TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 367, Number 3, March 2015, Pages 2137–2157
S 0002-9947(2014)06208-9
Article electronically published on August 12, 2014

RESONANCE OF AUTOMORPHIC FORMS FOR $GL(3)$

XIUMIN REN AND YANGBO YE

ABSTRACT. Let f be a Maass form for $SL_3(\mathbb{Z})$ with Fourier coefficients $A_f(m,n)$. A smoothly weighted sum of $A_f(m,n)$ against an exponential function $e(\alpha n^\beta)$ of fractional power n^β for $X \le n \le 2X$ is proved to have a main term of size $X^{2/3}$ when $\beta = 1/3$ and α is close to $3\ell^{1/3}$ for some integer $\ell \ne 0$. The sum becomes rapidly decreasing if $\beta < 1/3$. If such a sum is not smoothly weighted, the main term can only be detected under a conjectured bound toward the Ramanujan conjecture. The existence of such a main term manifests the vibration and resonance behavior of individual automorphic forms f for $GL(3)$. Applications of these results include a new modularity test on whether a two dimensional array $a(m,n)$ comes from Fourier coefficients $A_f(m,n)$ of a Maass form f for $SL_3(\mathbb{Z})$. Techniques used in the proof include a Voronoi summation formula, its asymptotic expansion, and the weighted stationary phase.

1. Introduction

Resonance is an important phenomenon which may occur between two vibration systems. Fixing one vibration system, one may change the second (testing) system to detect resonance frequencies of the first, and hence gain spectral information on the oscillation nature of the first vibration system. A classic example of this is the Fourier series expansion of a periodic function which is actually the $GL(1)$ theory (cf. Ren-Ye [20]). For the $GL(2)$ theory, Iwaniec-Luo-Sarnak [9] and Ren-Ye [20] proved that a cusp form f for $SL_2(\mathbb{Z})$ is resonant against an exponential function $e(\alpha n^\beta)$ if and only if $\beta = 1/2$ and α is close to $\pm 2\sqrt{q}$ for some positive integer q.

In this paper, we will study resonance behavior of certain Maass forms for $GL(3)$. Let f be a Maass form of type $\nu = (\nu_1, \nu_2)$ for $SL_3(\mathbb{Z})$. Then

$$\mu_f(1) = \nu_1 + 2\nu_2 - 1,\ \mu_f(2) = \nu_1 - \nu_2,\ \mu_f(3) = 1 - 2\nu_1 - \nu_2$$

are the Langlands' parameters for f which has a Fourier Whittaker expansion (cf. Goldfeld [4])

$$f(z) = \sum_{\gamma \in U_2(\mathbb{Z})\backslash SL_2(\mathbb{Z})} \sum_{m_1 \ge 1} \sum_{m_2 \ne 0} \frac{A_f(m_1,m_2)}{m_1|m_2|} W_J\Big(M\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} z, \nu, \psi_{1,1}\Big).$$

Here $U_2 = \Big\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\Big\}$, $W_J(z,\nu,\psi_{1,1})$ is the Jacquet-Whittaker function,

$$M = \operatorname{diag}(m_1|m_2|, m_1, 1),$$

Received by the editors October 4, 2012 and, in revised form, May 6, 2013 and June 5, 2013.
2010 *Mathematics Subject Classification.* Primary 11L07, 11F30.
Key words and phrases. Automorphic forms for $GL(3)$, resonance, Voronoi summation formula, modularity testing.

and $A_f(m_1, m_2)$ are Fourier coefficients of f. Note that $W_J(z, \nu, \psi_{1,1})$ represents an exponential decay in y_1 and y_2 for

$$z = \begin{pmatrix} 1 & x_{12} & x_{13} \\ & 1 & x_{23} \\ & & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix}.$$

It is known that (cf. Kim-Sarnak [12] and Sarnak [23] (21)) for automorphic cusp form f on GL_n over $\mathbb{Q}$ with $n \le 4$, the Fourier coefficient of f at p is bounded by $p^{1/2-1/(1+n(n+1)/2)}$. Thus in the case on hand

$$A_f(m,n) \ll |mn|^{5/14+\varepsilon}. \tag{1.1}$$

The Rankin-Selberg theory (cf. [6]) reveals that

$$\sum_{mn^2 \le N} |A_f(m,n)|^2 = O_f(N). \tag{1.2}$$

Since $A_f(m_2, m_1) = A_{\tilde{f}}(m_1, m_2)$, where $\tilde{f}$ is the contragredient form of f, there also holds

$$\sum_{m^2 n \le N} |A_f(m,n)|^2 = O_f(N). \tag{1.3}$$

These estimates lead to

$$\sum_{m \le N} \frac{|A_f(m,1)|}{m} \ll \log N, \qquad \sum_{n \le N} \frac{|A_f(1,n)|}{n} \ll \log N. \tag{1.4}$$

In this paper, we will first investigate the smoothly weighted exponential sum twisted with $A_f(m,n)$:

$$\sum_{n>0} A_f(m,n) \phi\Big(\frac{n}{X}\Big) e(\alpha n^\beta),$$

where ϕ is a C^∞ function supported on $[1,2]$ with bounded derivatives, $\beta > 0$ is a fixed parameter, $\alpha > 0$, $X > 1$ are the main parameters and $1 \le m < X$.

As is known, a Maass form f can be written as a finite sum of Hecke eigenforms, i.e.

$$f = \sum_{j=1}^{e_f} f_j, \tag{1.5}$$

where f_j, $j = 1,\ 2, \ldots, e_f$, are Hecke eigenforms. Our results are the following.

Theorem 1. *Suppose* $\max\{2^{\max\{\beta,1/3\}}\alpha\beta, 1\} m^{1/3} \le X^{1/3-\beta}$. *Then the estimate*

$$\sum_{n>0} A_f(m,n) e(\alpha n^\beta) \phi\Big(\frac{n}{X}\Big) \ll X^{-M} \tag{1.6}$$

holds for any $M > 0$*, where the implied constant may depend on* M, β *and* f *only. In particular for* $0 < \beta < 1/3$*,* (1.6) *holds for* $X > \max\{A, 1\}$ *and* $1 \le m \le (X/\max\{A,1\})^{(1-3\beta)}$ *where* $A = (2(\alpha\beta)^3)^{1/(1-3\beta)}$ *and the implied constant may depend only on* M *and* f.

Theorem 2. *For* $\beta \ne 1/3$*, the estimate*

$$\sum_{n>0} A_f(m,n) e(\alpha n^\beta) \phi\Big(\frac{n}{X}\Big) \ll m^{5/14+\varepsilon}\big(1 + (\alpha X^\beta)^{3/2} \log X\big)$$

holds for arbitrary $\varepsilon > 0$, where the implied constant may depend on f, β and ε only.

Theorem 3. *For arbitrary $\varepsilon > 0$, we have*

$$\sum_{n>0} A_f(m,n)e(\alpha\sqrt{n})\phi\Big(\frac{n}{X}\Big) \ll_{f,\phi,\varepsilon} m^{5/14+\varepsilon}(1+\alpha^{3/4+\varepsilon}X^{5/8+\varepsilon}+\alpha X^{1/2}\log X).$$

We remark that when α^2 is a rational number, the bound in Theorem 3 can be further reduced to $O(\alpha X^{1/2}\log X)$. See remarks after (3.18) below.

Theorem 4. *Let f be a Maass form as given in* (1.5). *Then we have*

$$\begin{aligned} &\sum_{n>0} A_f(m,n)e(\alpha n^{1/3})\phi\Big(\frac{n}{X}\Big) \\ = \;& -i3\sqrt{3}\alpha^{-1}X^{2/3}\sum_{d|m}\mathcal{E}_\alpha(d)\frac{\mu(d)}{d}J_\phi\Big(\frac{n_\alpha(d)}{d}\Big)\sum_{j=1}^{e_f}A_{f_j}\Big(\frac{m}{d},1\Big)A_{f_j}(1,n_\alpha(d)) \\ + \;& O_{f,\phi,\varepsilon}\Big(m^{5/14+\varepsilon}(1+\alpha^{29/14+\varepsilon}X^{1/3+\varepsilon})\Big), \end{aligned}$$

where

$$J_\phi(x) = \int_1^{2^{1/3}} u\phi(u^3)\, e\Big((\alpha-3x^{1/3})X^{1/3}u\Big)\,du, \tag{1.7}$$

and $\mathcal{E}_\alpha(d) = 1$ or 0 according to whether or not there is an integer $n_\alpha(d) \geq 1$ such that

$$|n_\alpha(d) - \alpha^3 d/27| < \min\{\alpha^2 d^{1-\varepsilon}X^{\varepsilon-1/3}, 1/10\}.$$

In particular, for any positive integer ℓ,

$$\begin{aligned} &\sum_{n>0} A_f(m,n)e(3(\ell n)^{1/3})\phi\Big(\frac{n}{X}\Big) \\ = \;& -ic(\phi)\sqrt{3}\ell^{-1/3}X^{2/3}\sum_{d|m}\frac{\mu(d)}{d}\sum_{j=1}^{e_f}A_{f_j}\Big(\frac{m}{d},1\Big)A_{f_j}(1,d\ell) \\ + \;& O_{f,\phi,\varepsilon}(m^{5/14+\varepsilon}\ell^{29/42+\varepsilon}X^{1/3+\varepsilon}), \end{aligned} \tag{1.8}$$

where $c(\phi) = \int_1^{2^{1/3}} u\phi(u^3)du$.

We remark that (1.8) generalizes Iwaniec-Luo-Sarnak's result ([9]) on $GL(2)$ to $GL(3)$.

In this paper we will also consider the sharp-cut sum

$$\sum_{X<n\leq 2X} A_f(m,n)e(\alpha n^\beta).$$

In this direction, Miller considered the linear case ($\beta = 1$) and proved that (cf. Miller [17]), uniformly for $\alpha \in \mathbb{R}$,

$$\sum_{1\leq m\leq T} A_f(m,n)e(\alpha m) \ll_{f,n,\varepsilon} T^{3/4+\varepsilon}. \tag{1.9}$$

In this paper, we will prove the following result which can be compared with the above estimate.

Theorem 5. (i) *For $\beta \neq 1/3$, we have*

$$\sum_{X<n\leq 2X} A_f(m,n)e(\alpha n^\beta) \ll_{f,\beta,\varepsilon} m^{5/14+\varepsilon}(X^{2/3}+(\alpha X^\beta)^{3/2})\log X.$$

(ii) *For $\beta = 1/3$, we have*

$$\sum_{X<n\leq 2X} A_f(m,n)e(\alpha n^\beta) \ll_{f,\varepsilon} m^{5/14+\varepsilon}(1+\alpha^{1/14+\varepsilon})(X^{2/3}\log X+\alpha^2 X^{1/3+\varepsilon}). \tag{1.10}$$

Note that (1.10) just fails to detect a main term as in Theorem 4. To get a main term and a smaller error term we need to assume a bound toward the Ramanujan conjecture

$$|A_f(m,n)| \ll |mn|^{\theta+\varepsilon} \tag{1.11}$$

for $\theta < 1/3$. The Ramanujan conjecture predicts that $\theta = 0$.

Theorem 6. *Let f be a Maass form as in* (1.5). *Assume the bound* (1.11) *toward the Ramanujan conjecture with $\theta < 1/3$. Then for $\beta \neq 1/3$ we have*

$$\sum_{X<n\leq 2X} A_f(m,n)e(\alpha n^\beta) \ll_{f,\beta,\varepsilon} m^{\theta+\varepsilon}\left(X^{(1+\theta)/2+\varepsilon}+(\alpha X^\beta)^{3/2}\log X\right).$$

For $\beta = 1/3$ we have

$$\begin{aligned}
&\sum_{X<n\leq 2X} A_f(m,n)e(\alpha n^\beta)\\
=\ & -i3\sqrt{3}X^{2/3}\sum_{d|m}\frac{\mathcal{E}_\alpha(d)\mu(d)}{d}J_1\Big(\frac{n_\alpha(d)}{d}\Big)\sum_{j=1}^{e_f}A_{f_j}\Big(\frac{m}{d},1\Big)A_{f_j}(1,n_\alpha(d))\\
+\ & O_{f,\varepsilon}\Big(m^{\theta+\varepsilon}(X^{(1+\theta)/2+\varepsilon}+\alpha^{1+3\theta+\varepsilon}X^{1/3})\Big),
\end{aligned}$$

where $J_1(x)$ is defined in (1.7) *with $\phi = 1$, and $\mathcal{E}_\alpha(d) = 1$ or* 0 *according to whether or not there is an integer $n_\alpha(d) \geq 1$ such that*

$$\Big|n_\alpha(d)-\alpha^3 d/27\Big| < \min\Big\{\alpha^2 dX^{-(1+3\theta)/6-\varepsilon}, 1/10\Big\}.$$

In particular for any positive integer ℓ, one has

$$\begin{aligned}
&\sum_{X<n\leq 2X} A_f(m,n)e(3(\ell n)^{1/3})\\
=\ & -i\frac{\sqrt{3}}{2}(2^{2/3}-1)\ell^{-1/3}X^{2/3}\sum_{d|m}\frac{\mu(d)}{d}\sum_{j=1}^{e_f}A_{f_j}\Big(\frac{m}{d},1\Big)A_{f_j}(1,d\ell)\\
+\ & O_{f,\varepsilon}\Big(m^{\theta+\varepsilon}(X^{(1+\theta)/2+\varepsilon}+\ell^{1/3+\theta+\varepsilon}X^{1/3})\Big).
\end{aligned} \tag{1.12}$$

Note that when $\alpha \to 0$, (1.6) reduces to

$$\sum_{n>0} A_f(m,n)\phi\Big(\frac{n}{X}\Big) \ll_{f,M} X^{-M} \tag{1.13}$$

for any $M > 0$ and $m \ll X^{1-\varepsilon}$. This bound (1.13) was proved by Booker in [2] and [3]. A similar bound was proved by the authors in [21] for $\beta = 1$ and α being a rational number or a transcendental number with its approximation index > 3.

The rapidly decreasing bound (1.13) played a crucial role in modularity testing in [2] and [3]. In Bian [1] a $GL(3)$ modularity test was carried out on a sequence a_n to see if $a_n = A_f(1,n)$ for some Maass form f for $SL_3(\mathbb{Z})$. Bian's test is based on the functional equation of the L-function attached to f twisted by a primitive Dirichlet character χ and he verified this functional equation numerically by replacing the degree-three L-function by a smooth sum

$$\sum_{n>0} a_n \chi(n) \phi\Big(\frac{n}{X}\Big). \tag{1.14}$$

This motivated an important application of our results. Instead of testing the functional equation for (1.14) twisted by χ, one may use the sum twisted by $e(\alpha n^{\beta})$ and test numerically (1.6), (1.8) and/or (1.12) with $A_f(m,n)$ replaced by a two dimensional sequence $a(m,n)$ which satisfies the usual multiplicativity as in (4.1) and [4] p. 168. If (1.6) decays rapidly but (1.8) or (1.12) remains of order $X^{2/3}$, these $a(m,n)$ are likely to be Fourier coefficients $A_f(m,n)$ of a Maass form f for $SL_3(\mathbb{Z})$.

This new modularity test scheme has an added advantage as the main term on the right side of (1.8) and (1.12) can be computed and compared with the sums on the left side. These sums will not decay rapidly in the case of (1.8) or (1.12), as opposite to

$$\sum_{n>0} a_n \chi(n) \phi\Big(\frac{n}{X}\Big) \ll_{f,M} X^{-M}.$$

One may also use the main terms to test whether the $SL_3(\mathbb{Z})$ form f is self-dual and comes from a symmetric-square lifting.

Xiaoqing Li and Matt Young proved in [16] bounds for similar sums for linear phase ($\beta = 1$), uniformly on f when the $SL_3(\mathbb{Z})$ form f is a symmetric-square lift of an $SL_2(\mathbb{Z})$ cusp form. Xiannan Li [13] extended their results to Maass forms for $SL_3(\mathbb{Z})$ which are not necessarily a symmetric-square lift from $SL_2(\mathbb{Z})$. In [10], [19], Kaczorowski and Perelli considered the following twists:

$$F_d(s,\alpha) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} e(-\alpha n^{1/d}), \quad \mathrm{Re}\ s = \sigma > 1, \quad \alpha > 0,$$

for $F \in S_d^{\#}$ which is denoted as the extended Selberg class of degree d, and proved analytic properties of $F_d(s,\alpha)$ (see [10] for details).

The main techniques we will use to prove the theorems include a Voronoi summation formula for $SL_3(\mathbb{Z})$ ([14], [18], [5], [6]), its asymptotic expansion ([8], [14], [22]), and a weighted stationary phase argument ([7], [15], [21]). These techniques when applied to sums with exponential functions of fractional powers played a crucial role in a recent proof of a subconvexity bound for automorphic L-functions for $SL_3(\mathbb{Z})$ ([14], [15]). These same techniques have been used for $SL_2(\mathbb{Z})$ in [20] in which we proved that resonance happens between a cusp form for $SL_2(\mathbb{Z})$ and exponential sums $e(\alpha n^{1/2})$. We expect that resonance occurs between a cusp form for $SL_m(\mathbb{Z})$ and exponential sums $e(\alpha n^{1/m})$. It is also an interesting question to generalize the resonance theory to automorphic representations of $GL(m)$ over a number field.

One can prove similar results for $\alpha < 0$, and the argument follows exactly the same line by replacing α by $-|\alpha|$. It is interesting to generalize the above results to the case when α and the $SL_3(\mathbb{Z})$ Maass form f both vary as well.

We will give some preliminaries in section 2 and prove Theorems 1-6 in sections 3-6. By (1.5) we only need to prove the theorems for a Hecke eigenform f, and then assertions for a general Maass form f follows easily.

In the following, $\varepsilon > 0$ is a small constant which may have different value in each occurence.

2. Voronoi formula for $SL_3(\mathbb{Z})$

Following [5], let $\psi(x) \in C_c^\infty(0,\infty)$, and set

$$\tilde{\psi}(s) = \int_0^\infty \psi(x) x^s \frac{dx}{x}.$$

Define

$$\Psi_k(x) = \int_{\mathrm{Re}\ s=\sigma} (\pi^3 x)^{-s} \prod_{j=1}^3 \frac{\Gamma\Big(\frac{1+s+2k+\mu_f(j)}{2}\Big)}{\Gamma\Big(-\frac{s+\mu_f(j)}{2}\Big)} \tilde{\psi}(-s-k) ds,$$

where

$$\sigma > \max(-1 - \mathrm{Re}\ \mu_f(1) - 2k, -1 - \mathrm{Re}\ \mu_f(2) - 2k, -1 - \mathrm{Re}\ \mu_f(3) - 2k).$$

Write

$$\Psi_{0,1}^0(x) = \Psi_0(x) + \frac{1}{i\pi^3 x}\Psi_1(x), \qquad \Psi_{0,1}^1(x) = \Psi_0(x) - \frac{1}{i\pi^3 x}\Psi_1(x). \tag{2.1}$$

Then the Voronoi type formula as first proved by Miller-Schmid [18] and then used in [14] is stated as follows.

Lemma 2.1 ([18], [14])**.** *Let $A_f(m,n)$ be the Fourier coefficients of the Maass cusp form f for $SL_3(\mathbb{Z})$. Suppose that $\psi \in C_c^\infty(0,\infty)$. Let $c,\ d$ be integers such that $c \geq 1$, $(c,d) = 1$ and $d\bar{d} \equiv 1(\mathrm{mod} c)$. Then*

$$\begin{aligned} &\sum_{n>0} A_f(m,n) e\Big(\frac{n\bar{d}}{c}\Big)\psi(n) \\ &= \frac{c\pi^{-5/2}}{4i} \sum_{n_1|cm} \sum_{n_2>0} \frac{A_f(n_2,n_1)}{n_1 n_2} S\left(md, n_2; mcn_1^{-1}\right) \Psi_{0,1}^0\Big(\frac{n_2 n_1^2}{c^3 m}\Big) \\ &+ \frac{c\pi^{-5/2}}{4i} \sum_{n_1|cm} \sum_{n_2>0} \frac{A_f(n_1,n_2)}{n_1 n_2} S\left(md, -n_2; mcn_1^{-1}\right) \Psi_{0,1}^1\Big(\frac{n_2 n_1^2}{c^3 m}\Big). \end{aligned} \tag{2.2}$$

Here $S(a,b;r)$ is the classical Kloosterman sum.

The asymptotic behaviors of $\Psi_0(x)$ and $\Psi_1(x)$ are included in the following lemma.

Lemma 2.2 ([14], [22])**.** *Suppose that ψ is a fixed smooth function of compact support on $[X,2X]$ where $X > 0$. Then for $x > 0$, $xX \gg 1$, $r \geq 2$ and $k = 0,\ 1$, we have*

$$\begin{aligned} \Psi_k(x) = (\pi^3 x)^{k+1} \sum_{j=1}^r \int_0^\infty \psi(y) \Big(a_k(j) e(3(xy)^{1/3}) + b_k(j) e(-3(xy)^{1/3})\Big) \frac{dy}{(\pi^3 xy)^{j/3}} \\ + O\Big((\pi^3 x)^k (\pi^3 xX)^{-r/3+1/2-k+\varepsilon}\Big), \end{aligned}$$

where the implied constant depends at most on r, $a_k(j)$ and $b_k(j)$ are constants with

$$a_0(1) = -\frac{2\sqrt{3\pi}}{3}, \quad b_0(1) = \frac{2\sqrt{3\pi}}{3}, \quad a_1(1) = b_1(1) = -\frac{2\sqrt{3\pi}}{3}i. \tag{2.3}$$

3. Proof of Theorems 1-4 for $m = 1$

In this section and the following sections we will assume that f is a Hecke eigenform for $SL_3(\mathbb{Z})$ and $A_f(m,n)$ its Fourier coefficients. Consider sums of the form

$$\sum_{n>0} A_f(1,n)e(\alpha n^\beta)\phi\Big(\frac{n}{X}\Big).$$

Let $\psi(x) = e(\alpha x^\beta)\phi(x/X)$ and take $m = c = d = 1$ in Lemma 2.1. Applying (2.1) and noting that $S(1, \pm n_2; 1) = 1$, we obtain

$$\sum_{n>0} A_f(1,n)e(\alpha n^\beta)\phi\left(\frac{n}{X}\right) = \sum_{n>0} A_f(1,n)\psi(n) = \mathcal{B}_0(X) + \mathcal{B}_1(X), \tag{3.1}$$

where for $k = 0,\ 1$

$$\mathcal{B}_k(X) \quad = \quad \frac{\pi^{-5/2}}{4i} \sum_{n>0} \frac{A_f(n,1) + (-1)^k A_f(1,n)}{n} \left(i\pi^3 n\right)^{-k} \Psi_k\left(n\right).$$

By Lemma 2.2 and making change of variable $y = Xu^3$ we get

$$\begin{aligned} \mathcal{B}_k(X) \quad &= \quad \frac{3\pi^{-5/2}}{4i^{k+1}} \sum_{n\geq 1} \frac{A_f(n,1) + (-1)^k A_f(1,n)}{n} \\ &\times \quad \sum_{j=1}^{r} \left(\pi^3 nX\right)^{1-j/3} \left(a_k(j) I_j^+\left(n\right) + b_k(j) I_j^-\left(n\right)\right) + E_r^{(k)}(X) \end{aligned} \tag{3.2}$$

with

$$I_j^{\pm}(x) = \int_1^{2^{1/3}} u^{2-j}\phi\left(u^3\right) e\left(\alpha X^\beta u^{3\beta} \pm 3\left(xX\right)^{1/3} u\right) du \tag{3.3}$$

and, for any integer $r \geq 2$,

$$\begin{aligned} E_r^{(k)}(X) \quad &\ll \quad X^{-r/3+1/2-k+\varepsilon} \sum_{n\geq 1} \frac{|A_f(n,1)| + |A_f(1,n)|}{n^{r/3+1/2+k-\varepsilon}} \\ &\ll_r \quad X^{-r/3+1/2-k+\varepsilon}. \end{aligned} \tag{3.4}$$

We next show that the contribution of $I_j^+(n)$ to $\mathcal{B}_k(X)$ is small. Write

$$F_{\pm}(u) = F_{\pm}(u,x) = \alpha X^\beta u^{3\beta} \pm 3\left(xX\right)^{1/3} u.$$

One has

$$F_{\pm}'(u) = 3(\alpha\beta X^\beta u^{3\beta-1} \pm (xX)^{1/3})$$

and

$$F_{\pm}^{(t)}(u) = 3\alpha\beta(3\beta-1)\cdots(3\beta-t+1)X^\beta u^{3\beta-t} \quad \text{for} \quad t \geq 2.$$

If $|F_{\pm}'(u)| \geq Q > 0$ for $u \in [1, 2^{1/3}]$, then by repeated integrating by part one obtains

$$I_j^{\pm}(x) = \int_1^{2^{1/3}} g_s(u) e\left(F_{\pm}(u)\right) du,$$

where

$$g_0(u) = u^{2-j}\phi\left(u^3\right), \quad g_t(u) = \left(\frac{g_{t-1}(u)}{2\pi i F'_\pm(u)}\right)', \quad t \geq 1.$$

Since $\phi^{(t)}(u) \ll 1$, one can see that $g_0(u)$ and all its derivatives are bounded on $[1, 2^{1/3}]$. Moreover $|F_\pm^{(t)}(u)| \ll |3\beta - 1| \cdots |3\beta - t + 1| \alpha\beta X^\beta$ for $t \geq 2$. Therefore

$$I_j^\pm(x) \ll \sum_{0 \leq t \leq s} \frac{(\alpha\beta X^\beta)^t}{Q^{s+t}}, \tag{3.5}$$

where the implied constant may depend on β, ϕ and s.

For $F'_+(u)$ one can take $Q = (xX)^{1/3} + \alpha\beta X^\beta$ and obtain by (3.5) that, for any $x > 0$,

$$I_j^+(x) \ll (xX)^{-s/3}.$$

Therefore the contribution of $I_j^+(n)$ to $\mathcal{B}_k(X)$ in (3.2) is, for any $s \geq 3$,

$$\ll \sum_{j=1}^r X^{1-(j+s)/3} \sum_{n \geq 1} \frac{|A_f(n,1)| + |A_f(1,n)|}{n^{(s+j)/3}} \ll X^{(2-s)/3}.$$

To estimate the contribution of $I_j^-(n)$ to $\mathcal{B}_k(X)$ we write

$$a^* = (\alpha\beta)^3 X^{3\beta-1} \min\{2^{3\beta-1}, 1\}, \tag{3.6}$$

$$b^* = (\alpha\beta)^3 X^{3\beta-1} \max\{2^{3\beta-1}, 1\}. \tag{3.7}$$

For $n \geq 2b^*$, one has $|F'_-(u,n)| \gg Q = (nX)^{1/3} \gg (b^*X)^{1/3} \gg \alpha\beta X^\beta$, and hence by (3.5),

$$I_j^-(n) \ll (nX)^{-s/3}.$$

Therefore the contribution of $I_j^-(n)$ with $n \geq 2b^*$ to $\mathcal{B}_k(X)$ is $O(X^{(2-s)/3})$. Taking $s = r + 3k$ $(r \geq 2)$ we get

$$\begin{aligned} \mathcal{B}_k(X) &= \frac{3\pi^{-5/2}}{4i^{k+1}} \sum_{j=1}^r \left(\pi^3 X\right)^{1-j/3} b_k(j) \sum_{n < 2b^*} \frac{A_f(n,1) + (-1)^k A_f(1,n)}{n^{j/3}} I_j^-(n) \\ &+ O(X^{-r/3+1/2-k+\varepsilon}), \end{aligned} \tag{3.8}$$

where the implied constant may depend on r, β and ϕ only.

Proof of Theorem 1 *for* $m = 1$. Note that the main term in (3.8) disappears when $2b^* \leq 1$ which is just

$$2^{\max\{\beta, 1/3\}} \alpha\beta \leq X^{1/3-\beta}. \tag{3.9}$$

In this case one obtains

$$\mathcal{B}_k(X) \ll_{r,\phi,\beta} X^{-r/3+1/2+\varepsilon}.$$

Back to (3.1) we get, for any integer $r \geq 3$,

$$\sum_{n>0} A_f(1,n) e(\alpha n^\beta) \phi\left(\frac{n}{X}\right) \ll_{r,\phi,\beta} X^{-r/3+1/2+\varepsilon}. \tag{3.10}$$

In particular when $0 < \beta < 1/3$, (3.9) is satisfied when $X > A = (2(\alpha\beta)^3)^{1/(1-3\beta)}$, and the implied constant in (3.10) depends on ϕ and r only in this situation. This proves Theorem 1 for $m = 1$ by choosing $r > 3M + 2$. □

Proof of Theorem 2 *for* $m = 1$. Without loss of generality, one can assume that $2b^* > 1$. Using trivial estimate $I_j^-(n) \ll 1$ and (1.4) one can derive that the contribution of $I_j^-(n)$ for $j \geq 2$ in (3.8) is

$$\ll (b^* X)^{1/3} \sum_{n<2b^*} \frac{|A_f(n,1)| + |A_f(1,n)|}{n} \ll_\beta \alpha X^\beta \log(2b^*).$$

In view of the definition of $b_k(1)$ in (2.3), the contribution of $j = 1$ in $\mathcal{B}_0(X)+\mathcal{B}_1(X)$ is

$$-\sqrt{3} i X^{2/3} \sum_{n<2b^*} \frac{A_f(1,n)}{n^{1/3}} I_1^-(n).$$

Thus we obtain

$$\begin{aligned} &\sum_{n>0} A_f(1,n) e(\alpha n^\beta) \phi\left(\frac{n}{X}\right) \\ &= -\sqrt{3} i X^{2/3} \sum_{n<2b^*} \frac{A_f(1,n)}{n^{1/3}} I_1^-(n) + O_\beta(1 + \alpha X^\beta \log(2b^*)). \end{aligned} \tag{3.11}$$

If $\beta \neq 1/3$, then

$$F_-''(u) = 3\alpha\beta(3\beta - 1) X^\beta u^{3\beta-2} \gg_\beta \alpha X^\beta.$$

By the second derivative test (cf. [24]) one obtains $I_1^-(n) \ll_\beta (\alpha X^\beta)^{-1/2}$. Hence the main term in (3.11) is

$$\ll (\alpha X^\beta)^{-1/2} (b^* X)^{2/3} \sum_{1 \leq n \leq 2b^*} \frac{|A_f(1,n)|}{n} \ll_\beta (\alpha X^\beta)^{3/2} \log(2b^*).$$

This proves

$$\sum_{n>0} A_f(1,n) e(\alpha n^\beta) \phi\left(\frac{n}{X}\right) \ll_\beta 1 + (\alpha X^\beta)^{3/2} \log X, \tag{3.12}$$

and hence finishes the proof of Theorem 2 for $m = 1$. □

Next we will show that when $\beta = 1/2$, the bound in (3.12) can be improved further.

Proof of Theorem 3 *for* $m = 1$. Letting $\beta = 1/2$ in (3.11), we get

$$\begin{aligned} &\sum_{n>0} A_f(1,n) e(\alpha \sqrt{n}) \phi\left(\frac{n}{X}\right) \\ &= -\sqrt{3} i X^{2/3} \sum_{n<2b^*} \frac{A_f(1,n)}{n^{1/3}} I(n) + O(1 + \alpha X^{1/2} \log X), \end{aligned} \tag{3.13}$$

where $2b^* = \sqrt{2}\alpha^3 X^{1/2}/4$, and

$$I(n) = \int_1^{2^{1/3}} u \phi\left(u^3\right) e\left(g(u,n)\right) du \quad \text{with} \quad g(u,n) = \alpha X^{1/2} u^{3/2} - 3\left(nX\right)^{1/3} u.$$

Note that for $n \leq a^*/2 = \alpha^3 X^{1/2}/16$, one has

$$g'(u,n) = \frac{3}{2} \alpha X^{1/2} u^{1/2} - 3(nX)^{1/3} \gg \alpha X^{1/2}.$$

By repeated integrating by parts one obtains $I(n) \ll \alpha^{-3}X^{-3/2} \ll (a^*X)^{-1}$. Hence the total contribution of the terms $1 \le n \le a^*/2$ in (3.13) is

$$\ll (a^*X)^{-1}(a^*X)^{2/3} \sum_{1\le n\le a^*/2} \frac{|A_f(1,n)|}{n} \ll (a^*X)^{-1/3}\log a^* \ll 1.$$

Thus the main term in (3.13) becomes

$$-\sqrt{3}iX^{2/3} \sum_{\alpha^3X^{1/2}/16<n\le\sqrt{2}\alpha^3X^{1/2}/4} \frac{A_f(1,n)}{n^{1/3}} I\left(n\right). \tag{3.14}$$

To bound $I(n)$, we will use weighted stationary phase integral as stated in the following lemma (cf. [7]).

Lemma 3.1 (Weighted stationary phase integral)**.** *Let $g \in C^4([a,b])$ and $f \in C^3([a,b])$ be real-valued functions with continuous fourth and third derivatives, respectively. Suppose that there are positive parameters P, N, Q, U such that*

$$P > b-a, \quad N \ge \frac{P}{\sqrt{Q}},$$

and positive constants C_j $(j=1,...,4)$ such that, for $a \le t \le b$,

$$|g''(t)| \ge \frac{Q}{C_2P^2}, \quad |g^{(r)}(t)| \le \frac{C_rQ}{P^r}, \quad |f^{(s)}(t)| \le \frac{C_sU}{N^s},\ 1\le r\le 4,\ 0\le s\le 3.$$

Suppose $g'(t)$ changes sign at a point $t=\gamma\in(a,b)$. Then

$$\begin{aligned} &\int_a^b f(t)e(g(t))dt \\ = \;& \operatorname{sgn}(g''(\gamma))\frac{f(\gamma)e\left(g(\gamma)+1/8\right)}{\sqrt{|g''(\gamma)|}} + \frac{f(b)e(g(b))}{2\pi i g'(b)} - \frac{f(a)e(g(a))}{2\pi i g'(a)} \\ + \;& O\Big(\frac{P^4U}{Q^2}\Big(1+\frac{P}{N}\Big)^2\Big(\frac{1}{(b-\gamma)^3}+\frac{1}{(\gamma-a)^3}\Big)\Big) + O\Big(\frac{PU}{Q^{3/2}}\Big(1+\frac{P}{N}\Big)^2\Big). \end{aligned}$$

Apply Lemma 3.1 to $f(t) = t\phi(t^3)$ and $g(t) = g(t,n)$. Note that $g'(t)$ has a unique zero at

$$\gamma = \gamma(n) = \frac{4}{\alpha^2}n^{2/3}X^{-1/3},$$

and $2^{-2/3} \le \gamma \le 2$ for n in the range

$$\alpha^3X^{1/2}/16 < n \le \sqrt{2}\alpha^3X^{1/2}/4. \tag{3.15}$$

Moreover

$$g''(t) = \frac{3}{4}\alpha X^{1/2}t^{-1/2}, \quad g^{(3)}(t) = -\frac{3}{8}\alpha X^{1/2}t^{-3/2}, \quad g^{(4)}(t) = \frac{9}{16}\alpha X^{1/2}t^{-5/2}.$$

Let $[a,b] = [1/2,4]$. Then for $a \le t \le b$ and $r \ge 1$ one has $|g^{(r)}(t)| \ll \alpha X^{1/2}$, and $g''(t) \ge \alpha X^{1/2}/4$. Moreover, $g'(a), g'(b) \gg \alpha X^{1/2}$, and

$$\left(f(t)\right)^{(s)} = \left(t\phi\left(t^3\right)\right)^{(s)} \ll_{\phi,s} 1, \quad \text{for} \quad s \ge 1.$$

Choosing $Q = \alpha X^{1/2}$, $N = U = 1$, $P > b-a$ and suitable constant C_j, we get

$$I(n) = \frac{\gamma\phi(\gamma^3)e\left(g(\gamma)+1/8\right)}{\sqrt{g''(\gamma)}} + O\left((\alpha X^{1/2})^{-1}\right).$$

Putting this in (3.14), the above O-term produces a contribution which is

$$\ll X^{2/3}(\alpha X^{1/2})^{-1} \sum_{\alpha^3 X^{1/2}/16<n\leq\sqrt{2}\alpha^3 X^{1/2}/4} \frac{|A_f(1,n)|}{n^{1/3}} \ll \alpha X^{1/2}\log X. \tag{3.16}$$

Note that $e(1/8)=(1+i)/\sqrt{2}$,

$$g(\gamma) \;=\; \alpha X^{1/2}\gamma^{3/2}-3(nX)^{1/3}\gamma=\lambda n \quad\text{with}\quad \lambda=-\frac{4}{\alpha^2}$$

and

$$\frac{\gamma\phi(\gamma^3)}{\sqrt{g''(\gamma)}}=\frac{8\sqrt{6}}{3\alpha^3}X^{-2/3}n^{5/6}\phi\left(\sigma n^2\right) \quad\text{with}\quad \sigma=\frac{64}{\alpha^6 X}.$$

Thus (3.14) can be written as

$$\frac{8}{\alpha^3}(1-i)\sum_{n>0}A_f(1,n)n^{1/2}\phi\left(\sigma n^2\right)e(\lambda n)+O(\alpha X^{1/2}\log X). \tag{3.17}$$

Here we have used the fact that ϕ is supported on $[1,2]$. By partial summation and applying (1.9) we get

$$\begin{aligned}
&\sum_{n>0}A_f(1,n)n^{1/2}\phi\left(\sigma n^2\right)e(\lambda n)\\
&= -\int_{1/\sqrt{\sigma}}^{\sqrt{2/\sigma}}\Big\{\sum_{n\leq u}A_f(1,n)e(\lambda n)\Big\}\left(u^{1/2}\phi\left(\sigma u^2\right)\right)' du\\
&\ll \int_{1/\sqrt{\sigma}}^{\sqrt{2/\sigma}} u^{3/4+\varepsilon}\left(u^{-1/2}|\phi\left(\sigma u^2\right)|+\sigma u^{3/2}|\phi'\left(\sigma u^2\right)|\right)du\\
&\ll \sigma^{-5/8-\varepsilon}\ll_{\phi,\varepsilon}\alpha^{15/4+\varepsilon}X^{5/8+\varepsilon}.
\end{aligned}$$

Back to (3.17), (3.14) and (3.13) we get

$$\sum_{n>0}A_f(1,n)e(\alpha\sqrt{n})\phi\left(\frac{n}{X}\right)\ll_{\phi,\varepsilon}1+\alpha^{3/4+\varepsilon}X^{5/8+\varepsilon}. \tag{3.18}$$

We remark that when α^2 is a rational number, the coefficient $\lambda=-4/\alpha^2$ is rational. In [20] we proved that, in this case, the sum in (3.17) decays rapidly. A problem to applying this argument to the case of square-root twist at hand, however, is that there seems no obvious way to reduce the error term in (3.17). The resulting bound is thus $O(\alpha X^{1/2}\log X)$ when α^2 is rational. □

Proof of Theorem 4 *for* $m=1$. Letting $\beta=1/3$ in (3.11), we get

$$\begin{aligned}
&\sum_{n>0}A_f(1,n)e(\alpha n^{1/3})\phi\left(\frac{n}{X}\right)\\
&= -\sqrt{3}iX^{2/3}\sum_{n<2\alpha^3/27}\frac{A_f(1,n)}{n^{1/3}}J_\phi(n)+O(1+\alpha X^{1/3}\log X)
\end{aligned} \tag{3.19}$$

with

$$J_\phi(x)=\int_1^{2^{1/3}}u\phi\left(u^3\right)e\left(F(u,x)\right)du,\quad F(u,x)=(\alpha-3x^{1/3})X^{1/3}u. \tag{3.20}$$

Note that the first term in (3.19) disappears when $\alpha \leq 3/2^{1/3}$. For $\alpha > 3/2^{1/3}$, let n_α (≥ 1) be the integer such that

$$(\alpha/3)^3 = n_\alpha + \lambda, \quad -1/2 < \lambda \leq 1/2.$$

Then for any positive integer n,

$$|3n^{1/3} - \alpha| = 3|n^{1/3} - \alpha/3| = \frac{3|n - n_\alpha - \lambda|}{n^{2/3} + n^{1/3}(\alpha/3) + (\alpha/3)^2}.$$

For $n \neq n_\alpha$, the right expression is $\gg |n - n_\alpha|\alpha^{-2}$, and hence

$$F'(u, n) \gg |n - n_\alpha|\alpha^{-2} X^{1/3}.$$

By integrating by parts one obtains

$$J_\phi(n) \ll \frac{\alpha^2 X^{-1/3}}{|n - n_\alpha|}.$$

The contribution of the terms $n \neq n_\alpha$ to (3.19) is, by applying (1.1),

$$\ll \alpha^2 X^{1/3} \sum_{\substack{1 \leq n \ll \alpha^3 \\ n \neq n_\alpha}} \frac{|A_f(1, n)|}{|n - n_\alpha| n^{1/3}} \ll \alpha^{29/14+\varepsilon} X^{1/3}.$$

The term corresponding to $n = n_\alpha$ in (3.19) is

$$-\sqrt{3} i X^{2/3} \frac{A_f(1, n_\alpha) J_\phi(n_\alpha)}{n_\alpha^{1/3}}. \tag{3.21}$$

One has

$$|F'(u, n_\alpha)| \asymp \alpha^{-2}|\lambda| X^{1/3}.$$

When $|\lambda| \geq 1/10$, one has $J_\phi(n_\alpha) \ll \alpha^2 X^{-1/3}$ by integrating by parts. Hence the expression in (3.21) is bounded by $\alpha^{29/14+\varepsilon} X^{1/3}$. If $\alpha^{-2}|\lambda| X^{1/3} \gg X^\varepsilon$, then by repeated integrating by parts one obtains $J_\phi(n_\alpha) \ll_\phi X^{-s\varepsilon}$, and hence the contribution of (3.21) is neglectful by choosing s large enough. This proves

$$\begin{aligned} &\sum_{n>0} A_f(1, n) e(\alpha n^{1/3}) \phi\left(\frac{n}{X}\right) \\ &= -\mathcal{E}_\alpha \sqrt{3} i X^{2/3} \frac{A_f(1, n_\alpha)}{n_\alpha^{1/3}} J_\phi(n_\alpha) + O_\phi(1 + \alpha^{29/14+\varepsilon} X^{1/3} \log X), \end{aligned} \tag{3.22}$$

where $\mathcal{E}_\alpha = 1$ if there is an integer $n_\alpha \geq 1$ satisfying

$$|(\alpha/3)^3 - n_\alpha| \leq \min\{X^{-1/3+\varepsilon}\alpha^2, 1/10\},$$

and $\mathcal{E}_\alpha = 0$ if else. Note that the above inequality implies $n_\alpha \asymp \alpha^3$ and

$$n_\alpha^{-1/3} = (\alpha/3)^{-1} + O(X^{-1/3+\varepsilon}\alpha^{-2}).$$

Replacing $n_\alpha^{-1/3}$ by $(\alpha/3)^{-1}$, the above error term produces $O(X^{1/3+\varepsilon})$ to (3.22). Thus we get

$$\begin{aligned} &\sum_{n>0} A_f(1, n) e(\alpha n^{1/3}) \phi\left(\frac{n}{X}\right) \\ &= -\mathcal{E}_\alpha 3\sqrt{3} i \alpha^{-1} X^{2/3} A_f(1, n_\alpha) J_\phi(n_\alpha) + O_{\phi,\varepsilon}(1 + \alpha^{29/14+\varepsilon} X^{1/3+\varepsilon}). \end{aligned} \tag{3.23}$$

In particular, if $(\alpha/3)^3 = \ell$ is an integer, then $\lambda = 0$, $n_\alpha = \ell$ and (3.23) becomes

$$\sum_{n>0} A_f(1,n)e(3(\ell n)^{1/3})\phi\left(\frac{n}{X}\right) = -\sqrt{3}ic(\phi)X^{2/3}\ell^{-1/3}A_f(1,l) + O_{\phi,\varepsilon}(\ell^{29/42+\varepsilon}X^{1/3+\varepsilon}),$$

where

$$c(\phi) = \int_1^{2^{1/3}} u\phi\left(u^3\right) du. \tag{3.24}$$

This proves Theorem 4 for $m = 1$. □

4. Proof of Theorems 1-4 for $m > 1$

Let f be a Hecke eigenform. Then there holds the multiplicative formula

$$A_f(m,n) = \sum_{d|(m,n)} \mu(d)A_f\Big(\frac{m}{d},1\Big)A_f\Big(1,\frac{n}{d}\Big). \tag{4.1}$$

Consequently

$$\begin{aligned}&\sum_{n>0} A_f(m,n)e(\alpha n^\beta)\phi\left(\frac{n}{X}\right)\\ = &\sum_{d|m}\mu(d)A_f\left(\frac{m}{d},1\right)\sum_{n>0}A_f(1,n)e(\alpha d^\beta n^\beta)\phi\Big(\frac{n}{X/d}\Big).\end{aligned} \tag{4.2}$$

By replacing α by αd^β and X by X/d in (3.9) and (3.10), we get

$$\sum_{n>0} A_f(1,n)e(\alpha d^\beta n^\beta)\phi\Big(\frac{n}{X/d}\Big) \ll (X/d)^{-r/3+1/2+\varepsilon}$$

provided that

$$2^{\max\{\beta,1/3\}}\alpha\beta m^{1/3} \le X^{1/3-\beta}.$$

Note that if $2^{\max\{\beta,1/3\}}\alpha\beta \le 1$ this inequality is satisfied when $m^{1/3} \le X^{1/3-\beta}$ which implies $(X/m)^{1/3} \ge X^\beta$; while $2^{\max\{\beta,1/3\}}\alpha\beta > 1$ the above inequality implies $(X/m)^{1/3} \ge X^\beta$. So if we suppose

$$\max\{2^{\max\{\beta,1/3\}}\alpha\beta, 1\}m^{1/3} \le X^{1/3-\beta},$$

then by (4.2) and (1.1) we get

$$\begin{aligned}&\sum_{n>0} A_f(m,n)e(\alpha n^\beta)\phi\left(\frac{n}{X}\right)\\ \ll\ &\sum_{d|m}(m/d)^{5/14+\varepsilon}(X/d)^{-r/3+1/2+\varepsilon} \ll_r (X/m)^{-r/3+1}\\ \ll\ & X^{-\beta(r-3)} \ll X^{-M}\end{aligned}$$

by choosing $r > M/\beta + 3$, where the implied constant may depend on ϕ, β and M.

In particular, when $0 < \beta < 1/3$, the above estimate holds when $X > \max\{A, 1\}$ and $1 \le m \le (X/\max\{A,1\})^{(1-3\beta)}$ with the implied constant depending at most on ϕ and M.

This proves Theorem 1 for a Hecke eigenform f, and hence for a general Maass form f.

Suppose $\beta \neq 1/3$. Replacing α by αd^β and X by X/d in (3.12), then putting in (4.2) and using (1.1) we get

$$\begin{aligned} & \sum_{n>0} A_f(m,n)e(\alpha n^\beta)\phi\left(\frac{n}{X}\right) \\ \ll_\beta \quad & \sum_{d|m}(m/d)^{5/14+\varepsilon}\left(1+((\alpha d^\beta)(X/d)^\beta)^{3/2}\log X\right) \\ \ll_{\beta,\varepsilon} \quad & m^{5/14+\varepsilon}(1+(\alpha X^\beta)^{3/2}\log X). \end{aligned}$$

This proves Theorem 2 for a Hecke eigenform f, and hence for a general Maass form f by applying (1.5).

When $\beta = 1/2$, by (3.18) and (4.2), we get

$$\begin{aligned} & \sum_{n>0} A_f(m,n)e(\alpha\sqrt{n})\phi\left(\frac{n}{X}\right) \\ \ll_{\phi,\varepsilon} \quad & \sum_{d|m}(m/d)^{5/14+\varepsilon}\left(1+(\alpha\sqrt{d})^{3/4+\varepsilon}(X/d)^{5/8+\varepsilon}\right) \\ \ll_{\phi,\varepsilon} \quad & m^{5/14+\varepsilon}(1+\alpha^{3/4+\varepsilon}X^{5/8+\varepsilon}). \end{aligned}$$

This proves Theorem 3 for a Hecke eigenform f, and hence for a general Maass form f.

When $\beta = 1/3$, by (4.2) and (3.23) one has

$$\begin{aligned} & \sum_{n>0} A_f(m,n)e(\alpha n^{1/3})\phi\left(\frac{n}{X}\right) \\ = \quad & -3\sqrt{3}i\alpha^{-1}X^{2/3}\sum_{d|m}\mathcal{E}_\alpha(d)\mu(d)\frac{A_f(m/d,1)A_f(1,n_\alpha(d))}{d}J_\phi\Big(\frac{n_\alpha(d)}{d}\Big) \\ + \quad & O\Big(\sum_{d|m}\Big|A_f\Big(m/d,1\Big)\Big|(1+(\alpha d^{1/3})^{29/14+\varepsilon}(X/d)^{1/3+\varepsilon})\Big), \end{aligned} \tag{4.3}$$

where $J_\phi(x)$ is defined by (3.20) and $\mathcal{E}_\alpha(d) = 1$ or 0 according to whether or not there is an integer $n_\alpha(d) \geq 1$ such that

$$|n_\alpha(d) - \alpha^3 d/27| < \min\{\alpha^2 d^{1-\varepsilon}X^{\varepsilon-1/3}, 1/10\}.$$

By (1.1), the O-term in (4.3) is

$$\begin{aligned} & \ll m^{5/14+\varepsilon}\Big(1+\alpha^{29/14+\varepsilon}X^{1/3}\sum_{d|m}d^{29/42-1/3-5/14+\varepsilon}\Big) \\ & \ll m^{5/14+\varepsilon}\Big(1+\alpha^{29/14+\varepsilon}X^{1/3+\varepsilon}\Big). \end{aligned}$$

This proves

$$\begin{aligned} & \sum_{n>0} A_f(m,n)e(\alpha n^{1/3})\phi\left(\frac{n}{X}\right) \\ = \quad & -3\sqrt{3}i\alpha^{-1}X^{2/3}\sum_{d|m}\mathcal{E}_\alpha(d)\mu(d)\frac{A_f(m/d,1)A_f(1,n_\alpha(d))}{d}J_\phi\Big(\frac{n_\alpha(d)}{d}\Big) \\ + \quad & O_{\phi,\varepsilon}\left(m^{5/14+\varepsilon}(1+\alpha^{29/14+\varepsilon}X^{1/3+\varepsilon})\right). \end{aligned} \tag{4.4}$$

In particular, if $(\alpha/3)^3 = \ell$ is an integer, then $\mathcal{E}_\alpha(d) = 1$, $n_\alpha(d) = d\ell$, and

$$J_\phi\Big(\frac{n_\alpha(d)}{d}\Big) = c(\phi) = \int_1^{2^{1/3}} u\phi(u^3)du.$$

This shows that

$$\begin{aligned} &\sum_{n>0} A_f(m,n)e(3(\ell n)^{1/3})\phi\left(\frac{n}{X}\right) \\ = \quad & -ic(\phi)\sqrt{3}\ell^{-1/3}X^{2/3}\sum_{d|m}\mu(d)\frac{A_f(m/d,1)A_f(1,d\ell)}{d} \\ & +O_{\phi,\varepsilon}\left(m^{5/14+\varepsilon}\ell^{29/42+\varepsilon}X^{1/3+\varepsilon}\right). \end{aligned}$$

This proves Theorem 4 for a Hecke eigenform f, and hence for a general Maass form f by using (1.5).

5. Proof of Theorem 5

Let $1 < \Delta \ll X$. Let $\phi(x)$ be a C^∞ function which is supported on $[1,2]$ and identically equal to 1 on $[1+\Delta^{-1}, 2-\Delta^{-1}]$ such that

$$\phi^{(r)}(x) \ll \Delta^r \quad \text{for integer } r \geq 0. \tag{5.1}$$

Then

$$\sum_{X<n\leq 2X} A_f(m,n)e(\alpha n^\beta) = \sum_{n>0} A_f(m,n)\phi\Big(\frac{n}{X}\Big)e(\alpha n^\beta) + E(\beta), \tag{5.2}$$

where

$$E(\beta) = \sum_{n>0} A_f(m,n)\left(1-\phi\Big(\frac{n}{X}\Big)\right)e(\alpha n^\beta).$$

By (4.1),

$$\begin{aligned} E(\beta) \ll\ & \sum_{d|m}\Big|A_f\Big(\frac{m}{d},1\Big)\Big|\sum_{X/d<n\leq X/d+X/(d\Delta)}|A_f(1,n)| \\ +\ & \sum_{d|m}\Big|A_f\Big(\frac{m}{d},1\Big)\Big|\sum_{2X/d-X/(d\Delta)<n\leq 2X/d}|A_f(1,n)|. \end{aligned} \tag{5.3}$$

By (1.1), the sum over n is $O((X/d)^{5/14+\varepsilon})$ for $d > X/\Delta$, and is $O(\sqrt{(X/d)X/(d\Delta)})$ for $d \leq X/\Delta$, by (1.3) and Cauchy's inequality. Thus one obtains

$$\begin{aligned} E(\beta) &\ll \sum_{d|m}(m/d)^{5/14+\varepsilon}\left((X/d)^{5/14+\varepsilon} + (X/d)\Delta^{-1/2}\right) \\ &\ll m^{5/14+\varepsilon}\left(X^{5/14+\varepsilon} + X\Delta^{-1/2}\right). \end{aligned} \tag{5.4}$$

The estimate of the main term in (5.2) follows the same line of the proof of Theorems 1-4 in sections 3 and 4. The difference is that the function ϕ here has large derivatives as described in (5.1). By (4.2), we first need to estimate

$$\sum_{n>0} A_f(1,n)e(\alpha n^\beta)\phi\left(\frac{n}{X}\right).$$

Following the argument from (3.1) to (3.4) and choosing $r = 2$, we get

$$\sum_{n>0} A_f(1,n)e(\alpha n^\beta)\phi\left(\frac{n}{X}\right) = \mathcal{D}_0(X) + \mathcal{D}_1(X) + O(1), \tag{5.5}$$

where for $k = 0, 1$

$$\begin{aligned} \mathcal{D}_k(X) &= \frac{3\pi^{-5/2}}{4i^{k+1}} \sum_{n\geq 1} \frac{A_f(n,1) + (-1)^k A_f(1,n)}{n} \\ &\times \sum_{j=1}^{2} \left(\pi^3 nX\right)^{1-j/3} \left(a_k(j)I_j^+(n) + b_k(j)I_j^-(n)\right). \end{aligned} \tag{5.6}$$

To bound $I_j^\pm(x)$, we first note that the estimate in (3.5) is not valid here when $s \geq 2$ because the $\phi(t)$ appearing in $I_j^\pm(x)$ has derivatives depending on Δ. Instead, for any integer $s \geq 1$ we have

$$I_j^\pm(x) \ll Q^{-s}\Delta^{s-1} \tag{5.7}$$

provided that $F'_\pm(u) \gg Q \gg \alpha\beta X^\beta$ for $u \in [1, 2^{1/3}]$. Here we have used the estimate

$$\int_0^\infty |\phi^{(t)}(u)|du = \Big\{\int_1^{1+\Delta^{-1}} + \int_{2-\Delta^{-1}}^2\Big\}|\phi^{(t)}(u)|du \ll \Delta^{t-1}, \quad t \geq 1.$$

Let b^* be defined by (3.7). Note that $F'_+(u,n) \gg (nX)^{1/3} + \alpha\beta X^\beta$ for $n \geq 1$, and $F'_-(u,n) \gg (nX)^{1/3} \gg \alpha\beta X^\beta$ for $n \geq 2b^*$. Thus (5.7) gives

$$I_j^+(n) \ll (nX)^{-s/3}\Delta^{s-1}, \qquad \text{for} \quad n \geq 1; \tag{5.8}$$

$$I_j^-(n) \ll (nX)^{-s/3}\Delta^{s-1}, \qquad \text{for} \quad n \geq 2b^*. \tag{5.9}$$

Let $B = \Delta^3 X^{-1} + 2b^*$. Taking $s = 4 - j$ in (5.8) and (5.9), one finds that the contribution of $I_j^\pm(n)$ with $n \geq B$ to (5.6) is

$$\ll \sum_{j=1}^{2}\sum_{n\geq B} \frac{|A_f(n,1)| + |A_f(1,n)|}{n} (nX)^{1-(j+s)/3}\Delta^{s-1} \ll (BX)^{-1/3}\Delta^2 \ll \Delta.$$

Taking $s = 3 - j$ in (5.8) and (5.9), one can see that the contribution of $I_j^-(n)$ with $2b^* \leq n < B$ to (5.6) and the contribution of $I_j^+(n)$ with $1 \leq n < B$ to (5.6) together is

$$\ll \sum_{j=1}^{2} \Delta^{2-j} \sum_{n\leq B} \frac{|A_f(n,1)| + |A_f(1,n)|}{n} \ll \Delta \log B.$$

Thus we get

$$\begin{aligned} \mathcal{D}_k(X) &= \frac{3\pi^{-5/2}}{4i^{k+1}} \sum_{n<2b^*} \frac{A_f(n,1) + (-1)^k A_f(1,n)}{n} \\ &\times \sum_{j=1}^{2} \left(\pi^3 nX\right)^{1-j/3} b_k(j) I_j^-(n) + O(\Delta \log B). \end{aligned} \tag{5.10}$$

Note that if $2b^* \leq 1$, then the main term disappears. Assume $2b^* > 1$. For $\beta \neq 1/3$, the second derivative test shows that $I_j^-(n) \ll (\alpha X^\beta)^{-1/2}$ for $1 \leq j \leq 2$. Thus the main term in (5.10) is

$$\ll (b^* X)^{2/3} (\alpha X^\beta)^{-1/2} \sum_{n \leq 2b^*} \frac{|A_f(1,n)| + |A_f(n,1)|}{n} \ll_\beta (\alpha X^\beta)^{3/2} \log(2b^*),$$

by (1.4). This proves

$$\sum_{n>0} A_f(1,n) e(\alpha n^\beta) \phi\left(\frac{n}{X}\right) \ll ((\alpha X^\beta)^{3/2} + \Delta) \log B. \tag{5.11}$$

For $\beta = 1/3$, one has $b^* = \alpha^3/27$. We will follow the argument from (3.19) to (3.22). Let n_α be a positive integer such that

$$\alpha^3/27 = n_\alpha + \lambda, \quad -1/2 < \lambda \leq 1/2.$$

If $n \neq n_\alpha$ or $n = n_\alpha$ but $|\lambda| \geq 1/10$ one can prove that the contribution of these terms to $\mathcal{D}_k(X)$ in (5.10) is $O(\alpha^{29/14+\varepsilon} X^{1/3})$. Note that there is at most one integer $n = n_\alpha$ such that $\left|\alpha^3/27 - n\right| < 1/10$. The contribution of this term to (5.10) is, by using (1.1) and the trivial estimate $I_j^-(n) \ll 1$,

$$\ll X^{2/3} (\alpha^3)^{5/14+\varepsilon-1/3} \ll \alpha^{1/14+\varepsilon} X^{2/3}.$$

This proves

$$\sum_{n>0} A_f(1,n) e(\alpha n^{1/3}) \phi\left(\frac{n}{X}\right) \ll \Delta \log B + \alpha^{1/14+\varepsilon} X^{2/3} + \alpha^{29/14+\varepsilon} X^{1/3+\varepsilon}. \tag{5.12}$$

Replace α by αd^β and X by X/d in (5.11) and (5.12). Set $\Delta = X^{2/3}$ and apply (4.2). The desired estimates in Theorem 5 follow easily from (5.2) and (5.4).

6. Proof of Theorem 6

We start from (5.2) and (5.3). By (1.11) we get

$$\begin{aligned} E(\beta) &\ll \sum_{d|m} \left(\frac{m}{d}\right)^{\theta+\varepsilon} \left(\frac{X}{d}\right)^{\theta+\varepsilon} \left(1 + \frac{X}{d\Delta}\right) \\ &\ll m^{\theta+\varepsilon} (X^{\theta+\varepsilon} + X^{1+\theta+\varepsilon} \Delta^{-1}). \end{aligned} \tag{6.1}$$

To estimate the smooth sum in (5.2), we follow the same arguments from (5.5) to (5.12), but pick up a single main term corresponding to $n = n_\alpha(d)$ such that

$$\left|\alpha^3 d/27 - n\right| < 1/10, \tag{6.2}$$

when $\beta = 1/3$ as in (4.4). Consequently one obtains, when $\beta \neq 1/3$,

$$\begin{aligned} &\sum_{n>0} A_f(m,n) e(\alpha n^\beta) \\ &\ll m^{\theta+\varepsilon} (X^{1+\theta+\varepsilon} \Delta^{-1} + X^{\theta+\varepsilon} + (\alpha X^\beta)^{3/2} + \Delta \log B), \end{aligned} \tag{6.3}$$

and when $\beta = 1/3$,

$$\begin{aligned}
&\sum_{X<n\le 2X} A_f(m,n)e(\alpha n^\beta) \\
&= -i\sqrt{3}X^{2/3}\sum_{d|m}\frac{\mathcal{E}_\alpha(d)\mu(d)}{n_\alpha(d)^{1/3}d^{2/3}}J_\phi\Big(\frac{n_\alpha(d)}{d}\Big)A_f\Big(\frac{m}{d},1\Big)A_f(1,n_\alpha(d)) \\
&+ O_{f,\varepsilon}\Big(m^{\theta+\varepsilon}\left\{X^{1+\theta+\varepsilon}\Delta^{-1}+X^{\theta+\varepsilon}+\Delta\log B+\alpha^{1+3\theta+\varepsilon}X^\beta\right\}\Big), \qquad (6.4)
\end{aligned}$$

where as before J_ϕ is given by (3.20) and

$$B = \Delta^3X^{-1} + 2b^* = \Delta^3X^{-1} + (\alpha\beta)^3X^{3\beta-1}\max(2^{3\beta},2),$$

and hence $\log B \ll \log\Delta + \log X$. Recall that $\mathcal{E}_\alpha(d) = 1$ if (6.2) has a solution, and $= 0$ otherwise.

By the construction of ϕ we get from (3.20) that

$$\begin{aligned}
J_\phi\Big(\frac{n_\alpha(d)}{d}\Big) &= \int_1^{2^{1/3}} u\phi(u^3)e\Big(\Big(\alpha-3\Big(\frac{n_\alpha(d)}{d}\Big)^{1/3}\Big)X^{1/3}u\Big)du \\
&= \int_1^{2^{1/3}} ue\Big(\Big(\alpha-3\Big(\frac{n_\alpha(d)}{d}\Big)^{1/3}\Big)X^{1/3}u\Big)du + O\Big(\frac{1}{\Delta}\Big) \\
&=: J_1\Big(\frac{n_\alpha(d)}{d}\Big) + O\Big(\frac{1}{\Delta}\Big).
\end{aligned}$$

When $\alpha - 3(n_\alpha(d)/d)^{1/3} \neq 0$, i.e., when $\alpha^3d/27$ is not an integer, we can compute $J_1(n_\alpha(d)/d)$ and show that

$$J_1\Big(\frac{n_\alpha(d)}{d}\Big) \ll \Big|\alpha-3\Big(\frac{n_\alpha(d)}{d}\Big)^{1/3}\Big|^{-1}X^{-1/3}, \qquad (6.5)$$

using integration by parts once. If

$$\Big|\alpha-3\Big(\frac{n_\alpha(d)}{d}\Big)^{1/3}\Big| \gg X^{-1/3-(1+\theta)/2-\varepsilon} = X^{-(1+3\theta)/6-\varepsilon}, \qquad (6.6)$$

then by (6.5)

$$X^{2/3}J_1\Big(\frac{n_\alpha(d)}{d}\Big) \ll X^{1/3+(1+3\theta)/6+\varepsilon} = X^{(1+\theta)/2+\varepsilon}.$$

Thus by (1.11) the main term on the right side of (6.4) is

$$\begin{aligned}
&\ll (X^{(1+\theta)/2+\varepsilon}+X^{2/3}\Delta^{-1})\sum_{d|m}\frac{1}{n_\alpha(d)^{1/3}d^{2/3}}\Big(\frac{m}{d}\Big)^{\theta+\varepsilon}n_\alpha(d)^{\theta+\varepsilon} \\
&\ll m^{\theta+\varepsilon}(X^{(1+\theta)/2+\varepsilon}+X^{2/3}\Delta^{-1})\sum_{d|m}\frac{1}{d^{2/3+\theta-\varepsilon}n_\alpha(d)^{1/3-\theta+\varepsilon}} \\
&\ll m^{\theta+\varepsilon}(X^{(1+\theta)/2+\varepsilon}+X^{2/3}\Delta^{-1}),
\end{aligned}$$

when $\beta = 1/3$ and (6.6) holds. When (6.6) does not hold, we may impose

$$\Big|n_\alpha(d)-\alpha^3d/27\Big| < \min\Big\{\alpha^2dX^{-(1+3\theta)/6-\varepsilon}, 1/10\Big\}, \qquad (6.7)$$

and a main term remains on the right side of (6.4) for each $d|m$. Note that (6.7) implies

$$n_a(d)^{-1/3} = 3\alpha^{-1}d^{-1/3} + O(\alpha^{-2}X^{-(1+3\theta)/6-\varepsilon}d^{-1/3}).$$

By (1.11) and using the fact that $J_\phi \ll 1$, $n_\alpha(d) \asymp \alpha^3 d$, the above O-term contributes to (6.4) the following:

$$O\Big(X^{2/3-(1+3\theta)/6-\varepsilon}\sum_{d|m}\frac{(m/d)^{\theta+\varepsilon}(n_\alpha(d))^{\theta+\varepsilon}}{\alpha^2 d}\Big) = O(m^{\theta+\varepsilon}X^{(1-\theta)/2-\varepsilon}).$$

Consequently when $\beta = 1/3$, (6.4) can be written as

$$\begin{aligned}
&\sum_{X<n\le 2X} A_f(m,n)e(\alpha n^\beta)\\
= &\ -i3\sqrt{3}\alpha^{-1}X^{2/3}\sum_{d|m}\frac{\mathcal{E}_\alpha(d)\mu(d)}{d}J_1\Big(\frac{n_\alpha(d)}{d}\Big)A_f\Big(\frac{m}{d},1\Big)A_f(1,n_\alpha(d))\\
+ &\ O_{f,\varepsilon}\Big(m^{\theta+\varepsilon}\left\{X^{1+\theta+\varepsilon}\Delta^{-1}+X^{\theta+\varepsilon}+\Delta\log B+\alpha^{1+3\theta+\varepsilon}X^\beta\right\}\Big)\\
+ &\ O_{f,\varepsilon}\Big(m^{\theta+\varepsilon}(X^{(1+\theta)/2+\varepsilon}+X^{2/3}\Delta^{-1})\Big),
\end{aligned}\tag{6.8}$$

where we redefine $\mathcal{E}_\alpha(d) = 1$ if there is an integer $n_\alpha(d)$ satisfying (6.7) and $= 0$ otherwise.

To minimize the error terms in (6.3) and (6.8), we take $\Delta = X^{(1+\theta)/2}$. Then we get, when $\beta \ne 1/3$,

$$\sum_{n>0} A_f(m,n)e(\alpha n^\beta) \ll m^{\theta+\varepsilon}\left(X^{(1+\theta)/2+\varepsilon}+(\alpha X^\beta)^{3/2}\log X\right),$$

and when $\beta = 1/3$, the O-terms in (6.8) become

$$O_{f,\varepsilon}\Big(m^{\theta+\varepsilon}(X^{(1+\theta)/2+\varepsilon}+\alpha^{1+3\theta+\varepsilon}X^{1/3})\Big).\tag{6.9}$$

Since θ is assumed to be $< 1/3$, we have $(1+\theta)/2 < 2/3$ in (6.9).

When $(\alpha/3)^3 = \ell$ is an integer, (6.7) implies $n_\alpha(d) = d\ell$, $\mathcal{E}_\alpha(d) = 1$, and

$$J_1\Big(\frac{n_\alpha(d)}{d}\Big) = \int_1^{2^{1/3}} u\,du = \frac{2^{2/3}-1}{2}.$$

Thus

$$\begin{aligned}
&\sum_{X<n\le 2X} A_f(m,n)e(3(\ell n)^{1/3})\\
= &\ -i\frac{\sqrt{3}}{2}(2^{2/3}-1)X^{2/3}\ell^{-1/3}\sum_{d|m}\mu(d)\frac{A_f(m/d,1)A_f(1,d\ell)}{d}\\
+ &\ O_{f,\varepsilon}\Big(m^{\theta+\varepsilon}(X^{(1+\theta)/2+\varepsilon}+\ell^{1/3+\theta+\varepsilon}X^{1/3})\Big).
\end{aligned}$$

This proves Theorem 6 for a Hecke eigenform f, and hence for a general Maass form f.

Acknowledgments

The first author was supported by the National Natural Science Foundation of China (Grant No. 10971119). The authors would like to thank an anonymous referee for helpful comments and a remark on Theorem 3.

References

[1] Ce Bian, *Computing* GL(3) *automorphic forms*, Bull. Lond. Math. Soc. **42** (2010), no. 5, 827–842, DOI 10.1112/blms/bdq038. MR2721743 (2012c:11108)

[2] Andrew R. Booker, *A test for identifying Fourier coefficients of automorphic forms and application to Kloosterman sums*, Experiment. Math. **9** (2000), no. 4, 571–581. MR1806292 (2001k:11077)

[3] Andrew R. Booker, *Numerical tests of modularity*, J. Ramanujan Math. Soc. **20** (2005), no. 4, 283–339. MR2193217 (2006k:11090)

[4] Dorian Goldfeld, *Automorphic forms and L-functions for the group* $\mathrm{GL}(n,\mathbf{R})$, Cambridge Studies in Advanced Mathematics, vol. 99, Cambridge University Press, Cambridge, 2006. With an appendix by Kevin A. Broughan. MR2254662 (2008d:11046)

[5] Dorian Goldfeld and Xiaoqing Li, *Voronoi formulas on* $\mathrm{GL}(n)$, Int. Math. Res. Not., posted on 2006, Art. ID 86295, 25, DOI 10.1155/IMRN/2006/86295. MR2233713 (2007f:11052)

[6] Dorian Goldfeld and Xiaoqing Li, *The Voronoi formula for* $\mathrm{GL}(n,\mathbb{R})$, Int. Math. Res. Not. IMRN **2** (2008), Art. ID rnm144, 39, DOI 10.1093/imrn/rnm144. MR2418857 (2009b:11077)

[7] M. N. Huxley, *Area, lattice points, and exponential sums*, London Mathematical Society Monographs. New Series, vol. 13, The Clarendon Press Oxford University Press, New York, 1996. Oxford Science Publications. MR1420620 (97g:11088)

[8] Aleksandar Ivić, *On the ternary additive divisor problem and the sixth moment of the zeta-function,* theory (Cardiff, 1995), London Math. Soc. Lecture Note Ser., vol. 237, Cambridge Univ. Press, Cambridge, 1997, pp. 205–243, DOI 10.1017/CBO9780511526091.015. MR1635762 (99k:11129)

[9] Henryk Iwaniec, Wenzhi Luo, and Peter Sarnak, *Low lying zeros of families of L-functions*, Inst. Hautes Études Sci. Publ. Math. **91** (2000), 55–131 (2001). MR1828743 (2002h:11081)

[10] J. Kaczorowski and A. Perelli, *On the structure of the Selberg class. VI. Non-linear twists*, Acta Arith. **116** (2005), no. 4, 315–341, DOI 10.4064/aa116-4-2. MR2110507 (2005i:11128)

[11] Henry H. Kim, *Functoriality for the exterior square of* GL_4 *and the symmetric fourth of* GL_2, J. Amer. Math. Soc. **16** (2003), no. 1, 139–183, DOI 10.1090/S0894-0347-02-00410-1. With appendix 1 by Dinakar Ramakrishnan and appendix 2 by Kim and Peter Sarnak. MR1937203 (2003k:11083)

[12] H. Kim and P. Sarnak, *Refined estimates towards the Ramanujan and Selberg conjectures*, Appendix to Kim [11].

[13] Xiannan Li, *Additive twists of Fourier coefficients of* $GL(3)$ *Maass forms*, Proc. Amer. Math. Soc. **142** (2014), no. 6, 1825–1836, DOI 10.1090/S0002-9939-2014-11909-5. MR3182004

[14] Xiaoqing Li, *The central value of the Rankin-Selberg L-functions*, Geom. Funct. Anal. **18** (2009), no. 5, 1660–1695, DOI 10.1007/s00039-008-0692-5. MR2481739 (2010a:11087)

[15] Xiaoqing Li, *Bounds for* $\mathrm{GL}(3)\times\mathrm{GL}(2)$ *L-functions and* $\mathrm{GL}(3)$ *L-functions*, Ann. of Math. (2) **173** (2011), no. 1, 301–336, DOI 10.4007/annals.2011.173.1.8. MR2753605 (2012g:11092)

[16] Xiaoqing Li and Matthew P. Young, *Additive twists of Fourier coefficients of symmetric-square lifts*, J. Number Theory **132** (2012), no. 7, 1626–1640, DOI 10.1016/j.jnt.2011.12.017. MR2903173

[17] Stephen D. Miller, *Cancellation in additively twisted sums on* $\mathrm{GL}(n)$, Amer. J. Math. **128** (2006), no. 3, 699–729. MR2230922 (2007k:11078)

[18] Stephen D. Miller and Wilfried Schmid, *Automorphic distributions, L-functions, and Voronoi summation for* $\mathrm{GL}(3)$, Ann. of Math. (2) **164** (2006), no. 2, 423–488, DOI 10.4007/annals.2006.164.423. MR2247965 (2007j:11065)

[19] A. Perelli, *Non-linear twists of L-functions: a survey*, Milan J. Math. **78** (2010), no. 1, 117–134, DOI 10.1007/s00032-010-0119-2. MR2684775 (2011h:11098)

[20] XiuMin Ren and YangBo Ye, *Resonance between automorphic forms and exponential functions*, Sci. China Math. **53** (2010), no. 9, 2463–2472, DOI 10.1007/s11425-010-3150-4. MR2718840 (2012a:11118)

[21] Xiumin Ren and Yangbo Ye, *Sums of Fourier coefficients of a Maass form for* $\mathrm{SL}_3(\mathbb{Z})$ *twisted by exponential functions*, Forum Math. **26** (2014), no. 1, 221–238, DOI 10.1515/form.2011.157. MR3176629

[22] Xiumin Ren and Yangbo Ye, *Asymptotic Voronoi's summation formulas and their duality for* $SL_3(\mathbb{Z})$, in Number Theory Arithmetic in Shangri-La Proceedings of the 6th China-Japan Seminar Shanghai, China 15-17 August 2011, edited by Shigeru Kanemitsu, Hongze

Li, and Jianya Liu, Series on Number Theory and Its Applications, volume **8**, 213–236, World Scientific, Singapore, 2013.

[23] Peter Sarnak, *Notes on the generalized Ramanujan conjectures*, Harmonic analysis, the trace formula, and Shimura varieties, Clay Math. Proc., vol. 4, Amer. Math. Soc., Providence, RI, 2005, pp. 659–685. MR2192019 (2007a:11067)

[24] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, 2nd ed., The Clarendon Press Oxford University Press, New York, 1986. Edited and with a preface by D. R. Heath-Brown. MR882550 (88c:11049)

Department of Mathematics, Shandong University, Jinan, Shandong 250100, People's Republic of China

E-mail address: `xmren@sdu.edu.cn`

Department of Mathematics, The University of Iowa, Iowa City, Iowa 52242-1419

E-mail address: `yangbo-ye@uiowa.edu`