k})} |u(z)|^p dz \right)^{1/p} \\ & \lesssim \|u\|_{L^p(\mathbb{R}^n)} |B(0, 2^{-k})|^{-s/n}. \end{aligned}$$

Moreover, for almost all $x \in B(0, 2^{-k})$, we claim that

$$(2.11) \quad |u(x) - m_u(B(x, 2^{-k}))| \lesssim \left(\int_{B(0, 2^{-k+2})} \frac{[g(z)]^{p/2}}{|z - x|^{n-sp/2}} dz \right)^{2/p}.$$

Assume that this holds for the moment. Denoting by $I_{sp/2}$ the Riesz potential of order $sp/2$ and by its boundedness from $L^2(\mathbb{R}^n)$ to $L^{2n/(n-sp)}(\mathbb{R}^n)$, we have

$$\begin{aligned} & \|u - m_u(B(\cdot, 2^{-k}))\|_{L^{np/(n-sp)}(B(0, 2^{-k}))} \\ & \lesssim \|I_{sp/2}(g^{p/2})\|_{L^{2n/(n-sp)}(\mathbb{R}^n)}^{2/p} \lesssim \|g^{p/2}\|_{L^2(\mathbb{R}^n)}^{2/p} \\ & \lesssim \|g\|_{L^p(\mathbb{R}^n)} \lesssim \|u\|_{W^{s,p}(\mathbb{R}^n)}. \end{aligned}$$

Thus,

$$\|u\|_{L^{np/(n-sp)}(B(0, 2^{-k}))} \lesssim \|u\|_{W^{s,p}(\mathbb{R}^n)} + \|u\|_{L^p(\mathbb{R}^n)} |B(0, 2^{-k})|^{-s/n},$$

which, when $k \rightarrow \infty$, implies that $\|u\|_{L^{np/(n-sp)}(\mathbb{R}^n)} \lesssim \|u\|_{W^{s,p}(\mathbb{R}^n)}$.

To see (2.11), for almost all $x \in B$, we have $u(x) = \lim_{j \rightarrow \infty} m_u(B(x, 2^{-j}))$ by Lemma 2.2 with $\Omega = \mathbb{R}^n$, and hence

$$\begin{aligned} & |u(x) - m_u(B(x, 2^{-k}))| \\ &= \left| \sum_{j \geq k} [m_u(B(x, 2^{-j-1})) - m_u(B(x, 2^{-j}))] \right| \\ &\leq \sum_{j \geq k} |m_u(B(x, 2^{-j-1})) - m_u(B(x, 2^{-j}))|. \end{aligned}$$

Choose a ball $B(x_j, 2^{-j-2}) \subset B(x, 2^{-j}) \setminus B(x, 2^{-j-1})$. For each $z \in B(x_j, 2^{-j-2})$, we have $B(x, 2^{-j}) \subset B(z, 2^{-j+2})$, and hence, by Lemma 2.2 again,

$$\begin{aligned} & |m_u(B(x, 2^{-j})) - m_u(B(x, 2^{-j-1}))| \\ &\lesssim |m_u(B(x, 2^{-j})) - m_u(B(z, 2^{-j+2}))| \\ &\quad + |m_u(B(z, 2^{-j+2})) - m_u(B(x, 2^{-j+2}))| \\ &\lesssim \sum_{\ell=j-1}^j \left(\int_{B(x, 2^{-\ell})} |u(w) - m_u(B(z, 2^{-j+2}))|^p dw \right)^{1/p} \\ &\lesssim \left(\int_{B(z, 2^{-j+2})} |u(w) - m_u(B(z, 2^{-j+2}))|^p dw \right)^{1/p} \\ &\lesssim 2^{-js} g(z), \end{aligned}$$

which implies that

$$|m_u(B(x, 2^{-j})) - m_u(B(x, 2^{-j-1}))| \lesssim 2^{-js} \left(\int_{B(x_j, 2^{-j-2})} [g(z)]^{p/2} dz \right)^{p/2}.$$

Observing that $|z - x| \sim 2^{-j}$ for all $z \in B(x_j, 2^{-j-2})$ and $\{B(x_j, 2^{-j-2})\}_{j \geq k}$ are pairwise disjoint, by the Cauchy-Schwarz inequality we arrive at

$$\begin{aligned} |u(x) - m_u(B(x, 2^{-k}))| &\lesssim \sum_{j \geq k} 2^{-js} \left(\int_{B(x_j, 2^{-j-2})} [g(z)]^{p/2} dz \right)^{p/2} \\ &\lesssim 2^{-ks} \left(\sum_{j \geq k} \int_{B(x_j, 2^{-j-2})} [g(z)]^{p/2} dz \right)^{p/2} \\ &\lesssim \left(\int_{B(0, 2^{-k+2})} \frac{[g(z)]^{p/2}}{|z - x|^{n-sp/2}} dz \right)^{p/2}. \end{aligned}$$

This gives (2.11).

Case $sp = n$. This relies on the following result.

Lemma 2.3. *Let $s \in (0, 1)$. Then there exist positive constants $C_4, C > 0$ such that for all balls $B \subset \mathbb{R}^n$ and $u \in W^{s, n/s}(8B)$,*

$$(2.12) \quad \int_B \exp \left(C_4 \frac{|u(x) - u_B|}{\|u\|_{W^{s, n/s}(8B)}} \right)^{n/(n-s)} dx \leq C|B|.$$

Now assume that Ω is a $W^{s, n/s}$ -extension domain and let $u \in W^{s, p}(\Omega)$. Then u has an extension $\tilde{u} \in W^{s, p}(\mathbb{R}^n)$ with $\|u\|_{W^{s, p}(\mathbb{R}^n)} \leq C_3 \|u\|_{W^{s, p}(\Omega)}$. For each ball $B = B(x, r)$ with $x \in \Omega$ and $r \in (0, 1]$, by Lemma 2.3, we have that (2.12) holds for $\tilde{u}$. Then $\|\tilde{u}\|_{W^{s, n/s}(8B)} \leq \|\tilde{u}\|_{W^{s, n/s}(\mathbb{R}^n)} \leq C \|u\|_{W^{s, n/s}(\Omega)}$ yields that

$$\inf_{c \in \mathbb{R}} \int_{B \cap \Omega} \exp \left(C_3 \frac{|u(x) - c|}{C \|u\|_{W^{s, n/s}(\Omega)}} \right)^{n/(n-s)} dx \lesssim |B|.$$

This gives (iv).

Proof of Lemma 2.3. Assume that $B \equiv B(x_0, 2^{-k_0})$ for some $x_0 \in \mathbb{R}^n$ and $k_0 \in \mathbb{Z}$. Let $u \in W^{s, p}(4B)$ and take

$$g(x) \equiv \sup_{j \geq k_0 - 2} 2^{js} \int_{B(x, 2^{-j})} |u(z) - u_{B(x, 2^{-j})}| dz$$

for all $x \in 2B$ and $g(x) = 0$ otherwise. Then $\|g\|_{L^{n/s}(8B)} \lesssim \|u\|_{W^{s, n/s}(8B)}$, which follows from

$$\begin{aligned} g(x) &\lesssim \sup_{j \geq k_0 - 2} \left(\int_{B(x, 2^{-j})} \frac{|u(z) - u(x)|^{n/s}}{|x - z|^{2n}} dz \right)^{n/s} \\ &\lesssim \left(\int_{8B} \frac{|u(z) - u(x)|^{n/s}}{|x - z|^{2n}} dz \right)^{n/s}. \end{aligned}$$

Moreover, for almost all $x \in B$, we have

$$(2.13) \quad |u(x) - u_B| \lesssim \int_{B(x_0, 2^{-k_0+2})} \frac{|g(y)|}{|y - x|^{n-s}} dy.$$

Indeed, by an argument similar to but easier than the case $p \in (0, 1)$, for every Lebesgue point $x \in B$ of u ,

$$|u(x) - u_{B(x, 2^{-k_0-1})}| \lesssim \int_{B(x_0, 2^{-k_0+2})} \frac{|g(y)|}{|y - x|^{n-s}} dy.$$

Similarly,

$$|u_B - u_{B(x, 2^{-k_0-1})}| \lesssim \int_B |u(z) - u_B| dz \lesssim \int_{B(x_0, 2^{-k_0+2})} \frac{|\mathcal{M}(g)(y)|}{|y - x|^{n-s}} dy.$$

This gives (2.13).

Applying [9, Lemma 7.2], for all $q \geq n/s$, we obtain

$$\|u - u_B\|_{L^q(B)} \leq q^{1-s/n+1/q} |B(0, 1)|^{1-s/n} |B|^{1/q} \|\mathcal{M}(g)\|_{L^{n/s}(4B)},$$

which, together with the $L^{n/s}(\mathbb{R}^n)$ -boundedness of $\mathcal{M}$ and the fact that

$$\|g\|_{L^{n/s}(8B)}^q \lesssim \|u\|_{W^{s, n/s}(8B)}^q,$$

implies that

$$\oint_B |u(z) - u_B|^q dz \lesssim q^{1+(n-s)/(nq)} |B(0, 1)|^{nq/(n-s)} \|u\|_{W^{s, n/s}(8B)}.$$

Thus when $q \geq n/s - 1$, we have

$$\oint_B |u(z) - u_B|^{qn/(n-s)} dz \lesssim \frac{nq}{n-s} \left(|B(0, 1)| \frac{nq}{n-s} \|u\|_{W^{s, n/s}(8B)}^{n/(n-s)} \right)^q.$$

Taking $\sigma > [e|B(0, 1)|n/(n-s)]^{(n-s)/n}$, this yields that

$$\begin{aligned} & \oint_B \sum_{j \geq \lfloor n/s \rfloor} \frac{1}{j!} \left(\frac{|u(x) - u_B|}{\sigma \|u\|_{W^{s, n/s}(8B)}} \right)^{jn/(n-s)} dx \\ & \lesssim \sum_{j \geq 1} \left(\frac{n|B(0, 1)|}{(n-s)\sigma^{n/(n-s)}} \right)^j \frac{j^j}{(j-1)!} \\ & \lesssim 1. \end{aligned}$$

Notice that by Hölder's inequality, we have

$$\begin{aligned} & \oint_B \sum_{j=0}^{\lfloor n/s \rfloor} \frac{1}{j!} \left(\frac{|u(x) - u_B|}{\sigma \|u\|_{W^{s, n/s}(8B)}} \right)^{jn/(n-s)} dx \\ & \lesssim \sum_{j=0}^{\lfloor n/s \rfloor} \left(\oint_B \frac{|u(x) - u_B|^{n/s}}{\|u\|_{W^{s, n/s}(8B)}^{n/s}} dx \right)^{(n-s)/(j-s)} \\ & \lesssim 1. \end{aligned}$$

This gives (2.12) and thus finishes the proof of Lemma 2.3. $\square$

(iii) $\Rightarrow$ (v). This case follows from the arguments that are exactly the same as in the case (ii) $\Rightarrow$ (iv).

(v) $\Rightarrow$ (i). We need the following estimates.

Lemma 2.4. *Let $s \in (0, 1)$ and $p \in (0, \infty)$. For $z \in \Omega$ and $0 < t < r \leq 1$, define a function u on Ω by setting*

$$(2.14) \quad u(x) \equiv \begin{cases} 1, & \text{if } x \in B(z, t) \cap \Omega, \\ \frac{r - |x - z|}{r - t}, & \text{if } x \in (B(z, r) \setminus B(z, t)) \cap \Omega, \\ 0, & \text{if } x \in \Omega \setminus B(z, r). \end{cases}$$

Then there exists a constant $C > 0$ independent of z, t, r such that

$$(2.15) \quad \|u\|_{W^{s, p}(\Omega)} \leq C \frac{|B(z, r) \cap \Omega|^{1/p}}{(r-t)^s}.$$

The proof of Lemma 2.4 will be given later. With the help of Lemma 2.4, we consider the following cases: *Case $sp < n$, Case $sp = n$ and Case $sp > n$* , separately.

Case $sp < n$. Take arbitrary $z \in \Omega$ and $r \in (0, 1]$. Notice that there always exists a unique $b \in (0, 1)$ such that

$$|B(z, br) \cap \Omega| = \frac{1}{2} |B(z, r) \cap \Omega|.$$

We claim that there exists a constant $C > 0$, independent of x and r , such that

$$(2.16) \quad r - br \leq C|B(z, r) \cap \Omega|^{1/n}.$$

Indeed, since Ω is a $W^{s,p}$ -imbedding domain, we know that $\|u\|_{L^{np/(n-sp)}(\Omega)} \lesssim \|u\|_{W^{s,p}(\Omega)}$, and hence by Lemma 2.4 and $\|u\|_{L^{np/(n-sp)}(\Omega)} \geq |B(z, br) \cap \Omega|^{(n-sp)/(np)}$, we further have

$$|B(z, br) \cap \Omega|^{(n-sp)/(np)} \lesssim \frac{|B(z, r) \cap \Omega|^{1/p}}{(r - br)^s},$$

which yields $r - br \lesssim |B(z, r) \cap \Omega|^{1/n}$. This gives (2.16). Moreover, let $b_0 = 1$ and $b_j \in (0, 1)$ for $j \in \mathbb{N}$ such that

$$(2.17) \quad |B(z, b_j r) \cap \Omega| = 2^{-1}|B(z, b_{j-1} r) \cap \Omega| = 2^{-j}|B(z, r) \cap \Omega|.$$

Then $b_j \rightarrow 0$ as $j \rightarrow \infty$, and hence

$$\begin{aligned} r &= \sum_{j \in \mathbb{N}} (b_{j-1} r - b_j r) \\ &\lesssim \sum_{j \in \mathbb{N}} |B(z, b_{j-1} r) \cap \Omega|^{1/n} \\ &\lesssim \sum_{j \in \mathbb{N}} 2^{-j/n} |B(z, r) \cap \Omega|^{1/n} \\ &\leq |B(z, r) \cap \Omega|^{1/n}, \end{aligned}$$

as desired.

Case $sp = n$. Take arbitrary $z \in \Omega$ and $r \in (0, 1]$. Let $b_0 = 1$ and $b_j \in (0, 1)$ for $j \in \mathbb{N}$ such that (2.17) holds. Considering the function u associated to $z, b_1 r, b_2 r$ as in (2.15), by Lemma 2.3, and applying (1.4) to the ball $B(z, r)$, we have

$$\inf_{c \in \mathbb{R}} \int_{B(z, r) \cap \Omega} \exp \left(C \frac{|u(x) - c|(b_1 r - b_2 r)^s}{|B(x, b_1 r) \cap \Omega|^{s/n}} \right)^{n/(n-s)} dx \leq C_2 r^n.$$

Observe that, for each $c \in \mathbb{R}$, $|u - c| \geq 1/2$ either on $(B(z, r) \setminus B(z, b_1 r)) \cap \Omega$ or on $B(z, b_2 r) \cap \Omega$. By (2.17), we have

$$|B(z, b_1 r) \cap \Omega| \exp \left(\frac{(b_1 r - b_2 r)^{sn/(n-s)}}{|B(x, b_1 r) \cap \Omega|^{s/(n-s)}} \right) \lesssim r^n,$$

which implies that

$$b_1 r - b_2 r \leq |B(x, b_1 r) \cap \Omega|^{1/n} \left[\log \left(\frac{Cr^n}{|B(z, b_1 r) \cap \Omega|} \right) \right]^{(n-s)/sn}.$$

Similar inequalities also hold for $b_j r - b_{j+1} r$ with $j \geq 2$. This leads to

$$\begin{aligned} b_1 r &= \sum_{j \in \mathbb{N}} (b_j r - b_{j+1} r) \leq \sum_{j \in \mathbb{N}} |B(x, b_j r) \cap \Omega|^{1/n} \left[\log \left(\frac{C(b_{j-1} r)^n}{|B(z, b_j r) \cap \Omega|} \right) \right]^{(n-s)/sn} \\ &\leq \sum_{j \in \mathbb{N}} 2^{-j/n} |B(x, r) \cap \Omega|^{1/n} \left[\log \left(2^j \frac{Cr^n}{|B(z, r) \cap \Omega|} \right) \right]^{(n-s)/sn} \\ &\lesssim |B(x, r) \cap \Omega|^{1/n} \left[\log \left(\frac{Cr^n}{|B(z, r) \cap \Omega|} \right) \right]^{(n-s)/sn}. \end{aligned}$$

If $b_1 \geq 1/10$, observing that $t(\log \frac{1}{t})^{(n-s)/s} \geq 1$ implies that $t \gtrsim 1$, we have $|B(x, r) \cap \Omega|^{1/n} \gtrsim r$, as desired. If $b_1 \leq 1/10$, we choose $R = 2r/5$ and a point $y \in B(z, r) \cap \Omega$ such that $|y - z| = b_1 r + R/2$. Then $B(z, b_1 r) \subset B(y, R) \subset B(z, r)$ but $B(y, R/2) \cap B(z, b_1 r) = \emptyset$. Therefore, if $|B(y, \tilde{b}_1 R) \cap \Omega| = \frac{1}{2}|B(y, R) \cap \Omega|$, then by $|B(z, b_1 r) \cap \Omega| \geq \frac{1}{2}|B(y, R) \cap \Omega|$ we have $\tilde{b}_1 \geq 1/2$. Applying the result when $b_1 \geq 1/10$, we conclude that $|B(y, R) \cap \Omega| \gtrsim R^n$, which implies that $|B(z, r) \cap \Omega| \gtrsim r^n$, as desired.

Case $sp > n$. For $z \in \Omega$ and $r \in (0, 1]$, take $t \in (0, r/4)$, and for such z, r, t , set u as in (2.14). Then for all $x, y \in \Omega$, by Lemma 2.4 and $r/2 \leq r - t \leq r$, we have

$$|u(x) - u(y)| \leq C \|u\|_{W^{s,p}(\Omega)} |x - y|^{s-n/p} \lesssim \frac{|B(z, r) \cap \Omega|^{1/p}}{r^s} |x - y|^{s-n/p}.$$

In particular, let $x \in B(z, t) \cap \Omega$ and $y \in (B(z, r + r/2) \cap \Omega) \setminus B(z, r)$. Then $|x - z| \leq r/4$, $r \leq |y - z| \leq 3r/2$, and hence $r/2 \leq |x - y| \leq 2r$. Therefore, $r^n \lesssim |B(z, r) \cap \Omega|$, as desired.

Proof of Lemma 2.4. Obviously, $\|u\|_{L^p(\Omega)} \lesssim |B(z, r) \cap \Omega|^{1/p}$. It then suffices to prove that

$$\int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dx dy \lesssim \frac{|B(z, r) \cap \Omega|}{(r - t)^{sp}}.$$

To this end, observing $u = 0$ on $\Omega \setminus B(z, r)$, we write

$$\begin{aligned} \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dx dy &= 2 \int_{B(z, r) \cap \Omega} \int_{\Omega \setminus B(z, r)} \frac{|u(x)|^p}{|x - y|^{n+sp}} dx dy \\ &\quad + \int_{B(z, r) \cap \Omega} \int_{B(z, r) \cap \Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dx dy \\ &\equiv H_1 + H_2. \end{aligned}$$

For each $x \in B(z, r) \cap \Omega$, we have $\Omega \setminus B(z, r) \subset \mathbb{R}^n \setminus B(x, r - |x - z|)$, and hence

$$\int_{\Omega \setminus B(z, r)} \frac{1}{|x - y|^{n+sp}} dy \leq \int_{\mathbb{R}^n \setminus B(x, r - |x - z|)} \frac{1}{|x - y|^{n+sp}} dy \leq \frac{1}{(r - |x - z|)^{sp}}.$$

Thus

$$\begin{aligned} H_1 &\leq \int_{(B(z, r) \setminus B(z, t)) \cap \Omega} \left(\frac{r - |x - z|}{r - t} \right)^p \frac{1}{(r - |x - z|)^{sp}} dx \\ &\quad + \int_{B(z, t) \cap \Omega} \frac{1}{(r - |x - z|)^{sp}} dx \\ &\lesssim \frac{|B(z, r) \cap \Omega|}{(r - t)^{sp}}. \end{aligned}$$

Write

$$\begin{aligned} H_2 &= \int_{B(z, r) \cap \Omega} \int_{B(x, r-t) \cap \Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dx dy \\ &\quad + \int_{B(z, r) \cap \Omega} \int_{(B(z, r) \setminus B(x, r-t)) \cap \Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dx dy \\ &\equiv H_{2,1} + H_{2,2}. \end{aligned}$$

Since $|\nabla u| \leq 1/(r-t)$, for all $x, y \in \Omega$, we have

$$|u(x) - u(y)| \leq \frac{1}{r-t} ||x - z| - |y - z|| \leq \frac{1}{r-t} |x - y|.$$

Then by $\int_{B(x, r-t)} \frac{1}{|x-y|^{n+sp-p}} dy \lesssim (r-t)^{p-sp}$ we obtain

$$\begin{aligned} H_{2,1} &\leq \int_{B(z, r) \cap \Omega} \int_{B(x, r-t)} \frac{1}{(r-t)^p} \frac{1}{|x-y|^{n+sp-p}} dy dx \\ &\lesssim \int_{B(z, r) \cap \Omega} \frac{(r-t)^{p-sp}}{(r-t)^p} dx \\ &\lesssim |B(z, r) \cap \Omega| \frac{1}{(r-t)^{sp}}. \end{aligned}$$

Observing $0 \leq u \leq 1$ and $\int_{\mathbb{R}^n \setminus B(x, r-t)} \frac{1}{|x-y|^{n+sp}} dy \lesssim (r-t)^{-sp}$, we also have

$$\begin{aligned} H_{2,2} &\lesssim \int_{B(z, r) \cap \Omega} \int_{\mathbb{R}^n \setminus B(x, r-t)} \frac{1}{|x-y|^{n+sp}} dy dx \\ &\lesssim \int_{B(z, r) \cap \Omega} \frac{1}{(r-t)^{sp}} dx \\ &\lesssim |B(z, r) \cap \Omega| \frac{1}{(r-t)^{sp}}. \end{aligned}$$

So $H_2 \lesssim \frac{|\Omega|}{(r-t)^{sp}}$, as desired. This finishes the proof of Lemma 2.4. $\square$

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