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LIFTS OF LONGEST ELEMENTS TO BRAID GROUPS ACTING ON DERIVED CATEGORIES

JOSEPH GRANT

ABSTRACT. If we have a braid group acting on a derived category by spherical twists, how does a lift of the longest element of the symmetric group act? We give an answer to this question, using periodic twists, for the derived category of modules over a symmetric algebra. The question has already been answered by Rouquier and Zimmermann in a special case. We prove a lifting theorem for periodic twists, which allows us to apply their answer to the general case.

Along the way we study tensor products in derived categories of bimodules. We also use the lifting theorem to give new proofs of two known results: the existence of braid relations and, using the theory of almost Koszul duality due to Brenner, Butler, and King, the result of Rouquier and Zimmermann mentioned above.

CONTENTS

1. INTRODUCTION

Actions of braid groups on derived categories were constructed by both Rouquier and Zimmermann [RZ] and Seidel and Thomas [ST] (see also [HK]). In both cases we get a group morphism from the braid group B_{n+1} on $n+1$ strands to the group of autoequivalences of a derived category $\mathcal{C}$:

$$\varphi_n : B_{n+1} \to \mathrm{Aut}(\mathcal{C}),$$

which sends the standard generators of B_{n+1} to spherical twists. In Rouquier and Zimmermann's setting, $\mathcal{C} = \mathrm{D}^{\mathrm{b}}(\Gamma_n)$, where Γ_n is the Brauer tree algebra of a line without multiplicity. This is the crucial example of an algebra whose derived category admits an action of B_{n+1}.

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The braid group B_{n+1} maps in an obvious way onto the symmetric group S_{n+1} on $n+1$ letters. S_{n+1} has a unique longest element which we denote $w_0^{(n+1)}$; this is the element of S_{n+1} which swaps i with $n+2-i$. We can lift this element to the braid group B_{n+1}. Rouquier and Zimmermann showed that the action of the positively lifted longest element on $\mathrm{D}^{\mathrm{b}}(\Gamma_n)$ via φ_n has a particularly nice description: it just acts by shifting complexes n places to the left and twisting by an algebra automorphism.

In [Gra1], a new construction of autoequivalences of derived categories of symmetric algebras, called periodic twists, was introduced, generalizing the construction of spherical twists. In Section 6 of that article it was noted that, given an action of B_3 on the derived category $\mathrm{D}^{\mathrm{b}}(\Gamma_3)$, the lift of w_0^3 to B_3 acts by a periodic twist. It is natural to ask whether this phenomenon holds in general and this question is the motivation for the current article.

Periodic twists are defined using a cone construction, so it is not surprising that studying their compositions leads naturally to studying tensor products of distinguished triangles. In Section 2 we consider tensor products of distinguished triangles in derived categories of bimodules in order to set up some machinery which will be useful later. We investigate May's axioms [May] using ideas of Keller and Neeman [KN] and the tools of enhanced triangulated categories due to Bondal and Kapranov [BoKa].

In Section 3 we will prove a lifting theorem, which allows us to lift relations between periodic twists from endomorphism algebras to our original algebra. Loosely, this says that relations between periodic twists that hold in the derived category of the endomorphism algebra of some projective module also hold in the derived category of the original algebra. Combined with the Rouquier-Zimmermann description of the action of $w_0^{(n+1)}$, this will allow us to prove that positive lifts of longest elements act by periodic twists in general. This is explained in Section 4, where we also give a new, explicit, proof of Rouquier-Zimmermann's result which uses the theory of almost Koszul duality due to Brenner, Butler, and King [BBK].

An early statement of the lifting theorem was presented at the 44th Symposium on Ring Theory and Representation Theory at Okayama University in September 2011 and was included in an article submitted to the proceedings of this symposium [Gra2].

1.1. **Conventions and notation.** All algebras are finite dimensional k-algebras, where k is an algebraically closed field of arbitrary characteristic. Moreover, for simplicity, all algebras are basic, i.e., their simple modules are 1 dimensional. By Morita theory, this is no restriction up to categorical equivalence. Modules are finitely generated left modules unless we say otherwise.

For an algebra A over a field k, or over a semisimple base ring S, and an A-module M, we write M^* to denote the dual A-module $\mathrm{Hom}_k(M,k)$ or $\mathrm{Hom}_S(M,S)$, as appropriate. This duality turns left modules into right modules, and vice versa.

In Sections 3 and 4 all algebras denoted A will be symmetric, i.e., we have an isomorphism $A \cong A^* = \mathrm{Hom}_k(A,k)$ of A-A-bimodules. An A-module denoted P or P_i will always be projective, but will not in general be indecomposable.

We write the composition of morphisms in a category as follows: $X \xrightarrow{f} Y \xrightarrow{g} Z$ is written $g \circ f$ (and not gf, due to our tensor product conventions, described below). However, for the composition of arrows $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$ in a quiver we write $\alpha\beta$.

For algebras A and B, we will say an A-B-bimodule when we mean an $A \otimes_k B^{\mathrm{op}}$-module. Let $\mathrm{D^b}(A)$ and $\mathrm{D^b}(A\text{-}B)$ denote the derived categories of A-modules and A-B-bimodules, respectively. Recall that the perfect category $\mathrm{per}(A)$ is the full subcategory of the bounded derived category generated by the compact objects. This means that $\mathrm{per}(A)$ is the full subcategory of $\mathrm{D^b}(A)$ which consists of objects quasi-isomorphic to bounded complexes of projective A-modules, and $\mathrm{per}(A\text{-}B)$ is the full subcategory of $\mathrm{D^b}(A\text{-}B)$ which consists of objects quasi-isomorphic to bounded complexes of projective A-B-bimodules. Note that the property of being a projective A-B-bimodule is stronger than the property of being projective as both a left A-module and a right B-module.

Derived categories are triangulated, and morphisms of triangulated categories are triangulated functors, which take distinguished triangles to distinguished triangles and commute with the shift functor. We denote the shift functor by Σ in Section 2 and by [1] in Sections 3 and 4. Distinguished triangles are written

$$X \to Y \to Z \to \Sigma X$$

or simply

$$X \to Y \to Z \rightsquigarrow,$$

so $Z \rightsquigarrow$ is a map from Z to ΣX.

A stalk complex is a complex where all but one constituent module is zero. We have a full and faithful embedding

$$A\text{-mod} \hookrightarrow \mathrm{D^b}(A)$$

which sends a module to the stalk complex concentrated in degree 0, and we will use this implicitly for both modules and bimodules.

We will make much use of tensor products, but it will be useful to suppress the tensor product sign at times, especially in large commutative diagrams. This should cause no confusion for objects as we will be clear about the categories in which our objects live. So, for example, if $P \in \mathrm{D^b}(A\text{-}E)$ and $Y \in \mathrm{D^b}(E\text{-}E)$, then we will sometimes write PY instead of $P \otimes_E Y$ for the object of $\mathrm{D^b}(A\text{-}E)$. This is reasonable in the same way that writing a composition of functions without specifying their domains and codomains is reasonable, as the objects we tensor can be seen as 1-morphisms in a 2-category. A more uncommon notational convention that we employ is to write two juxtaposed functions to denote a tensor product, not a composition. For example, if $f \in \mathrm{End}_{\mathrm{D^b}(A\text{-}E)}(P)$ and $g \in \mathrm{End}_{\mathrm{D^b}(E\text{-}E)}(Y)$, then $fg \in \mathrm{End}_{\mathrm{D^b}(A\text{-}E)}(PY)$. A tensor product of categories will always be over k, as per Definition 2.1.1.

Everything will be defined using cochain complexes (X, d) which are made up of modules X^i and a differential d which maps $X^i \to X^{i+1}$, but as we are working in an algebraic setting and so are interested in projective resolutions, we will often write $X_i := X^{-i}$ to avoid negative numbers.

We use the Koszul sign rule for tensor products: for $f : X \to X'$ and $g : Y \to Y'$,

$$(f \otimes g)(x \otimes y) = (-1)^{iq} f(x) \otimes g(y)$$

if $x \in X^i$ and g is homogeneous of degree q. We will often identify the isomorphic objects $A \otimes_A X$, X, and $X \otimes_B B$ in $\mathrm{D^b}(A\text{-}B)$ and will pretend that tensor products are strictly associative. This causes no serious problems.

2. Tensor products of distinguished triangles

May studied symmetric tensor products of maps in distinguished triangles and gave some axioms that well-behaved triangulated categories should satisfy [May]. He showed that these axioms are satisfied in certain nice cases. Keller and Neeman gave a nice interpretation of some of May's work based on the derived category of a commutative square and results of Happel on Dynkin quivers [Hap], and noted that the symmetry of the tensor product was not of crucial importance [KN, Remark 3.11].

We analyse the derived category of a commutative square in the spirit of Keller and Neeman, but our analysis is self-contained and does not use Happel's results. We would like a version of [KN, Lemma 3.8] for triangulated categories. To obtain this, we use the theory of DG-enhancements [BoKa].

In this section we will consider braid diagrams and braid axioms. There is no direct connection with the braid groups mentioned in the introduction and considered in later sections.

2.1. **Modules over a commutative square.** Keller and Neeman realized that to study tensor products of distinguished triangles we can work inside the derived category of the modules over the k-linear category of a commutative square. Instead of using Happel's description of this category [Hap] via the derived equivalence between a commutative square and an oriented graph of type D_4, we work directly with the commutative square. This approach has the advantage of being self-contained, and moreover we believe it is simpler. We will also be able to exhibit a direct connection between the derived category of the commutative square and May's braid axioms [May].

Let $\mathbf{Ar}$ be the path algebra of the quiver

$$Q_{\mathbf{Ar}} = 1 \xrightarrow{\alpha} 2$$

and, for $i \in \{1, 2\}$, let $P_i = \mathbf{Ar}e_i$ be the ith projective $\mathbf{Ar}$-module and $S_i \cong P_i/\operatorname{rad} P_i$ be the corresponding simple module. Then, up to isomorphism, there are three indecomposable objects in $\mathbf{Ar}$-mod: $P_1 \cong S_1$, P_2, and S_2. To save space and to produce diagrams which are more pleasing to the eye, we will write the modules P_1, P_2, and S_2 as simply 1, 2, and 3. Note that P_1 maps into P_2 by right multiplication by α. We will denote this map simply by $\alpha : P_1 \hookrightarrow P_2$.

Our three $\mathbf{Ar}$-modules fit in a short exact sequence

$$0 \to 1 \overset{\alpha}{\hookrightarrow} 2 \overset{\beta}{\twoheadrightarrow} 3 \to 0$$

which is, up to isomorphism, the only nonsplit indecomposable short exact sequence in $\mathbf{Ar}$-mod.

Let $\mathbf{Sq}$ be the algebra $\mathbf{Ar} \otimes_k \mathbf{Ar}$. We can write $\mathbf{Sq}$ as a quiver with relations in the following manner: $\mathbf{Sq} \cong Q_{\mathbf{Sq}}/I_{\mathbf{Sq}}$, where $Q_{\mathbf{Sq}}$ is the quiver

$$\begin{tikzcd} 11 \arrow[r, "a"] \arrow[d, "c"] & 21 \arrow[d, "b"] \\ 12 \arrow[r, "d"] & 22 \end{tikzcd}$$

and $I_{\mathbf{Sq}}$ is the ideal generated by $ab - cd$. Under the isomorphism, a, b, c, and d correspond to $\alpha \otimes_k \mathrm{id}_1$, $\mathrm{id}_2 \otimes_k \alpha$, $\mathrm{id}_1 \otimes_k \alpha$, and $\alpha \otimes_k \mathrm{id}_2$ which we abbreviate as $\alpha 1$, 2α, 1α, and $\alpha 2$, respectively.

We can view our algebra $\mathbf{Sq}$ as a preadditive category with one object. Then the k-linear category $\square$ studied by Keller and Neeman is the idempotent completion of our category $\mathbf{Sq}$. Hence we have equivalences

$$\mathbf{Sq}\text{-mod} \cong \square\text{-mod}$$

of module categories and

$$\mathrm{D}^{\mathrm{b}}(\mathbf{Sq}) \cong \mathrm{D}^{\mathrm{b}}(\square)$$

of derived categories. From now on we will not use the category $\square$ and will only consider $\mathbf{Sq}$.

Recall that there is a simple (naive) definition of the tensor product of two k-linear categories:

Definition 2.1.1. Given k-linear categories $\mathcal{C}$ and $\mathcal{D}$, the category $\mathcal{C} \otimes \mathcal{D}$ is defined as follows: the objects of $\mathcal{C} \otimes \mathcal{D}$ are the ordered pairs (c, d) for $c \in \mathcal{C}$ and $d \in \mathcal{D}$, and the formula

$$\mathrm{Hom}_{\mathcal{C}\otimes\mathcal{D}}((c,d),(c',d')) = \mathrm{Hom}_{\mathcal{C}}(c,c') \otimes_k \mathrm{Hom}_{\mathcal{D}}(d,d')$$

gives the hom-spaces.

Then given two algebras A and B we have a functor

$$i_{A,B} : A\text{-mod} \otimes B\text{-mod} \hookrightarrow A \otimes_k B\text{-mod}$$

defined in the obvious way: the tensor product of $M \in A$-mod and $N \in B$-mod is sent to $M \otimes_k N \in M \otimes_k N$-mod.

Lemma 2.1.2. *For $M_i \in A$-mod and $N_i \in B$-mod, $i = 1, 2$, there is a natural isomorphism of vector spaces:*

$$\mathrm{Hom}_{A\otimes_k B}(M_1 \otimes_k N_1, M_2 \otimes_k N_2) \cong \mathrm{Hom}_A(M_1, M_2) \otimes_k \mathrm{Hom}_B(N_1, N_2).$$

Proof. Recall the definition: for an algebra Λ and Λ-modules V and W,

$$\begin{aligned}\mathrm{Hom}_\Lambda(V,W) &= \{f \in \mathrm{Hom}_k(V,W) \mid f(\lambda v) = \lambda f(v) \text{ for all } \lambda \in \Lambda, v \in V\}\\ &\subset \mathrm{Hom}_k(V,W).\end{aligned}$$

Use the isomorphism $\mathrm{Hom}_k(V,W) \cong V^* \otimes_k W$ which sends f to $\sum_i v_i^* \otimes f(v_i)$ for some basis $\{v_i\}$ of V. Then $\mathrm{Hom}_\Lambda(V,W)$ corresponds to the subspace

$$\Big\{\sum g_i \otimes w_i \in V^* \otimes_k W \mid \sum g_i\lambda \otimes w_i = \sum g_i \otimes \lambda w_i \text{ for all } \lambda \in \Lambda\Big\} \subset V^* \otimes_k W.$$

Using the braiding in the symmetric monoidal category of vector spaces gives an isomorphism

$$(M_1 \otimes_k N_1)^* \otimes_k (M_2 \otimes_k N_2) \cong N_1^* \otimes_k M_1^* \otimes_k M_2 \otimes_k N_2 \cong (M_1^* \otimes_k M_2) \otimes_k (N_1^* \otimes_k N_2),$$

and we can check that the subspace corresponding to

$$\mathrm{Hom}_{A\otimes_k B}(M_1 \otimes_k N_1, M_2 \otimes_k N_2)$$

and the subspace corresponding to $\mathrm{Hom}_A(M_1, M_2) \otimes_k \mathrm{Hom}_B(N_1, N_2)$ agree. Naturality is clear. $\square$

So the functor $i_{A,B}$ is fully faithful. Note that in general A-mod $\otimes B$-mod is not an abelian category. It will be abelian, and the functor will in fact be an equivalence, if and only if at least one of A and B is semisimple.

Using the above functor we can immediately write nine nonisomorphic indecomposable $\mathbf{Sq}$-modules: this just comes down to tensoring one of the $\mathbf{Ar}$-modules

labelled 1, 2, or 3 with another. We see that they fit in the 3×3 commutative diagram of short exact sequences

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & 11 \arrow[r, "\alpha 1"] \arrow[d, "1\alpha"] & 21 \arrow[r, "\beta 1"] \arrow[d, "2\alpha"] & 31 \arrow[r] \arrow[d, "3\alpha"] & 0 \\
0 \arrow[r] & 12 \arrow[r, "\alpha 2"] \arrow[d, "1\beta"] & 22 \arrow[r, "\beta 2"] \arrow[d, "2\beta"] & 32 \arrow[r] \arrow[d, "3\beta"] & 0 \\
0 \arrow[r] & 13 \arrow[r, "\alpha 3"] \arrow[d] & 23 \arrow[r, "\beta 3"] \arrow[d] & 33 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

of **Sq**-modules, where we have suppressed the tensor product sign.

To list every object in **Sq**-mod, we can take the abelian hull of **Ar**-mod $\otimes$**Ar**-mod in **Sq**-mod: we start with our list $11, 21, \ldots, 33$ and add the kernel and cokernel of every map between these modules. It is easy to see that the only modules that are missing from our list are the kernel of the epimorphism $\beta \otimes \beta : 22 \twoheadrightarrow 33$ and the cokernel of the monomorphism $\alpha \otimes \alpha : 11 \hookrightarrow 22$, which we denote K and C, respectively. So **Sq** has finite representation type, and in fact its eleven modules fit into a large commutative diagram

$$\begin{tikzcd}
0 \arrow[dr] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[dl, bend left] & 0 \arrow[d] & 0 \arrow[dl, bend left] & 0 \arrow[dr, bend left] & \\
0 \arrow[r] & 11 \arrow[r, "\alpha 1"] \arrow[d, "1\alpha"] \arrow[dr] & 21 \arrow[r, "\beta 1"] \arrow[d, "2\alpha"] \arrow[dl, bend right] & 31 \arrow[d, "3\alpha"] \arrow[dl, bend right] \arrow[dr, bend left] & & & \\
 & K \arrow[dr] \arrow[uurr, bend left] & & & & 0 \arrow[dl, bend left] & \\
 & 12 \arrow[r, "\alpha 2"] \arrow[d, "1\beta"] & 22 \arrow[r, "\beta 2"] \arrow[d, "2\beta"] \arrow[dr] & 32 \arrow[d, "3\beta"] & & & \\
0 \arrow[ur, bend left] & & C \arrow[uurr, bend left] \arrow[dr] & & & & \\
0 \arrow[r] & 13 \arrow[r, "\alpha 3"] \arrow[d] \arrow[uurr, bend left] & 23 \arrow[r, "\beta 3"] \arrow[d] \arrow[dl, bend left] & 33 \arrow[d] \arrow[dr] & & & \\
0 \arrow[ur, bend left] & 0 \arrow[dl, bend left] & 0 \arrow[dl, bend left] & 0 & 0 & 0 & 0
\end{tikzcd}$$

consisting of many short exact sequences. But we will see that the situation is even clearer on the derived level. Note that, as well as the short exact sequences that can be read from the diagram, we have a short exact sequence

$$0 \to 11 \hookrightarrow 12 \oplus 21 \twoheadrightarrow K \to 0$$

which we call the *Mayer-Vietoris* short exact sequence.

The maps between modules, if not all the short exact sequences, are more clearly read from the Auslander-Reiten quiver of $\mathbf{Sq}$-mod:

$$\begin{tikzcd}
 & 21 \arrow[dr] & & 13 \arrow[dr] & & 32 \arrow[dr] & \\
11 \arrow[ur] \arrow[dr] & & K \arrow[ur] \arrow[r] \arrow[dr] & 22 \arrow[r] & C \arrow[ur] \arrow[dr] & & 33 \\
 & 12 \arrow[ur] & & 31 \arrow[ur] & & 23 \arrow[ur] &
\end{tikzcd}$$

Twisting this quiver and curving its arrows gives us the diagram

$$\begin{tikzcd}
 & 21 \arrow[rr, two heads] \arrow[dr, hook] & & 31 \arrow[rr, hook] \arrow[dr, hook] & & 32 \arrow[dr, two heads] \arrow[rrr, bend left] & & \\
11 \arrow[ur, hook] \arrow[dr, hook] \arrow[rrr, bend left] & & K \arrow[ur] \arrow[r, hook] \arrow[dr] & 22 \arrow[r, two heads] & C \arrow[ur] \arrow[dr] & & & 33 \\
 & 12 \arrow[ur, hook] \arrow[rr, two heads] & & 13 \arrow[ur, hook] \arrow[rr, hook] & & 23 \arrow[ur, two heads] \arrow[rrr, bend left] & &
\end{tikzcd}$$

which better illustrates the short exact sequences. The reader should imagine the arrows of the short exact sequences

$$0 \to 11 \hookrightarrow 22 \twoheadrightarrow C \to 0$$

and

$$0 \to K \hookrightarrow 22 \twoheadrightarrow 33 \to 0$$

as being threaded in and out of the diagram, going into and out of the piece of paper or computer screen.

2.2. **The derived category of a commutative square.** As $\mathbf{Ar}$ is a path algebra with no relations, it is hereditary and so has global dimension 1. As $\mathbf{Ar}$ is hereditary, the three objects $1, 2, 3$ are, up to isomorphism and shift, the only indecomposable objects in $\mathrm{D^b}(\mathbf{Ar})$, and

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \overset{\gamma}{\rightsquigarrow}$$

is, up to rotation and isomorphism, the only indecomposable distinguished triangle. Note that we have labelled the map $3 \to \Sigma 1$ as γ.

Using the identification of $\mathbf{Ar}$ with upper triangular 2×2 matrices with entries in k, we can identify $\mathbf{Sq}$ with upper triangular matrices with entries in $\mathbf{Ar}$, and so conclude [ERZ] that $\mathbf{Sq}$ has global dimension 2. This is also easy to see by directly calculating the projective resolutions of all simple modules.

We can check that the cones of the maps $\alpha \otimes \beta : 12 \to 23$ and $\beta \otimes \alpha : 21 \to 32$ are quasi-isomorphic and are not concentrated in degree zero. Let G denote a representative of the isomorphism class of objects containing the two cones above in the derived category $\mathrm{D^b}(\mathbf{Sq})$. As $\mathbf{Sq}$ has global dimension 2, all isomorphism classes of indecomposable objects in $\mathrm{D^b}(\mathbf{Sq})$ have a representative given by a chain complex of projective $\mathbf{Sq}$-modules with at most three nonzero terms, so we can quicky check that, up to isomorphism and shift, this is the only object of $\mathrm{D^b}(\mathbf{Sq})$ which is not concentrated in a single degree. So we have found all twelve isoclasses of indecomposable objects, up to shift, in $\mathrm{D^b}(\mathbf{Sq})$.

Lemma 2.2.1. *Suppose A and B are k-algebras and we have an A-module M and a distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

in $\mathrm{D}^{\mathrm{b}}(B)$. *Then the triangle defined by the bottom row of the commutative diagram*

$$\begin{tikzcd}
M \otimes_k X \arrow[r, "M\otimes f"] \arrow[d, equal] & M \otimes_k Y \arrow[r, "M\otimes g"] \arrow[d, equal] & M \otimes_k Z \arrow[r, "M\otimes h"] \arrow[d, equal] & M \otimes_k (\Sigma X) \arrow[d, "\sim"] \\
M \otimes_k X \arrow[r, "M\otimes f"] & M \otimes_k Y \arrow[r, "M\otimes g"] & M \otimes_k Z \arrow[r] & \Sigma(M \otimes_k X)
\end{tikzcd}$$

is distinguished in $\mathrm{D}^{\mathrm{b}}(A \otimes_k B)$.

Proof. Consider the canonical distinguished triangle

$$M \otimes_k X \overset{M\otimes f}{\to} M \otimes_k Y \to \mathrm{cone}(M \otimes f) \to \Sigma(M \otimes_k X)$$

in the homotopy category. By considering the first map in this triangle, it is clearly isomorphic to the triangle displayed in the lemma. Now localize to pass to the derived category. □

As in [May, Remark 4.2], we will implicitly use fixed isomorphisms

$$(\Sigma i) \otimes j \cong \Sigma(i \otimes j) \cong i \otimes (\Sigma j)$$

for $i, j \in \{1, 2, 3\}$. These come from the isomorphisms

$$(\Sigma X) \otimes Y \cong \Sigma(X \otimes Y) \cong X \otimes (\Sigma Y)$$
$$(-1)^{\deg(y)} x \otimes y \mapsto x \otimes y \mapsto (-1)^{\deg(x)} x \otimes y$$

of differential graded k-modules. The notation Σij will mean $\Sigma(i \otimes j)$.

We will sometimes be sloppy and write maps such as

$$31 \overset{\gamma 1}{\to} \Sigma(11)$$

when we really mean the composition

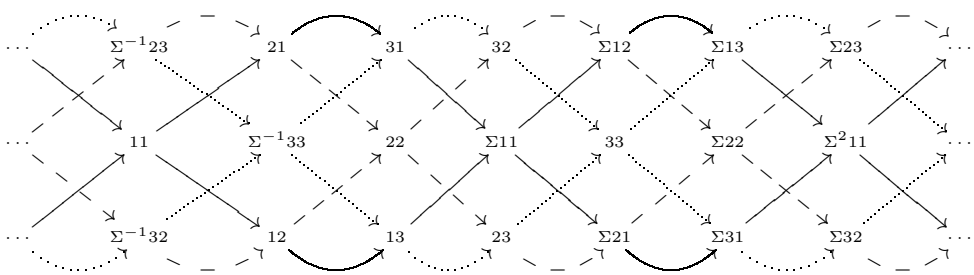
$$31 \overset{\gamma 1}{\to} (\Sigma 1)1 \overset{\sim}{\to} \Sigma(11)$$

for our fixed isomorphism $(\Sigma 1)1 \overset{\sim}{\to} \Sigma(11)$.

Now we must be very careful with our signs. We consider the diagram

$$\begin{tikzcd}
11 \arrow[r, "\alpha 1"] \arrow[d, "1\alpha"] & 21 \arrow[r, "\beta 1"] \arrow[d, "2\alpha"] & 31 \arrow[r, "\gamma 1"] \arrow[d, "3\alpha"] & \Sigma 11 \arrow[d, "\Sigma 1\alpha"] \\
12 \arrow[r, "\alpha 2"] \arrow[d, "1\beta"] & 22 \arrow[r, "\beta 2"] \arrow[d, "2\beta"] & 32 \arrow[r, "\gamma 2"] \arrow[d, "3\beta"] & \Sigma 12 \arrow[d, "\Sigma 1\beta"] \\
13 \arrow[r, "\alpha 3"] \arrow[d, "1\gamma"] & 23 \arrow[r, "\beta 3"] \arrow[d, "2\gamma"] & 33 \arrow[r, "\gamma 3"] \arrow[d, "3\gamma"] \arrow[dr, phantom, "(-)"] & \Sigma 13 \arrow[d, "-\Sigma 1\gamma"] \\
\Sigma 11 \arrow[r, "\Sigma\alpha 1"] & \Sigma 21 \arrow[r, "\Sigma\beta 1"] & \Sigma 31 \arrow[r, "-\Sigma\gamma 1"] & \Sigma^2 11
\end{tikzcd}$$

in $\mathrm{D}^{\mathrm{b}}(\mathbf{Sq})$. By Lemma 2.2.1 and the axioms for a triangulated category, each row and column is a distinguished triangle, and we can check directly that each square commutes except the bottom right square (marked $(-)$) which anticommutes. The diagram should be compared to Lemma 2.6 of [May]. This can also be viewed as a braid of morphisms of triangles, as follows:

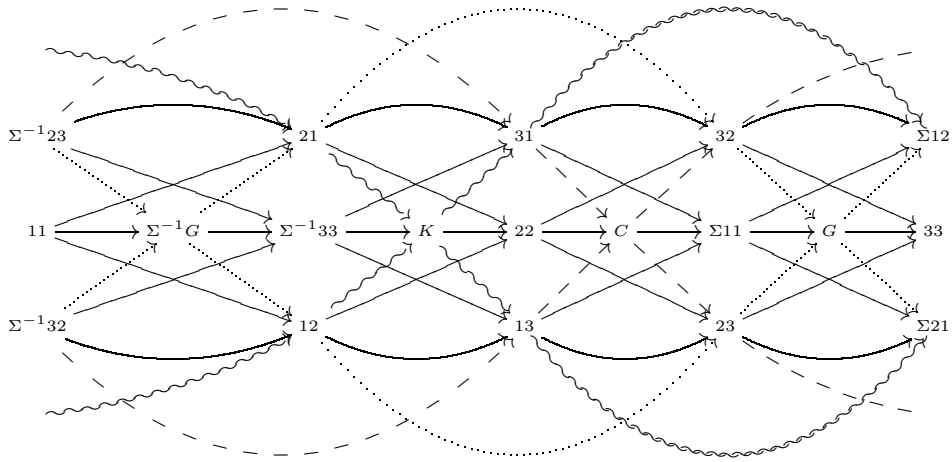


where all quadrilaterals commute except the square

$$\begin{tikzcd}
 & 31 \arrow[dr] & \\
\Sigma^{-1}33 \arrow[ur, dotted] \arrow[dr, dotted] & & \Sigma 11 \\
 & 13 \arrow[ur] &
\end{tikzcd}$$

and its shifts, which anticommute. These "braid diagrams" appear in, for example, [Ive, May]. We have used different types of arrows to help the reader see the different distinguished triangles.

We obtain a more complicated diagram



if we include our extra objects K, C, and G and six new distinguished triangles, such as

$$21 \to K \to 13 \rightsquigarrow$$

and

$$12 \to K \to 31 \rightsquigarrow$$

for K. Again, we have used different types of arrows to help the reader see the distinguished triangles.

It is important to note that this diagram does not commute, even up to sign. However, every subdiagram containing at most one of the objects K, C, and G will commute up to sign.

We note that the distinguished triangles

$$11 \to 22 \to C \rightsquigarrow ,$$

$$K \to 22 \to 33 \rightsquigarrow ,$$

and

$$\Sigma 11 \to G \to 33 \rightsquigarrow$$

are not visible in this diagram. Similarly to the case of the module category, one can imagine these distinguished triangles threading in and out of the diagram, lying in a plane which is perpendicular to the plane in which the rest of the diagram is embedded.

The above diagram is just a combination of May's braid axioms (TC3) for the object K and (TC3') for the object C [May, Section 4], together with the analogous braid for the object G.

Removing all but the irreducible morphisms from the above diagram gives the following graph:

$$\begin{tikzcd}
 & 21 \arrow[dr] & & 31 \arrow[dr] & & 32 \arrow[dr] & & \Sigma 12 \arrow[dr] & \\
\cdots & \Sigma^{-1}33 \arrow[r] & K \arrow[ur] \arrow[r] \arrow[dr] & 22 \arrow[r] & C \arrow[ur] \arrow[r] \arrow[dr] & \Sigma 11 \arrow[r] & G \arrow[ur] \arrow[r] \arrow[dr] & 33 & \cdots \\
 & 12 \arrow[ur] & & 13 \arrow[ur] & & 23 \arrow[ur] & & \Sigma 21 &
\end{tikzcd}$$

which one can see is a twisted version of the Auslander-Reiten quiver:

$$\begin{tikzcd}
 & 21 \arrow[dr] & & 13 \arrow[dr] & & 32 \arrow[dr] & & \Sigma 21 \arrow[dr] & \\
\cdots & \Sigma^{-1}33 \arrow[r] & K \arrow[ur] \arrow[r] \arrow[dr] & 22 \arrow[r] & C \arrow[ur] \arrow[r] \arrow[dr] & \Sigma 11 \arrow[r] & G \arrow[ur] \arrow[r] \arrow[dr] & 33 & \cdots \\
 & 12 \arrow[ur] & & 31 \arrow[ur] & & 23 \arrow[ur] & & \Sigma 12 &
\end{tikzcd}$$

of $\mathrm{D}^{\mathrm{b}}(\mathbf{Sq})$.

We note that there are other distinguished triangles in $\mathrm{D}^{\mathrm{b}}(\mathbf{Sq})$, the most important of which for us is the Mayer-Vietoris distinguished triangle

$$11 \to 12 \oplus 21 \to K \rightsquigarrow$$

coming from the associated short exact sequence in $\mathbf{Sq}$-mod. We refer the reader to [KN] for more information on the distinguished triangles we have not described here.

2.3. **Getting a morphism of triangulated categories.** We want to show that the results we obtained in the last subsection hold for arbitrary tensor products in derived categories. We will use the idea of Keller and Neeman [KN]: if we have a map of triangulated categories from $\mathrm{D}^{\mathrm{b}}(\mathbf{Sq})$ to another derived category, then because this map sends distinguished triangules to distinguished triangules we will know that the images of the objects in $\mathbf{Sq}$-mod fit into some nice distinguished triangules. Our objects in the derived category may not be concentrated in degree 0, so we approach the problem by working with enhancements of triangulated categories, following Bondal and Kapranov [BoKa]. This strategy is more technical than, though very similar in spirit to, Example 3.2 of [KN].

Our reference throughout is [BoKa].

Let $\mathcal{C}$ be a DG-category and let $\mathcal{C}^{\oplus}$ denote the DG-category obtained from $\mathcal{C}$ by adjoining finite formal direct sums of objects.

Definition 2.3.1 (Bondal-Kapranov)**.** A *twisted complex over* $\mathcal{C}$ is a set $\{(E_i)_{i\in\mathbb{Z}}, q_{ij} : E_i \to E_j\}$, where the E_i are objects in $\mathcal{C}^{\oplus}$, equal to 0 for almost

all i, and the q_{ij} are morphisms in $\mathcal{C}$ of degree $i-j+1$

$$\begin{tikzcd}
\cdots & E_{-2} \arrow[r, bend left] \arrow[rr, bend left] & E_{-1} \arrow[l, bend left] & E_0 \arrow[l, bend left] \arrow[r, bend left] \arrow[rr, bend left] \arrow[loop above] & E_1 \arrow[l, bend left] \arrow[r, bend left] \arrow[rr, bend left] \arrow[loop below] & E_2 \arrow[l, bend left] & E_3 \arrow[l, bend left] & \cdots
\end{tikzcd}$$

satisfying the Maurer-Cartan equation

$$dq_{ij} + \sum_k q_{kj} q_{ik} = 0.$$

Twisted complexes over $\mathcal{C}$ form a DG-category, denoted Pre-Tr($\mathcal{C}$), with morphism complexes made up of spaces

$$\mathrm{Hom}^k_{\text{Pre-Tr}(\mathcal{C})}(K, K') = \coprod_{i,j\in\mathbb{Z}} \mathrm{Hom}^{k+i-j}_{\mathcal{C}}(K_i, K'_j)$$

and a differential which is described in Section 1 of [BoKa]. Taking the category of twisted complexes is an endofunctor

$$\text{Pre-Tr} : \mathrm{dgCat} \to \mathrm{dgCat}$$

of the category of DG-categories. We call it the *pretriangulated completion* functor.

Any DG-category has an associated homology category $H(\mathcal{C})$ with the same objects as $\mathcal{C}$ but with hom spaces defined as the homology of the differential. In particular, we can take the 0th homology $\mathrm{H}_0(\mathcal{C})$ of $\mathcal{C}$. We denote $\mathrm{H}_0(\text{Pre-Tr}(\mathcal{C}))$ by $\mathrm{Tr}(\mathcal{C})$. It is a triangulated category [BoKa, Section 1, Proposition 1].

Recall that the DG-category of chain complexes of vector spaces has as morphisms $f : X \to Y$ of degree n all collections of maps $\{f_i : X_i \to Y_{i-n}\}$ with no requirement that these maps commute with the differentials of X and Y. The differential d on this DG category is such that the degree zero maps f which satisfy $df = 0$ are exactly the traditional maps of chain complexes, which do commute with the differentials of X and Y. Then, for any DG-category $\mathcal{C}$, a DG $\mathcal{C}$-module is a functor from $\mathcal{C}$ to the DG-category of chain complexes of vector spaces. We denote the category of DG $\mathcal{C}$-modules by $\mathcal{C}$-dgmod.

There is an embedding

$$\iota_{\mathcal{C}} : \text{Pre-Tr}(\mathcal{C}) \hookrightarrow \mathcal{C}\text{-dgmod}$$

defined as follows. Let $E = \{E_i, q_{ij}\}$ be a twisted complex over $\mathcal{C}$ and let $c \in \mathcal{C}$. For each $i \in \mathbb{Z}$, write the chain complex $\mathrm{Hom}_{\mathcal{C}^\oplus}(c, E_i)$ as

$$\cdots \xrightarrow{d_i} \mathrm{Hom}_{\mathcal{C}^\oplus}(c, E_i)_{-1} \xrightarrow{d_i} \mathrm{Hom}_{\mathcal{C}^\oplus}(c, E_i)_0 \xrightarrow{d_i} \mathrm{Hom}_{\mathcal{C}^\oplus}(c, E_i)_1 \xrightarrow{d_i} \cdots$$

with differential d_i of degree 1. Then $\iota_{\mathcal{C}}(E)$ is the functor which sends $c \in \mathcal{C}$ to the chain complex

$$\cdots \xrightarrow{\partial_E} \bigoplus_{i+j=-1} \mathrm{Hom}_{\mathcal{C}^\oplus}(c, E_i)_j \xrightarrow{\partial_E} \bigoplus_{i+j=0} \mathrm{Hom}_{\mathcal{C}^\oplus}(c, E_i)_j \xrightarrow{\partial_E} \bigoplus_{i+j=1} \mathrm{Hom}_{\mathcal{C}^\oplus}(c, E_i)_j \xrightarrow{\partial_E} \cdots$$

where

$$\partial_E|_{\mathrm{Hom}_{\mathcal{C}^\oplus}(c,E_i)_j} = d_i|_{\mathrm{Hom}_{\mathcal{C}^\oplus}(c,E_i)_j} + \sum_{k\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}^\oplus}(c, q_{ik}).$$

If $f \in \mathrm{Hom}^{\ell}_{\mathcal{C}}(E_i, E'_j)$ is a morphism of twisted complexes from $E = \{E_i, q_{ij}\}$ to $E' = \{E'_i, q'_{ij}\}$, then $\iota_{\mathcal{C}}(f)$ is the obvious degree ℓ map of chain complexes, which does not necessarily commute with the differentials ∂_E and $\partial_{E'}$.

Definition 2.3.2 (Bondal-Kapranov)**.** $\mathcal{C}$ is called *pretriangulated* if the image of every twisted complex under the above embedding is a representable functor. If $\mathcal{C}$ is pretriangulated, then every object in $\text{Pre-Tr}\,\mathcal{C}$ has a *convolution*, which is defined as the associated representing object of $\mathcal{C}$. A DG-functor $F : \mathcal{C} \to \mathcal{D}$ between pretriangulated categories is *pre-exact* if it commutes with the operation of taking convolutions of twisted complexes, i.e., the diagram

$$\begin{tikzcd} \text{Pre-Tr}(\mathcal{C}) \arrow[r, "\text{Pre-Tr}(F)"] \arrow[d, "\mathrm{conv}_{\mathcal{C}}"] & \text{Pre-Tr}(\mathcal{D}) \arrow[d, "\mathrm{conv}_{\mathcal{D}}"] \\ \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd}$$

commutes, where conv denotes the operation of taking the convolution of a twisted complex.

Note that if F is a pre-exact functor, then $\mathrm{H}_0(F)$ is a triangulated functor [BoKa, Section 3].

The following definition is based on, but different to, that of [BoKa]: we use projectives instead of injectives because we have algebraic and not geometric applications in mind. But by the usual duality everything still works.

Definition 2.3.3. Let A be an algebra and let $\mathcal{C}$ be the DG-category obtained by treating A as a preadditive category with one object. Let $\text{Pre-D}^{\mathrm{b}}(\mathcal{C})$ be the full DG-subcategory of chain complexes of $\mathcal{C}$-modules consisting of complexes where each module is projective over A and only finitely many modules in the complex are nonzero.

$\text{Pre-D}^{\mathrm{b}}(A)$ is a *DG-enhancement* of $\mathrm{D}^{\mathrm{b}}(A)$, i.e., there is an equivalence of triangulated categories $\mathrm{H}_0(\text{Pre-D}^{\mathrm{b}}(A)) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(A)$ [BoKa, Section 3, Example 3].

The following lemma states that the convolution for $\text{Pre-Tr}(A)$ is particularly simple: it is a generalized cone construction. This is surely well known to the experts. The proof is immediate from the definitions.

Lemma 2.3.4. *If $E = \{E_i, q_{ij}\}$ is a twisted complex over* $\text{Pre-D}^{\mathrm{b}}(A)$*, then its convolution is the chain complex*

$$\cdots \xrightarrow{\partial} \bigoplus_{i+j=-1} E_{i,j} \xrightarrow{\partial} \bigoplus_{i+j=0} E_{i,j} \xrightarrow{\partial} \bigoplus_{i+j=1} E_{i,j} \xrightarrow{\partial} \cdots$$

in $\text{Pre-D}^{\mathrm{b}}(A\text{-}A)$*, where E_i is the complex*

$$\cdots \xrightarrow{d_i} E_{i,-1} \xrightarrow{d_i} E_{i,0} \xrightarrow{d_i} E_{i,1} \xrightarrow{d_i} \cdots$$

and

$$\partial|_{\bigoplus_{i+j=k} E_{i,j}} = d_i|_{E_{i,j}} + \sum_{k\in\mathbb{Z}} q_{ik}.$$

Now we can start laying the first few bricks in the construction of our functor.

Lemma 2.3.5. *Let $f : X \to Y$ be a morphism in $\mathrm{D^b}(A\text{-}A)$. Then there is a pre-exact morphism of pretriangulated categories*

$$F : \text{Pre-D}^{\mathrm{b}}(\mathbf{Ar}) \to \text{Pre-D}^{\mathrm{b}}(A\text{-}A)$$

such that, on taking 0th homology, the map

$$\mathrm{H}_0(F) : \mathrm{D^b}(\mathbf{Ar}) \to \mathrm{D^b}(A\text{-}A)$$

is a triangulated functor which, up to isomorphism, takes the morphism $\alpha : P_1 \to P_2$ of stalk complexes to $f : X \to Y$.

Proof. We take projective A-A-bimodule resolutions $\mathbb{X}$ and $\mathbb{Y}$ of X and Y and lift f to a map $\varphi : \mathbb{X} \to \mathbb{Y}$ of chain complexes. Any object Z in $\text{Pre-D}^{\mathrm{b}}(\mathbf{Ar})$ is a complex made up of direct sums of P_1 and P_2 with differentials only consisting of multiples of α and identity maps. There is a functor $G : \mathbf{Ar}\text{-proj} \to \text{Pre-D}^{\mathrm{b}}(A \otimes_k A^{\mathrm{op}})$ defined on the additive category of projective $\mathbf{Ar}$-modules which sends P_1, P_2, and α to $\mathbb{X}$, $\mathbb{Y}$, and φ, respectively. So G induces a functor from bounded chain complexes of projective $\mathbf{Ar}$-modules into chain complexes in the category $\text{Pre-D}^{\mathrm{b}}(A \otimes_k A^{\mathrm{op}})$, and taking the total complex gives us a functor $F : \text{Pre-D}^{\mathrm{b}}(\mathbf{Ar}) \to \text{Pre-D}^{\mathrm{b}}(A \otimes_k A^{\mathrm{op}})$. In other words, F is defined by a cone construction. As we are working with chain complexes, this is functorial.

It is clear that F is a DG-morphism, and we now check that it is pre-exact, i.e., that the diagram

$$\begin{tikzcd}
\text{Pre-Tr}(\text{Pre-D}^{\mathrm{b}}(\mathbf{Ar})) \arrow[r, "\text{Pre-Tr}(F)"] \arrow[d, "\mathrm{conv}_{\text{Pre-D}^{\mathrm{b}}(\mathbf{Ar})}"] & \text{Pre-Tr}(\text{Pre-D}^{\mathrm{b}}(A \otimes_k A^{\mathrm{op}})) \arrow[d, "\mathrm{conv}_{\text{Pre-D}^{\mathrm{b}}(A \otimes_k A^{\mathrm{op}})}"] \\
\text{Pre-D}^{\mathrm{b}}(\mathbf{Ar}) \arrow[r, "F"] & \text{Pre-D}^{\mathrm{b}}(A \otimes_k A^{\mathrm{op}})
\end{tikzcd}$$

commutes. This comes down to a routine check which we will now describe.

Let $E = (E_i, q_{ij}) \in \text{Pre-Tr}(\text{Pre-D}^{\mathrm{b}}(\mathbf{Ar}))$, and for $M \in \mathbf{Ar}\text{-proj}$, write $G(M)$ as

$$\cdots \to G_{-1}(M) \to G_0(M) \to G_1(M) \to \cdots.$$

Using Lemma 2.3.4, the convolution of E is

$$\cdots \xrightarrow{\partial} \bigoplus_{i+j=-1} E_{i,j} \xrightarrow{\partial} \bigoplus_{i+j=0} E_{i,j} \xrightarrow{\partial} \bigoplus_{i+j=1} E_{i,j} \xrightarrow{\partial} \cdots,$$

and then to find the image of this complex under F we take the total complex of

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & \bigoplus_{i+j=-1} G_{-1}(E_{i,j}) \arrow[r] \arrow[d] & \bigoplus_{i+j=0} G_{-1}(E_{i,j}) \arrow[r] \arrow[d] & \bigoplus_{i+j=1} G_{-1}(E_{i,j}) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \bigoplus_{i+j=-1} G_{0}(E_{i,j}) \arrow[r] \arrow[d] & \bigoplus_{i+j=0} G_{0}(E_{i,j}) \arrow[r] \arrow[d] & \bigoplus_{i+j=1} G_{0}(E_{i,j}) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \bigoplus_{i+j=-1} G_{1}(E_{i,j}) \arrow[r] \arrow[d] & \bigoplus_{i+j=0} G_{1}(E_{i,j}) \arrow[r] \arrow[d] & \bigoplus_{i+j=1} G_{1}(E_{i,j}) \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

which gives the complex

$$\cdots \to \bigoplus_{i+j+k=-1} G_k(E_{i,j}) \to \bigoplus_{i+j+k=0} G_k(E_{i,j}) \to \bigoplus_{i+j+k=1} G_k(E_{i,j}) \to \cdots$$

in $\text{Pre-D}^{\text{b}}(A \otimes_k A^{\text{op}})$.

Now let's go the other way around the square. Applying $\text{Pre-Tr}(F)$ to E just gives the twisted complex $\{F(E_i), F(q_{ij})\}$. Unwinding the definitions, we see that $F(E_i)$ is the complex

$$\cdots \bigoplus_{j+k=-1} G_k(E_{i,j}) \to \bigoplus_{j+k=0} G_k(E_{i,j}) \to \bigoplus_{j+k=1} G_k(E_{i,j}) \to \cdots .$$

So, using Lemma 2.3.4 again, the convolution of $\{F(E_i), F(q_{ij})\}$ is

$$\cdots \to \bigoplus_{i+j=-1} F(E_i)_j \to \bigoplus_{i+j=0} F(E_i)_j \to \bigoplus_{i+j=1} F(E_i)_j \to \cdots ,$$

which we can rewrite as

$$\cdots \to \bigoplus_{i+j+k=-1} G_k(E_{i,j}) \to \bigoplus_{i+j+k=0} G_k(E_{i,j}) \to \bigoplus_{i+j+k=1} G_k(E_{i,j}) \to \cdots ,$$

and we see that this has the same terms as the complex above. It is simple to check that the differentials also agree.

So F is a pre-exact morphism and hence $\text{H}_0(F)$ is a triangulated functor. By definition it takes α to φ, which is isomorphic to f. □

We will need to find a way to tensor together two of the maps constructed using the previous lemma. The following result will be useful.

Lemma 2.3.6. *For two algebras A, B, the obvious inclusion*

$$i_{A,B} : \text{Pre-D}^{\text{b}}(A) \otimes \text{Pre-D}^{\text{b}}(B) \to \text{Pre-D}^{\text{b}}(A \otimes_k B)$$

becomes a quasi-equivalence

$$\text{Pre-Tr}(i_{A,B}) : \text{Pre-Tr}(\text{Pre-D}^{\text{b}}(A) \otimes \text{Pre-D}^{\text{b}}(B)) \to \text{Pre-Tr}(\text{Pre-D}^{\text{b}}(A \otimes_k B))$$

on applying the pretriangulated completion functor, i.e., the functor

$$\begin{aligned} \text{H}_0(\text{Pre-Tr}(i_{A,B})) : \text{Tr}(\text{Pre-D}^{\text{b}}(A) \otimes \text{Pre-D}^{\text{b}}(B)) &\to \text{Tr}(\text{Pre-D}^{\text{b}}(A \otimes_k B)) \\ &\cong \text{D}^{\text{b}}(A \otimes_k B) \end{aligned}$$

is an equivalence of triangulated categories.

Proof. If $\mathcal{C}$ is a pretriangulated category, then $\text{H}_0(\text{Pre-Tr}(\mathcal{C}))$ is equivalent to $\text{H}_0(\mathcal{C})$ [BoKa, Section 3, Proposition 1], so the codomain of our functor

$$\text{H}_0\left(\text{Pre-Tr}(\text{Pre-D}^{\text{b}}(A) \otimes_k \text{Pre-D}^{\text{b}}(B))\right) \to \text{H}_0\left(\text{Pre-Tr}(\text{Pre-D}^{\text{b}}(A \otimes_k B))\right)$$

is equivalent to $\text{D}^{\text{b}}(A \otimes_k B)$. The domain is the 0th homology of a pretriangulated category, and so is triangulated.

By extending Lemma 2.1.2 we see that our our original functor $i_{A,B}$ is fully faithful. From the definition, $\text{Pre-Tr}(i_{A,B})$ and therefore $\text{H}_0(\text{Pre-Tr}(i_{A,B}))$ are also both fully faithful, and so the domain of our functor is isomorphic to a full subcategory of $\text{D}^{\text{b}}(A \otimes_k B)$. But it clearly contains all simple $A \otimes_k B$-modules and so, as it is closed under taking cones, must be the whole category. □

We can now construct our morphism.

Proposition 2.3.7. *For $i \in \{1, 2\}$, let $F_i : \text{Pre-Tr}(\mathbf{Ar}) \to \text{Pre-Tr}(A \otimes_k A^{\text{op}})$ be exact morphisms of pretriangulated categories. Then there is a triangulated functor*

$$F : \text{D}^{\text{b}}(\mathbf{Sq}) \to \text{D}^{\text{b}}(A \otimes_k A^{\text{op}})$$

which acts as $\text{H}_0(F_1) \otimes \text{H}_0(F_2)$ on the subcategory $\text{D}^{\text{b}}(\mathbf{Ar}) \otimes \text{D}^{\text{b}}(\mathbf{Ar})$ of $\text{D}^{\text{b}}(\mathbf{Sq})$.

Proof. Tensor together our two functors F_i, $i \in \{1, 2\}$, to get a morphism

$$F_1 \otimes_k F_2 : \text{Pre-D}^{\text{b}}(\mathbf{Ar}) \otimes \text{Pre-D}^{\text{b}}(\mathbf{Ar}) \to \text{Pre-D}^{\text{b}}(A \otimes_k A^{\text{op}}) \otimes \text{Pre-D}^{\text{b}}(A \otimes_k A^{\text{op}})$$

of dg-categories. We compose this with the tensor product functor

$$\text{Pre-D}^{\text{b}}(A \otimes_k A^{\text{op}}) \otimes \text{Pre-D}^{\text{b}}(A \otimes_k A^{\text{op}}) \to \text{Pre-D}^{\text{b}}(A \otimes_k A^{\text{op}})$$

to get a morphism

$$F' : \text{Pre-D}^{\text{b}}(\mathbf{Ar}) \otimes \text{Pre-D}^{\text{b}}(\mathbf{Ar}) \to \text{Pre-D}^{\text{b}}(A \otimes_k A^{\text{op}}).$$

The codomain $\text{Pre-D}^{\text{b}}(\mathbf{Ar}) \otimes \text{Pre-D}^{\text{b}}(\mathbf{Ar})$ is not pretriangulated (for example, as in the abelian case, it doesn't contain the cone of $11 \to 22$), so we take the pretriangulated closure

$$\text{Pre-Tr}(F') : \text{Pre-Tr}(\text{Pre-D}^{\text{b}}(\mathbf{Ar}) \otimes \text{Pre-D}^{\text{b}}(\mathbf{Ar})) \to \text{Pre-Tr}(\text{Pre-D}^{\text{b}}(A \otimes_k A^{\text{op}})).$$

We also have the obvious map

$$\text{Pre-D}^{\text{b}}(\mathbf{Ar}) \otimes \text{Pre-D}^{\text{b}}(\mathbf{Ar}) \to \text{Pre-D}^{\text{b}}(\mathbf{Sq}).$$

Putting these together we get a commutative diagram

$$\begin{tikzcd}
\text{Pre-D}^{\text{b}}(\mathbf{Ar}) \otimes \text{Pre-D}^{\text{b}}(\mathbf{Ar}) \arrow[r, "F_1 \otimes F_2"] \arrow[d] \arrow[dr, "F'"] & \text{Pre-D}^{\text{b}}(A \otimes_k A^{\text{op}}) \otimes \text{Pre-D}^{\text{b}}(A \otimes_k A^{\text{op}}) \arrow[d, "- \otimes_A -"] \\
\text{Pre-D}^{\text{b}}(\mathbf{Sq}) & \text{Pre-D}^{\text{b}}(A \otimes_k A^{\text{op}})
\end{tikzcd}$$

of DG-categories which, on applying Pre-Tr$(-)$ everywhere, gives a commutative diagram

$$\begin{tikzcd}
\text{Pre-Tr}(\text{Pre-D}^{\mathrm{b}}(\mathbf{Ar}) \otimes \text{Pre-D}^{\mathrm{b}}(\mathbf{Ar})) \arrow[r] \arrow[d] \arrow[dr, "\text{Pre-Tr}(F')"] & \text{Pre-Tr}(\text{Pre-D}^{\mathrm{b}}(A \otimes_k A^{\mathrm{op}}) \otimes \text{Pre-D}^{\mathrm{b}}(A \otimes_k A^{\mathrm{op}})) \arrow[d] \\
\text{Pre-Tr}(\text{Pre-D}^{\mathrm{b}}(\mathbf{Sq})) & \text{Pre-Tr}(\text{Pre-D}^{\mathrm{b}}(A \otimes_k A^{\mathrm{op}}))
\end{tikzcd}$$

of pretriangulated categories. On applying H_0 to this diagram, the leftmost vertical map becomes an equivalence by Lemma 2.3.6, so we can define

$$F : \mathrm{D}^{\mathrm{b}}(\mathbf{Sq})) \to \mathrm{D}^{\mathrm{b}}(E\text{-}E))$$

as the unique map making the diagram

$$\begin{tikzcd}
\mathrm{Tr}(\text{Pre-D}^{\mathrm{b}}(\mathbf{Ar}) \otimes \text{Pre-D}^{\mathrm{b}}(\mathbf{Ar})) \arrow[r] \arrow[d, "\sim"] \arrow[dr, "\mathrm{Tr}(F')"] & \mathrm{Tr}(\text{Pre-D}^{\mathrm{b}}(E\text{-}E) \otimes \text{Pre-D}^{\mathrm{b}}(E\text{-}E)) \arrow[d] \\
\mathrm{D}^{\mathrm{b}}(\mathbf{Sq}) \arrow[r, dashed, "F"] & \mathrm{D}^{\mathrm{b}}(E\text{-}E))
\end{tikzcd}$$

commute. Then by lifting objects and maps of $\mathrm{D}^{\mathrm{b}}(\mathbf{Ar}) \otimes \mathrm{D}^{\mathrm{b}}(\mathbf{Ar})$ to Pre-$\mathrm{D}^{\mathrm{b}}(\mathbf{Ar}) \otimes$ Pre-$\mathrm{D}^{\mathrm{b}}(\mathbf{Ar})$, it is clear that F has the desired properties. $\square$

2.4. **Products of triangles.** The following statement is modelled on May's axiom (TC3), "The Braid Axiom for Products of Triangles" [May], but we have only recorded the properties that we will use later. The proof, based on [KN], makes it clear how to conclude that other properties which hold in $\mathrm{D}^{\mathrm{b}}(\mathbf{Sq})$ also hold in $\mathrm{D}^{\mathrm{b}}(A \otimes_k A^{\mathrm{op}})$. We point out that our notation is different from May's.

Corollary 2.4.1. *Given two distinguished triangles*

$$X_1 \xrightarrow{\alpha_X} X_2 \xrightarrow{\beta_X} X_3 \overset{\gamma_X}{\rightsquigarrow},$$

$$Y_1 \xrightarrow{\alpha_Y} Y_2 \xrightarrow{\beta_Y} Y_3 \overset{\gamma_Y}{\rightsquigarrow}$$

in $\mathrm{D}^{\mathrm{b}}(A\text{-}A)$ *we have an object* $\kappa \in \mathrm{D}^{\mathrm{b}}(A\text{-}A)$ *and three more distinguished triangles*

$$X_2Y_1 \to \kappa \to X_1Y_3 \overset{\alpha_X\gamma_Y}{\rightsquigarrow},$$

$$\kappa \to X_2Y_2 \xrightarrow{\beta_X\beta_Y} X_3Y_3 \rightsquigarrow,$$

$$X_1Y_2 \to \kappa \to X_3Y_1 \overset{\gamma_X\alpha_Y}{\rightsquigarrow}$$

such that the compositions

$$X_2Y_1 \to \kappa \to X_2Y_2,$$

$$X_1Y_2 \to \kappa \to X_2Y_2$$

are $X_2\alpha_Y$ *and* α_XY_2*, respectively.*

Proof. Using Lemma 2.3.5, let $F_X, F_Y : \text{Pre-Tr}(\mathbf{Ar}) \to \text{Pre-Tr}(E\text{-}E)$ send $\alpha : 1 \to 2$ to $\alpha_X : X_1 \to X_2$ and $\alpha_Y : Y_1 \to Y_2$ respectively, and then use Proposition 2.3.7 to construct $F : \mathrm{D}^{\mathrm{b}}(\mathbf{Sq}) \to \mathrm{D}^{\mathrm{b}}(E\text{-}E)$ which acts as $\mathrm{H}_0(F_1) \otimes \mathrm{H}_0(F_2)$ on the appropriate subcategory.

Applying $\mathrm{H}_0(F_X)$ to the distinguished triangle

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \overset{\gamma}{\rightsquigarrow}$$

gives a distinguished triangle

$$X_1 \overset{\alpha_X}{\to} X_2 \overset{F_X(\beta)}{\to} X_3' \overset{F_X(\gamma)}{\rightsquigarrow}$$

which, on comparison with the distinguished triangle

$$X_1 \overset{\alpha_X}{\to} X_2 \overset{\beta_X}{\to} X_3 \overset{\gamma_X}{\rightsquigarrow},$$

tells us that $X_3' := F_X(3) \cong X_3$. Similarly, $Y_3' := F_Y(3) \cong Y_3$. We conclude that $F(ij) \cong X_i Y_j$ for all $i, j \in \{1, 2, 3\}$.

The distinguished triangles

$$21 \to K \to 13 \rightsquigarrow,$$

$$12 \to K \to 31 \rightsquigarrow$$

are sent to distinguished triangles

$$X_2 Y_1 \to \kappa \to X_1 Y_3' \rightsquigarrow, \tag{1}$$

$$X_1 Y_2 \to \kappa \to X_3' Y_1 \rightsquigarrow \tag{2}$$

for some $\kappa \in \mathrm{D^b}(A\text{-}A)$. Then the commutativity of the diagrams

$$\begin{tikzcd} & \Sigma 11 \arrow[dr] & \\ 13 \arrow[ur] \arrow[rr] & & \Sigma 21 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & \Sigma 11 \arrow[dr] & \\ 31 \arrow[ur] \arrow[rr] & & \Sigma 12 \end{tikzcd}$$

in $\mathrm{D^b}(\mathbf{Sq})$ implies that their images under F also commute. From here it is easy to check that the last maps in the distinguished triangles

$$X_2 Y_1 \to \kappa \to X_1 Y_3 \rightsquigarrow,$$

$$X_1 Y_2 \to \kappa \to X_3 Y_1 \rightsquigarrow$$

obtained from distinguished triangles (1) and (2) are $\alpha_X \gamma_Y$ and $\gamma_X \alpha_Y$, respectively.

We also have the distinguished triangle

$$K \to 22 \to 33 \rightsquigarrow$$

in $\mathrm{D^b}(\mathbf{Sq})$, which is sent by F to a distinguished triangle

$$\kappa \to X_2 Y_2 \to X_3' Y_3' \rightsquigarrow$$

in $\mathrm{D^b}(A\text{-}A)$. As the diagrams

$$\begin{tikzcd} & K \arrow[dr] & \\ 21 \arrow[ur] \arrow[rr] & & 22 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & K \arrow[dr] & \\ 12 \arrow[ur] \arrow[rr] & & 22 \end{tikzcd}$$

commute, we have the final statement. It only remains to show that the composition

$$X_2 Y_2 \to X_3' Y_3' \xrightarrow{\sim} X_3 Y_3$$

is $\beta_X \beta_Y$, but this is easy to check given the corresponding statement in $\mathrm{D^b}(\mathbf{Sq})$. $\square$

Our derived bimodule category also has another nice property:

Corollary 2.4.2. *In the above situation, the Mayer-Vietoris triangle*

$$X_1Y_1 \to X_1Y_2 \oplus X_2Y_1 \to \kappa \rightsquigarrow$$

is distinguished.

For a more detailed analysis of such triangles, the interested reader should consult [Ive].

3. A lifting theorem

The aim of this section is to show that we can check that relations between periodic twists hold in a certain endomorphism algebra and conclude that they hold more generally. We will apply this result to braid relations in the following section.

3.1. **Periodic twists.** First we revise the construction of periodic twists from [Gra1]. The construction given here is the same, but we will apply it in a more general situation; this greater generality will be important in our proofs.

Recall that if we have an A-A-bimodule M and an algebra automorphism τ of A, then the twisted module M_τ is defined as the module with the same underlying vector space and left action as M, but with right action given by $m \cdot a := m\tau(a)$ for $m \in M$ and $a \in A$. Similarly, we can define the module ${}_\tau M$ which is twisted on the left instead of on the right.

Let P be a projective A-module and suppose that $E = \mathrm{End}_A(P)^{\mathrm{op}}$, so P is an A-E-bimodule. Recall that E is (twisted) periodic if there is a bounded complex Y of projective E-E-bimodules concentrated in degrees 0 to $n-1$, an algebra automorphism τ of E, and a short exact sequence

$$0 \to E_\tau[n-1] \hookrightarrow Y \twoheadrightarrow E \to 0$$

of chain complexes of E-E-bimodules. We say that E has period n and that Y is a truncated resolution of E.

Now suppose that we have a short exact sequence

$$0 \to F[-1] \hookrightarrow Y \twoheadrightarrow E \to 0$$

in $\mathrm{Ch}^{\mathrm{b}}(E\text{-}E)$, where $Y \in \mathrm{per}(E\text{-}E)$ and we impose no restriction on F. Denote the map $Y \twoheadrightarrow E$ by f. Then by applying the functor $P \otimes_E - \otimes_E P^\vee$ we obtain a map in $\mathrm{Ch}^{\mathrm{b}}(A\text{-}A)$, and we can consider the composition

$$P \otimes_E Y \otimes_E P^\vee \overset{P\otimes f \otimes P^\vee}{\to} P \otimes_E E \otimes_E P^\vee \overset{\sim}{\to} P \otimes_E P^\vee \overset{\mathrm{ev}}{\to} A,$$

where $P^\vee$ is the E-A-bimodule $\mathrm{Hom}_A(P, A)$ and the last map is an evaluation map. We denote this composition by $g : P \otimes_E Y \otimes_E P^\vee \to A$. Then the cone X of g is a bounded complex of A-A-bimodules which are projective both on the left and on the right, and so it induces an endofunctor of the derived category of A.

Definition 3.1.1. Given a short exact sequence

$$0 \to F[-1] \hookrightarrow Y \overset{f}{\twoheadrightarrow} E \to 0$$

we have a functor

$$\Psi_{P,f} := X \otimes_A - : \mathrm{D}^{\mathrm{b}}(A) \to \mathrm{D}^{\mathrm{b}}(A).$$

The main result of [Gra1] states that if E is periodic and Y is a truncated resolution, so that F is a shifted twisted copy of E, then $\Psi_{P,f}$ is an autoequivalence. We call such an autoequivalence a *periodic twist* and write it as $\Psi_{P,Y}$, or just Ψ_P when our truncated resolution is minimal. For an arbitrary F, $\Psi_{P,f}$ will not be an autoequivalence, but the construction still gives us a well-defined endofunctor.

3.2. **Periodic twists acting on different derived categories.** Let $P = P_1 \oplus \ldots \oplus P_\ell$ be a projective A-module which is basic, i.e., it has no two isomorphic nonzero direct summands. Associate the idempotent e to P and e_i to P_i. Let $E_i = \mathrm{End}_A(P_i)^{\mathrm{op}} \cong e_iAe_i$ and $E = \mathrm{End}_A(P)^{\mathrm{op}} \cong eAe$.

Let $Q_i = \mathrm{Hom}_A(P, P_i) \cong eAe_i \cong Ee_i$ for $1 \leq i \leq \ell$. Then Q_i is an E-E_i-bimodule and is projective as a left E-module. In fact, up to isomorphism, all indecomposable projective E-modules are obtained in this way.

We collect some basic but important properties of our different projective modules.

Lemma 3.2.1. $\mathrm{End}_E(Q_i) \cong \mathrm{End}_A(P_i)$ *and* $P \otimes_E Q_i \cong P_i$.

Proof. We see that $\mathrm{End}_E(Q_i) \cong e_iEe_i \cong e_i(eAe)e_i = e_iAe_i \cong E_i$, and so $\mathrm{End}_E(Q_i) \cong \mathrm{End}_A(P_i)$. For the second statement, note that $Ae \otimes_E Ee_i \cong Ae_i$. □

Suppose that for each $1 \leq i \leq \ell$ we are given a short exact sequence of chain complexes

$$0 \to F_i[-1] \hookrightarrow Y_i \overset{f_i}{\twoheadrightarrow} E_i \to 0,$$

where $Y_i \in \mathrm{per}(E_i\text{-}E_i)$. We therefore have a distinguished triangle

$$Y_i \xrightarrow{f_i} E_i \to F_i \rightsquigarrow$$

in $\mathrm{D^b}(E_i\text{-}E_i)$, where we have used f_i to denote both a map in $\mathrm{Ch^b}(E_i\text{-}E_i)$ and its image in $\mathrm{D^b}(E_i\text{-}E_i)$.

As both the A-module P_i and the E-module Q_i have endomorphism algebra E_i, for each $1 \leq i \leq \ell$ we obtain two periodic twists

$$\Psi_{P_i} = X_i \otimes_A - : \mathrm{D^b}(A) \to \mathrm{D^b}(A)$$

and

$$\Psi_{Q_i} = W_i \otimes_E - : \mathrm{D^b}(E) \to \mathrm{D^b}(E),$$

where X_i and W_i are defined as the cones of

$$P_i \otimes_{E_i} Y_i \otimes_{E_i} P_i^\vee \xrightarrow{g_i} A$$

and

$$Q_i \otimes_{E_i} Y_i \otimes_{E_i} Q_i^\vee \xrightarrow{g_i'} E$$

respectively. The following lemma, whose proof follows from Lemma 3.2.1 and the definition of g_i and g_i', will be useful later:

Lemma 3.2.2. *The diagram*

$$\begin{tikzcd}
P_i \otimes_{E_i} Y_i \otimes_{E_i} P_i^\vee \arrow[r, "g_i"] \arrow[d, "\sim"] & A \\
P \otimes_E Q_i \otimes_{E_i} Y_i \otimes_{E_i} Q_i^\vee \otimes_E P^\vee \arrow[r, "P \otimes g_i' \otimes P^\vee"] & P \otimes_E P^\vee \arrow[u, "\mathrm{ev}_P"']
\end{tikzcd}$$

in $\mathrm{Ch^b}(A\text{-}A)$ *commutes.*

The complexes X_i and W_i fit into distinguished triangles

$$P_i \otimes_{E_i} Y_i \otimes_{E_i} P_i^\vee \xrightarrow{g_i} A \xrightarrow{h_i} X_i \overset{i_i}{\rightsquigarrow}$$

in $\mathrm{D^b}(A\text{-}A)$ and

$$Q_i \otimes_{E_i} Y_i \otimes_{E_i} Q_i^\vee \xrightarrow{g_i'} E \xrightarrow{h_i'} W_i \overset{i_i'}{\rightsquigarrow}$$

in $\mathrm{D^b}(E\text{-}E)$, with maps denoted as labelled. We have used the symbol i to mean two different things here: in one context it denotes a map and in another it denotes an integer. But there should be no confusion as the map i will always have a subscript and will never be used as a subscript.

Sometimes, to save space, we will write

$$V_i = P_i \otimes_{E_i} Y_i \otimes_{E_i} P_i^\vee \in \mathrm{D^b}(A\text{-}A)$$

and

$$Z_i = Q_i \otimes_{E_i} Y_i \otimes_{E_i} Q_i^\vee \in \mathrm{D^b}(E\text{-}E)$$

so that our distinguished triangles look like

$$V_i \xrightarrow{g_i} A \xrightarrow{h_i} X_i \overset{i_i}{\rightsquigarrow}$$

in $\mathrm{D^b}(A\text{-}A)$ and

$$Z_i \xrightarrow{g_i'} E \xrightarrow{h_i'} W_i \overset{i_i'}{\rightsquigarrow}$$

in $\mathrm{D^b}(E\text{-}E)$.

3.3. **The lifting theorem.** We want to show that, loosely, periodic twists that decompose "downstairs" (i.e., on $\mathrm{D^b}(E)$) decompose in the same way "upstairs" (i.e., on $\mathrm{D^b}(A)$). We carry over the notation from the previous subsection.

Lemma 3.3.1. *There exists an object $Y_{1,2}$ and three triangles*

$$Y_{1,2} \xrightarrow{f_{1,2}} E \xrightarrow{h_1'h_2'} W_1W_2 \rightsquigarrow,$$

$$Z_1 \xrightarrow{p_1'} Y_{1,2} \xrightarrow{q_1'} W_1Z_2 \overset{i_1'g_2'}{\rightsquigarrow},$$

$$Z_2 \xrightarrow{p_2'} Y_{1,2} \xrightarrow{q_2'} Z_1W_2 \overset{g_1'i_2'}{\rightsquigarrow}$$

in $\mathrm{D^b}(E\text{-}E)$ such that $g_1' = f_{1,2} \circ p_1'$ and $g_2' = f_{1,2} \circ p_2'$. The second and third of these triangles live in $\mathrm{per}(E\text{-}E)$.

Proof. Apply Corollary 2.4.1 to the triangles

$$Z_i \xrightarrow{g_i'} E \xrightarrow{h_i'} W_i \overset{i_i'}{\rightsquigarrow}$$

for $i = 1$ and 2.

We now show the second and third triangles are perfect. As W_1 and W_2 both have the property of being bounded in nonnegative degrees with E being the only module in its underlying chain complex which is not a projective E-E-bimodule, so does their tensor product. So the cone of $h_1'h_2' : E \to W_1W_2$ is perfect and hence so is $Y_{1,2}$. The other objects in the second and third triangles are clearly perfect. $\square$

The next proposition should be compared to Proposition 3.3.3 of [Gra1].

Proposition 3.3.2. *There is an isomorphism of triangles $P \otimes_E \Delta \cong \nabla \otimes_A P$ in $\mathrm{D^b}(A\text{-}E)$, where Δ and ∇ are the triangles*

$$Q_i \otimes_{E_i} Y_i \otimes_{E_i} Q_i^\vee \xrightarrow{g_i'} E \xrightarrow{h_i'} W_i \overset{i_i'}{\rightsquigarrow}$$

and

$$P_i \otimes_{E_i} Y_i \otimes_{E_i} P_i^\vee \xrightarrow{g_i} A \xrightarrow{h_i} X_i \overset{i_i}{\rightsquigarrow}$$

in $\mathrm{D^b}(E\text{-}E)$ and $\mathrm{D^b}(A\text{-}A)$, respectively.

Proof. We want a commutative square

$$\begin{tikzcd}
P \otimes_E Q_i \otimes_{E_i} Y_i \otimes_{E_i} Q_i^\vee \arrow[r, "P\otimes g_i'"] \arrow[d, "\sim"] & P \otimes_E E \arrow[d, "\sim"] \\
P_i \otimes_{E_i} Y_i \otimes_{E_i} P_i^\vee \otimes_A P \arrow[r, "g_i\otimes P"] & A \otimes_A P
\end{tikzcd}$$

where the vertical maps are isomorphisms, and then the proposition will follow by the 5-lemma for triangulated categories. We define these maps, and show that the square commutes, by considering the following diagram:

$$\begin{tikzcd}
P \otimes_E Q_i \otimes_{E_i} Y_i \otimes_{E_i} Q_i^\vee \arrow[r] \arrow[d, "\sim"] & P \otimes_E Q_i \otimes_{E_i} E_i \otimes_{E_i} Q_i^\vee \arrow[r] \arrow[d, "\sim"] & P \otimes_E Q_i \otimes_{E_i} Q_i^\vee \arrow[r] \arrow[d, "\sim"] & P \otimes_E E \arrow[d, "\sim"] \\
P_i \otimes_{E_i} Y_i \otimes_{E_i} Q_i^\vee \arrow[r] \arrow[d, "\sim"] & P_i \otimes_{E_i} E_i \otimes_{E_i} Q_i^\vee \arrow[r] \arrow[d, "\sim"] & P_i \otimes_{E_i} Q_i^\vee \arrow[d, "\sim"] & P \arrow[d, "\sim"] \\
P_i \otimes_{E_i} Y_i \otimes_{E_i} P_i^\vee \otimes_A P \arrow[r] & P_i \otimes_{E_i} E_i \otimes_{E_i} P_i^\vee \otimes_A P \arrow[r] & P_i \otimes_{E_i} P_i^\vee \otimes_A P \arrow[r] & A \otimes_A P
\end{tikzcd}$$

The two leftmost squares commute by the naturality of tensoring with an isomorphism, and one can easily check in the module category that the remaining squares and the one hexagon commute. □

Corollary 3.3.3. $P^\vee \otimes_A \nabla \otimes_A P \cong \Delta$ *in* $\mathrm{D^b}(E\text{-}E)$.

Proof. $P^\vee \otimes_A \nabla \otimes_A P \cong P^\vee \otimes_A P \otimes_E \Delta \cong E \otimes_E \Delta \cong \Delta$. □

Lemma 3.3.4. *There are two triangles*

$$PY_{1,2}P^\vee \xrightarrow{s} A \to X_1X_2 \rightsquigarrow$$

and

$$V_1V_2 \xrightarrow{\left(\begin{smallmatrix} V_1 g_2 \\ -g_1 V_2 \end{smallmatrix}\right)} V_1 \oplus V_2 \xrightarrow{(p_1,p_2)} PY_{1,2}P^\vee \rightsquigarrow$$

in $\mathrm{D^b}(A\text{-}A)$ such that, for $i \in \{1,2\}$, $s \circ p_i = g_i$ and, after making the obvious identifications, $p_i = Pp_i'P^\vee$.

Proof. We use Corollary 2.4.1 again: take the two distinguished triangles

$$V_i \xrightarrow{g_i} A \xrightarrow{h_i} X_i \overset{i_i}{\rightsquigarrow}$$

for $i = 1, 2$ and we get distinguished triangles

$$\bar{Y} \xrightarrow{\bar{s}} AA \xrightarrow{h_1 h_2} X_1X_2 \rightsquigarrow,$$

$$V_1A \xrightarrow{\bar{p}_1} \bar{Y} \xrightarrow{\bar{q}_1} X_1V_2 \overset{i_1 g_2}{\rightsquigarrow},$$

$$AV_2 \xrightarrow{\bar{p}_2} \bar{Y} \xrightarrow{\bar{q}_2} V_1X_2 \overset{g_1 i_2}{\rightsquigarrow},$$

where we have labelled the map $\bar{Y} \to AA$ as $\bar{s}$.

Consider the distinguished triangle

$$V_1 A \xrightarrow{\bar{p}_1} \bar{Y} \xrightarrow{\bar{q}_1} X_1 V_2 \rightsquigarrow$$

from above. The triangle

$$Z_1 \xrightarrow{p_1'} Y_{1,2} \xrightarrow{q_1'} W_1 Z_2 \overset{r_1'}{\rightsquigarrow}$$

from Lemma 3.3.1 lives in per(E-E), so we can apply $P \otimes_E - \otimes_E P^\vee$ to it (without deriving any functors) to obtain a new distinguished triangle in $\mathrm{D^b}(A\text{-}A)$. Then we want to use isomorphisms as follows

$$\begin{tikzcd}
V_1 A \arrow[r, "\bar{p}_1"] \arrow[d, "\sim"] & \bar{Y} \arrow[r, "\bar{q}_1"] \arrow[d, dashed] & X_1 V_2 \arrow[r, "i_1 g_2"] \arrow[d, "\sim"] & V_1 A[1] \arrow[d, "\sim"] \\
PZ_1P^\vee \arrow[r, "Pp_1'P^\vee"] & P_1 Y_{1,2} P_1^\vee \arrow[r, "Pq_1'P^\vee"] & PW_1Z_2P^\vee \arrow[r, "Pi_1'g_2'P^\vee"] & PZ_1P^\vee[1]
\end{tikzcd}$$

in order to show $\bar{Y} \cong PY_{1,2}P^\vee$. This square lives in the full subcategory of $\mathrm{D^b}(A\text{-}A)$ generated by $P \otimes_k P^\vee$, which we will temporarily denote $\mathrm{D^b}(P\text{-}P^\vee)$. The functor

$$P^\vee \otimes_A - \otimes_A P : \mathrm{D^b}(A\text{-}A) \to \mathrm{D^b}(E\text{-}E)$$

restricts to an equivalence on $\mathrm{D^b}(P\text{-}P^\vee)$. By Corollary 3.3.3 we have that $P^\vee \otimes_A g_i \otimes_A P = g_i'$ and $P^\vee \otimes_A i_i \otimes_A P = i_i'$, so the square is sent to a commutative square after applying $P^\vee \otimes_A - \otimes_A P$; hence it must have been commutative to start with.

Now we have a map $\bar{s} : \bar{Y} \to A \otimes_A A$ and an isomorphism $\bar{Y} \cong PY_{1,2}P^\vee$, so we use s to label the composition

$$s : PY_{1,2}P^\vee \to \bar{Y} \to A \otimes_A A \xrightarrow{\sim} A.$$

Then we define the distinguished triangle

$$PY_{1,2}P^\vee \xrightarrow{s} A \to X_1 X_2 \rightsquigarrow$$

by the commutative diagram

$$\begin{tikzcd}
PY_{1,2}P^\vee \arrow[r, "s"] \arrow[d, "\sim"] & A \arrow[r] \arrow[d, "\sim"] & X_1X_2 \arrow[r, squiggly] \arrow[d, equal] & {} \\
\bar{Y} \arrow[r, "\bar{s}"] & AA \arrow[r] & X_1X_2 \arrow[r, squiggly] & {}
\end{tikzcd}$$

We also get the Mayer-Vietoris triangle

$$V_1 V_2 \xrightarrow{\left(\begin{smallmatrix} V_1 g_2 \\ -g_1 V_2 \end{smallmatrix}\right)} V_1 \oplus V_2 \xrightarrow{(p_1, p_2)} PY_{1,2}P^\vee \rightsquigarrow$$

from Corollary 2.4.2, where p_i is the composition

$$V_i \xrightarrow{\sim} V_i A \xrightarrow{\bar{p}_i} \bar{Y} \xrightarrow{\sim} PY_{1,2}P^\vee$$

and similarly for q_i.

One can check that the equations $s \circ p_i = g_i$ follow from $\bar{s} \circ \bar{p}_1 = g_1 \otimes_A A$ and $\bar{s} \circ \bar{p}_2 = A \otimes_A g_2$, and it is clear from our earlier isomorphism of triangles that

$p_i = Pp_i'P^\vee$, i.e., the diagram

$$\begin{tikzcd}
V_i \arrow[r, "p_i"] \arrow[d, "\sim"] & PY_{1,2}P^\vee \\
V_iA \arrow[r, "\bar{p}_i"] \arrow[d, "\sim"] & \bar{Y} \arrow[u, "\sim"'] \arrow[d, "\sim"] \\
PZ_1P^\vee \arrow[r, "Pp_1'P^\vee"] & P_1Y_{1,2}P_1^\vee
\end{tikzcd}$$

commutes. □

Proposition 3.3.5. $\Psi_{P_1,f_1} \circ \Psi_{P_2,f_2} = \Psi_{P,f_{1,2}}$.

Proof. Let $X_{1,2}$ be the cone of the composition $PY_{1,2}P^\vee \overset{Pf_{1,2}P^\vee}{\to} PP^\vee \overset{\mathrm{ev}}{\to} A$. Then we want to show that $X_{1,2} \cong X_1X_2$. As usual, we prove this by constructing a commutative diagram

$$\begin{tikzcd}
PY_{1,2}P^\vee \arrow[rr, "s"] \arrow[d, equal] & & A \arrow[d, equal] \\
PY_{1,2}P^\vee \arrow[dr, "Pf_{1,2}P^\vee"] & & A \\
& PP^\vee \arrow[ur, "\mathrm{ev}_P"] &
\end{tikzcd}$$

and appealing to the triangulated 5-lemma, so we need to show that the map

$$v = s - Pf_{1,2}P^\vee \circ \mathrm{ev}_P : PY_{1,2}P^\vee \to A$$

is zero. We argue more indirectly this time.

By the previous lemma, we have a distinguished triangle

$$\begin{tikzcd}
& V_1 \arrow[dr, "p_1"] & & \\
V_1V_2 \arrow[ur, "V_1g_2"] \arrow[dr, "-g_1V_2"'] & \oplus & PY_{1,2}P^\vee \arrow[r, squiggly] & {} \\
& V_2 \arrow[ur, "p_2"'] & &
\end{tikzcd}$$

in $\mathrm{D^b}(A\text{-}A)$. Suppose we can show that $(p_1, p_2) \circ v = 0$. Then the commutative diagram

$$\begin{tikzcd}
V_1 \arrow[dr, "p_1"] \arrow[ddd, bend right] & & & \\
\oplus & PY_{1,2}P^\vee \arrow[r] \arrow[dd, "v"] & V_1V_2[1] \arrow[r, squiggly] \arrow[dd, dashed] & {} \\
V_2 \arrow[ur, "p_2"'] \arrow[d] & & & \\
0 \arrow[r] & A \arrow[r, equal] & A \arrow[r, squiggly] & {}
\end{tikzcd}$$

and the completion axiom for triangulated categories shows that v must factor through $V_1V_2[1]$. But A is concentrated in degree 0, and $V_1V_2[1]$ is concentrated in strictly negative degrees, so v must be zero.

It remains to show that $(p_1, p_2) \circ v = 0$, or, equivalently, that for $i = 1, 2$,

$$p_i \circ s = p_i \circ P f_{1,2} P^\vee \circ \mathrm{ev}_P \,.$$

By the previous lemma, the left hand side is g_i and the right hand side is $P(p_i' \circ f_{1,2})P^\vee \circ \mathrm{ev}_P$, which by Lemma 3.3.1 is equal to $P g_i' P^\vee \circ \mathrm{ev}_P$. But this is equal to g_i, as Lemma 3.2.2 tells us that the corresponding statement is true in the chain complex category. □

If $\mathbf{i} = (i_1, \dots, i_r) \in \{1, \dots, \ell\}^r$, then we will write $\Psi_{\mathbf{i}} = \Psi_{Pi_1}\Psi_{Pi_2}\dots\Psi_{Pi_r} : \mathrm{D^b}(A) \to \mathrm{D^b}(A)$ and similarly for $\Psi'_{\mathbf{i}} : \mathrm{D^b}(E) \to \mathrm{D^b}(E)$. Recall that $P = P_1 \oplus \dots \oplus P_\ell$ and let $Q = Q_1 \oplus \dots \oplus Q_\ell$.

Theorem 3.3.6 (Lifting theorem)**.** *Suppose we have a short exact sequence*

$$0 \to F[-1] \hookrightarrow Y \overset{f}{\twoheadrightarrow} E \to 0$$

in $\mathrm{Ch^b}(E\text{-}E)$*, with* $Y \in \mathrm{per}(E\text{-}E)$*, and we have a natural isomorphism* $\Psi'_{\boldsymbol{i}} \cong F \otimes^{\boldsymbol{L}}_E -$ *of functors. Then*

$$\Psi_{\boldsymbol{i}} \cong \Psi_{P,f}.$$

In particular,

(i) *if* $\Psi'_{\boldsymbol{i}} \cong \Psi'_{\boldsymbol{j}}$ *for* $\boldsymbol{j} = (i_1, \dots, i_r) \in \{1, \dots, \ell\}^r$*, then* $\Psi_{\boldsymbol{i}} \cong \Psi_{\boldsymbol{j}}$*, and*

(ii) *if* $\Psi'_{\boldsymbol{i}} \cong E_\sigma[n] \otimes_E -$*, so* E *is twisted-periodic, then* $\Psi_{\boldsymbol{i}}$ *is isomorphic to the periodic twist* Ψ_P*.*

Proof. This follows from the previous proposition by induction. □

4. Braid relations and longest elements of symmetric groups

Suppose we have a braid group acting by spherical twists on the derived category of a symmetric algebra. Then using the lifting theorem developed in the previous section, we will show that lifts of longest elements from symmetric groups to braid groups act in the way suggested by the example at the end of [Gra1].

4.1. **Brauer tree algebras of lines without multiplicity.** We define a collection of algebras Γ_n, $n \geq 1$, as path algebras of quivers with relations. Let $\Gamma_1 = k[x]/\langle x^2 \rangle$ and let $\Gamma_2 = kQ_2/I_2$, where Q_2 is the quiver

$$Q_2 \quad = \quad \begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] & 2 \arrow[l, bend left, "\beta"] \end{tikzcd}$$

and I_2 is the ideal generated by $\alpha\beta\alpha$ and $\beta\alpha\beta$. For $n \geq 3$, let $\Gamma_n = kQ_n/I_n$ where Q_n is the quiver

$$Q_n \quad = \quad \begin{tikzcd} 1 \arrow[r, bend left, "\alpha_1"] & 2 \arrow[l, bend left, "\beta_2"] \arrow[r, bend left, "\alpha_2"] & \cdots \arrow[l, bend left, "\beta_3"] \arrow[r, bend left, "\alpha_{n-1}"] & n \arrow[l, bend left, "\beta_n"] \end{tikzcd}$$

and I_n is the ideal generated by $\alpha_{i-1}\alpha_i$, $\beta_{i+1}\beta_i$, and $\alpha_i\beta_{i+1} - \beta_i\alpha_{i-1}$ for $2 \leq i \leq n-1$. For ease of notation, let $\alpha = \sum \alpha_i$ and $\beta = \sum \beta_i$. Then we can write α_i and β_j as $e_i\alpha$ or αe_{i+1} and $e_j\beta$ or βe_{j-1}.

The algebras Γ_n have appeared in many contexts. Some examples are:

- Γ_n is the trivial extension algebra of the path algebra of a Dynkin quiver of type A_n with bipartite orientation [BBK, HK];
- Γ_n is the quadratic dual to the preprojective algebra of type A_n with the path-length grading, for $n > 2$ [BBK, HK];
- Γ_n is a Brauer tree algebra of a line

  $$\circ \text{———} \circ \text{———} \cdots \text{———} \circ \text{———} \circ$$

  with n edges and no exceptional vertex; up to derived equivalence, these are all Brauer tree algebras with multiplicity function the constant 1 [RZ];
- Γ_n is the zig-zag algebra of type A_n, which was used by Huerfano and Khovanov to categorify the adjoint representation of a type A quantum group [HK];
- Γ_n is isomorphic to the underlying ungraded algebra of the formal differential graded ext-algebras of an A_n-configuration of spherical objects [ST].

The relations for Γ_n are homogeneous, and there are various possible ways to put a grading on these algebras (see, for example, [ST, Section 4]). When we want to consider them as graded algebras, we will give x in Γ_1 degree 2, and for $n \geq 2$, all α_i and β_j will have degree 1, as in [HK].

Let e_i denote the primitive idempotent corresponding to the vertex i of Q_n. We have an algebra automorphism $\tau_n \in \mathrm{Aut}(\Gamma_n)$ of order 2: τ_1 sends x to $-x$, τ_2 swaps e_1 and e_2 and α and β, and for $n > 2$, τ_n sends the idempotent e_i to e_{n+1-i} and swaps α_i and β_{n+1-i}. Note that these automorphisms respect our grading.

Let P_i denote the projective Γ_n-module $\Gamma_n e_i$. With our grading conventions the following result, which is easy to prove, is true in both the graded and ungraded setting:

Lemma 4.1.1. *Let $1 \leq i \leq j \leq n$. Then*

$$\mathrm{End}_{\Gamma_n}(P_i \oplus P_{i+1} \oplus \ldots \oplus P_j)^{\mathrm{op}} \cong \Gamma_{j-i+1}.$$

In particular, for $1 \leq i \leq n$,

$$\mathrm{End}_{\Gamma_n}(P_i) \cong k[x]/\langle x^2 \rangle.$$

4.2. **Spherical twists for symmetric algebras.** Let A be a symmetric algebra.

Definition 4.2.1. We say that a projective A-module is *spherical* if $\mathrm{End}_A(P) \cong k[x]/\langle x^2 \rangle$.

By Lemma 4.1.1, for $n \geq 1$, each indecomposable projective Γ_n-module is spherical.

Theorem 4.2.2 ([RZ] for the algebras Γ_n; [ST] in general)**.** *If P is a projecive A-module which is spherical, then the functor F_P given by tensoring with the complex*

$$P \otimes_k P^\vee \xrightarrow{\mathrm{ev}} A$$

of A-A-bimodules concentrated in degrees 1 and 0 is a derived autoequivalence.

These equivalences are called spherical twists. Note that they are special cases of periodic twists.

Recall that the braid group B_{n+1} on $n+1$ letters has the presentation

$$B_{n+1} = \langle\, \sigma_1, \dots, \sigma_n \mid \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| > 1; \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } 1 \le i < n \,\rangle\,.$$

Definition 4.2.3 ([ST])**.** We say that a collection $\{P_1, \dots, P_n\}$ of projective A-modules is an A_n*-configuration* if each P_i is spherical and, for all $1 \le i, j \le n$,

$$\dim_k \operatorname{Hom}_A(P_i, P_j) = \begin{cases} 1 & \text{if } |i-j| = 1, \\ 0 & \text{if } |i-j| > 1. \end{cases}$$

We have the following observation, which is straightforward in our setting, and is a special case of the more general statement [ST, Lemma 4.10]:

Lemma 4.2.4. *Let A be a symmetric algebra. A collection $\{P_1, \dots, P_n\}$ of projective A-modules is an A_n-configuration if and only if*

$$\operatorname{End}_A(\bigoplus_{i=1}^{n} P_i)^{\mathrm{op}} \cong \Gamma_n.$$

Theorem 4.2.5 ([RZ] for the algebras Γ_n; [ST] in general)**.** *If the collection $\{P_1, \dots, P_n\}$ of projective A-modules is an A_n-configuration, then we have an action of the braid group on the derived category of A,*

$$B_{n+1} \to \operatorname{Aut}(\mathrm{D^b}(A)),$$

which sends the braid group generator σ_i to the spherical twist F_i associated to the projective A-module P_i.

As a corollary of the lifting theorem, Theorem 3.3.6, we obtain a new proof that the spherical twists associated to an A_n-configuration satisfy the braid relations.

Proof of Theorem 4.2.5. It is easy to see from the definitions that if $|i-j| > 1$, then $F_iF_j \cong F_jF_i$. The hard part is to show that $F_iF_{i+1}F_i \cong F_{i+1}F_iF_{i+1}$ for $1 \le i \le n-1$. But one can check directly that $F_1F_2F_1 \cong F_2F_1F_2$ on the derived category of $\Gamma_2 \cong \operatorname{End}_A(P_i \oplus P_{i+1})$ and, by the lifting theorem, this is enough. $\square$

The algebras Γ_n are of finite representation type, and hence are twisted periodic, but in fact we can say more.

Theorem 4.2.6 ([BBK])**.** *The algebra Γ_n is twisted periodic with period n and automorphism τ_n.*

A natural question is: what do the associated periodic twists look like? It was noted in [Gra1] that periodic twists on Γ_3 associated to the direct sum of the first two projectives, which have endomorphism algebra Γ_2, are isomorphic to the composition $F_1F_2F_1$ of spherical twists. We will show that this pattern continues.

4.3. **Longest elements.** The symmetric group S_{n+1} is the group of automorphisms of the set $\{1, 2, \dots, n+1\}$. It has a standard generating set consisting of the tranpositions s_i which interchange the numbers i and $i+1$. Note that these generators are involutions. There is a minimal number of letters from the alphabet $\{s_i\}_{i=1}^n$ needed to write a given element w of S_{n+1}; this number is called the length of w. There is a unique element of S_{n+1} of longest length, called the longest element. It sends $i \in \{1, 2, \dots, n+1\}$ to $n+2-i$. We write it as w_0 or, when we need to make explicit the dependence on n, as $w_0^{(n+1)}$.

A reduced expression for $w \in S_{n+1}$ is a way to write w using the minimal number of standard generators. There are different reduced expressions for the longest element. We give one inductive way to write the reduced expression: let $w_0^{(1)}$ be the identity element of the identity group S_1, and then

$$w_0^{(n+1)} = w_0^{(n)} s_n \dots s_2 s_1$$

is the longest element of S_{n+1}. Here, we have used the embedding $S_n \hookrightarrow S_{n+1}$ which sends $s_i \in S_n$ to $s_i \in S_{i+1}$ to consider $w_0^{(n)}$ as an element of S_{n+1}, and we are employing a slight abuse of notation in using $w_0^{(n)}$ to represent both an element of the group S_n and a word in the alphabet $\{s_i\}_{i=1}^{n-1}$.

The symmetric group has a presentation in terms of the generators $\{s_i\}_{i=1}^n$ as follows:

$$S_{n+1} = \langle\, s_1, \dots, s_n \mid s_i s_j = s_j s_i \text{ for } |i-j| > 1; \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \text{ for } 1 \le i < n;\ s_i^2 = 1 \,\rangle\,.$$

There is an obvious surjective group homomorphism $B_{n+1} \twoheadrightarrow S_{n+1}$ from the braid group to the symmetric group which sends σ_i to s_i. By sending s_i to σ_i we obtain a positive lift of each word in the alphabet $\{s_i\}_{i=1}^n$. By Matsumoto's Theorem [Mat], we can lift each element of S_{n+1} to B_{n+1} using reduced expressions, and this is independent of our particular choice of reduced expression. We denote the positive lift of $w_0^{(n+1)} \in S_{n+1}$ by t_{n+1}.

Consider the algebras Γ_n described in Subsection 4.1. By Lemma 4.1.1, each projective Γ_n-module P_i has an endomorphism algebra $k[x]/\langle x^2 \rangle$, and so we have a spherical twist given by the period 1 twisted resolution of $E_i = \mathrm{End}_{\Gamma_n}(P_i)^{\mathrm{op}}$, which we label $F_i : \mathrm{D^b}(\Gamma_n) \xrightarrow{\sim} \mathrm{D^b}(\Gamma_n)$. By Lemma 4.2.4 the indecomposable projective Γ_n-modules form an A_n-configuration, and so we have a braid group action

$$\varphi_n : B_{n+1} \to \mathrm{Aut}(\mathrm{D^b}(\Gamma_n))$$
$$\sigma_i \mapsto F_i.$$

It is natural to ask what the image of t_{n+1} is under φ_n. The answer was given by Rouquier and Zimmermann:

Theorem 4.3.1 ([RZ, Theorem 4.5])**.** *The longest element t_{n+1} of B_{n+1} acts on $\mathrm{D^b}(\Gamma_n)$ as the shift and twist $-_{\tau_n}[n]$.*

Note that a related result on the action of t_{n+1}^2 appears as Lemma 3.1 in [Sei].

We can combine the lifting theorem (Theorem 3.3.6) and Theorem 4.3.1 to answer the more general question for an arbitrary symmetric algebra A: given an A_n-configuration, how does the longest element t_{n+1} act on $\mathrm{D^b}(A)$?

Corollary 4.3.2. *For a symmetric algebra A and an A_n-configuration $\{P_1, \dots, P_n\}$, the longest element t_{n+1} acts as the periodic twist Ψ_P, where $P = P_1 \oplus \dots \oplus P_n$.*

4.4. **Quadratic algebras and Koszul complexes.** This section recounts well-known ideas and constructions from Koszul duality ([Pri], [BGS], [BG], [BuKi], ...), but we need to work with arbitrary quadratic algebras instead of only Koszul algebras (see [MOS]). Our aim is to use this theory, as well as the theory of almost Koszul duality due to Brenner-Butler-King [BBK], to further study the algebras Γ_n.

Let $\Lambda = \bigoplus_{i\in\mathbb{Z}} \Lambda_i$ be a $\mathbb{Z}$-graded k-algebra. We will assume that Λ is positively graded and generated in degree 1, i.e., $\Lambda = \bigoplus_{i\geq 0} \Lambda_i$ and, for $i > 0$, $\Lambda_i = \Lambda_1\Lambda_{i-1} = \Lambda_{i-1}\Lambda_1$. A Λ-module $M = \bigoplus_{i\in\mathbb{Z}} M_i$ is graded if $\Lambda_i M_j \subseteq M_{i+j}$ for all i, j. Let Λ-grmod denote the category of finitely generated graded Λ-modules, in which all maps $f : M \to N$ are homogeneous of degree 0, i.e., $f(M_i) \subseteq N_i$. For each $n \in \mathbb{Z}$ there is an autoequivalence $\{n\}$ of Λ-grmod which sends M to the module $M\{n\}$ with $M\{n\}_i = M_{n+i}$. If each M_i is finite dimensional over k, then we write M^* for $\bigoplus_{i\in\mathbb{Z}}(M_i)^*$. Note that $(M^*)_i = (M_{-i})^*$.

Definition 4.4.1. We say that Λ is *Frobenius of Gorenstein parameter* n if we have an isomorphism $\varphi : \Lambda \xrightarrow{\sim} {}_\nu(\Lambda^*)\{-n\}$ in the category Λ-grmod-Λ of finitely generated graded Λ-Λ-bimodules for some graded algebra automorphism $\nu \in \mathrm{Aut}(\Lambda)$, called the *Nakayama automorphism*. If moreover ν is the identity, we say that Λ is *symmetric of Gorenstein parameter* n.

Let S be a semisimple k-algebra S which, for simplicity, we will assume is basic, i.e., it is a product of copies of the field k. Let V be an S-S-bimodule. Then recall that the tensor algebra $\mathrm{Tens}_S(V)$ is a positively graded algebra

$$\mathrm{Tens}_S(V) = \bigoplus_{i\geq 0} V^{\otimes_S i},$$

where $V^{\otimes_S 0} = k$ and $V^{\otimes_S i} = V \otimes_S V \otimes_S \cdots \otimes_S V$ with i factors. The multiplication in $\mathrm{Tens}_S(V)$ is given by the obvious concatenation.

Definition 4.4.2. We say Λ is *quadratic* if there is a semisimple k-algebra S, an S-S-bimodule V, and a subset $R \subset V \otimes_S V$ such that $\Lambda \cong \mathrm{Tens}_S(V)/(R)$.

Note that as (R) is a homogeneous ideal, quadratic algebras inherit a positive grading from the tensor algebra.

For the rest of this subsection, assume Λ is quadratic. The quadratic dual $\Lambda^!$ is defined as

$$\Lambda^! = \mathrm{Tens}_S(V^*)/(R^\perp),$$

where $R^\perp = \{f \in (V \otimes_S V)^* \mid f(R) = 0\}$ is the perpendicular space to R and, for $V, W \in S$-grmod-S, we identify the S-S-bimodules $(V \otimes_S W)^*$ and $W^* \otimes_S V^*$. $\Lambda^!$ is also a quadratic algebra, and $(\Lambda^!)^! \cong \Lambda$.

If we have a graded algebra automorphism $\tau : \Lambda \xrightarrow{\sim} \Lambda$, then restricting to degrees 0 and 1 gives an algebra automorphism $\tau_0 : S \xrightarrow{\sim} S$ and a vector space automorphism $\tau_1 : V \xrightarrow{\sim} V$. We have a k-algebra map $\tau_0^! := \tau_0 : S \xrightarrow{\sim} S$, and taking the dual of the inverse of τ_1 gives us a vector space map $\tau_1^! : V^* \xrightarrow{\sim} V^*$. These maps generate an algebra automorphism $\tau^! : \Lambda^! \xrightarrow{\sim} \Lambda^!$, and we have $(\tau^!)^! = \tau$.

Let $M = \bigoplus M_i \in \Lambda^!$-grmod-$\Lambda^!$ be a graded $\Lambda^!$-$\Lambda^!$-bimodule, so we have left and right actions $\Lambda^! \otimes_S M \to M$ and $M \otimes_S \Lambda^! \to M$. Restricting to $(\Lambda^!)_1 = V^*$ gives maps $\ell_M : V^* \otimes_S M_{i-1} \to M_i$ and $r_M : M_{i-1} \otimes_S V^* \to M_i$, which have duals $\ell_M^* : (M_i)^* \to (M_{i-1})^* \otimes_S V$ and $r_M^* : (M_i)^* \to V \otimes_S (M_{i-1})^*$, where we have used the natural isomorphism $V^{**} \cong V$ to identify V and its double dual.

We say that $M \in \Lambda$-grmod is generated in degree i if there exists some subset $L \subset M_i$ such that $M = \Lambda L$. Let Λ-grproj-Λ denote the additive category of graded projective Λ-Λ-bimodules and let $\mathrm{lin}(\Lambda$-grproj-$\Lambda)$ denote the corresponding category of linear complexes: this is the category of chain complexes

$$\cdots \to X_1 \to X_0 \to X_{-1} \to \cdots,$$

where $X_i \in \Lambda\text{-grproj-}\Lambda$ is generated in degree i and all maps are homogeneous of degree 0.

Definition 4.4.3. There is a contravariant functor

$$Q : \Lambda^!\text{-grmod-}\Lambda^! \to \mathrm{lin}(\Lambda\text{-grproj-}\Lambda)$$

which is defined as follows: Q sends $M = \bigoplus_{i\in\mathbb{Z}} M_i \in \Lambda^!\text{-grmod-}\Lambda^!$ to the complex

$$\cdots \to Q(M)_1 \xrightarrow{d} Q(M)_0 \xrightarrow{d} Q(M)_{-1} \to \cdots$$

with $Q(M)_i = \Lambda \otimes_S (M_i)^* \otimes_S \Lambda\{-i\}$ (where we consider M_i as an S-S-bimodule concentrated in grade 0) and differential d given by the following composition:

$$\begin{tikzcd}
 & \Lambda(M_{i-1})^* V \Lambda\{-i+1\} \arrow[dr, "\Lambda(M_{i-1})^* m"] & \\
\Lambda(M_i)^*\Lambda\{-i\} \arrow[ur, "\Lambda \ell_M^* \Lambda"] \arrow[dr, "(-1)^i \Lambda r_M^* \Lambda"'] & & \Lambda(M_{i-1})^*\Lambda\{-i+1\} \\
 & \Lambda V(M_{i-1})^*\Lambda\{-i+1\} \arrow[ur, "m(M_{i-1})^*\Lambda"'] &
\end{tikzcd}$$

Here, $m : \Lambda \otimes_S \Lambda \to \Lambda$ is the algebra multiplication map and in applying m we have implicitly used the inclusion $V \hookrightarrow \Lambda$.

This construction does give a chain complex because the differential is defined so that, of the four ways an element can be mapped from $P_{i+1}\{i+1\}$ to $P_{i-1}\{i-1\}$, two of these are zero by definition of the quadratic dual and the other two cancel out due to the choice of sign. The contravariance of Q comes from the contravariance of $(-)^*$, and the functoriality comes from the fact that bimodule maps give commuting chain complex maps.

In the case when Q is Koszul, the functor Q can be extended to complexes of $\Lambda^!$-$\Lambda^!$-bimodules and we recover the Koszul duality functor of [BGS, Theorem 2.12.1].

Note that $\Lambda^!\text{-grmod-}\Lambda^!$ and $\mathrm{lin}(\Lambda\text{-grproj-}\Lambda)$ are both abelian categories.

Lemma 4.4.4. *Q is an exact functor; i.e., a short exact sequence*

$$0 \to K \hookrightarrow L \twoheadrightarrow M \to 0$$

in $\Lambda^!\text{-grmod-}\Lambda^!$ *is sent to a short exact sequence*

$$0 \to Q(M) \hookrightarrow Q(L) \twoheadrightarrow Q(K) \to 0$$

in $\mathrm{lin}(\Lambda\text{-grproj-}\Lambda)$.

Proof. This follows from the exactness of tensoring over the semisimple algebra S. □

Q plays nicely with various operations we can perform on bimodules.

Proposition 4.4.5. *Let $M \in \Lambda^!\text{-grmod-}\Lambda^!$. The functor Q has the following properties:*

(i) *For $i \in \mathbb{Z}$, $Q(M\{i\}) = Q(M)\{i\}[-i]$.*
(ii) *Let $\tau^! \in \mathrm{Aut}(\Lambda^!)$. Then $Q(M_{\tau^!}) \cong {}_\tau Q(M)$.*

(iii) *If Λ is a graded Frobenius algebra of Gorenstein parameter n, then there is a natural isomorphism $Q(M^*) \cong {}_\nu(Q(M)^*)_{\nu^{-1}}\{-2n\}$. In particular, if Λ is symmetric, then $Q(M^*) \cong Q(M)^*\{-2n\}$.*

Proof. (i) This is easy to check, and was noted in [BGS, Theorem 2.12.5(ii)].

(ii) For notational simplicity, let $N = M_{\tau^!}$. Then we have an isomorphism of Λ-Λ-bimodules

$$\tau \otimes_S \mathrm{id} \otimes_S \mathrm{id} : \Lambda \otimes_S (N_i)^* \otimes_S \Lambda \xrightarrow{\sim} {}_\tau\Lambda \otimes_S (M_i)^* \otimes_S \Lambda.$$

To see that this commutes with the differential, use the fact that τ is an algebra homomorphism and note that $r_N = r_M \circ (\mathrm{id} \otimes \tau_1^!)$, so $r_N^* = (\tau_1^{-1} \otimes \mathrm{id}) \circ r_M^*$.

(iii) First, assume that Λ is graded symmetric, i.e., we are given an isomorphism $\varphi : \Lambda \xrightarrow{\sim} \Lambda^*\{-n\}$ of Λ-Λ-bimodules. We have an isomorphism

$$\varphi \otimes \mathrm{id}\varphi^*\{-n\} : Q(M^*)_i \cong (Q(M)^*)_i\{-2n\}$$

for each $i \in \mathbb{Z}$, so we just need to show that the square

$$\begin{tikzcd}
\Lambda(M_i)^*\Lambda\{i\} \arrow[r, "d"] \arrow[d, "\varphi.1.\varphi^*\{-n\}"] & \Lambda(M_{i+1})^*\Lambda\{i+1\} \arrow[d, "\varphi.1.\varphi^*\{-n\}"] \\
\Lambda^*(M_i)^*\Lambda^*\{i-2n\} \arrow[r, "d^*"] & \Lambda^*(M_{i+1})^*\Lambda^*\{i+1-2n\}
\end{tikzcd}$$

commutes. All our maps will be homogeneous of degree 0, so we will drop the gradings from the notation.

Recall that $d = d^\ell + (-1)^i d^r : Q(M)_i \to Q(M)_{i-1}$, where $d^\ell = (1.1.m) \circ (1.\ell_M^*.1)$. We will show that $(\varphi.1.\varphi^*) \circ d^\ell = (d^r)^* \circ (\varphi.1.\varphi^*)$; the corresponding statement with d^ℓ and d^r interchanged is similar.

Let's draw the diagram for $(\varphi.1.\varphi^*) \circ d^\ell = (d^r)^* \circ (\varphi.1.\varphi^*)$:

$$\begin{tikzcd}
\Lambda(M_{-i})\Lambda \arrow[r, "1.\ell_{M^*}^*.1"] \arrow[d, "\varphi.1.\varphi^*"] & \Lambda(M_{-i+1})V\Lambda \arrow[r, "1.1.m"] & \Lambda(M_{-i+1})\Lambda \arrow[d, "\varphi.1.\varphi^*"] \\
\Lambda^*(M_{-i})\Lambda^* \arrow[r, "1.1.m^*"] & \Lambda^*(M_{-i})V^*\Lambda^* \arrow[r, "1.r_M.1"] & \Lambda^*(M_{-i+1})\Lambda^*
\end{tikzcd}$$

By definition, we have

$$\ell_{M^*} = (\mathrm{ev}_V.1) \circ (1.r_M^*) : V^* \otimes_S (M^*)_i \to (M^*)_{i+1},$$

and it is clear that the diagram commutes in the first tensor factor, so we need to show that

$$\begin{tikzcd}
(M_{-i})\Lambda \arrow[r, "1.\mathrm{ev}^*.1"] \arrow[dd, "1.\varphi^*"] & (M_{-i})V^*V\Lambda \arrow[rr, "r_M.1.1"] \arrow[dr, "1.1.m"] & & (M_{1-i})V\Lambda \arrow[dr, "1.m"] & \\
& & (M_{-i})V^*\Lambda \arrow[rr, "r_M.1"] \arrow[dl, "1.1.\varphi^*"] & & (M_{1-i})\Lambda \arrow[dl, "1.\varphi^*"] \\
(M_{-i})\Lambda^* \arrow[r, "1.m^*"] & (M_{-i})V^*\Lambda^* \arrow[rr, "r_M.1"] & & (M_{1-i})\Lambda^* &
\end{tikzcd}$$

commutes. The two squares on the right obviously commute, so it only remains to check the pentagon on the left. But, after removing the M_{-i}

tensor factor on the left, this is exactly dual to the commutative diagram which states that φ is a map of right A-modules.

Finally, if Λ is graded Frobenius but not symmetric, then we have an isomorphism $\varphi : \Lambda \xrightarrow{\sim} {}_\nu(\Lambda^*)$ of Λ-Λ-bimodules, where $\nu \in \mathrm{Aut}(\Lambda)$ is the Nakayama automorphism. So $\varphi^* : \Lambda_\nu \xrightarrow{\sim} \Lambda^*$, and we use the maps $\varphi.1.(\varphi^*)_{\nu^{-1}}$ as above to show $Q(M^*) \cong {}_\nu(Q(M)^*)_{\nu^{-1}}$. □

4.5. **Quotients and idempotents.** For an algebra Λ and an idempotent $e \in \Lambda$, we can define two algebras: $\Lambda/\Lambda(1-e)\Lambda$ and $e\Lambda e$. If Λ is semisimple then they are isomorphic but in general they are different. The first is, by definition, a quotient of Λ, while the second is, in general, not. We investigate the relation when Λ is quadratic.

Lemma 4.5.1. *Let Λ be a quadratic algebra and $e \in \Lambda_0 \subset \Lambda$ be an idempotent. Then $\Lambda/\Lambda(1-e)\Lambda$ is quadratic. Moreover, if the algebra $e\Lambda e$ is generated in degree 1 and is quadratic, then we have an isomorphism of graded algebras*

$$e\Lambda e \cong \left(\frac{\Lambda^!}{\Lambda^!(1-e)\Lambda^!}\right)^! .$$

Proof. Let $\Lambda \cong \mathrm{Tens}_S(V)/(R)$ as in Definition 4.4.2. First we will show that

$$\frac{\Lambda}{\Lambda(1-e)\Lambda} \cong \frac{\mathrm{Tens}_{eSe}(eVe)}{\langle \pi \otimes \pi(R)\rangle},$$

where $\pi : V \twoheadrightarrow eVe$ is the obvious map $v \mapsto eve$ of k-modules. This will, in particular, imply that $\Lambda/\Lambda(1-e)\Lambda$ is quadratic. Our degree 0 and degree 1 parts of the isomorphism, $S/S(1-e)S \cong eSe$ and $V/\left(V(1-e) \oplus (1-e)V\right) \cong eVe$, are clear. We now define a map from Λ to the tensor algebra on the right hand side by $v \mapsto eve$ for $v \in V$. This map is clearly surjective and well defined: if $\sum v_i w_i \in R$, then $\sum ev_i eew_i e \in \pi \otimes \pi(R)$. As $1 = e + (1-e)$ and $v = 1v1$ the kernel of this map is the ideal generated by $(1-e)$, and so we have our isomorphism.

Now, as $\Lambda^! = \mathrm{Tens}_S(V^*)/(R^\perp)$, we know that

$$\left(\frac{\Lambda^!}{\Lambda^!(1-e)\Lambda^!}\right)^! \cong \frac{\mathrm{Tens}_{eSe}((eV^*e)^*)}{\langle(\pi \otimes \pi(R^\perp))^\perp\rangle},$$

and so we need to show that, under our assumptions, the right hand side is isomorphic to $e\Lambda e$. We have equality in degrees 0 and 1, and so, as both algebras are quadratic, it suffices to show that we also have an isomorphism in degree 2, i.e.,

$$\frac{(eV^*e)^* \otimes_{eSe} (eV^*e)^*}{(\pi \otimes \pi(R^\perp))^\perp} \cong e\left(\frac{V \otimes_S V}{\langle R\rangle}\right)e.$$

We are assuming that $e\Lambda e$ is generated in degree 1, so the multiplication $(e\Lambda e)_1 \otimes_{eSe} (e\Lambda e)_1 \to (e\Lambda e)_2$, i.e., the composition

$$eVe \otimes_{eSe} eVe \hookrightarrow eV \otimes_S Ve \twoheadrightarrow \frac{eV \otimes_S Ve}{\langle eRe\rangle},$$

is surjective. Define a map $\psi : (eV^*e)^* \otimes_{eSe} (eV^*e)^* \to (e\Lambda e)_2$ by precomposing with $\psi \otimes \psi$, where $\psi : (e\Lambda e)_2 \to eVe$ is induced by the inverse of the evaluation isomorphism $V \xrightarrow{\sim} V^{**}$. Then we calculate the kernel of ψ: this is $(eV^*e)^* \otimes_{eSe} (eV^*e)^* \cap (eRe)^{**}$. Consider $(\ker\psi)^\perp$: after identifying V^{**} and V we see this is just $\pi \otimes \pi(R^\perp)$, and so $\ker\psi \cong (\pi \otimes \pi(R^\perp))^\perp$ and we have our isomorphism. □

Note that it is not automatically true that $e\Lambda e$ is generated in degree 1 or that it is quadratic. For counterexamples consider Γ_3 and the idempotents $e = e_1$ and $e = e_1 + e_2$ respectively, and, using the isomorphism $e\Lambda e \cong \mathrm{End}_\Lambda(\Lambda e)^{\mathrm{op}}$, apply Lemma 4.1.1.

Now let

$$\pi : \Lambda^! \twoheadrightarrow \frac{\Lambda^!}{\Lambda^!(1-e)\Lambda^!}$$

be the quotient map and let M' be a $\frac{\Lambda^!}{\Lambda^!(1-e)\Lambda^!}$-$\frac{\Lambda^!}{\Lambda^!(1-e)\Lambda^!}$-bimodule. Using π we can inflate M' on both sides to obtain a $\Lambda^!$-$\Lambda^!$-bimodule M.

Proposition 4.5.2. *Suppose $e\Lambda e$ is generated in degree 1 and quadratic. We have two functors*

$$Q : \Lambda^!\text{-grmod-}\Lambda^! \to \mathrm{lin}(\Lambda\text{-grproj-}\Lambda)$$

and

$$Q' : \frac{\Lambda^!}{\Lambda^!(1-e)\Lambda^!}\text{-grmod-}\frac{\Lambda^!}{\Lambda^!(1-e)\Lambda^!} \to \mathrm{lin}(e\Lambda e\text{-grproj-}e\Lambda e).$$

Then for $M \in \Lambda^!$-grmod-$\Lambda^!$ the inflation of M',

$$Q(M) = \Lambda e \otimes_{e\Lambda e} Q(M') \otimes_{e\Lambda e} e\Lambda.$$

Proof. Note that $M \cong eMe$ as $\Lambda^!$-$\Lambda^!$-bimodules, so in particular, $(M_i)^* \cong e(M_i)^*e$ for each $i \in \mathbb{Z}$. So in degree i

$$\begin{aligned} Q(M)_i &= \Lambda \otimes_S (M_i)^* \otimes_S \Lambda \cong \Lambda \otimes_S e(M_i)^*e \otimes_S \Lambda \cong \Lambda e \otimes_S (M_i)^* \otimes_S e\Lambda \\ &\cong \Lambda e \otimes_{eSe} (M_i')^* \otimes_{eSe} e\Lambda \end{aligned}$$

and

$$\begin{aligned} \Lambda e \otimes_{e\Lambda e} Q(M')_i \otimes_{e\Lambda e} e\Lambda &= \Lambda e \otimes_{e\Lambda e} e\Lambda e \otimes_{eSe} (M_i')^* \otimes_{eSe} e\Lambda e \otimes_{e\Lambda e} e\Lambda \\ &\cong \Lambda e \otimes_{eSe} (M_i')^* \otimes_{eSe} e\Lambda, \end{aligned}$$

and it is obvious that the differentials agree. □

4.6. **Preprojective algebras of type A.** Recall that the quadratic dual of Γ_n is the preprojective algebra Π_n of type A_n. It has an explicit description as follows: it is the quotient of the path algebra of the quiver

$$Q_n^* \quad = \quad \begin{tikzcd} 1 \arrow[r, bend left, "x_1"] & 2 \arrow[l, bend left, "y_2"] \arrow[r, bend left, "x_2"] & \cdots \arrow[l, bend left, "y_3"] \arrow[r, bend left, "x_{n-1}"] & n \arrow[l, bend left, "y_n"] \end{tikzcd}$$

by the ideal generated by x_1y_2, $x_iy_{i+1} - y_ix_{i-1}$ for $2 \le i \le n-1$, and y_nx_{n-1}. It is graded by path length, and we have identified x_i with β_{i+1}^* and y_j with $(-1)^{j-1}\alpha_{j-1}^*$ in the quiver Q_n of Γ_n.

Consider the algebra surjections

$$\pi^\ell : \Pi_n \twoheadrightarrow \frac{\Pi_n}{\Pi_n e_n \Pi_n} \cong \Pi_{n-1}$$

and

$$\pi^r : \Pi_n \twoheadrightarrow \frac{\Pi_n}{\Pi_n e_1 \Pi_n} \cong \Pi_{n-1}$$

with notation chosen to represent whether the left or right idempotents remain nonzero. These give us two ways to inflate a Π_{n-1}-module to a Π_n-module.

Let $\vec{A}_n$ denote the quiver

$$\vec{A}_n \quad = \quad 1 \to 2 \to \cdots \to n$$

of Dynkin type A_n with all arrows oriented $i \to i+1$. Then we have an algebra surjection $\Pi_n \twoheadrightarrow k\vec{A}_n$ defined by quotienting out by the ideal generated by all arrows y_i. This gives $k\vec{A}_n$ the structure of a Π_n-Π_n-bimodule.

Lemma 4.6.1. *There is a short exact sequence of graded Π_n-Π_n-bimodules*

$$0 \to {}_{\pi^r}(\Pi_{n-1})_{\pi^\ell}\{-1\} \hookrightarrow \Pi_n \twoheadrightarrow k\vec{A}_n \to 0,$$

where ${}_{\pi^r}(\Pi_{n-1})_{\pi^\ell}$ denotes the inflation of $\Pi_{n-1} \in \Pi_{n-1}$-grmod-Π_{n-1} using π^r on the left and π^ℓ on the right.

Proof. The map $\Pi_n \twoheadrightarrow k\vec{A}_n$ is given by the algebra surjection described above and its kernel is the submodule $\langle y_2, \ldots, y_n \rangle$ of Π_n, which is generated in degree 1. It is easy to check that $e_i \mapsto y_{i+1}$ defines a map ${}_{\pi^r}(\Pi_{n-1})_{\pi^\ell}\{-1\} \to \langle y_2, \ldots, y_n \rangle$ of right modules, and that it respects the left module structure. Similarly, we have an inverse bimodule map $y_j \mapsto e_{j-1}$, so we are done. □

The quadratic dual algebras Γ_n and Π_n (for $n \geq 3$) were studied by Brenner, Butler, and King using their theory of almost Koszul duality [BBK]. We will need the following result:

Proposition 4.6.2 (Brenner-Butler-King)**.** *The preprojective algebra Π_n is Frobenius of Gorenstein parameter $n-1$ with Nakayama automorphism $\tau_n^!$.*

Proof. We have an isomorphism of Π_n-Π_n-bimodules,

$$\Pi_n \xrightarrow{\sim} {}_{\tau_n^!}(\Pi_n)^*,$$

by Corollary 4.7 of [BBK], using the fact that $\tau_n^!$ is its own inverse, and it is easy to calculate the necessary grading shift. □

Note that the Gorenstein parameter $n-1$ corresponds to the fact that Π_n is $(n-1, 2)$-Koszul [BBK, Corollary 4.3].

We will also need a strengthening of Theorem 4.2.6, which is a combination of Theorem 3.15 and Proposition 5.1 of [BBK].

Theorem 4.6.3 (Brenner-Butler-King)**.** *The algebra Γ_n is twisted periodic with period n, automorphism τ_n, and truncated resolution $Y_n = Q(\Pi_n)$.*

4.7. **Explicit isomorphisms.** Theorem 4.3.1 told us how the longest elements of B_{n+1} act on the derived categories of the algebras Γ_n. The proof of this theorem proceeds by calculating the action of t_{n+1} on the indecomposable projective Γ_n-modules, showing that two-sided tilting complexes with isomorphic restrictions to one side can only differ by a twist [RZ, Proposition 2.3], and explicitly determining the outer automorphism groups of the algebras Γ_n [RZ, Proposition 4.4 and Remark 3].

In the rest of this section we present an alternative, explicit proof of Theorem 4.3.1. Our approach involves working directly with two-sided tilting complexes and uses the theory of almost Koszul duality due to Brenner, Butler, and King [BBK]. To simplify our notation, from now on, A will denote the algebra Γ_n.

Lemma 4.7.1. *The composition $F_m F_{m-1} \dots F_2 F_1$ is given by tensoring with the complex*

$$H_m = \mathrm{cone}(G_m \xrightarrow{\mathrm{ev}} A)$$

in $\mathrm{Ch}^{\mathrm{b}}(A\text{-}A)$*, where G_m is the complex*

$$G_m = \dots \to 0 \to G_{m,m-1} \to \dots \to G_{m,2} \to G_{m,1} \to G_{m,0} \to 0 \to \dots$$

of A-A bimodules where $G_{m,i}$ is in degree i and $G_{m,i} = \bigoplus_{j=1}^{m-i} P_{i+j,j}$. Here, $P_{i,j} = P_i \otimes_k P_i^\vee$, where $P_i = Ae_i$ is the ith indecomposable projective, so the evaluation map is $\mathrm{ev} : \bigoplus_{j=1}^{m} P_{j,j} \to A$*. The differential is, up to sign, induced by the maps $e_j A \to e_{j+1} A$ and $Ae_i \to Ae_{i-1}$ given by left multiplication by β_{j+1} and right multiplication by β_i, using $P_j^\vee \cong e_j A$.*

For example,

$$\begin{tikzcd}
 & & & P_{4,4} \arrow[dr] & \\
 & & P_{4,3} \arrow[ur] \arrow[dr] & \oplus & \\
 & P_{4,2} \arrow[ur] \arrow[dr] & \oplus & P_{3,3} \arrow[dr] & \\
H_4 = \mathrm{cone}(G_4 \xrightarrow{\mathrm{ev}} A) = P_{4,1} \arrow[ur] \arrow[dr] & \oplus & P_{3,2} \arrow[ur] \arrow[dr] & \oplus & A. \\
 & P_{3,1} \arrow[ur] \arrow[dr] & \oplus & P_{2,2} \arrow[ur] & \\
 & & P_{2,1} \arrow[ur] \arrow[dr] & \oplus & \\
 & & & P_{1,1} \arrow[uur] &
\end{tikzcd}$$

Proof. As A is symmetric, $P_j^\vee \cong P_j^* \cong e_j A^* \cong e_j A$.

We proceed by induction. Recall that F_i is given by tensoring with the complex $P_{i,i} \xrightarrow{\mathrm{ev}} A$ of A-A-bimodules, so the case $m = 1$ is clear.

Now suppose that we have shown the statement for $m-1$. For $1 \le i, j < m$, $P_{m,m} \otimes_A P_{i,j}$ is isomorphic to $P_m \otimes_k \langle \beta_m \rangle \otimes_k P_j^\vee \cong P_{m,j}$ if $i = m-1$, and to the zero module otherwise. So $P_{m,m} \otimes_A G(m-1)$ is the complex

$$P_{m,1} \to P_{m,2} \to \dots \to P_{m,m-1} \to P_{m,m}$$

with maps as we expect, and $A \otimes_A G(m-1)$ is just $G(m-1)$. As the differential in F_i is the evaluation map, the isomorphism $P_{m,m} \otimes_A P_{m-1,j} \cong P_m \otimes_k \langle \beta_m \rangle \otimes_k P_j^\vee$ tells us that these complexes glue together to give the complex $G(m+1)$. $\square$

Recall that a two-sided tilting complex is a chain complex $X \in \mathrm{D}^{\mathrm{b}}(A\text{-}B)$ such that $X \otimes_B^{\mathbf{L}} - : \mathrm{D}^{\mathrm{b}}(B) \to \mathrm{D}^{\mathrm{b}}(A)$ is an equivalence of triangulated categories [Ric]. All the autoequivalences of derived categories we consider in this article are given by tensoring with two-sided tilting complexes, so we can work inside the *derived Picard group*, which we denote $\mathrm{DPic}(A)$. This is the group of isomorphism classes of two-sided tilting complexes in $\mathrm{D}^{\mathrm{b}}(A\text{-}A)$, with group product given by taking the tensor product over A [RZ, Definition 3.1].

Our braid group action defines a group morphism

$$\psi_n : B_{n+1} \to \mathrm{DPic}(A),$$

$$\sigma_i \mapsto X_i.$$

Proof of Theorem 4.3.1. Recall that t_{n+1} is the positive lift of the longest element in B_{n+1}. We want to determine the action of t_{n+1} on $\mathrm{D^b}(A)$, where $A = \Gamma_n$, for all $n \geq 1$. We check this directly for $n = 1$: this is easy. With a little more work we can also check this directly for $n = 2$ and $n = 3$. Then we proceed by induction: assume that t_n acts on $\mathrm{D^b}(\Gamma_{n-1})$ as $-_{\tau_{n-1}}[n-1]$, and we will show the corresponding statement for $\mathrm{D^b}(A)$.

We want to show that two functors are naturally isomorphic: the shift and twist $-_{\tau_n}[n]$ and the image of t_{n+1} in the group morphism $\varphi_n : B_{n+1} \to \mathrm{DPic}(A)$. The first is naturally isomorphic to $A_{\tau_n}[n] \otimes_A -$, and the second to some complex of A-A-bimodules obtained by tensoring together complexes X_i. So it is enough to show that these bimodule complexes are isomorphic in $\mathrm{D^b}(A\text{-}A)$.

Let t'_n denote the image of t_n in the group monomorphism $B_n \hookrightarrow B_{n+1}$ which sends $\sigma_i \mapsto \sigma_i$. Let T_{n+1} and T'_n denote the image of t_{n+1} and t'_n in $\psi_n : B_{n+1} \to \mathrm{DPic}(A)$. Then we want to show that $T_{n+1} \cong A_{\tau_n}[n]$.

By the inductive description of t_{n+1}, we need to show that

$$T'_n X_n X_{n-1} \dots X_2 X_1 \cong A_{\tau_n}[n].$$

By our inductive hypothesis we know that $\varphi_{n-1}(t_n)$ acts as a shift and twist on $\mathrm{D^b}(\Gamma_{n-1})$, so by the lifting theorem, Theorem 3.3.6, $\varphi_n(t'_n)$ is given by a periodic twist: we have a distinguished triangle

$$PY_{n-1}P^\vee \to A \to T'_n \rightsquigarrow,$$

where $P = P_1 \oplus \dots \oplus P_{n-1} \in A$-mod, so by Lemma 4.1.1 we have $E = \mathrm{End}_A(P)^{\mathrm{op}} \cong \Gamma_{n-1}$ and Y_{n-1} is the truncated resolution of Γ_{n-1} by Theorem 4.6.3.

By Lemma 4.7.1 we know that $X_n X_{n-1} \dots X_2 X_1 \cong H_n$ and we have a distinguished triangle

$$G_n \to A \to H_n \rightsquigarrow,$$

so we want to show that

$$T'_n H_n \cong A_{\tau_n}[n]$$

which, as T'_n is a two-sided tilting complex, is equivalent to showing

$$H_n \cong ((T'_n)^*)_{\tau_n}[n].$$

We will show that there is a map between these two complexes in the derived category such that the cone is isomorphic to zero.

By Proposition 4.6.2 we have an isomorphism $\Pi_{n-1} \cong {}_{\tau^!_{n-1}}(\Pi_{n-1})^*\{2-n\}$ of Π_{n-1}-Π_{n-1}-bimodules, so, using that $\tau^!_{n-1} \circ \pi^r = \pi^\ell \circ \tau^!_n$, we have an isomorphism

$${}_{\pi^r}(\Pi_{n-1})_{\pi^\ell} \cong {}_{\tau^!_n}({}_{\pi^l}(\Pi^*_{n-1})_{\pi^\ell})\{2-n\}$$

of Π_n-Π_n-bimodules. Applying Proposition 4.4.5 (ii) we see that

$$Q({}_{\pi^r}(\Pi_{n-1})_{\pi^\ell}) \cong Q({}_{\tau^!_n}({}_{\pi^l}(\Pi^*_{n-1})_{\pi^\ell})\{2-n\}) \cong Q(({}_{\pi^l}(\Pi^*_{n-1})_{\pi^\ell})\{2-n\})_{\tau_n},$$

and Proposition 4.5.2, which is valid as $n-1 > 2$, and parts (i) and (iii) of Proposition 4.4.5 say that this is isomorphic to

$$PQ'(\Pi^*_{n-1}\{2-n\})P^\vee_{\tau_n} \cong PQ'(\Pi^*_{n-1})P^\vee_{\tau_n}[n-2]\{2-n\} \cong PY^*_{n-1}P^\vee_{\tau_n}[n-2]\{-n-2\}$$

as Γ_{n-1} is symmetric of Gorenstein parameter 2 because it is $(2, n-2)$-Koszul [BBK, Proposition 3.11 and Corollary 4.3].

Combining Lemmas 4.4.4 and 4.6.1 tells us that we have a short exact sequence

$$0 \to Q(k\vec{A}_n) \hookrightarrow Q(\Pi_n) \twoheadrightarrow Q({}_{\pi^r}(\Pi_{n-1})_{\pi^\ell}\{-1\}) \to 0$$

in $\mathrm{lin}(\Gamma_n\text{-grproj-}\Gamma_n)$. We know by Theorem 4.6.3 that $Q(\Pi_n) = Y_n$ and it is easy to see that $Q(k\vec{A}_n) \cong G_n$, so our short exact sequence is

$$0 \to G_n \hookrightarrow Y_n \twoheadrightarrow (PY_{n-1}^* P^\vee)_{\tau_n}[n-1]\{-n-1\} \to 0.$$

Forgetting the grading, we get the short exact sequence

$$0 \to G_n \hookrightarrow Y_n \twoheadrightarrow (PY_{n-1}^* P^\vee)_{\tau_n}[n-1] \to 0$$

of chain complexes of A-A-bimodules.

We build a diagram

$$\begin{tikzcd}
 & G_n \arrow[r, two heads] \arrow[d, hook] & A \arrow[d, equal] \\
A_{\tau_n}[n-1] \arrow[r, hook] \arrow[d, equal] & Y_n \arrow[r, two heads] \arrow[d, two heads] & A \\
A_{\tau_n}[n-1] \arrow[r] & (PY_{n-1}^* P^\vee)_{\tau_n}[n-1] &
\end{tikzcd}$$

which has exact columns: the only nontrivial column comes from the short exact sequence above. The map in the top row comes from Lemma 4.7.1 and the maps in the middle row come from Theorem 4.6.3, so it is clear that the top right square commutes.

We now describe the construction of the map $A_{\tau_n}[n-1] \to (PY_{n-1}^* P^\vee)_{\tau_n}[n-1]$. First, we use the usual periodic twist construction to define a map $PY_{n-1}P^\vee \to A$ from the map $Y_{n-1} \twoheadrightarrow E$ given by Theorem 4.6.3; recall that $E = \mathrm{End}_A(P)^{\mathrm{op}} \cong \Gamma_{n-1}$. Then take the dual of this map and pre- and post-compose maps as follows:

$$A \xrightarrow{\sim} A^* \to (PY_{n-1}P^\vee)^* \xrightarrow{\sim} PY_{n-1}^* P^\vee.$$

Finally, twist on the right by τ_n and apply the shift functor $[n-1]$.

Assume for a moment that our diagram commutes. Then the rows give a short exact sequence

$$0 \to H_n \hookrightarrow U \twoheadrightarrow ((T_n')^*)_{\tau_n}[n+1] \to 0$$

of chain complexes of A-A-bimodules where U is the complex

$$A_{\tau_n}[n-1] \hookrightarrow Y_n \twoheadrightarrow A$$

and so has zero homology. Therefore we have a distinguished triangle

$$H_n \to 0 \to ((T_n')^*)_{\tau_n}[n+1] \rightsquigarrow,$$

and so the map $((T_n')^*)_{\tau_n}[n] \xrightarrow{\sim} H_n$ in a rotation of this triangule must be an isomorphism.

Now we show the bottom left square commutes. Note that $A_{\tau_n}[n-1] \to (PY_{n-1}^* P^\vee)_{\tau_n}[n-1]$ is constructed from the dual of $Y_{n-1} \twoheadrightarrow \Gamma_{n-1}$, which is the

start of a bimodule resolution and so given by evaluation maps $\bigoplus P'_{i,i} \xrightarrow{\text{ev}} \Gamma_{n-1}$. By [Gra1, Lemma 4.3] ($\Delta' \cong \Delta^*$) we can, up to shift and twist, identify $A_{\tau_n}[n-1] \hookrightarrow Y_n$ with the dual of $Y_n \twoheadrightarrow A$, which is also given by evaluation maps $\bigoplus P_{j,j} \xrightarrow{\text{ev}} \Gamma_n$. So as $PP'_{i,i}P^\vee \cong P_{i,i}$ and $Y_n \twoheadrightarrow (PY^*_{n-1}P^\vee)_{\tau_n}[n-1]$ is surjective, the diagram commutes. □

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School of Mathematics, University of Leeds, Leeds, LS2 9JT, United Kingdom
E-mail address: j.s.grant@leeds.ac.uk