NONABELIAN POINCARÉ DUALITY AFTER STABILIZING

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ABSTRACT. We generalize the nonabelian Poincaré duality theorems of Salvatore and Lurie to the case of not necessarily grouplike E_n-algebras (in the category of spaces). We define a stabilization procedure based on McDuff's "bringing points in from infinity" maps. For open connected parallelizable n-manifolds, we prove that, after stabilizing, the topological chiral homology of M with coefficients in an E_n-algebra A, $\int_M A$, is homology equivalent to $Map^c(M, B^n A)$, the space of compactly supported maps to the n-fold classifying space of A.

1. INTRODUCTION

In this paper, we will be interested in two models of topological chiral homology. The first model that we will consider uses May's two-sided bar construction [11] and was communicated to the author by Ricardo Andrade. It is a simplification of his construction from [1] and is known to be equivalent to the definition of topological chiral homology introduced by Lurie in [10]. For an explanation of this equivalence, see Remark 3.15 of [5]. We will also be interested in an even earlier model, Salvatore's configuration spaces of particles with summable labels [15]. It is widely believed that this construction is equivalent to that of Lurie and Andrade. However, to the best of the author's knowledge, a proof of this equivalence has not been written up explicitly in the literature. We will prove a theorem for both models, which we call "nonabelian Poincaré duality after stabilizing". Even if one could prove that these two constructions are equivalent, it would still be interesting to have both proofs since the different nature of the proofs highlights the advantages and disadvantages of each construction. The two-sided bar construction has a natural filtration, and thus one can use spectral sequence arguments to study its homology. On the other hand, for Salvatore's configuration spaces, the notion of relative configuration space is easy to define so it is possible to mimic arguments used in the 1970s to study classical configuration spaces [12], [2].

Topological chiral homology is a collection of constructions which take as input an E_n-algebra A and a parallelized n-manifold M and produces a space often denoted $\int_M A$. There is a scanning map $s : \int_M A \longrightarrow Map^c(M, B^n A)$, the space of compactly supported functions from M to the n-fold classifying space of A. An E_n-algebra A is called grouplike if the induced monoid structure on $\pi_0(A)$ is a group. For grouplike E_n-algebras, the scanning map $s : \int_M A \longrightarrow Map^c(M, B^n A)$ is a homotopy equivalence [15], [10]. This fact is called nonabelian Poincaré duality since it is equivalent to Poincaré duality when $A = \mathbb{Z}$ after taking homotopy groups [9].

If A is not grouplike, then the scanning map is not a homotopy equivalence. For example, when $M = \mathbb{R}^n$, $\int_M A$ is homotopy equivalent to A, while $Map^c(M, B^n A)$ is homotopy equivalent to $\Omega^n B^n A$. From now on, we assume that n is at least 2 (see [13] for a discussion of the case $n = 1$). At the level of π_0, the scanning map is not an isomorphism but is instead the inclusion of a commutative monoid into its Grothendieck group. While the scanning map s is not a homotopy equivalence, it is a group completion. Let $\{a_i\}$ be representatives of generators of $\pi_0(A)$ and let $m_i : A \longrightarrow A$ be multiplication by a_i maps. The group completion theorem [13] states that the induced map $s : hocolim_{m_i} A \longrightarrow \Omega^n B^n A$ is a homology equivalence. We will say that after stabilizing, A is homology equivalent to $\Omega^n B^n A$. We shall generalize this to arbitrary open parallelizable manifolds M and prove that, after stabilizing, $\int_M A$ is homology equivalent to $Map^c(M, B^n A)$. By open, we mean the interior of a not necessarily compact manifold $\bar{M}$ with nonempty boundary. We prove this result both for Andrade's two-sided bar construction model of topological chiral homology as well as Salvatore's configuration spaces of particles with summable labels. The stabilization maps $t_i : \int_M A \longrightarrow \int_M A$ will be generalizations of McDuff's "bringing points in from infinity" maps introduced in [12]. The goal of this paper is to prove the following theorem.

Theorem 1.1. *Let M be the interior of a connected (not necessarily compact) n-manifold with nonempty boundary and with $n > 1$. There are stabilization maps $t_i : \int_M A \longrightarrow \int_M A$ and a scanning map $s : \int_M A \longrightarrow Map^c(M, B^n A)$ such that s induces a homology equivalence between $hocolim_{t_i} \int_M A$ and $Map^c(M, B^n A)$.*

In Salvatore's model, we are able to generalize Theorem 1.1 to not necessarily parallelizable manifolds. Since it is widely believed that the two models of topological chiral homology are equivalent, it is natural to conjecture that Theorem 1.1 for Andrade's model can be generalized to nonparallelizable manifolds. Unfortunately however, the methods of proof of this paper do not seem equipped to handle this generality. We also conjecture that similar theorems (nonabelian Poincaré duality after stabilizing) are true for bundles of E_n-algebras and E_n-algebras in categories other than spaces.

In Section 2, we prove Theorem 1.1 when $\int_M A$ is the model of topological chiral homology defined using the two-sided bar construction due to Andrade, and in Section 3, we prove the theorem when $\int_M A$ is Salvatore's configuration space of particles with summable labels. Since Andrade's model is known to be homotopy equivalent to Lurie's model, Theorem 1.1 will also be true when $\int_M A$ is interpreted to mean Lurie's definition of topological chiral homology. In Section 4, we make a conjecture regarding homological stability for the connected components of $\int_M A$.

2. TOPOLOGICAL CHIRAL HOMOLOGY
VIA THE TWO-SIDED BAR CONSTRUCTION

In this subsection, we will describe a definition of topological chiral homology derived from [1] employing the monadic two-sided bar construction. The monadic two-sided bar construction was introduced in [11] to prove the recognition principle, the theorem that all connected algebras over the little n-disks operad are homotopy equivalent to n-fold loop spaces. May's proof of the recognition principle immediately generalizes to show that the scanning map for Andrade's model of topological

chiral homology is a homotopy equivalence in the case when the E_n-algebra is connected. We will then generalize this proof to the case of nonconnected algebras using the Segal spectral sequence for the homology of the geometric realization of a proper simplicial space [16]. This will prove Theorem 1.1 when we interpret $\int_M A$ to mean Andrade's two-sided bar construction model of topological chiral homology. In Subsection 2.1, we review basic properties of operads and their modules and algebras. In Subsection 2.2, we recall the definition of monads and their algebras and functors. In Subsection 2.3, we describe the two-sided bar construction and Andrade's model of topological chiral homology. In Subsection 2.4, we review properties of simplicial spaces. In Subsection 2.5, we review classical theorems about configuration spaces of distinct points in a manifold. In Subsection 2.6, we prove nonabelian Poincaré duality for connected E_n-algebras. Finally, in Subsection 2.7, we prove that Andrade's model of topological chiral homology exhibits nonabelian Poincaré duality after stabilizing (Theorem 1.1).

2.1. **Symmetric sequences, operads, modules and algebras.** In this subsection we recall the definition of operads and their modules and algebras. An efficient way of defining operads and their modules is via Σ-spaces (called symmetric sequences in spoken language). Let $\mathbb{N}_0$ denote the nonnegative integers.

Definition 2.1. A Σ-space is a collection of spaces $X(k)$ for all $k \in \mathbb{N}_0$ such that $X(k)$ has an action of the symmetric group Σ_k. A map between Σ-spaces $f : X \longrightarrow Y$ is a collection of equivariant maps $f_k : X(k) \longrightarrow Y(k)$.

The category of Σ-spaces has a (nonsymmetric) monoidal structure defined as follows.

Definition 2.2. For X and Y Σ-spaces, $X \otimes Y$ is the Σ-space such that

$$(X \otimes Y)(k) = \bigsqcup_{j=0}^{\infty} X(j) \times_{\Sigma_j} \bigsqcup_{f \in Map(k,j)} \prod_{i=1}^{j} Y(|f^{-1}(i)|).$$

Here $Map(k,j)$ is the set of maps from $\{1, \ldots, k\}$ to $\{1, \ldots, j\}$ and Σ_k acts via precomposition.

Note that the unit with respect to this product is given by the Σ-space ι with

$$\iota(n) = \begin{cases} pt, & \text{if } n = 1, \\ \varnothing, & \text{if } n \neq 1. \end{cases}$$

Definition 2.3. An operad $\mathcal{O}$ is a monoid in the category of Σ-spaces.

In other words, an operad $\mathcal{O}$ is a Σ-space with maps $m : \mathcal{O} \otimes \mathcal{O} \longrightarrow \mathcal{O}$ and $i : \iota \longrightarrow \mathcal{O}$ satisfying the obvious compatibility relations. Denote the image of ι by $1 \in \mathcal{O}(1)$.

Definition 2.4. The data of a left module structure on a Σ-space $\mathcal{L}$ over an operad $\mathcal{O}$ is a map $p : \mathcal{O} \otimes \mathcal{L} \longrightarrow \mathcal{L}$ such that the following diagrams commute:

$$
\begin{array}{ccc}
\mathcal{O} \otimes \mathcal{O} \otimes \mathcal{L} & \xrightarrow{id \otimes p} & \mathcal{O} \otimes \mathcal{L} \\
{\scriptstyle m \otimes id} \downarrow & & \downarrow {\scriptstyle p} \\
\mathcal{O} \otimes \mathcal{L} & \xrightarrow{p} & \mathcal{L}
\end{array}
\qquad
\begin{array}{ccc}
\iota \otimes \mathcal{L} & \xrightarrow{i} & \mathcal{O} \otimes \mathcal{L} \\
& \searrow & \downarrow {\scriptstyle p} \\
& & \mathcal{L}
\end{array}
$$

We likewise define right modules over operads. There is a functor from spaces to Σ-spaces which sends a space X to the Σ-space with

$$X(n) = \begin{cases} X, & \text{if } n = 0, \\ \varnothing, & \text{if } n \neq 0. \end{cases}$$

We will ignore the distinction between a space and its image as a Σ-space.

Definition 2.5. For an operad $\mathcal{O}$, an $\mathcal{O}$-algebra is a space A with the structure of a left $\mathcal{O}$-module.

Note that if X is a Σ-space and Y is a space, then the formula for $X \otimes Y$ simplifies to $X \otimes Y = \bigsqcup_k X(k) \times_{\Sigma_k} Y^k$.

2.2. **Monads, right functors and algebras.** From now on, we will also assume that all operads $\mathcal{O}$ have $\mathcal{O}(0) = \{0\}$. All algebras that we will consider will have a base point $a_0 \in A$ equal to the image of $\mathcal{O}(0) \longrightarrow A$. We will recall the functor on the category of based spaces (Top_*) associated to a right module. See [11] for a more detailed treatment of the topics of this section.

Definition 2.6. For (X, x_0) a based space and $\mathcal{R}$ a right module over an operad $\mathcal{O}$, define a functor $R : Top_* \longrightarrow Top_*$ by $RX = \mathcal{R} \otimes X / \sim$. Here $\sim$ is the relation that if $r \in \mathcal{R}$, then $(r; x_1, \ldots, x_0, \ldots, x_n) \sim (r'; x_1, \ldots, x_n)$ with r' the composition of r with $(1, \ldots, 1, 0, 1, \ldots, 1)$.

We follow the convention of [11] and denote functors associated to operads or right modules by standard font letters. The functor O associated to an operad $\mathcal{O}$ (viewed as a right module over itself) has more structure than functors coming just from right modules. These types of functors are called monads.

Definition 2.7. A functor O and two natural transformations $\mu : OO \longrightarrow O$ and $\eta : Id \longrightarrow O$ are called a monad if the following diagrams commute for every based space X:

$$\begin{array}{ccccc}
OX & \xrightarrow{\eta O} & OOX & \xleftarrow{O\eta} & OX \\
& {}_{id} \searrow & {}_{\mu} \downarrow & {}_{id} \swarrow & \\
& & OX & &
\end{array}
\qquad
\begin{array}{ccc}
OOOX & \xrightarrow{O\mu} & OOX \\
{}_{\mu O} \downarrow & & {}_{\mu} \downarrow \\
OOX & \xrightarrow{\mu} & OX
\end{array}$$

Definition 2.8. Let O be a monad. A space A and a map $\xi : OA \longrightarrow A$ is called an O-algebra if the following diagrams commute:

$$\begin{array}{ccc}
A & \xrightarrow{\eta} & OA \\
{}_{id} \searrow & {}_{\xi} \downarrow & \\
& A &
\end{array}
\qquad
\begin{array}{ccc}
OOA & \xrightarrow{\mu} & OA \\
{}_{O\xi} \downarrow & & {}_{\xi} \downarrow \\
OA & \xrightarrow{\xi} & A
\end{array}$$

Note that the data of being an algebra over an operad is the same as the data of being an algebra over the monad associated to that operad.

Definition 2.9. Let O be a monad. An O-functor is a functor R and natural transformation $\xi : RO \longrightarrow R$ making the following diagrams commute for every based space X:

$$\begin{array}{ccc}
RX & \xrightarrow{R\eta} & ROX \\
{}_{id} \searrow & {}_{\xi} \downarrow & \\
& RX &
\end{array}
\qquad
\begin{array}{ccc}
ROOX & \xrightarrow{R\mu} & ROX \\
{}_{\xi} \downarrow & & {}_{\xi} \downarrow \\
ROX & \xrightarrow{\xi} & RX
\end{array}$$
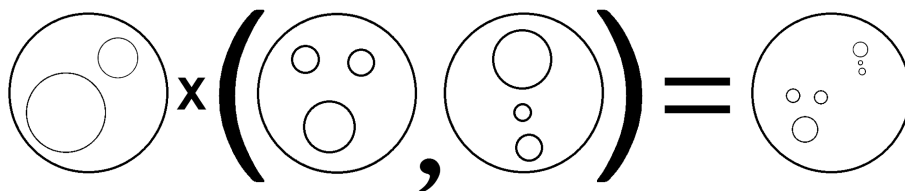
FIGURE 1. Little 2-disks operad

Note that if $\mathcal{R}$ is a right $\mathcal{O}$-module, then R is an O-functor. Unlike with algebras, there are O-functors which do not come from right $\mathcal{O}$-modules.

2.3. **Topological chiral homology via the two-sided bar construction.** In this subsection, we review a construction of [11] called the two-sided bar construction. May used this construction to prove the recognition principle, the theorem that connected D_n-algebras (see Definition 2.11) are homotopy equivalent to n-fold loop spaces. The two-sided bar construction is a construction which takes as inputs a monad O, an O-algebra A and an O-functor R and produces a simplicial space $B_*(R, O, A)$. We will then recall Andrade's definition of topological chiral homology which is the two-sided bar construction when O is the monad associated to the little n-disks operad and R is a right module defined using embeddings of disks in a manifold.

The space of k simplices of $B_*(R, O, A)$ will be $RO^k A$. The algebra composition map $OA \longrightarrow A$ induces a map $d_0 : B_k(R, O, A) \longrightarrow B_{k-1}(M, O, A)$. The monad composition map $OO \longrightarrow O$ gives $k - 1$ maps $d_i : B_k(R, O, A) \longrightarrow B_{k-1}(R, O, A)$ for $i = 1, \ldots, k - 1$ and the O-functor composition map $RO \longrightarrow R$ gives another map, $d_k : B_k(R, O, A) \longrightarrow B_{k-1}(R, O, A)$. These maps will be the face map of $B_*(R, O, A)$. The degeneracies $s_i : B_k(R, O, A) \longrightarrow B_{k+1}(R, O, A)$ are induced by the unit of the monad $Id \longrightarrow O$. In [11], May noted that these maps satisfy the axioms of face and degeneracy maps of a simplicial space.

Definition 2.10. For O a monad in based spaces, R an O-functor and A an O-algebra, let $B_*(R, O, A)$ be the simplicial space described above and let $B(R, O, A)$ denote its geometric realization.

Now we will recall the definition of the little n-disks operad D_n. We give examples of algebras, modules and functors over D_n. These operads, modules and functors will be used to define Andrade's model of topological chiral homology and the scanning map. Let $\mathbb{D}_n$ be the open unit ball in $\mathbb{R}^n$.

Definition 2.11. Let D_n be the Σ-space with $D_n(k)$ being the space of disjoint axis-preserving affine-linear embeddings of $\bigsqcup_{i=1}^{k} \mathbb{D}_n$ into $\mathbb{D}_n$. Topologize this space with the subspace topology inside the space of all continuous maps with the compact open topology. This forms an operad via composition of embeddings.

For orientable manifolds N and M, let $Emb(N, M)$ denote the space of smooth orientation-preserving embeddings.

Definition 2.12. Let M be a parallelized n-manifold. Using the parallelization, the derivative at a point of an orientation-preserving map between two framed
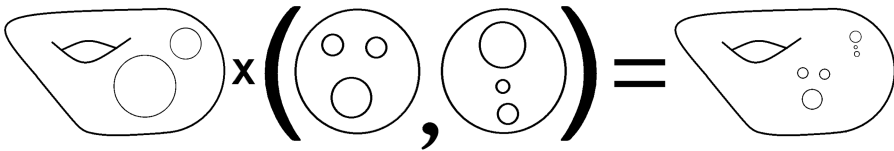
FIGURE 2. Embedding module

n-manifolds can be identified with a matrix in $GL_n(\mathbb{R})^+$ (+ denotes a positive determinant). Define $D(M)(k)$ to be the pullback of the following diagram:

$$
\begin{array}{ccc}
D(M)(k) & \longrightarrow & Emb(\bigsqcup_{i=1}^k \mathbb{D}_n, M) \\
\downarrow & & D_0 \downarrow \\
Mat_{\mathbb{R}}(n,n)^k & \overset{exp}{\longrightarrow} & GL_n^+(\mathbb{R})^k
\end{array}
$$

Here the map $D_0 : Emb(\bigsqcup_{i=1}^k \mathbb{D}_n, M) \longrightarrow GL_n^+(\mathbb{R})^k$ is the map which records the derivatives at $0 \in \mathbb{D}_n$ of all of the embeddings $\mathbb{D}_n \longrightarrow M$.

The spaces $D(M)(k)$ assemble to form a Σ-space denoted $D(M)$. The purpose of the matrices is to make $D(M)$ homotopy equivalent to the space of configurations of ordered distinct points in M. That is, the map which sends a collection of embeddings to the image of the centers of each disk induces a homotopy equivalence between $D(M)(k)$ and $M^k - \Delta_{fat}$, where Δ_{fat} is the fat diagonal. This map is a homotopy equivalence since the subspace of $Emb(\mathbb{D}_n, M)$ consisting of maps with a fixed value and derivative at $0 \in \mathbb{D}_n$ is contractible (see Chapter 4, Section 5 of [1]). This contrasts with the fact that the fibers of $Emb(\bigsqcup_{i=1}^k \mathbb{D}_n, M) \longrightarrow M^k - \Delta_{fat}$ are homotopic to $GL_n^+(\mathbb{R})^k$.

The space $D(M)$ has the structure of a right D_n-module as follows. Ignoring the matrices, the module structure is induced by composition of embeddings. To get the labeling matrices correct, use the following procedure. If $f : \mathbb{D}_n \longrightarrow M$ is an embedding with matrix A_0, and $e : \mathbb{D}_n \longrightarrow \mathbb{D}_n$ is an affine-linear axis-preserving embedding, pick a path in $D_n(1)$, e_t with $e_0 = id$ and $e_1 = e$. Next consider the following path of matrices: $B_t = D_{f \circ e_t}(0)$. Since $B_0 = exp(A_0)$, the path determines a branch of the logarithm and a unique continuous choice for A_t with $exp(A_t) = B_t$. Associate A_1 to the composition $f \circ e$.

To define the scanning map, we need to consider the following D_n-functor.

Example 2.13. Let Z be a paracompact space. The functor $Y \longrightarrow Map^c(Z, \Sigma^n Y)$ is a D_n-functor. Here $Map^c(Z, \Sigma^n Y)$ is defined to be the space of based maps from the one point compactification of Z (with ∞ as the basepoint) to Y.

Note that the standard definition of compactly supported maps is homotopic to this one. We use the convention that the one point compactification of a compact space is the space with a disjoint basepoint.

This is a D_n-functor because the functor $Y \longrightarrow \Sigma^n Y$ is a D_n-functor (page 128 of [11]). For M a parallelized n-manifold, there is a scanning natural transformation of D_n-functors $s : D(M) \longrightarrow Map^c(M, \Sigma^n \cdot)$ defined as follows. Let Y be a based space and $e_i : \mathbb{D}_n \longrightarrow M$ be embeddings. Map the pair $(e_1, \ldots, e_k; y_1, \ldots, y_k)$ to a map which is constant outside of the images of the e_i. For $m \in im(e_i)$, $(e_i^{-1}(m), y_i)$

defines a point inside $\Sigma^n Y$. Map m to this point. Here we view $\Sigma^n Y$ as Y smashed with the one point compactification of $\mathbb{D}_n$. This is a natural transformation of D_n-functors.

The scanning natural transformation induces the scanning map

$$s : B(D(M), D_n, A) \longrightarrow Map^c(M, B^n A)$$

as follows. The natural transformation induces a map $B(D(M), D_n, A) \longrightarrow B(Map^c(M, \Sigma^n \cdot), D_n, A)$. Let X be a based space and let A_* be a simplicial space. The spaces of based maps, $Map^\bullet(X, A_k)$, assemble to form a simplicial space $Map^\bullet(X, A_*)$. There is a natural map $|Map^\bullet(X, A_*)| \longrightarrow Map^\bullet(X, |A_*|)$. Taking X equal to the one point compactification of M gives a map

$$B(Map^c(M, \Sigma^n \cdot), D_n, A) \longrightarrow Map^c(M, B(\Sigma^n \cdot, D_n, A)).$$

The composition of these two maps is defined to be the scanning map

$$s : B(D(M), D_n, A) \longrightarrow Map^c(M, B^n A).$$

2.4. **Properties of simplicial spaces.** In this section, we recall some facts about simplicial spaces that we will need to prove that the scanning map induces a homotopy or homology equivalence. Most of these were proved in [11]. We will also use a lemma appearing in [8]. Recall that a levelwise weak homotopy equivalence between simplicial spaces does not always induce a weak homotopy equivalence between geometric realizations. The following sufficient condition is due to May in [11].

Definition 2.14. A simplicial space A_* is called proper if $\bigcup s_i(A_i) \longrightarrow A_{i+1}$ is a cofibration for each i.

Theorem 2.15. *A map between the proper simplicial space $f_* : A_* \longrightarrow B_*$ induces homology or weak homotopy equivalence on geometric realizations if it does levelwise.*

The fact that a levelwise homology equivalence induces a homology equivalence on geometric realizations follows from a spectral sequence introduced by Segal in [16]. Given a simplicial space A_*, let $E^0_{pq} = C_p(A_q)$. This is a double complex with the following two differentials. The first is induced by the differential on singular chains and the second induced by the alternating sum of the face maps. If A_* is proper, this gives a spectral sequence converging to the homology of $|A_*|$, the geometric realization of A_* with the following E_2 page. Let $\partial_{pq} : H_p(A_q) \longrightarrow H_p(A_{q-1})$ be the alternating sum of the maps in homology induced by the face maps. This forms a chain complex for each p. Call this chain complex $\mathcal{E}_p$. The Segal spectral sequence has $E_2^{pq} = H_q(\mathcal{E}_p)$.

To use the Segal spectral sequence or Theorem 2.15, one needs to be able to prove that a given simplicial space is proper. In [11], May gives the following criterion for $B_*(M, O, A)$ being a proper simplicial space. To state it, May introduces the following definitions.

Definition 2.16. A pair (X, A) of spaces is an NDR-pair if there exists a map $u : X \longrightarrow [0, 1]$ such that $A = u^{-1}(\{0\})$ and a homotopy $h : [0, 1] \times X \longrightarrow X$ with $h(0, x) = x$ for all $x \in X$, $h(t, a) = a$ for all $(t, a) \in [0, 1] \times A$, and $h(1, x) \in A$ for all $x \in u^{-1}([0, 1))$. The pair (h, u) is said to be a representation of (X, A) as an NDR-pair.

Definition 2.17. A functor $F : Top_* \longrightarrow Top_*$ is admissible if any representation (h, u) of (X, A) as an NDR-pair determines a representation (Fh, Fu) of (FX, FA) as an NDR-pair such that $(Fh)_t = F(h_t)$ on X and such that, for any map $g : X \longrightarrow X$ with $u(g(x)) < 1$ whenever $u(x) < 1$, the map $Fu : FX \longrightarrow [0, 1]$ satisfies $(Fu)((Fg)(y)) < 1$ whenever $Fu(y) < 1, y \in FX$.

The definition of an admissible functor is useful because of the following theorem in [11].

Proposition 2.18. *Let $\mathcal{O}$ be an operad with $\{id\} \longrightarrow \mathcal{O}(1)$ a cofibration of spaces. Let O be the monad associated to $\mathcal{O}$. Let A be an $\mathcal{O}$-algebra that is a well-based space (inclusion of the basepoint is a cofibration) and let M be an admissible O-functor. Then $B_*(M, O, A)$ is a proper simplicial space.*

May proved that Σ^n, Ω^n and monads associated to operads are admissible functors. He noted that composition of admissible functors is also admissible. We shall prove that the functor associated to $D(M)$ and $Map^c(M, \cdot)$ are admissible.

Lemma 2.19. *The functor associated to $D(M)$ and $Map^c(M, \cdot)$ are admissible.*

Proof. Let (h, u) represent (X, A) as an NDR-pair. Define

$$Map^c(M, \cdot)u : Map^c(M, X) \longrightarrow [0, 1]$$

by the formula $Map^c(M, \cdot)u(f) = Max_{m \in M} u(f(m))$. Define $D(M)u : D(M)X \longrightarrow [0, 1]$ by $(D(M)u)(e; x_1, \ldots, x_k) = max_i u(x_i)$. These functions satisfy the hypothesis of Definition 2.17.

Let X be a based space and let A_* be a simplicial space. The natural map $|Map^\bullet(X, A_*)| \longrightarrow Map^\bullet(X, |A_*|)$ is not always a homotopy equivalence. In [11], May proved that if each A_k is connected, A_* is proper, and X is a circle, then the map is a weak equivalence. In [8], they proved that the map is a homotopy equivalence if X is a simplicial complex, A_* is proper and A_k is $dimX$ connected for each k. We will generalize the last theorem by relaxing the connectivity of requirements on the spaces A_k by one. These proofs use the notion of a simplicial Hurewicz fibration introduced in [11]. A simplicial Hurewicz fibration is a condition on a map of simplicial spaces $f_* : E_* \longrightarrow B_*$ that generalizes the notion of fibration of spaces. This definition is somewhat involved, so we will not give it. However, we will state two theorems. In [8] (page 11), Hesselholt and Madsen observed that a particular class of maps are simplicial Hurewicz fibrations. In [11] (Theorem 12.7), May proved useful properties of simplicial Hurewicz fibrations.

Proposition 2.20. *Let Z and X be simplicial complexes and let $X = Z \cup e_l$ with e_l an l-cell. Then for any proper simplicial space A_*, $Map^\bullet(X, A_*) \longrightarrow Map^\bullet(Z, A_*)$ is a simplicial Hurewicz fibration with fiber $Map^\bullet(S^l, A_*)$.*

In [8], they also noted that $Map^\bullet(X, A_*)$ is proper if A_* is proper.

Proposition 2.21. *Let $E_* \longrightarrow B_*$ be a simplicial Hurewicz fibration with fiber F_*. If each B_k is connected and B_* is proper, then $|E_*| \longrightarrow |B_*|$ is a quasi-fibration with fiber $|F_*|$.*

Combining these two propositions, we get the following corollary.

Corollary 2.22. *Let X be a finite simplicial complex of dimension at most n. Assume that A_* is a proper simplicial space and that each A_k is $(n-1)$-connected. Then the map $|Map^\bullet(X, A_*)| \longrightarrow Map^\bullet(X, |A_*|)$ is a weak equivalence.*

Proof. Let X_{n-1} be the $n-1$-skeleton of X. The quotient X/X_{n-1} is a wedge of spheres $\bigvee S^n$. For any space Y,

$$Map^\bullet(\bigvee S^n, Y) \longrightarrow Map^\bullet(X, Y) \longrightarrow Map^\bullet(X_{n-1}, Y)$$

is a fibration sequence since $X_{n-1} \longrightarrow X$ is a cofibration. This is true in particular when $Y = |A_*|$. Since every A_k is $(n-1)$-connected and X_{n-1} is $(n-1)$-dimensional, the spaces $Map^\bullet(X_{n-1}, A_k)$ are connected. The sequence $|Map^\bullet(\bigvee S^n, A_*)| \longrightarrow |Map^\bullet(X, A_*)| \longrightarrow |Map^\bullet(X_{n-1}, A_*)|$ is a quasi-fibration sequence since

$$Map^\bullet(\bigvee S^n, A_*) \longrightarrow Map^\bullet(X, A_*) \longrightarrow Map^\bullet(X_{n-1}, A_*)$$

is a simplicial Hurewicz fibration, A_* is proper and $Map^\bullet(X_{n-1}, A_k)$ is connected for every k. Consider the following commuting diagram of quasi-fibrations:

$$\begin{array}{ccc}
|Map^\bullet(\bigvee S^n, A_*)| & \longrightarrow & Map^\bullet(\bigvee S^n, |A_*|) \\
\downarrow & & \downarrow \\
|Map^\bullet(X, A_*)| & \longrightarrow & Map^\bullet(X, |A_*|) \\
\downarrow & & \downarrow \\
|Map^\bullet(X_{n-1}, A_*)| & \longrightarrow & Map^\bullet(X_{n-1}, |A_*|)
\end{array}$$

The map $|Map^\bullet(\bigvee S^n, A_*)| \longrightarrow Map^\bullet(\bigvee S^n, |A_*|)$ is a weak equivalence by [11] (Theorem 12.3) and $|Map^\bullet(X_{n-1}, A_*)| \longrightarrow Map^\bullet(X_{n-1}, |A_*|)$ is a weak equivalence by [8] (Lemma 1.4). Thus, $|Map^\bullet(X, A_*)| \longrightarrow Map^\bullet(X, |A_*|)$ is a weak equivalence.

2.5. **Classical scanning theorems for configuration spaces.** To prove the recognition principle, May proved the so-called approximation theorem. Namely he proved that $s : D_n X \longrightarrow \Omega^n \Sigma^n X$ is a weak equivalence for X-connected and well based [11]. In order to prove nonabeilian Poincaré duality theorems, we will need similar results concerning the scanning map $s : D(M)X \longrightarrow Map(M, \Sigma^n X)$. Thus, we review some facts about classical configuration spaces proved in [12] and [2].

Definition 2.23. Let M be a parallelized n-manifold. Let $C(M)$ be the Σ-space with $C(M)(k) = M^k - \Delta_{fat}$ with Δ_{fat} denoting the fat diagonal.

Note that the map $c : D(M)(k) \longrightarrow C(M)(k)$ sending an embedding to its center is a homotopy equivalence, and so c induces a homotopy equivalence between $D(M)X \longrightarrow C(M)X$ for all well-based spaces X. In [2], Bödigheimer defined scanning maps s' making the following diagram homotopy commute:

$$\begin{array}{ccc}
D(M)X & \xrightarrow{c} & C(M)X \\
 & {\scriptstyle s} \searrow & {\scriptstyle s'} \downarrow \\
 & & Map^c(M, \Sigma^n X)
\end{array}$$

Bödigheimer also generalizes May's approximation theorem and proves the following theorem [2].

Theorem 2.24. *If X is connected and M a parallelizable n-manifold, then*

$$s' : C(M)X \longrightarrow Map^c(M, \Sigma^n X)$$

is a homotopy equivalence.

If X is not connected, then s and s' are not homotopy equivalences. However, if M is open, they are what we shall call a "stable" homology equivalence. The word stable does not mean stable in the sense of stable homotopy theory. For simplicity

we assume that M is connected and is the interior of a (not necessarily compact) manifold $\bar{M}$ with connected boundary ∂M. However, everything can be generalized to the case when $\pi_0(\partial M) \longrightarrow \pi_0(\bar{M})$ is onto.

Let $M' = \bar{M} \cup_{\partial M} \partial M \times [0, 1)$. Fix a diffeomorphism $d : M' \longrightarrow M$ whose inverse is isotopic to the inclusion $M \hookrightarrow M'$. Given $x \in X$ and $p \in \partial M \times [0, 1)$, there is an induced map $t_x : C(M)X \longrightarrow C(M)X$ defined as follows. Send a configuration $(m_1, \ldots, m_k; x_1, \ldots, x_k)$ to $(d(m_1), \ldots, d(m_k), d(p); x_1, \ldots, x_k, x)$. Up to homotopy, t only depends on $[x] \in \pi_0(X)$. Let $f_x : \partial M \times [0, 1) \longrightarrow \Sigma^n X$ be $s(p; x)$. Let $T_x : Map^c(M, \Sigma^n X) \longrightarrow Map^c(M, \Sigma^n X)$ be the following function:

$$T_x(f)(m) = \begin{cases} f(d^{-1}(m)) & \text{if } d^{-1}(m) \in M, \\ f_x(d^{-1}(m)) & \text{if } d^{-1}(m) \notin M. \end{cases}$$

Let $\{x_i\}$ be a sequence of not necessarily distinct elements of X such that each connected component of X has infinitely many terms of the sequence. The natural numbers $\mathbb{N}_0$ are a partially ordered set and hence a category. Let $\mathfrak{C} : \mathbb{N}_0 \longrightarrow Top$ be a functor that takes each object to $C(M)X$ and sends morphisms $j \longrightarrow j + 1$ to maps t_{x_j}. Likewise define $\mathfrak{M} : \mathbb{N}_0 \longrightarrow Top$ to be the functor which takes objects to $Map^c(M, \Sigma^n X)$ and morphisms to the maps T_{x_j}. In [12], McDuff proved the following theorem in the case that $X = S^0$.

Theorem 2.25. *If M is a connected parallelizable manifold, ∂M is nonempty and $\dim_{\mathbb{R}} M > 1$, then $s' : hocolim_{\mathbb{N}_0} \mathfrak{C} \longrightarrow hocolim_{\mathbb{N}_0} \mathfrak{M}$ is a homology equivalence.*

However, all of her arguments follow almost verbatim for general well based X. See [2] for more details. We shall describe the above theorem by saying that $s : C(M)X \longrightarrow Map^c(M, \Sigma^n X)$ is a stable homology equivalence. We call the maps t_x and T_x stabilization maps. Note that each T_x is a homotopy equivalence. Thus $hocolim_{\mathbb{N}_0} \mathfrak{M}$ is homotopy equivalent to $Map^c(M, \Sigma^n X)$.

2.6. **Scanning with a connected algebra.** The goal of this section is to prove that the scanning map $s : B(D(M), D_n, A) \longrightarrow Map^c(M, B^n A)$ is a weak homotopy equivalence when A is connected. The proof follows May's proof of the approximation theorem in [11]. This is a special case of Lurie's nonabelian Poincaré duality theorem from [10] since all connected D_n-algebras are grouplike. Despite being less general than theorems already appearing in the literature, we give this proof as warm up to the case when A is not necessarily grouplike.

Theorem 2.26. *If A is a connected D_n-algebra and M a parallelized n-manifold, then the scanning map $s : B(D(M), D_n, A) \longrightarrow Map^c(M, B^n A)$ is a weak homotopy equivalence.*

Proof. Consider the map of simplicial spaces:

$$B_*(D(M), D_n, A) \longrightarrow B_*(Map^c(M, \Sigma^n \cdot), D_n, A).$$

By Lemma 2.19 and Proposition 2.18, both simplicial spaces are proper. By Theorem 2.24 and the fact that $D(M)X$ is homotopic to $C(M)X$, the map is a levelwise homotopy equivalence. Thus by Theorem 2.15, it induces a weak equivalence on geometric realizations. Now consider the map

$$B(Map^c(M, \Sigma^n \cdot), D_n, A) \longrightarrow Map^c(M, B(\Sigma^n, D_n, A)).$$

Since $B_*(\Sigma^n, D_n, A)$ is a proper simplicial space and $\Sigma^n D_n^k A$ is n-connected (and hence $(n-1)$-connected) for every k, by Corollary 2.22 we can conclude that this

map is a weak homotopy equivalence. Since the scanning map is the composition of these two maps, it too is a weak homotopy equivalence.

2.7. **Scanning with an open manifold.** In this subsection, we no longer assume that the D_n-algebra A is connected or even grouplike. As before, we assume that M is a smooth, connected, open, parallelizable n-manifold with $n > 1$. We also assume that M is the interior of a (not necessarily compact) manifold $\bar{M}$ with connected boundary. Note that insisting that the boundary of $\bar{M}$ is connected is not a condition on M since we do not assume that $\bar{M}$ is compact. The goal of this section is to prove that the scanning map $s : B(D(M), D_n, A) \longrightarrow Map^c(M, B^n A)$ is a stable homology equivalence. Here stable is in the sense of Subsection 2.5. Intuitively, we want to use the same proof strategy as was used in the case where A is connected except using Theorem 2.25 instead of Theorem 2.24. See Subsection 2.5 for definitions of the stabilization maps $t_x : C(M)X \longrightarrow C(M)X$ and $T_x : Map^c(M; \Sigma^n X) \longrightarrow Map^c(M; \Sigma^n X)$.

For $\alpha \in B_k(D(\partial M \times (0,1)), D_n, A)$, we can define a stabilization map $t_\alpha : B_k(D(M), D_n, A) \longrightarrow B_k(D(M), D_n, A)$ in a similar way to the way $t_x : C(M)X \longrightarrow C(M)X$ was defined. That is, taking the "union" of an element $\xi \in B_k(D(M), D_n, A)$ with the element $\alpha \in B_k(D(\partial M \times (0,1)), D_n, A)$ gives an element of $B_k(D(\bar{M} \cup_{\partial M} \partial M \times [0,1)), D_n, A)$. Using a diffeomorphism $\bar{M} \cup_{\partial M} \partial M \times (0,1) \longrightarrow M$ produces an element of $B_k(D(M), D_n, A)$. For future use, we require that the inverse of this diffeomorphism is isotopic to the inclusion $M \hookrightarrow \bar{M} \cup_{\partial M} \partial M \times [0,1)$.

For $a \in A$, we shall define a stabilization map $t_a : B_*(D(M), D_n, A) \longrightarrow B_*(D(M), D_n, A)$ as follows. Let $a^0 \in D(\partial M \times (0,1))A$ be a disk labeled by a. Let $a^k \in B_k(D(\partial M \times (0,1)), D_n, A)$ be the image of a^0 under the composition of k degeneracy maps. Let $t_a : B_*(D(M), D_n, A) \longrightarrow B_*(D(M), D_n, A)$ be the map induced by $t_{a^k} : B_k(D(M), D_n, A) \longrightarrow B_k(D(M), D_n, A)$. Let $\{a_i\}$ be a sequence of not necessarily distinct elements of A such that each connected component of A has infinitely many terms of the sequence. Let $\mathfrak{C} : \mathbb{N}_0 \longrightarrow Top$ be a functor that takes each object to $B(D(M), D_n, A)$ and sends morphisms $j \longrightarrow j+1$ to maps t_{a_j}. Likewise define $\mathfrak{M} : \mathbb{N}_0 \longrightarrow Top$ to be the functor which takes objects to $Map^c(M, B^n X)$ and morphisms to the maps T_{a_j} (defined in the analogous way using the maps $T_{a_j^k} : B_k(Map^c(M, \Sigma^n \cdot), D_n, A) \longrightarrow B_k(Map^c(M, \Sigma^n \cdot), D_n, A)$).

Also define $\mathfrak{C}_k$ and $\mathfrak{M}_k$ to be the analogous functor sending every natural number to $B_k(D(M), D_n, A)$ and $B_k(Map^c(M, \Sigma^n \cdot), D_n, A)$ respectively. Let C_* be the simplicial space with $C_k = hocolim_{\mathbb{N}_0} \mathfrak{C}_k$ and face maps and degeneracies induced by the face maps and degeneracies of $B_*(D(M), D_n, A)$. Likewise define M_* to be the simplicial space with $M_k = hocolim_{\mathbb{N}_0} \mathfrak{M}_k$. We will show that $|C_*|$ is homology equivalent to $|M_*|$.

We will need to compare the maps t_α with the maps t_{a^k}. Let

$$\partial_k : H_*(B_k(D(M), D_n, A)) \longrightarrow H_*(B_{k-1}(D(M), D_n, A))$$

be the alternating sum of the facemaps.

Lemma 2.27. *For all $x \in ker(\partial_k)$ and $\alpha \in B_k(D(\partial M \times (0,1)), D_n, A)$, there exist $a \in A$ such that $t_{\alpha*}(x) - t_{a^k*}(x) \in im(\partial_{k+1})$. In other words, $t_{\alpha*}$ and t_{a^k*} induce the same map on the E_2 page of the Segal spectral sequence for the homologies of $|C_*|$ and $|M_*|$.*
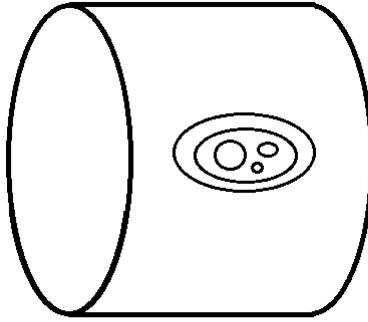
FIGURE 3. Two outer walls

Proof. For $\alpha, \alpha' \in B_k(D(\partial M \times (0,1)), D_n, A)$, we say $t_{\alpha*}$ and $t_{\alpha'*}$ are homologous to mean $t_{\alpha*}(x) - t_{\alpha'*}(x) \in im(\partial_{k+1})$ for all $x \in ker(\partial_k)$. An element $\alpha \in B_k(D(\partial M \times (0,1)), D_n, A)$ includes the data of a collection of embeddings $e \in D(\partial M \times (0,1))$ and elements $e_{ij} \in D_n$ for $0 \le i \le k$ and elements $a_l \in A$. We order the e_{ij} so that the elements $(e_{k1}, e_{k2}, \ldots; a_1, a_2, \ldots)$ define elements of $D_n A$ and $(e_{k-1,1}, e_{k-1,2}, \ldots; e_{k1}, e_{k2}, \ldots; a_1, a_2, \ldots)$ define elements of $D_n(D_n A)$, et cetera. For all $N > 0$, we say that $\alpha \in B_k(D(\partial M \times (0,1)), D_n, A)$ has at least N outer walls if $e \in D(\partial M \times (0,1))(1)$ and each $e_{ij} \in D_n(1)$ for $i < N$ (see Figure 3). We say that all elements have at least zero outer walls. For $\alpha, \alpha' \in B_k(D(\partial M \times (0,1)), D_n, A)$, we say that $\alpha \sim \alpha'$ if α and α' are in the same connected component of $B(D(\partial M \times (0,1)), D_n, A)$. If $\alpha \sim \alpha'$ and α and α' both have $k+1$ outer walls, then α and α' are in the same component of $B_k(D(\partial M \times (0,1)), D_n, A)$ and hence $t_{\alpha*} = t_{\alpha'*}$. Note that t_α is homotopic to t_{a^k} for some a if α has $k+1$ outer walls.

We shall assume for the purposes of induction that if $\alpha \sim \alpha'$ and α and α' both have $N+1$ outer walls, then $t_{\alpha*}$ and $t_{\alpha'*}$ are homologous. We shall now prove that if $\alpha \sim \alpha'$ and α and α' both have N outer walls, then $t_{\alpha*}$ and $t_{\alpha'*}$ are homologous. Let $\beta \in B_{k+1}(D(\partial M \times (0,1)), D_n, A)$ be α with an extra outer wall. Note that $d_i\beta = \alpha$ for $i \ge k+1-N$ and $d_i\beta$ has $N+1$ outer walls for $i < k+1-N$ (see Figure 4 for an illustration). Let $a \in A$ be the product of all of the elements of A labeling the disks comprising α. This is well defined in $\pi_0(A)$. Note that a^k has $k+1$ (and hence $N+1$) outer walls (see above for definition of a^k). Also observe that $a^k \sim d_i\beta$ for all i. By our induction hypothesis, t_{a^k*} is homologous to $t_{d_i\beta*}$ for $i < k+1-N$ since $d_i\beta$ has $N+1$ outer walls for $i < k+1-N$. Let $x \in ker\partial_k$ be arbitrary. Note that the following diagram commutes:

$$(2.1) \quad \begin{array}{ccc} B_{k+1}(D(M), D_n, A) & \xrightarrow{t_\beta} & B_{k+1}(D(M), D_n, A) \\ d_i \downarrow & & d_i \downarrow \\ B_k(D(M), D_n, A) & \xrightarrow{t_{d_i\beta}} & B_k(D(M), D_n, A) \end{array}$$

Let $s_l : B_k(D(M), D_n, A) \longrightarrow B_{k+1}(D(M), D_n, A)$ be a degeneracy map. Let $x \in ker\partial_k$ be arbitrary. From now on, we will write "=" for homologous and not distinguish between maps and their effects in homology. Note that for any

$z \in H_*(B_{k-1}(D(M), D_n, A))$, $s_l z$ is null homologous. We have

$$0 = \partial_{k+1} t_\beta s_l x \text{ (since boundaries are zero in homology)}$$

$$= \sum_{i=0}^{i=k+1} \pm d_i t_\beta s_l x \text{ (by the definition of } \partial\text{)}$$

$$= \sum_{i=0}^{i=k-N} \pm t_{a^k} d_i s_l x + \sum_{i=k+1-N}^{i=k+1} \pm t_\alpha d_i s_l x \text{ (by diagram (2.1))}$$

and

$$0 = \partial_{k+1} t_{a^{k+1}} s_l x \text{ (since boundaries are zero in homology)}$$

$$= t_{a^k} \partial_{k+1} s_l x \text{ (since } t_a \text{ is a chain map)}$$

$$= \sum_{i=0}^{i=k-N} \pm t_{a^k} d_i s_l x + \sum_{i=k+1-N}^{i=k+1} \pm t_{a^k} d_i s_l x \text{ (definition of } \partial\text{)}.$$

Subtracting, we see that $t_\alpha(y) = t_{a^k}(y)$ for $y = \sum_{i=k+1-N}^{i=k+1} \pm d_i s_l x$. Let $l = k - N$. We have

$$y = \pm d_{k+1-N} s_{k-N} x + \sum_{i=k+2-N}^{i=k+1} \pm d_i s_{k-N} x$$

$$= \pm x + \sum_{i=k+2-N}^{i=k+1} \pm s_{k-N} d_{i-1} x \text{ (simplicial identities)}$$

$$= \pm x \text{ (image of degeneracies are null homologous)}.$$

Thus $x = \pm y$ so $t_\alpha(x) = t_{a^k}(x)$. Since x was arbitrary, we can conclude that t_α and t_{a^k} are homologous. If $\alpha' \sim \alpha$, then $\alpha' \sim a^k$, and so t_α and $t_{\alpha'}$ are also homologous. The claim now follows by induction.

Theorem 2.28. *The scanning map* $s : B(D(M), D_n, A) \longrightarrow Map^c(M, B^n A)$ *induces a map of simplicial spaces* $s_* : C_* \longrightarrow M_*$ *which induces a homology equivalence on geometric realizations.*

Proof. First note that C_* and M_* are proper simplicial spaces. This follows from the fact that $B_*(D(M), D_n, A)$ and $B_*(Map(M, \Sigma^n \cdot), D_n, A)$ are proper and the following fact about cofibrations. Assume that the following diagram commutes and the vertical maps are cofibrations:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & W \end{array}$$

Then the inclusion of the mapping cylinder of $X \longrightarrow Y$ into the mapping cylinder of $Z \longrightarrow W$ is a cofibration.

It is not true that the scanning maps $s : C_k \longrightarrow M_k$ are homology equivalences. This would follow from Theorem 2.25 if we took the homotopy colimit with respect to stabilization maps t_α for α representing each component of $\pi_0(B_k(D(\partial M \times (0,1)), D_n, A))$. However, we are only using stabilization maps of the form $t_{a_i^k}$ (see above for notation). Fortunately this difference is not relevant on the E_2 page of the Segal spectral sequence (see Subsection 2.4). By Lemma 2.27, the difference between the effects in homology of t_α with α arbitrary and $t_{a_i^k}$ is in the image of the alternating sum of the face maps $\sum \pm d_i : H_*(C_{k+1}) \longrightarrow H_*(C_k)$.
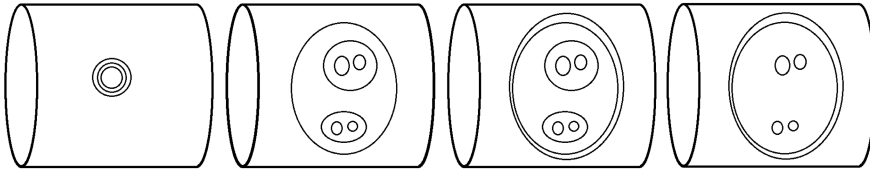
FIGURE 4. Example of a^k, α, β, and $d_i\beta$ with $i < k + 1 - N$

Thus by Theorem 2.25, the map, $s : C_* \longrightarrow M_*$ induces an isomorphism on the E_2 page of the Segal spectral sequence and hence $s : |C_*| \longrightarrow |M_*|$ is a homology equivalence.

We can now deduce Theorem 1.1 (nonabelian Poincaré duality after stabilizing) for Andrade's model of topological chiral homology, $\int_M A = B(D(M), D_n, A)$.

Corollary 2.29. *If M is an open parallelizable n-manifold, then the scanning map s induces a homology equivalence between $hocolim_{t_{a_i}} B(D(M), D_n, A)$ and $Map^c(M, B(\Sigma^n, D_n, A))$.*

Proof. By Theorem 2.28, $s : |C_*| \longrightarrow |M_*|$ is a homology equivalence. After interchanging colimits and using Corollary 2.22, we have that

$$|C_*| = hocolim_{t_{a_i}} B(D(M), D_n, A)$$

and

$$|M_*| = hocolim_{T_{a_i}} Map(M, B(\Sigma^n, D_n, A)).$$

Since the stabilization maps

$$T_{a_i} : Map^c(M, B(\Sigma^n, D_n, A)) \longrightarrow Map^c(M, B(\Sigma^n, D_n, A))$$

are homotopy equivalences, $Map^c(M, B(\Sigma^n, D_n, A))$ is weakly equivalent to $|M_*|$. Thus $hocolim_{t_{a_i}} B(D(M), D_n, A)$ is homology equivalent to $Map^c(M, B(\Sigma^n, D_n, A))$ and we have proven Theorem 1.1 for Andrade's model of topological chiral homology. ∎

3. Configuration spaces of particles with summable labels

The goal of this section is to prove Theorem 1.1 (nonabelian Poincaré duality after stabilizing) for Salvatore's model of topological chiral homology, configuration spaces of particles with summable labels [15]. We will no longer restrict our attention to the case of parallelizable manifolds. Thus we will consider framed E_n-algebras instead of unframed.

Unlike other models of topological chiral homology, Salvatore's model does not take as inputs an algebra over the little n-disks operad, but instead accepts algebras over the (framed) Fulton-MacPherson operad F_n. In the previous section, the arguments can be summarized as using May's two-sided bar construction to leverage classical results about configuration spaces to draw conclusions about topological chiral homology. In contrast, Salvatore's model of topological chiral homology is sufficiently close to classical configuration spaces that many of those arguments developed for classical configuration spaces directly apply. In Subsection 3.1, we recall the definition of the (framed) Fulton-MacPherson configuration spaces and operad

as well as review the definition of configuration spaces of particles with summable labels. In Subsection 3.2, we recall Salvatore's definition of relative configuration spaces. In Subsection 3.3, we extend Salvatore's nonabelian Poincaré duality theorem from [15] to more general relative configuration spaces using arguments from [2]. In Subsection 3.4, we use ideas similar to those used by McDuff in [12] regarding homology fibrations and configuration spaces to prove nonabelian Poincaré duality after stabilizing for Salvatore's model of topological chiral homology.

3.1. Fulton-Macpherson operad and configuration spaces.
In this subsection, we will recall the definition of the Fulton-MacPherson configuration space and the Fulton-MacPherson operad. Using these definitions, we will describe Salvatore's model of topological chiral homology. We follow the treatment in [15] in general but modify some notation in order not to conflict with notation from the previous section. The Fulton-MacPherson configuration space is a partial compactification of the configuration space of ordered distinct points in a manifold.

Definition 3.1. Let $Bl_\Delta M^k$ denote the real oriented blow-up of M^k along the small diagonal Δ.

There is a natural map from $C(M)(k) \longrightarrow Bl_\Delta(M^k)$. Given a subset $S \subset \{1, \ldots, k\}$, there is a natural map $C(M)(k) \longrightarrow C(M)(|S|)$. Let $j : C(M)(k) \longrightarrow \prod_{|S|>1} Bl_\Delta M^S$ be the product of these maps.

Definition 3.2. Let $C^{fm}(M)(k)$ be the closure of the image of j.

There is an obvious macroscopic location map $b : C^{fm}(M)(k) \longrightarrow M^k$.

Definition 3.3. As a space, the k'th space of operad F_n is the collection of points of $C^{fm}(\mathbb{R}^n)(k)$ macroscopically located at the origin, $b^{-1}(0, 0, \ldots)$.

See [6] or [15] for an operad structure on F_n and a proof that F_n is an E_n-operad. To define the framed Fulton-MacPherson operad, we first recall the definition of the semi-direct product of a group G and an operad $\mathcal{O}$ in G-spaces (an operad in spaces where each space has a G-action and all structure maps are G-equivariant).

Definition 3.4. Let G be a group and let $\mathcal{O}$ be an operad in G-spaces. Define $\mathcal{O} \rtimes G$ to be the topological operad with $(\mathcal{O} \rtimes G)(k) = \mathcal{O}(k) \times G^k$ and composition map $\tilde{m} : (\mathcal{O} \rtimes G) \otimes (\mathcal{O} \rtimes G) \longrightarrow \mathcal{O} \rtimes G$ given by

$$\tilde{m}((o, g_1, \ldots, g_k); (o_1, g_1^1, \ldots, g_1^{m_1}), \ldots, (o_k, g_k^1, \ldots, g_k^{m_k}))$$

$$= (m(o; g_1 o_1, \ldots, g_k o_k), g_1 g_1^1, \ldots, g_k g_k^{m_k})$$

with $m : \mathcal{O} \otimes \mathcal{O} \longrightarrow \mathcal{O}$ the composition map of the operad $\mathcal{O}$.

We now define the framed Fulton-MacPherson operad and the framed Fulton-MacPherson configuration space.

Definition 3.5. Define the framed Fulton-MacPherson operad fF_n as $F_n \rtimes GL_n(\mathbb{R})$ with the $GL_n(\mathbb{R})$ action on $F_n(k)$ induced by the action of $GL_n(\mathbb{R})$ on $\mathbb{R}^n$.

See [15] for more details on this action.

Definition 3.6. Let $P \longrightarrow M$ be the principle $GL_n(\mathbb{R})$ bundle associated to the tangent bundle of an n-manifold M. Let the framed Fulton-MacPherson configuration space of k points in M be the pullback of the following diagram:

$$
\begin{array}{ccc}
fC^{fm}(M)(k) & \longrightarrow & P^k \\
\downarrow & & \downarrow \\
C^{fm}(M)(k) & \overset{b}{\longrightarrow} & M^k
\end{array}
$$

with b the macroscopic location map. Let $fC^{fm}(M) = \bigsqcup_k fC^{fm}(M)(k)$. Note that the symmetric group actions on M^k induces a Σ-space structure on $fC^{fm}(M)$.

In [15], Salvatore describes a right fF_n-module structure on $fC^{fm}(M)$. This structure is similar to the D_n-module structure on $D(M)$. For M an n-manifold with corners, we define $fC^{fm}(M)$ as follows. Let W be an n-manifold (without boundary or corners) containing M. Let $fC^{fm}(M)$ be the subspace of $fC^{fm}(W)$ of particles macroscopically located in M. We can now give Salvatore's definition of the configuration space of particles with summable labels.

Definition 3.7. For an fF_n-algebra A and M an n-manifold (possibly with corners), let $C(M; A)$ be the coequalizer of the two natural maps $fC^{fm}(M) \otimes fF_n \otimes A \rightrightarrows fC^{fm}(M) \otimes A$.

3.2. **Relative configuration spaces of particles.** There is also a relative version of Salvatore's configuration spaces of particles with summable labels. For $N \subset M$, we will define a relative configuration space $C^{fm}(M, N; A)$. Intuitively it is the space of particles in M which vanish if they enter N. To define $C(M; A)$, we used a right fF_n-module $fC^{fm}(M)$. To define $C^{fm}(M, N; A)$, we will need to define a right functor $fC(M, N)$ over the monad fF_n. Throughout, we will assume that M and N are manifolds with corners, $N \hookrightarrow M$ is a cofibration, $dim M = n$ and that $M - N$ is open.

Let W be an open submanifold containing N. Let X be a space. All elements of $fC^{fm}(M)X$ can be uniquely described by an element of $fC^{fm}(M - N)X$ and an element of $fC^{fm}(W)X$ consisting of points macroscopically located in N. Let $\sim$ be the relation on $fC^{fm}(M)X$ given by identifying elements whose corresponding elements in $fC^{fm}(M - N)X$ are equal.

Definition 3.8. For a based space X, we define a space $fC^{fm}(M, N)X$ to be $C^{fm}(M)X / \sim$. Let $fC^{fm}(M, N)$ be the functor from based spaces to based spaces which sends a space X to $fC^{fm}(M, N)X$.

See [15] for a description of the right fF_n-functor structure on $C^{fm}(M, N)$.

Definition 3.9. For A an fF_n-algebra, let $C(M, N; A)$ denote the coequalizer of the two natural maps $fC^{fm}(M, N)fF_nA \rightrightarrows fC^{fm}(M, N)A$.

3.3. **Quasi-fibrations and scanning theorems.** In this subsection we will recall Salvatore's definition of the scanning map, review the nonabelian Poincaré duality theorems from [15], as well as review Salvatore's results concerning when the natural map $\pi : C(M; A) \longrightarrow C(M, N; A)$ is a quasi-fibration. All theorems without proof in this subsection are due to Salvatore in [15]. Before we can define Salvatore's scanning map, we need the following theorem regarding a model of B^nA and the following definition of a relative compactly supported space of sections.

Theorem 3.10. For an fF_n-algebra A, $C(S^n, pt; A) \simeq B^nA$ and $C(\mathbb{R}^n; A) \simeq A$.

For a manifold M with a metric, let TM, DM, and SM respectively denote the tangent bundle, closed unit disk bundle and sphere bundle of M. Given an fF_n-algebra A, let $E^A \longrightarrow M$ be the bundle whose fiber over a point $m \in M$ is $C(DM_m, SM_m; A)$. We will define a scanning map from $C(M; A)$ to a space of sections of E^A. Note that fibers of E^A are all models of $B^n A$.

Definition 3.11. Let W be an open n-manifold containing M containing N and let $E \longrightarrow W$ be a bundle with a preferred section s_0. Let $\Gamma^c_{(M,N)}(E)$ denote the space of sections of E over $W - N$ which agree with s_0 on $W - M$. For N empty, let $\Gamma^c_M(E) = \Gamma^c_{(M,N)}(E)$.

The scanning map will depend on a choice of metric on $M - N$ and function $\epsilon : M \longrightarrow \mathbb{R}_{>0}$. Require that the ball of radius $\epsilon(m)$ around $m \in M - N$ is inside the injectivity radius. For $m \in M - N$, the map $T_m M \longrightarrow M$ given by $v \longrightarrow exp_m(\epsilon(m)v)$ gives a diffeomorphism between $D_m M$ and $B_{\epsilon(m)}(m)$, the ball of radius $\epsilon(m)$ around the point m. Smooth maps of spaces induce maps of labeled configuration spaces with the maps acting on macroscopic locations of the particles and the derivatives of the maps acting on the labels. In this way, the inverse of the above function induces a map of configuration spaces $e_m : C(M, M - B_{\epsilon(m)}(m); A) \longrightarrow C(D_m M, S_m M; A)$. Using this map, we can construct the scanning map.

Definition 3.12. Let $s : C(M, N; A) \longrightarrow \Gamma^c_{(M,N)}(E^A)$ be the map defined as follows. For $m \in M - N$ and $\xi \in C(M, N; A)$, let $s(\xi)(m) \in E^A_m$ be the image of ξ under the maps

$$C(M, N; A) \xrightarrow{\pi} C(M, M - B_{\epsilon(m)}(m); A) \xrightarrow{e_m} C(D_m M, S_m M; A) = E^A_m.$$

Here the preferred section of E^A used in the compactly supported condition is the empty section s_0. That is, $s_0(m) \in E^A_m = C(D_m M, S_m M; A)$ is the empty configuration.

The following lemma is an important tool in Salvatore's proof of nonabelian Poincaré duality.

Lemma 3.13. *Let K be n-submanifold of M and assume that $\pi_0(N \cap K) \longrightarrow \pi_0(M)$ is onto. Then the projection map $\pi : C(M, N; A) \longrightarrow C(M, K \cup N; A)$ is a quasifibration with fiber $C(K, K \cap N; A)$.*

Theorem 3.14. *If $\pi_0(N) \longrightarrow \pi_0(M)$ is onto, then $s : C(M, N; A) \longrightarrow \Gamma^c_{(M,N)}(E^A)$ is a weak homotopy equivalence.*

Proof. The case when $N = \partial M$ is proven in [15]. Salvatore's proof is identical to Steps 1-4 of Proposition 2 of [2] after replacing labeled (without summing) configuration spaces with configuration spaces with summable labels. Steps 5-8 of Proposition 2 of [2] also apply to configuration spaces with summable labels to give the above theorem. The key tool in both Salvtore and Bödigheimer's proof is respectively Lemma 3.13 and the analogous statement for labeled configuration spaces without summable labels.

3.4. **Homology fibrations and scanning theorems.** In this subsection, we will consider the case that M is a connected open n-manifold which is the interior of a manifold with nonempty boundary. We will describe a stabilization procedure involving bringing points in from infinity and prove that the scanning map induces a homolology equivalence between the stabilization of $C(M; A)$ and $\Gamma^c_M(E^A)$.
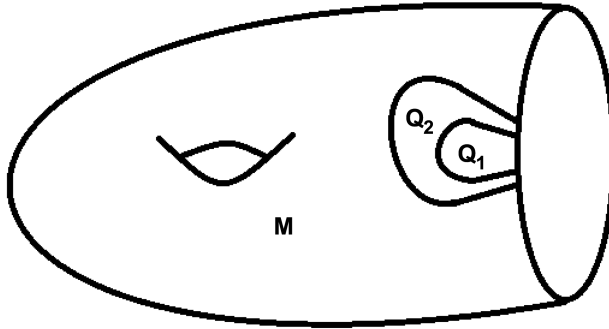
FIGURE 5. The subsets Q_1 and Q_2

In Andrade's model of topological chiral homology, the only point-set topological assumptions we made were that the D_n-algebras had nondegenerate basepoints. To prove nonabelian Poincaré duality after stabilizing for configuration spaces of particles with summable labels, we need to make additional point-set topological assumptions.

Definition 3.15. A space X is uniformly locally connected if there is a neighborhood $\mathcal{U}$ of the diagonal in $X \times X$ and a map $\lambda : \mathcal{U} \times [0,1] \longrightarrow X$ such that $\lambda(x,y,0) = x$, $\lambda(x,y,1) = y$ and $\lambda(x,x,t) = x$.

Definition 3.16. We call an fF_n-algebra A and a smooth n-manifold M nonpathological if A is uniformly locally connected and well based and both A and $C(M;A)$ are homotopy equivalent to CW complexes.

We will prove nonabelian Poincaré duality after stabilizing under the assumption that the pair (A, M) is nonpathological. Next we will describe the stabilization maps. For simplicity, we assume that M is connected. Since we assume that M is the interior of a manifold with nonempty boundary, we can find a submanifold with boundary Q_1, diffeomorphic to $[0,1) \times \mathbb{R}^{n-1}$ such that $M - Q_1$ is diffeomorphic to M by a diffeomorphism isotopic to the identity. For future use, we will fix another submanifold with boundary Q_2 containing Q_1 in its interior (see Figure 5) with the same properties as Q_1. Let $M_i = M - Q_i$. Fix a diffeomorphism $f : M_1 \longrightarrow M$ such that $f|_{M_2} = id$ and which is isotopic relative to M_2 to the standard inclusion $i : M_1 \longrightarrow M$. Also fix a point $q \in Q_1$.

Definition 3.17. For $a \in A$, let $t_a : C(M;A) \longrightarrow C(M;A)$ be defined as follows. The diffeomorphism f^{-1} induces a map $C(M;A) \longrightarrow C(M_1;A)$. Mapping a configuration of labeled points in $C(M_1;A)$ to the same configuration of points union $(q;a)$ gives a map $C(M_1;A) \longrightarrow C(M;A)$. Let t_a be the composition of these two maps.

Let T_a be the corresponding stabilization map for $\Gamma_M^c(E^A)$. The goal of this section is to prove the following theorem. It is nonabalian Poincaré duality after stabilizing for configuration spaces of particles with summable labels.

Theorem 3.18. *Let M be a connected parallelizable n-manifold with $n > 1$ which is the interior of a manifold with nonempty boundary. Assume that (A, M) is*

nonpathological. Let $\{a_i\}$ be a sequence of elements of A such that each connected component has an infinite number of terms of the sequence. The scanning map s induces a homology equivalence between $hocolim_{t_{a_i}} C(M; A)$ and $\Gamma_M^c(E^A)$.

By $hocolim_{t_{a_i}} C(M; A)$, we mean the mapping telescope of the following diagram:

$$C(M; A) \xrightarrow{t_{a_1}} C(M; A) \xrightarrow{t_{a_2}} C(M; A) \xrightarrow{t_{a_3}} C(M; A) \ldots.$$

All other similar homotopy colimits are understood to be taken over similar diagrams.

For M parallelizable, $\Gamma_M^c(E^A) \simeq Map^c(M, B^n A)$. In particular, Theorem 3.18 implies Theorem 1.1 for Salvatore's model of topological chiral homology. The rest of this section is devoted to proving Theorem 3.18. Consider the following commuting diagram:

$$
\begin{array}{ccc}
C(Q_2; A) & \xrightarrow{\ s\ } & \Gamma_{Q_2}^c(E^A) \\
\iota \downarrow & & \downarrow \\
C(M; A) & \xrightarrow{\ s\ } & \Gamma_M^c(E^A) \\
\pi \downarrow & & \downarrow \\
C(M, Q_2; A) & \xrightarrow{\ s\ } & \Gamma_{(M,Q_2)}^c(E^A)
\end{array}
$$

Note that the right-hand side is a fiber sequence. This follows from a generalization of the fact that the functor $Map(\cdot, X)$ takes cofiber sequences to fiber sequences (for example see the proof of Proposition 2 of [2]). By Theorem 3.14, the bottom row is a weak homotopy equivalence. When A is grouplike, Salvatore proved that the left-hand side is a quasi-fibration [15] and the top row is a weak equivalence. Thus, if A is grouplike, $s : C(M; A) \longrightarrow \Gamma_M^c(E^A)$ is a weak equivalence by the long exact sequence of homotopy groups and the five lemma. In the nongrouplike case, we will use a similar argument. First we will stabilize. Then we will show that the left-hand side is a homology fibration and that the top scanning map is a homology equivalence. Then we will use the spectral sequence comparison theorem. Since we assumed that f is the identity on M_2, the following diagram commutes:

$$
\begin{array}{ccc}
C(Q_2; A) & \xrightarrow{t_a} & C(Q_2; A) \\
\iota \downarrow & & \iota \downarrow \\
C(M; A) & \xrightarrow{t_a} & C(M; A) \\
\pi \downarrow & & \pi \downarrow \\
C(M, Q_2; A) & \xrightarrow{id} & C(M, Q_2; A)
\end{array}
$$

Thus, the following diagram commutes:

$$
\begin{array}{ccc}
hocolim_{t_{a_i}} C(Q_2; A) & \xrightarrow{\ s\ } & hocolim_{T_{a_i}} \Gamma_{Q_2}^c(E^A) \\
\iota \downarrow & & \downarrow \\
hocolim_{t_{a_i}} C(M; A) & \xrightarrow{\ s\ } & hocolim_{T_{a_i}} \Gamma_M^c(E^A) \\
\pi \downarrow & & \downarrow \\
C(M, Q_2; A) & \xrightarrow{\ s\ } & \Gamma_{(M,Q_2)}^c(E^A)
\end{array}
$$

(3.1)

To see that the top row of diagram (3.1) is a homology equivalence, we recall the following result from [15].

Theorem 3.19. *The scanning map $s : C(\mathbb{R}^n; A) \longrightarrow \Gamma_{\mathbb{R}^n}^c(E^A)$ is a group completion.*

Corollary 3.20. *For $n > 1$, the scanning map $s : C(Q_2; A) \longrightarrow \Gamma^c_{Q_2}(E^A)$ induces a homology equivalence, $s : \operatorname{hocolim}_{t_{a_i}} C(Q_2; A) \longrightarrow \operatorname{hocolim}_{T_{a_i}} \Gamma^c_{Q_2}(E^A)$.*

Proof. First note that a diffeomorpism $\mathbb{R}^n \longrightarrow Q_2$ induces homotopy equivalences $C(\mathbb{R}^n; A) \longrightarrow C(Q_2; A)$ and $\Gamma^c_{\mathbb{R}^n}(E^A) \longrightarrow \Gamma^c_{Q_2}(E^A)$. The space $C(\mathbb{R}^n; A)$ is a D_n-algebra. Thus for $n > 1$, it is homotopy equivalent to a homotopy commutative monoid. Therefore, it satisfies the hypothesis of the group completion theorem [13]. The monoid multiplication maps are homotopic to the stabilization maps $t_a : C(\mathbb{R}^n; A) \longrightarrow C(\mathbb{R}^n; A)$. By the group completion theorem, we can deduce that $s : \operatorname{hocolim}_{t_{a_i}} C(\mathbb{R}^n; A) \longrightarrow \operatorname{hocolim}_{t_{a_i}} \Gamma^c_{Q_2}(E^A)$ is a homology equivalence and the claim follows.

The left-hand side of diagram (3.1) is not always a quasi-fibration, but is a homology fibration when (A, M) is nonpathological.

Definition 3.21. A map $\pi : E \longrightarrow B$ is called a homology fibration if the inclusion of every fiber into the homotopy fiber is a homology equivalence.

This definition implies that the Serre spectral sequence can be used to study the homology of the total space of a homology fibration. In [12], McDuff states the following sufficient condition for a map being a homology fibration.

Proposition 3.22. *A map $r : Y \longrightarrow X$ is a homology fibration with fiber F if the following 5 conditions are satisfied. Let $X = \bigcup X_k$ with each X_i closed.*

(i) all spaces $X_k, X_k - X_{k-1}, r^{-1}(X_k), r^{-1}(X_k - X_{k-1})$ have the homotopy type of CW complexes;

(ii) each X_k is uniformly locally connected;

(iii) each $x \in X$ has a basis of contractible neighborhoods U such that the contraction of U lifts to a deformation retraction of $r^{-1}(U)$ into $r^{-1}(x)$;

(iv) each $r : r^{-1}(X_k - X_{k-1}) \longrightarrow X_k - X_{k-1}$ is a fibration with fiber F;

(v) for each k, there is an open subset U_k of X_k such that $X_{k-1} \subset U_k$, and there are homotopies $h_t : U_k \longrightarrow U_k$ and $H_t : r^{-1}(U_k) \longrightarrow r^{-1}(U_k)$ satisfying

(a) $h_0 = id, h_t(X_{k-1}) \subset X_{k-1}, h_1(U_k) \subset X_{k-1}$;

(b) $H_0 = id, r \circ H_t = h_t \circ r$;

(c) $H_1 : r^{-1}(x) \longrightarrow r^{-1}(h_1(x))$ induces an isomorphism on homology for all $x \in U_k$.

This is analogous to a theorem of Dold and Thom in [4] involving weak equivalences and quasi-fibrations.

Lemma 3.23. *If $dim M > 1$ and (A, M) is nonpathological, then the map*

$$\pi : \operatorname{hocolim}_{t_{a_i}} C(M; A) \longrightarrow C(M, Q_2; A)$$

is a homology fibration with fiber $\operatorname{hocolim}_{t_{a_i}} C(Q_2; A)$.

Proof. For simplicity of notation, we will assume that $\pi_0(A) = \mathbb{N}_0$. The general case is similar except with more indices. Since we also are assuming that M is connected, we have that $\pi_0(C(M; A)) = \pi_0(C(Q_2; A)) = \mathbb{N}_0$. The isomorphism $\pi_0(C(M; A)) \longrightarrow \pi_0(A)$ is given by multiplying together all of the elements of A labeling a particular configuration of points in M. Let A_k denote the k'th component of A. We will use Proposition 3.22 to show that π is a homology fibration by considering the following choice of filtration on $C(M, Q_2; A)$: let X_k

be the subset of points where the product of the labels of points macroscopically located in M_2 is in A_i with $i \leq k$.

Conditions (i) and (ii) are true since the pair (A, M) is nonpathological. To see that condition (iii) is satisfied, fix $x \in C(M, Q_2; A)$. Let $(m_1, \ldots, m_r)$ be the macroscopic locations of the points of x. Let $a_i \in A$ be the label of the configuration x at the point m_i and let $\{\mathcal{U}_i\}$ be a collection of disjoint open balls such that $m_i \in \mathcal{U}_i \subset M_2$. Let $\mathcal{A}_i \subset A$ be a contractible set deformation retracting onto the point a_i. Let $U \subset C(\bigcup \mathcal{U}_i; A) \subset C(M, Q_2; A)$ be the set of configurations of points such that the product of the labels of the points macroscopically located in $\mathcal{U}_i$ are in $\mathcal{A}_i$. Sets of this form give a basis of $C(M, Q_2; A)$ which satisfy the requirements of condition (iii). Condition (iv) is true since π is in fact a trivial fibration over each $X_k - X_{k-1}$ with fiber $hocolim_{t_{a_i}} C(Q_2; A)$.

We now check condition (v). Let M_3 be M minus a closed collar neighborhood of Q_2. Let $U_k \subset X_k$ be the subspace of configurations of points where the product of the labels of the points macroscopically located in M_3 is in A_i with $i < k$. Let g_t be a path of diffeomorphisms of M with $g_0 = id$, $g_t(Q_2) \subset Q_2$ for all t and $g_1(M - M_3) \subset Q_2$. The isotopy g_t induces homotopies $h_t : U_k \longrightarrow U_k$ and $H_t : \pi^{-1}(U_k) \longrightarrow \pi^{-1}(U_k)$ by applying g_t to each point in the configurations. To see that condition (a) is satisfied, first note that $h_0 = id$ since $g_0 = id$. We have that $h_t(X_{k-1}) \subset X_{k-1}$ since $g_t(Q_2) \subset Q_2$. Since $g_1(M - M_3) \subset Q_2$, $h_1(U_k) \subset X_{k-1}$. Condition (b) follows from the fact that $f_0 = id$, and H_t and h_t are both induced by f_t. The map on $hocolim_{t_{a_i}} C(Q_2; A)$ from condition (c) is homotopy equivalent to one induced by a product of stabilization maps t_{a_i}. The stabilization maps induce homology equivalences on $hocolim_{t_{a_i}} C(Q_2; A)$ provided that $n > 1$. When $n = 1$, there are two potentially nonhomotopic stabilization maps $C(Q_2; A) \longrightarrow C(Q_2; A)$ corresponding to adding points from the left or right. Thus π is a homology fibration.

Before we can prove Theorem 3.18, we need the following lemma.

Lemma 3.24. *The action of $\pi_1(C(M, Q_2; A))$ on $H_*(hocolim_{t_{a_i}} C(Q_2; A))$ induced by the map π is trivial for $\dim M > 1$. Also, the action of $\pi_1(\Gamma^c_{(M,Q_2)}(E^A))$ on $H_*(hocolim_{T_{a_i}} \Gamma^c_{Q_2}(E^A))$ is trivial.*

Proof. The way $\pi_1(C(M, Q_2; A))$ acts on $H_*(hocolim_{t_{a_i}} C(Q_2; A))$ is via conjugation by the maps on homology induced by the stabilization maps. Since $C(Q_2; A)$ is homotopy commutative for $n > 1$, conjugation is trivial. The same argument applies to the action of $\pi_1(\Gamma^c_{(M,Q_2)}(E^A))$ on $H_*(hocolim_{T_{a_i}} \Gamma^c_{Q_2}(E^A))$. ∎

The action could be nontrivial if we considered Q_2 whose interior is not of the form $\mathbb{R}^2 \times N$ with N a connected $(n-2)$-manifold. We can now prove Theorem 3.18, nonabelian Poincaré duality after stabilizing for Salvatore's configuration space of particles with summable labels.

Proof of Theorem 3.18. See diagram (3.1). The left row is a homology fibration and the right row is a quasi-fibration and hence also a homology fibration. The bottom row is a weak equivalence and the top row is a homology equivalence. The fundamental groups of the bases act trivially on the fibers. Consider the Serre spectral sequences for the two homology fibrations. The scanning map induces an isomorphism between the E_2 pages and hence an isomorphism between the E_∞

pages. Thus $s : hocolim_{t_{a_i}} C(M; A) \longrightarrow hocolim_{T_{a_i}} \Gamma_M^c(E^A)$ is a homology equivalence. Since each T_{a_i} is a homotopy equivalence, $hocolim_{T_{a_i}} \Gamma_M^c(E^A)$ is homotopy equivalent to $\Gamma_M^c(E^A)$. This completes the proof.

4. A CONJECTURE ON HOMOLOGICAL STABILITY

There are several natural questions raised by Theorem 1.1. In this section, we highlight one of them and make a conjecture regarding the effects in homology of the stabilization maps $t_{a_i} : \int_M A \longrightarrow \int_M A$. For simplicity of notation, assume that M is connected and open and $\pi_0(A) = \mathbb{N}_0$.

Definition 4.1. Let A be an E_n-algebra with $\pi_0(A) = \mathbb{N}_0$. Fix $b \in A_1$ and let $m_b : A_k \longrightarrow A_{k+1}$ be a multiplication with b map. We say that A has homological stability if there is a function $r : \mathbb{N}_0 \longrightarrow \mathbb{N}_0$ tending to ∞ such that $m_{b*} : H_i(A_k) \longrightarrow H_i(A_{k+1})$ is an isomorphism for $i \le r(k)$.

Examples of such E_n-algebras include: labeled (without summable labels) configuration spaces [17] and [14], rational functions [17] and [3], as well as the union of the classifying spaces of the mapping class groups of once punctured surfaces [7]. Let $(\int_M A)_k$ denote the k'th component of $\int_M A$.

Conjecture 4.2. *Let M be an open connected parallelizable n-manifold and let A be an E_n-algebra with $\pi_0(A) = \mathbb{N}_0$. Fix $b \in A_1$ and let $t_b : (\int_M A)_k \longrightarrow (\int_M A)_{k+1}$ be a stabilization map described in previous sections. If A has homological stability, we conjecture that there is a range of dimensions tending to infinity below in which t_b induces an isomorphism on homology. Moreover, this range should depend only on the homological stability range of A.*

This conjecture is trivially true when $M = \mathbb{R}^n$. It is also true when $A = C(\mathbb{R}^n)X$ for a based space X. If true, combining this conjecture with Theorem 1.1 would allow one to conclude that the scanning map $s : \int_M A \longrightarrow Map^c(M, B^n A)$ is a homology equivalence though a range of dimensions.

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