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CATEGORIFICATION OF QUANTUM KAC-MOODY SUPERALGEBRAS

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Abstract. We introduce a non-degenerate bilinear form and use it to provide a new characterization of quantum Kac-Moody superalgebras of anisotropic type. We show that the spin quiver Hecke algebras introduced by Kang, Kashiwara and Tsuchioka provide a categorification of half the quantum Kac-Moody superalgebras, using the recent work of Ellis-Khovanov-Lauda. A new idea here is that a supersign is categorified as spin (i.e., the parity-shift functor).

Contents

1. Introduction

1.1. **Background.** In recent years, quiver Hecke algebras (or, KLR algebras) were introduced independently by Khovanov-Lauda and Rouquier [KL1, KL2, Ro1]. These algebras are fundamental in the construction of Khovanov-Lauda-Rouquier 2-categories–categorical analogues of quantum Kac-Moody algebras (also see the earlier work [CR]). The KLR 2-categories and quiver Hecke algebras have implications in modular representation theory of symmetric groups and Hecke algebras, low-dimensional topology, algebraic geometry, and other areas [BK2, BKW, BS, CKL, HM, HS, KK, LV, VV, Ro2, Web]; see also the survey articles [Kle2, Kh1] for more references.

Several years ago, partly motivated by Nazarov's construction of affine Hecke-Clifford algebras [Naz], the second author [Wa] introduced *spin Hecke algebras* and studied them in a series of papers with Khongsap starting with [KW]. The spin Hecke algebras are associated to spin Weyl groups, and they afford many variations (e.g., affine, double affine, degenerate, nil, etc.). A distinct new feature in [Wa, KW] is the appearance of the skew-polynomial algebra as a subalgebra of the spin Hecke algebra, in contrast to the polynomial algebra for the stardard affine/double Hecke

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algebras. The spin Hecke algebras are naturally superalgebras, and are Morita super-equivalent to Hecke-Clifford algebras (though, we prefer to suppress the prefix "super" for such associative algebras in contrast to Lie superalgebras). A straightforward modification of the spin Hecke algebra is the spin nilHecke algebra, which has recently been rediscovered and studied in depth in [EKL] for the affine type A case.

Almost as recently, Kang-Kashiwara-Tsuchioka [KKT] utilized the spin nilHecke algebra to generalize the KLR construction to several new families of algebras, including the *spin quiver Hecke algebras* (called "quiver Hecke superalgebras" in *loc. cit.*) and quiver Hecke-Clifford algebras, starting from a generalized Cartan matrix (GCM) A parametrized by an index set $I = I_{\bar{0}} \cup I_{\bar{1}}$ subject to some natural conditions (see §2.1). Roughly speaking, to each $i \in I_{\bar{0}}$ is attached the usual nilHecke algebra and to each $i \in I_{\bar{1}}$ is attached the spin nilHecke algebra; when $I_{\bar{1}} = \emptyset$, the KKT construction reduces to the original KLR construction. It is suggested in [KKT] that these new algebras can be used to categorify the quantum Kac-Moody algebra associated to the GCM A with the $\mathbb{Z}_2$-parity forgotten (denoted by A^+ in this paper). Their expectation was partly motivated by [BK1] where an affine Kac-Moody algebra of type $A_{2\ell}^{(2)}$ arises (also cf. [Ts]).

1.2. **What to categorify?** It was known much earlier [Kac] that a Kac-Moody superalgebra can be associated to a GCM A exactly as specified in §2.1. This class of Kac-Moody superalgebras is distinguished among the Lie superalgebras in the following sense: the odd simple roots are all non-isotropic, the notion of integrable modules is defined as usual, and the super Weyl-Kac character formula for integrable modules holds. Note however that the only finite-dimensional simple Lie superalgebras in this class, which are not Lie algebras, are $\mathfrak{osp}(1|2n)$. This class of Lie superalgebra and beyond have been quantized in [Ya, BKM], and a generalization of Lusztig's theorem [Lu1] on deforming integrable modules was obtained in [BKM].

We propose a connection between the spin quiver Hecke algebras and quantum Kac-Moody superalgebras, despite the major difficulty caused by additional signs appearing in the superalgebras. These signs show up in the superquantum integers (3.1), which may specialize to zero if one naively takes $q \mapsto 1$, and also appear in the super quantum Serre relations (3.2). Another major conceptual obstacle is that no canonical basis à la Lusztig-Kashiwara has ever been constructed (even conjecturally) for superalgebras, in spite of various works generalizing the corresponding crystal basis theory.

1.3. **The main results.** In this paper, we introduce a twisted bialgebra ${}_{\mathcal{A}}\mathbf{f}^{\pi}$, called the *(quantum) covering Kac-Moody algebra*, with an additional parameter π satisfiying $\pi^2 = 1$. This algebra is associated to the GCM A with an extra natural condition (C6) in §2.2 which is assumed throughout this paper unless otherwise specified.

The assumption (C6) plays an essential role in the paper. First, we introduce an apparently novel bar-involution on ${}_{\mathcal{A}}\mathbf{f}^{\pi}$, which is the identity on the Chevalley generators: $\bar{\theta}_i = \theta_i$ $(i \in I)$, as usual, but $\bar{q} = \pi q^{-1}$. Assumption (C6) guarantees that the quantum integers in ${}_{\mathcal{A}}\mathbf{f}^{\pi}$, and therefore the divided powers, $\theta_i^{(a)}$ $(i \in I)$, are bar invariant. Additionally, following Rouquier [Ro1], we define a family of (skew-) polynomials $\mathsf{Q} = (\mathsf{Q}_{ij}(u, v))_{i,j \in I}$ from a *quiver with compatible automorphism* and show in Lemma 2.1 that assumption (C6) implies that these polynomials satisfy the necessary conditions to construct an associated spin quiver Hecke algebra, [KKT, (3.1)].

The two specializations, $\pi \mapsto 1$ and $\pi \mapsto -1$, of ${}_{\mathcal{A}}\mathbf{f}^{\pi}$ become half of the Kac-Moody algebra associated to A^+ and half of the Kac-Moody superalgebra associated to A, respectively. The main result of this paper is that the spin quiver Hecke algebras defined by the family of polynomials, $\mathtt{Q}$, naturally categorify the algebra ${}_{\mathcal{A}}\mathbf{f}^{\pi}$, and, consequently, we obtain a categorification of halves of the corresponding anisotropic type Kac-Moody bialgebra and bisuperalgebra simultaneously.

Our key new idea in this paper is to distinguish two types of signs occuring in (quantum) Kac-Moody superalgebras. Signs common to Lie algebras and superalgebras, or ordinary signs, are denoted by (-1) as usual, while the signs arising from exchanges of odd elements are replaced by the parameter π. The ordinary signs are categorified using complexes as usual following [KL1, KL2, Ro1], while π is the shadow of a parity-shift functor (called *spin*). Just as the parameter q is categorified by an integer grading shift, π is categorified by a spin. Forgetting the spin corresponds to the specialization $\pi \mapsto 1$, and in this way one ends up with the usual quantum Kac-Moody algebras (as suggested in [KKT]).

We introduce a bilinear form on the algebra ${}_{\mathcal{A}}\mathbf{f}^{\pi}$ (which specializes to a form on the superalgebra ${}_{\mathcal{A}}\mathbf{f}^{-}$) and establish its non-degeneracy, following [Lu2]. With this in place, the necessary categorical constructions can be obtained within the framework of [KL1, KL2] using the spin quiver Hecke algebras. In the process we find the detailed structures of spin nilHecke algebras worked out in [EKL] handy to use. As a consequence of our categorification, we prove a conjecture of [KKT] that all simple modules of spin quiver Hecke algebras are of type $\mathtt{M}$ (that is, they remain simple with the $\mathbb{Z}_2$-grading forgotten).

In the simplest case when $I = I_{\bar{1}}$ consists of an (odd) singleton, the spin quiver Hecke algebra reduces to the spin nilHecke algebra of the second author. In this case, our assertion is that the spin nilHecke algebras categorify half of the quantum $\mathfrak{osp}(1|2)$ (which is new) as well as half of the quantum $\mathfrak{sl}(2)$ (which was already proved in [EKL]; see also [KKT]).

1.4. **Future work.** The ideas of this paper are expected to have several ramifications. The results here can be rephrased in terms of 2-Kac-Moody superalgebras in the sense of [Ro1, Ro2]. One can also imitate [Web] to formulate a (conjectural) categorification of tensor products of integrable modules of quantum Kac-Moody superalgebras. Following the algebraic construction of [KK] it should allow one to show that the cyclotomic spin quiver Hecke algebras categorify the integrable modules of the quantum Kac-Moody (super)algebras. The idea here can also be combined with [KOP] to categorify the more general quantum Borcherds superalgebras and their integrable modules studied in [BKM].

Another main point of this paper is the introduction for the first time of a bar-involution on quantum superalgebras such that

$$\bar{q} = -q^{-1},$$

and the assumption (C6) is again perfect for this purpose. A remarkable property of this bar-involution is its compatibility with the categorification. The canonical basis for the modified or idempotented quantum $\mathfrak{osp}(1|2)$ is constructed in [CW].

In a joint work with Sean Clark [CHW], we are undertaking a construction of the canonical bases for quantum Kac-Moody superalgebras. It will be interesting to compare these canonical bases with those coming from categorification.

We hope that our work helps to clarify the right framework for categorifying the odd Khovanov homology (cf. [EKL] and the references therein). A new idea was suggested in [Kh2] on how to categorify a superalgebra with an isotropic odd simple root. It is natural to expect, though it remains highly non-trivial, that the categorification of the more general Kac-Moody superalgebras will have to combine all these ideas of categorifying the even simple roots, the isotropic odd simple roots, and the non-isotropic odd simple roots.

1.5. **Organization.** The layout of this paper is as follows. In Section 2 we collect the necessary Lie theoretic data. In Section 3 we define the covering Kac-Moody algebra, realize it as the quotient of a free algebra by the radical of a bilinear form, and define a new bar-involution. In Section 4 we recall the definition of the spin quiver Hecke algeba and describe some of its basic properties. In Section 5 we introduce the category of finitely generated graded projective modules over the spin quiver Hecke algebra that will categorify the covering Kac-Moody algebra, and then establish the categorical Serre relations. From these results, we deduce in Section 6 the categorification of the covering Kac-Moody algebra.

Conventions. A module over a superalgebra R in this paper is understood as a *left* module with $\mathbb{Z}_2$-grading compatible with that of R.

2. Root data

2.1. **Generalized Cartan matrices.** Let $I = I_{\bar{0}} \cup I_{\bar{1}}$ be a $\mathbb{Z}_2$-graded finite set of size ℓ. Let $A = (a_{ij})_{i,j \in I}$ be a generalized Cartan matrix (GCM) such that

(C1) $a_{ii} = 2$, for all $i \in I$;
(C2) $a_{ij} \in \mathbb{Z}_{\leq 0}$, for $i \neq j \in I$;
(C3) $a_{ij} = 0$ if and only if $a_{ji} = 0$;
(C4) $a_{ij} \in 2\mathbb{Z}$, for all $i \in I_{\bar{1}}$ and all $j \in I$;
(C5) there exists an invertible matrix $D = \mathrm{diag}(s_1, \dots, s_r)$ with DA symmetric.

We further assume $s_i \in \mathbb{Z}_{>0}$ and $\gcd(s_1, \dots, s_r) = 1$. Introduce the parity function $p(i) = 0$ for $i \in I_{\bar{0}}$ and $p(i) = 1$ for $i \in I_{\bar{1}}$.

Condition (C4) was imposed first in [Kac] so that the corresponding Kac-Moody superalgebras possess similar favorable properties as $\mathfrak{osp}(1|2n)$, i.e., the odd simple roots are all non-isotropic and the Weyl-Kac character formula for integrable modules expressed in terms of the Weyl group holds.

2.2. **The assumption (C6).** We will impose an additional condition on a GCM A introduced in §2.1 for the Kac-Moody superalgebras considered in this paper:

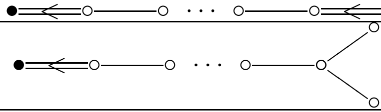
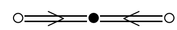
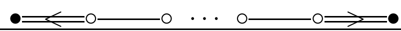
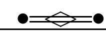
(C6) $I_{\bar{1}} \neq \emptyset$, and the integer s_i is odd if and only if i is odd (i.e., $i \in I_{\bar{1}}$).

The case $I_{\bar{1}} = \emptyset$ has been well studied, and we have nothing new to add.

There is precisely one Kac-Moody superalgebra of finite type satisfying (C1)–(C6). Namely, $\mathfrak{osp}(1|2n)$ (or, $B(0,n)$):

In Table 1, we list the affine Dynkin diagrams satisfying the parity assumption (C6). The nodes labeled by $I_{\bar{0}}$ are drawn as hollow circles $\circ$; the nodes labeled by $I_{\bar{1}}$ are drawn as solid dots $\bullet$. A complete list of affine Lie superalgebras and Dynkin diagrams can be found in [vdL], and we observe that there is exactly one family of affine superalgebras excluded by (C6).

TABLE 1. Affine Dynkin diagrams satisfying (C1)-(C6)

$B^{(1)}(0,n)$		$B^{(1)}(0,1)$	
$A^{(2)}(0,2n-1)$		$A^{(2)}(0,3)$	
$C^{(2)}(n+1)$		$C^{(2)}(2)$	

2.3. **Quivers with compatible automorphism.** Let $\mathbb{K}$ be a field, char$\mathbb{K} \neq 2$. We continue to work under the assumptions of §2.1 and §2.2 throughout the paper.

Let $\widetilde{\Gamma}$ be a graph without loops. We construct a Dynkin diagram Γ by giving $\widetilde{\Gamma}$ the structure of a *graph with compatible automorphism* in the sense of [Lu2, §12, 14]. To define the quiver Hecke algebra, we will use the notion of a *quiver with compatible automorphism* as described in [Ro1, §3.2.4].

Let $\widetilde{I}$ be the labelling set for $\widetilde{\Gamma}$, and $\widetilde{H}$ be the (multi)set of edges. An automorphism $a : \widetilde{\Gamma} \to \widetilde{\Gamma}$ is said to be *compatible* with $\widetilde{\Gamma}$ if, whenever $(i,j) \in \widetilde{H}$ is an edge, i is not in the orbit of j under a (so the quotient graph has no loops).

Fix a compatible automorphism $a : \widetilde{\Gamma} \to \widetilde{\Gamma}$, and set I to be a set of representatives of the obits of $\widetilde{I}$ under a and let $\Gamma = \widetilde{\Gamma}/a$ be the corresponding diagram with nodes labeled by I. For each $i \in I$, let $\alpha_i \in \widetilde{I}/a$ be the corresponding orbit. For $i, j \in I$ with $i \neq j$, let

$$s_i = |\alpha_i|, \tag{2.1}$$

$$(\alpha_i, \alpha_i) = 2s_i, \tag{2.2}$$

$$(\alpha_i, \alpha_j) = -|\{(i', j') \in \tilde{H} \,|\, i' \in \alpha_i, j' \in \alpha_j\}|. \tag{2.3}$$

For all $i, j \in I$, let $a_{ij} = (\alpha_i, \alpha_j)/s_i$. Then, by [Lu2, Proposition 14.1.2] $A = (a_{ij})_{i,j \in I}$ is a GCM and every GCM arises in this way. Moreover, the symmetrizing constants s_i are non-negative integers by definition, and we may assume that $\gcd\{s_i | i \in I\} = 1$ (otherwise, let $\ell = \mathrm{lcm}\{s_i \mid i \in I\}/\gcd\{s_i \mid i \in I\}$, and repeat the construction above with $\widetilde{\Gamma}/a^{\ell}$ instead). Define a $\mathbb{Z}_2$-grading on I by setting $I_{\bar{0}} = \{i \in I \mid |\alpha_i| \in 2\mathbb{Z}\}$ and $I_{\bar{1}} = I \backslash I_{\bar{0}}$. Then, (C6) is satisfied. Among the diagrams obtained from this construction, we will work only with those satisfying (C4).

Then, A is a GCM as in §2.1, and Γ is its Dynkin diagram. We additionally have the data:

$$\text{Simple roots: } \{\alpha_i | i \in I\}; \tag{2.4}$$

$$\text{Root lattice: } Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i,\ Q^+ = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i; \tag{2.5}$$

$$\text{Bilinear pairing: } (\cdot,\cdot) : Q \times Q \longrightarrow \mathbb{Z}. \tag{2.6}$$

Assume further that $\widetilde{\Gamma}$ is a quiver. That is, we have a pair of maps $s : \widetilde{H} \to \widetilde{I}$ and $t : \widetilde{H} \to \widetilde{I}$ (the source and the target). We assume that a is compatible with the quiver structure in the sense that it is equivariant with respect to the source and target maps: $s(a(h)) = a(s(h))$ and $t(a(h)) = a(t(h))$ for all $h \in \widetilde{H}$.

Set

$$d_{ij} = \left| \{h \in \widetilde{H} \mid s(h) \in \alpha_i \text{ and } t(h) \in \alpha_j\}/a \right| \tag{2.7}$$

and let $m(i,j) = \mathrm{lcm}\{(\alpha_i,\alpha_i),(\alpha_j,\alpha_j)\} = 2s_is_j/\gcd(s_i,s_j)$. As noted in [Ro1],

$$d_{ij} + d_{ji} = -2(\alpha_i,\alpha_j)/m(i,j). \tag{2.8}$$

Following [KKT, §3.1], for $i,j \in I$, define a ring

$$\mathbb{K}_{ij}\{u,v\} = \mathbb{K}\langle u,v\rangle / \langle uv - (-1)^{p(i)p(j)}vu\rangle.$$

The data above defines a matrix $\mathsf{Q} = (\mathsf{Q}_{ij}(u,v))_{i,j\in I}$; see [Ro1]. Each $\mathsf{Q}_{ij}(u,v) \in \mathbb{K}_{ij}\{u,v\}$, and the (skew-)polynomial entries in Q are defined by $\mathsf{Q}_{ii}(u,v) = 0$, and for $i \neq j$,

$$\mathsf{Q}_{ij}(u,v) = (-1)^{d_{ij}}\left(u^{m(i,j)/(\alpha_i,\alpha_i)} - v^{m(i,j)/(\alpha_j,\alpha_j)}\right)^{-2(\alpha_i,\alpha_j)/m(i,j)}. \tag{2.9}$$

Lemma 2.1. *For $i,j \in I$, we have*

(a) $\mathsf{Q}_{ij}(u,v) \in \mathbb{K}_{ij}\{u,v\}$,
(b) $\mathsf{Q}_{ii}(u,v) = 0$,
(c) $\mathsf{Q}_{ij}(u,v) = \mathsf{Q}_{ji}(v,u)$,
(d) $\mathsf{Q}_{ij}(-u,v) = \mathsf{Q}_{ij}(u,v)$ *whenever* $i \in I_{\bar{1}}$.

In particular, the entries in Q *satisfy the properties in* [KKT, (3.1)].

Proof. Properties (a), (b), and (c) are clear by definition.

To prove (d), first assume $j \in I_{\bar{0}}$. Note that by assumption (C6) s_i is odd and s_j is even. Therefore, $m(i,j)/(\alpha_i,\alpha_i) = s_j/\gcd(s_i,s_j)$ is even, proving the lemma in this case.

Next, assume $j \in I_{\bar{1}}$. In this case, both $m(i,j)/(\alpha_i,\alpha_i) = s_j/\gcd(s_i,s_j)$ and $m(i,j)/(\alpha_j,\alpha_j) = s_i/\gcd(s_i,s_j)$ are odd by assumption (C6). However,

$$-2(\alpha_i,\alpha_j)/m(i,j) = a_{ij}\gcd(s_i,s_j)/s_j$$

is even, since a_{ij} is even by (C4) and both s_i and s_j are odd (again, by (C6)). Since $uv = -vu \in \mathbb{K}_{ij}\{u,v\}$,

$$(u^a - v^b)^{2c} = (u^{2a} + v^{2b})^c$$

whenever a and b are both odd, and the result follows. □

3. Quantum superalgebras and bilinear forms

3.1. **Kac-Moody superalgebras.** Associated to a GCM A as in §2.1 and §2.2 is a Kac-Moody superalgebra $\mathfrak{g} = \mathfrak{g}(A)$ (see [Kac]) and a quantized enveloping superalgebra (see [BKM]). Let $\mathbf{f}^-$ be the superanalogue of Lusztig's algebra $\mathbf{f}$, generated over $\mathbb{Q}(q)$ by $\{\theta_i | i \in I\}$ and subject to a signed analogue of Serre relations (cf. (3.2) below). The θ_i parameterized by $i \in I_{\bar{0}}$ (respectively, $i \in I_{\bar{1}}$) are *even* (respectively, *odd*). For $k \geq 0$ and $i \in I$, we shall denote by $\theta_i^{(k)} = \theta_i^k/[k]_i^-!$ the divided powers, where

$$q_i = q^{s_i}, \qquad [k]_i^- = \frac{(-1)^{kp(i)}q_i^k - q_i^{-k}}{(-1)^{p(i)}q_i - q_i^{-1}}, \qquad [k]_i^-! = \prod_{a=1}^{k}[a]_i^-. \tag{3.1}$$

We shall denote by $\mathcal{A} = \mathbb{Z}[q,q^{-1}] \subset \mathbb{Q}(q)$ and by ${}_\mathcal{A}\mathbf{f}^-$ the $\mathcal{A}$-subalgebra of $\mathbf{f}^-$ generated by all divided powers $\theta_i^{(k)}$, for $k \geq 1, i \in I$. Besides the standard relation among divided powers, the signed Serre relations in ${}_\mathcal{A}\mathbf{f}^-$ can be written as

$$\sum_{k=0}^{1-a_{ij}} (-1)^{k+p(k;i,j)}\theta_i^{(1-a_{ij}-k)}\theta_j\theta_i^{(k)} = 0, \tag{3.2}$$

where

$$p(k;i,j) = kp(i)p(j) + \frac{1}{2}k(k-1)p(i). \tag{3.3}$$

Definition 3.1. Assume a GCM A satisfies (C1)-(C6). We define a bar-involution $^- : \mathbf{f}^- \to \mathbf{f}^-$ by letting

$$\overline{q} = -q^{-1}, \qquad \overline{\theta_i} = \theta_i \quad (\forall i \in I). \tag{3.4}$$

Note that all the divided powers $\theta_i^{(k)}$ are bar-invariant under the assumption (C6), and hence we have $^- : {}_\mathcal{A}\mathbf{f}^- \to {}_\mathcal{A}\mathbf{f}^-$.

Remark 3.2. With respect to this apparently new bar-involution, we will develop a theory of canonical basis in a forthcoming work with Sean Clark.

3.2. **Kac-Moody algebras.** If we forget the parity on I in §2.1, we shall write the corresponding GCM matrix by A^+. Associated to the GCM A^+ is the usual Kac-Moody algebra $\mathfrak{g}^+ = \mathfrak{g}(A^+)$, and write $\mathbf{f}^+$ for Luszig's algebra $\mathbf{f}$. For $k \geq 0$ and $i \in I$, we shall abuse notation slightly and denote by $\theta_i^{(k)} = \theta_i^k/[k]_i^+!$ the divided powers as before, where now

$$[k]_i^+ = \frac{(+1)^{kp_i}q_i^k - q_i^{-k}}{(+1)^{p(i)}q_i - q_i^{-1}}, \qquad [k]_i^+! = \prod_{a=1}^{k}[a]_i^+. \tag{3.5}$$

Let ${}_\mathcal{A}\mathbf{f}^+$ be the $\mathcal{A}$-subalgebra of $\mathbf{f}^+$ generated by all divided powers $\theta_i^{(k)}$, for $k \geq 1$ and $i \in I$, subject to the usual Serre relations, which we suggestively write as

$$\sum_{k=0}^{1-a_{ij}} (-1)^k(+1)^{p(k;i,j)}\theta_i^{(1-a_{ij}-k)}\theta_j\theta_i^{(k)} = 0. \tag{3.6}$$

3.3. **Covering Kac-Moody algebras.** Finally, we present a common framework to describe the presentations in §3.1 and §3.2, and in the process justify our seemingly inconsistent use of notation therein. To this end, fix an indeterminant π, and for a ring R, we introduce a new ring

$$R^\pi = R[\pi]/(\pi^2 - 1).$$

Denote by $'\mathbf{f}^\pi = {'\mathbf{f}}_{\bar{0}}^\pi \oplus {'\mathbf{f}}_{\bar{1}}^\pi$ the free associative algebra over $\mathbb{Q}(q)^\pi$ generated by even generators θ_i for $i \in I_{\bar{0}}$ and odd generators θ_i for $i \in I_{\bar{1}}$. We have parity $p(x) = 0$ for $x \in {'\mathbf{f}}_{\bar{0}}^\pi$ and $p(x) = 1$ for $x \in {'\mathbf{f}}_{\bar{1}}^\pi$. Letting the weight of θ_i be $\alpha_i \in Q^+$, the algebra $'\mathbf{f}^\pi$ has an induced weight space decomposition $'\mathbf{f}^\pi = \bigoplus_{\nu \in Q^+} {'\mathbf{f}}_\nu^\pi$. For $x \in {'\mathbf{f}}_\nu^\pi$, we set $|x| = \nu$.

For $a \geq t \geq 0$ and $i \in I$, we shall denote

$$q_i = q^{s_i}, \quad \pi_i = \pi^{p(i)},$$

and

$$[a]_i = \frac{\pi_i^a q_i^a - q_i^{-a}}{\pi_i q_i - q_i^{-1}}, \quad [a]_i! = \prod_{k=1}^{a} [k]_i, \quad \begin{bmatrix} a \\ t \end{bmatrix}_i = \frac{[a]_i!}{[t]_i![a-t]_i!}. \tag{3.7}$$

We denote by $\theta_i^{(a)} = \theta_i^a / [a]_i!$ the divided powers.

Define an algebra homomorphism $r : {}'\mathbf{f}^\pi \to {}'\mathbf{f}^\pi \otimes {}'\mathbf{f}^\pi$ by letting

$$r(\theta_i) = \theta_i \otimes 1 + 1 \otimes \theta_i, \qquad \forall i \in I.$$

The algebra structure on ${}'\mathbf{f}^\pi \otimes {}'\mathbf{f}^\pi$ is given by

$$(x_1 \otimes x_2)(x_1' \otimes x_2') = \pi^{p(x_2)p(x_1')} q^{-(|x_2|,|x_1|)} x_1 x_1' \otimes x_2 x_2', \tag{3.8}$$

for homogeneous $x_1, x_2, x_1', x_2' \in {}'\mathbf{f}^\pi$, where $(|x_2|, |x_1'|)$ is defined in (2.6). In particular, we note that

$$(1 \otimes \theta_i) \cdot (\theta_i \otimes 1) = \pi_i q_i^{-2} (\theta_i \otimes 1) \cdot (1 \otimes \theta_i). \tag{3.9}$$

The following is a superanalogue of [Lu2, Proposition 1.2.3]. Note that the v in Lusztig corresponds to our q^{-1}. Though the identities below look almost identical to those in loc. cit., we give a detailed proof where some super signs show up.

Proposition 3.3. *There exists a unique bilinear form* $(\cdot,\cdot)$ *on* ${}'\mathbf{f}^\pi$ *with values in* $\mathbb{Q}(q)^\pi$ *such that* $(1,1) = 1$,

(a) $(\theta_i, \theta_j) = \delta_{ij}(1 - \pi_i q_i^2)^{-1} \quad (\forall i, j \in I)$,
(b) $(x, y'y'') = (r(x), y' \otimes y'') \quad (\forall x, y', y'' \in {}'\mathbf{f}^\pi)$,
(c) $(xx', y'') = (x' \otimes x'', r(y'')) \quad (\forall x', x'', y'' \in {}'\mathbf{f}^\pi)$,
(d) *the induced bilinear form* $({}'\mathbf{f}^\pi \otimes {}'\mathbf{f}^\pi) \times ({}'\mathbf{f}^\pi \otimes {}'\mathbf{f}^\pi) \to \mathbb{Q}(q)$ *is given by*

$$(x' \otimes x'', y' \otimes y'') := (x', y')(x'', y''). \tag{3.10}$$

Moreover, the bilinear form $(\cdot,\cdot)$ *is symmetric.*

Proof. We follow [Lu2, 1.2.3] to define an associative algebra structure on ${}'\mathbf{f}^* := \bigoplus_\nu {}'\mathbf{f}_\nu^*$ by dualizing the coproduct $r : {}'\mathbf{f} \to {}'\mathbf{f} \otimes {}'\mathbf{f}$. Let $\xi_i \in {}'\mathbf{f}_i^*$ be defined by $\xi_i(\theta_j) = \delta_{ij}(1 - \pi_i q_i^2)^{-1}$, for all $j \in I$. Let $\phi : {}'\mathbf{f} \to {}'\mathbf{f}^*$ be the unique algebra homomorphism such that $\phi(\theta_i) = \xi_i$ for all i. We may identify ${}'\mathbf{f}^* \otimes {}'\mathbf{f}^* \cong ({}'\mathbf{f} \otimes {}'\mathbf{f})^*$ so that the functional $(\phi \otimes \phi)(y \otimes y')$, for $y, y' \in {}'\mathbf{f}$, is given by

$$(\phi(y) \otimes \phi(y'))(x \otimes x') = (\phi(y)(x))(\phi(y')(x')), \quad x, x' \in {}'\mathbf{f}. \tag{3.11}$$

Note that the maps ϕ and $\phi \otimes \phi$ preserve the $(Q \times \mathbb{Z}_2)$-grading.

Define $(x, y) = \phi(y)(x)$, for $x, y \in {}'\mathbf{f}$. Property (a) follows directly from the definition, and (d) follows from (3.11).

Clearly $(x, y) = 0$ unless (homogeneous) x, y have the same weight in Q, which implies they must have the same parity. All elements involved below will be assumed to be homogeneous.

Now, write $r(x) = \sum x_1 \otimes x_2$. We have

$$\begin{aligned}(x, y'y'') &= \phi(y'y'')(x) = (\phi(y')\phi(y''))(x)\\ &= (\phi(y') \otimes \phi(y''))(r(x))\\ &= \sum (\phi(y') \otimes \phi(y''))(x_1 \otimes x_2)\\ &= \sum \phi(y')(x_1)\phi(y'')(x_2)\\ &= \sum \phi(y')(x_1)\phi(y'')(x_2) = (r(x), y' \otimes y'').\end{aligned}$$

This proves (b).

It remains to prove (c). The cases when y'' is 1 or θ_j can be checked directly. Assume that (c) is known for y'' replaced by y or y' and for any x, x'. We then prove that (c) holds for $y'' = yy'$. Write

$$\begin{aligned} r(x) &= \sum x_1 \otimes x_2, \quad r(x') = \sum x'_1 \otimes x'_2,\\ r(y) &= \sum y_1 \otimes y_2, \quad r(y') = \sum y'_1 \otimes y'_2.\end{aligned}$$

Then

$$\begin{aligned} r(xx') &= \sum q^{(|x_2|,|x'_1|)}\pi^{p(x_2)p(x'_1)} x_1x'_1 \otimes x_2x'_2,\\ r(yy') &= \sum q^{(|y_2|,|y'_1|)}\pi^{p(y_2)p(y'_1)} y_1y'_1 \otimes y_2y'_2.\end{aligned}$$

We have

$$\begin{aligned}(xx', yy') &= (\phi(y)\phi(y'))(xx') = (\phi(y) \otimes \phi(y'))(r(xx'))\\ &= \sum q^{-(|x_2|,|x'_1|)}\pi^{p(x_2)p(x'_1)}(x_1x'_1, y)(x_2x'_2, y')\\ &= \sum q^{-(|x_2|,|x'_1|)}\pi^{p(x_2)p(x'_1)}(x_1 \otimes x'_1, r(y))(x_2 \otimes x'_2, r(y'))\\ &= \sum q^{-(|x_2|,|x'_1|)}\pi^{p(x_2)p(x'_1)}(x_1, y_1)(x'_1, y_2)(x_2, y'_1)(x'_2, y'_2). \qquad (3.12)\end{aligned}$$

On the other hand,

$$\begin{aligned}(x \otimes x', r(yy')) &= \sum q^{-(|y_2|,|y'_1|)}\pi^{p(y_2)p(y'_1)}(x \otimes x', y_1y'_1 \otimes y_2y'_2)\\ &= \sum q^{-(|y_2|,|y'_1|)}\pi^{p(y_2)p(y'_1)}(x, y_1y'_1)(x', y_2y'_2)\\ &= \sum q^{-(|y_2|,|y'_1|)}\pi^{p(y_2)p(y'_1)}(r(x), y_1 \otimes y'_1)(r(x'), y_2 \otimes y'_2)\\ &= \sum q^{-(|y_2|,|y'_1|)}\pi^{p(y_2)p(y'_1)}(x_1, y_1)(x'_1, y_2)(x_2, y'_1)(x'_2, y'_2). \qquad (3.13)\end{aligned}$$

For a summand to make a non-zero contribution, we must have $|x'_1| = |y_2|$ and $|x_2| = |y'_1|$ in Q and, therefore, both $p(x'_1) = p(y_2)$ and $p(x_2) = p(y'_1)$. It follows that the powers of q and π in (3.12) and (3.13) match perfectly. Hence the two sums in (3.12) and (3.13) are equal, proving (c). □

We then define (half of) the *covering Kac-Moody algebra* to be the quotient algebra $'\mathbf{f}^\pi$ by the radical as

$$\mathbf{f}^\pi = {'\mathbf{f}^\pi}/\mathrm{Rad}(\cdot,\cdot).$$

The bilinear form $(\cdot,\cdot)$ on $'\mathbf{f}$ descends to a non-degenerate bilinear form $(\cdot,\cdot)$ on $\mathbf{f}^\pi$ satisfying the same properties as in Proposition 3.3, and we also have an induced algebra homomorphism $r : \mathbf{f}^\pi \to \mathbf{f}^\pi \otimes \mathbf{f}^\pi$.

3.4. **Binomial identities.** For an indeterminate v, let

$$[a]_v = \frac{v^a - v^{-a}}{v - v^{-1}}, \qquad \begin{bmatrix} a \\ t \end{bmatrix}_v = \frac{[a]_v!}{[t]_v![a-t]_v!}, \quad \text{for } 0 \le t \le a.$$

Recall from [Lu2, 1.3.5] the quantum binomial formula: for any $a \ge 0$ and x, y in a $\mathbb{Q}(v)$-algebra such that $xy = v^2 yx$,

$$(x+y)^a = \sum_{t=0}^{a} v^{t(a-t)} \begin{bmatrix} a \\ t \end{bmatrix}_v y^t x^{a-t}. \tag{3.14}$$

Now we convert (3.14) to notations in our supersetting.

Lemma 3.4. *For any $a \ge 0$ and x, y in a $\mathbb{Q}(q)$-algebra such that $xy = \pi_i q_i^{-2} yx$,*

$$(x+y)^a = \sum_{t=0}^{a} (\pi_i q_i)^{-t(a-t)} \begin{bmatrix} a \\ t \end{bmatrix}_i y^t x^{a-t}. \tag{3.15}$$

Proof. For i even, $p(i) = 0$, and so (3.15) is simply (3.14) with $v = q_i^{-1}$.

Now assume i is odd, i.e., $p(i) = 1$, and so $\pi_i = \pi$. Fix a square root $\sqrt{\pi}$ of π once for all. We have $xy = v^2 yx$, if we introduce a new indeterminate v by letting

$$v := \sqrt{\pi} q_i^{-1}. \tag{3.16}$$

Hence (3.14) is applicable, and we have

$$(x+y)^a = \sum_{t=0}^{a} (\sqrt{\pi} q_i^{-1})^{t(a-t)} \begin{bmatrix} a \\ t \end{bmatrix}_v y^t x^{a-t}.$$

This identity can then converted to (3.15) by using the following identities: for $a \ge t \ge 0$,

$$[a]_v = \sqrt{\pi}^{a-1} [a]_i, \qquad [a]_v! = \sqrt{\pi}^{\frac{a(a-1)}{2}} [a]_i!, \qquad \begin{bmatrix} a \\ t \end{bmatrix}_v = \sqrt{\pi}^{t(a-t)} \begin{bmatrix} a \\ t \end{bmatrix}_i. \tag{3.17}$$

Since $\pi = \pi^{-1}$, this proves the lemma. ☐

3.5. **Bilinear form.** We now study the divided powers in relation to the homomorphism r and the bilinear form $(\cdot, \cdot)$.

Lemma 3.5. *For any $a \in \mathbb{Z}_{\ge 0}$ and any $i \in I$, we have*

$$r(\theta_i^{(a)}) = \sum_{t+t'=a} (\pi_i q_i)^{-tt'} \theta_i^{(t)} \otimes \theta_i^{(t')}.$$

Proof. When $i \in I_{\bar{0}}$, this is simply [Lu2, Lemma 1.4.2], which was proved directly using (3.14).

Now assume $i \in I_{\bar{1}}$. Thanks to the identity (3.9), the assumption of Lemma 3.4 is satisfied with $x = 1 \otimes \theta_i$ and $y = \theta_i \otimes 1$. Hence this lemma follows directly from (3.15) by the definition of the divided power based on (3.7). ☐

Lemma 3.6. *For any $a \in \mathbb{Z}_{\ge 0}$ and any $i \in I$, we have*

$$\begin{aligned}(\theta_i^{(a)}, \theta_i^{(a)}) &= \pi_i^{a(a-1)/2} \prod_{s=1}^{a} (1 - \pi_i^s q_i^{2s})^{-1} \\ &= (-1)^a \pi_i^{a(a-1)/2} q_i^{-a(a+1)/2} (\pi_i q_i - q_i^{-1})^{-a} ([a]_i!)^{-1}.\end{aligned}$$

Proof. We will only prove the first identity, as the second identity is elementary.

The argument is similar to the proof of [Lu2, Lemma 1.4.4], which corresponds to the case for $i \in I_{\bar{0}}$. We proceed by induction. Note that the lemma holds for $a = 0$ or 1. Assume the lemma holds for a and also for a'. Using Lemma 3.5 and (3.10) we have

$$\begin{aligned}(\theta_i^{(a+a')}, \theta_i^{(a+a')}) &= \begin{bmatrix} a+a' \\ a \end{bmatrix}_i^{-1} \left(r(\theta_i^{(a+a')}), \theta_i^{(a)} \otimes \theta_i^{(a')} \right) \\ &= (\pi_i q_i)^{-aa'} \begin{bmatrix} a+a' \\ a \end{bmatrix}_i^{-1} (\theta_i^{(a)}, \theta_i^{(a)})(\theta_i^{(a')}, \theta_i^{(a')}) \\ &= (\pi_i q_i)^{-aa'} \begin{bmatrix} a+a' \\ a \end{bmatrix}_i^{-1} \pi_i^{\binom{a}{2}} \prod_{s=1}^{a} (1 - \pi_i^s q_i^{2s})^{-1} \cdot \pi_i^{\binom{a'}{2}} \prod_{s=1}^{a'} (1 - \pi_i^s q_i^{2s})^{-1} \\ &= \pi_i^{aa' + a(a-1)/2 + a'(a'-1)/2} \prod_{s=1}^{a+a'} (1 - \pi_i^s q_i^{2s})^{-1}.\end{aligned}$$

Since $aa' + \binom{a}{2} + \binom{a'}{2} = \binom{a+a'}{2}$, this gives the result. □

3.6. **Quantum Serre relations.** We first formulate the following superanalogue of the last formula in the proof of [Lu2, Lemma 1.4.5].

Lemma 3.7. *Assume $i \in I$ is odd. Let $n \in \mathbb{Z}_{\geq 0}$ and let $a, a', b, b' \in \mathbb{Z}_{\geq 0}$ such that $a + a' = b + b' = n$. Then,*

$$(3.18) \qquad \left(\theta_i^{(a)} \theta_j \theta_i^{(a')}, \theta_i^{(b)} \theta_j \theta_i^{(b')}\right) = \sum \frac{(-1)^{n+1} q_i^{-\clubsuit} \pi^{\heartsuit}}{(\pi q_i - q_i^{-1})^n (1 - \pi_j q_j^2) [s]_i! [s']_i! [t]_i! [t']_i!},$$

where the sum is taken over all t, t', s, s' in $\mathbb{Z}_{\geq 0}$ such that

$$(3.19) \qquad t + s = b, t' + s' = b', t + t' = a, s + s' = a',$$

and we have denoted

$$\begin{aligned}\spadesuit &= \frac{1}{2}\big(s(s-1) + s'(s'-1) + t(t-1) + t'(t'-1)\big), \\ \clubsuit &= ss' + tt' + ts + t's' + 2t's + (\spadesuit + n) + (t' + s)a_{ij}, \\ \heartsuit &= ss' + tt' + ts + t's' + t's + (s + t')p(j) + \spadesuit.\end{aligned}$$

Proof. We first compute by Lemma 3.5 that

(3.20)

$$\begin{aligned}& r(\theta_i^{(b)} \theta_j \theta_i^{(b')}) = r(\theta_i^{(b)}) r(\theta_j) r(\theta_i^{(b')}) \\ &= \Big(\sum_{t+s=b} (\pi q_i)^{-ts} \theta_i^{(t)} \otimes \theta_i^{(s)} \Big) \cdot (\theta_j \otimes 1 + 1 \otimes \theta_j) \cdot \Big(\sum_{t'+s'=b'} (\pi q_i)^{-t's'} \theta_i^{(t')} \otimes \theta_i^{(s')} \Big) \\ &= \sum q_i^{-(ts+t's'+2t's)} q^{-s(\alpha_i, \alpha_j)} \pi^{ts+t's'+t's+sp(j)} \theta_i^{(t)} \theta_j \theta_i^{(t')} \otimes \theta_i^{(s)} \theta_i^{(s')} \\ &\quad + \sum q_i^{-(ts+t's'+2t's)} q^{-t'(\alpha_i, \alpha_j)} \pi^{ts+t's'+t's+t'p(j)} \theta_i^{(t)} \theta_i^{(t')} \otimes \theta_i^{(s)} \theta_j \theta_i^{(s')}.\end{aligned}$$

Using (3.20) and $q^{(\alpha_i,\alpha_j)} = q_i^{a_{ij}}$, we further compute that

$$(3.21)\quad \begin{aligned}&(\theta_i^{(a)}\theta_j\theta_i^{(a')}, \theta_i^{(b)}\theta_j\theta_i^{(b')})\\ &= (\theta_i^{(a)}\theta_j \otimes \theta_i^{(a')}, r(\theta_i^{(b)}\theta_j\theta_i^{(b')}))\\ &= \sum q_i^{-(ts+t's'+2t's)} q^{-s(\alpha_i,\alpha_j)} \pi^{t's+ts+t's'+sp(j)} (\theta_i^{(a)}\theta_j \otimes \theta_i^{(a')}, \theta_i^{(t)}\theta_j\theta_i^{(t')} \otimes \theta_i^{(s)}\theta_i^{(s')})\\ &= \sum q_i^{-(ts+t's'+2t's+sa_{ij})} \pi^{ts+t's'+t's+sp(j)} (\theta_i^{(a)}\theta_j, \theta_i^{(t)}\theta_j\theta_i^{(t')})(\theta_i^{(a')}, \theta_i^{(s)}\theta_i^{(s')}).\end{aligned}$$

It follows by Lemma 3.5 that

$$(3.22)\quad \begin{aligned}(\theta_i^{(a')}, \theta_i^{(s)}\theta_i^{(s')}) &= (r(\theta_i^{(a')}), \theta_i^{(s)} \otimes \theta_i^{(s')})\\ &= q_i^{-ss'} \pi^{ss'} (\theta_i^{(s)}, \theta_i^{(s)})(\theta_i^{(s')}, \theta_i^{(s')}).\end{aligned}$$

Also using $q^{(\alpha_i,\alpha_j)} = q_i^{a_{ij}}$, we have

$$(3.23)\quad \begin{aligned}(\theta_i^{(a)}\theta_j, \theta_i^{(t)}\theta_j\theta_i^{(t')}) &= (r(\theta_i^{(a)})r(\theta_j), \theta_i^{(t)}\theta_j \otimes \theta_i^{(t')})\\ &= q_i^{-(tt'+t'a_{ij})} \pi^{tt'+t'p(j)} (\theta_j, \theta_j)(\theta_i^{(t)}, \theta_i^{(t)})(\theta_i^{(t')}, \theta_i^{(t')}).\end{aligned}$$

Inserting (3.22) and (3.23) into (3.21), we obtain

$$(3.24)\quad \begin{aligned}&(\theta_i^{(a)}\theta_j\theta_i^{(a')}, \theta_i^{(b)}\theta_j\theta_i^{(b')})\\ &= q_i^{-(ss'+tt'+ts+t's'+2t's+(s+t')a_{ij})} \pi^{ss'+tt'+ts+t's'+t's+(s+t')p(j)}\\ &\quad \times (\theta_j, \theta_j)(\theta_i^{(s)}, \theta_i^{(s)})(\theta_i^{(s')}, \theta_i^{(s')})(\theta_i^{(t)}, \theta_i^{(t)})(\theta_i^{(t')}, \theta_i^{(t')}).\end{aligned}$$

The right-hand side of (3.24) can then be converted to (3.18) using Lemma 3.6 repeatedly and noting that

$$\frac{1}{2}\big(s(s+1) + s'(s'+1) + t(t+1) + t'(t'+1)\big) = \spadesuit + n.$$

This proves the lemma. □

Now we are ready to state and prove the following fundamental result, called the *quantum Serre relations*. Recall $p(k; i, j)$ from (3.3).

Theorem 3.8. *Let A be a GCM satisfying (C1)-(C5). For any $i \neq j$ in I, the following identities hold in $\mathbf{f}^\pi$:*

$$(3.25)\quad \sum_{a+a'=1-a_{ij}} (-1)^{a'} \pi^{p(a';i,j)} \theta_i^{(a)}\theta_j\theta_i^{(a')} = 0.$$

Proof. The strategy is to show the element on the left-hand side of (3.25) orthogonal to $\mathbf{f}^\pi$ with respect to $(\cdot,\cdot)$. For i even, the same proof for [Lu2, Proposition 1.4.3] applies here without any change, regardless of the parity of j.

Now assume that i is odd. Hence $p(i) = 1$ and $\pi_i = \pi$. We still proceed as in [Lu2] and keep track of the supersigns carefully in the meantime.

Using (3.19) to get rid of t and s', we can rewrite ♣ in Lemma 3.7 as

$$(3.26)\quad \clubsuit = b(b+1)/2 + b'(b'+1)/2 - s(b-1) - t'(b'-1) + (t'+s)(a_{ij}+n-1).$$

(This agreed with the power of v^{-1} in the corrected [Lu2, Lemma 1.4.5].)

Recall the new indeterminate v from (3.16). It follows by (3.16), (3.17), (3.18) and (3.26) that

$$\begin{aligned}(3.27)\quad &\pi^{p(a';i,j)} \cdot \big(\theta_i^{(a)}\theta_j\theta_i^{(a')}, \theta_i^{(b)}\theta_j\theta_i^{(b')}\big)\\ &= \frac{(-1)^{n+1}v^{b(b+1)/2+b'(b'+1)/2}}{(\pi q_i - q_i^{-1})^n(1-q_j^2)} \sum \frac{v^{-s(b-1)-t'(b'-1)+(t'+s)(a_{ij}+n-1)}\sqrt{\pi}^{\diamondsuit}}{[s]_v![s']_v![t]_v![t']_v!},\end{aligned}$$

where the sum is taken as in Lemma 3.7. Here we have

$$(3.28)\qquad \diamondsuit = 2p(a';i,j) + 2\heartsuit - \clubsuit + \spadesuit,$$

where $\spadesuit$ arises from the conversion (3.17). Recall $p(a';i,j) = a'p(j) + \frac{1}{2}a'(a'-1)$, and note that $a_{ij} = 1-n$ is an even integer by (C4). Some elementary and lengthy manipulation using (3.19) allows us to rewrite $\diamondsuit$ in (3.28) (in terms of s, t', b, b') as

$$\begin{aligned}\diamondsuit &= p(j)\big(2a' + 2(s+t')\big) + a'(a'-1) + ss' + tt' + ts + t's' + 2\spadesuit - n - (s+t')a_{ij}\\ &= p(j)(4s+2b') + 2a'(a'-1) + b^2 - b + (1-n-a_{ij})s + (n-1-a_{ij})t' - n\\ &\equiv 2p(j)b' + b^2 - b - n \mod 4.\end{aligned}$$

Recalling $\pi^2 = 1$, it follows that $\sqrt{\pi}^{\diamondsuit}$ is independent of s, s', t, t' and can be moved to the front of $\sum$ in (3.27).

To prove (3.25), it suffices to show that $\sum_{a+a'=1-a_{ij}} (-1)^{a'}\pi^{p(a',i,j)}\theta_i^{(a)}\theta_j\theta_i^{(a')} \in \mathbf{f}^\pi$ is orthogonal with respect to $(\cdot,\cdot)$ to $\theta_i^{(b)}\theta_j\theta_i^{(b')}$ for all b, b' such that $b+b' = 1-a_{ij}$. (These elements $\theta_i^{(b)}\theta_j\theta_i^{(b')}$ span $\mathbf{f}^\pi{}_{\alpha_j+(1-a_{ij})\alpha_i}$.) To that end, recalling (3.27) and noting $a_{ij}+n-1 = 0$, it remains to verify the identity

$$\sum (-1)^{s+s'} v^{-s(b-1)-t'(b'-1)}([s]_v![s']_v![t]_v![t']_v!)^{-1} = 0,$$

where the sum is taken as in Lemma 3.7 again. We can factorize the sum on the left-hand side as

$$\Big(\sum_{t+s=b} (-1)^s v^{-s(b-1)}([s]_v![t]_v!)^{-1}\Big)\Big(\sum_{t'+s'=b'} (-1)^{s'} v^{-s'(b'-1)}([s']_v![t']_v!)^{-1}\Big).$$

Since at least one of b and b' is positive, one of the two factors must be zero thanks to a classical binomial identity (cf. [Lu2, 1.3.4(a)]). ☐

3.7. **The bar-involution.** It is shown in [BKM] that the integrable modules of the quantum Kac-Moody superalgebras have the same characters as their classical counterparts, generalizing the earlier work of [Lu1]. Now based on this fact and proceeding as in [Lu2, 33.1], we can establish Theorem 3.9(a) below as a superanalogue of [Lu2, Theorem 33.1.3] (which is reformulated as Theorem 3.9(b) below). Recall that the algebras $\mathbf{f}^-, \mathbf{f}^+, \mathbf{f}^\pi$ were introduced in earlier subsections.

Theorem 3.9. *There exist isomorphisms of $\mathbb{Q}(q)$-algebras:*

(a) $\mathbf{f}^\pi/(\pi+1) \cong \mathbf{f}^-$,
(b) $\mathbf{f}^\pi/(\pi-1) \cong \mathbf{f}^+$.

Corollary 3.10. Rad$(\cdot,\cdot)$ *is generated by Serre relations.*

Proof. When $\pi = 1$ this is proved in [Lu2, 33.1]. Using [BKM], a similar argument proves this in the case $\pi = -1$. ☐

Remark 3.11. Theorem 3.9(a) identifies half of the quantum Kac-Moody superalgebra associated to the generalized Cartan matrix A in §2.1 with the quotient of a free algebra by the radical of an analogue of Lusztig's bilinear form. A similar result for quantum $\mathfrak{osp}(1|2n)$ was also obtained in [Ya] using a different normalization of bilinear form such that $(\theta_i, \theta_i) = 1$ for i odd and with additional signs appearing in the definition of the form on $\mathbf{f}^- \otimes \mathbf{f}^-$ (see also [Gr]).

We will write ${}_{\mathcal{A}}\mathbf{f}^\pi$ for the $\mathcal{A}^\pi$-subalgebra of $\mathbf{f}^\pi$ generated by the divided powers $\theta_i^{(k)} = \theta_i^k/[k]_i!$ (see (3.7) for $[k]_i$), subject to the quantum Serre relation (3.25).

Definition 3.12. Under the assumption (C1)-(C6) for the GCM A, we define the bar-involution $^- : \mathbf{f}^\pi \to \mathbf{f}^\pi$ by letting

$$\overline{\pi} = \pi, \qquad \overline{q} = \pi q^{-1}, \qquad \overline{\theta_i} = \theta_i \quad (\forall i \in I). \tag{3.29}$$

We have that

$$\overline{q_i} = \pi^{s_i} q_i^{-1},$$

and a calculation gives

$$\overline{[k]}_i = \pi_i^{(k-1)} \pi^{s_i(k-1)} [k]_i = \pi^{(s_i + p(i))(k-1)} [k]_i.$$

It now follows from (C6) that the quantum integers $[k]_i$ are bar-invariant, and so the divided powers $\theta_i^{(k)}$ are bar-invariant as well. Thus, this induces a bar-involution $^- : {}_{\mathcal{A}}\mathbf{f}^\pi \to {}_{\mathcal{A}}\mathbf{f}^\pi$.

4. Spin quiver Hecke algebras

4.1. **Generators and relations.** Fix an $\ell \times \ell$ GCM A as in §2.1, and continue to assume (C6) as usual. Let $n \in \mathbb{Z}_{\geq 0}$, and assume for $\nu \in Q^+$ (see (2.5)) that $\nu = n_1\alpha_1 + \cdots + n_\ell \alpha_\ell$ and $n_1 + \cdots + n_\ell = n$ (i.e. ν has *height* $\mathrm{ht}(\nu) = n$). Let $I^\nu \subset I^n$ be the S_n-orbit of the element

$$(\underbrace{1, \ldots, 1}_{n_1}, \ldots, \underbrace{\ell, \ldots, \ell}_{n_\ell}),$$

where S_n acts on I^n by place permutation: $w \cdot (i_1, \ldots, i_n) = (i_{w(1)}, \ldots, i_{w(n)})$. Equivalently,

$$I^\nu = \{\underline{i} = (i_1, \ldots, i_n) \in I^n \mid \alpha_{i_1} + \cdots + \alpha_{i_n} = \nu\}. \tag{4.1}$$

We now define an algebra based on the data above in terms of generators and relations. When $I_{\bar{1}} = \emptyset$, the algebra is nothing but the quiver Hecke algebra of Khovanov-Lauda and Rouquier [KL1, Ro1]. In the general case $I_{\bar{1}} \neq \emptyset$ we are considering, these algebras were recently defined in [KKT]. We refer to them as *spin* quiver Hecke algebras as explained in the introduction. In the special case when $I = I_{\bar{1}}$ is a singleton, the algebra is the spin nilHecke algebra, a nil version of the spin Hecke algebra first introduced in [Wa, 3.3]. The spin quiver Hecke algebra is defined to be

$$\mathcal{H}^- = \bigoplus_{\nu \in Q^+} \mathcal{H}^-(\nu),$$

where $\mathcal{H}^-(\nu)$ is the unital $\mathbb{K}$-algebra, with identity 1_ν, given by generators and relations as described below.

The generators of $\mathcal{H}^-(\nu)$ are

$$\{e(\underline{i}) | \underline{i} \in I^\nu\} \cup \{y_1, \ldots, y_n\} \cup \{\tau_1, \ldots, \tau_{n-1}\}.$$

We refer to the $e(\underline{i})$ as *idempotents*, the y_r as *(skew) Jucys-Murphy elements*, and the τ_r as *intertwining elements*. Indeed, these generators are subject to the following relations for all $\underline{i},\underline{j} \in I^\nu$ and all admissible r,s:

$$e(\underline{i})e(\underline{j}) = \delta_{\underline{i},\underline{j}}e(\underline{i}); \tag{4.2}$$

$$\sum_{\underline{i}\in I^\nu} e(\underline{i}) = 1_\nu; \tag{4.3}$$

$$y_r e(\underline{i}) = e(\underline{i})y_r; \tag{4.4}$$

$$\tau_r e(\underline{i}) = e(s_r\cdot\underline{i})\tau_r; \tag{4.5}$$

$$y_r y_s e(\underline{i}) = (-1)^{p(i_r)p(i_s)}y_s y_r e(\underline{i}); \tag{4.6}$$

$$\tau_r y_s e(\underline{i}) = (-1)^{p(i_r)p(i_{r+1})p(i_s)}y_s\tau_r e(\underline{i}) \qquad \text{if } s\neq r,r+1; \tag{4.7}$$

$$\tau_r\tau_s e(\underline{i}) = (-1)^{p(i_r)p(i_{r+1})p(i_s)p(i_{s+1})}\tau_s\tau_r e(\underline{i}) \qquad \text{if } |s-r|>1; \tag{4.8}$$

$$\tau_r y_{r+1}e(\underline{i}) = \begin{cases} ((-1)^{p(i_r)p(i_{r+1})}y_r\tau_r+1)e(\underline{i}) & i_r=i_{r+1},\\ (-1)^{p(i_r)p(i_{r+1})}y_r\tau_r e(\underline{i}) & i_r\neq i_{r+1};\end{cases} \tag{4.9}$$

$$y_{r+1}\tau_r e(\underline{i}) = \begin{cases} ((-1)^{p(i_r)p(i_{r+1})}\tau_r y_r+1)e(\underline{i}) & i_r=i_{r+1},\\ (-1)^{p(i_r)p(i_{r+1})}\tau_r y_r e(\underline{i}) & i_r\neq i_{r+1}.\end{cases} \tag{4.10}$$

Additionally, the intertwining elements satisfy the quadratic relations

$$\tau_r^2 e(\underline{i}) = \mathsf{Q}_{i_r,i_{r+1}}(y_r,y_{r+1})e(\underline{i}) \tag{4.11}$$

for all $1\leq r\leq n-1$, and the braid-like relations

$$(\tau_r\tau_{r+1}\tau_r - \tau_{r+1}\tau_r\tau_{r+1})e(\underline{i}) \tag{4.12}$$

$$= \begin{cases} \left(\frac{\mathsf{Q}_{i_r,i_{r+1}}(y_{r+2},y_{r+1})-\mathsf{Q}_{i_r,i_{r+1}}(y_r,y_{r+1})}{y_{r+2}-y_r}\right)e(\underline{i}) & \text{if } i_r=i_{r+2}\in I_{\bar 0},\\ (-1)^{p(i_{r+1})}(y_{r+2}-y_r) & \\ \quad\times\left(\frac{\mathsf{Q}_{i_r,i_{r+1}}(y_{r+2},y_{r+1})-\mathsf{Q}_{i_r,i_{r+1}}(y_r,y_{r+1})}{y_{r+2}^2-y_r^2}\right)e(\underline{i}) & \text{if } i_r=i_{r+2}\in I_{\bar 1},\\ 0 & \text{otherwise,}\end{cases}$$

for $1\leq r\leq n-2$. A subtle point, explained in [KKT], is that in the case $i_r = i_{r+2}\in I_{\bar 1}$ above, y_r^2 and y_{r+2}^2 are even and, consequently, commute with $y_1,\dots,y_n$. Therefore, there is no ambiguity in the corresponding formula.

Finally, this algebra is bigraded, with $\mathbb{Z}$-grading given by

$$\deg e(\underline{i})=0,\quad \deg y_r e(\underline{i}) = (\alpha_{i_r},\alpha_{i_r}),\quad\text{and}\quad \deg\tau_r e(\underline{i}) = -(\alpha_{i_r},\alpha_{i_{r+1}}), \tag{4.13}$$

and $\mathbb{Z}_2$-grading given by

$$p(e(\underline{i}))=0,\quad p(y_r e(\underline{i}))=p(i_r),\quad\text{and}\quad p(\tau_r e(\underline{i})) = p(i_r)p(i_{r+1}). \tag{4.14}$$

4.2. **An automorphism and antiautomorphism.** The following two propositions can be verified directly by definition.

Proposition 4.1. *There is a unique* $\mathbb{K}$*-linear automorphism* $\phi:\mathcal{H}^-(\nu)\to\mathcal{H}^-(\nu)$ *given by*

$$\phi(e(\underline{i})) = e(w_0\cdot\underline{i}),\quad \phi(y_r)=y_{n-r+1},$$
$$\phi(\tau_r e(\underline{i})) = (-1)^{1+p(i_r)r(i_{r+1})}\tau_{n-r}e(s_r w_0\cdot\underline{i}),$$

where $w_0\in S_n$ *is the longest element.*

Proposition 4.2. *There is a unique $\mathbb{K}$-linear anti-automorphism $\psi : \mathcal{H}^-(\nu) \to \mathcal{H}^-(\nu)$ defined by $\psi(e(\underline{i})) = e(\underline{i})$, $\psi(y_r) = y_r$, and $\psi(\tau_s) = \tau_s$ for all $\underline{i} \in I^\nu$ and admissible r, s.*

4.3. **The rank 1 case.** Note that the diagram of a single node $\bullet$ corresponds to Lie superalgebra $\mathfrak{osp}(1|2)$. In this section, we will also consider the diagram $\circ$ which corresponds to $\mathfrak{sl}(2)$. These are quivers with a compatible automorphism (the trivial one) and the corresponding GCM is $A = (2)$, and there is only one polynomial $\mathsf{Q}(u, v) \equiv 0$.

In the even case $\circ$, the quiver Hecke algebra is the nilHecke algebra $\mathcal{NH}_n$ generated by subalgebras $\mathbb{K}[Y]$ for $Y = \{y_1, \ldots, y_n\}$ and the nil-Coxeter algebra $\mathcal{NC}_n$. The nil-Coxeter algebra is generated by the divided difference operators ∂_r^+, $1 \le r < n$, which are subject to the relations

$$
\begin{aligned}
(\partial_r^+)^2 &= 0,\\
\partial_r^+\partial_s^+ &= \partial_s^+\partial_r^+, \quad |r - s| > 1,\\
\partial_r^+\partial_{r+1}^+\partial_r^+ &= \partial_{r+1}^+\partial_r^+\partial_{r+1}^+.
\end{aligned}
\tag{4.15}
$$

The nil-Coxeter algebra act faithfully on $\mathbb{K}[Y]$ via

$$
\partial_r^+ \mapsto \frac{1 - s_r}{y_{r+1} - y_r},
\tag{4.16}
$$

where the simple transposition acts on polynomials by permuting the variables as usual.

For the odd case, the corresponding spin (or odd) nilHecke algebra $\mathcal{NH}_n^-$ is a nil version of the spin Hecke algebra in [Wa, 3.3], and we follow the presentation in [EKL, 2.2] below. Let $\mathbb{K}[Y]^-$ be a skew-polynomial ring in n variables (that is, the quotient of the free algebra on $Y = \{y_1, \ldots, y_n\}$ by the relation $y_r y_s = -y_s y_r$ for $r \ne s$). Then, $\mathcal{NH}_n^-$ is generated by subalgebras $\mathbb{K}[Y]^-$ and the spin (or odd) nil-Coxeter algebra $\mathcal{NC}_n^-$. The spin nil-Coxeter algebra is generated by an odd analogue of divided difference operators ∂_r^-, $1 \le r < n$, which are subject to the relations

$$
\begin{aligned}
(\partial_r^-)^2 &= 0,\\
\partial_r^-\partial_s^- &= -\partial_s^-\partial_r^-, \quad |r - s| > 1,\\
\partial_r^-\partial_{r+1}^-\partial_r^- &= \partial_{r+1}^-\partial_r^-\partial_{r+1}^-.
\end{aligned}
\tag{4.17}
$$

The spin nil-Coxeter algebra acts faithfully on $\mathbb{K}[Y]^-$ as in the even case, where

$$
\partial_r^-(y_s) = \begin{cases} 1 & \text{if } s = r, r+1,\\ 0 & \text{otherwise,} \end{cases}
\tag{4.18}
$$

and the action of ∂_r extends to $\mathbb{K}[Y]^-$ using the *Leibnitz rule*: for $f, g \in \mathbb{K}[Y]^-$,

$$
\partial_r^-(fg) = \partial_r^-(f)g + s_r(f)\partial_r^-(g).
\tag{4.19}
$$

Here the action of S_n on $\mathbb{K}[Y]^-$ is given by $s_r(y_k) = -y_{s_r(k)}$.

4.4. **A basis theorem.** We now show that $\mathcal{H}^-(\nu)$ satisfies the PBW property. More precisely, we have the following.

Theorem 4.3. *For each $w \in S_n$, fix a reduced expression $w = s_{i_1} \cdots s_{i_r}$ for w, and let $\tau_w = \tau_{i_1} \cdots \tau_{i_r}$. Set*

$$\mathcal{B} = \{\tau_w y_1^{a_1} \cdots y_n^{a_n} e(\underline{i}) \,|\, \underline{i} \in I^\nu,\ (a_1, \dots, a_n) \in \mathbb{Z}_{\geq 0}^n,\ w \in S_n\}. \tag{4.20}$$

Then, $\mathcal{H}^-(\nu)$ is free over $\mathbb{K}$ with basis $\mathcal{B}$.

Note that by (4.12) this depends on the choice of reduced expression. However, we have the following proposition, the proof of which is almost identical to [BKW, Proposition 2.5].

Proposition 4.4. *Let $\underline{i} \in I^\nu$ and $w \in S_n$. Assume that $w = s_{k_1} \cdots s_{k_t}$ is a presentation of w as a product of simple transpositions.*

(a) *If the presentation $w = s_{k_1} \cdots s_{k_t}$ is reduced, and $w = s_{\ell_1} \cdots s_{\ell_t}$ is another reduced presentation of w, then*

$$\tau_{k_1} \cdots \tau_{k_t} e(\underline{i}) = \pm \tau_{\ell_1} \cdots \tau_{\ell_t} e(\underline{i}) + \sum_{u<w} \tau_u f_u(y) e(\underline{i}),$$

where the sum is over $u < w$ in the Bruhat order, $f_u(y)$ is a polynomial in $y_1, \dots, y_n$, and $\deg \tau_u f_u(y) e(\underline{i}) = \deg \tau_w e(\underline{i})$ for all u.

(b) *If the presentation $w = s_{k_1} \cdots s_{k_t}$ is not reduced, then $\tau_{k_1} \cdots \tau_{k_t} e(\underline{i})$ can be written as a linear combination of words of the form $\tau_{k_{r_1}} \cdots \tau_{k_{r_s}} f(y) e(\underline{i})$, such that $1 \leq r_1 < \cdots < r_s \leq t$, $s < t$, $w = s_{k_{r_1}} \cdots s_{k_{r_s}}$ is reduced, $f(y)$ is a polynomial in $y_1, \dots y_n$, and*

$$\deg \tau_{k_1} \cdots \tau_{k_t} e(\underline{i}) = \deg \tau_{k_{r_1}} \cdots \tau_{k_{r_s}} f(y) e(\underline{i}).$$

We now turn to the proof of Theorem 4.3. To this end, let $\mathcal{P}^-(\nu)$ be the subalgebra of $\mathcal{H}^-(\nu)$ generated by the elements $y_1, \dots, y_n$ and $e(\underline{i})$, $i \in I^\nu$. Fix a total ordering on I, and define a family of polynomials $\mathsf{P} = (\mathsf{P}_{ij}(u,v))_{i,j \in I}$ by the formula

$$\mathsf{P}_{ij}(u,v) = \begin{cases} 0 & \text{if } i = j, \\ \mathsf{Q}_{ij}(u,v) & \text{if } i < j, \\ 1 & \text{if } i > j. \end{cases} \tag{4.21}$$

Then, Theorem 4.3 follows immediately from the following proposition.

Proposition 4.5. *There is a faithful action of the algebra $\mathcal{H}^-(\nu)$ on $\mathcal{P}^-(\nu)$, where y_s and $e(\underline{i})$ act by left multiplication ($s = 1, \dots, n$, $\underline{i} \in I^\nu$), and*

$$\tau_r e(\underline{i}) \mapsto \begin{cases} \partial_r^+ e(\underline{i}) & \text{if } i_r = i_{r+1} \in I_{\bar{0}}, \\ \partial_r^- e(\underline{i}) & \text{if } i_r = i_{r+1} \in I_{\bar{1}}, \\ \mathsf{P}_{i_r, i_{r+1}}(y_{r+1}, y_r) e(s_r \cdot \underline{i}) s_r & \text{otherwise,} \end{cases}$$

where S_n acts on $\mathcal{P}^-(\nu)$ according to

$$s_k(y_r e(\underline{i})) = (-1)^{p(i_k)p(i_{k+1})p(i_r)} y_{s_k(r)} e(s_k \cdot \underline{i}). \tag{4.22}$$

Proof. The proof can be adapted from either [KL1, Proposition 2.3, Theorem 2.5], or [Ro1, Proposition 3.12]. □

4.5. **The center.** For $\nu=\sum_{i\in I} n_i\alpha_i$, let $n_{\bar 1}=\sum_{i\in I_{\bar 1}} n_i$. Let

$$Y_\nu=\{y_r^{1+p(i_r)}e(\underline{i})|r=1,\dots,n,\underline{i}\in I^\nu\}, \tag{4.23}$$

and let $\mathcal{S}^-(\nu)=\mathbb{K}[Y_\nu]^{S_n}\subset\mathcal{P}^-(\nu)$, where the symmetric group S_n acts on $\mathcal{P}^-(\nu)$ by (4.22). Then, we obtain the following by combining the proofs of [KL1, Proposition 2.7] and [EKL, Proposition 2.15].

Proposition 4.6. *The center of $\mathcal{H}^-(\nu)$ is $\mathcal{S}^-(\nu)$. Moreover, $\mathcal{H}^-(\nu)$ is free of rank $2^{n_{\bar 1}}(n!)^2$ over $\mathcal{S}^-(\nu)$.*

5. Categorification of quantum Serre relations

5.1. **Module categories.** Throughout the rest of the paper, we will primarily work inside the abelian category $\mathrm{Mod}^-(\nu):=\mathrm{Mod}\mathcal{H}^-(\nu)$ of finitely generated $(\mathbb{Z}\times\mathbb{Z}_2)$-*graded* left $\mathcal{H}^-(\nu)$-modules, with morphisms being $\mathcal{H}^-(\nu)$-homomorphisms that preserve *both* the $\mathbb{Z}$-grading and $\mathbb{Z}_2$-grading. Let $\mathrm{Hom}_\nu(M,N)$ denote the $\mathbb{K}$-vector space of morphisms between M and N.

Note that $\mathcal{H}^-(\nu)$ is *almost* positively $\mathbb{Z}$-graded; cf. [KL1, p. 24]. Indeed, it is nontrivial only in degrees greater than, or equal to, $-\sum_i n_i(n_i-1)$, for $\nu=\sum_{i\in I}n_i\alpha_i$. In particular, for any $M\in\mathrm{Mod}^-(\nu)$, we may define its (q,π)-*dimension*

$$\dim_q^\pi M=\sum_{a\in\mathbb{Z}}(\dim M_{\bar 0}[a]+\pi\dim M_{\bar 1}[a])q^a\in\mathbb{Z}((q))^\pi. \tag{5.1}$$

The specialization $\pi\mapsto 1$ recovers the usual graded dimension of the module M, while $\pi\mapsto -1$ produces the graded *super*dimension. Additionally, we define the *graded character* of $M\in\mathrm{Mod}^-(\nu)$ by

$$\mathrm{ch}_q^\pi M=\sum_{\underline{i}\in I^\nu}(\dim_q^\pi e(\underline{i})M)\cdot\underline{i}. \tag{5.2}$$

This is an element of the free $\mathbb{Z}((q))^\pi$-module with basis labelled by I^ν. Define

$$\mathrm{ch}_q^+M=\mathrm{ch}_q^\pi M|_{\pi=1}\quad\text{and}\quad\mathrm{ch}_q^-M=\mathrm{ch}_q^\pi M|_{\pi=-1}. \tag{5.3}$$

Note that we may have $\mathrm{ch}_q^-M=0$.

There exists a parity shift functor

$$\Pi:\mathrm{Mod}^-(\nu)\longrightarrow\mathrm{Mod}^-(\nu)$$

which is defined on an object M by letting $\Pi M=M$ as $\mathcal{H}^-(\nu)$-modules, but with $(\Pi M)_{\bar 0}=M_{\bar 1}$ and $(\Pi M)_{\bar 1}=M_{\bar 0}$. We set

$$\mathrm{Hom}_\nu^-(M,N)=\mathrm{Hom}_\nu(M,N)\oplus\mathrm{Hom}_\nu(M,\Pi N). \tag{5.4}$$

Then, $\mathrm{Hom}_\nu^-(M,N)$ is $\mathbb{Z}_2$-graded, with

$$\mathrm{Hom}_\nu^-(M,N)_{\bar 0}=\mathrm{Hom}_\nu(M,N),\quad\mathrm{Hom}_\nu^-(M,N)_{\bar 1}=\mathrm{Hom}_\nu(M,\Pi N).$$

For each $a\in\mathbb{Z}$, there is a functor $\mathsf{q}:\mathrm{Mod}^-(\nu)\to\mathrm{Mod}^-(\nu)$ which shifts the $\mathbb{Z}$-grading: $\mathsf{q}M=M\{1\}$ where, for $a\in\mathbb{Z}$, the module $M\{a\}$ equals M as an $\mathcal{H}^-(\nu)$-module, but has a kth graded component $(M\{a\})[k]=M[k-a]$ (the $(k-a)$th graded component of M). Define the space of Π-twisted *enhanced* homomorphisms

$$\mathrm{HOM}_\nu^-(M,N)=\bigoplus_{a\in\mathbb{Z}}\mathrm{Hom}_\nu^-(M,\Pi^aN\{a\}). \tag{5.5}$$

The category $\mathrm{Mod}^-(\nu)$ contains the full subcategories $\mathrm{Rep}^-(\nu) := \mathrm{Rep}\mathcal{H}^-(\nu)$ and $\mathrm{Proj}^-(\nu) := \mathrm{Proj}\,\mathcal{H}^-(\nu)$ of finite-dimensional and finitely generated projective modules, respectively. The category $\mathrm{Rep}^-(\nu)$ is abelian, while $\mathrm{Proj}^-(\nu)$ is additive. The corresponding Grothendieck groups $[\mathrm{Rep}^-(\nu)]$ and $[\mathrm{Proj}^-(\nu)]$ are naturally $\mathcal{A}^\pi$-modules. The functors Π and q induce an action of $\mathcal{A}^\pi$ on both $[\mathrm{Proj}^-(\nu)]$ and $[\mathrm{Rep}^-(\nu)]$ via $q[M] = [\mathsf{q}M] = [M\{1\}]$ and $\pi[M] = [\Pi M]$, where M is an object in the relevant category (recall that $\mathrm{Hom}_\nu(M, N)$ is the space of morphisms, and NOT $\mathrm{Hom}_\nu^-(M, N)$).

Recall the anti-automorphism ψ from Proposition 4.2. For $P \in \mathrm{Proj}^-(\nu)$, define

$$P^\# = \mathrm{HOM}_\nu^-(P, \mathcal{H}^-(\nu))^\psi,$$

where for a finitely generated graded *right* (resp. *left*) $\mathcal{H}^-(\nu)$-module M, M^ψ is the *left* (resp. *right*) module obtained by twisting the action on M by the anti-automorphism ψ: $y.m = m.\psi(y)$ (resp. $m.y = \psi(y).m$) for $m \in M$ and $y \in \mathcal{H}^-(\nu)$. Let $\mathbf{1}$ denote the unique simple $\mathcal{H}^-(0)$ module. It follows by (5.5) that

$$(\mathsf{q}\mathbf{1})^\# \cong \Pi\mathsf{q}^{-1}\mathbf{1}. \tag{5.6}$$

For each $\underline{i} \in I^\nu$, define the projective module

$$P_{\underline{i}} = \mathcal{H}^-(\nu)e(\underline{i}). \tag{5.7}$$

Then, the mapping which sends $e(\underline{i})$ to the natural embedding $\{P_{\underline{i}} \hookrightarrow \mathcal{H}^-(\nu)\}$ defines an isomorphism

$$P_{\underline{i}} \cong P_{\underline{i}}^\#. \tag{5.8}$$

Moreover, $(P_{\underline{i}}\{a\})^\# \cong \Pi^a P_{\underline{i}}\{-a\}$. Let

$$^- : [\mathrm{Proj}^-(\nu)] \longrightarrow [\mathrm{Proj}^-(\nu)] \tag{5.9}$$

be the $\mathbb{Z}$-linear involution: $\overline{\pi} = \pi$, $\overline{q} = \pi q^{-1}$, and $\overline{[P]} = [P^\#]$.

There is a bilinear form

$$(\cdot, \cdot) : [\mathrm{Proj}^-(\nu)] \times [\mathrm{Proj}^-(\nu)] \longrightarrow \mathbb{Z}((q))^\pi \tag{5.10}$$

given by

$$([P], [Q]) = \dim_q^\pi (P^\psi \otimes_{\mathcal{H}^-(\nu)} Q). \tag{5.11}$$

Note that $(P_{\underline{i}}, P_{\underline{j}}) = \dim_q^\pi e(\underline{j})\mathcal{H}^-(\nu)e(\underline{i})$. For future reference, we also note that the natural form on $[\mathrm{Proj}\ \mathcal{H}^-(\mu) \otimes \mathcal{H}^-(\nu)]$ is given by

$$([P \otimes Q], [P' \otimes Q']) = \dim_q^\pi ((P \otimes Q)^\psi \otimes_{\mathcal{H}^-(\mu)\otimes\mathcal{H}^-(\nu)} (P' \otimes Q')). \tag{5.12}$$

Lemma 5.1. *For $P, Q \in \mathrm{Proj}^-(\nu)$, we have $([P], [Q]) = \dim_q^\pi \mathrm{HOM}_\nu^-(P^\#, Q)$.*

Proof. For $P, Q \in \mathrm{Proj}^-(\nu)$, we compute

$$\begin{aligned} ([P^\#], [Q]) &= \dim_q^\pi (P^\#)^\psi \otimes_{\mathcal{H}^-(\nu)} Q \\ &= \dim_q^\pi \mathrm{HOM}_\nu^-(P, \mathcal{H}^-(\nu)) \otimes_{\mathcal{H}^-(\nu)} Q \\ &= \dim_q^\pi \mathrm{HOM}_\nu^-(P, Q). \end{aligned}$$

The lemma is proved. □

The simple objects in $\mathrm{Mod}^-(\nu)$ belong to the category $\mathrm{Rep}^-(\nu)$. Let $\mathcal{S}_0^-(\nu)$ be the unique maximal graded ideal in $\mathcal{S}^-(\nu)$, spanned by S_n-invariant polynomials without constant term. We have the following.

Proposition 5.2. *$\mathcal{S}_0^-(\nu)$ acts by 0 on any simple module. Hence, there are only finitely many simple $\mathcal{H}^-(\nu)$-modules up to isomorphism and grading/parity shift.*

Proof. Recall the commuting family of elements Y_ν from (4.23). These elements are positively graded, and therefore act nilpotently on any finite-dimensional module. Given a simple module, we can find a simultaneous eigenvector for the action of Y_ν (necessarily with eigenvalue 0). By definition, $\mathcal{S}_0^-(\nu)$ acts as 0 on this vector. But, by Proposition 4.6, $\mathcal{S}_0^-(\nu)$ is contained in the center of $\mathcal{H}^-(\nu)$, and so this determines its action on the entire simple module. It now follows that every simple module factors through the $2^{n_{\bar{1}}}(n!)^2$-dimensional algebra $\mathcal{H}^-(\nu)/\mathcal{S}_0^-(\nu)\mathcal{H}^-(\nu)$, so there can only be finitely many. □

For each $M \in \mathrm{Rep}^-(\nu)$, we associate $M^{\circledast} = \mathrm{HOM}_{\mathbb{K}}^-(M, \mathbb{K})$ (Π-twisted linear maps), where $\mathbb{K} = \mathbb{K}_{\bar{0}}$. There is an $\mathcal{H}^-(\nu)$-action on $M^{\circledast}$ given by $(xf)(m) = f(\psi(x)m)$, for $x \in \mathcal{H}^-(\nu)$, $f \in M^{\circledast}$, and $m \in M$, making $M^{\circledast}$ an object in $\mathrm{Rep}^-(\nu)$. Using the definitions, we deduce that for any simple module L, $L^{\circledast} \cong \Pi^a L\{a\}$ for some $a \in \mathbb{Z}$. Replacing L with $\Pi^a L\{a\}$, we obtain $L^{\circledast} \cong L$.

This duality defines an involution

$$\bar{\ } : [\mathrm{Rep}^-(\nu)] \longrightarrow [\mathrm{Rep}^-(\nu)] \tag{5.13}$$

given by $\overline{\pi} = \pi$, $\overline{q} = \pi q^{-1}$, and $\overline{[M]} = [M^{\circledast}]$.

Define the $\mathcal{A}^\pi$-sesquilinear (i.e., antilinear in the first variable) Cartan pairing

$$\langle \cdot, \cdot \rangle : [\mathrm{Proj}^-(\nu)] \times [\mathrm{Rep}^-(\nu)] \longrightarrow \mathcal{A}^\pi \tag{5.14}$$

given by

$$\langle [P], [M] \rangle = \dim_q^\pi \mathrm{HOM}_\nu^-(P^\#, M). \tag{5.15}$$

For each simple module $L \in \mathrm{Rep}^-(\nu)$, there exists a (unique up to $(\mathbb{Z} \times \mathbb{Z}_2)$-homogeneous isomorphism) projective indecomposable cover $P_L \in \mathrm{Proj}^-(\nu)$. The modules L and P_L are dual with respect to (5.14). Moreover, we have

$$P_L^\# \cong P_{L^{\circledast}}.$$

5.2. **More rank 1 calculations.** In this subsection, we will consider the algebra $\mathcal{H}^-(n\alpha_i)$, which is isomorphic to either the nilHecke algebra or spin nilHecke algebra, depending on whether i is even or odd, respectively. Accordingly, we will revert to the notation of §4.3 to better facilitate comparison with [La, Section 3] and [EKL, Section 2]. Indeed, this section is essentially a review of results from [La, EKL] which we need, adapted for computations with *left* modules (cf. (5.28) below) and enhanced by the insertion of π to keep track of the parity. This amounts to reading their diagrams from top to bottom, as opposed to bottom to top (alternatively, applying the antiautomorphism ψ). Additional signs also appear, caused by (4.9) which differs from the corresponding formula in loc. cit., but is commonly used in the literature; cf. [BK2].

The even and odd cases will be treated simultaneously by writing $\partial_r^\pm$ as necessary for the relevant divided difference operator. Of course, both $\mathcal{NH}_n (= \mathcal{NH}_n^+)$ and $\mathcal{NH}_n^-$ can be viewed as $(\mathbb{Z}, \mathbb{Z}_2)$-graded algebras as in §4.1. Our first observation is that (q, π)-dimension of these algebras is

$$\dim_q^\pi \mathcal{NH}_n^\pm = \frac{(\pi_i q_i)^{-\binom{n}{2}} [n]_i!}{(1 - \pi_i q_i^2)^n}. \tag{5.16}$$

This can be established exactly as in [La, §3.1], keeping track of the parity when the superscript is $-$ (see also [EKL, Proposition 2.11]).

Following [EKL, (2.12)], define the algebra of (spin) symmetric polynomials

$$\Lambda_n^{\pm} = \bigcap_{r=1}^{n-1} \ker(\partial_r^{\pm}) \subset \mathbb{K}[Y]^{\pm}. \tag{5.17}$$

This algebra has a $(\mathbb{Z} \times \mathbb{Z}_2)$-homogeneous basis given by the elementary (odd) symmetric functions $\epsilon_\lambda^{\pm} = \epsilon_{\lambda_1}^{\pm} \cdots \epsilon_{\lambda_h}^{\pm}$, where $\lambda = (\lambda_1 \geq \cdots \geq \lambda_h)$ is a partition of n and

$$\epsilon_k^{\pm}(y_1, \ldots, y_n) = \sum_{1 \leq r_1 < \ldots < r_k \leq n} (\pm 1)^{r_1 + \cdots + r_k - k} y_{r_1} \cdots y_{r_k}; \tag{5.18}$$

see [EKL, (2.21), Lemma 2.3, Remark 2.4]. Then, $\epsilon_k^{\pm}$ has bidegree $(2k, p(i)k) \in \mathbb{Z} \times \mathbb{Z}_2$, and a straightforward computation analogous to [EKL, (2.18)] proves that

$$\dim_q^{\pi} \Lambda_n^{\pm} = \frac{(\pi_i q_i)^{-\binom{n}{2}}}{[n]_i!(1 - \pi_i q_i^2)^n}. \tag{5.19}$$

Below is a π-enhanced version of [La, Proposition 3.5] and [EKL, Corollary 2.14].

Proposition 5.3. *The natural action of $\mathcal{NH}_n^{\pm}$ on $\mathbb{K}[Y]^{\pm}$ defines an isomorphism*

$$\mathcal{NH}_n^{\pm} \cong \mathrm{End}_{\Lambda_n^{\pm}}(\mathbb{K}[Y]^{\pm}) \cong \mathrm{Mat}\left((\pi_i q_i)^{\binom{n}{2}}[n]_i!,\, \Lambda_n^{\pm}\right).$$

Define the Demazure operator

$$\overline{\partial}_r^{\pm} = -\partial_r^{\pm} y_r. \tag{5.20}$$

Then, using (4.9),

$$(\overline{\partial}_r^{\pm})^2 = (-\partial_r^{\pm} y_r)(-\partial_r^{\pm} y_r) = \partial_r^{\pm}(\pm \partial_r^{\pm} y_{r+1} - 1) y_r = -\partial_r^{\pm} y_r = \overline{\partial}_r^{\pm}.$$

Moreover, it is straightforward to verify that these elements satisfy the type A braid relations:

$$\overline{\partial}_r^{\pm} \overline{\partial}_s^{\pm} = \overline{\partial}_s^{\pm} \overline{\partial}_r^{\pm}, \qquad |r - s| > 1,$$
$$\overline{\partial}_r^{\pm} \overline{\partial}_{r+1}^{\pm} \overline{\partial}_r^{\pm} = \overline{\partial}_{r+1}^{\pm} \overline{\partial}_r^{\pm} \overline{\partial}_{r+1}^{\pm}.$$

In particular, for each $w \in S_n$, the element $\overline{\partial}_w^{\pm}$ is well defined in terms of any reduced expression of w. Set

$$\mathbf{e}_n^{\pm} = \overline{\partial}_{w_0}^{\pm}. \tag{5.21}$$

Since the spin nil-Coxeter algebra satisfies the braid relations for the spin symmetric group as opposed to the standard braid relations, the elements ∂_w^-, $w \in S_n$, are only well defined up to sign, and, therefore, we will fix the reduced expressions $w = s_{r_1} \cdots s_{r_k}$ for each element $w \in S_n$ and define $\partial_w^- = \partial_{r_1}^- \cdots \partial_{r_k}^-$ to remove this ambiguity. For the moment, let $w_0\langle n \rangle$ denote the longest word in S_n. We define the fixed reduced expression for $w_0\langle n \rangle$ inductively by $w_0\langle 1 \rangle = 1$, and $w_0\langle n \rangle = s_1 s_2 \cdots s_{n-1} w_0\langle n-1 \rangle$, for $n > 1$. Then, we have a well-defined element

$$\partial_{w_0}^{\pm} = \partial_{w_0\langle n \rangle}^{\pm} = \partial_1^{\pm} \partial_2^{\pm} \cdots \partial_{n-1}^{\pm} \partial_{w_0\langle n-1 \rangle}^{\pm} = \cdots. \tag{5.22}$$

For $k < n$, let ${}^{\dagger}S_k \leq S_n$ be the subgroup of permutations of $\{n-k+1, \ldots, n\}$ and, for $w \in S_k$, let ${}^{\dagger}w \in {}^{\dagger}S_k$ denote the corresponding element. The following useful fact is proved in [EKL].

Lemma 5.4 ([EKL, Lemma 3.2])**.** *The following holds in* $\mathcal{NH}_n^-$*:*

$$\partial_{w_0}^- = \partial_1^- \partial_2^- \cdots \partial_{n-1}^- \partial_{{}^\dagger w_0 \langle n-1 \rangle}^- .$$

Set

$$y^{\delta_n} = (-1)^{\binom{n-1}{2}} y_1^{n-1} y_{n+2}^{n-2} \cdots y_{n-1}. \tag{5.23}$$

Lemma 5.5 ([EKL, Proposition 3.5], [KL2, p. 2688])**.** *We have*

$$\mathbf{e}_n^\pm = (\pm 1)^{\binom{n}{3}} \partial_{w_0}^\pm y^{\delta_n} \in \mathcal{NH}_n^\pm. \tag{5.24}$$

Note that we have adapted this idempotent for *left* modules.

Lemma 5.6 ([KL2, Lemma 5], [EKL, Proposition 3.6])**.** *The following identity holds in* $\mathcal{NH}_n^\pm$*:* $\mathbf{e}_n^\pm \partial_{w_0}^\pm = \partial_{w_0}^\pm$.

Identify $\mathbf{e}_{n-1}^\pm$ with the image of $\mathbf{e}_{n-1}^\pm \otimes 1$ under the natural inclusion $\mathcal{NH}_{n-1}^\pm \otimes \mathcal{NH}_1^\pm \hookrightarrow \mathcal{NH}_n^\pm$ and let ${}^\dagger\mathbf{e}_{n-1}^\pm$ denote the image of $1 \otimes \mathbf{e}_{n-1}^\pm$ under the inclusion $\mathcal{NH}_1^\pm \otimes \mathcal{NH}_{n-1}^\pm \hookrightarrow \mathcal{NH}_n^\pm$. The following identities are [KL2, (10)-(11)].

Lemma 5.7. *The following hold in* $\mathcal{NH}_n^\pm$*:*

$$\mathbf{e}_{n-1}^\pm \mathbf{e}_n^\pm = \mathbf{e}_n^\pm, \quad {}^\dagger\mathbf{e}_{n-1}^\pm \mathbf{e}_n^\pm = \mathbf{e}_n^\pm, \tag{5.25}$$

$$\mathbf{e}_n^\pm \partial_1^\pm \cdots \partial_{n-1}^\pm \mathbf{e}_{n-1}^\pm = \partial_1^\pm \cdots \partial_{n-1}^\pm \mathbf{e}_{n-1}^\pm, \tag{5.26}$$

$$\mathbf{e}_n^\pm \partial_1^\pm \cdots \partial_{n-1}^\pm {}^\dagger\mathbf{e}_{n-1}^\pm = \partial_1^\pm \cdots \partial_{n-1}^\pm {}^\dagger\mathbf{e}_{n-1}^\pm. \tag{5.27}$$

Proof. Using the definition (5.20), formulae (5.25) reduce to standard properties of the Demazure operators. Formula (5.26) is immediate from Lemmas 5.5 and 5.6 since

$$\begin{aligned}\mathbf{e}_n^\pm \partial_1^\pm \cdots \partial_{n-1}^\pm \mathbf{e}_{n-1}^\pm &= \mathbf{e}_n^\pm \partial_1^\pm \cdots \partial_{n-1}^\pm \partial_{w_0\langle n-1\rangle}^\pm y^{\delta_{n-1}} \\ &= \mathbf{e}_n^\pm \partial_{w_0\langle n\rangle}^\pm y^{\delta_{n-1}} = \partial_{w_0\langle n\rangle}^\pm y^{\delta_{n-1}} = \partial_1^\pm \cdots \partial_{n-1}^\pm \mathbf{e}_{n-1}^\pm.\end{aligned}$$

The proof of (5.27) is the same, except inserting Lemma 5.4 for $\partial_{w_0}^\pm$. ☐

Lemma 5.8 ([KL2, (12)-(13)])**.** *The following hold in* $\mathcal{NH}_n^\pm$*:*

$$\partial_{n-1}^\pm \cdots \partial_2^\pm \partial_1^\pm y_1^a \mathbf{e}_n^\pm = \begin{cases} (-1)^{n-1}\mathbf{e}_n^\pm & \text{if } a = n-1, \\ 0 & \text{if } a < n-1, \end{cases}$$

and

$$\partial_1^\pm \partial_2^\pm \cdots \partial_{n-1}^\pm y_n^a \mathbf{e}_n^\pm = \begin{cases} \mathbf{e}_n^\pm & \text{if } a = n-1, \\ 0 & \text{if } a < n-1. \end{cases}$$

Proof. The proof follows by induction using (4.9) and the fact that $\partial_r^\pm \mathbf{e}_n^\pm = 0$. ☐

5.3. **Categorical Serre relations.** From now on, we write $\mathbf{e}_{i,n} = \mathbf{e}_n^\pm$ since the $\pm$ notation can be recovered from the parity of i, and translate $\mathcal{NH}_n^\pm$ to the notation of $\mathcal{H}^-(n\alpha_i)$. This defines the unique projective indecomposable $\mathcal{H}^-(n\alpha_i)$-module

$$P_{i^{(n)}} := \Pi^{p(i)\binom{n}{2}} \mathcal{H}^-(n\alpha_i)\mathbf{e}_{i,n}\left\{-\binom{n}{2}\right\}, \tag{5.28}$$

and

$$\mathcal{H}^-(n\alpha_i) \cong \bigoplus_{[n]_i!} P_{i^{(n)}}.$$

Here for $f(q,\pi)=\sum_{k\in\mathbb{Z}}(a_{k,\bar 0}+\pi a_{k,\bar 1})q^k\in\mathbb{Z}_{\geq 0}[q,q^{-1}]^\pi$ and for a $(\mathbb{Z}\times\mathbb{Z}_2)$-graded vector space M, we have set

$$\bigoplus_{f(q,\pi)} M = M^{\oplus f} = \bigoplus_{k\in\mathbb{Z}}\left(M^{\oplus a_{k,\bar 0}}\oplus(\Pi M)^{\oplus a_{k,\bar 1}}\right)\{k\}.$$

For example, if $i\in I_{\bar 1}$, $H(2\alpha_i)\cong P_{i^{(2)}}^{\oplus[2]_i!}=(\Pi P_{i^{(2)}})\{1\}\oplus P_{i^{(2)}}\{-1\}$.

Now, let $\nu\in Q^+$ be arbitrary. For $\underline{i}\in I^\nu$, we may consider any grouping of the form

$$\underline{i}=(\underbrace{i_1,\ldots,i_1}_{k_1},\ldots,\underbrace{i_t,\ldots,i_t}_{k_t}). \tag{5.29}$$

Identify $\mathcal{H}^-(k_1\alpha_{i_1})\otimes\cdots\otimes\mathcal{H}^-(k_t\alpha_{i_t})$ with its image in $\mathcal{H}^-(\nu)$ under the natural embedding, and define

$$\begin{aligned}\mathbf{e}_{\underline{i},\underline{k}} &:= \mathbf{e}_{i_1,k_1}\otimes\cdots\otimes\mathbf{e}_{i_t,k_t},\\ p(\underline{i},\underline{k}) &:= p(i_1)\binom{k_1}{2}+\ldots+p(i_t)\binom{k_t}{2},\\ \binom{\underline{k}}{2} &:= \sum_{a=1}^{t}\binom{k_a}{2}.\end{aligned}$$

We further define the projective module

$$P_{\underline{i}^{(\underline{k})}} := P_{i_1^{(k_1)}\cdots i_t^{(k_t)}} = \Pi^{p(\underline{i},\underline{k})}\mathcal{H}^-(\nu)\mathbf{e}_{\underline{i},\underline{k}}\left\{-\binom{\underline{k}}{2}\right\}. \tag{5.30}$$

Recall $p(k;i,j)$ from (3.3).

Theorem 5.9. *For $i,j\in I$, $i\neq j$, let $N=1-a_{ij}$. There exists a split exact sequence of $\mathcal{H}^-(N\alpha_i+\alpha_j)$-modules:*

$$0\longrightarrow \Pi^{p(0;i,j)}P_{i^{(N)}j}\longrightarrow \Pi^{p(1;i,j)}P_{i^{(N-1)}ji}\longrightarrow\cdots\longrightarrow \Pi^{p(N;i,j)}P_{ij^{(N)}}\longrightarrow 0\,.$$

In particular, there is an isomorphism

$$\bigoplus_{k=0}^{\lfloor\frac{N+1}{2}\rfloor}\Pi^{p(2k;i,j)}P_{i^{(N-2k)}ji^{(2k)}}\cong\bigoplus_{k=0}^{\lfloor\frac{N}{2}\rfloor}\Pi^{p(2k+1;i,j)}P_{i^{(N-2k-1)}ji^{(2k+1)}}.$$

Proof. Assume i is even, so $p(k;i,j)\equiv 0$. Then, this result is proved in [KL2] (all the relevant maps being even). The main technical tools needed are Lemma 5.6 and Lemma 5.8. From now on, we assume i is odd. The necessary technical facts carry over and, once we keep careful track of the parity, essentially the same proof as in [KL2] works.

In the special case $a_{ij}=0$, the theorem states that there is an isomorphism $P_{ji}\cong\Pi^{p(j)}P_{ij}$. The relevant map is given by right multiplication by $\tau_1 e(ij)\in\mathcal{H}^-(\alpha_i+\alpha_j)$. In order to be a morphism in $\mathrm{Proj}^-(\alpha_i+\alpha_j)$, this map must preserve the $\mathbb{Z}_2$-grading. To see this, note that $\tau_1 e(ij)$ is odd when j is odd, and even when j is even. In particular, the corresponding morphism is a parity preserving isomorphism $P_{ji}\to\Pi^{p(j)}P_{ij}$, as required.

Now consider the case $a_{ij}\neq 0$. Let $n=N+1$, and write $P_k=P_{i^{(k)}ji^{(n-k-1)}}$, and $\mathbf{e}(k)=\mathbf{e}_{i^{(k)}ji^{(n-k-1)}}$ $(0\leq k<n)$. Let

$$\alpha_{k,k+1}=\tau_{n-1}\cdots\tau_{k+2}\tau_{k+1}\mathbf{e}(k+1),$$

and

$$\alpha_{k+1,k} = \tau_1\tau_2\cdots\tau_{k+1}\mathbf{e}(k),$$

for $0 \le k < n$. Right multiplication by $\alpha_{k,k+1}$ and $\alpha_{k+1,k}$ define elements of

$$\mathrm{Hom}^-_{N\alpha_i+\alpha_j}(\Pi^{p(N-k;i,j)}P_k, \Pi^{p(N-k-1;i,j)}P_{k+1}),$$

and

$$\mathrm{Hom}^-_{N\alpha_i+\alpha_j}(\Pi^{p(N-k-1;i,j)}P_{k+1}, \Pi^{p(N-k;i,j)}P_k),$$

respectively.

Assume $0 \le k < n-1$. Via the graphical calculus developed in [KL2] one readily shows by Lemma 5.6 that

$$\begin{aligned}\alpha_{k,k+1}\alpha_{k+1,k} &= \tau_{n-1}\cdots\tau_{k+1}\mathbf{e}(k+1)\tau_1\cdots\tau_{k+1}\mathbf{e}(k)\\ &= \tau_{n-1}\cdots\tau_{k+1}\tau_1\cdots\tau_{k+1}\mathbf{e}(k)\\ &= (-1)^{k-1}(\tau_{n-1}\cdots\tau_{k+2})(\tau_1\cdots\tau_{k-1})(\tau_{k+1}\tau_k\tau_{k+1})\mathbf{e}(k)\\ &= (-1)^{(k-1)(n-k-1)}(\tau_1\cdots\tau_{k-1})(\tau_{n-1}\cdots\tau_{k+2})(\tau_{k+1}\tau_k\tau_{k+1})\mathbf{e}(k),\end{aligned}$$

where the minus signs are due to (4.8). Similarly, for $0 < k \le n-1$,

$$\begin{aligned}\alpha_{k,k-1}\alpha_{k-1,k} &= \tau_1\cdots\tau_k\tau_{n-1}\cdots\tau_k\mathbf{e}(k)\\ &= (-1)^{(n-1)-(k+2)+1}(\tau_1\cdots\tau_{k-1})(\tau_{n-1}\cdots\tau_{k+2})(\tau_k\tau_{k+1}\tau_k)\mathbf{e}(k);\end{aligned}$$

cf. [KL2, p. 2693]. Using (C4), $n-1 = N = 1-a_{ij}$ is odd, so $(n-1)-(k+2)+1 \equiv k$ and $(k-1)(n-k-1) \equiv (k-1) \pmod 2$. Now, we have

(5.31)

$$\begin{aligned}&(-1)^k(\alpha_{k,k-1}\alpha_{k-1,k} + \alpha_{k,k+1}\alpha_{k+1,k})\\ &\quad=(-1)^k\alpha_{k,k-1}\alpha_{k-1,k} - (-1)^{k-1}\alpha_{k,k+1}\alpha_{k+1,k}\\ &\quad=\tau_1\cdots\tau_{k-1}\tau_{n-1}\cdots\tau_{k+2}(\tau_k\tau_{k+1}\tau_k - \tau_{k+1}\tau_k\tau_{k+1})\mathbf{e}(k)\\ &\quad=\tau_1\cdots\tau_{k-1}\tau_{n-1}\cdots\tau_{k+2}\\ &\qquad\times\left((-1)^{p(j)}(y_{k+2}-y_k)\frac{\mathsf{Q}_{ij}(y_{k+2},y_{k+1}) - \mathsf{Q}_{ij}(y_k,y_{k+1})}{y_{k+2}^2 - y_k^2}\right)\mathbf{e}(k).\end{aligned}$$

Set $\xi = (-1)^{1+p(j)}$. Then, it follows from the proof of Lemma 2.1 that

$$\mathsf{Q}_{ij}(u,v) = (-1)^d(u^{2a} + \xi v^{2b})^c = (-1)^d\sum_{m=0}^{c}\xi^{c-m}\binom{c}{m}u^{2am}v^{2b(c-m)}$$

for some $a,b,c,d \in \mathbb{Z}_{\ge 0}$, where $d = d_{ij}$, $2ac = -a_{ij}$ and $2bc = -a_{ji}$. Now, a calculation gives

(5.32)

$$\begin{aligned}&(-1)^{p(j)+d}(y_{k+2}-y_k)\frac{\mathsf{Q}_{ij}(y_{k+2},y_{k+1}) - \mathsf{Q}_{ij}(y_k,y_{k+1})}{y_{k+2}^2 - y_k^2}\\ &\quad= (-1)^{d+1}\sum_{m=0}^{c}\sum_{l=1}^{am}\xi^{c-m+1}\binom{c}{m}(y_{k+2}^{2(am-l)+1}y_k^{2l-2} - y_{k+2}^{2(am-l)}y_k^{2l-1})y_{k+1}^{2b(c-m)}.\end{aligned}$$

Since $\mathbf{e}(k) = \mathbf{e}_{i,k} \otimes 1_{\alpha_j} \otimes \mathbf{e}_{i,n-k-1}$, $y_{k+1}\mathbf{e}(k) = \mathbf{e}(k)y_{k+1}\mathbf{e}(k)$, and we see from Lemma 5.8 that the only terms in (5.32) which are non-zero in (5.31) involve the monomial $y_{k+2}^{n-k-2}y_k^{k-1}$ (see [KL2, p. 2694] for a graphical interpretation). By degree considerations, the terms in (5.32) with $m < c$ do not contribute to (5.31), and so

$$\begin{aligned}(-1)^k&(\alpha_{k,k-1}\alpha_{k-1,k} + \alpha_{k,k+1}\alpha_{k+1,k})\\ &=(-1)^{d+1}\xi\tau_1\cdots\tau_{k-1}\tau_{n-1}\cdots\tau_{k+2}\sum_{l=1}^{-a_{ij}/2}(y_{k+2}^{-a_{ij}-2l+1}y_k^{2l-2} - y_{k+2}^{-a_{ij}-2l}y_k^{2l-1})\mathbf{e}(k).\end{aligned}$$

The monomial $y_{k+2}^{n-k-2}y_k^{k-1}$ occurs when $l = \frac{k+1}{2}$ or $l = \frac{k}{2}$, depending on whether k is even or odd. In either case, we arrive at

(5.33)
$$(-1)^k(\alpha_{k,k-1}\alpha_{k-1,k} + \alpha_{k,k+1}\alpha_{k+1,k}) = (-1)^{d+1}\xi(-1)^{n-1}\mathbf{e}(k) = (-1)^d\xi\mathbf{e}(k),$$

where we have used the fact that $n-1 = N = 1 - a_{ij}$ is odd by (C4).

For $k = n-1$, we have

$$\begin{aligned}\alpha_{n-1,n-2}\alpha_{n-2,n-1} &= \tau_1\cdots\tau_{n-2}\tau_{n-1}^2\mathbf{e}(n-1)\\ &= \tau_1\cdots\tau_{n-2}\mathsf{Q}_{ij}(y_{n-1}, y_n)\mathbf{e}(n-1)\\ &= (-1)^d\sum_{m=0}^{c}\xi^{c-m}\binom{c}{m}y_n^{2b(m-c)}(\tau_1\cdots\tau_{n-2})y_{n-1}^{2am}\mathbf{e}(n-1).\end{aligned}$$

Using Lemma 5.8 applied to $\mathcal{H}^-((n-1)\alpha_i) \subset \mathcal{H}^-(n\alpha_i)$, the only non-zero term in the sum above corresponds to $m = c$ (so the exponent of y_n is $2ac = -a_{ij} = (n-1)-1$), and gives

$$\alpha_{n-1,n-2}\alpha_{n-2,n-1} = (-1)^{d_{ij}}\mathbf{e}(n-1).$$

Using a similar argument, adapted to the copy ${}^{\dagger}\mathcal{H}^-((n-1)\alpha_i) \subset \mathcal{H}^-(n\alpha_i)$ of $\mathcal{H}^-((n-1)\alpha_i)$ *embedded on the right*, we can show that

$$\alpha_{0,1}\alpha_{1,0} = (-1)^{d_{ij}}\mathbf{e}(0).$$

The maps $\alpha_{k+1,k}$ are going to be maps in a chain complex. Therefore, we prove by induction that, for *any* $N \geq 2$, $\alpha_{k+1,k}\alpha_{k,k-1} = 0$ in $\mathcal{H}^-(N\alpha_i + \alpha_j)$, for $1 \leq k \leq N-1$. We want to emphasize that this particular statement holds independently of the meaning of N given in the statement of the theorem (as will be necessary when making the inductive step). To that end, let

(5.34)
$$\underline{i}_k = (\underbrace{i,\ldots,i}_{k}, j, \underbrace{i,\ldots,i}_{n-k-1}).$$

Then, we have $\mathbf{e}(k) = e(\underline{i}_k)\mathbf{e}(k)$.

When $k = 1$, we calculate as above that

$$\alpha_{2,1}\alpha_{1,0} = \tau_1\tau_2\mathbf{e}(1)\tau_1\mathbf{e}(0) = \tau_1\tau_2\tau_1\mathbf{e}(0).$$

Recalling that $\mathbf{e}(0) = \mathbf{e}(0) = 1_{\alpha_j}\otimes e_{i,n-1}$, we deduce from properties of the spin nilHecke algebra that $\tau_2\mathbf{e}(0) = 0$. Therefore, applying (4.12),

$$\alpha_{2,1}\alpha_{1,0} = \tau_1\tau_2\tau_1 e(\underline{i}_0)\mathbf{e}(0) = \tau_2\tau_1\tau_2\mathbf{e}(0) = 0,$$

proving the base case. For the inductive step,

$$\begin{aligned}\alpha_{k+1,k}\alpha_{k,k-1} &= \tau_1\cdots\tau_{k+1}\mathbf{e}(k)\tau_1\cdots\tau_k\mathbf{e}(k-1)\\ &= \tau_1\cdots\tau_{k+1}\tau_1\cdots\tau_k e(\underline{i}_{k-1})\mathbf{e}(k-1)\\ &= \tau_1\cdots\tau_{k+1}\tau_1 e(\underline{i}_k)\tau_2\cdots\tau_k\mathbf{e}(k-1)\\ &= (-1)^{p(j)+k-2}(\tau_1\tau_2\tau_1)e(\underline{i}_{k+1})(\tau_3\cdots\tau_{k+1}\tau_2\cdots\tau_k)\mathbf{e}(k-1)\\ &= (-1)^{p(j)+k-2}(\tau_2\tau_1\tau_2)(\tau_3\cdots\tau_{k+1}\tau_2\cdots\tau_k)\mathbf{e}(k-1),\end{aligned}$$

where we have again used (4.12). By Lemma 5.7, we have that

$$\mathbf{e}(k-1) = {}^{\dagger}\mathbf{e}(k-2)\mathbf{e}(k-1),$$

where ${}^{\dagger}\mathbf{e}(k-2) = 1_{\alpha_i}\otimes e_{i,k-2}\otimes 1_{\alpha_j}\otimes e_{i,n-k}$. Therefore, we may apply induction using ${}^{\dagger}\mathcal{H}^-((N-1)\alpha_i+\alpha_j)\subset\mathcal{H}^-(N\alpha_i+\alpha_j)$ to show that

$$\tau_2\cdots\tau_{k+1}\tau_2\cdots\tau_k\mathbf{e}(k-1) = 0.$$

Finally, we may define isomorphisms

$$\bigoplus_{k=0}^{\frac{n-2}{2}}\Pi^{p(N-2k-1;i,j)}P_{2k+1} \underset{\alpha''}{\overset{\alpha'}{\rightleftarrows}} \bigoplus_{k=0}^{\frac{n}{2}}\Pi^{p(N-2k;i,j)}P_{2k}\ ,$$

where α',α'' are given by the sum of maps

$$\alpha' = \sum_{k=0}^{n/2-1}(-\xi)(\alpha_{2k+1,2k}+\alpha_{2k+1,2k+2})+\alpha_{n-1,n-2}$$

and

$$\alpha'' = \sum_{k=0}^{n/2}(-1)^{d_{ij}}(\alpha_{2k,2k+1}+\alpha_{2k,2k-1})+(-1)^{d_{ij}}\alpha_{n-2,n-1}.$$

For the last step, we need to check that these are actually morphisms in our category. That is, we need to show that these are *even* elements of the respective Hom^--spaces. Recall $\underline{i}_k$ from (5.34). We may rewrite

$$\alpha_{k,k+1} = \tau_{n-1}\cdots\tau_{k+1}e(\underline{i}_{k+1})\mathbf{e}_{k+1} = \tau_{n-1}e(\underline{i}_k)\cdots\tau_{k+2}e(\underline{i}_k)\tau_{k+1}e(\underline{i}_{k+1})\mathbf{e}(k+1),$$

and

$$\alpha_{k+1,k} = \tau_1\cdots\tau_{k+1}e(\underline{i}_k)\mathbf{e}_k = \tau_1 e(\underline{i}_{k+1})\cdots\tau_k e(\underline{i}_{k+1})\tau_{k+1}e(\underline{i}_k)\mathbf{e}(k).$$

Therefore, the parity of $\alpha_{k+1,k}$ is $k+p(j)$ (mod 2) and the parity of $\alpha_{k,k+1}$ is $(n-1)-(k+2)+1+p(j)\equiv k+p(j)$ (mod 2), where we have again used the fact that $n-1=N=1-a_{ij}$ is odd.

When $j\in I_{\bar{0}}$, this means we have the following sequence of morphisms in our category:

$$0\longrightarrow P_N\longrightarrow P_{N-1}\longrightarrow \Pi P_{N-2}\longrightarrow\cdots\longrightarrow \Pi^{\binom{N}{2}}P_0\longrightarrow 0\ ,$$

where the maps are

$$\alpha_{N-k,N-k-1}\in\mathrm{Hom}_{N\alpha_i+\alpha_j}(\Pi^{\binom{k}{2}}P_{N-k},\Pi^{\binom{k+1}{2}}P_{N-k-1}).$$

For $j\in I_{\bar{1}}$,

$$0\longrightarrow P_N\longrightarrow \Pi P_{N-1}\longrightarrow \Pi P_{N-2}\longrightarrow\cdots\longrightarrow \Pi^{\binom{N+1}{2}}P_0\longrightarrow 0$$

is a sequence of morphisms in our category, where the maps are given by

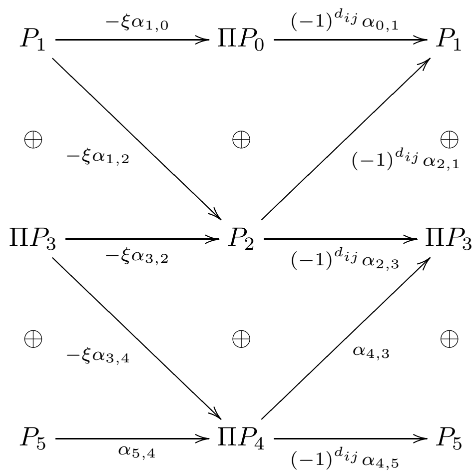
$$\alpha_{N-k,N-k-1} \in \mathrm{Hom}_{N\alpha_i+\alpha_j}(\Pi^{\binom{k+1}{2}} P_{N-k}, \Pi^{\binom{k+2}{2}} P_{N-k-1}).$$

Now, the proof follows from the simple computation

$$p(k;i,j) = \begin{cases} \binom{k}{2} & \text{if } j \in I_{\bar{0}}, \\ \binom{k+1}{2} & \text{if } j \in I_{\bar{1}}. \end{cases}$$

□

Example 5.10. Assume $a_{ij} = -4$ (so $n = 6$), where i is odd and j is even, say. Then, the maps α' and α'' are defined by summing the maps down the left and right columns, respectively, in the following diagram:



In particular, it follows by (5.33) that the composition is

$$\begin{aligned}\alpha''\alpha' &= (-1)^{d_{ij}}\xi(\alpha_{1,0}\alpha_{0,1} + \alpha_{1,2}\alpha_{2,1}) + (-1)^{d_{ij}}\xi(\alpha_{3,2}\alpha_{2,3} + \alpha_{3,4}\alpha_{4,3}) + (-1)^{d_{ij}}\alpha_{5,4}\alpha_{4,5} \\ &= \mathbf{e}(1) + \mathbf{e}(3) + \mathbf{e}(5).\end{aligned}$$

6. Categorification of quantum superalgebras

6.1. **Induction and restriction.** Let $\mu, \nu \in Q^+$. Assume throughout the section that $\mathrm{ht}\,\mu = m$ and $\mathrm{ht}\,\nu = n$, and let $D_{m,n}$ be the set of minimal left $S_m \times S_n$-coset representatives in S_{m+n}. The natural embedding $\mathcal{H}^-(\mu) \otimes \mathcal{H}^-(\nu) \hookrightarrow \mathcal{H}^-(\mu+\nu)$ maps $e(\underline{i}) \otimes e(\underline{j})$ to $e(\underline{ij})$ for all $\underline{i} \in I^\mu$ and $\underline{j} \in I^\nu$, and $\underline{ij} \in I^{\mu+\nu}$ is obtained by concatenation. The image of the identity element $1_\mu \otimes 1_\nu$ is the idempotent

$$1_{\mu,\nu} = \sum_{\underline{i}\in I^\mu,\, \underline{j}\in I^\nu} e(\underline{ij}).$$

Define the functor

$$\mathrm{Res}^{\mu+\nu}_{\mu,\nu} : \mathrm{Mod}^-(\mu+\nu) \longrightarrow \mathrm{Mod}^-(\mu) \otimes \mathrm{Mod}^-(\nu), \tag{6.1}$$

by

$$\mathrm{Res}^{\mu+\nu}_{\mu,\nu} M = 1_{\mu,\nu} M,$$

and the functor

$$\mathrm{Ind}^{\mu+\nu}_{\mu,\nu} : \mathrm{Mod}^-(\mu) \otimes \mathrm{Mod}^-(\nu) \longrightarrow \mathrm{Mod}^-(\mu+\nu), \tag{6.2}$$

by

$$\mathrm{Ind}_{\mu,\nu}^{\mu+\nu}(M\otimes N)=\mathcal{H}^-(\mu+\nu)1_{\mu,\nu}\bigotimes_{\mathcal{H}^-(\mu)\otimes\mathcal{H}^-(\nu)}(M\otimes N).$$

Proposition 6.1. *The module $1_{\mu,\nu}\mathcal{H}^-(\mu+\nu)$ is a free graded left $\mathcal{H}^-(\mu)\otimes\mathcal{H}^-(\nu)$-module.*

Proof. The set $\{1_{\mu,\nu}\tau_w|w\in D_{m,n}\}$ is a basis for $1_{\mu,\nu}\mathcal{H}^-(\mu+\nu)$ as a free graded left $\mathcal{H}^-(\mu)\otimes\mathcal{H}^-(\nu)$-module. □

Corollary 6.2. *The functors $\mathrm{Res}_{\mu,\nu}^{\mu+\nu}$ and $\mathrm{Ind}_{\mu,\nu}^{\mu+\nu}$ take finitely generated projective modules to finitely generated projective modules.*

We have

$$\mathrm{Ind}_{\mu,\nu}^{\mu+\nu}(P_{\underline{i}}\otimes P_{\underline{j}})=P_{\underline{ij}}.$$

Passing to direct summands, we deduce that the same holds if we replace $P_{\underline{i}}$ and $P_{\underline{j}}$ by $P_{\underline{i}^{(\underline{k})}}$ and $P_{\underline{j}^{(\underline{l})}}$, respectively.

For any $\nu\in Q^+$, define the parity

$$p(\nu)=p(i_1)+\cdots+p(i_n), \tag{6.3}$$

where $\underline{i}=(i_1,\dots,i_n)\in I^\nu$.

Theorem 6.3 (Mackey Theorem). *Let $\mu_1,\mu_2,\nu_1,\nu_2\in Q^+$ be such that $\mu_1+\nu_1=\mu_2+\nu_2=\alpha$. Then, the graded $(\mathcal{H}^-(\mu_1)\otimes\mathcal{H}^-(\nu_1),\mathcal{H}^-(\mu_2)\otimes\mathcal{H}^-(\nu_2))$-bimodule*

$$\mathrm{Res}_{\mu_1,\nu_1}^{\alpha}\mathrm{Ind}_{\mu_2,\nu_2}^{\alpha}(\mathcal{H}^-(\mu_2)\otimes\mathcal{H}^-(\nu_2))$$

has a filtration by graded bimodules isomorphic to

$$\Pi^{p(\lambda)p(\nu_1+\lambda-\nu_2)}\Big((1_{\mu_1}\mathcal{H}^-(\alpha)1_{\mu_1-\lambda,\lambda}\otimes 1_{\nu_1}\mathcal{H}^-(\alpha)1_{\nu_1+\lambda-\nu_2,\nu_2-\lambda})$$
$$\otimes_H(1_{\mu_1-\lambda,\mu_2-\lambda-\mu_1}\mathcal{H}^-(\alpha)1_{\mu_2}\otimes 1_{\lambda,\nu_2-\lambda}\mathcal{H}^-(\alpha)1_{\nu_2})\Big)\{-(\lambda,\nu_1+\lambda-\nu_2)\}$$

over all $\lambda\in Q^+$ such that all the terms above are in Q^+. In this expression, we have denoted

$$H=\mathcal{H}^-(\mu_1-\lambda)\otimes\mathcal{H}^-(\lambda)\otimes\mathcal{H}^-(\nu_1+\lambda-\nu_2)\otimes\mathcal{H}^-(\nu_2-\lambda).$$

Proof. The proof is exactly the same as [KL1, Proposition 2.18], with the parity corresponding to the parity of the diagram appearing in the proof. □

The following formula for restriction follows from Theorem 6.3.

Proposition 6.4. *For $\underline{k}\in I^{\mu+\nu}$, we have*

$$\mathrm{Res}_{\mu,\nu}^{\mu+\nu}P_{\underline{k}}=\bigoplus_{\substack{w\in D_{m,n}\\ \underline{k}=w^{-1}(\underline{ij})}}\Pi^{p(\tau_w e(\underline{ij}))}(P_{\underline{i}}\otimes P_{\underline{j}})\{\deg(\tau_w e(\underline{ij}))\}.$$

6.2. **Grothendieck group as a bialgebra.** The exact functors (6.1) and (6.2) give rise to exact functors

$$\mathrm{Ind} = \bigoplus_{\mu,\nu} \mathrm{Ind}_{\mu,\nu}^{\mu+\nu} : \bigoplus_{\mu,\nu\in Q^+} \mathrm{Mod}^-(\mu)\otimes \mathrm{Mod}^-(\nu) \longrightarrow \bigoplus_{\lambda\in Q^+} \mathrm{Mod}^-(\lambda),$$

and

$$\mathrm{Res} = \bigoplus_{\substack{\lambda,\mu,\nu \\ \mu+\nu=\lambda}} \mathrm{Res}_{\mu,\nu}^{\lambda} : \bigoplus_{\lambda\in Q^+} \mathrm{Mod}^-(\lambda) \longrightarrow \bigoplus_{\mu,\nu\in Q^+} \mathrm{Mod}^-(\mu)\otimes \mathrm{Mod}^-(\nu).$$

We set

$$[\mathrm{Proj}^-] = \bigoplus_{\nu\in Q^+} [\mathrm{Proj}^-(\nu)] \quad \text{and} \quad [\mathrm{Rep}^-] = \bigoplus_{\nu\in Q^+} [\mathrm{Rep}^-(\nu)].$$

For $x, y \in [\mathrm{Proj}^-]$, we will simply write $xy = [\mathrm{Ind}](x, y)$. Define multiplication on $[\mathrm{Proj}^-]\otimes[\mathrm{Proj}^-]$ by

$$(x_1\otimes x_2)(y_1\otimes y_2) = \pi^{p(\mu)p(\nu)} q^{-(\mu,\nu)}(x_1y_1\otimes x_2y_2), \tag{6.4}$$

for $x_1, x_2, y_1, y_2 \in [\mathrm{Proj}^-]$ such that $x_2 \in [\mathrm{Proj}^-(\mu)]$ and $y_1 \in [\mathrm{Proj}^-(\nu)]$, and the notation $p(\nu)$ is given in (6.3).

Proposition 6.5. $[\mathrm{Proj}^-]$ *and* $[\mathrm{Rep}^-]$ *with* $[\mathrm{Ind}]$ *as multiplication are associative unital* $\mathcal{A}^\pi$*-algebras.* $[\mathrm{Proj}^-]$ *and* $[\mathrm{Rep}^-]$ *with* $[\mathrm{Res}]$ *are coassociative counital* $\mathcal{A}^\pi$*-coalgebras. Together,* $([\mathrm{Proj}^-], [\mathrm{Ind}], [\mathrm{Res}])$ *is a bialgebra.*

Proof. The first two claims are clear from the properties of induction and restriction functors. The third one means that [Res] is an algebra homomorphism, which follows from Theorem 6.3. □

Recall from (5.10) the bilinear pairing $(\cdot,\cdot)$ on $[\mathrm{Proj}^-(\nu)]$. This extends naturally to a pairing $(\cdot,\cdot)$ on $[\mathrm{Proj}^-]$ by letting $[\mathrm{Proj}^-(\nu)]$ be orthogonal for different ν.

Proposition 6.6. *The pairing* $(\cdot,\cdot)$ *on* $[\mathrm{Proj}^-]$ *satisfies* $(1,1) = 1$*, and*

(a) $(P_i, P_j) = \delta_{ij}(1-\pi_i q_i^2)^{-1}$;
(b) $(x, yy') = ([\mathrm{Res}](x), y\otimes y')$;
(c) $(xx', y) = (x\otimes x', [\mathrm{Res}](y))$;
(d) $(x\otimes x', y\otimes y') = (x,y)(x',y')$, *for all* $x, x', y, y' \in [\mathrm{Proj}^-]$.

Proof. This is exactly as in [KL1, Proposition 3.3]. Indeed, (a) is calculated as the graded dimension of $e(i)\mathcal{H}^-(\alpha_i)e(j) = \delta_{ij}\mathcal{H}^-(\alpha_i)$. Property (b) is calculated as in [KL1, Proposition 3.3] now using (5.12): For $X \in \mathrm{Proj}^-(\mu+\nu)$, $Y \in \mathrm{Proj}^-(\mu)$ and $Y' \in \mathrm{Proj}^-(\nu)$,

$$\begin{aligned}
([X],[Y][Y']) &= ([X], [\mathrm{Ind}_{\mu,\nu}^{\mu+\nu} Y\otimes Y'])\\
&= \dim_q^\pi(X^\psi \otimes_{\mathcal{H}^-(\mu+\nu)} \mathcal{H}^-(\mu+\nu)1_{\mu,\nu} \otimes_{\mathcal{H}^-(\mu)\otimes\mathcal{H}^-(\nu)} Y\otimes Y')\\
&= \dim_q^\pi(X^\psi 1_{\mu,\nu} \otimes_{\mathcal{H}^-(\mu)\otimes\mathcal{H}^-(\nu)} Y\otimes Y')\\
&= \dim_q^\pi((1_{\mu,\nu}X)^\psi \otimes_{\mathcal{H}^-(\mu)\otimes\mathcal{H}^-(\nu)} Y\otimes Y')\\
&= ([\mathrm{Res}_{\mu,\nu}^{\mu+\nu} X], [Y\otimes Y']) = ([\mathrm{Res}_{\mu,\nu}^{\mu+\nu} X], [Y]\otimes[Y']).
\end{aligned}$$

A nearly identical calculation gives (c). Finally, to prove (d), it is enough to observe that there is an even isomorphism of $(\mathbb{Z}\times\mathbb{Z}_2)$-graded vector spaces

$$(P\otimes Q)^{\psi}\otimes_{\mathcal{H}^-(\mu)\otimes\mathcal{H}^-(\nu)}(P'\otimes Q')\cong(P^{\psi}\otimes_{\mathcal{H}^-(\mu)}P')\otimes(Q^{\psi}\otimes_{\mathcal{H}^-(\nu)}Q')$$

for any $P,P'\in\mathrm{Proj}^-(\mu)$ and $Q,Q'\in\mathrm{Proj}^-(\nu)$. □

Example 6.7. We compute $([P_{ii}],[P_{ii}])$ in two ways. Note that $P_{ii}=\mathcal{H}^-(2\alpha_i)$. First, by definition and using (5.16), we have

$$([P_{ii}],[P_{ii}])=\dim_q^{\pi}\mathcal{H}^-(2\alpha_i)=\frac{\pi_i q_i^{-1}[2]_i}{(1-\pi_i q_i^2)^2}.$$

On the other hand, using Propositions 6.4 and 6.6(b), we deduce that

$$\begin{aligned}([P_{ii}],[P_{ii}])&=([\mathrm{Res}^{2\alpha_i}_{\alpha_i,\alpha_i}P_{ii}],[P_i]\otimes[P_i])\\&=((1+\pi^{p(\tau_1)}q^{\deg(\tau_1)})[P_i]\otimes[P_i],[P_i]\otimes[P_i])\\&=(1+\pi_i q_i^{-2})([P_i],[P_i])^2.\end{aligned}$$

By Proposition 6.6(a), the two answers agree, since $\pi_i q_i^{-1}[2]_i=1+\pi_i q^{-2}$.

6.3. **The homomorphism γ^{π}.** Recall the algebra ${}_{\mathcal{A}}\mathbf{f}^{\pi}$ from Section 3. The following is a superanalogue of [KL1, Proposition 3.4].

Proposition 6.8. *There exists an injective $\mathcal{A}^{\pi}$-homomorphism $\gamma^{\pi}:{}_{\mathcal{A}}\mathbf{f}^{\pi}\to[\mathrm{Proj}^-]$ defined by*

$$\gamma^{\pi}(\theta_{\underline{i}}^{(\underline{k})})=P_{\underline{i}^{(\underline{k})}},$$

for each $\theta_{\underline{i}}^{(\underline{k})}:=\theta_{i_1}^{(k_1)}\cdots\theta_{i_h}^{k_h}\in{}_{\mathcal{A}}\mathbf{f}^{\pi}$. The following properties hold under γ^{π}:

(a) *The comultiplication map $({}_{\mathcal{A}}\mathbf{f}^{\pi})_{\mu+\nu}\to({}_{\mathcal{A}}\mathbf{f}^{\pi})_{\mu}\otimes({}_{\mathcal{A}}\mathbf{f}^{\pi})_{\nu}$ corresponds to the exact functor $\mathrm{Res}^{\mu+\nu}_{\mu,\nu}$.*
(b) *The bar involution on ${}_{\mathcal{A}}\mathbf{f}^{\pi}$ corresponds to the duality functor #.*
(c) *The bilinear form $(\cdot,\cdot)$ on ${}_{\mathcal{A}}\mathbf{f}^{\pi}$ corresponds to the bilinear form $(\cdot,\cdot)$ on $[\mathrm{Proj}^-]$, i.e., $(x,y)=(\gamma^{\pi}(x),\gamma^{\pi}(y))$, for all $x,y\in{}_{\mathcal{A}}\mathbf{f}^{\pi}$.*

Proof. We start with a $\mathbb{Q}(q)^{\pi}$-homomorphism $\gamma^{\pi}_{\mathbb{Q}}$ from the free algebra $'\mathbf{f}$ to $[\mathrm{Proj}^-]$ which sends each θ_i to $P_{\underline{i}}$. By Theorem 3.8 on quantum Serre and Theorem 5.9 on categorical Serre, $\gamma^{\pi}_{\mathbb{Q}}$ descends to a homomorphism $\gamma^{\pi}:{}_{\mathcal{A}}\mathbf{f}^{\pi}\to[\mathrm{Proj}^-]$ as defined in the theorem. Property (c) on the compatibility of bilinear forms follows from Proposition 3.3 and Proposition 6.6. The injectivity of γ^{π} follows from the non-degeneracy of $(\cdot,\cdot)$ on ${}_{\mathcal{A}}\mathbf{f}^{\pi}$. Property (a) follows from Proposition 6.5 and then checking that the homomorphisms r and [Res] are compatible on the generators θ_i and $[P_i]$ under γ^{π}. The bar-involution on $[\mathrm{Proj}^-]$ fixes each P_i and satisfies (5.6), and (b) follows. □

Let $[\mathrm{Proj}^-(\nu)]_{\pi=1}$ (resp. $[\mathrm{Proj}^-(\nu)]_{\pi=-1}$) be the quotient of $[\mathrm{Proj}^-(\nu)]$ by the relations

$$[M]=[\Pi M]\qquad(\text{respectively, }[M]=-[\Pi M]),$$

for every $M\in\mathrm{Proj}^-(\nu)$. Recall the algebra ${}_{\mathcal{A}}\mathbf{f}^{+}$ and ${}_{\mathcal{A}}\mathbf{f}^{-}$ from Section 3.

Corollary 6.9. *There exist injective homomorphisms $\gamma^{+}:{}_{\mathcal{A}}\mathbf{f}^{+}\to[\mathrm{Proj}^-]_{\pi=1}$ and $\gamma^{-}:{}_{\mathcal{A}}\mathbf{f}^{-}\to[\mathrm{Proj}^-]_{\pi=-1}$ satisfying the analogous properties of Proposition 6.8.*

Theorem 6.10. *The map* $\gamma^+ : {}_{\mathcal{A}}\mathbf{f}^+ \to [\mathrm{Proj}^-]_{\pi=1}$ *is an isomorphism of bialgebras.*

Proof. Recall ${}_{\mathcal{A}}\mathbf{f}^+$ is the usual half of the quantum Kac-Moody algebra. The surjectivity of γ^+ can be established following [Kle1, Chapter 5], exactly as was done in [KL1, §3.2]. □

6.4. **The isomorphisms γ^- and γ^π.** The arguments in [Kle1, Chapter 5] are insufficient to derive the surjectivity of γ^- directly. In particular, the proof of the independence of characters [Kle1, Theorem 5.3.1] fails since we have $\mathrm{ch}_q^- L = 0$ if $\mathrm{ch}_q^\pi L = \mathrm{ch}_q^\pi \Pi L$ (see (5.2) and (5.3)). We now give an alternative argument based on representation theory of quantum Kac-Moody superalgebras to show this never happens.

Denote by $\mathrm{Ch} M = \sum_\mu \dim M_\mu e^\mu$ the (formal) character of an algebra or a module $M = \bigoplus_\mu M_\mu$ which is graded by Q^+ and free over the base ring $\mathcal{A}$ or $\mathbb{Q}(q)$. We have a natural partial order $\geq$ on the collection of characters: for characters g, h, we have $g \geq h$ if and only if $g - h$ is a non-negative integer linear combination of e^μ for $\mu \in Q^+$.

Lemma 6.11. *We have* $\mathrm{Ch}_{\mathcal{A}}\mathbf{f}^+ = \mathrm{Ch}_{\mathcal{A}}\mathbf{f}^-$.

Proof. By [Kac], the super Weyl-Kac character formula for integrable modules holds and therefore so does the super Weyl-Kac denominator formula. According to [BKM], the integrable modules of quantum Kac-Moody superalgebras have the same characters as their classical counterpart at $q \mapsto 1$ (generalizing Lusztig's work for quantum Kac-Moody [Lu1]). Hence the super-Weyl-Kac denominator formula holds, and so

$$\mathrm{Ch}_{\mathcal{A}}\mathbf{f}^- = \Big(\sum_{w \in W} \mathrm{sgn}(w) e^{w\rho - \rho}\Big)^{-1},$$

where ρ and the Weyl group W for the Kac-Moody superalgebra associated to the GCM A [Kac] coincide with the counterparts for the usual Kac-Moody algebra associated to the GCM A^+. On the other hand, it is well known that $\mathrm{Ch}_{\mathcal{A}}\mathbf{f}^+$ is given by exactly the same formula (by a combination of the Weyl-Kac denominator formula and Lusztig's result for quantum Kac-Moody). Hence the lemma follows. □

Recall that a finite-dimensional simple module S of an associative superalgebra A is of type `M` if it remains to be simple with the $\mathbb{Z}_2$-grading forgotten, and is of type `Q` otherwise. Recall the parity-shift functor Π. It is known (cf. for example [Kle1, Chapter 12]) that a simple A-module S is of type `Q` if and only if there exists an even isomorphism of A-modules: $\Pi S \cong S$.

Proposition 6.12. *We have*

$$\mathrm{Ch}_{\mathcal{A}}\mathbf{f}^- = \mathrm{Ch}[\mathrm{Proj}^-]_{\pi=-1} = \mathrm{Ch}[\mathrm{Proj}^-]_{\pi=1} = \mathrm{Ch}_{\mathcal{A}}\mathbf{f}^+.$$

Proof. Using the Cartan pairing (5.14), $[\mathrm{Proj}^-]_{\pi=1}$ has a basis of projective indecomposable modules P_L labeled by all simple modules L of the spin quiver Hecke algebra (of both type `M` and type `Q`). On the other hand, since a type `Q` simple module L always admits an even isomorphism $\Pi L \cong L$, therefore $[\mathrm{Proj}^-]_{\pi=-1}$ has a basis labeled by the type M simple modules. Hence, we have $\mathrm{Ch}[\mathrm{Proj}^-]_{\pi=1} \geq \mathrm{Ch}[\mathrm{Proj}^-]_{\pi=-1}$. By Theorem 6.10, the map γ^+ is an isomorphism. Combining this with the injection γ^- gives us a sequence of inequalities of formal characters:

$$\mathrm{Ch}_{\mathcal{A}}\mathbf{f}^- \leq \mathrm{Ch}[\mathrm{Proj}^-]_{\pi=-1} \leq \mathrm{Ch}[\mathrm{Proj}^-]_{\pi=1} = \mathrm{Ch}_{\mathcal{A}}\mathbf{f}^+.$$

All inequalities must be equalities by Lemma 6.11. □

The equality $\mathrm{Ch}_{\mathcal{A}}\mathbf{f}^- = \mathrm{Ch}[\mathrm{Proj}^-]_{\pi=-1}$ above implies the following.

Corollary 6.13. *The map $\gamma^-_{\mathbb{Q}(q)} : \mathbf{f}^- \to \mathbb{Q}(q) \otimes_{\mathcal{A}} [\mathrm{Proj}^-]_{\pi=-1}$ is an isomorphism of $\mathbb{Z} \times \mathbb{Z}_2$-graded algebras.*

Arguing as in [KL1, Proposition 3.20] and using Corollary 6.13, we now deduce the part (a) of the following theorem. Part (b) then follows from (a), Proposition 6.8 and Theorem 6.10.

Theorem 6.14. (a) *The map $\gamma^- : {}_{\mathcal{A}}\mathbf{f}^- \to [\mathrm{Proj}^-]_{\pi=-1}$ is an isomorphism of $\mathbb{Z} \times \mathbb{Z}_2$-graded $\mathcal{A}$-algebras.*
(b) *The map $\gamma^\pi : {}_{\mathcal{A}}\mathbf{f}^\pi \to [\mathrm{Proj}^-]$ is an isomorphism of $\mathbb{Z} \times \mathbb{Z}_2$-graded $\mathcal{A}^\pi$-algebras.*

6.5. **The type M phenomenon.** From the proof of Proposition 6.12, the identity

$$\mathrm{Ch}[\mathrm{Proj}^-]_{\pi=-1} = \mathrm{Ch}[\mathrm{Proj}^-]_{\pi=1}$$

holds, and it implies (and is indeed equivalent to) the following property of simple modules of a spin quiver Hecke algebra (which was conjectured in [KKT, page 3]).

Proposition 6.15. *Each simple module of a spin quiver Hecke algebra is of type M.*

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After the completion of this paper, Kang-Kashiwara-Oh [KKO] used [KKT] to categorify the Kac-Moody algebra $\mathfrak{g} = \mathfrak{g}(A^+)$ and its integrable representations following [KK], but the connection with Kac-Moody superalgebras was not pursued.

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