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LAWS OF THE ITERATED LOGARITHM FOR SELF-NORMALISED LÉVY PROCESSES AT ZERO

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ABSTRACT. We develop tools and methodology to establish laws of the iterated logarithm (LILs) for small times (as $t \downarrow 0$) for the "self-normalised" process $(X_t - at)/\sqrt{V_t}$, $t > 0$, constructed from a Lévy process $(X_t)_{t\geq 0}$ having quadratic variation process $(V_t)_{t\geq 0}$, and an appropriate choice of the constant a. We apply them to obtain LILs when X_t is in the domain of attraction of the normal distribution as $t \downarrow 0$, when X_t is symmetric and in the Feller class at 0, and when X_t is a strictly $\alpha-$stable process. When X_t is attracted to the normal distribution, an important ingredient in the proof is a Cramér-type theorem which bounds above the distance of the distribution of the self-normalised process from the standard normal distribution.

1. INTRODUCTION

Let $(X_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}$ with canonical triplet (γ, σ^2, Π), where $\gamma \in \mathbb{R}$, $\sigma^2 \geq 0$, and Π is the Lévy measure of X, satisfying

$$\int_{\mathbb{R}} \min(x^2, 1)\Pi(\mathrm{d}x) < \infty. \tag{1.1}$$

See Bertoin [3] and Sato [27] for basic properties. Suppose X_t has jump process $\Delta X_t := X_t - X_{t-}$, $t > 0$, $\Delta X_0 = 0$, and define its quadratic variation process by

$$V_t = \sigma^2 t + \sum_{0<s\leq t} (\Delta X_s)^2, \ t \geq 0. \tag{1.2}$$

The small time (as $t \downarrow 0$) behavior of X_t, and, in particular, types of iterated logarithm behavior as $t \downarrow 0$, have been of interest since the work of Khintchine in the last century. See Sato [27, Sect. 47] for an overview of the results obtained. More recently, the small time behavior of the *self-normalised process*[1]

$$\frac{X_t - at}{\sqrt{V_t}}, \ t > 0, \tag{1.3}$$

where a is an appropriate constant, has been studied in a variety of contexts; see, for example, Maller and Mason [22], and their references. Maller and Mason give conditions for the convergence in distribution and, more generally, for some kinds of

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[1]We use the convention $0/0 = 0$ in (1.3) and throughout.

compactness behavior of the self-normalised process, related to the Feller stochastic compactness classes, as $t \downarrow 0$.

A classic result of Khintchine [20] (see Sato [27, p. 358]) tells us that

$$\limsup_{t\downarrow 0} \frac{|X_t|}{\sqrt{2t\log\log(1/t)}} = \sigma, \text{ a.s.} \tag{1.4}$$

This implies that whenever X_t has no normal component ($\sigma^2 = 0$), the limsup in (1.4) is almost surely zero. We shall show, perhaps somewhat unexpectedly, that if $\sigma^2 = 0$ and the first t in the denominator of (1.4) is replaced by V_t, then the limsup can be finite and positive. Thus normalisation by V_t can have a remarkable stabilising effect. We shall call such limit results *self-normalised laws of the iterated logarithm [LILs] at zero* for X_t.

Our aim in this paper is to develop tools and methodology to establish LILs for small times (as $t \downarrow 0$) for the self-normalised process $(X_t - at)/\sqrt{V_t}$, $t > 0$, and apply them when X_t is in the domain of attraction of the normal distribution as $t \downarrow 0$, when X_t is symmetric and in the Feller class at 0, and when X_t is a strictly $\alpha-$stable process. These results can be viewed as analogues of large-time LIL results of Griffin and Kuelbs [14, 15], Shao [28], Giné and Mason [12] and Jing, Shao and Zhou [17], for self-normalised random walks. We refer also to the monograph of de la Peña, Lai and Shao [6] for background and an overview.

Our three LILs are stated in Section 2. After some necessary tools are gathered together in Section 3, they are proved in Section 4 using a general procedure developed there. A crucial ingredient when X_t is attracted to normality is a Cramér-type theorem which bounds above the distance of the distribution of the self-normalised process from the standard normal distribution, for small t. This result is established in Section 5 and should be of separate interest. Our LIL in this case is the most technically involved to establish. Some other needed technical results are collected in an appendix.

2. LILs at 0 for the self-normalised Lévy process

In order to state our LILs, we need to introduce some notions concerning domains of attraction and the Feller stochastically compact classes at 0. To do this we must establish some notation. Define the tail functions based on the Lévy measure Π,

$$\overline{\Pi}^-(x) = \Pi\{(-\infty, -x)\},\ \overline{\Pi}^+(x) = \Pi\{(x,\infty)\},\ \overline{\Pi}(x) = \overline{\Pi}^+(x) + \overline{\Pi}^-(x), \qquad x > 0. \tag{2.1}$$

$\overline{\Pi}(x)$ and $\overline{\Pi}^{\pm}(x)$ are nonincreasing right continuous functions on $(0,\infty)$, possibly taking the value $+\infty$ as $x \to 0+$. Truncated mean and variance functions are defined by

$$\nu(x) = \gamma - \int_{x<|y|\le 1} y\Pi(\mathrm{d}y) \quad \text{and} \quad V(x) = \sigma^2 + \int_{0<|y|\le x} y^2\Pi(\mathrm{d}y),\ x > 0. \tag{2.2}$$

Also let

$$U(x) = \sigma^2 + 2\int_0^x y\overline{\Pi}(y)\,\mathrm{d}y = x^2\overline{\Pi}(x) + V(x),\ x > 0. \tag{2.3}$$

These are all right continuous functions on $(0,\infty)$ (continuous in the case of $U(x)$), and finite for all $x > 0$ by virtue of property (1.1) of the Lévy measure Π, which further implies that $\lim_{x\downarrow 0} x^2\overline{\Pi}(x) = 0$. The function $\nu(x)$ may be unbounded as

$x \to 0+$, but $V(x)$ and $U(x)$ are always finite, with $V(0+) = U(0+) = \sigma^2 \in [0, \infty)$, and $V(+\infty) < \infty$ if and only if $EX_1^2 < \infty$.

Note that $(V_t)_{t\geq 0}$ in (1.2) is a subordinator with drift $\sigma^2 \geq 0$ and Lévy measure having tail function $\overline{\Pi}_V(x) = \overline{\Pi}(\sqrt{x})$, $x > 0$. It's easy to check that V_t remains positive, a.s., for all $t > 0$, if and only if $\sigma^2 > 0$ or $\Pi(\mathbb{R}) = \infty$, which we will assume throughout. The assumption $\sigma^2 > 0$ or $\Pi(\mathbb{R}) = \infty$ also implies $V(y) > 0$ for all $y > 0$.

We say that the Lévy process X is in the *domain of attraction of the normal distribution* as $t \downarrow 0$, denoted $X_t \in D(N)$ at 0, when there are deterministic functions $a(t)$ and $b(t) > 0$ such that

$$\frac{X_t - a(t)}{b(t)} \xrightarrow{\mathrm{D}} N(0,1), \text{ as } t \downarrow 0.$$

($N(0,1)$ is a standard normal rv.) See Doney and Maller [7] for various analytical equivalences for $X_t \in D(N)$ at 0, among them being that

$$\lim_{x\downarrow 0} \frac{x^2\overline{\Pi}(x)}{V(x)} = 0, \tag{2.4}$$

or, equivalently, $V(x)$ is slowly varying at 0. Maller and Mason [22] show that we may always take the centering function $a(t)$ as 0, when $X_t \in D(N)$ at 0. This means that

$$\lim_{x\downarrow 0} \frac{x|\nu(x)|}{V(x)} = \lim_{x\downarrow 0} \frac{x|\nu(x) - \gamma|}{V(x)} = 0. \tag{2.5}$$

In fact Maller and Mason [22] prove that $X_t \in D(N)$ at 0 if and only if $X_t/\sqrt{V_t} \xrightarrow{\mathrm{D}} N(0,1)$, as $t \downarrow 0$.

By the *Feller class at 0* we will mean the class of Lévy processes X_t which are stochastically compact at 0 after norming and centering; that is, those for which there are nonstochastic functions $a(t)$, $b(t) > 0$ such that every sequence $t_k \downarrow 0$ contains a subsequence $t_{k'} \downarrow 0$ with

$$\frac{X_{t_{k'}} - a(t_{k'})}{b(t_{k'})} \xrightarrow{\mathrm{D}} \mathcal{I}', \text{ as } t_{k'} \downarrow 0, \tag{2.6}$$

where $\mathcal{I}'$ is a finite rv, a.s., not degenerate at a constant. (The prime on $\mathcal{I}'$ denotes that in general it will depend on the choice of subsequence $t_{k'}$.) We describe this kind of convergence as "$X_t \in FC$ at 0". An analytic equivalence for $X_t \in FC$ at 0 is

$$\limsup_{x\downarrow 0} \frac{x^2\overline{\Pi}(x)}{U(x)} = c_0 < 1. \tag{2.7}$$

See Maller and Mason [22] for this and some other results on the Feller class at 0.

Note that if $X_t \in D(N)$, then (2.7) holds with $c_0 = 0$. Our first LIL concerns this case.

Theorem 2.1. *Assume $\sigma^2 > 0$ or $\overline{\Pi}(0+) = \infty$ and $X_t \in D(N)$ at 0. Then*

$$\limsup_{t\downarrow 0} \frac{\pm(X_t - t\gamma)}{\sqrt{2V_t \log\log(1/t)}} = 1, \text{ a.s.} \tag{2.8}$$

Remark 2.1. (i) Note that (2.7) holds with $c_0 = 0$ when $\sigma^2 > 0$, since $\lim_{x\downarrow 0} x^2\overline{\Pi}(x) = 0$. In this case we have $\lim_{t\downarrow 0} V_t/t = \sigma^2$, a.s., by a result of Shtatland [30] (see also Sato [27, Thm. 43.20, p. 323]). (2.8) then reduces to the constant norming LIL in (1.4).

(ii) Apart from the case $\sigma^2 > 0$, we stipulate $\overline{\Pi}(0+) = \infty$ (equivalently, $\Pi(\mathbb{R}) = \infty$) in Theorem 2.1 since results like (2.8) cannot hold when $\Pi(\mathbb{R}) < \infty$ (i.e., X_t is compound Poisson); see the proof of Theorem 2.1.

(iii) We show following the proof of Theorem 2.1 that the centering term $t\gamma$ in (2.8) can be deleted.

The interesting case in Theorem 2.1 is thus when $\sigma^2 = 0$. To illustrate, we give two examples of Lévy processes in $D(N)$ at 0 which satisfy $\sigma^2 = 0$ and $\overline{\Pi}(0+) = \infty$.

Example 1. Let X_t be symmetric with $\sigma^2 = 0$. Assume that for some $\beta > 0$ and all small positive y,

$$\Pi\,(\mathrm{dy}) = (\beta/2)\, y^{-3}\, |\log y|^{-1}\, (\log|\log y|)^{-1-\beta}\, \mathrm{dy}.$$

For this example Fristedt [9] shows that

$$\limsup_{t\downarrow 0} \frac{\pm X_t}{\sqrt{2t}\,(\log\log(1/t))^{(1-\beta)/2}} = 1, \text{ a.s.}$$

Example 2. Let X_t be symmetric with $\sigma^2 = 0$. Assume that for some $\beta > 0$ and all small positive y,

$$\Pi\,(\mathrm{dy}) = (\beta/2)\, y^{-3}\, |\log y|^{-1-\beta}\, \mathrm{dy}.$$

Let h be a positive, continuous, strictly increasing function on $(0,\infty)$ with inverse function h^{-1} such that $h^3(t)/t \downarrow 0$ as $t \downarrow 0$. For this example, it follows from Theorem 1 of Fristedt [9], as he points out, that $\int_0^1 h^{-1}(y)\Pi\,(\mathrm{dy}) = \infty$ implies

$$\limsup_{t\downarrow 0} \frac{|X_t|}{h(t)} = \infty, \text{ a.s.},$$

whereas if $\int_0^1 h^{-1}(y)\Pi\,(\mathrm{dy}) < \infty$ and h satisfies two additional technical conditions, then

$$\lim_{t\downarrow 0} \frac{|X_t|}{h(t)} = 0, \text{ a.s.}$$

On the other hand, each of these examples satisfies (2.4), hence (2.8) with $\gamma = 0$. This illustrates how $\sqrt{V_t}$ can stabilise the behavior of X_t near zero.

For the next result we need a slight modification of the Feller classes. Say that X_t is in the *centered Feller class*, written $X_t \in FC_0$ at 0, when there exists a nonstochastic function $b(t) > 0$ such that $X_t/b(t)$ is stochastically bounded as $t \downarrow 0$ and each subsequential limit random variable Y' is nondegenerate. In particular, this occurs when X_t is symmetric about 0 and $X_t \in FC$ at 0.

Results like (2.8), even with the $\sqrt{V_t \log\log(1/t)}$ norming, are not restricted to $X_t \in D(N)$ at 0 (though we expect that the constant "2" in (2.8) is specific to the normal distribution). The methods we shall develop can also be applied to obtain bounds of the form

$$\limsup_{t\downarrow 0} \frac{X_t - t\gamma}{\sqrt{V_t \log\log(1/t)}} \le c_1, \text{ a.s.},$$

and

$$\limsup_{t\downarrow 0} \frac{X_t - t\gamma}{\sqrt{V_t \log\log(1/t)}} \geq c_2, \text{ a.s.},$$

where $c_1 \geq c_2 > 0$, in cases when $X_t \in FC_0$ at 0; in particular, when X is symmetric and in a domain of attraction of a stable law as $t \downarrow 0$.

The following theorem is a small-time Lévy process version of the LILs given in Theorems 2 and 3 of Giné and Mason [12] for self-normalised random walks.

Theorem 2.2. *Suppose $\overline{\Pi}(0+) = \infty$ and X_t is symmetric about 0. Then*

$$\limsup_{t\downarrow 0} \frac{\pm X_t}{\sqrt{V_t \log\log(1/t)}} \leq \sqrt{2}, \text{ a.s.} \tag{2.9}$$

If in addition X_t is in the Feller class at zero, then for some constant $c > 0$,

$$\limsup_{t\downarrow 0} \frac{\pm X_t}{\sqrt{V_t \log\log(1/t)}} \geq c, \text{ a.s.} \tag{2.10}$$

Remark 2.2. We expect that a deeper analysis of the sort performed by Jing et al. [17] in proving their Theorem 1.1 would yield a moderate deviation result that would allow us to specify the constant in (2.10) under added regularity conditions. However, this is well beyond the scope of the present paper.

The next result, Theorem 2.3, should be compared to Proposition 47.16 in Sato [27] (see Khintchine [19] and Theorem 11.2 of Fristedt [11]) which gives: suppose X_t is a strictly α-stable process on $\mathbb{R}$ with $0 < \alpha < 2$ and $\Pi\{(0,\infty)\} > 0$. Let h be a positive, continuous, increasing function on $(0,\delta)$, for some $\delta > 0$. Then

$$\limsup_{t\downarrow 0} \frac{|X_t|}{h(t)} = 0, \text{ or } = \infty, \text{ a.s.}, \tag{2.11}$$

according to whether

$$\int_0^\delta \frac{\mathrm{dt}}{(h(t))^\alpha} < \infty \text{ or } = \infty. \tag{2.12}$$

Theorem 2.3. *Suppose X_t is a strictly $\alpha-$stable process with $0 < \alpha < 2$ and $\Pi\{(0,\infty)\} = \infty$. Then*

$$\limsup_{t\downarrow 0} \frac{|X_t|}{\sqrt{V_t \log\log(1/t)}} < \infty, \text{ a.s.}, \tag{2.13}$$

and for some finite constant $c > 0$,

$$\limsup_{t\downarrow 0} \frac{X_t}{\sqrt{V_t \log\log(1/t)}} = c, \text{ a.s.} \tag{2.14}$$

Remark 2.3. The constant c in (2.14) will be identified in the proof of Theorem 2.3. When X_t is a strictly $\alpha-$stable subordinator with $0 < \alpha < 1$, we have (2.14) holding, while the version of (2.14) with limsup replaced by liminf is 0, a.s.

3. Tools for proving LILs

In this section we shall establish a number of tools needed to prove our self-normalised LILs. Write the Lévy-Itô decomposition of X (cf. Sato [27, Thm. 19.2, p. 120], Doney and Maller [7, Lemma 6.1, p. 770]) in the form

$$X_t = t\nu(h) + \sigma Z_t + X_t^{(S,h)} + X_t^{(B,h)},\ t \geq 0,\ h > 0, \tag{3.1}$$

where $(Z_t)_{t\geq 0}$ is a standard Brownian motion, $(X_t^{(S,h)})_{t\geq 0}$, is a compensated sum of jumps of X_t smaller than or equal to h in the modulus

$$X_t^{(S,h)} = \text{a.s.}\lim_{\varepsilon\downarrow 0}\left(\sum_{0<s\leq t}\Delta X_s 1_{\{\varepsilon<|\Delta X_s|\leq h\}} - t\int_{\varepsilon<|x|\leq h} x\Pi(\mathrm{d}x)\right),\ t>0,$$

and $X_t^{(B,h)}$ is the big jump part of X_t, defined as

$$X_t^{(B,h)} = \sum_{0<s\leq t}\Delta X_s 1_{\{|\Delta X_s|>h\}}.$$

The corresponding quadratic variation processes are

$$V_t^{(S,h)} = \sum_{0<s\leq t}(\Delta X_s)^2 1_{\{0<|\Delta X_s|\leq h\}} \quad\text{and}\quad V_t^{(B,h)} = \sum_{0<s\leq t}(\Delta X_s)^2 1_{\{|\Delta X_s|>h\}},\ t>0. \tag{3.2}$$

As shown by Sato, all three components $(Z_t)_{t\geq 0}$, $(X_t^{(S,h)})_{t\geq 0}$, and $(X_t^{(B,h)})_{t\geq 0}$ in (3.1) are independent of each other, as are $(V_t^{(S,h)})_{t\geq 0}$ and $(V_t^{(B,h)})_{t\geq 0}$ in (3.2).

All moments of $X_t^{(S,h)}$ and $V_t^{(S,h)}$ are finite. Observe that, for any $t>0$, $h>0$,

$$E\left(\sigma Z_t + X_t^{(S,h)}\right)^2 = \sigma^2 t + t\int_{0<|x|\leq h} x^2\Pi(\mathrm{d}x) = tV(h). \tag{3.3}$$

Define, for $t \geq 0$,

$$\mathcal{F}_t = \sigma\{X_y : y \geq t\},$$

with an analogous definition for $\mathcal{F}_t^{(S,h)}$ in terms of $X_y^{(S,h)}$. Note that the σ–fields $\mathcal{F}$ and $\mathcal{F}^{(S,h)}$ concern future values of the process.

Lemma 3.1. *Let X_t be a Lévy process with $E|X_1| < \infty$. Then*

$$E\left(X_t\big|\mathcal{F}_{t+s}\right) = \frac{t}{t+s}X_{t+s},\ \textit{a.s.},\ t>0,\ s>0. \tag{3.4}$$

In particular, when $X_t^{(S,h)}$ is the small jump process,

$$E\left(X_t^{(S,h)}\big|\mathcal{F}_{t+s}^{(S,h)}\right) = \frac{t}{t+s}X_{t+s}^{(S,h)},\ \textit{a.s.},\ t>0,\ s\geq 0,\ h>0. \tag{3.5}$$

Proof of Lemma 3.1. (3.4) is in Lemma 10.3.5, p. 432 of Rolski et al. [26]. Since $|X_1^{(S,h)}|$ has finite expectation we can apply (3.4) to get (3.5). □

Lemma 3.2. *For $s>0$ let $(S_t,\mathcal{A}_t)_{t\geq s}$ be a right-continuous nonnegative submartingale, and fix $0<c<1$, $x>0$, and $p>1$, $q>1$ with $1/p+1/q=1$. Assume $ES_t^p<\infty$, $t\geq s$. Then for all $t\geq s$,*

$$P\left(\sup_{s\leq y\leq t} S_y > x\right) \leq \frac{1}{x(1-c)}\int_{\{S_t>cx)} S_t\mathrm{d}P \leq \frac{P^{1/q}\left(S_t>cx\right)E^{1/p}(S_t^p)}{x(1-c)}. \tag{3.6}$$

Proof of Lemma 3.2. Shorack and Wellner [29, p. 870] give the following discrete time version of (3.6): let $(T_n, \mathcal{A}_n)$, $n \geq k \geq 1$, be a submartingale in discrete time. Then for any $x > 0$ and $0 < c < 1$,

$$P\left(\max_{k \leq m \leq n} T_m > x\right) \leq \frac{\int_{\{T_n > cx)} T_n \mathrm{d}P}{x(1-c)}.$$

Using this we get

$$P\left(\max_{\lfloor ns \rfloor \leq m/n \leq \lfloor nt \rfloor} S_m > x\right) \leq \frac{\int_{\{S_{\lfloor nt \rfloor/n} > cx)} S_{\lfloor nt \rfloor/n} \mathrm{d}P}{x(1-c)}.$$

Since, for $s < t$, $S_{\lfloor nt \rfloor/n} \leq \sup_{s \leq y \leq t} S_y$ is integrable by Doob's inequality, letting $n \to \infty$ gives the left-hand inequality in (3.6). The right-hand inequality in (3.6) follows from Hölder's inequality. □

Lemma 3.3. *Let $\varphi(z)$ be a symmetric convex function which is an increasing function of $|z|$ such that, for each $t > 0$, $E|\varphi(X_t/\sqrt{V_t})| < \infty$, and for each $t > 0$ define*

$$M_u = \varphi\left(\frac{t}{t+s-u} \frac{X_{t+s-u}}{\sqrt{V_{t+s-u}}}\right) \quad \text{and} \quad \mathcal{G}_u = \sigma\left\{(X_y, V_y) : y \geq t+s-u\right\}, \text{ for } 0 \leq u \leq s.$$

Then $(M_u, \mathcal{G}_u)_{0 \leq u \leq s}$ is a submartingale. This implies, for $x > 0$, $0 < c < 1$, $t > 0$, $s > 0$, the inequalities

(3.7)
$$P\left(\sup_{0 \leq y \leq s} \frac{t}{t+s-y} \left|\frac{X_{t+s-y}}{\sqrt{V_{t+s-y}}}\right| > x\right) \leq \frac{P^{1/q}\left(\left|X_t/\sqrt{V_t}\right| > cx\right) E^{1/p}\left|X_t/\sqrt{V_t}\right|^p}{x(1-c)}$$

and

(3.8)
$$P\left(\sup_{0 \leq y \leq s} \frac{t}{t+s-y} \frac{X_{t+s-y}}{\sqrt{V_{t+s-y}}} > x\right) \leq \frac{P^{1/q}\left(X_t/\sqrt{V_t} > cx\right) E^{1/p}\left|X_t/\sqrt{V_t}\right|^p}{x(1-c)}.$$

Proof of Lemma 3.3. To check the submartingale property, write

$$\begin{aligned} E\left(\varphi\left(\frac{X_t}{\sqrt{V_t}}\right)\middle|\mathcal{F}_{t+s}\right) &\geq E\left(\varphi\left(\frac{X_t}{\sqrt{V_{t+s}}}\right)\middle|\mathcal{F}_{t+s}\right) \\ &\geq \varphi\left(E\left(\frac{X_t}{\sqrt{V_{t+s}}}\middle|\mathcal{F}_{t+s}\right)\right) \\ &= \varphi\left(\frac{t}{t+s}\frac{X_{t+s}}{\sqrt{V_{t+s}}}\right). \end{aligned} \tag{3.9}$$

Here the first inequality holds since $\varphi(z)$ is a symmetric increasing function of $|z|$, the second inequality holds by Jensen's inequality since φ is a convex function, and the equality holds by (3.4). By applying (3.6) to $(M_u, \mathcal{G}_u)_{0 \leq u \leq s}$, with the convex functions $\varphi(z) = |z|$ and $\varphi(z) = \max(0, z)$, we get (3.7) and (3.8). □

Remark 3.1. We can replace $(X_t)_{t \geq 0}$ by $(X_t - tc)_{t \geq 0}$, for any $c \in \mathbb{R}$, in (3.4), (3.5), (3.7), and (3.8), because $(X_t - tc)_{t \geq 0}$ is a Lévy process, also with quadratic variation process $(V_t)_{t \geq 0}$.

Lemma 3.4 (Moment bounds for Poissonized sums)**.** *Suppose $Y, Y_1, Y_2, \ldots$ are independent identically distributed (i.i.d.) random variables (rvs), with $E|Y|^p < \infty$, $p \geq 2$, and N is an independent Poisson random variable. Then, for every $p \geq 2$,*

$$E\left|\sum_{i=1}^{N} Y_i - ENEY\right|^p \leq \left(\frac{15p}{\log p}\right)^p \max\left[\left(ENEY^2\right)^{p/2},\ ENE|Y|^p\right]. \tag{3.10}$$

Moreover, specializing to $Y \equiv 1$, we have for every $p \geq 2$,

$$E\left|N - EN\right|^p \leq \left(\frac{15p}{\log p}\right)^p \max\left[(EN)^{p/2},\ EN\right]. \tag{3.11}$$

Proof of Lemma 3.4. This is Lemma 2.3 of Giné, Mason and Zaitsev [13]. □

Remark 3.2. For $0 < p \leq 2$, we get the following bound from Jensen's inequality:

$$E\left|N - EN\right|^p = E[(N - EN)^2]^{p/2} \leq (EN)^{p/2}.$$

Now define, for $p \geq 2$ and $x > 0$

$$V_p(x) = \int_{0<|y|\leq x} |y|^p \Pi(\mathrm{d}y) \quad \text{and} \quad U_p(x) = p\int_0^x y^{p-1}\overline{\Pi}(y)\,\mathrm{d}y = x^p\overline{\Pi}(x) + V_p(x).$$

Also, for $h > 0$, $p > 0$, define $c_p = E|Z_1|^p$, where $Z_1 \sim N(0,1)$, and

$$C_p = 1_{\{1\leq p\leq 2\}} + (15p/\log p)^p\, 1_{\{p>2\}},$$

and let

$$M_p(t,h) = \begin{cases} \max\{2^{1-p/2}, 2^{p/2}\}\,(tV(h))^{p/2}, & 0 < p \leq 2, \\ 2^{2p-2}\left(c_p\sigma^p t^{p/2} + C_p \max\left[(tV_2(h))^{p/2},\ tV_p(h)\right]\right), & p > 2, \end{cases} \tag{3.12}$$

and

$$\pi_p(t,h) = \begin{cases} \max\{2^{2p-1}, 1\}\left(t\overline{\Pi}(h)\right)^p, & 0 < p \leq 1, \\ 2^{2p-1}\left[t\overline{\Pi}(h) + \left(t\overline{\Pi}(h)\right)^2\right]^{p/2}, & 1 < p \leq 2, \\ 2^{3p-2}\left[C_p \max\left[\left(t\overline{\Pi}(h)\right)^{p/2},\ t\overline{\Pi}(h)\right] + \left(t\overline{\Pi}(h)\right)^p\right], & p > 2. \end{cases} \tag{3.13}$$

Lemma 3.5. *For any $p > 0$, $t > 0$, $h > 0$, and $c \in \mathbb{R}$,*

$$\begin{aligned} E\left(|X_t - tc|/\sqrt{V_t}\right)^p \leq{}& \max\{2^{2p-2}, 1\}\ \left(M_p(t,h) + (t|\nu(h) - c|)^p\right) E(V_{t/2}^{-p/2}) \\ &+ \pi_{p/2}(t,h). \end{aligned} \tag{3.14}$$

Proof of Lemma 3.5. Use the Lévy-Itô decomposition (see (3.1) and the definitions following it) to write, for $t > 0$, $h > 0$,

$$\frac{|X_t - tc|}{\sqrt{V_t}} \leq \frac{|t(\nu(h) - c) + \sigma Z_t + X_t^{(S,h)}|}{\sqrt{V_t}} + \frac{|X_t^{(B,h)}|}{\sqrt{V_t^{(B,h)}}}. \tag{3.15}$$

First we deal with the small jump part. Write

(3.16)

$$
\begin{aligned}
&E\left(\left|\sigma Z_t + X_t^{(S,h)}\right| / \sqrt{V_t}\right)^p \\
&\le E\left(\left(\left|\sigma Z_{t/2} + X_{t/2}^{(S,h)}\right| + \left|\sigma(Z_t - Z_{t/2}) + X_t^{(S,h)} - X_{t/2}^{(S,h)}\right|\right) / \sqrt{V_t}\right)^p \\
&\le E\left(\left(\left|\sigma Z_{t/2} + X_{t/2}^{(S,h)}\right| / \sqrt{V_t - V_{t/2}} + \left|\sigma(Z_t - Z_{t/2}) + X_t^{(S,h)} - X_{t/2}^{(S,h)}\right| / \sqrt{V_{t/2}}\right)\right)^p \\
&\le 2\max\{2^{p-1}, 1\}\, E\left|\sigma Z_{t/2} + X_{t/2}^{(S,h)}\right|^p E\left(V_{t/2}^{-p/2}\right),
\end{aligned}
$$

where in (3.16) the last inequality is implied by the c_r-inequality

$$
E|\xi + \eta|^p \le \max\{2^{p-1}, 1\}\,(E|\xi|^p + E|\eta|^p) \qquad (\text{when } p > 0)\,. \tag{3.17}
$$

For $0 < p \le 2$ we have by Jensen's inequality

$$
E\left|\sigma Z_{t/2} + X_{t/2}^{(S,h)}\right|^p \le \left(E\left(\sigma Z_{t/2} + X_{t/2}^{(S,h)}\right)^2\right)^{p/2} = 2^{-p/2}(tV(h))^{p/2}. \tag{3.18}
$$

For the case $p > 2$, recall again the Lévy-Itô decomposition in (3.1). Now for any $0 < \varepsilon < h \le 1$,

$$
\sum_{0<s\le t} \Delta X_s 1_{\{\varepsilon<|\Delta X_s|\le h\}} - t\int_{\varepsilon<|x|\le h} x\Pi(\mathrm{d}x) \overset{\mathrm{D}}{=} \sum_{i=1}^{N_t} J_i - EN_t EJ_1,
$$

where the J_i are i.i.d. random variables, $i = 1, 2, \ldots$, with distribution

$$
\frac{\Pi(\mathrm{d}x)}{\overline{\Pi}(\varepsilon) - \overline{\Pi}(h)} \mathbf{1}_{\{\varepsilon<|x|\le h\}},
$$

and N_t is a Poisson random variable with expectation $t\left(\overline{\Pi}(\varepsilon) - \overline{\Pi}(h)\right)$, independent of the J_i. Lemma 3.4 then gives, for any $p > 2$,

$$
\begin{aligned}
&E\left|\sum_{0<s\le t} \Delta X_s 1_{\{\varepsilon<|\Delta X_s|\le h\}} - t\int_{\varepsilon<|x|\le h} x\Pi(\mathrm{d}x)\right|^p \\
&\le C_p \max\left[\left(EN_t EJ_1^2\right)^{p/2},\ EN_t E|J_1|^p\right] \\
&= C_p \max\left[\left(t\int_{\varepsilon<|x|\le h} x^2\Pi(\mathrm{d}x)\right)^{p/2},\ t\int_{\varepsilon<|x|\le h} |x|^p\Pi(\mathrm{d}x)\right] \\
&\le C_p \max\left[(tV_2(h))^{p/2}, tV_p(h)\right]. \qquad (3.19)
\end{aligned}
$$

Letting $\varepsilon \downarrow 0$ and using Fatou's lemma, we get from the inequalities in (3.19) that

$$
E\left|X_t^{(S,h)}\right|^p \le C_p \max[(tV_2(h))^{p/2}, tV_p(h)] \quad (\text{when } p > 2), \tag{3.20}
$$

and, thus, by the c_r-inequality again,

$$
\begin{aligned}
E\left|\sigma Z_{t/2} + X_{t/2}^{(S,h)}\right|^p &\le 2^{p-1}\left(E|\sigma Z_{t/2}|^p + E|X_{t/2}^{(S,h)}|^p\right) \\
&\le 2^{p-2}\left(c_p\sigma^p t^{p/2} + C_p \max[(tV_2(h))^{p/2}, tV_p(h)]\right) \quad (\text{when } p > 2). \qquad (3.21)
\end{aligned}
$$

To summarise, for the small jump process, we get from (3.16), (3.18) and (3.21) that

$$E\left(\left| t(\nu(h)-c)+\sigma Z_t + X_t^{(S,h)}\right| / \sqrt{V_t}\right)^p \tag{3.22}$$
$$\leq \max\{2^{p-1},1\}\ \left(M_p(t,h)+(t|\nu(h)-c|)^p\right)\ E(V_{t/2}^{-p/2}),$$

for all $p>0$, $t>0$, $h>0$.

Next, for the big jump component, note that

$$\frac{\left|\sum_{0<s\leq t}\Delta X_s \mathbf{1}_{\{|\Delta X_s|\geq h\}}\right|}{\sqrt{\sum_{0<s\leq t}(\Delta X_s)^2\mathbf{1}_{\{|\Delta X_s|\geq h\}}}} \leq \sqrt{\sum_{0<s\leq t}\mathbf{1}_{\{|\Delta X_s|\geq h\}}}. \tag{3.23}$$

Thus when $0<p\leq 2$, by Jensen's inequality,

$$E\left(|X_t^{(B,h)}|/\sqrt{V_t^{(B,h)}}\right)^p \leq \left(E\left[\sum_{0<s\leq t}\mathbf{1}_{\{|\Delta X_s|\geq h\}}\right]\right)^{p/2} = \left(t\overline{\Pi}(h)\right)^{p/2}, \tag{3.24}$$

and when $2<p\leq 4$, by Jensen's inequality,

$$\begin{aligned} E\left(|X_t^{(B,h)}|/\sqrt{V_t^{(B,h)}}\right)^p &\leq E\left(\sum_{0<s\leq t}\mathbf{1}_{\{|\Delta X_s|\geq h\}}\right)^{p/2} \\ &\leq \left(E\left(\sum_{0<s\leq t}\mathbf{1}_{\{|\Delta X_s|\geq h\}}\right)^2\right)^{p/4} \\ &= \left(t\overline{\Pi}(h)+\left(t\overline{\Pi}(h)\right)^2\right)^{p/4}. \end{aligned} \tag{3.25}$$

When $q>2$, using (3.11),

$$E\,|N_t-EN_t|^q \leq C_q \max[(EN_t)^{q/2}, EN_t]$$

and

$$EN_t^q \leq 2^{q-1}\left(E\,|N_t-EN_t|^q+(EN_t)^q\right),$$

we get

$$EN_t^q \leq 2^{q-1}C_q\left(\max[(EN_t)^{q/2},EN_t]+(EN_t)^q\right). \tag{3.26}$$

Hence when $p>4$, using (3.23) and (3.26) with q replaced by $q=p/2$, we get

(3.27)

$$\begin{aligned} E\left(|X_t^{(B,h)}|/\sqrt{V_t^{(B,h)}}\right)^p &\leq E\left(\sum_{0<s\leq t}1_{\{|\Delta X_s|\geq h\}}\right)^{p/2} \\ &\leq 2^{(p-2)/2}\left(C_{p/2}\max\left[\left(t\overline{\Pi}(h)\right)^{p/4}, t\overline{\Pi}(h)\right]+\left(t\overline{\Pi}(h)\right)^{p/2}\right). \end{aligned}$$

To summarise, we get from (3.24), (3.25) and (3.27) that, for all $p,t,h>0$,

$$\max\{2^{p-1},1\}\ E\left(|X_t^{(B,h)}|/\sqrt{V_t^{(B,h)}}\right)^p \leq \pi_{p/2}(t,h). \tag{3.28}$$

In view of (3.17), bounds (3.15), (3.22), and (3.28) together prove (3.14). □

Now we need the following construction. Assume $\overline{\Pi}(0+) = \infty$. Recall the definition of U in (2.3). The function $x \mapsto x^{-2}U(x)$ is absolutely continuous with strictly negative derivative on $(0,\infty)$. Hence it is strictly decreasing on $(0,\infty)$ and tends to ∞ as $x \downarrow 0$ and to 0 as $x \to \infty$. Let φ be its inverse function. This is a continuous strictly decreasing function with $\varphi(0+) = \infty$ and $\varphi(\infty) = 0$, and $\varphi(1/t)$ solves

$$tU(\varphi(1/t)) = \varphi^2(1/t), \text{ for } t > 0. \tag{3.29}$$

Lemma 3.6. *Assume* $\overline{\Pi}(0+) = \infty$ *and*

$$\limsup_{y\to\infty} \frac{y|\nu(y) - c|}{U(y)} < \infty \tag{3.30}$$

for some $c \in \mathbb{R}$. *Then for* $t > 0$, $p > 0$, *and some finite constants* $A_p > 0$, $\widetilde{A}_p > 0$,

$$E\left(|X_t - tc|/\sqrt{V_t}\right)^p \le A_p + \widetilde{A}_p\left(\sigma^p t^{p/2} + \varphi^p(1/t)\right)E(V_{t/2}^{-p/2}). \tag{3.31}$$

Proof of Lemma 3.6. Since $x^2\overline{\Pi}(x) \le U(x)$, (3.29) implies $t\overline{\Pi}(\varphi(1/t)) \le 1$, which implies, for $p > 0$,

$$\pi_{p/2}(t, \varphi(1/t)) \le \begin{cases} 24, & 0 < p/2 \le 2, \\ 2^{3p}(C_{p/2} + 1), & p/2 > 2 \end{cases} \tag{3.32}$$

(see (3.13)). By (3.30) we have for some $D > 0$ and all $0 < t \le 1$,

$$t|\nu(\varphi(1/t)) - c| \le \frac{tD}{\varphi(1/t)}U(\varphi(1/t)) = D\varphi(1/t).$$

Thus for $p > 0$,

$$(t|\nu(\varphi(1/t)) - c|)^p \le D^p\varphi^p(1/t). \tag{3.33}$$

Now for $p \ge 2$ and $t > 0$,

$$\begin{aligned} tV_p(\varphi(1/t)) &\le tU_p(\varphi(1/t)) \\ &= tp\int_0^{\varphi(1/t)} x^{p-1}\overline{\Pi}(x)\,\mathrm{d}x \\ &\le tp\varphi^{p-2}(1/t)\,U(\varphi(1/t)) \\ &= p\varphi^p(1/t). \end{aligned}$$

Hence (see (3.12))

$$M_p(t, \varphi(1/t)) \le \begin{cases} 2\varphi^p(1/t), & 0 < p \le 2, \\ 2^{2p-2}c_p\sigma^p t^{p/2} + 2^{2p-2}\max[2^{p/2}, p]C_p\varphi^p(1/t), & p > 2. \end{cases} \tag{3.34}$$

Putting together (3.14), (3.32), (3.33), and (3.34), we get (3.31). □

We also need:

Lemma 3.7. *Suppose* (2.7) *holds with the given* c_0. *Then, with* $\alpha = 2(1 - c_0) \in (0, 2]$, *we have*

$$\liminf_{x\downarrow 0} \frac{U(\mu x)}{U(x)} \ge \mu^{2-\alpha}, \text{ for all } 0 < \mu < 1, \tag{3.35}$$

and for all $\varepsilon > 0$ with $c_0 + \varepsilon/2 < 1$, i.e., $0 < \varepsilon < \alpha$, there is an $x_0 = x_0(\varepsilon) > 0$ such that $0 < x \le x_0$ and $0 < \mu < 1$ imply

$$\frac{U(\mu x)}{U(x)} \ge \mu^{2-\alpha+\varepsilon}. \tag{3.36}$$

Further, for some $c_1 = c_1(\varepsilon) > 0$ and all $0 < y \le x_0$,

$$U(y) \ge c_1 y^{2-\alpha+\varepsilon}. \tag{3.37}$$

Proof of Lemma 3.7. If $\sigma^2 > 0$, then (3.35)–(3.37) are obvious, so take $\sigma^2 = 0$.

Assume (2.7) and choose $\varepsilon > 0$ with $c_0+\varepsilon/2 < 1$. Then there is an $x_0 = x_0(\varepsilon) > 0$ such that $0 < x \le x_0$ implies $x^2\overline{\Pi}(x) \le (c_0 + \varepsilon/2)U(x)$. Hence for $0 < \mu < 1$ and $0 < x \le x_0$,

$$\log\left(\frac{U(x)}{U(\mu x)}\right) = \int_{x\mu}^{x} \frac{2y\overline{\Pi}(y)}{U(y)}\mathrm{d}y \le 2(c_0 + \varepsilon/2)\int_{x\mu}^{x} \frac{\mathrm{d}y}{y} = \log \mu^{-2c_0-\varepsilon}.$$

Thus

$$\frac{U(\mu x)}{U(x)} \ge \mu^{2c_0+\varepsilon} = \mu^{2-\alpha+\varepsilon},$$

where $\alpha = 2(1 - c_0) \in (0, 2]$, proving (3.36). Letting $\varepsilon \downarrow 0$ in (3.36) gives (3.35). Further, take $x = x_0$ in (3.36) to get $U(\mu x_0) \ge \mu^{2-\alpha+\varepsilon}U(x_0)$. Then set $y = \mu x_0$ to write this as the lower bound in (3.37), valid for $0 < y \le x_0$, because then $\mu = y/x_0 \le 1$. □

Remark 3.3. (i) It's easy to show that (3.35) is also equivalent to

$$\limsup_{x\downarrow 0} \frac{U(\lambda x)}{U(x)} \le \lambda^{2-\alpha}, \text{ for all } \lambda > 1, \tag{3.38}$$

with $\alpha \in (0, 2]$. Further, (3.35) and (3.38) imply (2.7).

(ii) Versions of relations (3.35)–(3.38) hold with $V(x)$ in place of $U(x)$ if (2.7) is replaced by the equivalent condition

$$\limsup_{x\downarrow 0} \frac{x^2\overline{\Pi}(x)}{V(x)} = c < \infty. \tag{3.39}$$

(To be precise, the bounds in (3.35) and (3.38) then have to be multiplied by positive constants, in general.)

Let $L_V(z) = Ee^{-zV_t}$, $z > 0$, be the Laplace transform of the subordinator V_t. Then

$$L_V(z) = e^{-t\Psi_V(z)},$$

where

$$\Psi_V(z) = \sigma^2 z + \int_0^\infty (1 - e^{-zx})\mathrm{d}\Pi_V(x).$$

Lemma 3.8. *(i) We have the identity*

$$E\left(\frac{1}{V_t^{p/2}}\right) = \frac{1}{\Gamma(p/2)}\int_0^\infty e^{-t\Psi_V(z)} z^{p/2-1}\mathrm{d}z, \ t > 0, \ p > 0. \tag{3.40}$$

(ii) Suppose $X_t \in FC$ at 0. Then for all $p > 0$ and some $c_\pm > 0$ there is a $t_0 > 0$ such that

$$\frac{c_-}{\varphi^p(1/t)} \le E\left(\frac{1}{V_t^{p/2}}\right) \le \frac{c_+}{\varphi^p(1/t)}, \text{ for } 0 < t \le t_0. \tag{3.41}$$

Proof of Lemma 3.8. (i) For $p > 0$,

$$E\left(\frac{1}{V_t^{p/2}}\right) = \frac{1}{\Gamma(p/2)}\int_0^\infty L_V(z)z^{p/2-1}\mathrm{d}z,$$

which gives (3.40).

(ii) By Bertoin [3, Prop. 1, p. 74], there are positive constants c_1, c_2 such that

$$c_1 z(\sigma^2 + I_V(1/z)) \le \Psi_V(z) \le c_2 z(\sigma^2 + I_V(1/z))$$

for all $z > 0$, where

$$I_V(1/z) = \int_0^{1/z}\overline{\Pi}_V(y)\mathrm{d}y = \int_0^{1/z}\overline{\Pi}(\sqrt{y})\mathrm{d}y = 2\int_0^{1/\sqrt{z}} y\overline{\Pi}(y)\mathrm{d}y.$$

Thus

$$c_1 zU(1/\sqrt{z}) \le \Psi_V(z) \le c_2 zU(1/\sqrt{z}) \tag{3.42}$$

for all $z > 0$, and so from (3.40),

$$\begin{aligned}\frac{1}{\Gamma(p/2)}&\int_0^\infty e^{-c_2 tzU(1/\sqrt{z})}z^{p/2-1}\mathrm{d}z\\ &\le E\left(\frac{1}{V_t^{p/2}}\right) \le \frac{1}{\Gamma(p/2)}\int_0^\infty e^{-c_1 tzU(1/\sqrt{z})}z^{p/2-1}\mathrm{d}z,\end{aligned}$$

or, equivalently,

$$\frac{2}{\Gamma(p/2)}\int_0^\infty e^{-c_2 tU(x)/x^2}x^{-p-1}\mathrm{d}x \le E\left(\frac{1}{V_t^{p/2}}\right) \le \frac{2}{\Gamma(p/2)}\int_0^\infty e^{-c_1 tU(x)/x^2}x^{-p-1}\mathrm{d}x. \tag{3.43}$$

Recalling that φ is the inverse function to $x^{-2}U(x)$, we get from (3.43) that

$$\frac{2}{\Gamma(p/2)}\int_0^\infty e^{-c_2 ts}\frac{|\mathrm{d}\varphi(s)|}{\varphi^{p+1}(s)} \le E\left(\frac{1}{V_t^{p/2}}\right) \le \frac{2}{\Gamma(p/2)}\int_0^\infty e^{-c_1 ts}\frac{|\mathrm{d}\varphi(s)|}{\varphi^{p+1}(s)}. \tag{3.44}$$

We can invert the inequality (3.36) as follows. (3.36) implies, for all $0 < \mu < 1$ and $0 < x \le x_0(\varepsilon)$,

$$\frac{U(\mu x)}{(\mu x)^2} \ge \mu^{-\alpha+\varepsilon}\frac{U(x)}{x^2};$$

hence $\mu x \le \varphi(\mu^{-\alpha+\varepsilon}y)$, where $y = U(x)/x^2$, and thus, $x = \varphi(y)$. Now $x \le x_0$ iff $y = U(x)/x^2 \ge U(x_0)/x_0^2 =: y_0 = y_0(\varepsilon)$. Then for all $0 < \mu < 1$ and $y \ge y_0(\varepsilon)$, we have

$$\mu\varphi(y) \le \varphi(\mu^{-\alpha+\varepsilon}y).$$

Let $\lambda = \mu^{-\alpha+\varepsilon}$ in this to get

$$\varphi(y) \le \lambda^{\frac{1}{\alpha-\varepsilon}}\varphi(\lambda y), \tag{3.45}$$

valid for all $\lambda \ge 1$ and $y \ge y_0(\varepsilon)$. Note that y_0 depends on ε only, not on λ.

Also, (3.37) gives $\varphi(y) \geq cy^{-\frac{1}{\alpha-\varepsilon}}$, for large y and some $c > 0$. Thus, integrating by parts in (3.44), we find, for $t > 0$,

$$
\begin{aligned}
E\left(\frac{1}{V_t^{p/2}}\right) &\leq \frac{2}{\Gamma(p/2)} \int_0^\infty e^{-c_1 ts} \frac{|\mathrm{d}\varphi(s)|}{\varphi^{p+1}(s)} \\
&= \frac{2c_1 t}{p\Gamma(p/2)} \int_0^\infty e^{-c_1 ts} \frac{\mathrm{d}s}{\varphi^p(s)} \\
&= \frac{c_1}{\Gamma(1+p/2)} \int_0^\infty e^{-c_1 s} \frac{\mathrm{d}s}{\varphi^p(s/t)} \\
&= \frac{c_1}{\Gamma(1+p/2)} \left(\int_0^1 + \int_1^\infty\right) e^{-c_1 s} \frac{\mathrm{d}s}{\varphi^p(s/t)}. \qquad (3.46)
\end{aligned}
$$

Then

$$
\begin{aligned}
\int_0^1 e^{-c_1 s} \frac{\mathrm{d}s}{\varphi^p(s/t)} &\leq \frac{1}{\varphi^p(1/t)} \int_0^1 e^{-c_1 s} \mathrm{d}s \\
&\leq \frac{1}{c_1 \varphi^p(1/t)}, \qquad (3.47)
\end{aligned}
$$

while by (3.45) we have for $0 < t \leq t_0(\varepsilon) := 1/y_0(\varepsilon)$ that

$$
\begin{aligned}
\int_1^\infty e^{-c_1 s} \frac{\mathrm{d}s}{\varphi^p(s/t)} &= \frac{1}{\varphi^p(1/t)} \int_1^\infty e^{-c_1 s} \left(\frac{\varphi(1/t)}{\varphi(s/t)}\right)^p \mathrm{d}s \\
&\leq \frac{1}{\varphi^p(1/t)} \int_1^\infty s^{\frac{p}{\alpha-\varepsilon}} e^{-c_1 s} \mathrm{d}s \\
&\leq \frac{\Gamma(\frac{p}{\alpha-\varepsilon}+1)}{c_1^{\frac{p}{\alpha-\varepsilon}+1} \varphi^p(1/t)}. \qquad (3.48)
\end{aligned}
$$

Together, (3.46), (3.47) and (3.48) prove the right-hand inequality in (3.41). For the left-hand inequality in (3.41), use (3.44) and integration by parts to write

$$
\begin{aligned}
E\left(\frac{1}{V_t^{p/2}}\right) &\geq \frac{c_2}{\Gamma(1+p/2)} \int_1^\infty e^{-c_2 s} \frac{\mathrm{d}s}{\varphi^p(s/t)} \\
&\geq \frac{e^{-c_2}}{\Gamma(1+p/2)\varphi^p(1/t)}.
\end{aligned}
$$

This completes Lemma 3.8. □

Lemma 3.9. *Assume $\sigma^2 > 0$ or $\Pi(\mathbb{R}) = \infty$, $E|X_t| < \infty$, $EX_t = t\mu$, $t > 0$, and write*

$$
X_t - \mu t =: \sum_{i=1}^n Y(i,t,n), \ t > 0, n \geq 1, \qquad (3.49)
$$

where for each $t > 0$,

$$
Y(i,t,n) := X_{it/n} - X_{(i-1)t/n} - t\mu/n.
$$

Then

(3.50)
$$\left(\sum_{i=1}^{n} Y(i,t,n),\ \sum_{i=1}^{n} Y^2(i,t,n)\right) = \left(X_t - t\mu,\ \sum_{i=1}^{n} Y^2(i,t,n)\right) \xrightarrow{\text{ucp}} (X_t - t\mu,\ V_t)\,,$$

where the convergence is ucp (in probability, uniform on compact subsets of t*; Protter* [24, p. 57]*).*

Proof of Lemma 3.9. The $Y(i,t,n)$ are i.i.d., $1 \le i \le n$, distributed as $X_{t/n} - t\mu/n$, and

$$\begin{aligned}\sum_{i=1}^{n} Y^2(i,t,n) &= \sum_{i=1}^{n}\left(X_{it/n}^2 - X_{(i-1)t/n}^2 - 2X_{(i-1)t/n}(X_{it/n} - X_{(i-1)t/n})\right)\\ &\qquad - 2t\mu\sum_{i=1}^{n}(X_{it/n} - X_{(i-1)t/n})/n + t^2\mu^2/n\\ &= X_t^2 - 2\sum_{i=1}^{n} X_{(i-1)t/n}(X_{it/n} - X_{(i-1)t/n}) - 2t\mu X_t/n + t^2\mu^2/n\\ &\xrightarrow{\text{ucp}} X_t^2 - 2\int_0^t X_{s-}\mathrm{d}X_s,\ \text{ as } n\to\infty, \end{aligned} \tag{3.51}$$

where the convergence holds by Theorem 21, p. 64, of Protter [24]. The last quantity is the quadratic variation of X, namely, V_t. So we have (3.50). □

4. Self-normalised LIL proofs

In this section we prove Theorems 2.1–2.3. We begin by developing a general methodology. This is then applied in the following subsections to establish the upper and lower bound parts of each of the three LIL theorems.

Proposition 4.1 (Upper bound)**.** *Suppose that for some constants* $\gamma_1 > 0$ *and* $\varepsilon_0 > 0$*, for all* $0 < \varepsilon < \varepsilon_0$*,*

$$\limsup_{t\downarrow 0} \frac{\log P\left((X_t - t\gamma)/\sqrt{V_t} > \sqrt{\gamma_1(1+\varepsilon)\log\log(1/t)}\right)}{\log\log(1/t)} \le -(1+\varepsilon)\,, \tag{4.1}$$

and for every $p > 1$ *there exist* $t_p > 0$ *and* $0 < B_p < \infty$ *such that*

$$\sup_{0<t\le t_p} E\left|(X_t - t\gamma)/\sqrt{V_t}\right|^p \le B_p. \tag{4.2}$$

Then

$$\limsup_{t\downarrow 0} \frac{X_t - t\gamma}{\sqrt{\gamma_1 V_t \log\log(1/t)}} \le 1,\ \textit{a.s.} \tag{4.3}$$

Proof of Proposition 4.1. Define for $r \ge 2$

$$\delta_r = \exp(-r/(\log r)^2),$$

and let

$$\Lambda_r = \sup_{\delta_{r+1}<t\le\delta_r} \frac{X_t - t\gamma}{\sqrt{V_t}}.$$

Note that since $\delta_r/\delta_{r+1} \to 1$, we have for all $\varepsilon_0 > \varepsilon > \varepsilon' > 0$ and all large r,

$$P\left(\Lambda_r > \left(\gamma_1 (1+\varepsilon)\log r\right)^{1/2}\right)$$
$$\leq P\left(\sup_{0\leq s\leq \delta_r-\delta_{r+1}} \frac{\delta_{r+1}}{\delta_{r+1}+s}\frac{X_{\delta_{r+1}+s}-(\delta_{r+1}+s)\gamma}{\sqrt{V_{\delta_{r+1}+s}}} > \left(\gamma_1(1+\varepsilon')\log r\right)^{1/2}\right),$$

which by inequality (3.8) does not exceed, for any $p > 1$ and $0 < c < 1$,

(4.4)
$$\frac{P^{1/q}\left(\left(X_{\delta_{r+1}}-\delta_{r+1}\gamma\right)/\sqrt{V_{\delta_{r+1}}} > c\left(\gamma_1(1+\varepsilon')\log r\right)^{1/2}\right)E^{1/p}\left(\left|\left(X_{\delta_{r+1}}-\delta_{r+1}\gamma\right)/\sqrt{V_{\delta_{r+1}}}\right|^p\right)}{\left(\gamma_1(1+\varepsilon')\log r\right)^{1/2}(1-c)}$$

where $q = 1 - 1/p$. (Recall that in inequality (3.8) we can replace $X_\cdot$ by $X_\cdot - \cdot\gamma_\cdot$.) Now since

$$(\log r)/\log\log(1/\delta_{r+1}) \to 1, \text{ as } r \to \infty,$$

we see by (4.1) and (4.2) that for any $0 < \varepsilon'' < \varepsilon'$ and for all large enough r, the bound in (4.4) is

$$\leq \exp\left(-q^{-1}c^2(1+\varepsilon'')\log r\right)B_p^{1/p}. \tag{4.5}$$

Clearly by choosing $p > 0$ large enough and $0 < c < 1$ close enough to 1, so that $q^{-1}c^2(1+\varepsilon'') > 1$, we get from the bound (4.5) that

$$\sum_{r=2}^{\infty} P\left(\Lambda_r > \left(\gamma_1(1+\varepsilon)\log r\right)^{1/2}\right) < \infty,$$

which by the Borel-Cantelli lemma implies (4.3), since $\varepsilon > 0$ can be chosen as close to 0 as desired. □

Proposition 4.2 (Lower bound)**.** *Suppose*

$$\limsup_{t\downarrow 0} \frac{|X_t - t\gamma|}{\sqrt{V_t \log\log(1/t)}} < \infty, \text{ a.s.}, \tag{4.6}$$

and there are constants $\gamma_2 > 0$ and $1 > \varepsilon_0 > 0$ such that, for all $0 < \varepsilon < \varepsilon_0$,

$$\liminf_{t\downarrow 0} \frac{\log P\left((X_t - t\gamma)/\sqrt{V_t} > \sqrt{\gamma_2(1-\varepsilon)\log\log(1/t)}\right)}{\log\log(1/t)} \geq -(1-\varepsilon). \tag{4.7}$$

Then

$$\limsup_{t\downarrow 0} \frac{X_t - t\gamma}{\sqrt{\gamma_2 V_t \log\log(1/t)}} \geq 1, \text{ a.s.} \tag{4.8}$$

Proof of Proposition 4.2. For any $\beta > 1$ and $k > (\log 2)^{-1/\beta}$, define

$$\eta_k = 2^{-k^\beta} \text{ and } \lambda_k = \sum_{j=k}^{\infty} \eta_j.$$

Observe that, by our choice of k, we have $-\log\eta_k = k^\beta \log 2 > 1$. For future use, note that since for $j \geq k$,

$$j^\beta - k^\beta \geq \beta(j-k)k^{\beta-1} \geq \beta(j-k),$$

we have

$$2^{-k^\beta} \le \lambda_k \le 2^{-k^\beta} \sum_{j=k}^{\infty} 2^{-j^\beta + k^\beta} \le 2^{-k^\beta} \sum_{j=0}^{\infty} 2^{-\beta j} = 2^{-k^\beta} / \left(1 - 2^{-\beta}\right), \tag{4.9}$$

which implies, in particular,

$$\log\log\left(1/\lambda_{k-1}\right) / \left(\beta \log k\right) \to 1, \text{ as } k \to \infty. \tag{4.10}$$

Next, for $k > (\log 2)^{-1/\beta} + 1 =: \kappa_\beta$, set

$$S(k) = X_{\lambda_{k-1}} - X_{\lambda_k} - (\lambda_{k-1} - \lambda_k)\gamma \quad \text{and} \quad V(k) = \sum_{\lambda_k < t \le \lambda_{k-1}} (\Delta X_t)^2 .$$

We see that for each $k > \kappa_\beta$,

$$S(k) / \sqrt{V(k)} \overset{\mathrm{D}}{=} \left(X_{\eta_{k-1}} - \eta_{k-1}\gamma\right) / \sqrt{V_{\eta_{k-1}}},$$

and $\left\{S(k)/\sqrt{V(k)}\right\}_{k>\kappa_\beta}$ are independent random variables. As $\log\log\left(1/\eta_{k-1}\right) / \log k \to \beta$ when $k \to \infty$, given $0 < \varepsilon' < \varepsilon < \varepsilon_0 < 1 < \beta$ with $(1-\varepsilon')\beta < 1$, we have by (4.7)

$$P\left(S(k) / \sqrt{V(k)} > \sqrt{\gamma_2 (1-\varepsilon)\beta \log k}\right) \ge \exp\left(-(1-\varepsilon')\beta \log k\right),$$

for all large enough k. Since $(1-\varepsilon')\beta < 1$,

$$\sum_{k>\kappa_\beta} P\left(S(k) / \sqrt{V(k)} > \sqrt{\gamma_2 (1-\varepsilon)\beta \log k}\right) = \infty,$$

which implies by the Borel-Cantelli lemma that

$$\limsup_{k\to\infty} \frac{S(k)}{\sqrt{\gamma_2 \beta V(k) \log k}} \ge \sqrt{1-\varepsilon}, \text{ a.s.} \tag{4.11}$$

Applying (3.5) to V_t we get for $h > 0$, $t > 0$, $s > 0$,

$$E\left(\frac{V_t^{(S,h)}}{V_{t+s}^{(S,h)}} \middle| \sigma(V_y^{(S,h)}, y \ge t+s)\right) = \frac{t}{t+s},$$

which, since $\mathcal{G}_1 := \sigma(V_y^{(S,h)}, y \ge t+s)$ and $\mathcal{G}_2 := \sigma(V_y^{(B,h)}, y \ge t+s)$ are independent σ-fields and $\sigma(V_y, y \ge t+s) = \sigma(\mathcal{G}_1 \cup \mathcal{G}_2)$, gives

$$E\left(\frac{V_t^{(S,h)}}{V_{t+s}^{(S,h)}} \middle| \sigma(V_y^{(S,h)}, y \ge t+s)\right) = E\left(\frac{V_t^{(S,h)}}{V_{t+s}^{(S,h)}} \middle| \sigma(V_y, y \ge t+s)\right) = \frac{t}{t+s}.$$

Now by letting $h \to \infty$ we get, by dominated convergence,

$$E\left(\frac{V_t}{V_{t+s}} \middle| \sigma(V_y, y \ge t+s)\right) = \frac{t}{t+s}.$$

It follows that

$$E\left(\frac{V_{\lambda_k}}{V_{\lambda_{k-1}}}\right) = \frac{\lambda_k}{\lambda_{k-1}} \le \frac{2^{-k^\beta} \sum_{j=k}^{\infty} 2^{-j^\beta + k^\beta}}{2^{-(k-1)^\beta}},$$

which by (4.9) and $(k-1)^\beta - k^\beta \le -\beta (k-1)^{\beta-1}$ is bounded above by

$$2^{-\beta (k-1)^{\beta-1}} / (1 - 2^{-\beta}).$$

Thus $\sum_{k>\kappa_\beta} E\left(V_{\lambda_k}/V_{\lambda_{k-1}}\right) < \infty$, which implies that

$$\frac{V_{\lambda_k}}{V_{\lambda_{k-1}}} \to 0, \text{ a.s.} \tag{4.12}$$

By assumption (4.6) and (4.10),

$$\limsup_{k\to\infty} \frac{|X_{\lambda_k} - \lambda_k\gamma|}{\sqrt{V_{\lambda_k}\log k}} < \infty.$$

Hence by (4.12),

$$\lim_{k\to\infty} \frac{|X_{\lambda_k} - \lambda_k\gamma|}{\sqrt{V_{\lambda_{k-1}}\log k}} = 0, \text{ a.s.} \tag{4.13}$$

We now get by (4.10), (4.11) and (4.13) that

$$\begin{aligned}
&\limsup_{k\to\infty} \frac{X_{\lambda_{k-1}} - \lambda_{k-1}\gamma}{\sqrt{\gamma_2 V_{\lambda_{k-1}}\log\log\left(1/\lambda_{k-1}\right)}} \\
&\geq \limsup_{k\to\infty} \frac{S\left(k\right)}{\sqrt{\gamma_2\beta V\left(k\right)\log k}} - \lim_{k\to\infty} \frac{|X_{\lambda_k} - \lambda_k\gamma|}{\sqrt{\gamma_2\beta V_{\lambda_{k-1}}\log k}} \\
&\geq \sqrt{1-\varepsilon}, \text{ a.s.},
\end{aligned}$$

which implies (4.8) since $\varepsilon > 0$ can be chosen arbitrarily close to 0. □

4.1. **Proof of Theorem 2.1.** Assume $\sigma^2 > 0$ or $\overline{\Pi}(0+) = \infty$ and $X_t \in D(N)$ at 0.

We begin with the observation that the random time $\tau := \inf\{t > 0 : |\Delta X_t| > 1\}$ is exponential with expectation $1/\overline{\Pi}(1)$ if $\overline{\Pi}(1) > 0$ and $+\infty$ otherwise; hence $\tau > 0$ a.s., and we have $X_t^{(B,1)} = V_t^{(B,1)} = 0$ for $0 \leq t \leq \tau$. So by (3.1) and (3.2), to prove an LIL for

$$\frac{X_t - t\gamma}{\sqrt{V_t}}$$

as $t \downarrow 0$, it suffices to establish one for $(\sigma Z_t + X_t^{(S,1)})/\sqrt{V_t^{(S,1)}}$. Therefore in the remainder of the proof of Theorem 2.1 we shall assume without loss of generality that

$$(X_t - t\gamma, V_t) = \left(\sigma Z_t + X_t^{(S,1)}, V_t^{(S,1)}\right),$$

equivalently, $\overline{\Pi}(1) = 0$. Then, in particular, X_t and V_t have finite moments of all orders.

Since $X_t \in D(N)$ at 0, (2.4) and (2.5) hold. Thus (3.30) holds with $c = \gamma$, and (2.7) holds with $c_0 = 0$, so $X_t \in FC$ at 0 and (3.41) of Lemma 3.8 is applicable. In (3.41), $\varphi(1/t)$ is in $RV(1/2)$ as $t \downarrow 0$, because $U(x)$ is slowly varying at 0; this follows immediately from (3.35) with $c_0 = 0$ and from (3.29). (Note that if $\sigma^2 > 0$, $\varphi(1/t) \sim \sigma\sqrt{t}$ as $t \downarrow 0$, and then (3.41) of Lemma 3.8 gives $EV_t^{-p/2} \asymp t^{-p/2}$ as $t \downarrow 0$.) We now deduce from (3.31) that for each $p > 1$ there is a $0 < B_p < \infty$ with

$$\limsup_{t\downarrow 0} E\left(|X_t - t\gamma|/\sqrt{V_t}\right)^p \leq B_p. \tag{4.14}$$

Next we apply the Cramér-type bound given in Theorem 5.2. (Theorem 5.2 is proved independently. Refer to Section 5 for its statement and proof.) From (5.9)

it is easy to conclude that (4.1) holds with $\gamma_1 = 2$ both for $(X_t - t\gamma)/\sqrt{V_t}$ and for $-(X_t - t\gamma)/\sqrt{V_t}$. Thus we can apply Proposition 4.1 to get

$$\limsup_{t\downarrow 0} \frac{|X_t - t\gamma|}{\sqrt{2V_t \log\log(1/t)}} \le 1, \text{ a.s.} \tag{4.15}$$

Also from (5.9) we get that (4.7) holds with $\gamma_2 = 2$. Hence by Proposition 4.2,

$$\limsup_{t\downarrow 0} \frac{X_t - t\gamma}{\sqrt{2V_t \log\log(1/t)}} \ge 1, \text{ a.s.} \tag{4.16}$$

This completes the proof of Theorem 2.1. □

The next lemma allows us to delete the centering in Theorem 2.1.

Lemma 4.1. *Suppose $\sigma^2 > 0$ or $\overline{\Pi}(0+) = \infty$ and (3.37) holds with $\alpha \in (0,2]$. Then*

$$\lim_{t\downarrow 0} \frac{V_t}{t^\beta} = \infty, \text{ a.s., for all } \beta > 2/\alpha. \tag{4.17}$$

Proof of Lemma 4.1. The result can be deduced quickly from Theorem 3.3 of Bertoin, Doney and Maller [4]. We omit the details. □

4.2. **Proof of Theorem 2.2.** Assume $\overline{\Pi}(0+) = \infty$, $\overline{\Pi}(1) = 0$ and X_t is symmetric about 0. First we shall show that for some $\varepsilon_0 > 0$, (4.1) holds with $\gamma_1 = 2$ for all $0 < \varepsilon < \varepsilon_0$. We have, for all $x > 0$,

$$P\left(\pm X_t/\sqrt{V_t} > x\right) \le e^{-x^2/2}. \tag{4.18}$$

To see this, define

$$X_t = \sum_{i=1}^n Y(i,t,n), \ t > 0, n \ge 1,$$

where $Y(i,t,n) := X_{it/n} - X_{(i-1)t/n}$. Since X_t is a symmetric Lévy process,

$$\frac{\pm\sum_{i=1}^n Y(i,t,n)}{\sqrt{\sum_{i=1}^n Y^2(i,t,n)}} \stackrel{\mathrm{D}}{=} \frac{\sum_{i=1}^n s_i Y(i,t,n)}{\sqrt{\sum_{i=1}^n Y^2(i,t,n)}},$$

where $s, s_1, \ldots, s_n$, are i.i.d. with $P\{s = -1\} = P\{s = 1\} = 1/2$, independent of $Y(i,t,n)$, $i = 1, 2, \ldots, n$. Conditioning on $Y(1,t,n), \ldots, Y(n,t,n)$ and applying Hoeffding's [16] inequality, we get

$$P\left(\frac{\sum_{i=1}^n s_i Y(i,t,n)}{\sqrt{\sum_{i=1}^n Y^2(i,t,n)}} > x\right) \le e^{-x^2/2}.$$

Now apply Lemma 3.9 in which we can take $\mu = 0$ by the symmetry to get

$$\lim_{n\to\infty} P\left(\frac{\pm\sum_{i=1}^n Y(i,t,n)}{\sqrt{\sum_{i=1}^n Y^2(i,t,n)}} > x\right) = P\left(\frac{\pm X_t}{\sqrt{V_t}} > x\right).$$

This proves (4.18). Thus the first upper bound condition (4.1) of Proposition 4.1 holds with $\gamma_1 = 2$ and any $\varepsilon_0 > 0$. Moreover, from inequality (4.18) we can also infer that for every $p > 1$ there exists a $B_p \in (0,\infty)$ such that (4.2) holds with $\gamma = 0$, independently of $t > 0$. Hence (2.9) holds.

We shall now show that when X_t is also in the Feller class at zero, condition (4.7) of Proposition 4.2 is satisfied. This will follow from the next two lemmas. Note that symmetry is not assumed in these two lemmas.

Lemma 4.2. *Suppose there exists a nonstochastic function $b(t) > 0$ such that every sequence $t_k \downarrow 0$ contains a subsequence $t_{k'} \downarrow 0$ such that*

$$X_{t_{k'}}/b(t_{k'}) \xrightarrow{\mathrm{D}} Y',$$

where Y' is a finite rv which does not place positive mass at 0. Then

$$\lim_{c\downarrow 0} \limsup_{t\downarrow 0} P\left(|X_t/b(t)| \le c\right) = 0. \tag{4.19}$$

Proof of Lemma 4.2. Suppose on the contrary that there exists an $\varepsilon > 0$ such that

$$\lim_{c\downarrow 0} \limsup_{t\downarrow 0} P\left(|X_t/b(t)| \le c\right) > \varepsilon. \tag{4.20}$$

Let $c_1 > 0$ and $t_1 > 0$ be such that

$$P\left(|X_{t_1}/b(t_1)| \le c_1\right) > \varepsilon.$$

Now let $0 < t_2 < t_1/2$ be such that

$$P\left(|X_{t_2}/b(t_2)| \le c_1/2\right) > \varepsilon.$$

Continuing, we select $0 < t_{m+1} < t_m/2^m$ such that

$$P\left(\left|X_{t_{m+1}}/b(t_{m+1})\right| \le c_1/2^m\right) > \varepsilon.$$

We can find a subsequence $t_{m'}$ of t_{m+1} such that

$$X_{t_{m'}}/b(t_{m'}) \xrightarrow{\mathrm{D}} Y'.$$

Since, by assumption, Y' does not place positive mass at 0, 0 is a continuity point of G, the cumulative distribution of $|Y'|$. Therefore we can find a continuity point $y > 0$ of G as close to 0 as desired such that $0 < y < c_1$ and $P\left(|Y'| \le y\right) < \varepsilon/2$. Now since

$$P\left(\left|X_{t_{m'}}/b(t_{m'})\right| \le y\right) \to P\left(|Y'| \le y\right)$$

we see that $P\left(\left|X_{t_{m'}}/b(t_{m'})\right| \le y\right) \le \varepsilon$ for all large enough m', which contradicts the fact that

$$P\left(\left|X_{t_{m'}}/b(t_{m'})\right| \le c_1/2^{m'}\right) > \varepsilon \text{ for all } m'. \qquad \square$$

Lemma 4.3. *Suppose $X_t \in FC_0$ at 0. Then there exist $\delta > 0$ and $0 < t_0 < 1$ such that for all integers $r > 1$ and $0 < t \le t_0$,*

$$P\left(|X_{tr}|/\sqrt{V_{rt}} > \delta\sqrt{r}\right) \ge \exp\left(-r\right). \tag{4.21}$$

Proof of Lemma 4.3. Suppose $X_t \in FC_0$ with norming function $b(t)$. Choose any integer $r > 1$. By Proposition 6.2 in the Appendix, every subsequential limit law of $X_t/b(t)$ has a density and hence does not place positive mass at 0. Thus we can apply Lemma 4.2 to infer the existence of $c > 0$ and $0 < t_0 < 1$ such that for all $0 < t \le t_0$,

$$P\left(|X_t/b(t)| > c\right) > \frac{18}{20}.$$

Clearly for each such t either $P\left(X_t/b(t) > c\right) > 9/20$ or $P\left(X_t/b(t) < -c\right) > 9/20$. Suppose

$$P\left(X_t/b(t) > c\right) > 9/20. \tag{4.22}$$

By Theorem 2.1 of Maller and Mason [22], $X_t \in FC_0$ at 0 implies that $V_t/b^2(t)$ is stochastically bounded as $t \downarrow 0$. Thus there exist $M > 0$ and $0 < s_0 < 1$ such that for all $0 < t \leq s_0$,

$$P\left(V_t/b^2(t) > M^2\right) < 9/20 - e^{-1}. \tag{4.23}$$

Thus if $0 < t \leq t_0 \wedge s_0$ and (4.22) holds, then

$$P\left(X_t/b(t) > c, \sqrt{V_t}/b(t) \leq M\right) \geq P\left(X_t/b(t) > c\right) - P\left(V_t/b^2(t) > M^2\right) > e^{-1}. \tag{4.24}$$

Set $\Delta X_t(i) = X_{it} - X_{(i-1)t}$ and $\Delta V_t(i) = V_{it} - V_{(i-1)t}$ for $i = 1, \ldots, r$. We have by independence

$$\begin{aligned} P\left(|X_{tr}|/\sqrt{V_{rt}} > c\sqrt{r}/M\right) &\geq P\left(X_{tr}/b(t) > cr, \sqrt{V_{rt}}/b(t) \leq M\sqrt{r}\right) \\ &\geq P\left(\Delta X_t(i)/b(t) > c, \sqrt{\Delta V_t(i)}/b(t) \leq M, i = 1, \ldots, r\right) \\ &\geq e^{-r}. \end{aligned}$$

The same inequality holds if $P\left(X_t/b(t) < -c\right) > 9/20$. □

Remark 4.1. Suppose $X_t \in FC$ at 0 and X_t is symmetric about 0. The same proof shows that there exist $\delta > 0$ and $0 < t_0 < 1$ such that for all integers $r > 1$ and $0 < t \leq t_0$,

$$P\left(X_{tr}/\sqrt{V_{rt}} > \delta\sqrt{r}\right) \geq \exp(-r). \tag{4.25}$$

Returning to the proof of Theorem 2.2, choose any $0 < \varepsilon < 1$. Set for $0 < t < e^{-2}$

$$r(t) = \lceil (1-\varepsilon)\log\log(1/t)\rceil.$$

We get by Lemma 4.3, for some $t_0 > 0$ and $\delta > 0$, and for all $0 < t/r(t) \leq t_0$ (note that we write $t = (t/r(t))\, r(t)$), that

$$\begin{aligned} \frac{\log P\left(|X_t|/\sqrt{V_t} > \delta\sqrt{(1-\varepsilon)\log\log(1/t)}\right)}{\log\log(1/t)} &\geq \frac{\log P\left(|X_t|/\sqrt{V_t} > \delta\sqrt{r(t)}\right)}{\log\log(1/t)} \\ &\geq -\frac{\lceil (1-\varepsilon)\log\log(1/t)\rceil}{\log\log(1/t)}, \end{aligned}$$

which implies

$$\liminf_{t\downarrow 0} \frac{\log P\left(|X_t|/\sqrt{V_t} > \delta\sqrt{(1-\varepsilon)\log\log(1/t)}\right)}{\log\log(1/t)} \geq -(1-\varepsilon).$$

The same observation holds with $|X_t|/\sqrt{V_t}$ replaced by $X_t/\sqrt{V_t}$ when X_t is symmetric about 0. This shows that condition (4.7) of Proposition 4.2 is satisfied with $\gamma_2 = \delta^2$ and any $1 > \varepsilon_0 > 0$. This completes the proof of Theorem 2.2. □

4.3. **Proof of Theorem 2.3.** Assuming X_t is a strictly $\alpha-$stable process, one can verify, as in the appendix of Maller and Mason [23], that for each $t > 0$,

$$\left(t^{-1/\alpha}X_t, t^{-2/\alpha}V_t\right) \stackrel{\mathrm{D}}{=} (X_1, V_1),$$

which of course implies

$$T_t := X_t/\sqrt{V_t} \stackrel{\mathrm{D}}{=} X_1/\sqrt{V_1} =: T,\ t > 0. \tag{4.26}$$

(Note that X_1 is strictly stable with index $0 < \alpha < 2$.) Moreover, it can be shown that if $\xi_1, \xi_2, \ldots$ are i.i.d. as X_1, then as $n \to \infty$,

$$T_n := \sum_{i=1}^{n} \xi_i / \sqrt{\sum_{i=1}^{n} \xi_i^2} \xrightarrow{\mathrm{D}} X_1/\sqrt{V_1}.$$

(See the arguments in Maller and Mason [23].) From the results in Section 5 of Logan et al. [21] and the assumption $\Pi\{(0,\infty)\} = \infty$, we can conclude that T has a density that is positive on $(0,\infty)$. Further, by applying Theorem 3.2 of Shao [28], we get that for any sequence of positive constants $x_n \to \infty$ at the rate $x_n = o(\sqrt{n})$, (6.8) of Lemma 6.2 in the Appendix holds for a positive constant $\tau > 0$ whose value depends on the tails of X_1. (See Shao [28] for a description of this constant.) Thus (6.9) of Lemma 6.2 is satisfied. In particular, by the distributional identity (4.26) we get, for all $0 < \varepsilon < 1$ with $c = \tau^{-1}$,

$$\lim_{t \downarrow 0} \frac{\log P\left(X_t/\sqrt{V_t} > \sqrt{c\,(1+\varepsilon)\log\log(1/t)}\right)}{\log\log(1/t)} = -(1+\varepsilon) \tag{4.27}$$

and

$$\lim_{t \downarrow 0} \frac{\log P\left(X_t/\sqrt{V_t} > \sqrt{c\,(1-\varepsilon)\log\log(1/t)}\right)}{\log\log(1/t)} = -(1-\varepsilon). \tag{4.28}$$

We also have, for some constants $d > 0$ and $D > 0$, and for all $x \geq 0$,

$$P\{T > x\} \leq D e^{-d^2 x^2}. \tag{4.29}$$

If $\Pi\{(-\infty,0)\} = \infty$ we obtain by the same argument that (4.27), (4.28) and (4.29) hold with $X_t/\sqrt{V_t}$ and T replaced by $-X_t/\sqrt{V_t}$ and $-T$, respectively, and with c, d and D replaced by suitable $c_1 > 0$, $d_1 > 0$ and $D_1 > 0$, respectively. Whereas if $\Pi\{(-\infty,0)\} = 0$, then necessarily $0 < \alpha < 1$ and $X_t \geq 0$ for all $t > 0$. This provides us with the ingredients to apply Propositions 4.1 and 4.2, with $\gamma = 0$, so as to conclude (2.14). □

5. A Cramér bound for the self-normalised process

A Cramér-type bound for the self-normalised process is a crucial component of the proof of Theorem 2.1. In this section we prove such a theorem by transferring to continuous time a discrete time result of Robinson and Wang [25]. To state their result, let $Y, (Y_i)_{i=1,2,\ldots,n}$ be i.i.d. random variables, and for each $x > 0$ and $n \geq 1$ define

$$\kappa_n(x) := \sup\left\{y > 0 : \frac{EY^2 \mathbf{1}_{\{|Y| \leq y\}}}{y^2} \geq \frac{1+x^2}{n}\right\}. \tag{5.1}$$

Here and elsewhere we define the supremum of the empty set to be zero. Then, with $Y^\kappa = Y\mathbf{1}_{\{|Y| \leq \kappa_n(x)\}}$, set

$$\Delta_{n,x} := nP(|Y| > \kappa_n(x)) + \frac{n|EY^\kappa|}{\kappa_n(x)} + \frac{nE|Y^\kappa|^3}{\kappa_n^3(x)}. \tag{5.2}$$

(We suppress the dependence on n in Y^κ.) Let $\Phi(x)$ be the cdf of the standard normal distribution.

Robinson and Wang [25, Theorem 2] prove:

Theorem 5.1. *Let $Y, (Y_i)_{i=1,2,\dots,n}$ be i.i.d. rvs with $E|Y| < \infty$ and $EY = 0$. Then there is an absolute constant $A > 0$ such that*

$$e^{-A\Delta_{n,x}} \le \frac{P\left(\sum_{i=1}^n Y_i \ge x\sqrt{\sum_{i=1}^n Y_i^2}\right)}{1-\Phi(x)} \le e^{A\Delta_{n,x}}, \tag{5.3}$$

for each $n = 1, 2, \dots$ and $x \ge 0$ satisfying $\Delta_{n,x} \le (1+x^2)/A$.

Remark 5.1. Robinson and Wang state their theorem under the hypothesis that Y is in the domain of attraction of the normal distribution as $n \to \infty$ (denoted $Y \in D(N)$), meaning that $\sum_{i=1}^n Y_i$ is asymptotically normal, after centering and norming. However, this assumption is not actually used in their derivation of (5.3). Note, however, that (5.3) is only sharp when $\lim_{n\to\infty} \Delta_{n,x} = 0$, and this occurs iff $Y \in D(N)$ (Bentkus and Götze [1]).

Now let $(X_t)_{t\ge 0}$ be a Lévy process with $E|X_t| < \infty$, $EX_t = t\mu$, $t > 0$, quadratic variation process V_t as defined in (1.2), and with tails $\overline{\Pi}(x)$ and $\overline{\Pi}^{\pm}(x)$ of Π as defined in (2.1). Recall that V_t is positive, a.s., for all $t > 0$ when $\sigma^2 > 0$ or $\Pi(\mathbb{R}) = \infty$, and this also implies $V(y) > 0$ for all $y > 0$. For $t > 0$, $x > 0$, let

$$b(t,x) := \sup\left\{y > 0 : \frac{V(y)}{y^2} \ge \frac{1+x^2}{t}\right\}. \tag{5.4}$$

Then $0 \le b(t,x) < \infty$ for all $t > 0$, $x > 0$, $b(t,x)$ is nondecreasing in t, for all $x > 0$, and $\lim_{t\to 0} b(t,x) = 0$ for all $x > 0$. Further, Lemma 6.1 in the Appendix shows that $b(t,x)$ is a point of continuity of the function $x \mapsto x^{-2}V(x)$, thus satisfying

$$(1+x^2)b^2(t,x) = tV(b(t,x)),\ t > 0,\ x > 0. \tag{5.5}$$

Take $b > 0$ and define

$$\Delta(b) := \overline{\Pi}(b) + \frac{|\nu(b)-\mu|}{b} + \frac{\int_{0<|y|\le b} |y|^3 \Pi(\mathrm{d}y)}{b^3}. \tag{5.6}$$

Our main theorem in this section is:

Theorem 5.2. *Suppose $(X_t)_{t\ge 0}$ has $E|X_t| < \infty$ and $EX_t = t\mu$, $t > 0$, and assume $\sigma^2 > 0$ or $\Pi(\mathbb{R}) = \infty$. For a given $\eta \in (0,1)$, assume there is a $y_0 = y_0(\eta)$ such that*

$$y^2\overline{\Pi}(y) \le (1-\eta)V(y),\ \textit{whenever}\ 0 < y \le y_0. \tag{5.7}$$

(i) Then there is an absolute constant $A > 0$ such that

$$e^{-3At\Delta(b(\eta t,x))} \le \frac{P\left(X_t - t\mu \ge x\sqrt{V_t}\right)}{1-\Phi(x)} \le e^{3At\Delta(b(\eta t,x))} \tag{5.8}$$

for all $t > 0$ and $x \ge 0$ satisfying $t\Delta(b(\eta t,x)) \le (1+x^2)/(6A)$ and $b(t,0) \le y_0$.

(ii) In particular, (5.8) holds with $\mu = \gamma$ if X_t is the small jump component in (3.1) with $h = 1$, i.e., $X_t = \gamma t + \sigma Z_t + X_t^{(S,1)}$, and $V_t = \sigma^2 t + V_t^{(S,1)}$.

(iii) Suppose, in addition, in part (ii) that $X_t \in D(N)$ at 0. Then, for a given $\varepsilon \in (0,1)$, there is a $t_0 = t_0(\varepsilon)$ such that, for all $t > 0$ and $x \in \mathbb{R}$ satisfying $0 < t \le (1+x^2)t_0$, we have

$$\left|P\left(\sigma Z_t + X_t^{(S,1)} \le x\sqrt{V_t^{(S,1)}}\right) - \Phi(x)\right| \le \frac{4e^{-(1-\varepsilon)x^2/2}}{1+|x|}. \tag{5.9}$$

Remark 5.2. (i) When $\sigma^2 = 0$ and $\Pi(\mathbb{R}) < \infty$, X and V_t are compound Poisson processes and hence are zero in a random neighbourhood of 0. In this case we do not expect (5.8) to hold.

(ii) The constant A in (5.8) is the same as in (5.3).

(iii) The assumption (5.7) holds with η arbitrarily close to 1 when $X_t \in D(N)$ at 0, by (2.4). (5.7) as stated implies X_t is in a Feller compactness class at 0, with index $\alpha \in (1, 2]$; see (3.39) and (3.38).

Before proving Theorem 5.2 we need a preliminary proposition and a lemma.

Proposition 5.1. *Suppose $X_t \in D(N)$ at 0 with $E|X_t| < \infty$ and $EX_t = t\mu$, $t > 0$, and assume $\sigma^2 > 0$ or $\Pi(\mathbb{R}) = \infty$, so that $b(t, x) > 0$ for all $t > 0$, $x > 0$, and so $\Delta(b(t, x))$ is well defined (see (5.6)). Take $\varepsilon > 0$. Then there is a $t_0 = t_0(\varepsilon)$ such that for all $t > 0$ and $x \in \mathbb{R}$ satisfying $0 < t \le (1 + x^2)t_0$, we have*

$$t\Delta(b(t, x)) \le \varepsilon(1 + x^2); \tag{5.10}$$

thus $\lim_{t\to 0} t\Delta(b(t, x)) = 0$ for each $x > 0$. Conversely, $\lim_{t\to 0} t\Delta(b(t, x)) = 0$ for some $x > 0$ implies $X_t \in D(N)$ at 0.

When $X_t \in D(N)$ at 0, for every $0 < c < 1$ and $\beta > 0$ there is a $t_1(c, \beta) > 0$ such that $0 < t \le t_1$ implies

$$(1 + x^2)(b(t, x))^{2-\beta} \ge ct, \text{ for all } x > 0. \tag{5.11}$$

Proof of Proposition 5.1. Under the assumption $\sigma^2 > 0$ or $\Pi(\mathbb{R}) = \infty$, (5.7) implies $\lim_{y\to 0} y^{-2}V(y) = \infty$, and this means that $b(t, x) > 0$ for all $t > 0$, $x > 0$. Thus $\Delta(b(t, x))$ is well defined.

Now suppose $X_t \in D(N)$ at 0 so (2.4) and (2.5) hold and $V(y)$ is slowly varying at 0. Assume also that $E|X_t| < \infty$ and $EX_t = t\mu$, $t > 0$. Given $\varepsilon = 3\delta \in (0, 1)$ choose $y_0(\varepsilon) > 0$ such that

$$y^2\overline{\Pi}(y) + y|\nu(y) - \mu| \le \delta^4 V(y), \text{ for } 0 < y \le y_0. \tag{5.12}$$

Now fix $x > 0$. Note that $b(t, x)$ decreases in x for each $t > 0$, so $b(t, x) \le b(t, 0)$. Thus for $t \le t_0(\varepsilon)$ such that $b(t, 0) \le y_0$, we have by (5.12) that

$$t\overline{\Pi}(b(t, x)) + \frac{t|\nu(b(t, x)) - \mu|}{b(t, x)} \le \frac{\delta^4 tV(b(t, x))}{b^2(t, x)} = \delta^4(1 + x^2) < \delta(1 + x^2). \tag{5.13}$$

Next, still with $\varepsilon = 3\delta \in (0, 1)$, write

$$\begin{aligned}
\frac{t\int_{0<|y|\le b(t,x)} |y|^3\Pi(\mathrm{d}y)}{b^3(t, x)} &\le \frac{\delta t\int_{0<|y|\le\delta b(t,x)} y^2\Pi(\mathrm{d}y)}{b^2(t, x)} + \frac{t\int_{\delta b(t,x)<|y|\le b(t,x)} |y|^3\Pi(\mathrm{d}y)}{b^3(t, x)} \\
&\le \frac{\delta tV(b(t, x))}{b^2(t, x)} + t\overline{\Pi}(\delta b(t, x)) \\
&= \delta(1 + x^2) + \left(\frac{(\delta b(t, x))^2\overline{\Pi}(\delta b(t, x))}{V(\delta b(t, x))}\right)\left(\frac{tV(\delta b(t, x))}{\delta^2 b^2(t, x)}\right) \\
&\le \delta(1 + x^2) + \delta^4(1 + x^2)/\delta^2 \\
&\le 2\delta(1 + x^2), \qquad (5.14)
\end{aligned}$$

provided we keep $b(t, 0) \le y_0$, since then $\delta b(t, x) \le y_0$. Combining (5.13) and (5.14), recalling (5.6), and recalling that $\varepsilon = 3\delta$ gives, for $0 < t \le t_0$,

$$t\Delta(b(t, x)) \le \varepsilon(1 + x^2). \tag{5.15}$$

To extend this range, let $t_x = t/(1+x^2)$ and note that $b(t,x) = b(t_x, 0)$, and so

$$t_x \Delta(b(t,x)) = (1+x^2) t \Delta(b(t_x, 0)).$$

Then by (5.15), $t_x \Delta(b(t_x, x)) \le \varepsilon(1+x^2)$ if $0 < t_x \le t_0(\varepsilon)$, giving (5.10) for $0 < t \le (1+x^2)t_0$. Of course this also means $\lim_{t\to 0} t\Delta b((t,x)) = 0$ for each fixed $x > 0$.

Conversely, assume $\lim_{t\to 0} t\Delta(b(t,x)) = 0$ for some $x > 0$. Then for $\varepsilon \in (0,1)$,

$$\begin{aligned}\frac{t\int_{\varepsilon b(t,x)<|y|\le b(t,x)} |y|^3 \Pi(\mathrm{d}y)}{b^3(t,x)} &\ge \varepsilon^3 t \left(\overline{\Pi}(\varepsilon b(t,x)) - \overline{\Pi}(b(t,x))\right) \\ &= \varepsilon^3 t \overline{\Pi}(\varepsilon b(t,x)) + o(1)\end{aligned}$$

shows that $\lim_{t\to 0} t\overline{\Pi}(\varepsilon b(t,x)) = 0$ for all $\varepsilon \in (0,1)$ and hence for all $\varepsilon > 0$. Then (5.5) gives, for all $\varepsilon > 0$,

$$\begin{aligned}\frac{tV(\varepsilon b(t,x))}{b^2(t,x)} &= 1 + x^2 - \frac{t\int_{\varepsilon b(t,x)<|y|\le b(t,x)} y^2 \Pi(\mathrm{d}y)}{b^2(t,x)} \\ &\to 1 + x^2, \text{ as } t \to 0.\end{aligned}$$

Then by Theorem 15.14, p. 295, of Kallenberg [18],

$$\frac{X_t - a(t,x)}{b(t,x)} \xrightarrow{\mathrm{D}} N(0, 1+x^2), \text{ as } t \to 0,$$

for some $a(t,x)$. Thus $X_t \in D(N)$ at 0.

When V is slowly varying at 0, then by a Potter bound (Bingham, Goldie and Teugels [2, p. 25]), for every $0 < c < 1$ and $\beta > 0$ there is a $y_0(c,\beta) > 0$ such that $V(y) \ge cy^\beta$ for all $y \le y_0(\beta)$. Then by (5.5), for $b(t,0) \le y_0$ (so $b(t,x) \le y_0$),

$$(1+x^2) b^2(t,x) = tV(b(t,x)) \ge ctb^\beta(t,x),$$

giving (5.11). □

Lemma 5.1. *Let X_t be an arbitrary Lévy process with canonical triplet (γ, σ^2, Π). Fix $x > 0$, a continuity point of $\overline{\Pi}^\pm$.*

(i) We have

$$\lim_{t\downarrow 0} t^{-1} P(X_t > x) = \overline{\Pi}^+(x), \ \lim_{t\downarrow 0} t^{-1} P(X_t < -x) = \overline{\Pi}^-(x), \tag{5.16}$$

$$\lim_{t\downarrow 0} t^{-1} E\left(X_t \mathbf{1}_{\{|X_t|\le x\}}\right) = \gamma - \int_{x<|y|\le 1} y\Pi(\mathrm{d}y) = \nu(x), \tag{5.17}$$

$$\lim_{t\downarrow 0} t^{-1} E\left(X_t^2 \mathbf{1}_{\{|X_t|\le x\}}\right) = \sigma^2 + \int_{0<|y|\le x} y^2 \Pi(\mathrm{d}y) = V(x), \tag{5.18}$$

and, for all $\beta > 2$,

$$\lim_{t\downarrow 0} t^{-1} E|X_t|^\beta \mathbf{1}_{\{|X_t|\le x\}} = \int_{0<|y|\le x} |y|^\beta \Pi(\mathrm{d}y). \tag{5.19}$$

(ii) Let $c \in \mathbb{R}$ be any real number. Equations (5.16), (5.18), *and* (5.19) *remain true if X_t is replaced by $X_t - ct$ on the left-hand side, while in this case* (5.17) *takes the modified form*

$$\lim_{t\downarrow 0} t^{-1} E\left((X_t - ct)\mathbf{1}_{\{|X_t - ct|\le x\}}\right) = \nu(x) - c. \tag{5.20}$$

Proof of Lemma 5.1. Part (i), (5.16)–(5.19), can be deduced from the work of Figueroa-Lopez [8], and part (ii) follows easily from these. □

Proof of Theorem 5.2. Assume $\sigma^2 > 0$ or $\Pi(\mathbb{R}) = \infty$, $E|X_t| < \infty$, $EX_t = t\mu$, $t > 0$, and (5.7). We shall first establish part (i). As in Lemma 3.9, write

$$X_t - \mu t =: \sum_{i=1}^{n} Y(i,t,n),\ t > 0, n \geq 1,$$

where for each $t > 0$,

$$Y(i,t,n) := X_{it/n} - X_{(i-1)t/n} - t\mu/n.$$

The assumption $EX_t = t\mu$ implies $EY(1,t,n) = 0$, $t > 0$, $n \geq 1$. Apply (5.3) to get

$$e^{-A\Delta_{n,x}(t)} \leq \frac{P\left(\sum_{i=1}^{n} Y(i,t,n) \geq x\sqrt{\sum_{i=1}^{n} Y^2(i,t,n)}\right)}{1-\Phi(x)} \leq e^{A\Delta_{n,x}(t)}, \tag{5.21}$$

for all $n \geq 1$ and $x \geq 0$ satisfying $\Delta_{n,x} \leq (1+x^2)/A$, where A is an absolute constant,

(5.22)

$$\Delta_{n,x}(t) := nP(|Y(1,t,n)| > b_n(t,x)) + \frac{n|E\left(Y(1,t,n)\mathbf{1}_{\{|Y(1,t,n)|\leq b_n(t,x)\}}\right)|}{b_n(t,x)} + \frac{nE\left(|Y(1,t,n)|^3\mathbf{1}_{\{|Y(1,t,n)|\leq b_n(t,x)\}}\right)}{b_n^3(t,x)},$$

and $b_n(t,x)$ is defined by

$$b_n(t,x) := \sup\left\{y > 0 : \frac{E\left(Y^2(1,t,n)\mathbf{1}_{\{|Y(1,t,n)|\leq y\}}\right)}{y^2} \geq \frac{1+x^2}{n}\right\}.$$

Lemma 6.1 in the Appendix shows that

$$(1+x^2)b_n^2(t,x) = nE\left(Y^2(1,t,n)\mathbf{1}_{\{|Y(1,t,n)|\leq b_n(t,x)\}}\right). \tag{5.23}$$

Our aim is to let $n \to \infty$ in (5.21) so as to obtain (5.8), for the stated ranges of t and x, via (3.50). This will follow from the inequality

$$\limsup_{n\to\infty} \Delta_{n,x}(t) \leq 3t\Delta(b(\eta t, x)) \tag{5.24}$$

and our assumption that $t\Delta(b(\eta t,x)) \leq (1+x^2)/(6A)$. To prove (5.24), recall the definition of $b(t,x)$ in (5.4) and note from Lemma 5.1 that we have

$$\begin{aligned}\lim_{n\to\infty} nE\left(Y^2(1,t,n)\mathbf{1}_{\{|Y(1,t,n)|\leq y\}}\right) &= \lim_{n\to\infty} nE\left(\left(X_{t/n} - t\mu/n\right)^2\mathbf{1}_{\{|X_{t/n}-t\mu/n|\leq y\}}\right)\\ &= tV(y),\ t > 0,\ y > 0.\end{aligned} \tag{5.25}$$

Notice that (5.25) implies that the sequence $\{b_n(t,x)\}_{n=1,2,\ldots}$ is bounded above. Thus, given $\varepsilon \in (0,1)$, we can apply (5.25) with $y = b((1-\varepsilon)t,x)$ to see that there is an $n_0(t,y,x,\varepsilon)$ such that $n \geq n_0$ implies

$$\frac{nE\left(\left(X_{t/n} - t\mu/n\right)^2\mathbf{1}_{\{|X_{t/n}-t\mu/n|\leq b((1-\varepsilon)t,x)\}}\right)}{y^2} \geq \frac{(1-\varepsilon)tV(b((1-\varepsilon)t,x))}{b^2((1-\varepsilon)t,x)} = 1+x^2.$$

Hence from (5.4), for all $n \geq n_0$,

$$b_n(t,x) \geq b((1-\varepsilon)t, x).$$

Since $b(t,x)$ is nondecreasing in t, for all $x > 0$, we can take $\varepsilon \downarrow 0$ in this to get

$$\liminf_{n\to\infty} b_n(t,x) \geq b(t-,x), \tag{5.26}$$

which as we argued in Proposition 5.1 is positive for all $x > 0$, $t > 0$. Now, since the sequence $\{b_n(t,x)\}_{n=1,2,\ldots}$ is bounded above, given any sequence $n'' \to \infty$, we can choose a subsequence $n' = n'(t,x) \to \infty$ such that

$$b_{n'}(t,x) \to b'(t,x) \in [b(t-,x), \infty).$$

Clearly, to establish (5.24) it suffices to show that for any such sequence n' we have

$$\lim_{n'\to\infty} \Delta_{n',x}(t) = t\Delta(b'(t,x)) \leq 3t\Delta(b(\eta t, x)). \tag{5.27}$$

To this end, given $\varepsilon \in (0, b'(t,x))$, choose n' so large that $|b_{n'}(t,x) - b'(t,x)| \leq \varepsilon$, and use (5.23) to write

$$\begin{aligned}(1+x^2)b_{n'}^2(t,x) &= n'E\left(\left(X_{t/n'} - t\mu/n'\right)^2 \mathbf{1}_{\{|X_{t/n'} - t\mu/n'| \leq b_{n'}(t,x)\}}\right)\\ &\leq n'E\left(\left(X_{t/n'} - t\mu/n'\right)^2 \mathbf{1}_{\{|X_{t/n'} - t\mu/n'| \leq b'(t,x)+\varepsilon\}}\right).\end{aligned} \tag{5.28}$$

Thus, by Lemma 5.1, for any $\varepsilon > 0$ such that $b'(t,x) + \varepsilon$ is a continuity point of $V(\cdot)$,

$$\begin{aligned}(1+x^2)(b'(t,x))^2 &= (1+x^2)\lim_{n'\to\infty} b_{n'}^2(t,x)\\ &\leq \lim_{n'\to\infty} n'E\left(\left(X_{t/n'} - t\mu/n'\right)^2 \mathbf{1}_{\{|X_{t/n'} - t\mu/n'| \leq b'(t,x)+\varepsilon\}}\right)\\ &= tV(b'(t,x)+\varepsilon),\end{aligned}$$

in which we can let $\varepsilon \downarrow 0$ to deduce, by right-continuity of $V(\cdot)$,

$$(1+x^2)(b'(t,x))^2 \leq tV(b'(t,x)). \tag{5.29}$$

By the definition of $b(t,x)$ this means $b'(t,x) \leq b(t,x)$; hence by (5.26)

$$b(t-,x) \leq b'(t,x) \leq b(t,x). \tag{5.30}$$

Now if $b(t-,x) = b(t,x)$, then $\lim_{n'\to\infty} b_{n'}(t,x) = b(t,x)$, and, as argued previously, $b(t,x)$ is a point of continuity of $V(\cdot)$. Alternatively, $b(t-,x) < b(t,x)$. This can only be the case if $\sigma^2 = 0$ and $x^{-2}V(x)$ is constant on the interval $[b(t-,x), b(t,x)]$. If this is so, then

$$\frac{V(b(t-,x))}{b^2(t-,x)} = \frac{V(b'(t,x))}{(b'(t,x))^2} = \frac{V(b(t,x))}{b^2(t,x)} = \frac{1+x^2}{t}. \tag{5.31}$$

By (5.5) we have for $\varepsilon \in (0,t)$ that

$$(1+x^2)b^2(t-\varepsilon,x) = tV(b(t-\varepsilon,x)),$$

and letting $\varepsilon \downarrow 0$ shows that

$$\begin{aligned}(1+x^2)b^2(t-,x) &= tV(b(t-,x)-)\\ &= tV(b(t-,x)) \quad \text{(by (5.31))}.\end{aligned}$$

Thus $b(t-,x)$ is a point of continuity of $V(\cdot)$. Since $V(\cdot)$ is also continuous at $b(t,x)$, $V(\cdot)$ is continuous at all points of $[b(t-,x), b(t,x)]$. Thus $V(\cdot)$ is continuous at $b'(t,x)$ and, by (5.31), satisfies

$$(1+x^2)(b'(t,x))^2 = tV(b'(t,x)).$$

This is true for any subsequential limit of $b_n(t,x)$, in particular, if $\lim_{n\to\infty} b_n(t,x) = b(t,x)$ exists.

Taking a limit through n' in (5.22), and keeping in mind that $V(\cdot)$ is continuous at $b'(t,x)$, we obtain from Lemma 5.1 that

$$\begin{aligned}\lim_{n'\to\infty} \Delta_{n',x}(t) &= t\overline{\Pi}(b'(t,x)) + \frac{t|\nu(b'(t,x)) - \mu|}{b'(t,x)} + \frac{t\int_{0<|y|\le b'(t,x)} |y|^3 \Pi(\mathrm{d}y)}{(b'(t,x))^3} \\ &= t\Delta(b'(t,x)) \quad \text{(by (5.6))}. \end{aligned} \tag{5.32}$$

Now, for the η and $y_0(\eta)$ specified in (5.7), choose $t>0$ so small that $b(t,0)\le y_0$. Then $b(t-,x)\le y_0$ for all $x>0$, and we have

$$\begin{aligned} 1 - \frac{b^2(t-,x)}{b^2(t,x)} &= \frac{V(b(t,x)) - V(b(t-,x))}{V(b(t,x))} \quad \text{(by (5.31))} \\ &\le b^2(t,x)\left(\frac{\overline{\Pi}(b(t-,x)) - \overline{\Pi}(b(t,x))}{V(b(t,x))}\right) \\ &\le \frac{b^2(t-,x)\overline{\Pi}(b(t-,x))}{V(b(t-,x))} \quad \text{(using (5.31) again)} \\ &\le 1-\eta. \end{aligned}$$

Thus we have by (5.30) that

$$(b'(t,x))^2 \ge b^2(t-,x) \ge \eta b^2(t,x). \tag{5.33}$$

Note further that, by (5.5),

$$\begin{aligned}(1+x^2)b^2(\eta t,x) &= \eta t V(b(\eta t,x)) \\ &\le \eta t V(b(t,x)) \\ &= \eta(1+x^2)b^2(t,x),\end{aligned}$$

implying $b(\eta t,x)\le\sqrt{\eta}b(t,x)$. This, (5.30) and inequality (5.33) imply

$$b(t,x)\ge b'(t,x)\ge b(\eta t,x).$$

This allows us to replace $\Delta(b'(t,x))$ in (5.32) by $\Delta(b(\eta t,x))$, as follows. First,

$$\begin{aligned}\frac{|\nu(b'(t,x)) - \mu|}{b'(t,x)} &= \frac{1}{b'(t,x)}\left|\left(\gamma - \int_{b'(t,x)<|x|\le 1} x\Pi(\mathrm{d}x)\right) - \left(\gamma - \int_{|x|>1} x\Pi(\mathrm{d}x)\right)\right| \\ &= \frac{1}{b'(t,x)}\left|\int_{|x|>b'(t,x)} x\Pi(\mathrm{d}x)\right| \\ &\le \frac{1}{b(\eta t,x)}\left|\int_{|x|>b(\eta t,x)} x\Pi(\mathrm{d}x)\right| + \overline{\Pi}(b(\eta t,x)) \\ &\le \frac{|\nu(b(\eta t,x)) - \mu|}{b(\eta t,x)} + \overline{\Pi}(b(\eta t,x)).\end{aligned}$$

Second,

$$\frac{1}{(b'(t,x))^3}\int_{|x|\le b'(t,x)}|x|^3\Pi(\mathrm{d}x)\le\frac{1}{(b(\eta t,x))^3}\int_{|x|\le b(\eta t,x)}|x|^3\Pi(\mathrm{d}x)+\overline{\Pi}(b(\eta t,x)).$$

Hence, using (5.6),

$$\begin{aligned}\Delta(b'(t,x))&\le 3\overline{\Pi}(b(\eta t,x))+\frac{|\nu(b(\eta t,x))-\mu|}{b(\eta t,x)}+\frac{1}{(b(\eta t,x))^3}\int_{|x|\le b(\eta t,x)}|x|^3\Pi(\mathrm{d}x)\\ &\le 3\Delta(b(\eta t,x)).\end{aligned}\tag{5.34}$$

Thus we conclude from (5.32) that, for $t>0$ such that $b(t,0)\le y_0$, and all $x>0$,

$$\limsup_{n\to\infty}\Delta_{n,x}(t)\le 3t\Delta(b(\eta t,x)).\tag{5.35}$$

Now suppose $t>0$ and $x\ge 0$ further satisfy $t\Delta(b(\eta t,x))\le(1+x^2)/6A$. Then by (5.35), for some $n_0=n_0(x,A)$ and all $n\ge n_0$, we have $\Delta_{n,x}(t)\le(1+x^2)/A$. Thus (5.21) holds for such n and x. Letting $n\to\infty$ in (5.21) through subsequences n', keeping in mind (5.35), then proves (5.8).

Let us note at this stage that (5.8) holds under the same conditions if the ratio in (5.8) is replaced by

$$\frac{P\left(X_t-t\mu\le -x\sqrt{V_t}\right)}{\Phi(-x)},\ x>0.$$

To see this, apply (5.8) to the Lévy process $Y_t:=-X_t$, which has canonical characteristics $(-\mu,\sigma^2,\Pi(-\mathrm{d}x))$, and, in an obvious notation, $V_t^Y=V_t$, $\overline{\Pi}_Y(x)=\overline{\Pi}(x)$, $\nu_Y(x)=-\nu(x)$, $V_Y(x)=V(x)$, $b_Y(t,x)=b(t,x)$, and $\Delta_Y(b(t,x))=\Delta(b(t,x))$. Since

$$\frac{P\left(X_t-t\mu\le -x\sqrt{V_t}\right)}{\Phi(-x)}=\frac{P\left(Y_t+t\mu\ge x\sqrt{V_t}\right)}{1-\Phi(x)},\tag{5.36}$$

we get the indicated result. This completes the proof of part (i).

(ii) When $h=1$, the small jump component of X in (3.1) equals $\gamma t+\sigma Z_t+X_t^{(S,1)}$, which has expectation γt, and we can apply part (i) of the theorem to $\gamma t+\sigma Z_t+X_t^{(S,1)}$ and $t\sigma^2+V_t^{(S,1)}$.

(iii) From (5.8) with $\eta=1/2$ applied to $\gamma t+\sigma Z_t+X_t^{(S,1)}$ and $\sigma^2t+V_t^{(S,1)}$, we can deduce that

$$\left|P\left(\sigma Z_t+X_t^{(S,1)}\le x\sqrt{V_t^{(S,1)}}\right)-\Phi(x)\right|\le\left(e^{3A\Delta(b(t/2,x))}-1\right)(1-\Phi(x))\tag{5.37}$$

for $t>0$, $x\ge 0$ satisfying $t\Delta(b(t/2,x))\le(1+x^2)/(6A)$ and $b(t,0)\le y_0(1/2)$. Given $\varepsilon\in(0,1/2)$ there is, by Proposition 5.1 with t replaced by $t/2$ and ε replaced by $\varepsilon/(6(A\vee 1))$, a $t_0=t_0(\varepsilon)$ such that $0<t\le(1+x^2)t_0$ implies $t\Delta(b(t/2,x))\le\varepsilon(1+x^2)/(6A)$. Thus (5.37) applies in this situation. Then, using the inequality $1-\Phi(x)\le 2e^{-x^2/2}/(1+x)$, we get

$$\begin{aligned}\left|P\left(X_t-t\gamma\le x\sqrt{V_t^{(S,1)}}\right)-\Phi(x)\right|&\le\left(e^{\varepsilon(1+x^2)/2}-1\right)\frac{2e^{-x^2/2}}{1+x}\\ &\le\frac{4e^{-(1-\varepsilon)x^2/2}}{1+|x|}.\end{aligned}$$

We can replace $x>0$ by $x<0$ in this using (5.36). □

6. Appendix: Some technical results

Recall that the canonical triplet of X_t is denoted by (γ, σ^2, Π), and the tails $\overline{\Pi}(x)$ and $\overline{\Pi}^{\pm}(x)$ and truncated mean and variance functions are defined in (2.1) and (2.2).

We use the following fact, proved as for U in Lemma 3.7:

Proposition 6.1. *$X_t \in FC$ at 0 iff there exist constants $c > 0$ and $\alpha \in (0, 2]$ such that, for each $\mu \in (0, 1)$ and $\varepsilon \in (0, \alpha)$, there is an $x_0 = x_0(\varepsilon)$ such that*

$$\frac{V(\mu x)}{V(x)} \geq c\mu^{2-\alpha+\varepsilon}, \text{ for all } 0 < x \leq x_0. \tag{6.1}$$

Remarks. Just as for U, (6.1) implies that $V(y)$ is bounded away from 0 by a power of y, for small y, i.e.,

$$V(y) \geq cy^{2-\alpha+\varepsilon}, \tag{6.2}$$

for some $c > 0$, for $0 < y \leq x_0(\varepsilon)$, with $\varepsilon \in (0, \alpha)$.

Suppose $X_t \in FC$ at 0, with centering and norming functions $a(t)$ and $b(t) > 0$. Let us call any a.s. finite rv $\mathcal{I}$ obtained as the limit in the distribution of $(X_{t_k} - a(t_k))/b(t_k)$, for a sequence $t_k \to 0$ as $k \to \infty$, a "subsequential limit rv".

Proposition 6.2. *Let $X_t \in FC$ at 0, having a subsequential limit rv $\mathcal{I}$. Then the distribution of $\mathcal{I}$ is absolutely continuous, in fact, is infinitely differentiable at each point in $\mathbb{R}$.*

Proof of Proposition 6.2. Let $X_t \in FC$ at 0. Denote the Lévy triplet of $\mathcal{I}$ by $(\gamma_{\mathcal{I}}, \sigma^2_{\mathcal{I}}, \Pi_{\mathcal{I}}(\cdot))$, and let $\overline{\Pi}^{\pm}_{\mathcal{I}}$ and $\overline{\Pi}_{\mathcal{I}}$ be the tails of $\Pi_{\mathcal{I}}$.

As $\mathcal{I}$ is a subsequential limit rv, we know that there exist nonstochastic functions $a(t)$ and $b(t) > 0$, where necessarily $b(t) \to 0$ as $t \downarrow 0$, and a sequence $t_k \to 0$ as $k \to \infty$, for which $(X_{t_k} - a(t_k))/b(t_k) \xrightarrow{\mathrm{D}} \mathcal{I}$. Thus, for each $x > 0$ which is a point of continuity of $\Pi^{\pm}_{\mathcal{I}}$, we have

$$\overline{\Pi}^{\pm}_{\mathcal{I}}(x) = \lim_{k\to\infty} t_k \overline{\Pi}^{\pm}(xb(t_k)) \tag{6.3}$$

and

$$V_{\mathcal{I}}(x) := \sigma^2_{\mathcal{I}} + \int_{|y|\leq x} y^2 \Pi_{\mathcal{I}}(\mathrm{d}y) = \lim_{k\to\infty} \frac{t_k V(xb(t_k))}{b^2(t_k)}. \tag{6.4}$$

We can apply (6.1) to show that there is a $c > 0$ such that, whenever $\mu \in (0, 1)$ and $x > 0$, and both μx and x are continuity points of $\Pi^{\pm}_{\mathcal{I}}$,

$$\frac{V_{\mathcal{I}}(\mu x)}{V_{\mathcal{I}}(x)} = \lim_{k\to\infty} \frac{V(\mu x b(t_k))}{V(xb(t_k))} \geq \liminf_{y\downarrow 0} \frac{V(\mu y)}{V(y)} \geq c\mu^{2-\alpha}. \tag{6.5}$$

When passing to the limit in (6.5), we keep in mind that $b(t) \to 0$ and $\varepsilon \in (0, \alpha)$ can be made arbitrarily small in (6.1).

Just as for V, (6.5) implies that $V_{\mathcal{I}}(y)$ is bounded away from 0 by a power of y, for small y, i.e.,

$$V_{\mathcal{I}}(y) \geq cy^{2-\alpha+\varepsilon}, \tag{6.6}$$

for some $c > 0$, for $0 < y \leq x_0(\varepsilon)$, with $\varepsilon \in (0, \alpha)$.

The conclusion of Proposition 6.2 is obvious if $\sigma_{\mathcal{I}}^2 > 0$, since then $\mathcal{I}$ has the normal distribution as a convolution component, so we can assume $\sigma_{\mathcal{I}}^2 = 0$. Then the characteristic function of $\mathcal{I}$ satisfies

$$\begin{aligned}|Ee^{\mathrm{i}\theta\mathcal{I}}| &= \left|e^{\mathrm{i}\theta\gamma_{\mathcal{I}}+\int_R(e^{\mathrm{i}\theta y}-1-\mathrm{i}\theta y I_{\{|y|\le 1\}})\Pi_{\mathcal{I}}(\mathrm{d}y)}\right| \\ &= e^{-\int_R(1-\cos(\theta y))\Pi_{\mathcal{I}}(\mathrm{d}y)}.\end{aligned}$$

Since $\sigma_{\mathcal{I}}^2 = 0$ we have $\Pi_{\mathcal{I}} \neq 0$, and then for $|\theta| > 1$,

$$\begin{aligned}\int_R(1-\cos(\theta y))\Pi_{\mathcal{I}}(\mathrm{d}y) &\ge \frac{\theta^2}{3}\int_{0<|y|\le|\theta|^{-1}} y^2\Pi_{\mathcal{I}}(\mathrm{d}y) \\ &= \frac{\theta^2}{3}V_{\mathcal{I}}(|\theta|^{-1}) \\ &\ge c|\theta|^{\alpha-\varepsilon}/3,\end{aligned}$$

by (6.6). Thus we get, for $|\theta| > 1$,

$$|Ee^{\mathrm{i}\theta\mathcal{I}}| \le e^{-c|\theta|^{\alpha-\varepsilon}/3}, \tag{6.7}$$

which shows that $\mathcal{I}$ has an absolutely integrable characteristic function. This implies the infinite differentiability of the distribution of $\mathcal{I}$. □

The following lemma is useful in defining norming functions as in (5.4). Let the function $\phi : (0,\infty) \to (0,\infty)$ be a right continuous function not identically equal to zero and set

$$b = \sup\{\phi(y) : y > 0\}.$$

Note that b may be infinity and necessarily $b > 0$. Assume that

(i) $\sup\{\phi(y) : y \ge c\} < \infty$ for all $c > 0$;

(ii) $\phi(y) \to 0$, as $y \to \infty$;

(iii) for all $0 < y < b$, $\phi(y) - \phi(y-) \ge 0$.

For any $0 < s < b$, define

$$p(s) = \sup\{y > 0 : \phi(y) \ge s\}.$$

Notice that by assumption (ii) $p(s)$ is finite. Assumption (ii) also implies that

$$b = \sup\{\phi(y) : 0 < y \le a\}$$

for all $a > 0$ sufficiently large. Thus there exists $a > 0$, $\kappa \in [0,a]$, and a sequence $z_m \in (0,a)$ such that $\lim_{m\to\infty}\phi(z_m) = b$, $\phi(z_m) \le b$ for all $m \ge 1$ and $z_m \to \kappa$. Therefore for all $0 < s < b$, the set $\{y > 0 : \phi(y) \ge s\}$ is nonempty, and hence $p(s) > 0$.

Lemma 6.1. *Under assumptions (i), (ii) and (iii), for any* $0 < s < b$, $\phi(p(s)) = s$.

Proof of Lemma 6.1. We always have $p(s) > 0$. Suppose $\phi(p(s)) < s$. In this case, by the definition of $p(s)$ we can find a sequence $y_m \in \{y > 0 : \phi(y) \ge s\}$ such that $y_m \uparrow p(s)$, as $m \to \infty$, which since $\phi(y_m) \ge s$, implies $\phi(p(s)) - \phi(p(s)-) < 0$. This contradicts (iii), so $\phi(p(s)) < s$ is impossible.

Now suppose that $\phi(p(s)) > s$. Then necessarily $\phi(p(s)+\varepsilon) < s$ for all $\varepsilon > 0$. However, by right continuity of ϕ, $\phi(p(s)) - \phi(p(s)+) = 0$. Thus $\phi(p(s)) > s$ is also impossible. Hence $\phi(p(s)) = s$. □

Example a. Set for $y > 0$, $V(y) = \sigma^2 + \int_{0<|u|\le y} u^2 \Pi(\mathrm{d}u)$ and

$$\phi(y) = V(y)/y^2.$$

It is easy to check that (i) and (ii) hold. In this example $\phi(y) - \phi(y-) = \overline{\Pi}(y-) - \overline{\Pi}(y)$, and thus (iii) is also satisfied.

Example b. Let X be any random variable. Set for $y > 0$

$$\phi(y) = E\left(X^2 \mathbf{1}_{\{|X|\le y\}}\right)/y^2.$$

Notice that $\phi(y)$ converges to 0 as $y \to \infty$, and also $\phi(y) \le 1$. Thus (i) and (ii) hold, and $b \le 1$. Further, it is easily checked that $\phi(y) - \phi(y-) = P\{X = y\}$, so that (iii) is fulfilled too.

The proof of Theorem 2.3 requires the following lemma.

Lemma 6.2. *Let $\{T_n\}_{n\ge1}$ be a sequence of random variables such that $T_n \xrightarrow{\mathrm{D}} T$, where $P(T > x) > 0$ for all $x > 0$ and $P(T > x)$ is continuous on $(0, \infty)$. Assume there exists a constant $\tau > 0$ such that for any sequence of positive constants $x_n \to \infty$ satisfying $x_n = o(\sqrt{n})$, we have*

$$-x_n^{-2} \log P(T_n > x_n) \to \tau, \text{ as } n \to \infty. \tag{6.8}$$

Then

$$-x^{-2} \log P(T > x) \to \tau, \text{ as } x \to \infty. \tag{6.9}$$

Proof of Lemma 6.2. Choose any $2 > \lambda > 1$. Set $M_0 = N_0 = 1$. Select $M_1 \ge 1$ such that for all $n \ge M_1$,

$$2^{-1} P(T > \lambda) \le P(T_n > \lambda) \le 2P(T > \lambda),$$

and let $N_1 = \max\{M_1, 2\}$. Then for any $k > 1$ choose $M_k \ge 1$ such that for all $n \ge M_k$,

$$2^{-1} P(T > \lambda^k) \le P(T_n > \lambda^k) \le 2P(T > \lambda^k),$$

and set, for $k \ge 2$,

$$N_k = \max\left\{M_k, 2^{k^2}, M_{k-1} + 1\right\}.$$

Now for $N_k \le n < N_{k+1}$, $k \ge 0$, let $x_n = \lambda^k$. We see that $x_n \to \infty$ and for $N_k \le n < N_{k+1}$, $k \ge 1$,

$$\frac{x_n}{\sqrt{n}} = \frac{\lambda^k}{\sqrt{n}} \le \frac{\lambda^k}{\sqrt{2^{k^2}}} \le \frac{\lambda^k}{\sqrt{\lambda^{k^2}}},$$

which implies that $x_n = o(\sqrt{n})$ and thus that (6.8) holds. Further, by construction, for any $N_k \le n < N_{k+1}$,

$$-\frac{\log P(T_n > \lambda^k)}{\lambda^{2k}} = -\frac{\log P(T_n > x_n)}{x_n^2}$$

and

$$-\frac{\log\left(2P(T > \lambda^k)\right)}{\lambda^{2k}} \le -\frac{\log P(T_n > \lambda^k)}{\lambda^{2k}} \le -\frac{\log\left(2^{-1}P(T > \lambda^k)\right)}{\lambda^{2k}}.$$

Thus

$$-\lambda^{-2k} \log P(T > \lambda^k) \to \tau, \text{ as } k \to \infty.$$

Given any $x \geq 2$ choose an integer $k(x)$ such that $\lambda^{k(x)} \leq x < \lambda^{k(x)+1}$. Then

$$\lambda^{-2} \lim_{x\to\infty} -\log P(T > \lambda^{k(x)})/\lambda^{2k(x)} = \tau\lambda^{-2}$$
$$\leq \liminf_{x\to\infty} -\log P(T > x)/x^2 \leq \limsup_{x\to\infty} -\log P(T > x)/x^2$$
$$\leq \lambda^2 \lim_{x\to\infty} -\log P(T > \lambda^{k(x)+1})/\lambda^{2k(x)+2} = \tau\lambda^2.$$

Since $2 > \lambda > 1$ can be chosen arbitrarily close to 1 we see that (6.9) is satisfied. □

References

[1] V. Bentkus and F. Götze, *The Berry-Esseen bound for Student's statistic*, Ann. Probab. **24** (1996), no. 1, 491–503, DOI 10.1214/aop/1042644728. MR1387647 (97f:62021)

[2] V. Bentkus, M. Bloznelis, and F. Götze, *A Berry-Esséen bound for Student's statistic in the non-i.i.d. case*, J. Theoret. Probab. **9** (1996), no. 3, 765–796, DOI 10.1007/BF02214086. MR1400598 (97e:60036)

[3] Jean Bertoin, *Lévy processes*, Cambridge Tracts in Mathematics, vol. 121, Cambridge University Press, Cambridge, 1996. MR1406564 (98e:60117)

[4] J. Bertoin, R. A. Doney, and R. A. Maller, *Passage of Lévy processes across power law boundaries at small times*, Ann. Probab. **36** (2008), no. 1, 160–197, DOI 10.1214/009117907000000097. MR2370602 (2009d:60141)

[5] N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular variation*, Encyclopedia of Mathematics and its Applications, vol. 27, Cambridge University Press, Cambridge, 1987. MR898871 (88i:26004)

[6] Victor H. de la Peña, Tze Leung Lai, and Qi-Man Shao, *Self-normalized processes*, Probability and its Applications (New York), Springer-Verlag, Berlin, 2009. Limit theory and statistical applications. MR2488094 (2010d:60002)

[7] R. A. Doney and R. A. Maller, *Stability and attraction to normality for Lévy processes at zero and at infinity*, J. Theoret. Probab. **15** (2002), no. 3, 751–792, DOI 10.1023/A:1016228101053. MR1922446 (2003g:60076)

[8] José E. Figueroa-López, *Small-time moment asymptotics for Lévy processes*, Statist. Probab. Lett. **78** (2008), no. 18, 3355–3365, DOI 10.1016/j.spl.2008.07.012. MR2479503 (2010b:60140)

[9] Bert Fristedt, *Upper functions for symmetric processes with stationary, independent increments*, Indiana Univ. Math. J. **21** (1971/1972), 177–185. MR0288841 (44 #6036)

[10] Bert Fristedt, *Corrections to: "Upper functions for symmetric processes with stationary, independent increments"*, Indiana Univ. Math. J. **23** (1973/74), 445. MR0324780 (48 #3129)

[11] Bert Fristedt, *Sample functions of stochastic processes with stationary, independent increments*, Advances in probability and related topics, Vol. 3, Dekker, New York, 1974, pp. 241–396. MR0400406 (53 #4240)

[12] Evarist Giné and David M. Mason, *On the LIL for self-normalized sums of IID random variables*, J. Theoret. Probab. **11** (1998), no. 2, 351–370, DOI 10.1023/A:1022675620729. MR1622575 (99e:60082)

[13] Evarist Giné, David M. Mason, and Andrei Yu. Zaitsev, *The L_1-norm density estimator process*, Ann. Probab. **31** (2003), no. 2, 719–768, DOI 10.1214/aop/1048516534. MR1964947 (2003m:60046)

[14] Philip S. Griffin and James D. Kuelbs, *Self-normalized laws of the iterated logarithm*, Ann. Probab. **17** (1989), no. 4, 1571–1601. MR1048947 (91k:60036)

[15] Philip Griffin and James Kuelbs, *Some extensions of the LIL via self-normalizations*, Ann. Probab. **19** (1991), no. 1, 380–395. MR1085343 (92e:60062)

[16] Wassily Hoeffding, *Probability inequalities for sums of bounded random variables*, J. Amer. Statist. Assoc. **58** (1963), 13–30. MR0144363 (26 #1908)

[17] Bing-Yi Jing, Qi-Man Shao, and Wang Zhou, *Towards a universal self-normalized moderate deviation*, Trans. Amer. Math. Soc. **360** (2008), no. 8, 4263–4285, DOI 10.1090/S0002-9947-08-04402-4. MR2395172 (2009d:60065)

[18] Olav Kallenberg, *Foundations of modern probability*, 2nd ed., Probability and its Applications (New York), Springer-Verlag, New York, 2002. MR1876169 (2002m:60002)

[19] A. Khintchine, *Zwei Sätze über stochastische Prozesse mit stabilen Verteilungen* (Russian with German summary), Akad. Nauk. SSSR, Izvestia, Ser. Math. **3** (1938), 487-508.

[20] A. Khintchine, *Sur la croissance locale des processus stochastiques homogènes à accroissements indépendants* (Russian with French resumé), Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **3** (1939), 487–508.

[21] B. F. Logan, C. L. Mallows, S. O. Rice, and L. A. Shepp, *Limit distributions of self-normalized sums*, Ann. Probability **1** (1973), 788–809. MR0362449 (50 #14890)

[22] Ross Maller and David M. Mason, *Small-time compactness and convergence behavior of deterministically and self-normalised Lévy processes*, Trans. Amer. Math. Soc. **362** (2010), no. 4, 2205–2248, DOI 10.1090/S0002-9947-09-05032-6. MR2574893 (2011b:60078)

[23] Ross Maller and David M. Mason, *A characterization of small and large time limit laws for self-normalized Lévy processes*, in Limit Theorems in Probability, Statistics and Number Theory - in Honor of Friedrich Götze, Eichelsbacher, P.; Elsner, G.; Kösters, H.; Löwe, M.; Merkl, F.; Rolles, S. (Eds.), Springer Proceedings in Mathematics & Statistics, Vol. 42, 2013, VIII. To appear.

[24] Philip E. Protter, *Stochastic integration and differential equations*, Stochastic Modelling and Applied Probability, vol. 21, Springer-Verlag, Berlin, 2005. Second edition. Version 2.1; Corrected third printing. MR2273672 (2008e:60001)

[25] John Robinson and Qiying Wang, *On the self-normalized Cramér-type large deviation*, J. Theoret. Probab. **18** (2005), no. 4, 891–909, DOI 10.1007/s10959-005-7531-5. MR2300002 (2008h:60090)

[26] Tomasz Rolski, Hanspeter Schmidli, Volker Schmidt, and Jozef Teugels, *Stochastic processes for insurance and finance*, Wiley Series in Probability and Statistics, John Wiley & Sons Ltd., Chichester, 1999. MR1680267 (2000a:62273)

[27] K. Sato, *Lévy Processes and Infinitely Divisible Distributions.* Cambridge University Press, Cambridge, 1999.

[28] Qi-Man Shao, *Self-normalized large deviations*, Ann. Probab. **25** (1997), no. 1, 285–328, DOI 10.1214/aop/1024404289. MR1428510 (98b:60056)

[29] Galen R. Shorack and Jon A. Wellner, *Empirical processes with applications to statistics*, Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, John Wiley & Sons, Inc., New York, 1986. MR838963 (88e:60002)

[30] E. S. Štatland, *On local properties of processes with independent increments* (Russian, with English summary), Teor. Verojatnost. i Primenen. **10** (1965), 344–350. MR0183022 (32 #504)

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