TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 367, Number 4, April 2015, Pages 2339–2410
S 0002-9947(2014)06091-1
Article electronically published on November 24, 2014

COLLAPSING THREE-DIMENSIONAL CLOSED ALEXANDROV SPACES WITH A LOWER CURVATURE BOUND

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ABSTRACT. In the present paper, we determine the topologies of three-dimensional closed Alexandrov spaces which converge to lower dimensional spaces in the Gromov-Hausdorff topology.

CONTENTS

1. INTRODUCTION

The purpose of the present paper is to determine the topologies of collapsing three-dimensional Alexandrov spaces.

Alexandrov spaces are complete length spaces with the notion of curvature bounds. In this paper, we deal with finite dimensional Alexandrov spaces with a lower curvature bound (see Definition 2.2). Alexandrov spaces naturally appear in convergence and collapsing phenomena of Riemannian manifolds with a lower curvature bound ([SY00], [Y 4-dim]), and have played important roles in the study of collapsing Riemannian manifolds with a lower curvature bound.

For a positive integer n, $D > 0$, $\kappa \in \mathbb{R}$, let us consider the following two families: $\mathcal{M}^n(D,\kappa)$ is the family of all isometry classes of complete n-dimensional Riemannian manifolds M whose diameters and sectional curvatures satisfy $\mathrm{diam}\,(M) \leq D$ and $sec(M) \geq \kappa$. $\mathcal{A}^n(D,\kappa)$ is the family of all isometry classes of n-dimensional Alexandrov spaces with diam $\leq D$ and curvature $\geq \kappa$. It follows from the definition of Alexandrov spaces that $\mathcal{M}^n(D,\kappa) \subset \mathcal{A}^n(D,\kappa)$. By Gromov's precompactness theorem, $\mathcal{A}^n(D,\kappa)$ has a nice property that $\bigcup_{k\leq n} \mathcal{A}^k(D,\kappa)$ is compact in the Gromov-Hausdorff topology, while $\bigcup_{k\leq n} \mathcal{M}^k(D,\kappa)$ is precompact. Therefore,

Received by the editors June 29, 2012.
2010 *Mathematics Subject Classification.* Primary 53C20, 53C23.

it is quite natural to study the convergence and collapsing phenomena in $\mathcal{A}^n(D,\kappa)$. Thus, the following problem naturally appears:

Problem 1.1. Let $\{M_i^n\}_{i=1}^{\infty}$ be a sequence in $\mathcal{A}^n(D,\kappa)$ converging to an Alexandrov space X. Can one describe the topological structure of M_i by using the geometry and topology of X for large i?

In this paper, we consider Problem 1.1 for $n=3$ when M_i has no boundary. We exhibit previously known results related to Problem 1.1. Let us fix the following setting: $M_i := M_i^n \in \mathcal{A}^n(D,\kappa)$ converges to X as $i\to\infty$, and fix a sufficiently large integer i.

If the non-collapsing case arises, i.e. $\dim X = n$, Perelman's stability theorem [Per II] (cf. [Kap Stab]) shows that M_i is homeomorphic to X.

In the collapsing case, we know the following results in the general dimension: If M_i and X are Riemannian manifolds, then Yamaguchi proved that there is a locally trivial fiber bundle (smooth submersion) $f_i : M_i \to X$ whose fiber is a quotient of torus by some finite group action ([Y91], [Y conv]). Fukaya and Yamaguchi proved that if M_i are Riemannian manifolds and X is a single-point set, then $\pi_1(M_i)$ has a nilpotent subgroup of finite index [FY]. This statement also goes through even if M_i is an Alexandrov space ([Y conv]).

In the lower dimensional cases, we know the following conclusive results: In dimension three, Shioya and Yamaguchi [SY00] gave a complete classification of three-dimensional closed (orientable) Riemannian manifolds M_i collapsing in $\mathcal{M}^3(D,\kappa)$. It is also proved that volume collapsed closed orientable Riemannian three-manifolds M_i with no diameter bound are graph-manifolds or have small diameters and finite fundamental groups ([SY05], [Per Ent]). For more recent works, see Morgan and Tian [MT], Cao and Ge [CaGe], Kleiner and Lott [KL]. In dimension four, Yamaguchi [Y 4-dim] gave a classification of four-dimensional orientable closed Riemannian manifolds M_i collapsing in $\mathcal{M}^4(D,\kappa)$.

1.1. **Main results.** To state our results, we fix notation in this paper. D^n is a closed n-disk. D^1 is written as I, called a (bounded closed) interval. P^n is an n-dimensional real projective space. T^n is an n-dimensional torus. K^2 is a Klein bottle, Mö is a Mobius band. $K^2\tilde{\times}I$ is an orientable (non-trivial) I-bundle over K^2. $K^2\hat{\times}I$ is a non-orientable non-trivial I-bundle over K^2. A solid Klein bottle $S^1\tilde{\times}D^2$ is obtained by $\mathbb{R}\times D^2$ with identification $(t,x)=(t+1,\bar{x})$. Here, we consider D^2 as the unit disk on the complex plane and $\bar{x}$ is the complex conjugate of x. Note that a solid Klein bottle is homeomorphic to Mö $\times I$.

A compact Alexandrov space without boundary is called *closed.* We classify all three-dimensional closed Alexandrov spaces collapsing to lower dimensional ones. It turns out that there is a strange phenomenon which does not occur in the manifold case. This phenomenon can be typically seen in the following example.

Example 1.2. Let $S^1\times\mathbb{R}^2$ be a flat manifold with product metric. For the isometric involution α defined by

$$\alpha(e^{i\theta},x)=(e^{-i\theta},-x),$$

we consider the quotient space $M_{\rm pt} := S^1\times\mathbb{R}^2/\langle\alpha\rangle$ which is an Alexandrov space with non-negative curvature. This space $M_{\rm pt}$ has the two topologically singular points, i.e. non-manifold points, $p_+ := [(1,0)]$ and $p_- := [(-1,0)]$, which correspond to fixed points $(1,0)$ and $(-1,0)$ of α. We consider a standard projection

$p : M_{\mathrm{pt}} \to \mathbb{R}^2/x \sim -x = K(S^1_\pi)$ from M_{pt} to the cone $K(S^1_\pi)$ over the circle S^1_π of length π. This is an S^1-fiber bundle over $K(S^1_\pi)$ except the vertex $o \in K(S^1_\pi)$. Remark that the fiber $p^{-1}(\partial B(o,r))$ over a metric circle at o is topologically a Klein bottle. The fiber $p^{-1}(o)$ over the origin is an interval joining the topologically singular points p_+ and p_-. Thus, we may regard M_{pt} as a circle fibration, with the singular fiber $p^{-1}(o)$, over the cone $K(S^1_\pi)$. We rescale the "circle orbits" of M_{pt} as $M_{\mathrm{pt}}(\varepsilon) := (\varepsilon S^1) \times \mathbb{R}^2/\langle\alpha\rangle$. Then, as $\varepsilon \to 0$, $M_{\mathrm{pt}}(\varepsilon)$ collapse to the cone $K(S^1_\pi)$.

We obtain the following results.

An *essential singular point* of an Alexandrov space is a point at which the space of directions has radius not greater than $\pi/2$.

Theorem 1.3. *Let M_i^3 be a sequence of three-dimensional closed Alexandrov spaces with curvature ≥ -1 and $\operatorname{diam} M_i \leq D$. Suppose that M_i converges to an Alexandrov surface X without boundary. Then, for sufficiently large i, M_i is homeomorphic to a generalized Seifert fiber space over X. Further, singular orbits may occur over essential singular points in X.*

Here, a *generalized* Seifert fiber space is a Seifert fiber space in a generalized sense, which possibly has singular interval fibers just as in Example 1.2. For the precise definition, see Definition 2.48.

To describe the topologies of M_i^3 converging to an Alexandrov surface with nonempty boundary, we define the notion of generalized solid tori and generalized solid Klein bottles. Let $K(A)$ be the cone over a topological space A, obtained from $A \times [0,+\infty)$ smashing $A \times \{0\}$ to a point. Let $K_1(A)$ be the closed cone over A, obtained from $A \times [0,1]$ smashing $A \times \{0\}$ to a point. We put $\partial K_1(A) := A \times \{1\}$.

Definition 1.4. We will construct a certain three-dimensional topological orbifold whose boundary is homeomorphic to a torus or a Klein bottle.

We first observe that the closed cone $K_1(P^2)$ over P^2 can be regarded as a "fibration"[1] over I as follows. Let $\Gamma \cong \mathbb{Z}_2$ be the group generated by the involution γ on $\mathbb{R}^3$ defined by $\gamma(v) = -v$. Then $\mathbb{R}^3/\Gamma = K(P^2)$.

We consider the following families of surfaces in $\mathbb{R}^3$,

$$A(t) := \{v = (x,y,z) \mid x^2+y^2-z^2 = t^2, |z| \leq 1\},$$
$$B(t) := \{v = (x,y,z) \mid x^2+y^2-z^2 = -t^2, x^2+y^2 \leq 1\},$$

and set

$$D(t) := \begin{cases} A(t)/\Gamma & \text{if } t > 0, \\ B(t)/\Gamma & \text{if } t \leq 0. \end{cases}$$

Then $D(t)$ is homeomorphic to a Mobius band for $t > 0$ and is homeomorphic to a disk for $t \leq 0$. Remark that $\bigcup_{t\in[-1,1]} \partial D(t)$ is homeomorphic to $S^1 \times I$. The union $D(1) \cup \bigcup_{t\in[-1,1]} \partial D(t) \cup D(-1)$ corresponds to $P^2 \times \{1\} = \partial K_1(P^2) \subset K_1(P^2)$. Define a projection

$$\pi : K_1(P^2) \approx \bigcup_{t\in[-1,1]} D(t) \to [-1,1] \text{ as } \pi(D(t)) = t. \tag{1.1}$$

This is a "fibration" stated as above.

[1] In fact, it is NOT a Serre fibration, because the fibers D^2 and Mö are not weak homotopy equivalence.

For a positive integer $N \geq 1$, let us consider a circle $S^1 = [0, 2N]/\{0\} \sim \{2N\}$. Let I_j be a sub-arc in S^1 corresponding to $[j-1, j] \subset [0, 2N]$ for $j = 1, \dots, 2N$. We consider a sequence B_j of topological spaces such that each B_j is homeomorphic to $K_1(P^2)$. We take a sequence of projections $\pi_j : B_j \to I_j$ obtained as above such that there are homeomorphisms $\phi_j : \pi_j^{-1}(j) \approx \pi_{j+1}^{-1}(j)$ for all $j = 1, \dots, 2N$. Then we obtain a topological space $Y := \bigcup_{j=1}^{2N} B_j$ glued by ϕ_j's. Define a projection

$$\pi : Y = \bigcup_{j=1}^{2N} B_j \to S^1 \text{ by } \pi(\pi_j^{-1}(t)) = t \tag{1.2}$$

for any $t \in S^1$. By the construction, Y has $2N$ topologically singular points. Remark that the restriction $\pi|_{\partial Y} : \partial Y \to S^1$ is a usual S^1-fiber bundle. Then we obtain a topological orbifold Y whose boundary ∂Y is homeomorphic to a torus or a Klein bottle. If ∂Y is a torus, then Y is called a *generalized solid torus of type N*. If ∂Y is a Klein bottle, then Y is called a *generalized solid Klein bottle of type N*. We regard a solid torus $S^1 \times D^2$ and the product $S^1 \times$ Mö as generalized solid tori of type 0. We also regard a solid Klein bottle $S^1 \tilde{\times} D^2$ and non-trivial Mö-bundle $S^1 \tilde{\times}$Mö over S^1 as generalized solid Klein bottles of type 0. Note that $S^1 \tilde{\times}$Mö is homeomorphic to a non-orientable I-bundle $K^2 \hat{\times} I$ over K^2.

For a two-dimensional Alexandrov space X, a boundary point $x \in \partial X$ is called a *corner* point if $\operatorname{diam} \Sigma_x \leq \pi$, in other words, if it is an essential singular point.

Theorem 1.5. *Let $\{M_i\}_{i=1}^{\infty}$ be a sequence of three-dimensional closed Alexandrov spaces with curvature ≥ -1 and $\operatorname{diam} M_i \leq D$. Suppose that M_i converges to an Alexandrov surface X with non-empty boundary. Then, for large i, there exist a generalized Seifert fiber space $\operatorname{Seif}_i(X)$ over X and generalized solid tori or generalized solid Klein bottles $\pi_{i,k} : Y_{i,k} \to (\partial X)_k$ over each component $(\partial X)_k$ of ∂X such that M_i is homeomorphic to a union of $\operatorname{Seif}_i(X)$ and $Y_{i,k}$'s glued along their boundaries, where the fibers of $\operatorname{Seif}_i(X)$ over a boundary points $x \in (\partial X)_k$ are identified with $\partial \pi_{i,k}^{-1}(x) \approx S^1$.*

It should be remarked that in Theorem 1.5, the fiber of $\pi_{i,k} : Y_{i,k} \to (\partial X)_k$ may change at a corner point of $(\partial X)_k$ and that the type of $Y_{i,k}$ is less than or equal to half of the number of corner points in $(\partial X)_k$.

Corollary 1.6. *Under the same assumption and notation of Theorem 1.5, for large i, there exists a continuous surjection $f_i : M_i \to X$ which is a $\theta(i)$-approximation satisfying the following:*

1. *$f_i : f_i^{-1}(\operatorname{int} X) \to \operatorname{int} X$ is a generalized Siefert fibration.*
2. *For $x \in \partial X$, $f_i^{-1}(x)$ is homeomorphic to a one-point set or a circle. The fiber of f_i may change over a corner point in ∂X.*
3. *For any collar neighborhood $\varphi : (\partial X)_k \times [0,1] \to X$ of a component $(\partial X)_k$ of ∂X, which contains no interior essential singular points, $f_i^{-1}(\operatorname{image} \varphi)$ is a generalized solid torus or a generalized solid Klein bottle.*

Using the same notation as in Corollary 1.6, we remark that, for $x \in (\partial X)_k$,

$$f_i^{-1}(\varphi(\{x\} \times [0,1])) \approx D^2 \text{ if } f_i^{-1}(x) \approx \{\text{pt}\},$$
$$f_i^{-1}(\varphi(\{x\} \times [0,1])) \approx \text{Mö if } f_i^{-1}(x) \approx S^1.$$

The structure of M_i collapsing to one-dimensional space is determined as follows.

Theorem 1.7. *Let M_i^3 be a sequence of three-dimensional closed Alexandrov spaces with curvature ≥ -1 and* $\operatorname{diam} M_i \leq D$. *Suppose that M_i^3 converges to a circle. Then, for large i, M_i is homeomorphic to a total space of an F_i-fiber bundle over S^1, where the fiber F_i is homeomorphic to one of S^2, P^2, T^2 and K^2.*

To describe the structures of M_i converging to an interval I, we prepare certain topological orbifolds. First, we provide

$$B(\mathrm{pt}) := S^1 \times D^2/\langle\alpha\rangle.$$

Here, the involution α is the restriction of the one provided in Example 1.2. Remark that $\partial B(\mathrm{pt}) \approx S^2$. We also need to consider three-dimensional open Alexandrov spaces L_2 and L_4 with two-dimensional souls S_2 and S_4 respectively, where S_2 (resp. S_4) is homeomorpshic to S^2 or P^2 (resp. to S^2). For their definition, see Example 2.63. The space L_i $(i = 2, 4)$ has i topologically singular points, which are contained in S_i. We denote by $B(S_i)$ a metric ball around S_i in L_i. Here we point out that $\partial B(S_2) \approx S^2$ (resp.$\approx K^2$) if $S_2 \approx S^2$ (resp. if $S_2 \approx P^2$), and $\partial B(S_4) \approx T^2$.

Theorem 1.8. *Let M_i^3 be a sequence of three-dimensional closed Alexandrov spaces with curvature ≥ -1 and* $\operatorname{diam} M_i \leq D$. *Suppose that M_i^3 converges to an interval. Then, for large i, M_i is the union of $B_i \cup B_i'$ glued along their boundaries. ∂B_i is homeomorphic to one of S^2, P^2, T^2 and K^2. The topologies of B_i (and B_i') are determined as follows:*

(1) *If $\partial B_i \approx S^2$, then B_i is homeomorphic to one of D^3, $P^3 - \operatorname{int} D^3$, $B(S_2)$ with $S_2 \approx S^2$.*
(2) *If $\partial B_i \approx P^2$, then B_i is homeomorphic to $K_1(P^2)$.*
(3) *If $\partial B_i \approx T^2$, then B_i is homeomorphic to one of $S^1 \times D^2$, $S^1 \times$ Mö, $K^2 \tilde{\times} I$, and $B(S_4)$.*
(4) *If $\partial B_i \approx K^2$, then B_i is homeomorphic to one of $S^1 \tilde{\times} D^2$, $K^2 \hat{\times} I$, $B(\mathrm{pt})$, and $B(S_2)$ with $S_2 \approx P^2$.*

Corollary 1.9. *Let M_i be a sequence of three-dimensional closed Alexandrov spaces with curvature ≥ -1 and diameter $\leq D$. Suppose M_i converges to a point. Then, for large i, M_i is homeomorphic to one of*

- *generalized Seifert fiber spaces in the conclusion of Theorem* 1.3 *with a base Alexandrov surface having non-negative curvature,*
- *spaces in the conclusion of Theorem* 1.5 *with a base Alexandrov surface having non-negative curvature,*
- *spaces in the conclusion of Theorems* 1.7 *and* 1.8, *and*
- *closed Alexandrov spaces with non-negative curvature having finite fundamental groups.*

We remark that all spaces appearing in the conclusions of Theorems 1.3, 1.5, 1.7 and 1.8 and Corollary 1.9 actually have sequences of metrics as Alexandrov spaces collapsing to such respective limit spaces described there.

By Corollary 1.9, to achieve a complete classification of the topologies of collapsing three-dimensional closed Alexandrov spaces, we provide a version of the "Poincaré conjecture" for three-dimensional closed Alexandrov spaces with non-negative curvature.

For Alexandrov spaces A and A' having boundaries isometric to each other, $A \cup_\partial A'$ denotes the gluing of $A \cup A'$ via an isometry $\phi : \partial A \to \partial A'$. Note that $A \cup_\partial A'$ is an Alexandrov space (see [Pet Appl]).

Conjecture 1.10. *A simply connected three-dimensional closed Alexandrov space with non-negative curvature is homeomorphic to an isometric gluing $A \cup_\partial A'$ for A and A' chosen in the following list* (1.3) *of non-negatively curved Alexandrov spaces:*

$$D^3, K_1(P^2), B(\mathrm{pt}), B(S_2), B(S_4). \tag{1.3}$$

We also remark that any connected sum of those spaces admits a metric of Alexandrov space having a lower curvature bound by some constant.

Conjecture 1.11. *A simply connected three-dimensional closed Alexandrov space with curvature ≥ 1 is homeomorphic to a three-sphere S^3 or a suspension $\Sigma(P^2)$ over P^2.*

The organization of this paper and basic ideas of the proofs of our results are as follows:

In Section 2, we review some basic notation and results on Alexandrov spaces. We provide a three-dimensional topological orbifold having a circle fiber structure with singular arc fibers, and call it a generalized Seifert fiber space. At the end of this section, we prove fundamental properties on the topologically singular point set.

In Section 3, for any $n \in \mathbb{N}$, we consider n-dimensional closed Alexandrov spaces M_i^n collapsing to a space X^{n-1} of co-dimension one. Assume that all points in X are almost regular, except finite points $x_1, \dots, x_m$. For any fixed $p \in \{x_\alpha\}$, we take a sequence $p_i \in M_i$ converging to p. By Yamaguchi's Fibration Theorem 2.25, for large i, there is a fiber bundle $\pi_i : A_i \to A$, where A is a small metric annulus $A = A(p; r, R)$ around p and A_i is some corresponding domain. Here, r and R are small positive numbers so that $r \ll R$.

Although A_i is not a metric annulus in general, it is expected that A_i is homeomorphic to a standard annulus $A(p_i; r, R)$. Moreover, we may expect that there exist an isotopy $\phi : M_i \times [0,1] \to M_i$ such that, putting $\phi_t := \phi(\cdot, t)$,

$$\begin{cases} \phi_0 = id_{M_i}, \\ \phi_1\left(B\left(p_i, \dfrac{r+R}{2}\right) \cup A_i\right) = B(p_i, R), \text{ and} \\ \phi_1(x) = x \text{ if } x \notin B(p_i, R+\delta) \end{cases} \tag{1.4}$$

for any fixed $\delta > 0$.

If we consider the case that all M_i are Riemannian manifolds, then we can obtain a smooth flow Φ_t of a gradient-like vector field V of the distance function dist_{p_i} from p_i. Then, by using integral curves of V, we can obtain such an isotopy ϕ from id_{M_i} satisfying the property (1.4).

We will prove that such an argument of flow goes through on Alexandrov spaces M_i as well. To do this, we first prove a main result, Flow Theorem 3.2, in this section. Theorem 3.2 implies the existence of an integral flow Φ_t of a gradient-like vector field of a distance function dist_{p_i} on $A(p_i; r, R)$ in a suitable sense. This flow leads to an isotopy ϕ satisfying the property (1.4). Theorem 3.2 is important throughout the paper.

In Sections 4 – 8, we prove Theorems 1.3, 1.5, 1.7 and 1.8 and Corollaries 1.6 and 1.9. To explain the arguments used in those proofs, let us fix a sequence $M_i = M_i^3$ of three-dimensional closed Alexandrov spaces in $\mathcal{A}^3(-1, D)$ converging to X of dimension ≤ 2.

In Section 4, we consider the case that $\dim X = 2$ and $\partial X = \emptyset$. Let $p_1, \dots, p_m$ be all δ-singular points in X for a fixed small $\delta > 0$. Let us take a converging sequence $p_{i,\alpha} \to p_\alpha$ $(i \to \infty)$ for each $\alpha = 1, \dots, m$. Let us fix any α and set $p := p_\alpha$, $p_i := p_{i,\alpha}$. We take $r = r_p > 0$ such that all points in $B(p, 2r) - \{p\}$ are $(2, \varepsilon)$-strained. Then, all points in an annulus $A(p_i; \varepsilon_i, 2r - \varepsilon_i)$ are $(3, \theta(i, \varepsilon))$-strained. Here, ε_i is a sequence of positive numbers converging to zero. Then, by Fibration Theorem 2.25, we have an S^1-fiber bundle $\pi_i : A_i \to A(p; r, 2r)$. On the other hand, by the rescaling argument 2.27, we obtain the conclusion that $B_i := B(p_i, r)$ is homeomorphic to a solid torus or $B(\text{pt})$. Here, we can exclude the possibility that B_i is topologically a solid Klein bottle. Theorem 3.2 implies that there exists an isotopy carrying the fiber $\pi_i^{-1}(\partial B(p, r))$ to ∂B_i. If $B_i \approx S^1 \times D^2$ then we can prove an argument similar to [SY00] that B_i has the structure of a Seifert fibered torus in the usual sense, extending π. If $B_i \approx B(\text{pt})$, then by some new observation on the topological structure of $B(\text{pt})$, we can prove that B_i has the standard "circle fibration" structure provided in Example 1.2, compatible with π. In this way, we obtain the structure of a generalized Seifert fiber space on M_i.

In Section 5, we consider the case that $\dim X = 2$ and $\partial X \neq \emptyset$. Take a decomposition of ∂X to connected components $\bigcup_\beta (\partial X)_\beta$. Put $X_0 := X - U(\partial X, r)$ for some small $r > 0$. By Theorem 1.3, we have a generalized Seifert fibration $\pi_i : M_{i,0} \to X_0$ for some closed domain $M_{i,0} \subset M_i$. For any fixed β, we take points p_α in $(\partial X)_\beta$ so fine that $\{p_\alpha\}$ contains all ε-singular points in $(\partial X)_\beta$. Let $p_{i,\alpha} \in M_i$ be a sequence converging to p_α. Deform a metric ball $B(p_{i,\alpha}, r)$ to a neighborhood $B_{i,\alpha}$ of $p_{i,\alpha}$ by an isotopy obtained in Theorem 3.2. Because of the existence of ∂X, we need a bit more complicated construction of flows of gradient-like vector fields of distance functions.

In Section 6, we consider the case that X is isometric to a circle $S^1(\ell)$ of length ℓ. If M_i has no ε-singular points, by Fibration Theorem 2.25, we obviously obtain the conclusion of Theorem 1.7. But, in general, M_i has ε-singular points. Therefore, we use Perelman's Morse theory to construct a fibration over S^1.

In Section 7, we consider the case that X is isometric to an interval $[0, \ell]$ of some length ℓ. We use rescaling arguments around the end points of interval X and an argument similar to Theorem 1.7 to prove Theorem 1.8.

In Section 8, we consider the case of $\dim X = 0$ and prove Corollary 1.9.

For three-dimensional Alexandrov spaces with non-empty boundary collapsing to lower dimensional spaces, considering their doubles, one could make use of the results in the present paper to obtain the structure of collapsing in that case. This will appear in a forthcoming paper.

2. Preliminaries

2.1. **Definitions, conventions and notation.** In the present paper, we use the following notation.

- $\theta(\delta)$ is a function depending on $\delta = (\delta_1, \dots, \delta_k)$ such that $\lim_{\delta \to 0} \theta(\delta) = 0$. $\theta(i, \delta)$ is a function depending on $\delta \in \mathbb{R}^k$ and $i \in \mathbb{N}$ such that $\lim_{i \to \infty, \delta \to 0} \theta(i, \delta) = 0$. When we write $A < \theta(\delta)$ for a non-negative number A, we always assume that $\theta(\delta)$ is taken to be non-negative.
- $X \approx Y$ means that X is homeomorphic to Y. For metric spaces X and Y, $X \equiv Y$ means that X is isometric to Y.

- For metric spaces X and Y, the direct product $X \times Y$ has the product metric if nothing is stated.
- For continuous mappings $f_1 : X_1 \to Y$, $f_2 : X_2 \to Y$ and $g : X_1 \to X_2$, we say that g *represents* f_1 *and* f_2 if $f_1 = f_2 \circ g$ holds.
- Denote by $d(x, y)$, $|x, y|$, and $|xy|$ the distance between x and y in a metric space X. Sometimes we mark X as lower index $|x, y|_X$.
- For a subset S of a topological space, $\overline{S}$ is the closure of S in the whole space.
- For a metric space $X = (X, d)$ and $r > 0$, denote the rescaling metric space $rX = (X, rd)$.
- For a subset Y of a metric space, denote by dist_Y the distance function from Y. When $Y = \{x\}$ we denote $\mathrm{dist}_x := \mathrm{dist}_{\{x\}}$. For a subset Y of a metric space X and a subset I of $\mathbb{R}_+$, define a subset $B(Y; I) := B_X(Y; I) := \mathrm{dist}_Y^{-1}(I) \subset X$. For special cases, we denote and call those sets in the following way: $B(Y, r) := B(Y; [0, r])$ the closed ball, $U(Y, r) := B(Y; [0, r))$ the open ball, $A(Y; r', r) := B(Y; [r', r])$ the annulus, and $\partial B(Y, r) := B(Y; \{r\})$ the metric sphere. For $Y = \{x\}$, we set $B(x, r) := B(\{x\}, r)$, $U(x, r) := U(\{x\}, r)$ and $A(x; r', r) := A(\{x\}; r', r)$.
- For a topological space X, the *cone* $K(X)$ over X is obtained from $X \times [0, \infty)$ by smashing $X \times \{0\}$ to a point. An equivalent class $[(x, a)] \in K(X)$ of $(x, a) \in X \times [0, +\infty)$ is denoted by ax or often simply written by (x, a). A special point $(x, 0) = 0x \in K(X)$ is denoted by o or o_X, called the *origin* of $K(X)$. A point $v \in K(X)$ is often called a vector. $K_1(X)$ denotes the (unit) closed cone over X, i.e.
$$K_1(X) := \{ax \in K(X) \,|\, x \in X, 0 \le a \le 1\}.$$
$K_1(X)$ is homeomorphic to the join between X and a single-point.
- For a metric space X, $K(X)$ often denotes the *Euclidean metric cone*, which is equipped with the following metric: for two points $(x_1, r_1), (x_2, r_2) \in X \times [0, \infty)$ the distance between them is defined by
$$d((x_1, r_1), (x_2, r_2))^2 := r_1^2 + r_2^2 - 2r_1 r_2 \cos \min\{d(x_1, x_2), \pi\}.$$
And for $v \in K(X)$, we put $|v| := d(x, o)$ and call it the *norm* of v. Define an *inner product* $\langle v, w\rangle$ of $v, w \in K(X)$ by $\langle v, w\rangle := |v||w| \cos \angle vow$.
- When we write M^n marked upper index n, this means that M is an n-dimensional Alexandrov space.

For a curve $\gamma : [0, 1] \to X$ in a metric space X, the *length* $L(\gamma)$ of γ is defined by
$$L(\gamma) := \sup_{0=t_0<t_1<\cdots<t_m=1} \sum_{i=1}^{m} d(\gamma(t_{i-1}), \gamma(t_i)) \in [0, +\infty].$$
A metric space X is called a *length space* if for any x, $y \in X$ and $\varepsilon > 0$, there exists a curve $\gamma : [0, 1] \to X$ such that $\gamma(0) = x$, $\gamma(1) = y$ and $0 \le L(\gamma) - d(x, y) \le \varepsilon$. A curve is called a *geodesic* if it is an isometric embedding from some interval. Sometimes a geodesic γ defined on a bounded closed interval $[0, \ell]$ is called a geodesic *segment*. A geodesic defined on $\mathbb{R}$ is called a *line*; a geodesic defined on $[0, +\infty)$ is called a *ray*. For a geodesic $\gamma : I \to X$ in a metric space X, we often regard γ itself as the subset $\gamma(I) \subset X$.

2.2. **Alexandrov spaces.** From now on, throughout this paper, we always assume that a metric space is *proper*, namely, any closed bounded subset is compact. A proper length space is a *geodesic space*, namely, any two points are jointed by a geodesic.

For three points x_0, x_1, x_2 in a metric space, the *size* of (x_0, x_1, x_2) is size $(x_0, x_1, x_2) := |x_0x_1| + |x_1x_2| + |x_2x_0|$. The *size* of four points $(x_0; x_1, x_2, x_3)$ (centered at x_0) is defined by the maximum of $\mathrm{size}\,(x_0, x_i, x_j)$ for $1 \leq i \neq j \leq 3$, denoted by $\mathrm{size}\,(x_0; x_1, x_2, x_3)$.

Definition 2.1. For three points x_0, x_1, x_2 in a metric space X with $\mathrm{size}\,(x_0, x_1, x_2) < 2\pi/\sqrt{\kappa}$, the *$\kappa$-comparison angle* of $(x_0; x_1, x_2)$, written by $\tilde{\angle}_\kappa x_1x_0x_2$ or $\tilde{\angle}_\kappa(x_0; x_1, x_2)$, is defined as follows: Take three points $\tilde{x}_i$ $(i = 0, 1, 2)$ in κ-plane $\mathbb{M}^2_\kappa$, which is a simply connected complete surface with constant curvature $= \kappa$, such that $d(x_i, x_j) = d(\tilde{x}_i, \tilde{x}_j)$ for $0 \leq i, j \leq 2$ and put $\tilde{\angle}_\kappa x_1x_0x_2 := \angle\tilde{x}_1\tilde{x}_0\tilde{x}_2$. Sometimes we write $\tilde{\angle}$ omitting κ in the notation $\tilde{\angle}_\kappa$.

Definition 2.2. For $\kappa \in \mathbb{R}$, a complete metric space X is called an *Alexandrov space with curvature* $\geq \kappa$ if X is a length space and, for every four points $x_0, x_1, x_2, x_3 \in X$ (with $\mathrm{size}\,(x_0; x_1, x_2, x_3) < 2\pi/\sqrt{\kappa}$ if $\kappa > 0$), we have the inequality

$$\tilde{\angle}_\kappa x_1x_0x_2 + \tilde{\angle}_\kappa x_2x_0x_3 + \tilde{\angle}_\kappa x_3x_0x_1 \leq 2\pi.$$

The *dimension* of an Alexandrov space means its Hausdorff dimension. The Hausdorff dimension and the topological dimension are equal to each other ([BGP], [PP QG], [Pl]). Throughout this paper, we always assume that an Alexandrov space is finite dimensional.

Remark 2.3. If X is an Alexandrov space with curvature $\geq \kappa$, then the rescaling space rX is an Alexandrov space with curvature $\geq \kappa/r^2$.

For two geodesics $\alpha, \beta : [0, \varepsilon] \to X$ emanating at $\alpha(0) = \beta(0) = p \in X$ in an Alexandrov space X, the *angle* $\angle(\alpha, \beta)$ at p is defined by

$$\angle(\alpha, \beta) := \angle_p(\alpha, \beta) := \lim_{s,t\to 0} \tilde{\angle}(p; \alpha(t), \beta(s)).$$

The set of all non-trivial geodesics emanating at p in an Alexandrov space X is denoted by Σ'_pX. The angle $\angle_p$ at p satisfies the triangle inequality on this set. Its metric completion is denoted by $\Sigma_p = \Sigma_pX$, called the *space of directions* at p. For a geodesic $\gamma : [0, \ell] \to X$ starting from $x = \gamma(0)$ to $y = \gamma(\ell)$, we denote $\gamma^+(0) = \gamma'(0) = \gamma'_x = \gamma^+_x = \uparrow^y_x$ the direction of γ at x. By xy, we denote some segment $xy = \gamma : [0, |xy|] \to X$ joining from $\gamma(0) = x$ to $\gamma(|xy|) = y$. For a subset $A \subset X$, the closure of a set of all directions from x to A is denoted by A'_x, i.e.

$$A'_x := \{\xi \in \Sigma_x \,|\, \exists a_i \in A \text{ such that } \lim_{i\to\infty} |xa_i| = |x, A| \text{ and } \lim_{i\to\infty} \uparrow^{a_i}_x = \xi\}.$$

When $x \in A$, we put $\Sigma_x(A) := A'_x$. For $x, y \in X$, we denote as $y'_x := \{y\}'_x$. Or sometimes we denote by y'_x an element that belongs with y'_x. For $x \in X$ and $y, z \in X - \{x\}$, we denote by $\angle yxz$ the angle $\angle(xy, xz) = \angle(\uparrow^y_x, \uparrow^z_x)$ between some fixed segments xy, xz.

Definition 2.4. A (k,δ)-*strainer* at $x \in M$ is a collection of points $\{p_\alpha^\pm\}_{\alpha=1}^k = \{p_\alpha^+, p_\alpha^- \mid \alpha = 1,\dots,k\}$ satisfying the following:

$$\tilde{\angle} p_\alpha^+ x p_\beta^+ > \pi/2 - \delta \tag{2.1}$$

$$\tilde{\angle} p_\alpha^+ x p_\beta^- > \pi/2 - \delta \tag{2.2}$$

$$\tilde{\angle} p_\alpha^- x p_\beta^- > \pi/2 - \delta \tag{2.3}$$

$$\tilde{\angle} p_\alpha^+ x p_\alpha^- > \pi - \delta \tag{2.4}$$

for all $1 \le \alpha \ne \beta \le k$.

The *length* of a strainer $\{p_\alpha^\pm\}$ at x is $\min_{1\le\alpha\le k}\{|p_\alpha^+, x|, |p_\alpha^-, x|\}$. The (k,δ)-*strained radius* of x, denoted by (k,δ)-str.rad x, is the supremum of lengths of (k,δ)-strainers at x. A (k,δ)-strained radius (k,δ)-str.rad A of a subset $A \subset M$ is defined by

$$(k,\delta)\text{-str.rad}\, A := \inf_{x\in A}\, (k,\delta)\text{-str.rad}\, x.$$

If there is a (k,δ)-strainer at x, then x is called (k,δ)-*strained*. Denote by $R_{k,\delta}(M)$ the set of all (k,δ)-strained points in M. $R_{k,\delta}(M)$ is an open subset. Put $S_{k,\delta}(M) := M - R_{k,\delta}(M)$. Any point in $S_{k,\delta}(M)$ is called a (k,δ)-*singular point*. When we consider an n-dimensional Alexandrov space M^n and δ is sufficiently small with respect to $1/n$, we simply say δ-strained, δ-singular, etc., instead of (n,δ)-strained, (n,δ)-singular, etc., and we omit writing $R_\delta(M)$, $S_\delta(M)$ instead of $R_{n,\delta}(M)$, $S_{n,\delta}(M)$. For an n-dimensional Alexandrov space M^n, put $R(M^n) := \bigcap_{\delta>0} R_\delta(M^n)$ and $S(M^n) := \bigcup_{\delta>0} S_\delta(M^n) = M^n - R(M^n)$.

Theorem 2.5 ([BGP], [OS])**.** *For any n-dimensional Alexandrov space M^n, we have* $\dim_H S(M) \le n-1$ *and* $\dim_H S(M) - \partial M \le n-2$.

Here, the boundary ∂M of an Alexandrov space M is defined inductively in the following manner.

Definition 2.6. A one-dimensional Alexandrov space M^1 is a manifold, and the boundary of M^1 is the boundary of M^1 as a manifold. Now let M^n be an n-dimensional Alexandrov space with $n > 1$. A point p in M^n is called a *boundary point* if Σ_p has a boundary point. The set of all boundary points is denoted by ∂M^n, called the *boundary* of M^n. Its complement is denoted by $\operatorname{int} M^n = M^n - \partial M^n$, called the *interior* of M^n. A point in $\operatorname{int} M^n$ is called an *interior point* of M^n. ∂M^n is a closed subset in M^n ([BGP], [Per II]).

A compact Alexandrov space without boundary is called a *closed* Alexandrov space, and a non-compact Alexandrov space without boundary is called an *open* Alexandrov space.

Definition 2.7. For an n-dimensional Alexandrov space M^n, we say that $p \in M$ is a *topologically regular point* (or a *manifold-point*) if there is a neighborhood of p which is homeomorphic to $\mathbb{R}^n$ or $\mathbb{R}^{n-1} \times [0,\infty)$. p is called a *topologically singular point* if p is not a topologically regular point. We denote by $S_{\mathrm{top}}(M)$ the set of all topologically singular points.

Definition 2.8. For an Alexandrov space M, a point $p \in M$ is called an *essential singular point* if $\operatorname{rad} \Sigma_p \le \pi/2$. A set of whole essential singular points in M is denoted by $\operatorname{Ess}(M)$. We define the set of interior (resp. boundary) essential singular points $\mathrm{Ess}(\mathrm{int}\, M)$ (resp. $\mathrm{Ess}(\partial M)$) as follows:

$$\mathrm{Ess}(\mathrm{int}\, M) := \mathrm{Ess}(M) \cap \mathrm{int}\, M,$$
$$\mathrm{Ess}(\partial M) := \mathrm{Ess}(M) \cap \partial M.$$

Remark that if $\dim M = 1$, then $\mathrm{Ess}(\mathrm{int}\, M) = \emptyset$ and $\mathrm{Ess}(\partial M) = \partial M$.

Remark 2.9. By Theorem 2.36 and Stability Theorem 2.34, we can check the following:

$$S_{\mathrm{top}}(M) \subset \mathrm{Ess}(M) \subset S(M).$$

For small $\delta \ll 1/n$, any (n, δ)-regular point in an n-dimensional Alexandrov space M^n is an interior point.

Theorem 2.10 ([BGP, Corollary 12.8]). *An $(n-1, \delta)$-regular interior point in an n-dimensional Alexandrov space is an (n, δ')-regular point. Here, $\delta' \to 0$ as $\delta \to 0$.*

The boundary of an Alexandrov space is determined by its topology:

Theorem 2.11 ([BGP, Theorem 13.3(a)], [Per II]). *Let M_1, M_2 be n-dimensional Alexandrov spaces with homeomorphism $\phi : M_1 \to M_2$. Then $\phi(\partial M_1) = \partial M_2$.*

2.3. **The Gromov-Hausdorff convergence.** For metric spaces X and Y, and $\varepsilon > 0$, an *ε-approximation* f from X to Y is a map $f : X \to Y$ such that

(1) $|d(x, x') - d(f(x), f(x'))| \le \varepsilon$ for any $x, x' \in X$,
(2) $Y = B(\mathrm{Image}\,(f), \varepsilon)$.

The *Gromov-Hausdorff distance* $d_{GH}(X, Y)$ between X and Y is defined by the infimum of those $\varepsilon > 0$ so that there exist ε-approximations from X to Y and from Y to X. We say that a sequence of metric spaces X_i, $i = 1, 2, \dots$, converges to a metric space X as $i \to \infty$ if $d_{GH}(X_i, X) \to 0$ as $i \to \infty$.

For two pointed metric spaces (X, x), (Y, y), a *pointed ε-approximation* f from (X, x) to (Y, y) is a map $f : B_X(x, 1/\varepsilon) \to Y$ such that

(1) $f(x) = y$,
(2) $|d(x', x'') - d(f(x'), f(x''))| \le \varepsilon$ for $x', x'' \in B_X(x, 1/\varepsilon)$,
(3) $B_Y(y, 1/\varepsilon) \subset B(\mathrm{Image}\,(f), \varepsilon)$.

The *pointed Gromov-Hausdorff distance* $d_{GH}((X, x), (Y, y))$ between (X, x) and (Y, y) is defined by the infimum of those $\varepsilon > 0$ so that there exist pointed ε-approximations from (X, x) to (Y, y) and from (Y, y) to (X, x).

For an n-dimensional Alexandrov space X^n, the (Gromov-Hausdorff) *tangent cone T_xX of X at x* is defined by the pointed Gromov-Hausdorff limit of $(1/r_iX, x)$ for some sequence (r_i) converging to zero. Thus, T_xX is an n-dimensional noncompact Alexandrov space with non-negative curvature. Also, T_xX is isometric to the metric cone $K(\Sigma_x)$ over the space of directions Σ_x.

For a locally Lipschitz map $f : X \to M$ between Alexandrov spaces and a curve $\gamma : [0, a] \to X$ starting at $p = \gamma(0)$ with direction γ^+ at p, we say that f has the *directional derivative $df(\gamma^+)$ in the direction γ^+* if there exists the limit

$$df(\gamma^+) := (f \circ \gamma)^+ := \frac{d}{dt} f \circ \gamma(0+).$$

A distance function on an Alexandrov space has the directional derivative in any direction.

For a local Lipschitz function f on a metric space, the *absolute gradient* $|\nabla f|_p$ of f at p is defined by

$$|\nabla f|_p := |\nabla f|(p) := \max\left\{ \limsup_{x\to p} \frac{f(x)-f(p)}{d(x,p)}, 0 \right\}.$$

Definition 2.12. f is called *regular* at p if $|\nabla f|_p > 0$. Such a point p is a *regular point* for f. Otherwise, f is called *critical* at p.

Let X be an Alexandrov space and U be an open subset of X. Let $f : U \to \mathbb{R}$ be a locally Lipschitz function. For $\lambda \in \mathbb{R}$, f is said to be λ-*concave* if for every segment $\gamma : [0,\ell] \to U$, the function

$$f\circ\gamma(t) - \frac{\lambda}{2}t^2$$

is concave in t. A 0-concave function is said to be concave. f is said to be *semi-concave* if for every $x \in U$ there are an open neighborhood V of x in U and a constant $\lambda \in \mathbb{R}$ such that $f|_V$ is λ-concave.

For a semiconcave function f on a finite dimensional Alexandrov space, the gradient vector ∇f of f is defined in the tangent cone:

Definition 2.13 ([PP QG])**.** Let X be a finite dimensional Alexandrov space. Let $f : U \to \mathbb{R}$ be a semiconcave function defined on an open neighborhood U of p. A vector $v \in T_pX$ is called the *gradient* of f at p if the following hold:

(i) For any $w \in T_pX$, we have $d_pf(w) \le \langle v, w\rangle$.
(ii) $d_pf(v) = |v|^2$.

The gradient of f at p is denoted by $\nabla_p f$ for short.

Remark that $\nabla_p f$ is uniquely determined in the following manner: If $|\nabla f|_p = 0$, then $\nabla_p f = o_p$, and otherwise

$$\nabla_p f = d_pf(\xi_{\max})\xi_{\max},$$

where $\xi_{\max} \in \Sigma_p$ is the uniquely determined unit vector such that $d_pf(\xi_{\max}) = \max_{\xi\in\Sigma_p} d_pf(\xi)$.

We can show that the absolute gradient $|\nabla f|(p)$ of f is equal to the norm $|\nabla_p f|$ of gradient vector $\nabla_p f$ in T_pX.

2.4. **Ultraconvergence.** We will recall the notion of ultrafilters and ultralimits. For more details, we refer to [BH]. A (non-principle) *ultrafilter* ω on the set of natural numbers $\mathbb{N}$ is a finitely additive measure on the power set $2^{\mathbb{N}}$ of $\mathbb{N}$ that has values 0 or 1 and contains no atoms. For each sequence $\{y_i\} = \{y_i\}_{i\in\mathbb{N}}$ in a compact Hausdorff space Y, an *ultralimit* $\lim_\omega y_i = y \in Y$ of this sequence is uniquely determined by the requirement $\omega(\{i \in \mathbb{N} \mid y_i \in U\}) = 1$ for all neighborhoods U of y. If $f : Y \to Z$ is a continuous map between topological spaces, then $\lim_\omega f(y_i) = f(\lim_\omega y_i)$.

For a sequence $\{(X_i, x_i)\}$ of pointed metric spaces, consider the set of all sequences $\{y_i\}$ of points $y_i \in X_i$ with $\lim_\omega |x_iy_i| < \infty$ and provide the pseudometric $|\{y_i\}\{z_i\}| = \lim_\omega |y_iz_i|$ on the set. The *ultralimit* $(X,x) = \lim_\omega(X_i,x_i)$ of $\{(X_i,x_i)\}$ is defined to be the metric space arising from this pseudometric, and the equivalence class of a sequence $\{y_i\}$ is denoted by (y_i). The ultralimit of a constant sequence $\{(X,x)\}$ of a metric space (X,x) is called the *ultrapower* of (X,x) and is denoted by

$X^\omega = (X^\omega, x)$. The natural map $X \ni y \mapsto (y) = (y, y, y, \dots) \in X^\omega$ is an isometric embedding.

We review a relation between the ultraconvergence and the usual convergence. A sequence (ε_i) of positive numbers is said to be a *scale* if $\lim_{i\to\infty} \varepsilon_i = 0$.

Lemma 2.14. *For a real number A and a function $h : \mathbb{R}_+ \to \mathbb{R}$, the following are equivalent:*

(*i*) $\liminf_{t\searrow 0} h(t) \geq A$.

(*ii*) *For any scale $(o) = (t_i)$, we have* $\lim_\omega h(t_i) \geq A$.

Proof. ((*i*) $\Rightarrow$ (*ii*)). We assume (*i*). Then, for any $\varepsilon > 0$, there is $t_0 > 0$ such that

$$\inf_{0<t\leq t_0} h(t) > A - \varepsilon.$$

Let us take any scale (t_i). Then there is i_0 such that, for all $i \geq i_0$, we have

$$h(t_i) \geq \inf_{0<t\leq t_0} h(t).$$

Therefore, taking an ultralimit, we have

$$\lim_\omega h(t_i) \geq A - \varepsilon.$$

The above inequality holds for all $\varepsilon > 0$. Then we obtain (*ii*).

((*ii*) $\Rightarrow$ (*i*)). We assume (*ii*). We take a sequence (t_i) tending to 0 such that

$$\lim_{i\to\infty} h(t_i) = \liminf_{t\searrow 0} h(t).$$

Then, taking an ultralimit, we obtain (*i*):

$$A \leq \lim_\omega h(t_i) = \lim_{i\to\infty} h(t_i) = \liminf_{t\searrow 0} h(t).$$

□

Let (X_i, x_i) and (Y_i, y_i) be sequences of pointed metric spaces and let $f_i : (X_i, x_i) \to (Y_i, y_i)$ be a sequence of maps. Then the *ultralimit* $f_\omega = \lim_\omega f_i$ of $\{f_i\}$ is defined by

$$\lim_\omega X_i \ni a_\omega = (a_i) \mapsto f_\omega(a_\omega) := (f_i(a_i)) \in \lim_\omega Y_i$$

if it is well-defined. For instance, if f_i is an L_i-Lipschitz map with $L_\omega := \lim_\omega L_i < \infty$, then the ultralimit f_ω is well-defined and L_ω-Lipschitz. If $f_i : (X_i, x_i) \to (Y_i, y_i)$ is a pointed τ_i-approximation with $\tau_\omega := \lim_\omega \tau_i < \infty$, then the ultralimit f_ω is well-defined and a τ_ω-approximation. Remark that if $f_i : (X_i, x_i) \to (Y_i, y_i)$ and $g_i : (Y_i, y_i) \to (Z_i, z_i)$ have the ultralimits $f_\omega := \lim_\omega f_i$ and $g_\omega := \lim_\omega g_i$, then $\lim_\omega (g_i \circ f_i) = g_\omega \circ f_\omega$. For $a_\omega = (a_i), a'_\omega = (a'_i) \in \lim_\omega X_i$, we have $|f_\omega(a_\omega), f_\omega(a'_\omega)| = \lim_\omega |f_i(a_i), f_i(a'_i)|$.

For a pointed metric space (X, x) and a scale $(o) = (\varepsilon_i)$, we define the *blow-up* $X_x^{(o)} = (X_x^{(o)}, o_x)$ of (X, x) by

$$(X_x^{(o)}, o_x) := \lim_\omega (1/\varepsilon_i X, x).$$

For a map $f : (X, x) \to (Y, y)$ between pointed metric spaces, we consider a sequence $\{f_i\}$ of maps defined by

$$f_i = f : (1/\varepsilon_i X, x) \to (1/\varepsilon_i Y, y).$$

The *blow-up* $f_x^{(o)} : X_x^{(o)} \to Y_y^{(o)}$ of f is defined by $f_x^{(o)} := \lim_\omega f_i$ if it is well-defined.

Let X be an Alexandrov space and $x \in X$, and let $(o) = (\varepsilon_i)$ be a scale. We consider the exponential map at x:

$$\exp_x : (\mathrm{dom}(\exp_x), o_x) \ni (\gamma, t) \mapsto \exp_x(\gamma, t) := \gamma(t) \in (X, x).$$

Here, $\mathrm{dom}(\exp_x) \subset T_xX$ is the domain of $\exp_x$. Since $\exp_x$ is locally Lipschitz, the blow-up of $\exp_x$ is well-defined and written by

$$\exp_x^{(o)} := (\exp_x)_{o_x}^{(o)} : (T_xX, o_x) \to (X_x^{(o)}, o_x).$$

The domain of $\exp_x^{(o)}$ is the blow-up of $(\mathrm{dom}(\exp_x), o_x)$, which is identified as (T_xX, o_x).

Lemma 2.15 ([L], [BGP]). *Let $(o) = (\varepsilon_i)$ be an arbitrary scale.*

- (i) *Let X be a (possibly infinite dimensional) Alexandrov space. Then $\exp_x^{(o)}$ is an isometric embedding.*
- (ii) *If X be a finite dimensional Alexandrov space. Then $\exp_x^{(o)} : K(\Sigma_x) \to X_x^{(o)}$ is surjective, for any $x \in X$.*

Proof. (i) By the definition of the angle between geodesics, for any (γ, s) and $(\eta, t) \in \Sigma'_x \times [0, \infty)$, we have

$$\frac{|\gamma(s\varepsilon_i), \eta(t\varepsilon_i)|_X}{\varepsilon_i} \stackrel{i\to\infty}{\longrightarrow} |s\gamma, t\eta|_{K(\Sigma_x)}.$$

(ii) By [BGP], the Gromov-Hausdorff tangent cone T_xX and the cone $K(\Sigma_x)$ over a space of directions are isometric to each other. More precisely, the scaled logarithmic map

$$\log_x = \exp_x^{-1} : \left(\frac{1}{\varepsilon_i}X, x\right) \to \left(\frac{1}{\varepsilon_i}T_xX, o_x\right)$$

is a τ_i-approximation for some sequence $\{\tau_i\}$ of positive numbers converging to zero, and $\exp_x \circ \log_x = id$. Then we have, for each $(x_i) \in X_x^{(o)}$,

$$\exp_x^{(o)}(\log_x(x_i)) = (\exp_x \circ \log_x(x_i)) = (x_i).$$

Therefore, $\exp_x^{(o)}$ is surjective. □

2.5. **Preliminaries from the geometry of Alexandrov spaces.** In this subsection, we review the basic facts on the geometry and topology of Alexandrov spaces. We refer mainly to [BGP], [Per II].

2.5.1. *Local structure around an almost regular point.* Burago, Gromov and Perelman proved that a neighborhood of an almost regular point is almost isometric to an open subset of Euclidean space.

Theorem 2.16 ([BGP], [OS]). *For $n \in \mathbb{N}$, there exists a positive number $\delta_n > 0$ satisfying the following: Let X be an n-dimensional Alexandrov space with curvature ≥ -1. For $0 < \delta \leq \delta_n$, if $x \in X$ is an (n, δ)-strained point with a strainer $\{p_\alpha\}_{\alpha=\pm1,\dots,\pm n}$ of length ℓ, then the two maps*

$$\varphi := (d(p_\alpha, \cdot))_{\alpha=1,\dots,n} \tag{2.5}$$

$$\tilde\varphi := \left(\frac{1}{\mathcal{H}^n(B(p_\alpha, r))}\int_{B(p_\alpha,\varepsilon)} d(y, \cdot)\, d\mathcal{H}^n(y)\right)_{\alpha=1,\dots,n} \tag{2.6}$$

on $B(x,r)$ for small $r > 0$ are both $(\theta_n(\delta) + \theta_n(r/\ell))$-almost isometries, where ε is so small with $\varepsilon \ll r/\ell$. Here, $\theta_n(\delta)$ is a positive function depending on n and δ such that $\lim_{\delta\to 0}\theta_n(\delta) = 0$.

Lemma 2.17 ([Y conv, Lemma 1.8])**.** *Let M be an n-dimensional Alexandrov space and δ be taken in Theorem 2.16. For any (n,δ)-strained point $p \in M$, there exists $r > 0$ satisfying the following: For every $q \in B(p,r/2)$ and $\xi \in \Sigma_q$ there exists $x, y \in B(p,r)$ such that*

$$|xq|, |yq| \geq r/4, \tag{2.7}$$

$$|x'_q, \xi| \leq \theta(\delta, r), \tag{2.8}$$

$$\tilde{\angle} xqy \geq \pi - \theta(\delta, r). \tag{2.9}$$

Lemma 2.18 ([Y conv, Lemma 1.9])**.** *Let M, p, r and δ be taken in Lemma 2.17. For every $q \in M$ with $r/10 \leq |pq| \leq r$ and for every $x \in M$ with $|px| \ll r$, we have*

$$|\angle xpq - \tilde{\angle} xpq| < \theta(\delta, r, |px|/r).$$

2.5.2. *Splitting Theorem.* The Splitting Theorem is an important tool to study the structure of non-negatively curved spaces.

Theorem 2.19 (Splitting Theorem [Milka])**.** *Let X be an Alexandrov space of curvature ≥ 0. Suppose that there exists a line $\gamma : \mathbb{R} \to X$. Then there exists an Alexandrov space Y of curvature ≥ 0 such that X is isometric to the product $Y \times \mathbb{R}$.*

Theorem 2.20. *If an Alexandrov space Σ of curvature ≥ 1 has the maximal diameter π, then Σ is isometric to the metric suspension $\Sigma(\Lambda)$ over some Alexandrov space Λ of curvature ≥ 1.*

Corollary 2.21. *If an n-dimensional Alexandrov space Σ of curvature ≥ 1 has the maximal radius π, then Σ is isometric to a unit n-sphere of constant curvature $= 1$.*

Remark 2.22 ([M]). The Splitting Theorem and Corollary 2.21 hold even for infinite dimensional Alexandrov spaces.

2.5.3. *Convergence and collapsing theory.* Yamaguchi proved the following two theorems (Theorems 2.24 and 2.25) for Alexandrov spaces converging to an almost regular Alexandrov space, which are counterparts of the Fibration Theorem [Y91] in the Riemannian geometry.

Definition 2.23. A surjective map $f : X \to Y$ between Alexandrov spaces is called an ε-*almost Lipschitz submersion* if f is an ε-approximation, and for any $x, y \in X$ setting $\theta := \angle_x(y'_x, \Sigma_x\Pi_x)$, we have

$$\left| \frac{|f(x)f(y)|}{|xy|} - \sin\theta \right| < \varepsilon$$

where $\Pi_x := f^{-1}(f(x))$.

A surjective map $f : X \to Y$ is called an ε-*almost isometry* if for any x, $y \in X$ we have

$$\left| \frac{|f(x)f(y)|}{|xy|} - 1 \right| < \varepsilon.$$

Theorem 2.24 (Lipschitz submersion theorem [Y conv])**.** *For $n \in \mathbb{N}$ and $\eta > 0$, there exist δ_n, $\varepsilon_n(\eta) > 0$ satisfying the following. Let M^n, X^k be Alexandrov spaces with curvature ≥ -1, $\dim M^n = n$, and $\dim X^k = k$. Suppose that δ-strain* *radius of $X > \eta$. Then if the Gromov-Hausdorff distance between M and X is less than $\varepsilon \le \varepsilon_n(\eta)$, there is a $\theta(\delta,\varepsilon)$-almost Lipschitz submersion $f : M \to X$. Here, $\theta(\delta,\varepsilon)$ denotes a positive constant depending on n, η and δ, ε and satisfying* $\lim_{\delta,\varepsilon\to 0}\theta(\delta,\varepsilon) = 0$.

When M is almost regular (and X has non-empty boundary), Theorem 2.24 deforms as Theorem 2.25 below. Let X be a k-dimensional complete Alexandrov space with curvature ≥ -1 having nonempty boundary. Let X^* be another copy of X. Take the double $\mathrm{dbl}\,(X) = X \cup X^*$ of X. The double $\mathrm{dbl}\,(X)$ is also an Alexandrov space of curvature ≤ -1. A (k,δ)-strainer $\{(a_i, b_i)\}$ of $\mathrm{dbl}(X)$ at $p \in X$ is called *admissible* if $a_i, b_j \in X$ for $1 \le i \le k$, $1 \le j \le k-1$ (b_k may be in X^* if $p \in \partial X$ for instance). Let $R^D_\delta(X)$ be the set of all admissible (k,δ)-strained points in X.

Let Y be a closed domain of $R^D_\delta(X)$. For a small $\nu > 0$, we put

$$Y_\nu := \{x \in Y \mid d(x, \partial X) \ge \nu\},$$

and we put

$$\partial_0 Y_\nu := Y_\nu \cap \{d_{\partial X} = \nu\},\ \mathrm{int}_0 Y_\nu := Y_\nu - \partial_0 Y_\nu.$$

The admissible δ-strained radius δ^D-str.rad x at $p \in X$ is the supremum of the length of all admissible δ-strainers at p. The admissible δ-strained radius δ^D-str.rad$\,(Y)$ of a subset $Y \subset X$ is

$$\delta^D\text{-str.rad}\,(Y) := \inf_{p\in Y} \delta^D\text{-str.rad}\, p.$$

Theorem 2.25 (Fibration Theorem ([Y 4-dim, Theorem 1.2]))**.** *Given k and $\mu > 0$, there exist positive numbers $\delta = \delta_k$, $\varepsilon_k(\mu)$ and $\nu = \nu_k(\mu)$ satisfying the following: Let X^k be an Alexandrov space with curvature ≥ -1 of dimension k. Let $Y \subset R^D_\delta(X)$ be a closed domain such that δ_D-str.rad$\,(Y) \ge \mu$. Let M^n be an Alexandrov space with curvature ≥ -1 of dimension n. Suppose that $R_{\delta_n}(M^n) = M^n$ for some small $\delta_n > 0$. If $d_{GH}(M, X) < \varepsilon$ for some $\varepsilon \le \varepsilon_k(\mu)$, then there exist a closed domain $N \subset M$ and a decomposition*

$$N = N_{\mathrm{int}} \cup N_{\mathrm{cap}}$$

of N into two closed domains glued along their boundaries and a Lipschitz map $f : N \to Y_\nu$ such that

(1) *N_{int} is the closure of $f^{-1}(\mathrm{int}_0\, Y_\nu)$ and $N_{\mathrm{cap}} = f^{-1}(\partial_0 Y_\nu)$;*
(2) *both the restrictions $f_{\mathrm{int}} := f|_{N_{\mathrm{int}}} : N_{\mathrm{int}} \to Y_\nu$ and $f_{\mathrm{cap}} := f|_{N_{\mathrm{cap}}} : N_{\mathrm{cap}} \to \partial_0 Y_\nu$ are*
   (a) *locally trivial fiber bundles (see Definition* 2.37*);*
   (b) *$\theta(\delta, \nu, \varepsilon/\nu)$-Lipschitz submersions.*

Remark 2.26. If $\partial X = \emptyset$, then $N_{\mathrm{cap}} = \emptyset$ in the statement of Theorem 2.25.

The following theorem is a fundamental and important tool to study a local structure of collapsing Alexandrov spaces.

Theorem 2.27 (Rescaling Argument [Y ess], [SY00], [Y 4-dim])**.** *Let M_i, $i = 1, 2, \dots$, be a sequence of Alexandrov spaces of dimension n with curvature ≥ -1 and let X be an Alexandrov space of dimension k with curvature ≥ -1 and $k < n$.*

Let $p_i \in M_i$ and $p \in X$. Assume that (M_i, p_i) converges to (X, p), and $r > 0$ is a small number depending on p. Assume the following:

Assumption 2.28. *For any $\tilde{p}_i$ with $d(p_i, \tilde{p}_i) \to 0$ and for any sufficiently large i, $B(\tilde{p}_i, r)$ has a critical point for $\mathrm{dist}_{\tilde{p}_i}$*

Then there exist a sequence $\delta_i \to 0$ of positive numbers and $\hat{p}_i \in M_i$ such that

- *$d(p_i, \hat{p}_i) \to 0$ as $i \to \infty$;*
- *for any limit Y of $(\frac{1}{\delta_i} M_i, \hat{p}_i)$, we have $\dim Y \geq k + 1$;*
- *$\dim S \leq \dim Y - \dim X$, where S is a soul of Y.*

Remark 2.29. If a sequence of $B(p_i, r)$ metric balls does not satisfy Assumption 2.28, then by the Stability Theorem 2.34, $B(\tilde{p}_i, r)$ (resp. $U(\tilde{p}_i, r)$) is homeomorphic to the closed cone $K_1(\Sigma_{\tilde{p}_i})$ (resp. the open cone $K(\Sigma_{\tilde{p}_i})$) over the space of directions $\Sigma_{\tilde{p}_i}$ for some $\tilde{p}_i \in M_i$ with $d(p_i, \tilde{p}_i)$ tending to zero.

Fukaya and Yamaguchi proved the following.

Theorem 2.30 ([FY], [Y conv]). *For $n \in \mathbb{N}$, there exists $\varepsilon_n > 0$ satisfying the following. Suppose that an n-dimensional Alexandrov space M^n with curvature ≥ -1 and $\operatorname{diam} M^n < \varepsilon_n$. Then, the fundamental group $\pi_1(M^n)$ is almost nilpotent; i.e. $\pi_1(M^n)$ has a nilpotent subgroup of finite index.*

Remark 2.31 ([Y conv]). In Fibration Theorems 2.24 and 2.25, the fiber is connected and has an almost nilpotent fundamental group.

2.5.4. *Perelman's Morse theory and stability theorem.* In this section, we mainly refer to [Per II].

Definition 2.32 ([Per II]). Let $f = (f_1, \dots f_m) : U \to \mathbb{R}^m$ be a map on an open subset U of an Alexandrov space X defined by $f_i = d(A_i, \cdot)$ for compact subsets $A_i \subset X$. The map f is said to be (c, ε)*-regular at* $p \in U$ if there is a point $w \in X$ such that:

(1) $\angle((A_i)'_p, (A_j)'_p) > \pi/2 - \varepsilon$.
(2) $\angle(w'_p, (A_i)'_p) > \pi/2 + c$.

Theorem 2.33 ([Per II]). *Let X be an finite dimensional Alexandrov space, let $U \subset X$ be an open subset, and let f be (c, ε)-regular at each point of U. If ε is small compared with c, then we have:*

(1) *f is a topological submersion (see Definition 2.37).*
(2) *If f is proper in addition, then the fibers of f are MCS-spaces. Hence f is a fiber bundle over its image.*

Here, a metrizable space X is called an *n-dimensional MCS-space* if any point $p \in X$ has an open neighborhood U and there exists an $(n-1)$-dimensional compact MCS-space Σ such that (U, p) is a pointed homeomorphic to the cone $(K(\Sigma), o)$, where o is the apex of the cone. Here, we regarded the (-1)-dimensional MCS-space as the empty-set and its cone as the single-point set.

Perelman proved the Stability Theorem:

Theorem 2.34 (Stability Theorem [Per II] (cf. [Kap Stab])). *Let X^n be a compact n-dimensional Alexandrov space with curvature $\geq \kappa$. Then there exists $\delta > 0$ depending on X such that if Y^n is an n-dimensional Alexandrov space with curvature $\geq \kappa$ and $d_{GH}(X, Y) < \delta$, then Y is homeomorphic to X.*

In addition, let $A \subset X$ be a compact subset, and let $A' \subset Y$ be a compact subset. Then there exists $\delta > 0$ depending (X, A) satisfying the following. Suppose that there is a δ-approximation $f : Y \to X$ such that $f(A') \subset A$ and $f|_{A'}$ is a δ-approximation. If $t \in (0, \sup d_A)$ is a regular value of d_A, then $S(A, t)$ is homeomorphic to $S(A', t)$. Here, we say that t is a regular value if d_A is regular on $S(A, t)$.

In particular, every point in a finite dimensional Alexandrov space has a cone neighborhood over its spaces of directions.

Theorem 2.35 ([Per II])**.** *If an n-dimensional Alexandrov space Σ^n of curvature ≥ 1 has diameter greater than $\pi/2$, then Σ is homeomorphic to a suspension over an $(n-1)$-dimensional Alexandrov space of curvature ≥ 1.*

Theorem 2.36 ([Per II], [Pet Appl], [GP])**.** *If an n-dimensional Alexandrov space Σ^n of curvature ≥ 1 has radius $> \pi/2$, then Σ is homeomorphic to an n-sphere.*

2.5.5. *Preliminaries from Siebenmann's theory in* [Sie].

Definition 2.37. A continuous map $p : E \to X$ between topological spaces is called a *topological submersion* (or called a *locally trivial fiber bundle*) if for any $y \in E$ there are an open neighborhood U of y in the fiber $p^{-1}(p(y))$, an open neighborhood N of $p(y)$ in X, and an open embedding $f : U \times N \to E$ such that $p \circ f$ is the projection $U \times N \to N$. We call the embedding $f : U \times N \to E$ a *product chart* about U for p, and the image $f(U \times N)$ a *product neighborhood* around y.

A surjective continuous map $p : E \to X$ of topological spaces is called a *topological fiber bundle* if there exists an open covering $\{U_\alpha\}$ of X, a family $\{F_\alpha\}$ of topological spaces, and a family $\{\varphi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times F_\alpha\}$ of homeomorphisms such that $\mathrm{proj}_{U_\alpha} \circ \varphi_\alpha = p|_{p^{-1}(U_\alpha)}$ holds for each α. Here, proj_{U_α} is the projection from $U_\alpha \times F_\alpha$ to U_α.

A finite dimensional topological space Y is said to be a *WCS-set* [Sie, §5] if it satisfies both (1) and (2):

(1) Y is stratified into topological manifolds; i.e. it has a stratification
$$Y \supset \cdots \supset Y^{(n)} \supset Y^{(n-1)} \supset \cdots \supset Y^{(-1)} = \emptyset,$$
such that $Y^{(n)} - Y^{(n-1)}$ is a topological n-manifold without boundary.
(2) For each $x \in Y^{(n)} - Y^{(n-1)}$ there are a cone C with a vertex v and a homeomorphism $\rho : \mathbb{R}^n \times C \to Y$ onto an open neighborhood of x in Y such that $\rho^{-1}(Y^{(n)}) = \mathbb{R}^n \times \{v\}$.

From the definition, we can see that an MCS-space is a WCS-set.

Theorem 2.38 (Union Lemma [Sie])**.** *Let $p : E \to X$ be a topological submersion and $F = p^{-1}(x_0)$ the fiber over $x_0 \in X$. We assume that F is a WCS-space. Let A_1 and A_2 be compact sets in F. Let $\varphi_i : U_i \times N_i \to E$ be a product chart about U_i for an open neighborhood U_i of A_i in F, and $i = 1, 2$. Then there exists a product chart $\varphi : U \times N \to E$ about $U \supset A_1 \cup A_2$ in F such that*
$$\varphi = \begin{cases} \varphi_1 \text{ near } A_1 \times \{x_0\}, \\ \varphi_2 \text{ near } (A_2 - U_1) \times \{x_0\}. \end{cases}$$

Theorem 2.39 ([Sie])**.** *Let $p : E \to X$ be a topological submersion. We assume that p is proper and all fibers of p are WCS-spaces. Then p is a topological fiber bundle over $p(E)$.*

We provide the following lemma that will be used in Section 5.

Lemma 2.40. *Let $f : E \to [0,1]$ be a fiber bundle and the fiber $F := f^{-1}(0)$ be a WCS-space. Let $U \subset F$ be an open subset and $A \subset U$ be a closed subset. Suppose that $\varphi : U \times [0,1] \to E$ is a product chart about U for f. Then there exists a product chart $\chi : F \times [0,1] \to E$ such that*

$$\chi = \varphi \text{ on } A \times [0,1].$$

In particular, $E - \varphi(A \times [0,1])$ is homeomorphic to $(F - A) \times [0,1]$.

Proof. We may assume that $E = F \times [0,1]$ and f is the projection onto $[0,1]$. Let $\varphi : U \times [0,1] \to F \times [0,1]$ be a product chart about U. Using Union Lemma 2.38 and the compactness of $[0,1]$, we will construct an extension of $\varphi|_{A\times[0,1]}$ to a product chart defined on $F \times [0,1]$.

By Union Lemma 2.38, for any $t \in [0,1]$, there exist an open neighborhood N_t of t in $[0,1]$ and a product chart

$$\psi^{(t)} : F \times N_t \to E$$

such that

$$\psi^{(t)}|_{A\times N_t} = \varphi|_{A\times N_t}.$$

By the Lebesgue number lemma, there is $n \in \mathbb{N}$ such that, setting $I_k := [k/n, (k+1)/n]$, $\{I_k\}_{k=0,1,\dots,n-1}$ is a refinement of an open covering $\{N_t\}_{t\in[0,1]}$ of $[0,1]$. Namely, for $k = 0, 1, \dots, n-1$, there is $t_k \in [0,1]$ such that $I_k \subset N_{t_k}$. Let us set

$$\psi^k := \psi^{(t_k)}|_{F\times I_k}.$$

For $t \in I_k$, let us define a homeomorphism $\psi^k_t : F \to F$ by the equality

$$\psi^k(x,t) = (\psi^k_t(x), t).$$

Gluing these local product charts ψ^k, we construct the required product chart χ as follows. We inductively define a homeomorphism $\chi^k_t : F \to F$ by

$$\begin{aligned} \chi^0_t &= \psi^0_t && \text{for } t \in I_0, \\ \chi^k_t &= \psi^k_t \circ (\psi^k_{k/n})^{-1} \circ \chi^{k-1}_{k/n} && \text{for } t \in I_k, k \geq 1. \end{aligned}$$

For $k = 0, 1, \dots, n-1$ and $(x,t) \in F \times I_k$, we define

$$\chi(x,t) := (\chi^k_t(x), t).$$

One can easily check that

$$\chi = \varphi \text{ on } A \times [0,1].$$

Namely, $\chi : F \times [0,1] \to E$ satisfies the conclusion of the lemma. $\square$

2.6. **Differentiable structures of Alexandrov spaces.** Otsu and Shioya [OS] proved that any Alexandrov space has a differential structure and a Riemannian structure in a weak sense.

Definition 2.41 ([Per DC])**.** Let $U \subset M^n$ be an open subset of an Alexandrov space M. A locally Lipschitz function $f : U \to \mathbb{R}$ is called a *DC-function* if for any $x \in U$ there exist two (semi-)concave functions g and h on some neighborhood V of x in U such that $f = g-h$ on V. A locally Lipschitz map $f = (f_1, \dots, f_m) : U \to \mathbb{R}^m$ is called a *DC-map* if each f_i is a *DC*-function.

In [KMS, §2.6], the authors formulated a general concept of structure on topological spaces.

Definition 2.42 ([KMS])**.** For an integer $n \geq 0$, we consider the family

$$\mathcal{F} = \{\, \mathcal{F}(U; A) \,|\, U \subset \mathbb{R}^n \text{ is an open subset and } A \subset U \text{ a subset} \,\}$$

such that

(i) each $\mathcal{F}(U; A)$ is a class of maps from U to $\mathbb{R}^n$;
(ii) if $A \supset B$, then $\mathcal{F}(U; A) \subset \mathcal{F}(U; B)$;
(iii) if $f \in \mathcal{F}(U; A)$, $g \in \mathcal{F}(V; B)$, and $f(U) \subset V$, then

$$g \circ f \in \mathcal{F}(U; A \cap f^{-1}(B)).$$

The following are examples of $\mathcal{F} = \{\mathcal{F}(U; A)\}$.

(Class C^1) Let $C^1(U; A)$ be the class of maps from U to $\mathbb{R}^n$ which are C^1 on A; i.e. they are differentiable on A and their derivatives are continuous on A.

(Class DC) Let $DC(U; A)$ be the class of maps from U to $\mathbb{R}^n$ which are DC on some open subset $O \subset \mathbb{R}^n$ with $A \subset O \subset U$.

Let X be a paracompact Hausdorff space, $Y \subset X$ a subset, and $\mathcal{F}$ as above. We call a pair (U, φ) a *local chart of* X if U is an open subset of X and if φ is a homeomorphism from U to an open subset of $\mathbb{R}^n$. A family $\mathcal{A} = \{(U, \varphi)\}$ of local charts of X is called an $\mathcal{F}$*-atlas on* $Y \subset X$ if the following (i) and (ii) hold:

(i) $Y \subset \bigcup_{(U,\varphi)\in\mathcal{A}} U$.
(ii) If two local charts (U, φ), $(V, \psi) \in \mathcal{A}$ satisfy $U \cap V \neq \emptyset$, then

$$\psi \circ \varphi^{-1} \in \mathcal{F}(\varphi(U \cap V); \varphi(U \cap V \cap Y)).$$

Two $\mathcal{F}$-atlases $\mathcal{A}$ and $\mathcal{A}'$ on $Y \subset X$ are said to be *equivalent* if $\mathcal{A} \cup \mathcal{A}'$ is also an $\mathcal{F}$-atlas on $Y \subset X$. We call each equivalent class of $\mathcal{F}$-atlases on $Y \subset X$ an $\mathcal{F}$*-structure* on $Y \subset X$.

Assume that $Y = X$. Then, an $\mathcal{F}$-structure on $Y \subset X$ is simply called an $\mathcal{F}$*-structure on* X. If there is an $\mathcal{F}$-structure on X, then X is a topological manifold. We call a space equipped with an $\mathcal{F}$-structure an $\mathcal{F}$*-manifold*. Notice that $\mathcal{F}$-manifolds for $\mathcal{F} = C^1$ are nothing more than C^1-differentiable manifolds in the usual sense.

Let M^n be an n-dimensional Alexandrov space. Fix a number $\delta > 0$ with $\delta \ll 1/n$. By Theorem 2.16, for any $x \in M - S_\delta(M)$, we obtain a local chart $(U, \tilde{\varphi})$, $U = U(x, r)$. The family $\mathcal{A}_0$ of all the $(U, \tilde{\varphi})$'s on M induces:

Theorem 2.43 ([OS])**.** *There exists a C^1-structure on $M - S(M) \subset M$ containing $\mathcal{A}_0$.*

Theorem 2.44 ([Per DC])**.** *There exists a DC-structure on $M - S_\delta(M) \subset M$ containing $\mathcal{A}_0$.*

Thus, $M - S_\delta(M)$ is a DC^1-manifold with singular set $S(M)$ in the following sense.

Definition 2.45 ([KMS, §5])**.** A paracompact topological manifold V with a subset $S \subset V$ is said to be a DC^1*-manifold with singular set* S if V possesses a DC-atlas $\mathcal{A}$ on V which is also a C^1-atlas on $V - S \subset V$. We say that each local chart compatible with the atlas $\mathcal{A}$ is a DC^1*-local chart*.

Let V' be an another DC^1-manifold with singular set S'. A map $f : V \to V'$ is called a DC^1*-map* if for any DC^1-local chart (U', φ') of V', $(f^{-1}(U'), \varphi' \circ f)$ is a DC^1-local chart of V. A homeomorphism $f : V \to V'$ is called a DC^1*-homeomorphism* if f and f^{-1} are DC^1-maps.

Using Otsu's method [O], Kuwae, Machigashira and Shioya [KMS] proved that an almost regular Alexandrov space has a smooth approximation by a Riemannian manifold.

Theorem 2.46 ([KMS]; cf. [O]). *For any $n \in \mathbb{N}$, there exists a positive number $\varepsilon_n > 0$ depending only on n satisfying the following: If C is a compact subset in an n-dimensional Alexandrov space M with curvature ≥ -1 and it is ε-strained for $\varepsilon \leq \varepsilon_n$, then there exist an open neighborhood $U(C)$, a C^∞-Riemannian n-manifold $N(C)$ with C^∞-Riemannian metric $g_{N(C)}$, and a $\theta(\varepsilon)$-isometric DC^1-homeomorphism $f : U(C) \to N(C)$ such that $g_{N(C)}(df(v), df(w)) = \langle v, w\rangle + \theta(\varepsilon)$ for any $v, w \in \Sigma_x U(C)$ and $x \in U(C)$. Here, $\langle \cdot, \cdot \rangle$ is the inner product of $T_x M$.*

Remark 2.47. Otsu [O] proved this theorem for any Riemannian manifold M with a lower sectional curvature bound and having small excess.

We will review the proof of Theorem 2.46 in the proof of Theorem 3.2 in Section 3. It is important and needed in our proof of Theorem 3.2.

2.7. **Generalized Seifert fiber spaces.** To describe results obtained in the present paper we define the notion of a generalized Seifert fiber space.

Definition 2.48. Let M^3 and X^2 be, respectively, a three-dimensional and a two-dimensional topological orbifold possibly with boundaries. A continuous map $f : M \to X$ is called a *generalized Seifert fibration* if there exists a family $\{c_x\}_{x\in X}$ of subsets of M such that the following properties hold:

- The index set of $\{c_x\}$ is X. Each $x \in X$, $f^{-1}(x) = c_x$.
- Each c_x is homeomorphic to a circle or a bounded closed interval. c_x are disjoint and
$$\bigcup_{x\in X} c_x = M.$$
- For each $x \in X$, there exists a closed neighborhood U_x of x such that U_x is homeomorphic to a disk, and putting $V_x := f^{-1}(U_x)$, V_x satisfies the following.
  - (i) If c_x is topologically a circle, then $f|_{V_x} : V_x \to U_x$ is a Seifert fibered solid torus in the usual sense.
  - (ii) If c_x is topologically a bounded closed interval, then there exist homeomorphisms $\tilde\phi_x : V_x \to B(\mathrm{pt})$ and $\phi_x : U_x \to K_1(S^1_\pi)$, which preserve the structure of circle fibration with singular fiber. Namely, the following diagram commutes:
$$\begin{array}{ccc}(V_x, c_x) & \xrightarrow{\tilde\phi_x} & (B(\mathrm{pt}), p^{-1}(o)) \\ {\scriptstyle f|_{V_x}}\downarrow & & \downarrow{\scriptstyle p} \\ (U_x, x) & \xrightarrow{\phi_x} & (K_1(S^1_\pi), o)\end{array}$$
    Here, $B(\mathrm{pt}) = S^1 \times D^2/\mathbb{Z}_2$ is the topological orbifold defined after Theorem 1.7 and p is a canonical projection.
- If ∂X has a compact component C, then there is a collar neighborhood N of C in X such that $f|_{f^{-1}(N)}$ is a usual circle fiber bundle over N.

We say that a three-dimensional topological orbifold M is a *generalized Seifert fiber space over* X if there exists a generalized Seifert fibration $f : M \to X$. Each fiber $f^{-1}(x) = c_x$ of f is often called an *orbit* of M. An orbit c_x is called *singular* if V_x is a usual Seifert solid torus of (μ, ν)-type with $\mu > 1$ or if c_x is homeomorphic to an interval.

2.8. **Soul Theorem from** [SY00] **with complete classification.** In this subsection, we recall the Soul Theorem for open three-dimensional Alexandrov spaces of non-negative curvature, obtained in [SY00]. Also, we classify the geometry and topology of open three-dimensional Alexandrov spaces of non-negative curvature having two-dimensional soul together with some new precise arguments. The Soul Theorem is very important to determine the topology of a neighborhood around a singular point in a collapsing three-dimensional Alexandrov space.

Definition 2.49. Let M^n be an n-dimensional non-compact Alexandrov space with non-negative curvature. For a ray $\gamma : [0, \infty) \to M$ in M, we define the *Busemann function* $b_\gamma : M \to \mathbb{R}$ *with respect to* γ as follows:

$$b_\gamma(x) := \lim_{t\to\infty} d(\gamma(t), x) - t$$

for $x \in M$. Fix a point $p \in M$ and define the *Busemann function* $b : M \to \mathbb{R}$ *with respect to* p by

$$b(x) := \inf_\gamma b_\gamma(x)$$

for $x \in M$. Here, γ runs over all the rays emanating from p. The Busemann functions b_γ and b are concave on M.

We denote by $C(0)$ the set of all points attaining the maximum value of b:

$$C(0) := b^{-1}(\max_M b).$$

Since b is concave, $C(0)$ is an Alexandrov space possibly with boundary of dimension less than n. If $C(0)$ has no boundary, we call it a *soul* of M. Inductively, if $C(k)$, $k \geq 0$, has the non-empty boundary, we define $C(k + 1)$, the set of all points attaining the maximum value of the distance function $\mathrm{dist}_{\partial C(k)}$ from the boundary $\partial C(k)$:

$$C(k + 1) := \mathrm{dist}^{-1}_{\partial C(k)}(\max_{C(k)} \mathrm{dist}_{\partial C(k)}).$$

Since $\mathrm{dist}_{\partial C(k)}$ is concave on $C(k)$, $C(k+1)$ is also an Alexandrov space of dimension $< \dim C(k)$. Since M has finite dimension, this construction stops, i.e. $\partial C(k) = \emptyset$ for some $k \geq 0$. Then we call such $C(k)$ a *soul* of M.

Proposition 2.50 ([Per II]; cf. [Pet Semi, §2])**.** *For any open Alexandrov space M of non-negative curvature and its soul S, there is a Sharafutdinov retraction from M to S. In particular, S is homotopic to M.*

2.8.1. *Soul Theorem.* We recall that a non-compact Alexandrov space without boundary is called open. In this section, we state the Soul Theorem for open three-dimensional Alexandrov spaces of non-negative curvature obtained in [SY00]. We also define examples of open three-dimensional Alexandrov spaces of non-negative curvature which are not topological manifolds and study those topologies.

First, we shall prove a rigidity result for the case that a soul has codimension one. This is a generalization of [SY00, Theorem 9.8(2)].

Theorem 2.51. *Let M be an n-dimensional open Alexandrov space and let S be a soul of M. Suppose that* $\dim S = n - 1$ *and S has a one-normal point. Let $B = B(S, t)$ be a metric ball around S of radius $t > 0$. Then, the metric sphere*

$\hat{S} := \partial B$ equipped with the induced intrinsic metric is an Alexandrov space of non-negative curvature. Also, $\hat{S}$ has an isometric involution σ such that $\hat{S}/\sigma$ is isometric to S and M is isometric to $\hat{S} \times \mathbb{R}/(x,t) \sim (\sigma(x), -t)$.

Proof. Let us denote by

$$\pi : M \to S$$

a canonical projection. Namely, for $x \in M$, we set $\pi(x) \in S$ to be the nearest point from x in S. We use rigidity facts on the π, referring to [SY00, §9] and [Y 4-dim, §2], for proving the theorem.

Assertion 2.52. *$\hat{S}$ satisfies the following convexity property: For $x, y \in \hat{S}$ with $|xy| < 2t$, any geodesic γ between x and y in M is contained in $\hat{S}$. In particular, $\hat{S}$ with the induced intrinsic metric is an Alexandrov space of non-negative curvature.*

Proof of Assertion 2.52. Since $|xy| < 2t$, γ does not intersect S. From the total convexity of B, we have $\gamma \subset B$. Let us consider a curve $\bar{\gamma} := \pi \circ \gamma$ on S. Let σ_s denote a unique ray emanating from $\bar{\gamma}(s)$ containing $\gamma(s)$. By [Y 4-dim, Proposition 2.1],

$$\Pi := \bigcup_{s \in [0, |xy|]} \sigma_s$$

is a flatly immersed surface in M. Moving γ along with Π, we obtain a curve $\hat{\gamma}$ contained in $\hat{S}$. This is a lift of $\bar{\gamma}$ via $\pi : \hat{S} \to S$. Therefore, we obtain $L(\hat{\gamma}) = L(\bar{\gamma}) \leq L(\gamma) = |xy|$. Suppose that γ is not contained in $\hat{S}$. From the construction of $\hat{\gamma}$ and [Y 4-dim, Proposition 2.1], one can show that $L(\hat{\gamma}) < L(\gamma)$. This is a contradiction. Therefore, $\hat{\gamma}$ must coincide with γ. □

Now, we denote by $\hat{d}$ the induced intrinsic metric on $\hat{S}$. Assertion 2.52 says that $(\hat{S}, \hat{d})$ is an Alexandrov space of non-negative curvature. Let us denote by $\hat{\pi} : \hat{S} \to S$ the restriction of π on $\hat{S}$. Let $S_{\rm two}$ (resp. $S_{\rm one}$) denote the set of all two-normal (resp. one-normal) points in S. We set $\hat{S}_{\rm two} := \hat{\pi}^{-1}(S_{\rm two})$ and $\hat{S}_{\rm one} := \hat{\pi}^{-1}(S_{\rm one})$. Then, $\hat{\pi} : \hat{S}_{\rm two} \to S_{\rm two}$ is a two-to-one map, and $\hat{\pi} : \hat{S}_{\rm one} \to S_{\rm one}$ is a one-to-one map.

Let us consider $S_{\rm reg} := S \cap M^{\rm reg}_{\delta}$ for a small $\delta > 0$, which is open dense in S. Note that since any one-normal point is an essentially singular point [SY00], $S_{\rm reg}$ is contained in $S_{\rm two}$. By [Pet Para], $S_{\rm reg}$ is convex, and hence, it is connected. We set $\hat{S}_{\rm reg} := \hat{\pi}^{-1}(S_{\rm reg})$. The restriction

$$\hat{\pi} : \hat{S}_{\rm reg} \to S_{\rm reg}$$

is a double covering. We define an involution σ on $\hat{S}_{\rm reg}$ as the non-trivial deck transformation of $\hat{\pi} : \hat{S}_{\rm reg} \to S_{\rm reg}$. By using [Y 4-dim, Proposition 2.1], we conclude that σ is a local isometry. Hence, there is a continuous extension of σ on the whole $\hat{S}$. We denote it by the same notation σ. Then, σ on $(\hat{S}, \hat{d})$ is also a local isometric involution. We note that σ on $\hat{S}_{\rm one}$ is defined as the identity. From the construction, σ is bijective. Therefore, σ is an isometry on $\hat{S}$ with respect to $\hat{d}$. We now fix the metric $\hat{d}$ on $\hat{S}$. By construction, $\hat{S}/\sigma$ and S are isometric to each other.

Let us consider the quotient space $N := \hat{S} \times \mathbb{R}/(x,s) \sim (\sigma(x), -s)$, which is an open Alexandrov space of non-negative curvature. We define $\varphi : N \to M$ as sending $[x,t] \in N$ to $x \in \hat{S}$. By construction, φ is an isometry. □

Example 2.53 ([SY00, p. 39]). For a non-negatively curved closed Alexandrov surface S and $p_1, p_2 \dots, p_k \in S$ ($k \in \mathbb{Z}_{\geq 0}$), we denote by $L(S;k) = L(S; p_1, p_2 \dots, p_k)$ an open three-dimensional Alexandrov space of non-negative curvature (if it exists) satisfying the following:

(1) $p_1, p_2, \dots, p_k$ are essential singular points in S, and S is isometric to a soul of $L(S;k)$. Hereafter, S is identified as a soul of $L(S;k)$.
(2) $\{p_1, \dots, p_k\}$ is the set of all topological singular points in $L(S;k)$.
(3) There is a continuous surjection $\pi : L(S;k) \to S$ such that for $x \in S - \{p_1, \dots, p_k\}$, $\pi^{-1}(x)$ is the union of two rays emanating from x perpendicular to S; and for $x \in \{p_1, \dots, p_k\}$, $\pi^{-1}(x)$ is the unique ray emanating from x perpendicular to S.
(4) The restriction $\pi : \pi^{-1}(S - \{p_1, \dots, p_k\}) \to S - \{p_1, \dots, p_k\}$ is a line bundle.

Proposition 2.54 ([SY00, Proposition 9.5]; cf. [Y 4-dim, §17]). *If $k \geq 1$, then any space $L(S;k)$ is one of $L(S^2;2)$, $L(P^2;2)$ and $L(S^2;4)$.*

Remark 2.55. There is an error in Proposition 9.5 (and Theorem 9.6) in [SY00]. Actually, a space $L(S;1)$ cannot exist, and a space $L(S;2)$ can have a soul homeomorphic to P^2. See [Y 4-dim, §17].

Proof of Proposition 2.54. Since $k \geq 1$, by Theorem 2.68, S is homeomorphic to S^2 or P^2. Moreover, if $S \approx S^2$, then we have $k \leq 4$; and if $S \approx P^2$, then $k \leq 2$.

We consider the case that $S \approx P^2$. Suppose that $k = 1$. Let $p \in S$ be a unique topological singular point in $L(S;1)$. Let $\pi : L(S;1) \to S$ be a surjection obtained in Example 2.53. For a neighborhood B of p in S homeomorphic to D^2, the restriction

$$\pi : \pi^{-1}(B) \to B$$

is fiber-wise isomorphic to $\pi_0 : D^2 \times \mathbb{R}/\mathbb{Z}_2 \to D^2/\mathbb{Z}_2$ such that $p \in B$ corresponds to the origin of $D^2/\mathbb{Z}_2$. Here, $D^2 \times \mathbb{R}/\mathbb{Z}_2$ denotes the quotient space of $D^2 \times \mathbb{R}$ by an involution $(x,t) \mapsto (-x,-t)$, $D^2/\mathbb{Z}_2$ denotes the quotient space of D^2 by an involution $x \mapsto -x$ which is homeomorphic to a disk, and π_0 is a canonical projection $\pi_0 : [x,t] \mapsto [x]$. In particular, $\partial \pi^{-1}(B)$ is homeomorphic to a Mobius strip $S^1 \tilde{\times} \mathbb{R}$. On the other hand, $B' := S - \mathrm{int}\, B$ is homeomorphic to Mö. Then, the restriction $\pi : \pi^{-1}(B') \to B'$ is a line bundle over Mö. In particular, it is trivial over $\partial B'$. Namely, we have $\partial \pi^{-1}(B') \approx S^1 \times \mathbb{R}$. This contradicts $\partial \pi^{-1}(B) \approx S^1 \tilde{\times} \mathbb{R}$. Therefore, we obtain that if $S \approx P^2$, then $k = 2$.

By a gluing argument as above, if $S \approx S^2$, then $k = 2$ or 4. □

Explicitly, we determine the topology of $L(S;k)$.

Corollary 2.56. *$L(S^2;2)$ is isometric to $\hat{S}^2 \times \mathbb{R}/(x,s) \sim (\sigma(x), -s)$, where $\hat{S}^2$ is a sphere of non-negative curvature in the sense of Alexandrov with an isometric involution σ such that $\hat{S}^2/\sigma$ is isometric to the soul S^2 of $L(S^2;2)$.*

$L(P^2;2)$ is isometric to $K^2 \times \mathbb{R}/(x,s) \sim (\sigma(x), -s)$, where K^2 is a flat Klein bottle with an isometric involution σ such that K^2/σ is isometric to the soul P^2 of $L(P^2;2)$.

$L(S^2;4)$ is isometric to $T^2 \times \mathbb{R}/(x,s) \sim (\sigma(x), -s)$, where T^2 is a flat torus with an isometric involution σ such that T^2/σ is isometric to the soul S^2 of $L(S^2;4)$.

Proof. To prove this, it suffices to determine the topology of a metric sphere around the soul of any $L(S;k)$. For any $L(S;k)$, we denote by $B(S;k)$ a metric ball around

S. Let us denote by π a canonical projection

$$\pi : B(S; k) \to S.$$

Namely, for $x \in S$, $\pi(x)$ is the nearest point from x in S.

We consider the case that $S \approx S^2$ and $k = 2$. Let $p_1, p_2 \in S$ be the topological singular points of $L(S^2; 2)$ in S. We divide S into D_1 and D_2 such that each D_i is a disk neighborhood of p_i and $D_1 \cap D_2$ is homeomorphic to a circle. Then, for $i = 1, 2$, there is a homeomorphism $\varphi_i : \pi^{-1}(D_i) \to D^2 \times [-1, 1]/(x, s) \sim (-x, -s)$. The gluing part $\pi^{-1}(D_1 \cap D_2)$ of $\pi^{-1}(D_1)$ and $\pi^{-1}(D_2)$ is homeomorphic to a Mobuis band Mö. Since the space $D^2 \times [-1, 1]/\sim$ is homeomorphic to $K_1(P^2)$, we obtain that $B(S^2; 2) = \pi^{-1}(D_1) \cup \pi^{-1}(D_2)$ is homeomorphic to $K_1(P^2) \cup_{\text{Mö}} K_1(P^2)$ (see Remark 2.62 below). Then, $\partial B(S^2; 2)$ is homeomorphic to a gluing of two copies of $P^2 - \text{int(Mö)} \approx D^2$. Therefore, $\partial B(S^2; 2) \approx S^2$.

We consider the case that $S \approx P^2$ and $k = 2$. Let $p_1, p_2 \in S$ be the topological singular points of $L(P^2; 2)$ in S. We take a disk neighborhood D of $\{p_1, p_2\}$ in S. Let us divide D into D_1 and D_2 such that each D_i is a disk neighborhood of p_i and $D_1 \cap D_2$ is homeomorphic to an interval. Then, $\pi^{-1}(D_1 \cap D_2)$ is homeomorphic to D^2. Hence, $\pi^{-1}(D) = \pi^{-1}(D_1) \cup \pi^{-1}(D_2)$ is homeomorphic to $K_1(P^2) \cup_{D^2} K_1(P^2)$ (see Lemma 2.61). By Lemma 2.61, $\partial\pi^{-1}(D)$ is homeomorphic to a Klein bottle. Since π is a non-trivial I-bundle over ∂D_i for $i = 1, 2$, it is a trivial I-bundle over ∂D. Then, $\pi^{-1}(\partial D) \approx S^1 \times I$. Let us set $A := \partial B(P^2; 2) \cap \pi^{-1}(D)$. Since D has singular points p_1 and p_2 of the projection π, A is connected, and hence A is homeomorphic to $S^1 \times I$.

Let us set $D' := S - \operatorname{int} D$ which is homeomorphic to Mö. Then, $\pi^{-1}(D')$ is homeomorphic to a total space of an I-bundle over Mö, which is Mö $\times I$ or Mö$\tilde{\times} I$. Let us set A' to be $\partial B(P^2; 2) \cap \pi^{-1}(D')$. Therefore, if $\pi^{-1}(D') \approx$ Mö $\times I$, then A' is a disjoint union of two Mobius bands; and if $\pi^{-1}(D') \approx$ Mö$\tilde{\times} I$, then A' is homeomorphic to $S^1 \times I$. Then, $\partial B(P^2; 2) = A \cup A'$ is homeomorphic to a Klein bottle if $\pi^{-1}(D') \approx$ Mö$\times I$ and is homeomorphic to $S^1 \times I \cup_\partial S^1 \times I$, which is a torus or a Klein bottle if $\pi^{-1}(D') \approx$ Mö$\tilde{\times} I$. Suppose that $\partial B(P^2; 2)$ is homeomorphic to T^2. By Theorem 2.51, there is an involution on T^2 having only two fixed points. This is a contradiction (see [N, Lemma 3]). Therefore, $\partial B(P^2; 2) \approx K^2$.

We consider the case that $S \approx S^2$ and $k = 4$. Let $p_1, p_2, p_3, p_4 \in S$ be all topological singular points of $L(S^2; 4)$. Let D and D' be domains in S homeomorphic to a disk such that $\operatorname{int} D$ (resp. $\operatorname{int} D'$) contains p_1 and p_2 (resp. p_3 and p_4), $D \cap D'$ is homeomorphic to a circle and $S = D \cup D'$. Let us denote $\partial B(S^2; 4) \cap \pi^{-1}(D)$ (resp. $\partial B(S^2; 4) \cap \pi^{-1}(D')$) by A (resp. A'). By repeating an argument similar to the case that $L(S; k) = L(P^2; 2)$, we obtain that A and A' are homeomorphic to $S^1 \times I$. Then, $\partial B(S^2; 4) = A \cup A'$ is homeomorphic to a torus or a Klein bottle. Suppose that $\partial B(S^2; 4)$ is homeomorphic to K^2. By Theorem 2.51, there is an involution on K^2 having only four fixed points. This is a contradiction (see [N, Lemma 2]). Therefore, $\partial B(S^2; 4) \approx T^2$. □

Remark 2.57. Since involutions on closed surfaces are completely classified [N], the topology of each $L(S; k)$ is unique.

For any space $L(S; k)$, we denote a metric ball around S in $L(S; k)$ by $B(S; k)$. The topology of any $B(S; k)$ is as follows.

Corollary 2.58. *$B(S^2;2)$ is homeomorphic to $S^2 \times [-1,1]/\mathbb{Z}_2$, $B(P^2;2)$ is homeomorphic to $K^2 \times [-1,1]/\mathbb{Z}_2$, and $B(S^2;4)$ is homeomorphic to $T^2 \times [-1,1]/\mathbb{Z}_2$. Here, all $\mathbb{Z}_2$-actions correspond to ones of Corollary* 2.56.

Theorem 2.59 (Soul Theorem (Theorem 9.6 in [SY00]))**.** *Let Y be a three-dimensional open Alexandrov space and S be its soul. Then we have the following:*

(1) *If $\dim S = 0$, then Y is homeomorphic to $\mathbb{R}^3$, or the cone $K(P^2)$ over the projective plane P^2, or M_{pt}, which is defined in Example* 1.2.
(2) *If $\dim S = 1$, then Y is isometric to a quotient $(\mathbb{R} \times N)/\Lambda$, where N is an Alexandrov space with non-negative curvature homeomorphic to $\mathbb{R}^2$ and Λ is an infinite cyclic group. Here, the Λ-action is diagonal.*
(3) *If $\dim S = 2$, then Y is isometric to one of the normal bundle $N(S) = L(S;0)$ over S, $L(S;2)$ and $L(S;4)$.*

We will define examples of $L(S^2;2)$, $L(P^2;2)$ and $L(S^2;4)$ in Example 2.63.

Example 2.60 ([SY00, Example 9.3])**.** Let Γ be a group of isometries generated by γ and σ on $\mathbb{R}^3$. Here, γ and σ are defined by $\gamma(x,y,z) = -(x,y,z)$ and $\sigma(x,y,z) = (x+1,y,z)$. Then we obtain an open non-negatively curved Alexandrov space $\mathbb{R}^3/\Gamma$. This space is isometric to M_{pt} in Example 1.2.

We denote by $B(\mathrm{pt})$ a metric ball $B(p_0,R)$ around a soul p_0 of $M_{\mathrm{pt}} = \mathbb{R}^3/\Gamma$ for large $R > 0$. Remark that $B(\mathrm{pt})$ is homeomorphic to $S^1 \times D^2/(x,v) \sim (\bar{x},-v)$. We can check that $B(\mathrm{pt})$ is one of $K_1(P^2) \cup_{D^2} K_1(P^2)$. Here, $K_1(P^2) \cup_{D^2} K_1(P^2)$ denotes the gluing $K_1(P^2) \cup_\varphi K_1(P^2)$ of two copies $K_1(P^2)$ along domains A_1 and A_2 homeomorphic to D^2 contained in $\partial K_1(P^2) \approx P^2$ via a homeomorphism $\varphi : A_1 \to A_2$. We show that the topology of $K_1(P^2) \cup_{D^2} K_1(P^2)$ does not depend on the choice of the gluing map.

Lemma 2.61. *For any domains A_1 and A_2 which are homeomorphic to D^2 contained in $\partial K_1(P^2)$ and any homeomorphism $\varphi : A_1 \to A_2$, there is a homeomorphism*
$$\tilde{\varphi} : K_1(P^2) \cup_\varphi K_1(P^2) \to K_1(P^2) \cup_{id} K_1(P^2).$$
Here, $id : A_0 \to A_0$ is the identity of a domain A_0 which is homeomorphic to D^2 contained in $\partial K_1(P^2)$. In particular, any such gluing is homeomorphic to $B(\mathrm{pt})$.

Proof. Let X_1, X_2 and $Y_1 = Y_2$ be spaces homeomorphic to $K_1(P^2)$. Let us take domains $A_1 \subset \partial X_1$, $A_2 \subset \partial X_2$ and $A_0 \subset \partial Y_1 = \partial Y_2$ which are homeomorphic to D^2. Let us take any homeomorphism $\varphi : A_1 \to A_2$.

Now let us fix a homeomorphism $\varphi_1 : A_1 \to A_0$. Then there is a homeomorphism $\hat{\varphi}_1 : \partial X_1 \to \partial Y_1$ which is an extension of φ_1. By using the cone structures of X_1 and Y_1, we obtain a homeomorphism $\tilde{\varphi}_1 : X_1 \to Y_1$ which is an extension of $\hat{\varphi}_1$. Let us set $\varphi_2 := \varphi_1 \circ \varphi^{-1} : A_2 \to A_0$. By an argument similar to the above, we obtain a homeomorphism $\tilde{\varphi}_2 : X_2 \to Y_2$ which is an extension of φ_2. We define a map $\tilde{\varphi} : X_1 \cup_\varphi X_2 \to Y_1 \cup_{id_{A_0}} Y_2$ by
$$\tilde{\varphi}(x) = \begin{cases} \tilde{\varphi}_1(x) \text{ if } x \in X_1, \\ \tilde{\varphi}_2(x) \text{ if } x \in X_2. \end{cases}$$
This map is well-defined and a homeomorphism. $\square$

Remark 2.62. We define a space $K_1(P^2)\cup_{\text{Mö}} K_1(P^2)$ in a way similar to $K_1(P^2)\cup_{D^2} K_1(P^2)$. Let us consider domains A_1, $A_2 \subset \partial K_1(P^2) \approx P^2$ which are homeomorphic to a Mobius band Mö, and take a homeomorphism $\varphi : A_1 \to A_2$. Then, we denote $K_1(P^2) \cup_\varphi K_1(P^2)$ by the gluing $K_1(P^2) \cup_{\text{Mö}} K_1(P^2)$ for some gluing map φ. By an argument similar to the proof of Lemma 2.61, the topology of $K_1(P^2)\cup_{\text{Mö}} K_1(P^2)$ does not depend on the choice of the gluing map. We can show that any such gluing is homeomorphic to $S^2 \times [-1,1]/(v,t) \sim (\sigma(v), -t)$. Here, S^2 is regarded as $\{v = (x,y,z) \in \mathbb{R}^3 \mid |v| = 1\}$ and σ is an involution defined as $\sigma : (x,y,z) \mapsto (-x,-y,z)$. Further, it is homeomorphic to $B(P^2;2)$ (see Corollary 2.58).

$K_1(P^2) \cup_\partial K_1(P^2)$ denotes the gluing of two copies of $K_1(P^2)$ via a homeomorphism on $\partial K_1(P^2)$. This space has the same topology as $K_1(P^2)\cup_{id} K_1(P^2)$, where id is the identity on $\partial K_1(P^2)$, which is homeomorphic to the suspension $\Sigma(P^2)$ over P^2. The proof is done by using the cone structure as in the proof of Lemma 2.61.

Example 2.63. We will define open Alexandrov spaces L_2 and L_4 as follows. Later, we show that L_k is isometric to an $L(S;k)$ for $k = 2,4$.

Recall that M_{pt} is defined as

$$M_{\text{pt}} := S^1 \times \mathbb{R}^2/(x,y) \overset{\alpha}{\sim} (\bar{x}, -y)$$

in Example 1.2. We consider a closed domain M'_{pt} of M_{pt} as

$$M'_{\text{pt}} := S^1 \times [-\ell, \ell] \times \mathbb{R}/\alpha$$

for some $\ell > 0$. Then, M'_{pt} is a convex subset of M_{pt}, and hence it is an Alexandrov space of non-negative curvature with boundary $\partial M'_{\text{pt}} \equiv S^1 \times \mathbb{R}$.

We denote by L_4 one of the open Alexandrov spaces of non-negative curvature defined as

$$L_4(\varphi) = M'_{\text{pt}} \cup_\varphi M'_{\text{pt}}$$

for an isometry φ on $\partial M'_{\text{pt}}$. Here, we use the following notation: For Alexandrov spaces A and A' whose boundaries are isometric to each other in the induced inner metric with an isometry $\varphi : \partial A \to \partial A'$, $A \cup_\varphi A'$ denotes the gluing of A and A' via φ.

We will show that L_4 is $L(S^2;4)$ (Lemma 2.64).

Let $U_{2,1}$ be the Alexandrov space defined by

$$U_{2,1} := S^1 \times \mathbb{R}^2/(x,y) \overset{\beta}{\sim} (-x,-y).$$

Let us set

$$U'_{2,1} := S^1 \times [-\ell,\ell] \times \mathbb{R}/\beta \subset U_{2,1}$$

which is a convex subset of $U_{2,1}$, and hence it is an Alexandrov space of non-negative curvature with boundary $\partial U'_{2,1} \equiv S^1 \times \mathbb{R}$. Let us set $S(U'_{2,1}) := S^1 \times [-\ell,\ell] \times \{0\}/\beta$. Note that $S(U'_{2,1})$ is isometric to a Mobius band Mö and $U'_{2,1}$ is isomorphic to an $\mathbb{R}$-bundle over $S(U'_{2,1})$.

We define open Alexandrov spaces $L_{2,1}$, $L_{2,2}$ and $L_{2,3}$ of non-negative curvature as

$$\begin{aligned}
L_{2,1} &:= L_{2,1}(\varphi) = M'_{\text{pt}} \cup_\varphi U'_{2,1}, \\
L_{2,2} &:= L_{2,2}(\varphi) = M'_{\text{pt}} \cup_\varphi D^2 \times \mathbb{R}, \text{ and} \\
L_{2,3} &:= L_{2,3}(\varphi) = M'_{\text{pt}} \cup_\varphi \text{Mö} \times \mathbb{R}.
\end{aligned}$$

Here, φ denotes a gluing isometry between the corresponding boundaries, D^2 denotes a two-disk of non-negative curvature, and Mö is a flat Mobius band.

Let us define an Alexandrov space A of non-negative curvature

$$A := [-a, a] \times [-b, b] \times \mathbb{R}/(v, s) \sim (-v, -s).$$

Here, $v \in [-a, a] \times [-b, b]$ and $s \in \mathbb{R}$. The boundary ∂A is isometric to $S^1 \times \mathbb{R}$. We define an open Alexandrov space $L_{2,4}$ of non-negative curvature as

$$L_{2,4} = L_{2,4}(\varphi) = A \cup_\varphi A$$

for some isometry φ on ∂A.

We will prove that $L_{2,1}$ and $L_{2,3}$ are $L(P^2; 2)$ and $L_{2,2}$ and $L_{2,4}$ are $L(S^2; 2)$ (Lemma 2.65).

From now on throughout this paper, we denote by L_2 one of $L_{2,1}$, $L_{2,2}$, $L_{2,3}$ and $L_{2,4}$.

Lemma 2.64. *L_4 is $L(S^2; 4)$.*

Proof. Recall that $L_4 = L_4(\varphi) = M'_{\mathrm{pt}} \cup_\varphi M'_{\mathrm{pt}}$. We identify $\partial M'_{\mathrm{pt}}$ as $S^1 \times \mathbb{R}$ via an isometry $[\xi, \ell, s] \mapsto [\xi, s]$. The isometry $\varphi : \partial M'_{\mathrm{pt}} \to \partial M'_{\mathrm{pt}}$ is written as

$$\varphi[\xi, \ell, s] = [f(\xi), \ell, g(s)]$$

for some isometries f on S^1 and g on $\mathbb{R}$. Then, $g(s) = (\pm 1) \cdot s + g(0)$.

Let us define $E := [-\ell, \ell] \times \mathbb{R}/(s, t) \sim (-s, -t)$. Obviously, there is a canonical projection $\pi : M'_{\mathrm{pt}} \to E$ defined by $[\xi, s, t] \mapsto [s, t]$. Here, $\xi \in S^1$, $s \in [-\ell, \ell]$ and $t \in \mathbb{R}$. The map π is a line bundle over $E - \{[0, 0]\}$.

For $a \in \mathbb{R}$, let us define $S'_{\mathrm{pt}}(a) \subset M'_{\mathrm{pt}}$ as

$$S'_{\mathrm{pt}}(a) := S^1 \times \{(t, at/\ell) \mid t \in [-\ell, \ell]\}/\alpha.$$

$S'_{\mathrm{pt}}(a)$ is homeomorphic to a disk. Then, by using the fibration $\pi : M'_{\mathrm{pt}} \to E$, we obtain that M'_{pt} is homotopic to $S'_{\mathrm{pt}}(a)$ for any $a \in \mathbb{R}$.

By choosing a with respect to $g(0)$, we obtain that L_4 is homotopic to the gluing $S'_{\mathrm{pt}}(a) \cup_\partial S'_{\mathrm{pt}}(-a)$ which is homeomorphic to S^2. Thus, a soul of L_4 is homeomorphic to a sphere. Since M'_{pt} has only two topological singular points in its interior, L_4 has only four topological singular points. Therefore, L_4 is $L(S^2; 4)$. ☐

Lemma 2.65. *$L_{2,1}$ and $L_{2,3}$ are $L(P^2; 2)$, and $L_{2,2}$ and $L_{2,4}$ are $L(S^2; 2)$.*

Proof. We will use the same notation as in the proof of Lemma 2.64.

Let us consider $L_{2,1} = L_{2,1}(\varphi) = M'_{\mathrm{pt}} \cup_\varphi U_{2,1}$. Recall that $U'_{2,1} \subset U_{2,1}$ is isomorphic to a line bundle over $S(U'_{2,1})$, where $S(U'_{2,1})$ is a subset of $U'_{2,1}$ homeomorphic to Mö. By using the bundle structure of $U'_{2,1}$ and the fibration π, we obtain that $L_{2,1}$ is homotopic to the gluing $S'_{\mathrm{pt}} \cup_\partial S(U'_{2,1})$, which is homeomorphic to P^2. Since $L_{2,1}$ has only two topological singular points, it follows that $L_{2,1}$ is $L(P^2; 1)$.

Let us take $L_{2,2} = L_{2,2}(\varphi) = M'_{\mathrm{pt}} \cup_\varphi D^2 \times \mathbb{R}$. By using the fibration π, we obtain that $L_{2,2}$ is homotopic to the gluing $S'_{\mathrm{pt}}(a) \cup_\partial D^2$ for some a, which is homeomorphic to S^2. Also, $L_{2,2}$ has only two topological singular points. This implies that $L_{2,2}$ is $L(S^2; 2)$.

Let us take $L_{2,3} = L_{2,3}(\varphi) = M'_{\mathrm{pt}} \cup_\varphi \text{Mö} \times \mathbb{R}$. By using π, we obtain that $L_{2,3}$ is homotopic to the gluing $S'_{\mathrm{pt}}(a) \cup_\partial \text{Mö}$ for some a, which is homeomorphic to P^2. Since $L_{2,3}$ has only two topological singular points, it follows that $L_{2,3}$ is $L(P^2; 2)$.

Let us take $L_{2,4} = L_{2,4}(\varphi) = A \cup_\varphi A$. Recall that $A = [-a,a] \times [-b,b] \times \mathbb{R}/(x,y,s) \sim (-x,-y,-s)$. Let us consider a subset $S' := [-a,a] \times [-b,b] \times \{0\}/\sim$ of A, which is homeomorphic to a disk. Let us set $E := [-b,b] \times \mathbb{R}/(y,s) \sim (-y,-s)$. There is a canonical projection $\pi' : A \to E$ defined by $\pi'([x,y,s]) = [y,s]$. By using it, we obtain that $L_{2,4}$ is homotopic to $S' \cup_\partial S'$, which is homeomorphic to S^2. Since $L_{2,4}$ has only two topological singular points, it follows that $L_{2,4}$ is $L(S^2;2)$. □

2.9. **Classification of Alexandrov surfaces from** [SY00]**.** We recall a result for a classification of Alexandrov surfaces by quoting [SY00].

Proposition 2.66 (The Gauss-Bonnet Theorem [SY00, Proposition 14.1])**.** *If X is a compact Alexandrov surface, then we have*

$$\omega(X) + \kappa(\partial X) = 2\pi\chi(X).$$

Proposition 2.67 (The Cohn-Vossen Theorem [SY00, Proposition 14.2])**.** *If X is a non-compact Alexandrov surface, then we have*

$$2\pi\chi(X) - \pi\chi(\partial X) - \omega(X) - \kappa(\partial X) \geq 0.$$

Theorem 2.68 ([SY00, Corollary 14.4])**.** *Let X be a non-negatively curved Alexandrov surface. Then, the following hold:*

(1) *X is homeomorphic to either $\mathbb{R}^2$, $\mathbb{R}_{\geq 0} \times \mathbb{R}$, S^2, P^2, D^2 or isometric to $[0,\ell] \times \mathbb{R}$, $[0,\ell] \times S^1(r)$, $\mathbb{R}_{\geq 0} \times S^1(r)$, $\mathbb{R} \times S^1(r)$, $\mathbb{R} \times S^1(r)/\mathbb{Z}_2$, a flat torus, or a flat Klein bottle for some $\ell, r > 0$.*

(2) $\operatorname{int} X$ *contains at most four essential singular points, and denoting by n the number of essential singular points in* $\operatorname{int} X$*, we have the following for some $\ell, r > 0$.*
   - (a) *If $n \geq 1$, X is either homeomorphic to $\mathbb{R}^2$, S^2, P^2, D^2 or isometric to* $\operatorname{dbl}(\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}) \cap \{(x,y) \mid y \leq h\}$.
   - (b) *If $n \geq 2$, X is either homeomorphic to S^2 or isometric to* $\operatorname{dbl}(\mathbb{R}_{\geq 0} \times [0,h])$, $\operatorname{dbl}(\mathbb{R}_{\geq 0} \times [0,h]) \cap \{(x,y) \mid x \leq \ell\}$ *or* $\operatorname{dbl}([0,\ell] \times [0,h])/\mathbb{Z}_2$.
   - (c) *If $n \geq 3$, then X is homeomorphic to S^2.*
   - (d) *If $n = 4$, X is isometric to $A \cup_\phi B$, where A and B are isometric to* $\operatorname{dbl}([0,\ell] \times [0,\infty)) \cap \{(x,y) \mid y \leq a\}$ *and* $\operatorname{dbl}([0,\ell] \times [0,\infty)) \cap \{(x,y) \mid y \leq b\}$ *for some $a, b > 0$, respectively, and $\phi : \partial A \to \partial B$ is some isometry.*

2.10. **A fundamental observation.** In this subsection, we prove fundamental propositions on the sets of topologically singular points of Alexandrov spaces.

First, we note the following proposition on the number of topologically singular points of a three-dimensional closed Alexandrov space. Let us consider a $(2n+1)$-dimensional manifold X such that its boundary ∂X is homeomorphic to the disjoint union $\bigsqcup_{i=1}^m P^{2n}$ of the projective spaces. Then we see that m is even. Indeed, we consider the double $\operatorname{dbl}(X)$ and its Euler number:

$$0 = \chi(\operatorname{dbl}(X)) = 2\chi(X) - \chi(\partial X) = 2\chi(X) - m.$$

Proposition 2.69. *Let M be a three-dimensional closed Alexandrov space. Then the number of topologically singular points of M is even.*

Proof. Since M is compact, $S_{\mathrm{top}}(M)$ is a finite set. By Theorem 2.34, there exists $r > 0$ such that for any $p \in S_{\mathrm{top}}(M)$ we have $(B(p,r),p) \approx (K_1(P^2), o)$. Therefore,

$$M_0 := M - \bigcup_{p \in S_{\mathrm{top}}(M)} U(p,r)$$

is a manifold with boundary $\partial M_0 \approx \bigsqcup_{p\in S_{\rm top}(M)} P^2$. By the above argument, $\sharp S_{\rm top}(M)$ is even. □

We also prepare the following proposition.

Proposition 2.70. *Let (M_i, p_i) be a sequence of n-dimensional pointed Alexandrov spaces of curvature ≥ -1 converging to (X, p). If $\operatorname{diam}\Sigma_p > \pi/2$, then Σ_{p_i} is homeomorphic to a suspension over an Alexandrov space of curvature ≥ 1, for large i.*

Proof. Suppose that the conclusion fails. Then we have some sequence $\{M_i^n\}$ such that (M_i, p_i) converges to (X, p) and each Σ_{p_i} does not have topological suspension structure over any Alexandrov space of curvature ≥ 1. It follows from Theorem 2.35 that $\operatorname{diam}(\Sigma_{p_i}) \leq \pi/2$. The convergence of spaces of directions is lower semi-continuous:

$$\liminf_{i\to\infty} \Sigma_{p_i} \geq \Sigma_p.$$

Then we have $\operatorname{diam}(\Sigma_p) \leq \pi/2$. This is a contradiction. □

3. Smooth approximations and flow arguments

3.1. **Flow Theorem.**

A bijective map $f : X \to Y$ between metric spaces is called bi-Lipschitz if both f and f^{-1} are Lipschitz.

Definition 3.1. Let M be a topological space. A continuous map $\Phi : M \times \mathbb{R} \to M$ is called a *flow* if it satisfies

$$\Phi(x, 0) = x,$$
$$\Phi(x, s + t) = \Phi(\Phi(x, s), t)$$

for any $x \in M$ and $s, t \in \mathbb{R}$. Remark that, for each $t \in \mathbb{R}$, the map

$$\Phi_t = \Phi(\cdot, t) : M \to M$$

has the inverse map Φ_{-t}.

Let M be a metric space. If a flow Φ is a Lipschitz map from $M \times \mathbb{R}$ to M, then we call it a *Lipschitz flow*. Remark that for any Lipschitz flow Φ, $\Phi(\cdot, t)$ is bi-Lipschitz for each $t \in \mathbb{R}$.

By using the proof of Theorem 2.46, we obtain the following theorem. This is a main tool for the proof of our results throughout the present paper.

Theorem 3.2 (Flow Theorem)**.** *For any $n \in \mathbb{N}$, there exists a positive number ε_n depending only on n satisfying the following: Let C be a compact subset and S be a closed subset in an n-dimensional Alexandrov space M with curvature ≥ -1. Suppose that $C \cap S = \emptyset$ and C is ε-strained and* dist_S *is $(1-\delta)$-regular on C for $\delta > 0$, where $0 < \varepsilon \leq \varepsilon_n$ and δ is smaller than some constant. Then there exist a neighborhood $U(C)$ of C and a Lipschitz flow $\Phi : M \times \mathbb{R} \to M$ satisfying the following:*

(*i*) *For any $x \in U(C)$, putting $I_x := \{t \in \mathbb{R} \mid \Phi(x, t) \in U(C)\}$, $\Phi(x, t)$ is a $5\sqrt{\delta} + \theta(\varepsilon)$-isometric embedding in $t \in I_x$.*

(ii) The speed with which Φ leaves S is almost one. Namely,

$$\frac{d}{dt}\bigg|_{t=0+} \mathrm{dist}_S \circ \Phi(x,t) > 1 - 5\sqrt{\delta} - \theta(\varepsilon) \tag{3.1}$$

at any $x \in U(C)$.

Proof of Theorem 3.2. To prove this, we must remember the proof of Theorem 2.46 in reference to [KMS] and [O].

For a while, x denotes an arbitrary point in C. We set

$$v(x) := \frac{\nabla \mathrm{dist}_S}{|\nabla \mathrm{dist}_S|} \in \Sigma_x M.$$

Since d_S is $(1-\delta)$-regular, we have

$$(\mathrm{dist}_S)'_x(v(x)) = -\cos\angle(S'_x, v(x)) > 1 - \delta. \tag{3.2}$$

We fix a point $q(x) \in S$ such that

$$|xq(x)| = |xS|.$$

Then, by (3.2), we have

$$\angle(q(x)', v(x)) \geq \angle(S', v(x)) > \pi - \delta'.$$

Here, $\delta' := \pi - \cos^{-1}(-1+\delta)$. Note that $\lim_{\delta\to 0} \frac{\delta'}{\sqrt{\delta}} = \frac{1}{2\sqrt{2}}$.

We put $\ell := \min\{\varepsilon\text{-str.}\,\mathrm{rad}\,(C), d(S,C)\}$. We fix positive numbers s and t with $s \ll t \ll \ell$. Take a maximal $0.2s$-net $\{x_j \mid j = 1, \dots, N\}$ of C. Fix any $j \in \{1, \dots, N\}$. We take ε-strainer $\{q_j^\alpha \mid \alpha = \pm 1, \dots, \pm n\}$ at x_j of length $\geq \ell$. We may assume that $\{q_j^\alpha\}$ satisfies the following:

$$q_j^{+1} = q(x_j). \tag{3.3}$$

Since $t \ll \ell$, $\{q_j^\alpha\}$ is also $\theta(\varepsilon)$-strainer at any $x \in B(x_j, 10t)$. It follows from $s \ll t$ and [Y conv, Lemma 1.9] that

$$|\tilde{\angle} q_j^\alpha xy - \angle q_j^\alpha xy| < \theta(\varepsilon) \tag{3.4}$$

for any $x \in B(x_j, s)$ and $y \in B(x, s)$.

We denote by E_j the standard n-dimensional Euclidean space. Define a map

$$f_j = (f_j^\alpha)_{\alpha=1}^n : B(x_j, 10t) \to E_j$$

by

$$f_j^\alpha(y) = \frac{1}{\mathcal{H}^n(B(q_j^\alpha, \varepsilon'))} \int_{z \in B(q_j^\alpha, \varepsilon')} d(y,z) - d(x_j, z)\, d\mathcal{H}^n(z) \tag{3.5}$$

where $\varepsilon' \ll \varepsilon$. This map is a $\theta(\varepsilon)$-almost isometric DC^1-homeomorphism, which is actually a DC^1-coordinate system.

Lemma 3.3 ([O, Lemma 5]). *There is an isometry $F_j^k : E_k \to E_j$ satisfying the following:*

$$|F_j^k \circ f_k(y) - f_j(y)| < \theta(\varepsilon)s, \tag{3.6}$$

$$|dF_j^k \circ df_k(\xi) - df_j(\xi)| < \theta(\varepsilon) \tag{3.7}$$

for any j and k, and $y \in B(x_j, s) \cap B(x_k, s)$ and $\xi \in \Sigma_y$.

Remark that each f_j has the directional derivative df_j.

Proof of Lemma 3.3. We first recall how to define F_j^k's. The property (3.6) is proved in the same way to the original proof of [O, Lemma 5] in our situation. We only prove (3.7).

Fix any j and k. For $\alpha = 1, \dots, n$, take $y_k^\alpha \in x_k q_k^{-\alpha}$ and $y_k^{-\alpha} \in x_k q_k^\alpha$ such that

$$|x_k y_k^\alpha| = |x_k y_k^{-\alpha}| = s.$$

Then we have

$$\langle f_k(y_k^\alpha), f_k(y_k^\beta)\rangle = s^2 \delta_{\alpha\beta} + \theta(\varepsilon, s/\ell)$$

for all $\alpha, \beta = 1, \dots, n$. Here, $\langle \cdot, \cdot \rangle$ is the standard inner product on E_k. Since $s \ll \ell$, $\theta(\varepsilon, s/\ell) = \theta(\varepsilon)$. Then, we have

$$|f_k(y_k^\alpha) - s e_k^\alpha| < \theta(\varepsilon).$$

Here, $\{e_k^\alpha\}_{\alpha=1}^n$ is an o.n.b on E_k. In a similar way, we have

$$|f_k(y_k^{-\alpha}) + s e_k^\alpha| < \theta(\varepsilon).$$

We define vectors v^α, $w^\alpha \in E_j$ $(\alpha = 1, \dots, n)$ by

$$v^\alpha := \frac{1}{2s}\{f_j(y_k^\alpha) - f_j(y_k^{-\alpha})\}.$$

Then, we have

$$\langle v^\alpha, v^\beta\rangle = \delta_{\alpha\beta} + \theta(\varepsilon).$$

Then, $\{v_\alpha\}$ is an almost orthonormal basis. By Schmidt's orthogonalization we obtain an orthonormal basis $\{\tilde{e}_\alpha\}$ of E_j such that

$$|\tilde{e}_\alpha - v_\alpha| < \theta(\varepsilon).$$

We now define an isometry $F_j^k : E_k \to E_j$ by changing the orthonormal basis and the translation:

$$F_j^k(v) = f_j(x_k) + \sum_{\alpha=1}^n \langle v, e_k^\alpha\rangle \tilde{e}_\alpha.$$

Then, we have

$$F_j^k(f_k(x)) = f_j(x) + s\vec{v}(\theta(\varepsilon))$$

for all $x \in B(x_j, s)$. Here, $\vec{v}(c)$ is a vector whose norm is less than or equal to $|c|$.

We prove (3.7). For any $y \in B(x_j, s) \cap B(x_k, s)$ and $\xi \in \Sigma_y$, by Lemma 2.17, there exists $z \in M$ such that

$$|yz| = t \text{ and } \angle(\xi, \uparrow_y^z) = \theta(\varepsilon). \tag{3.8}$$

Then, we have

$$\tilde{\angle} q_j^\alpha y z = \angle((q_j^\alpha)', z') + \theta(s/t).$$

Since $s \ll t$, we have $\theta(s/t) = \theta(\varepsilon)$. Therefore,

$$d_y f_j^\alpha(\xi) = \frac{1}{\mathcal{H}^n(B(q_j^\alpha, \varepsilon'))} \int_{w \in B(q_j^\alpha, \varepsilon')} -\cos\angle(w_y', \xi)\, d\mathcal{H}^n(w) \tag{3.9}$$

$$= -\cos \tilde{\angle} q_j^\alpha y z + \theta(\varepsilon). \tag{3.10}$$

On the other hand,

$$dF_j^k \circ df_k(\xi) = dF_j^k\Big(\big(-\cos\tilde{\angle} q_j^\alpha yz\big)_{\alpha=1,\dots,n}\Big) + \vec{v}(\theta(\varepsilon)) \tag{3.11}$$

$$= \sum_\alpha -\cos\tilde{\angle} q_j^\alpha yz \cdot \tilde{e}_\alpha + \vec{v}(\theta(\varepsilon)). \tag{3.12}$$

Therefore, we have (3.7). □

Set $V_j := B(0, 0.4s) \subset E_j$ for all j.

Next, we perturb $\{F_j^k\}$ to a family $\{\tilde{F}_j^k\}$ satisfying the following.

Lemma 3.4 ([O, Lemma 6]). *For any j and k with $d(x_j, x_k) < 0.9s$, there exists a $\theta(\varepsilon)$-almost isometric C^∞ map $\tilde{F}_j^k : V_k \to E_j$ satisfying the following:*

$$\tilde{F}_j^j = \mathrm{id} \text{ on } E_j \text{ and} \tag{3.13}$$

$$\tilde{F}_j^l(v) = \tilde{F}_j^k \circ \tilde{F}_k^l(v) \tag{3.14}$$

for any j and k with $d(x_j, x_k) < 0.9s$ and $v \in V_l \cap \tilde{F}_l^k(V_k) \cap \tilde{F}_l^j(V_j)$.

Moreover, we can obtain this perturbed $\{\tilde{F}_j^k\}$ also satisfying (3.6) and (3.7). That is, we have

$$|\tilde{F}_j^k \circ f_k(y) - f_j(y)| < \theta(\varepsilon)s, \tag{3.15}$$

$$|d\tilde{F}_j^k \circ df_k(\xi) - df_j(\xi)| < \theta(\varepsilon) \tag{3.16}$$

for any j and k, and $y \in B(x_j, s) \cap B(x_k, s)$ and $\xi \in \Sigma_y$.

Proof. We only review the first step of construction of $\tilde{F}_j^k$'s by induction referring to the proof of [O].

Let us first review how to construct $\tilde{F}_j^k$'s. Let $\phi : [0,\infty) \to [0,\infty)$ be a C^∞-function such that

$$\begin{aligned} \phi &= 1 \text{ on } [0, 1/2], \\ \phi &= 0 \text{ on } [1, \infty), \text{ and} \\ -4 &\le \phi' \le 0. \end{aligned}$$

Set

$$\psi_j(v) := \phi(|v|/0.8s)$$

for $v \in V_j$.

We set $\tilde{F}_j^1 = F_j^1$ and $\tilde{F}_1^j = (\tilde{F}_j^1)^{-1}$, and define $\tilde{F}_j^2 : U_2 \to \mathbb{R}^n$ for $j \ge 2$ by

$$\tilde{F}_j^2(v) := \psi_1 \circ \tilde{F}_1^2(v) \cdot \tilde{F}_j^1 \circ \tilde{F}_1^2(v) + (1 - \psi_1 \circ \tilde{F}_1^2(v)) \cdot F_j^2(v)$$

for $v \in V_2$. By construction, $\tilde{F}_j^2$ is smooth and satisfies (3.15) and (3.16).

For $v \in V_2$, we have

$$\begin{aligned} \phi_1 \circ \tilde{F}_1^2(v) &= 1 + \theta(\varepsilon)|v|, \\ \|d(\phi_1 \circ \tilde{F}_1^2)\|_{C^1} &= \theta(\varepsilon). \end{aligned}$$

Therefore, we have, for any $v \in V_2$ and $w \in T_vE_2$ with $|w| = 1$,

$$\begin{aligned} d\tilde{F}_j^2(w) &= d(\tilde{F}_j^1 \circ \tilde{F}_1^2)(w) + \theta(\varepsilon) \\ &= d(F_j^1 \circ F_1^2)(w) + \theta(\varepsilon) \\ &= dF_j^2(w) + \theta(\varepsilon). \end{aligned}$$

Thus, we have

$$\|d\tilde{F}_j^2 - dF_j^2\| < \theta(\varepsilon)$$

at any $v \in V_2$.

Therefore, for a segment $c : [0, t_0] \to V_2$ between v and y, we have

$$|\tilde{F}_j^2(v) - \tilde{F}_j^2(w)| = \left| \int_0^{t_0} d\tilde{F}_j^2(c'(t))dt \right| \geq 0.9|v - w|.$$

Thus, $\tilde{F}_j^2$ is injective. □

By the chain rule (3.13), an equivalence relation $\sim$ on the disjoint union $\bigsqcup_j V_j$ is defined in the following natural way: $V_j \ni y \sim y' \in V_k \iff \tilde{F}_j^k(y') = y$. Set $N := \bigsqcup V_j/\sim$. We denote by π the projection

$$\pi : \bigsqcup_{j=1}^{N} V_j \to N.$$

We denote by $\tilde{V}_j := \pi(V_j)$ the subset of N corresponding to V_j, and by π_j the restriction of π

$$\pi_j : V_j \to \tilde{V}_j.$$

We define $\tilde{f}_j := \pi_j^{-1}$. Then N is a C^∞-manifold with atlas $\{(\tilde{V}_j, \tilde{f}_j)\}_j$, and $\tilde{F}_j^k : \tilde{f}_k(\tilde{V}_k \cap \tilde{V}_j) \to \tilde{f}_j(\tilde{V}_k \cap \tilde{V}_j)$ is the associate transformation.

Define maps $f^{(j)} : B(x_j, s) \to E_j$ $(j = 1, \dots, N)$ by

$$\begin{aligned} f^{(1)}(x) &:= f_1(x), \\ f^{(2)}(x) &:= \psi_1 \circ f^{(1)}(x) \cdot \tilde{F}_2^1 \circ f^{(1)}(x) + (1 - \psi_1 \circ f^{(1)}(x)) f_2(x), \\ &\dots \end{aligned}$$

Set $\hat{V}_j := f^{(j)-1}(V_j)$. Then we have

$$f^{(j)} = \tilde{F}_j^k \circ f^{(k)}$$

on $\hat{V}_j \cap \hat{V}_k$. Indeed, for instance, $f^{(2)} = \tilde{F}_2^1 \circ f^{(1)}$ on $B_1 \cap B_2$. For the general case, we refer to [O, pp. 1272-1273]. Set $U := \bigcup_j \hat{V}_j$. A homeomorphism $f : U \to N$ is defined to be the inductive limit of $\pi \circ f^{(j)}$.

By [O, Lemma 8], we obtain the following properties of $f^{(j)}$:

$$|f_j(x) - f^{(j)}(x)| < \theta(\varepsilon)s, \tag{3.17}$$

$$|df_j(\xi) - df^{(j)}(\xi)| < \theta(\varepsilon) \tag{3.18}$$

for all $x \in B(x_j, 0.4s)$ and $\xi \in \Sigma_x$.

Let $\{\chi_j\}_j$ be a smooth partition of unity such that $\mathrm{supp}\,(\chi_j) \subset \tilde{V}_j$. The desired Riemannian metric g_N on N is defined by

$$(g_N)_x(v, w) := \sum_j \chi_j(x) \langle d\tilde{f}_j(v), d\tilde{f}_j(w) \rangle \tag{3.19}$$

for any $x \in N$ and $v, w \in T_xN$.

Up to here, we reviewed the construction of a smooth approximation $f : U \to N$ by [KMS] (and [O]). Next, we construct the desired flow.

We first remark that

Lemma 3.5. *For each j, $f^{(j)-1} : V_j \to \hat{V}_j$ is differentiable. Hence, f and f^{-1} are also differentiable.*

Proof. Since $f^{(j)}$ is differentiable, for any scale (o), the following diagram commutes:

$$\begin{tikzcd} \left(\hat{V}_j\right)^{(o)}_x \arrow[r, "\left(f^{(j)}\right)^{(o)}_x"] \arrow[d, "\hat{\rho}^{(o)}"'] & (V_j)^{(o)}_y \arrow[d, "\rho^{(o)}"] \\ T_x M \arrow[r, "d_x f^{(j)}"'] & T_y E_j \end{tikzcd}$$

where $y := f^{(j)}(x)$ and $\hat{\rho}^{(o)}$ and $\rho^{(o)}$ are canonical isometries. We will omit the symbol (o) to write $\hat{\rho} := \hat{\rho}^{(o)}$ and $\rho := \rho^{(o)}$.

Since $f^{(j)}$ is $\theta(\varepsilon)$-isometric, $\left(f^{(j)}\right)^{(o)}_x$and $\left(f^{(j)-1}\right)^{(o)}_y$ are also. We define a map $A : T_y E_j \to T_x M$ by

$$A := \hat{\rho} \circ \left(f^{(j)-1}\right)^{(o)}_y \circ \rho^{-1}.$$

Then we have

$$\begin{aligned} A \circ d_x f^{(j)} &= \mathrm{id}_{T_x M}, \\ d_x f^{(j)} \circ A &= \mathrm{id}_{T_y E_j}. \end{aligned}$$

Namely, $A = \left(d_x f^{(j)}\right)^{-1}$ is determined independently of the choice of (o). By its construction, $A = d_y(f^{(j)-1})$ is well-defined. Thus $f^{(j)-1}$ is differentiable.

Since f is the composition of differentiable map $f^{(j)}$ and smooth map π_j, f and f^{-1} are also differentiable. $\square$

Set $y_j := y_j^{+1}$. Remark that y_j can be taken to satisfy the following:

$$\begin{cases} |x_j y_j| = t, \\ \angle Sxy_j \geq \tilde{\angle} Sxy_j > \pi - \delta' - \theta(\varepsilon, s/t) \end{cases} \tag{3.20}$$

for all $x \in B(x_j, s)$.

Now, let us forget the construction of f_j above. We will use the following notation:

$$f_j := f^{(j)}. \tag{3.21}$$

We set

$$\begin{aligned} Y_j(x) &:= \Uparrow_x^{y_j} \in \Sigma_x M, \\ Z_j(x) &:= \frac{f_j(y_j^{+1}) - f_j(x)}{|f_j(y_j^{+1}) - f_j(x)|} \in E_j, \end{aligned}$$

for all $x \in B(x_j, s)$.

We recall that f_j is $\theta(\varepsilon)$-isometric on $B(x_j, t)$. It follows from (3.10) that we have

$$df_j(Y_j(x)) = Z_j(x) + \vec{v}(\theta(\varepsilon)) \tag{3.22}$$

for any $x \in B(x_j, 0.4s)$.

Since

$$\angle_x(S', Y_j) + \angle_x(S', Y_k) + \angle_x(Y_j, Y_k) \le 2\pi,$$

we have

$$\angle(Y_j(x), Y_k(x)) < 2\delta' + \theta(\varepsilon)$$

for all $x \in B(x_j, s) \cap B(x_k, s)$. Then, we have

$$\begin{aligned} d(Y_j, Y_k)^2 &< 2(1 - \cos(2\delta' + \theta(\varepsilon))) \\ &\le 2\delta'^2 + \theta(\varepsilon). \end{aligned}$$

Therefore, we obtain

$$df_j(Y_k(x)) = df_j(Y_j(x)) + \vec{v}(\sqrt{2}\delta' + \theta(\varepsilon)) \tag{3.23}$$

$$= Z_j(x) + \vec{v}(\sqrt{2}\delta' + \theta(\varepsilon)). \tag{3.24}$$

Note that Z_j is smooth on $V_j \subset E_j$. We define a smooth vector field $\tilde{W}_j$ on $\tilde{V}_j \subset N$ by

$$\tilde{W}_j(x) := d\tilde{f}_j^{-1}(Z_j(x)).$$

We next prove the following.

Lemma 3.6. *For any $x \in \tilde{V}_j \cap \tilde{V}_k$, we have*

$$|\tilde{W}_j(x) - \tilde{W}_k(x)|_N < 4\sqrt{2}\delta' + \theta(\varepsilon).$$

Proof. At first, we see

$$\begin{aligned} |\tilde{W}_j - \tilde{W}_k|_N^2 &= \sum_{\ell=1}^{N} \chi_\ell \left| d\tilde{f}_\ell \left(\tilde{W}_j - \tilde{W}_k \right) \right|_{E_\ell}^2 \\ &= \sum_{\ell=1}^{N} \chi_\ell \left| d\tilde{F}_\ell^j(Z_j) - d\tilde{F}_\ell^k(Z_k) \right|_{E_\ell}^2 . \end{aligned}$$

By (3.24), we have

$$\begin{aligned} d\tilde{F}_\ell^k(Z_k) &= d\tilde{F}_\ell^k(df_k(Y_k)) + \vec{v}(\theta(\varepsilon)) \\ &= df_\ell(Y_k) + \vec{v}(\theta(\varepsilon)) \\ &= Z_\ell + \vec{v}(2\sqrt{2}\delta' + \theta(\varepsilon)). \end{aligned}$$

Therefore, Lemma 3.6 is proved. □

We next define a smooth vector field $\tilde{W}$ on N by

$$\tilde{W}(x) := \sum_{j=1}^{N} \chi_j(x)\, \tilde{W}_j(x). \tag{3.25}$$

By Lemma 3.6, we have

$$\begin{aligned} |\tilde{W} - \tilde{W}_j| &= \left| \sum_\ell \chi_\ell \cdot \tilde{W}_\ell - \tilde{W}_j \right| \\ &\le \sum_\ell \chi_\ell |\tilde{W}_\ell - \tilde{W}_j| < 4\sqrt{2}\delta' + \theta(\varepsilon) \end{aligned}$$

on $\tilde{V}_j \subset N$.

We consider an integral flow $\tilde\Phi$ of $\tilde W$, namely,

$$\frac{d\tilde\Phi}{dt}(x,t) = \tilde W(\tilde\Phi(x,t)).$$

We now define a flow Φ on U by

$$\Phi(x,t) := f^{-1}\big(\tilde\Phi(f(x),t)\big).$$

Lemma 3.7. *The conclusion (ii) of Theorem 3.2 holds:*

$$\left.\frac{d}{dt}\right|_{t=0+} \mathrm{dist}_S \circ \Phi(x,t) > 1 - 5\sqrt{\delta} - \theta(\varepsilon),$$
$$(\tilde d_S)'(\tilde W(\tilde x)) > 1 - 5\sqrt{\delta} - \theta(\varepsilon)$$

for all $x \in U$.

Proof. By Lemma 3.5, f is differentiable. Therefore, the flow curve

$$\Phi(x,\cdot) : I_x \to U$$

is differentiable for any $x \in U$. Then $x \in V_j$ for some j. Then, we have

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} d_S \circ \Phi(x,t) &= (d_S \circ f^{-1})'\left(\frac{d}{dt} f \circ \Phi(x,t)\right)\\
&= (d_S \circ f^{-1})'\left(\tilde W(f(x))\right) = (d_S \circ f_j^{-1})' \circ d\tilde f_j(\tilde W)\\
&= (d_S \circ f_j^{-1})' \circ d\tilde f_j(\tilde W_j + 4\sqrt{2}\vec v(\delta' + \theta(\varepsilon)))\\
&= (d_S \circ f_j^{-1})'(Z_j + 4\sqrt{2}\vec v(\delta' + \theta(\varepsilon)))\\
&= (d_S \circ f_j^{-1})'(df_j(Y_j) + 4\sqrt{2}\vec v(\delta' + \theta(\varepsilon)))\\
&> d_S'(Y_j) - 4\sqrt{2}\delta' - \theta(\varepsilon)\\
&> 1 - \delta - 4\sqrt{2}\delta' - \theta(\varepsilon)\\
&> 1 - 5\sqrt{\delta} - \theta(\varepsilon).
\end{aligned}$$

□

Lemma 3.8. *The conclusion (i) of Theorem 3.2 holds: For any $x \in U$, $\Phi(x,t)$ is a $5\sqrt{\delta} + \theta(\varepsilon)$-isometric embedding in $t \in I_x$. Here, $I_x := \{t \in \mathbb{R} \mid \Phi(x,t) \in U\}$.*

Proof. By the construction of $\tilde W$, we have $|\tilde W| \le 1 + \theta(\varepsilon)$. Indeed, for any $t, t' \in I_x$ with $t < t'$, we obtain

$$\begin{aligned}
d\left(\tilde\Phi(f(x),t'), \tilde\Phi(f(x),t)\right) &\le \int_t^{t'} \left|\tilde W\left(\tilde\Phi(f(x),s)\right)\right| ds\\
&\le |1 + \theta(\varepsilon)|(t' - t).
\end{aligned}$$

Then we have

$$\begin{aligned}
\frac{d\left(\Phi(x,t'), \Phi(x,t)\right)}{t' - t} &= \frac{|\Phi(x,t'), \Phi(x,t)|}{\left|\tilde\Phi(f(x),t'), \tilde\Phi(f(x),t)\right|} \cdot \frac{\left|\tilde\Phi(f(x),t'), \tilde\Phi(f(x),t)\right|}{t' - t}\\
&\le 1 + \theta(\varepsilon).
\end{aligned}$$

By Lemma 3.7, for $t < t'$ in I_x, we obtain

$$\begin{aligned} d\left(\Phi(x,t'),\Phi(x,t)\right) &\geq d(S,\Phi(x,t')) - d(S,\Phi(x,t)) \\ &= \int_t^{t'} (d_S)'(W)ds \geq (1 - 5\sqrt{\delta} - \theta(\varepsilon))(t'-t). \end{aligned}$$

This completes the proof of Lemma 3.8. $\square$

Combining Lemmas 3.8 and 3.7, we obtain the conclusions of Theorem 3.2. $\square$

Definition 3.9. Let M be an Alexandrov space, $f : M \to \mathbb{R}$ be a Lipschitz function and $\Phi : M \times \mathbb{R} \to M$ be a Lipschitz flow. Let M' be a subset of M. We say that Φ is *gradient-like for f on M'* if there exists a constant $c > 0$ such that for any $x \in M'$ we have

$$\liminf_{t\to 0} \frac{f(\Phi(x,t)) - f(x)}{t} > c.$$

We denote by

$$\Phi \pitchfork f \text{ on } M'$$

this situation.

In this notation, we obtained in Theorem 3.2 a gradient-like flow Φ for dist_S on $U(C)$ with a constant $c = 1 - 5\sqrt{\delta} - \theta(\varepsilon)$.

3.2. **Flow and fibration.** We will find a nice relation between Fibration Theorems 2.24 and 2.25 and Flow Theorem 3.2. We first recall an important property of Yamaguchi's fibration.

Proposition 3.10 (cf. Lemma 4.6 in [Y conv])**.** *Let M and X be Alexandrov spaces and $\pi : M \to X$ be a $\theta(\delta,\varepsilon)$-Lipschitz submersion as in Theorem 2.24. Let $(o) = (\varepsilon_i)$ be an arbitrary scale. We denote by H_x a set of horizontal directions to the fiber $\pi^{-1}(\pi(x))$ at x. Then for any $x \in M$, the restriction of the blow-up*

$$\pi_x^{(o)} \circ \exp_x^{(o)} : H_x \to X^{(o)}_{\pi(x)}$$

satisfies the following: For any Y, $Z \in H_x$, we have

$$\left| \left| \pi_x^{(o)} \circ \exp_x^{(o)}(Y), \pi_x^{(o)} \circ \exp_x^{(o)}(Z) \right| - |Y,Z| \right| < \theta(\delta,\varepsilon).$$

Here, the set of horizontal directions is defined in [Y conv, §4] as

$$H_x := \{\xi \in y'_x \,|\, |xy| \geq \sigma\}$$

for some small number $\sigma > 0$ with $\varepsilon \ll \sigma$.

Proof of Proposition 3.10. We will use the following notation: θ denotes a variable constant $\theta(\delta,\varepsilon)$. We set $\bar{x} = \pi(x)$ for any $x \in M$.

Let us take $Y \in H_x$. By the definition of H_x, there is a point $y \in M$ such that

$$|xy| \geq \sigma, \ \angle(y', Y) < \theta.$$

Then, by Lemma 4.6 in [Y conv], for any $\bar{Y} \in \bar{y}' \subset \Sigma_{\bar{x}} X$, we have

$$\frac{|\pi(\gamma_Y(t)), \gamma_{\bar{Y}}(t)|}{t} < \theta \tag{3.26}$$

for any small $t > 0$. Here, γ_ξ denotes the geodesic from $\gamma_\xi(0)$ tangent to $\xi \in \Sigma_{\gamma(0)}$. Let $(o) = (\varepsilon_i)$ be an arbitrary scale. From (3.26), we have

$$|\pi_x^{(o)} \circ \exp_x^{(o)}(Y), \exp_{\bar{x}}^{(o)}(\bar{Y})| = \lim_\omega \frac{|\pi(\gamma_Y(\varepsilon_i)), \gamma_{\bar{Y}}(\varepsilon_i)|}{\varepsilon_i} < \theta. \tag{3.27}$$

We next take any $Z \in H_x$. Then there exists $z \in M$ such that

$$|xz| \geq \sigma,\ \angle(z', Z) < \theta.$$

Then, for any $\bar{Z} \in \bar{z}' \subset \Sigma_{\bar{x}} X$, we have

$$|\pi_x^{(o)} \circ \exp_x^{(o)}(Z), \exp_{\bar{x}}^{(o)}(\bar{Z})| < \theta. \tag{3.28}$$

On the other hand, by Lemma 4.7 in [Y conv], we have

$$|\angle(Y, Z) - \angle(\bar{Y}, \bar{Z})| < \theta.$$

It follows together with (3.26), (3.27) and (3.28) that we obtain

$$||\pi_x^{(o)} \circ \exp_x^{(o)}(Y), \pi_x^{(o)} \circ \exp_x^{(o)}(Z)| - |Y, Z|| < \theta.$$

This completes the proof. □

Theorem 3.11. *For any $n \in \mathbb{N}$, there is a positive number $\epsilon = \epsilon_n$ satisfying the following: Let M^n be an n-dimensional Alexandrov space without boundary with curvature ≥ -1 and p be a point of M^n. Let X^{n-1} be an $(n-1)$-dimensional non-negatively curved Alexandrov space. Assume that X is given by the Euclidean cone $K(\Sigma)$ over a closed Riemannian manifold Σ of curvature ≥ 1. If $d_{GH}((M, p), (X, p_0)) < \epsilon$, where p_0 is the origin of the cone X, then there exists a small $r = r_p > 0$ such that a metric sphere $\partial B(p, r)$ is homeomorphic to an S^1-fiber bundle over Σ.*

Proof. $d_{GH}((M, p), (X, p_0)) < \varepsilon$ implies $d_{GH}(B_M(p, 1/\varepsilon), B_X(p_0, 1/\varepsilon)) < \varepsilon$. Take a small number $r > 0$ such that $r \ll 1 \ll 1/\varepsilon$. Since Σ is a closed Riemannian manifold, $A(p_0; r/2, 2)$ is a Riemannian manifold $\approx \Sigma \times [r/2, 2]$ with boundary $\approx \Sigma \times \{r/2, 2\}$.

Let C be an annulus $C := A(p; r/2, 2)$. Since $d_{GH}(C, A(p_0; r/2, 2)) < \varepsilon$, C is $(n-1, \varepsilon)$-strained. Since M has no boundary points, Theorem 2.10 implies that C is $(n, \theta(\varepsilon))$-strained. Therefore by Theorem 2.25, there exists a $\theta(\varepsilon)$-Lipschitz submersion $\pi : M_1 \to A(p_0; r/2, 2)$ which is actually an S^1-fiber bundle. Here, M_1 is some closed domain in M near C containing $A(p; r, 1)$.

Set $S := \pi^{-1}(\partial B(p_0, r))$. Let $\Phi = \Phi(x, t)$ be a gradient-like flow for dist_p obtained by Theorem 3.2 on an annulus around p.

We are going to prove

Lemma 3.12. *The flow Φ is gradient-like for $\mathrm{dist}_{p_0} \circ \pi$. Namely, we obtain the following:*

$$\liminf_{t \to 0+} \frac{\mathrm{dist}_{p_0} \circ \pi \circ \Phi(x, t) - \mathrm{dist}_{p_0} \circ \pi(x)}{t} > 1 - \theta(\varepsilon) \tag{3.29}$$

for any $x \in M_1$.

If it is proved, then S is homeomorphic to $\partial B(p, r)$ by a standard flow argument.

Proof of Lemma 3.12. Let us set $\bar{x} := \pi(x)$ for any $x \in M_1$. We set

$$V := \frac{d}{dt}\Big|_{t=0+} \Phi(x,t) \in T_x M.$$

By Theorem 3.2 (ii), we have

$$V \doteqdot \nabla \mathrm{dist}_p$$

and $|V| \doteqdot 1$. Here, $A \doteqdot A'$ means that $d(A, A') < \theta(\varepsilon)$.

We set $\xi := V/|V|$ and recall that $\xi \in H_x$. It follows together with (3.27) that there exists $q \in M$ with $|xq| \geq \sigma$ such that any $\bar{\xi} \in \bar{q}' \subset \Sigma_{\bar{x}} X$ satisfies

$$\pi_x^{(o)} \circ \exp_x^{(o)}(\xi) \doteqdot \exp_{\bar{x}}^{(o)}(\bar{\xi})$$

for each scale (o).

Let us take $\eta \in p_x' \subset \Sigma_x M$. Then we have

$$\angle(\xi, \eta) > \pi - \theta(\varepsilon).$$

Since $\eta \in H_x$, there exists $\bar{\eta} \in \Sigma_{\bar{x}} X$ such that

$$\pi_x^{(o)} \circ \exp_x^{(o)}(\eta) \doteqdot \exp_{\bar{x}}^{(o)}(\bar{\eta}).$$

By Proposition 3.10, we obtain

$$\angle(\bar{\xi}, \bar{\eta}) > \pi - \theta(\varepsilon). \tag{3.30}$$

On the other hand, from Lemma 4.3 in [Y conv], π is $\theta(\varepsilon)$-close to an ε-approximation from (M, p) to (X, p_0). This implies

$$\tilde{\angle} \bar{q} \bar{x} p_0 > \pi - \theta(\varepsilon).$$

We take an arbitrary direction $\bar{\zeta} \in p_0' \subset \Sigma_{\bar{x}} X$. Then, we have

$$\angle(\bar{\zeta}, \bar{\xi}) > \pi - \theta(\varepsilon). \tag{3.31}$$

By (3.30) and (3.31), we have

$$\bar{\xi} \doteqdot \nabla \mathrm{dist}_{p_0}.$$

Summarizing the above arguments, we obtain

$$\begin{aligned} \lim_{\omega} \frac{d(p_0, \pi \circ \Phi(x, \varepsilon_i)) - d(p_0, \bar{x})}{\varepsilon_i} &= (\mathrm{dist}_{p_0})'_{\bar{x}} \circ (\exp_{\bar{x}}^{(o)})^{-1} \circ \pi_x^{(o)} \circ \exp_x^{(o)}(V) \\ &\doteqdot (\mathrm{dist}_{p_0})'_{\bar{x}}(\bar{\xi}) \\ &\doteqdot (\mathrm{dist}_{p_0})'_{\bar{x}}(\nabla \mathrm{dist}_{p_0}) \\ &= 1. \end{aligned}$$

It follows from Lemma 2.14 that we obtain (3.29). □

As mentioned above, by Lemma 3.12, we have $\partial B(p, r) \approx S$. This completes the proof of Theorem 3.11. □

Remark 3.13. Kapovitch proved a statement similar to Theorem 3.11 for collapsing Riemannian manifolds M ([Kap Rest, Theorem 7.1]).

Perelman and Petrunin proved the existence and uniqueness of a gradient flow of any semiconcave function, especially of any distance function ([Pet QG], [PP QG]). Note that the gradient "flow" is not a flow in the sense of Definition 3.1, because the gradient flow is defined on $M \times [0, \infty)$.

Remark 3.14. One might ask why we do not use the gradient flow of a distance function to prove Theorem 3.11. The reason is that the gradient flow may not be injective.

For instance, we consider the cone $X = K(S^1_\theta)$ over a circle S^1_θ with length $\theta < 2\pi$. X is expressed by the quotient of a set

$$X_0 = \{re^{it} \in \mathbb{C} \mid r \geq 0, t \in [0, \theta]\}$$

by a relation $r \sim re^{i\theta}$ for $r \geq 0$. By $[re^{it}] \in X$ we denote the equivalent class of $re^{it} \in X_0$. We fix $r > 0$ and take $p := re^{i\theta/2}$. Let $a > 0$ be a sufficiently small number such that $S_a \cap \partial X_0 = \emptyset$. Here, we denote by S_a the circle centered at p with radius a in $\mathbb{C}$. We take b with $a < b < r$ such that $S_b \cap \partial X_0 \neq \emptyset$ and take x_1, x_2 with $x_1 \neq x_2$ in $S_b \cap \partial X_0$ near p. Then $[x_1] = [x_2]$ in X. We put points $y_i \in px_i \cap S_a$ in X_0 and set geodesics $\gamma_i := [y_i][x_i]$ in X for $i = 1, 2$. In particular, γ_i $(i = 1, 2)$ are the gradient curves for $d_{[p]}$ in X. This case says that $d_{[p]}$-flow does not injectively send $[S_a] := \{[z] \in X \mid x \in S_a\}$ to $[S_b]$.

3.3. **Flow arguments.**

Theorem 3.15. *For a positive integer n, there is a positive constant ε_n satisfying the following: Let M^n be an n-dimensional Alexandrov space with curvature ≥ -1. Let $A_1, A_2, \dots, A_m \subset M$ be closed subsets and $C \subset M$ be an (n, ε)-strained compact subset with $A_i \cap C = \emptyset$ for all $i = 1, 2, \dots, m$ and for $\varepsilon \leq \varepsilon_n$. Suppose the following: For each $x \in C$ and $1 \leq i \leq m$, there is a point $w = w(x) \in M$ such that*

$$\angle_x(A_i', w') > \pi - \delta. \tag{3.32}$$

Here, $c\,(< \pi/2)$ is a positive constant bigger than some constant. Then there exist an open neighborhood U of C and a Lipschitz flow Φ on M such that

$$\left.\frac{d}{dt}\right|_{t=0+} d_{A_i}(\Phi(x,t)) > 1 - 5\delta - \theta(\varepsilon) \tag{3.33}$$

for all $i = 1, \dots, m$ and $x \in U$.

Proof. We can show the following: There exists a precompact open neighborhood U of C such that each $x \in U$ is $(n, \theta(\varepsilon))$-strained, and there is a point $w = w(x) \in M$ such that

$$|xw(x)| > \ell, \tag{3.34}$$

$$\tilde{\angle} A_i y w(x) > \pi - \delta - \theta(\varepsilon), \tag{3.35}$$

for all $y \in B(x, r)$ and $i = 1, \dots, m$. Here, r and ℓ are positive numbers with $r \ll \ell$.

Since U is $(n, \theta(\varepsilon))$-strained, there is a smooth approximation $f : U \to N$ which is a $\theta(\varepsilon)$-isometry for some Riemannian manifold N. By an argument similar to the proof of Theorem 3.2, we can construct a smooth vector field $\tilde{W}$ and its integral flow $\tilde{\Phi}$ on N such that

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0+} \mathrm{dist}_{A_i} \circ f^{-1}(\tilde{\Phi}(x,t)) &= \mathrm{dist}'_{A_i} \circ df^{-1}(\tilde{W}) \\ &> 1 - 5\delta - \theta(\varepsilon).\end{aligned}$$

Then, the pull-back flow $\Phi_t := \tilde{\Phi}_t \circ f$ satisfies the conclusion of Theorem 3.15. $\square$

Corollary 3.16. *Let M^n, A_1, A_2, $\dots$, A_m and C be the same as in the assumption of Theorem* 3.15 *and satisfy the following: All d_{A_i} are $(1-\delta)$-regular at x, $m \le n$ and*

$$|\angle_x(A_i', A_j') - \pi/2| < \mu \tag{3.36}$$

for any $x \in C$ and $1 \le i \ne j \le m$.

If $\nu := \delta + \mu$ is smaller than some constant depending on m, then there are a Lipschitz flow Φ and a neighborhood U of C satisfying the following:

$$\left.\frac{d}{dt}\right|_{t=0+} d_{A_i}(\Phi(x,t)) > 1 - 5\sqrt{\delta} - \theta(\varepsilon) - \theta_m(\nu) \tag{3.37}$$

for any $x \in U$ and $i = 1, \dots, m$.

Proof. Let us consider a smooth approximation

$$f : U \to N$$

for some neighborhood U of C and a Riemannian manifold N. By Lemmas 3.7 and 3.8, we obtain smooth vector fields $\tilde{W}_i$ on N such that

$$|\tilde{W}_i| \le 1 + \theta(\varepsilon), \tag{3.38}$$

$$(d_{A_i})'(W_i) > 1 - 5\sqrt{\delta} - \theta(\varepsilon) \tag{3.39}$$

on N for all $i = 1, \cdots, m$. Here, $W_i := df^{-1}(\tilde{W}_i)$.

Let us define $\varphi_m(\nu) \in (\pi/2, \pi)$ by

$$\cos\varphi_m(\nu) = \frac{1 - (m-1)\cos(\pi/2 - \nu)}{\sqrt{m}\sqrt{1 + (m-1)\cos(\pi/2 - \nu)}}.$$

Note that $\cos\varphi_m(\nu) \to 1/\sqrt{m}$ as $\nu \to 0$.

Let us consider a vector field

$$\tilde{W} := (\tilde{W}_1 + \tilde{W}_2 + \cdots + \tilde{W}_m)/|\tilde{W}_1 + \tilde{W}_2 + \cdots + \tilde{W}_m|.$$

Since $|\angle(A_i', A_j') - \pi/2| < \mu$, we have

$$|\angle(W_i, W_j)| < 10\delta + \mu + \theta(\varepsilon).$$

Putting $W := df^{-1}(\tilde{W})$, we obtain

$$\cos\angle(W_i, W) \ge \cos(\varphi_m(\nu) + \theta(\varepsilon))$$

for $\nu = 10\delta + \mu$. Then we have

$$\begin{aligned}(d_{A_i})'(W) &\ge (d_{A_i})'(W_i) - |W, W_i| \\ &\ge 1 - 5\sqrt{\delta} - \cos(\varphi_m(\nu)) - \theta(\varepsilon).\end{aligned}$$

We consider the gradient flow Φ of the vector field W on U, which is the desired flow. $\square$

4. The case that $\dim X = 2$ and $\partial X = \emptyset$

In this and the next sections, we study the topologies of three-dimensional closed Alexandrov spaces which collapse to Alexandrov surfaces. First, we exhibit examples of three-dimensional Alexandrov spaces (which are closed or open) collapsing to Alexandrov surfaces.

We denote a circle of length ε by S^1_ε. We often regard S^1_ε as $\{x \in \mathbb{C} \mid \|x\| = \varepsilon/2\pi\}$, and $\bar{x}$ denotes the complex conjugate for $x \in \mathbb{C}$.

Example 4.1. Recall that M_{pt} is obtained by the quotient space $M_{\mathrm{pt}} := S^1 \times \mathbb{R}^2/(x,y) \sim (\bar{x}, -y)$. M_{pt} have collapsing metrics d_ε and ρ_ε as follows.

Recall that a collapsing metric provided Example 1.2. The quotient $(M_{\mathrm{pt}}, d_\varepsilon) := S^1_\varepsilon \times \mathbb{R}^2/(x,y) \sim (\bar{x}, -y)$ has a metric d_ε of non-negative curvature collapsing to $K(S^1_\pi) = \mathbb{R}^2/y \sim -y$ as $\varepsilon \to 0$.

We consider an isometry defined by

$$K(S^1_\varepsilon) \ni [t,v] \mapsto [t,-v] \in K(S^1_\varepsilon).$$

Here, $t \geq 0$ and $v \in S^1_\varepsilon$. Note that $K(S^1_\varepsilon)$ collapses to $\mathbb{R}_+$ as $\varepsilon \to 0$. We consider a metric ρ_ε on M_{pt} of non-negative curvature defined by taking the quotient of the direct product $S^1 \times K(S^1_\varepsilon)$:

$$(M_{\mathrm{pt}}, \rho_\varepsilon) := S^1 \times K(S^1_\varepsilon)/(x,t,v) \sim (\bar{x}, t, -v).$$

Then, $(M_{\mathrm{pt}}, \rho_\varepsilon)$ collapses to $[0,\pi] \times \mathbb{R}_+$ as $\varepsilon \to 0$. Here, $[0,\pi]$ is provided as $S^1/x \sim \bar{x}$.

Example 4.2. Let $\Sigma(S^1_\varepsilon)$ be the spherical suspension of S^1_ε, which has curvature ≥ 1. Any point in $\Sigma(S^1_\varepsilon)$ is expressed as $[t,v]$ parametrized by $t \in [0,\pi]$ and $v \in S^1_\varepsilon$. We consider an isometry

$$\alpha : \Sigma(S^1_\varepsilon) \ni [t,v] \mapsto [\pi - t, -v] \in \Sigma(S^1_\varepsilon).$$

Then, we obtain a metric d_ε on P^2 of curvature ≥ 1 defined by taking the quotient $\Sigma(S^1_\varepsilon)/\langle\alpha\rangle$. We set $P^2_\varepsilon := (P^2, d_\varepsilon)$. Note that P^2_ε collapses to $[0,\pi/2]$ as $\varepsilon \to 0$. Then, $K(P^2_\varepsilon)$ collapses to $K([0,\pi/2]) \equiv \mathbb{R}_+ \times \mathbb{R}_+$ as $\varepsilon \to 0$.

Remark that $K(P^2_\varepsilon)$ is isometric to the quotient space $\mathbb{R} \times K(S^1_\varepsilon)/\langle\sigma\rangle$ defined as follows: Let σ be an involution defined by

$$\sigma(x, tv) \mapsto (-x, t(-v))$$

for $x \in \mathbb{R}$, $t \geq 0$ and $v \in S^1_\varepsilon$. We sometimes use this expression in the paper.

Example 4.3. Let us consider the direct product $S^1 \times \Sigma(S^1_\varepsilon)$ and an isometry

$$\beta : S^1 \times \Sigma(S^1_\varepsilon) \ni (x,t,v) \mapsto (\bar{x}, t, -v) \in S^1 \times \Sigma(S^1_\varepsilon).$$

Then, the quotient space $N_\varepsilon := S^1 \times \Sigma(S^1_\varepsilon)/\langle\beta\rangle$ has non-negative curvature, and N_ε collapses to $[0,\pi] \times [0,\pi]$ as $\varepsilon \to 0$.

Let us start the proof of Theorem 1.3.

Proof of Theorem 1.3. Fix a sufficiently small $\delta > 0$. Then there are only finitely many $(2,\delta)$-singular points $x_1, \dots, x_k$ in X. For sufficiently small $r > 0$, consider the set $X' := X - (U(x_1, r) \cup \cdots \cup U(x_k, r))$. By Theorem 2.10, there exists a $(3, \theta(i,\delta))$-strained closed domain $M_i' \subset M_i$ which is converging to X'. From Theorem 2.25, we may assume that there exists a circle fiber bundle $\pi_i' : M_i' \to X'$ which is a $\theta(i,\delta)$-almost Lipschitz submersion. Here, $\theta(i,\delta)$ is a positive constant such that $\lim_{i\to\infty,\delta\to 0}\theta(i,\delta)=0$. We fix a large i and use the notation $\theta(\delta)=\theta(i,\delta)$ for simplicity.

Fix any $(2,\delta)$-singular point $p\in\{x_1,\dots,x_k\}\subset X$ and take a sequence $p_i\in M_i$ converging to p. Since the Flow Theorem implies that $B(p_i,r)$ is not contractible, applying the rescaling argument Theorem 2.27, we have points $\hat{p}_i\in B(p_i,r)$ with $d(p_i,\hat{p}_i)\to 0$ and a scaling constant δ_i such that any limit space (Y,y_0) of $\lim_{i\to\infty}(\frac{1}{\delta_i}B(\hat{p}_i,r),\hat{p}_i)$ is a three-dimensional open Alexandrov space of non-negative curvature. We may assume that $p_i=\hat{p}_i$. We denote by S a soul of Y. By Theorem 2.27, we have $\dim S\le 1$.

From Theorem 3.11, the boundary $\partial B(p_i,r)$ is homeomorphic to a torus T^2 or a Klein bottle K^2. It follows from the Soul Theorem 2.59 and the Stability Theorem 2.34 that $B(p_i,r)$ is homeomorphic to the orbifold $B(\mathrm{pt})$ if $\dim S=0$ or a solid torus $S^1\times D^2$ or a solid Klein bottle $S^1\tilde{\times}D^2$ if $\dim S=1$.

We first consider the case of $\dim S=1$; namely, S is a circle. In this case, we obtain the following conclusion.

Lemma 4.4. *If $\dim S=1$, then $B(p_i,r)$ is homeomorphic to $S^1\times D^2$.*

Proof. Put $B_i:=B(p_i,r)$, $B:=B(p,r)$ and $\varepsilon_i:=d_{GH}(B_i,B)$. Suppose that B_i is homeomorphic to a solid Klein bottle $S^1\tilde{\times}D^2$. Take $r_i\to 0$ with $\varepsilon_i/r_i\to 0$ such that $\lim(\frac{1}{r_i}B_i,p_i)=(T_pX,o)$. Let $\pi_i:\tilde{B}_i\to B_i$ be a universal covering and $\tilde{p}_i\in\pi_i^{-1}(p_i)$. Let $\Gamma_i\cong\mathbb{Z}$ be the deck transformation group of π_i. Passing to a subsequence, we have a limit triple (Z,z,G) of a sequence of triples of pointed spaces and isometry groups $(\frac{1}{r_i}\tilde{B}_i,\tilde{p}_i,\Gamma_i)$ in the equivariant pointed Gromov-Hausdorff topology (cf. [FY]). Z is an Alexandrov space of non-negative curvature because of $r_i\to 0$, and G is abelian. Note that $Z/G=\lim(\frac{1}{r_i}B_i,p_i)=(T_pX,o)$. Using the G-action, we find a line in Z ([ChGr]). Then, by the Splitting Theorem, there is some non-negatively curved Alexandrov space Z_0 such that Z is isometric to the product $\mathbb{R}\times Z_0$. We may assume that Z_0 is a cone by taking a suitable rescaling $\{r_i\}$. We denote by G_0 the identity component of G. By [FY, Lemma 3.10], there is a subgroup Γ_i^0 of Γ_i such that:

(1) $(\frac{1}{r_i}\tilde{B}_i,\tilde{p}_i,\Gamma_i^0)$ converges to (Z,z,G_0).
(2) $\Gamma_i/\Gamma_i^0\cong G/G_0$ for large i.

Since $\dim T_pX=2$ and $\dim Z=3$, we have $\dim G=1$. This implies $G\cong\mathbb{R}\times H$ for some finite abelian group H. Since $T_pX=Z_0/H$, H must be cyclic. Here, G-action is component-wise: $G_0\cong\mathbb{R}$ acts by translation of the line $\mathbb{R}$ and H acts on Z_0 independently. By Stability Theorem 2.34, Z is simply-connected. Therefore, Z_0 is homeomorphic to $\mathbb{R}^2$.

Take a generator γ_i of Γ_i. From our assumption, γ_i is orientation reversing on $\tilde{B}_i$. Consider $\Gamma_i':=\langle\gamma_i^2\rangle\cong\mathbb{Z}$. Then Γ_i' acts on $\tilde{B}_i$ preserving orientation. Taking a subsequence, we have a limit triple (Z,z,G') of a sequence $\{(\frac{1}{r_i}\tilde{B}_i,\tilde{p}_i,\Gamma_i')\}$. By an argument similar to the above, $G'\cong\mathbb{R}\times H'$ for some finite cyclic group H'. Let $\lim_{i\to\infty}\gamma_i=\gamma_\infty\in G$, which implies that $\gamma_i(x_i)\to\gamma_\infty(x_\infty)$ under the Gromov-Hausdorff convergence $\frac{1}{r_i}\tilde{B}_i\ni x_i\to x_\infty\in Z$. Then γ_∞ is represented by $(0,\phi)\in\mathbb{R}\times H$. Then, for large i, we have

$$Z/G=(Z/G')/(G/G')=(Z_0/H')/\langle[\phi]\rangle=T_pX. \tag{4.1}$$

Since Z_0/H' is the flat cone over a circle or an interval, and $[\phi] \in H/H'$ acts on a Z_0/H' reversing orientation, $(Z_0/H')/\langle\phi\rangle$ cannot be T_pX. This is a contradiction. $\square$

By Lemma 4.4, $B(p_i, r/2)$ must be homeomorphic to a solid torus. From Flow Theorem 3.2, $(\pi_i')^{-1}(\partial B(p,r))$ and $\partial B(p_i, r/2)$ bound a closed domain homeomorphic to $T^2 \times [0,1]$, and this provides a circle fiber structure on $\partial B(p_i, r/2)$. By [SY00, Lemma 4.4], it extends to a topological Seifert structure on $B(p_i, r/2)$ over $B(p, r/2)$ which is compatible to the circle bundle structure on $A(p_i; r/2, r)$.

In the case of $\dim S = 0$, B_i is homeomorphic to $B(\mathrm{pt})$. We must prove that

Lemma 4.5. *If $\dim S = 0$, then B_i has the structure of circle fibration with a singular arc fiber satisfying:*

(1) *it is isomorphic to the standard fiber structure on $B(pt) = S^1 \times D^2/\mathbb{Z}_2$;*
(2) *it is compatible to the structure of circle fiber bundle π_i' near the boundary.*

Proof. Recall that $B(pt) = S^1 \times D^2/\mathbb{Z}_2$, where $\mathbb{Z}_2$-action on $S^1 \times D^2$ is given by the involution $\hat\sigma$ defined by $\hat\sigma(x,y) = (\bar x, -y)$. Let $p_+ := (1,0)$, $p_- := (-1,0)$ be the fixed points of $\hat\sigma$. Putting $\hat U := S^1 \times D^2 \setminus \{p_+, p_-\}$, and $U := \hat U/\mathbb{Z}_2$, let $\hat\pi : \hat U \to U$ be the projection map. Fix a homeomorphism $f_i : S^1 \times D^2/\mathbb{Z}_2 \to B_i$, and set $U_i := f_i(U)$. Take a $\mathbb{Z}_2$-covering $\hat\pi_i : \hat U_i \to U_i$ such that there is a homeomorphism $\hat f_i : \hat U \to \hat U_i$ together with the following commutaitve diagram:

$$\begin{array}{ccc} \hat U & \xrightarrow{\hat f_i} & \hat U_i \\ {\scriptstyle\hat\pi}\downarrow & & \downarrow{\scriptstyle\hat\pi_i} \\ U & \xrightarrow[f_i]{} & U_i \end{array}$$

Consider the length-metric on $\hat U_i$ induced from that of U_i via $\hat\pi_i$, and the length-metrics of U and $\hat U$ for which both f_i and $\hat f_i$ become isometries. Note that $U_i = \hat U_i/\hat\sigma_i$, where $\hat\sigma_i := \hat f_i \circ \hat\sigma \circ (\hat f_i)^{-1}$. $\hat\sigma_i$ extends to an isometry on the completion $\hat B_i$ of $\hat U_i$. Let $\hat\pi_i : \hat B_i \to B_i$ also denote the the projection. Then we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{R} \times D^2 & \xrightarrow{\tilde f_i} & \tilde B_i \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi_i} \\ S^1 \times D^2 & \xrightarrow{\hat f_i} & \hat B_i \\ {\scriptstyle\hat\pi}\downarrow & & \downarrow{\scriptstyle\hat\pi_i} \\ S^1 \times D^2/\mathbb{Z}_2 & \xrightarrow[f_i]{} & B_i, \end{array} \tag{4.2}$$

where $\pi_i : \tilde B_i \to \hat B_i$ is the universal covering, and $\tilde f_i$ is an isometry covering $\hat f_i$. Here we consider the metric on $\mathbb{R} \times D^2$ induced by that of $S^1 \times D^2$. Let $\sigma, \lambda : \mathbb{R} \times D^2 \to \mathbb{R} \times D^2$ be defined as

$$\sigma(x,y) = (-x, -y), \quad \lambda(x,y) = (x+1, y).$$

Since σ covers $\hat\sigma$, σ is an isometry. Put

$$\sigma_i := \tilde f_i \circ \sigma \circ (\tilde f_i)^{-1}, \quad \lambda_i := \tilde f_i \circ \lambda \circ (\tilde f_i)^{-1}.$$

From construction, the group Λ_i generated by λ_i is the deck transformation group of $\pi_i : \tilde{B}_i \to \hat{B}_i$. Let Λ be the group generated by λ. Let Γ_i (resp. Γ) be the group generated by σ_i and Λ_i (resp. by σ and Λ). Obviously we have an isomorphism $(\Gamma_i, \Lambda_i) \simeq (\Gamma, \Lambda)$. Note that

$$\sigma\lambda\sigma^{-1} = \lambda^{-1}. \tag{4.3}$$

Let us consider the limit of the action of (Γ_i, Λ_i) on $\tilde{B}_i$. We may assume that $(\tilde{B}_i, \tilde{p}_i, \Gamma_i, \Lambda_i)$ converges to $(Z, z_0, \Gamma_\infty, \Lambda_\infty)$, where $Z = \mathbb{R} \times L$, $\Lambda_\infty = \mathbb{R} \times H$, L is a flat cone and H is a finite cyclic group acting on L. Let $\sigma_\infty \in \Gamma_\infty$ and $\lambda_\infty \in \Lambda_\infty$ be the limits of σ_i and λ_i under the above convergence. Note that $\sigma_\infty : \mathbb{R} \times L \to \mathbb{R} \times L$ can be expressed as $\sigma_\infty(x, y) = (-x, \sigma'_\infty(y))$, where σ'_∞ is a rotation of angle $\ell/2$ and ℓ is the length of the space of directions at the vertex of the cone L. Note that $T_pX = (L/H)/\sigma'_\infty$.

As discussed above, from the action of H on L, we can put a Seifert fibered torus structure on $\partial\hat{B}_i$. Namely, if $\lambda_\infty(re^{i\theta}) = re^{i(\theta+\nu\ell/\mu)}$, then $\partial\hat{B}_i$ has a Seifert fibered torus structure of type (μ, ν) that is $\hat{\sigma}_i$-invariant (see [SY00, Lemma 4.4]). From (4.3), we have $\sigma_\infty\lambda_\infty\sigma_\infty = \lambda_\infty^{-1}$. This yields that $\lambda_\infty^2 = 1$. Thus (μ, ν) is equal to $(1, 1)$ or $(2, 1)$.

We shall show $(\mu, \nu) = (1, 1)$ and extend the fiber structure on $\partial\hat{B}_i$ to a $\hat{\sigma}_i$-invariant fiber structure on $\hat{B}_i$ which projects down to the generalized Seifert bundle structure on B_i.

Let B and $\hat{B}$ be the r-balls in the cone $T_pX = (L/H)/\sigma'_\infty$ and L/H around the vertices o_p and $\hat{o}_p$ respectively. Consider the metric annuli

$$A := A(o_p; r/4, r), \ \hat{A} := A(\hat{o}_p; r/4, r).$$

Applying the equivariant Fibration Theorem (Theorem 18.4 in [Y 4-dim]), we have a $\mathbb{Z}_2$-equivariant S^1-fibration $\hat{g}_i : \hat{A}_i \to \hat{A}$ for some closed domain $\hat{A}_i$ of $\hat{B}_i$, which gives rise to an S^1-fibration $g_i : A_i \to A$ for some closed domain A_i of B_i.

We denote by $B(\pi'_i)$ and $B(g_i)$ the closed domain bounded by $(\pi'_i)^{-1}(S(p, r))$ and $(g_i)^{-1}(S(o_p, r/2))$ respectively, and set

$$A(\pi'_i, g_i) := \overline{B(\pi'_i) \setminus B(g_i)}.$$

By Flow Theorem 3.2, there is a Lipschitz flow $\Phi : \partial B(\pi'_i) \times [0, 1] \to A(\pi'_i, g_i)$ such that $\Phi(x, 0) = x$. Let $\Phi_1 : \partial B(\pi'_i) \to \partial B(g_i)$ be the homeomorpshism defined by $\Phi_1(x) = \Phi(x, 1)$. Obviously the π'_i-fibers of $(\pi'_i)^{-1}(S(p, r))$ and the $(\Phi_1)^{-1}$-images of g_i-fibers of $(g_i)^{-1}(S(o_p, r/2))$ are isotopic to each other. Namely, we have an isotopy φ_t of $\partial B(\pi_i)$, $0 \le t \le 1$, such that $\varphi_0 = id$ and φ_1 sends every π'_i-fiber to the $(\Phi_1)^{-1}$-image of a g_i-fiber. Define $\Psi : A(\pi'_i, g_i) \to A(\pi'_i, g_i)$ by

$$\Psi(\Phi(x, t)) = \Phi(\varphi_t(x), t).$$

This joins the two fiber structures of π'_i and g_i. Thus we obtain a circle fibration $\pi'' : M''_i \to X''$ gluing the fibrations $\pi'_i : M'_i \to X'$ and g_i, where $X'' = X - (U(x_1, r/4) \cup \cdots \cup U(x_k, r/4))$.

Let $V_{\mu,\nu} = S^1 \times D^2$ denote the fibered solid torus of type (μ, ν).

From now on, for simplicity, we denote $B(g_i)$ by B_i, and use the same notation as in (4.2). In particular, we have the $\mathbb{Z}_2$-equivariant homeomorphism $\hat{f}_i : V_{\mu,\nu} \to \hat{B}_i$. Using $\hat{f}_i$, we have a fiber structure on $\partial V_{\mu,\nu}$ induced from the $\hat{g}_i$-fibers which is isotopic to the standard fiber structure of type (μ, ν).

Assertion 4.6. *$(\mu,\nu) = (1,1)$ and there is a $\hat\sigma$-equivariant isotopy of $\partial V_{1,1}$ joining the two fiber structures on $\partial V_{1,1}$.*

Proof. First suppose $(\mu,\nu) = (1,1)$. On the torus $\partial V_{1,1} = S^1 \times \partial D^2$, let $m = m(t) = (1, e^{it})$ and $\ell = \ell(t) = (e^{it}, 1)$ denote the meridian and the longitude. Fix a meridian m_i and a longitude ℓ_i of $\partial \hat B_i$ such that each fiber of $\hat g_i$ transversally meets m_i. Here we may assume that all the longitudes of $\partial \hat B_i$ discussed below are $\hat g_i$-fibers.

Set $h_i := (\hat f_i)^{-1}$ for simplicity. We now show that $h_i(\ell_i)$ is $\hat\sigma$-equivariantly ambient isotopic to ℓ. Recall that $\hat\pi : \partial V_{1,1} = S^1 \times \partial D^2 \to K^2 = S^1 \times \partial D^2/\hat\sigma$ is the projection. Since $h_i(\ell_i)$ is homotopic to ℓ, $\hat\pi(h_i(\ell_i))$ is homotopic to $\hat\pi(\ell)$, and hence is ambient isotopic to $\hat\pi(\ell)$. Namely, there exists an isotopy φ_t, $0 \le t \le 1$, of K^2 such that

$$\varphi_0 = id, \;\; \varphi_1(\hat\pi(h_i(\ell_i))) = \hat\pi(\ell).$$

Let $\hat\varphi_t : \partial V_{1,1} \to \partial V_{1,1}$ be the lift of φ_t such that $\hat\varphi_0 = id$. Note that $\hat\varphi_1(h_i(\ell_i)) = \ell$. Therefore we may assume that $h_i(\ell_i) = \ell$ from the beginning.

Next we claim that $h_i(m_i)$ is $\hat\sigma$-equivariantly ambient isotopic to m while keeping ℓ fixed. Namely, we show that there exists an isotopy $\hat\varphi_t$, $0 \le t \le 1$, of $\partial V_{1,1}$ such that

$$\hat\varphi_0 = id, \;\; \hat\varphi_1(h_i(m_i)) = m, \;\; \hat\varphi|_\ell = 1_\ell.$$

To show this, we proceed in a way similar to the above. Since $h_i(m_i)$ is homotopic to m, $\hat\pi(h_i(m_i))$ is homotopic to $\hat\pi(m)$, and hence is ambient isotopic to $\hat\pi(m)$. Here the construction of isotopy is local (see [E]). Hence approximating m near the intersection point $\ell \cap m$ via a PL-arc for instance, we can choose such an isotopy φ_t, $0 \le t \le 1$, of K^2 such that

$$\varphi_0 = id, \;\; \varphi_1(\hat\pi(h_i(m_i))) = \hat\pi(m), \;\; \varphi_t|_{\hat\pi(\ell)} = 1_{\hat\pi(\ell)}.$$

Let $\hat\varphi_t : \partial V_{1,1} \to \partial V_{1,1}$ be the lift of φ_t such that $\hat\varphi_0 = id$. Note that $\hat\varphi_1$ sends $h_i(m_i)$ to m and $\hat\varphi_t$ is the required isotopy. Therefore we may assume that $h_i(m_i) = m$ from the beginning.

For a small $\epsilon > 0$, let $\ell' = (e^{it}, e^{i\epsilon})$ and $\ell'' = (e^{it}, e^{-i\epsilon})$ (resp. $m' = (e^{i\epsilon}, e^{it})$ and $m'' = (e^{-i\epsilon}, e^{it})$) be longitudes near ℓ (resp. meridians near m). Let ℓ_i' and ℓ_i'' (resp. m_i' and m_i'') be longitudes (resp. meridians) near ℓ_i (resp. near m_i) such that ℓ_i', ℓ_i'', m_i' and m_i'' bound a regular neighborhood of $\ell_i \cup m_i$. In a way similar to the above, taking a $\hat\sigma$-equivariant ambient isotopy, we may assume that $h_i(\ell_i') = \ell'$, $h_i(\ell_i'') = \ell''$, $h_i(m_i') = m'$ and $h_i(m_i'') = m''$.

Let D (resp. D_i) be the small domain bounded by ℓ, ℓ', m and m' (resp. ℓ_i, ℓ_i', m_i and m_i'). Identify $D = I_0 \times [0,1]$, $D_i = I_i \times [0,1]$, where $I_0 \subset m$, $I_i \subset m_i$ are arcs, and define $k_i : D \to D$ by $k_i(x,t) = h_i(\hat f_i(x), t)$. From what we have discussed above, $k_i|_{\partial D} = 1_{\partial D}$. It is then standard to obtain an isotopy ψ_t of D which sends k_i to 1_D keeping ∂D fixed. Extending ψ_t $\hat\sigma$-equivariantly, we obtain a $\hat\sigma$-equivariant isotopy of $\partial V_{1,1}$ which sends the h_i-image of I-fibers of D_i to I-fibers of D keeping the outside D fixed. Applying this argument to the other domains bounded by longitudes ℓ', ℓ'', $\hat\sigma(\ell')$, $\hat\sigma(\ell'')$ and meridians m', m'' of $\partial V_{1,1}$, we finally construct a $\hat\sigma$-equivariant ambient isotopy φ_t of $\partial V_{1,1}$ sending the h_i-images of the $\hat g_i$-fibers in ∂B_i to the corresponding longitudes of $\partial V_{1,1}$.

Finally we show that the case $(\mu,\nu) = (2,1)$ never happens. Let us fix a g_i-fiber, say k_i, and a standard $(2,1)$-fiber, say k, on the fibered torus $\partial V_{2,1}$ of type $(2,1)$. Since $h_i(k_i)$ is homotopic to k in T^2, in a way similar to the above discussion, we have a $\hat{\sigma}$-equivariant ambient isotopy $\hat{\varphi}_t$ of $\partial V_{2,1}$ such that $\hat{\varphi}_0 = id$ and $\hat{\varphi}_1$ sends $h_i(k_i)$ to k. In $S^1 \times \partial D^2$, k is described as $k(t) = (e^{2it}, e^{it})$, and hence $\hat{\sigma} \circ k(t) = (e^{-2it}, e^{i(t+\pi)})$. Therefore the images $\mathrm{Im}(\hat{\sigma} \circ k)$, $\mathrm{Im}(k)$ of $\hat{\sigma} \circ k$ and k respectively must meet at $\hat{\sigma} \circ k(-\pi) = k(2\pi)$. On the other hand,

$$\hat{\sigma} \circ k = \hat{\sigma} \circ \hat{\varphi}_1(h_i(k_i)) = \hat{\varphi}_1 \circ \hat{\sigma}(h_i(k_i)), \ \ k = \hat{\varphi}_1(h_i(k_i)).$$

It turns out that $\mathrm{Im}(\hat{\sigma}(h_i(k_i))) = \mathrm{Im}(h_i(\hat{\sigma}_i(k_i)))$ meets $\mathrm{Im}(h_i(k_i))$. This implies that $\mathrm{Im}(\hat{\sigma}_i(k_i))$ meets $\mathrm{Im}(k_i)$, a contradiction to the fact that g_i is a $\mathbb{Z}_2$-equivariant fibration.

This completes the proof of the assertion. □

Obviously the standard fiber structure on $\partial V_{1,1}$ extends to a standard $\hat{\sigma}$-invariant fiber structure on $V_{1,1}$. Now it becomes easy to extend the fiber structure defined by $\hat{g}_i$-fibers on $\hat{\partial} B_i$ to a $\hat{\sigma}_i$-equivariant fiber structure on $\hat{B}_i$ of type $(1,1)$ via h_i, which projects down to a generalized Seifert bundle structure on B_i and on M_i for large i which is compatible to the fiber structure of π_i'. This completes the proof of Lemma 4.5. □

This completes the proof of Theorem 1.3. □

5. The case that $\dim X = 2$ and $\partial X \neq \emptyset$

Let $\{M_i \,|\, i = 1, 2, \dots\}$ be a sequence of three-dimensional closed Alexandrov spaces with curvature ≥ -1 having a uniform diameter bound. Suppose that M_i converges to an Alexandrov surface X with non-empty boundary.

In this section, we provide decompositions of X into $X' \cup X''$ and of M_i into $M_i' \cup M_i''$ such that M_i' fibers over X' in the sense of a generalized Seifert fiber space and M_i'' is the closure of the complement of M_i'. We will prove that each component of M_i'' has the structure of a generalized solid torus or a generalized solid Klein bottle, and the circle fiber structure on its boundary is compatible to the circle fiber structure induced by the generalized Seifert fibration.

From now on, we denote by C one of the components of ∂X. Since a two-dimensional Alexandrov space is a manifold, C is homeomorphic to a circle. Let us fix a small positive number ε. To construct the desired decompositions of X and M_i, we define a notion of an ε-regular covering of C.

Definition 5.1. Let $\{B_\alpha, D_\alpha\}_{1 \leq \alpha \leq n}$ be a covering of C by closed subsets in X. We say that $\{B_\alpha, D_\alpha\}_{1 \leq \alpha \leq n}$ is *ε-regular* if it satisfies the following:

(1) $\bigcup_{1 \leq \alpha \leq n} B_\alpha \cup D_\alpha - C$ is $(2, \varepsilon)$-strained.

(2) Each B_α is the closed metric ball $B_\alpha = B(p_\alpha, r_\alpha)$ centered at p_α with radius $r_\alpha > 0$ such that

$$|\nabla d_{p_\alpha}| > 1 - \varepsilon \text{ on } B(p_\alpha, 2r_\alpha) - \{p_\alpha\},$$
$$B_\alpha \cap B_{\alpha'} = \emptyset \text{ for all } \alpha \neq \alpha'.$$

Also, the sequence $p_1, p_2, \dots, p_n$ is consecutive in C.

(3) D_α forms

$$D_\alpha := B(\gamma_\alpha, \delta) - \mathrm{int}(B_\alpha \cup B_{\alpha+1}),$$

where $\gamma_\alpha := \widehat{p_\alpha p_{\alpha+1}}$ with $p_{n+1} := p_1$. Here, $\delta > 0$ is a small positive number with $\delta \ll \min_\alpha r_\alpha$.

(4) For any $x \in D_\alpha$, we have

$$\tilde{\angle} p_\alpha x p_{\alpha+1} > \pi - \varepsilon.$$

For $x \in D_\alpha - C$ and $y \in C$ with $|xy| = |xC|$, we have

$$|\nabla d_C|(x) > 1 - \varepsilon,$$
$$|\tilde{\angle} p_\alpha x y - \pi/2| < \varepsilon, \text{ and}$$
$$|\tilde{\angle} p_{\alpha+1} x y - \pi/2| < \varepsilon.$$

The existence of an ε-regular covering of C will be proved in Section 9. We fix an ε-regular covering

$$\{B_\alpha, D_\alpha \mid \alpha = 1, 2, \dots, n\}$$

of C.

We consider a closed neighborhood X''_C of C defined as

$$X''_C := \bigcup_{\alpha=1}^{n} B_\alpha \cup D_\alpha. \tag{5.1}$$

We set

$$X'' := \bigcup X''_C \text{ and } X' := \text{ the closure of } X - X''.$$

This is our decomposition $X = X' \cup X''$.

Since $\operatorname{int} X'$ has all interior $(2, \varepsilon)$-singular points of X, by Theorem 1.3 we obtain a generalized Seifert fibration

$$\pi'_i : M'_i \to X' \tag{5.2}$$

for some closed domain $M'_i \subset M_i$. Let us denote by X^{reg} the complement of a small neighborhood of the union of ∂X and the set of all interior $(2, \varepsilon)$-singular points in X. By Theorem 2.25, we may assume that π'_i is both a circle fibration and a $\theta(\varepsilon)$-Lipschitz submersion on X^{reg}. Recall that ${\pi'_i}^{-1}(X^{\mathrm{reg}})$ is $(3, \theta(\varepsilon))$-regular, for large i.

We set $M''_i := M_i - \operatorname{int} M'_i$. We will determine the topology of M''_i in the following subsections.

5.1. **Decomposition of M''_i.**

Let us denote by M_i^{reg} a $(3, \theta(\varepsilon))$-regular closed domain of M_i which contains ${\pi'_i}^{-1}(X^{\mathrm{reg}})$. By Theorem 2.46, we obtain a smooth approximation

$$f_i : U(M_i^{\mathrm{reg}}) \to N(M_i^{\mathrm{reg}}) \tag{5.3}$$

for a neighborhood $U(M_i^{\mathrm{reg}})$ of M_i^{reg} and some Riemannian manifold $N(M_i^{\mathrm{reg}})$.

Let us take $p_{\alpha,i} \in M_i$ converging to $p_\alpha \in C \subset \partial X$, and $\gamma_{\alpha,i}$ a simple arc joining $p_{\alpha,i}$ and $p_{\alpha+1,i}$ converging to γ_α. By the definition of regular covering, we may assume that

$$A\left(\bigcup_{\alpha=1}^{N} \gamma_{\alpha,i}; \delta/100, 10 \max r_\alpha\right)$$

is $(3, \theta(\varepsilon))$-regular.

From now on, we fix any index $\alpha \in \{1, \dots, N\}$ and use the following notation: $p := p_\alpha$, $p' := p_{\alpha+1}$, $B := B_\alpha$, $B' := B_{\alpha+1}$, $\gamma := \gamma_\alpha$ and $\gamma'' := \gamma_{\alpha-1}$; and $p_i := p_{\alpha,i}$, $p'_i := p_{\alpha+1,i}$, $\gamma_i := \gamma_{\alpha,i}$ and $\gamma''_i := \gamma_{\alpha-1,i}$. To avoid a disordered notation, we assume that all r_α are equal to each other, and set $r := r_\alpha$.

Let δ' be a small positive number with $\delta' \ll \delta$. We will construct an isotopy of $B(p_i, r+\delta')$ which deforms the metric ball $B(p_i, r-\delta')$ to some domain B_i such that

$$B_i \approx B(p_i, r); \tag{5.4}$$

$$\partial B_i - U(\gamma_i \cup \gamma_i'', 3\delta/2) = {\pi_i'}^{-1}(\partial B(p, r) - U(\gamma \cup \gamma'', 3\delta/2)); \tag{5.5}$$

$$\partial B_i \cap B(\gamma_i \cup \gamma_i'', \delta) = \partial B(p_i, r-\delta') \cap B(\gamma_i \cup \gamma_i'', \delta). \tag{5.6}$$

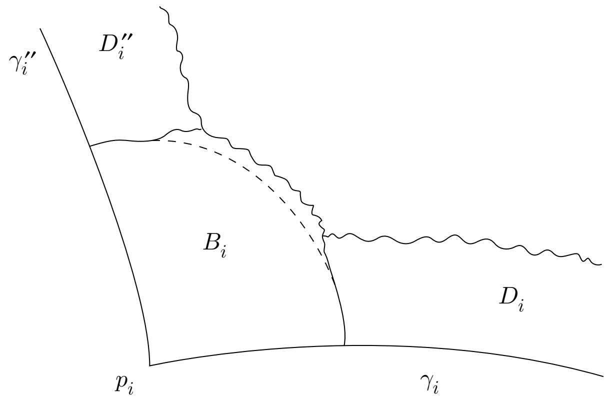

Figure 1. A domain near the corner

In Figure 1, the broken line denotes the metric sphere $S(p_i, r-\delta')$, and the wavy line denotes the pull-back of metric levels with respect to γ'', γ and p in X by π_i'.

Suppose that we construct such an isotopy and obtain a domain $B_i = B_{\alpha,i}$ satisfying (5.5) and (5.6) for a moment. We consider the domain

$$D_i = D_{\alpha,i} := B(\gamma_{\alpha,i}, 3\delta/2) \cup {\pi_i'}^{-1}(A(\gamma_\alpha; \delta, 2\delta)) - \operatorname{int}(B_{\alpha,i} \cup B_{\alpha+1,i}). \tag{5.7}$$

Then we obtain a decomposition of M_i'':

$$M_{i,C}'' := \bigcup_{\alpha=1}^{N} B_{\alpha,i} \cup D_{\alpha,i}, \tag{5.8}$$

$$M_i'' = \bigcup_{C \subset \partial X} M_{i,C}''. \tag{5.9}$$

Now we construct an isotopy which deforms $B(p_i, r-\delta')$ to B_i satisfying (5.5) and (5.6). From now on throughout this paper, we use the following notation. For any set $A \subset M_i^{\mathrm{reg}}$, we set $\tilde{A} := f_i(A)$. We denote by $U(A)$ a neighborhood of A in $U(M_i^{\mathrm{reg}})$ and by $N(A)$ the image of $U(A)$ by the approximation f_i. Namely, $N(A) = \widetilde{U(A)}$. For any point $x \in A$, we set $\tilde{x} := f_i(x) \in \tilde{A}$. For any function $\phi : A \to \mathbb{R}$, we define $\tilde{\phi} : \tilde{A} \to \mathbb{R}$ by

$$\tilde{\phi} := \phi \circ f_i^{-1}. \tag{5.10}$$

Let $\tilde{V}$ be a gradient-like smooth vector field for a Lipschitz function $\widetilde{\mathrm{dist}}_{p_i}$ on $N((B(p_i, r) \cup B(p_i', r) \cup B(\gamma_i, 2\delta)) \cap M_i^{\mathrm{reg}})$ obtained by Lemma 3.7.

We take a Lipschitz function h defined on $B(p_i, r+\delta')$ such that

$$\begin{aligned}&\tilde{h} \text{ is smooth},\\ &0 \le h \le 1,\\ &\operatorname{supp}(h) \subset B(p_i, r+\delta') - U(\gamma_i \cup \gamma_i'', \delta/2),\\ &h \equiv 1 \text{ on } B(p_i, r+\delta') - U(\gamma_i \cup \gamma_i'', \delta).\end{aligned}$$

We consider a smooth vector field $\tilde{h} \cdot \tilde{V}$ and its integral flow $\tilde{\Phi}$, and we define the pull-back flow $\Phi_t := f_i^{-1} \circ \tilde{\Phi}_t \circ f_i$. Then by construction and Theorem 3.11, the flow Φ transversally intersects $\pi_i'^{-1}(\partial B(p,r) - U(\gamma \cup \gamma'', \delta))$. Then we can construct an isotopy by using the flow Φ, which provides a closed neighborhood B_i of p_i satisfying (5.4), (5.5) and (5.6).

5.2. **The topologies of the balls near corners.** We first prove that ∂B_i is homeomorphic to a closed 2-manifold.

Lemma 5.2. *$\partial B_i \approx \partial B(p_i, r)$ is a closed 2-manifold.*

Proof. If B_i does not satisfy Assumption 2.28, we have some sequence $\hat{p}_i$ with $|\hat{p}_i p_i| \to 0$ such that $\partial B(\hat{p}_i, r) \approx \Sigma_{\hat{p}_i}$, where we may assume that $\hat{p}_i = p_i$. Since M_i has no boundary, $\partial B(p_i, r)$ is homeomorphic to S^2 or P^2.

If B_i satisfies Assumption 2.28, there exist a sequence $\delta_i \to 0$ and $\hat{p}_i$ with $|\hat{p}_i p_i| \to 0$ such that the limit (Y, y_0) of $(\frac{1}{\delta_i} B(\hat{p}_i, r), \hat{p}_i)$ has dimension three. Here, we may assume that $\hat{p}_i = p_i$. Then, by Soul Theorem 2.59 and Stability Theorem 2.34, $\partial B(p_i, r)$ is homeomorphic to S^2, P^2, T^2 or K^2. □

From (5.5) and the construction of B_i, we have

$$\partial B_i - U(\gamma_i \cup \gamma_i'', \delta) \approx S^1 \times I. \tag{5.11}$$

Now, we put F_i and F_i'' as follows:

$$F_i := \partial B_i \cap B(\gamma_i, \delta) \text{ and } F_i'' := \partial B_i \cap B(\gamma_i'', \delta). \tag{5.12}$$

Then, by Lemma 5.2, F_i and F_i'' are 2-manifolds with boundaries homeomorphic to S^1. By the generalized Margulis lemma [FY], F_i has an almost nilpotent fundamental group. Hence F_i is homeomorphic to D^2 or Mö.

Therefore, we obtain the following assertion:

Lemma 5.3. *∂B_i is homeomorphic to S^2, P^2 or K^2.*

We now determine the topology of B_i.

Lemma 5.4. *B_i is homeomorphic to D^3, Mö$\times I$ or $K_1(P^2)$. Moreover, if $\operatorname{diam} \Sigma_p > \pi/2$, then B_i is not homeomorphic to $K_1(P^2)$.*

Proof. We first consider the case that $\operatorname{diam} \Sigma_p > \pi/2$. Then by Proposition 2.70, Σ_{p_i} is topologically a suspension over a one-dimensional Alexandrov space Λ of curvature ≥ 1. Since $\partial \Sigma_{p_i} = \emptyset$, Λ is a circle. Hence p_i is a topologically regular point. Note that, in this situation, any $x \in B(p, r)$ has $\operatorname{diam} \Sigma_x > \pi/2$. Therefore, $\operatorname{int} B_i$ is topologically a manifold, and B_i is not homeomorphic to $K_1(P^2)$.

From now on we assume that $\operatorname{diam} \Sigma_p \le \pi/2$. If B_i does not satisfy Assumption 2.28, then there exists $\hat{p}_i$ such that $\lim |p_i \hat{p}_i| = 0$ and $B(\hat{p}_i, r) \approx K_1(\Sigma_{\hat{p}_i})$ which is homeomorphic to D^3 or $K_1(P^2)$, where we may assume that $p_i = \hat{p}_i$.

Suppose that B_i satisfies Assumption 2.28. By Theorem 2.27, there is a sequence δ_i of positive numbers tending to zero and points $\hat{p}_i$ (where we may assume that $\hat{p}_i = p_i$) such that

- any limit (Y, y_0) of $(\frac{1}{\delta_i} B_i, p_i)$ as $i \to \infty$ is a three-dimensional open Alexandrov space of non-negative curvature;
- denoting by S a soul of Y, we obtain $\dim S \le 1$.

Then, by Soul Theorem 2.59, Y is homeomorphic to $\mathbb{R}^3$, $K(P^2)$ or M_{pt} if $\dim S = 0$ or an $\mathbb{R}^2$-bundle over S^1 if $\dim S = 1$. Therefore, B_i is homeomorphic to D^3, $K_1(P^2)$ or $B(\mathrm{pt})$ if $\dim S = 0$ or $S^1 \times D^2$ or $S^1 \tilde{\times} D^2 \approx \mathrm{M\ddot{o}} \times I$ if $\dim S = 1$. By the boundary condition (Lemma 5.3), B_i is actually not homeomorphic to $S^1 \times D^2$. It remains to show that

$$B_i \text{ is not homeomorphic to } B(\mathrm{pt}). \tag{5.13}$$

We prove (5.13) by contradiction. Suppose that there is a homeomorphism $f_i : B(\mathrm{pt}) \to B_i$. We will use the notation in the proof of Lemma 4.5. Recall that $B(\mathrm{pt})$ is obtained by the quotient space of $S^1 \times D^2$ by the involution $\hat{\sigma}$. We consider the corresponding space $\hat{B}_i$ with an involution $\hat{\sigma}_i$ such that its quotient is B_i. By the argument of the proof of Lemma 4.5, we obtain the following commutating diagram:

$$\begin{tikzcd}
\mathbb{R} \times D^2 \arrow[r, "\tilde{f}_i"] \arrow[d, "\pi"'] & \tilde{B}_i \arrow[d, "\pi_i"] \\
S^1 \times D^2 \arrow[r, "\hat{f}_i"] \arrow[d, "\hat{\pi}"'] & \hat{B}_i \arrow[d, "\hat{\pi}_i"] \\
B(\mathrm{pt}) \arrow[r, "f_i"'] & B_i
\end{tikzcd}$$

Here, the horizontal arrows are homeomorphisms, π and π_i are the universal coverings, and $\hat{\pi}$ and $\hat{\pi}_i$ are the projections by involutions $\hat{\sigma}$ and $\hat{\sigma}_i$, respectively. We may assume that $(\tilde{B}_i, \tilde{p}_i, \Gamma_i, \Lambda_i)$ converges to $(Z, z_0, \Gamma_\infty, \Lambda_\infty)$ with $Z = \mathbb{R} \times L$, $\Lambda_\infty = \mathbb{R} \times H$, L is a flat cone over a circle and H is a finite abelian group acting on L. Note that all elements of H are orientation preserving on L. Recall that σ_∞ is expressed as $\sigma_\infty(x, y) = (-x, \sigma'_\infty(y))$ and σ_∞ is orientation preserving on L. Therefore, $[\sigma'_\infty]$ is orientation preserving on L/H. We remark that $(L/H)/[\sigma'_\infty] = T_p X$. Then, L/H has no boundary. Indeed, to check this, we suppose that L/H has non-empty boundary. Then L/H is the cone over an arc. Since $[\sigma'_\infty]$ is a non-trivial isometry on L/H, $[\sigma'_\infty]$ is the reflection with respect to the center line. Therefore, $[\sigma'_\infty]$ does not preserve orientation. This is a contradiction.

Thus, L/H is the cone over a circle. It turns out that σ'_∞ is a half rotation of L, and hence so is $[\sigma'_\infty]$ for L/H. This implies $T_p X$ has no boundary, and we obtain a contradiction. We conclude (5.13), and complete the proof of Lemma 5.4 □

Next, we will divide D_i into two pieces $D_i = H_i \cup K_i$ depending on the topology of F_i. We will also determine the topology of H_i, K_i, and D_i.

5.3. **The case that F_i is a disk.** We consider the case that $F_i \approx D^2$. Then, we divide D_i into H_i and K_i as follows:

$$H_i := D_i - U(\gamma_i, \delta),$$
$$K_i := D_i \cap B(\gamma_i, \delta).$$

5.3.1. *The topology of K_i.* We prove that

Assertion 5.5. *K_i is homeomorphic to D^3.*

K_i is contained in a domain L_i defined by

$$L_i := A(p_i; r - \delta', |pp'| - r/2) \cap B(\gamma_i, \delta). \tag{5.14}$$

Since (d_{p_i}, d_{γ_i}) is $(c, \theta(\varepsilon))$-regular near $L_i \cap S(\gamma_i, \delta)$, by Theorem 2.33 and Lemma 2.40, L_i is homeomorphic to $F_i \times [0,1] \approx D^3$. On the other hand, we can take a closed domain $A_i \subset \operatorname{int} K_i$ such that $A_i \approx D^3$ and

$$K_i^0 := B(\gamma_i, \delta/2) - (U(p_i, 2r) \cup U(p_i', 2r)) \subset \operatorname{int} A_i. \tag{5.15}$$

By Theorem 2.33 and Lemma 2.40, $K_i \approx K_i^0$. Remark that $F_i' := \partial B_i' \cap B(\gamma_i, \delta)$ is homeomorphic to D^2. Indeed, if we assume that $F_i' \approx$ Mö, then $\partial K_i \approx P^2$. Then, by the embedding (5.15), we have

$$P^2 \approx \partial K_i^0 \subset \operatorname{int} A_i \approx \mathbb{R}^3.$$

This is a contradiction. Therefore, $F_i' \approx D^2$ and $\partial K_i^0 \approx \partial K_i \approx S^2$. By Theorem 2.33, ∂K_i^0 is locally flatly embedded in $A_i \approx D^3$. Therefore, by the generalized Schoenflies theorem, we conclude $K_i \approx K_i^0 \approx D^3$.

5.3.2. *The topology of H_i.*

Assertion 5.6. *H_i is homeomorphic to $S^1 \times D^2$ and the circle fiber structure on H_i induced by the standard one on $S^1 \times D^2$ is compatible to π_i'.*

Let us define a domain $Q \subset X$ by

$$Q := A(\gamma; \delta - \delta', 2\delta + \delta') - (U(p, r - 2\delta') \cup U(p', r - 2\delta')). \tag{5.16}$$

Note that Q is homeomorphic to a two-disk without $(2, \varepsilon)$-singular points. Then $Q_i := {\pi_i'}^{-1}(Q)$ is topologically a solid torus, and H_i is contained in the interior of Q_i.

We will construct an isotopy $\varphi : Q_i \times [0,1] \to Q_i$ satisfying

$$\varphi(\cdot, 0) = \mathrm{id}_{Q_i}, \tag{5.17}$$
$$\varphi(Q_i, 1) = H_i, \tag{5.18}$$
$$\varphi : \partial Q_i \times [0,1] \to Q_i - \operatorname{int} H_i \text{ is bijective.} \tag{5.19}$$

If we obtain such a φ, then by (5.18), we conclude $H_i \approx Q_i \approx S^1 \times D^2$. And by (5.19), we can obtain the circle fiber structure of H_i over Q which is compatible to the generalized Seifert fibration π_i'.

Next we use the conventions as in (5.10).

Lemma 5.7. *There is a smooth vector field $\tilde{X}$ on $N(Q_i - H_i)$ such that it is gradient-like:*

- *for $\tilde{d}_{p_i}$ and $\widetilde{d_p \circ \pi_i'}$ on $N(B(p_i, r + \delta') \cap Q_i - H_i)$,*
- *for $\tilde{d}_{p_i'}$ and $\widetilde{d_{p'} \circ \pi_i'}$ on $N(B(p_i', r + \delta') \cap Q_i - H_i)$,*

- *for* $\widetilde{d}_{\gamma_i}$ *and* $\widetilde{d_\gamma \circ \pi'_i}$ *on* $N(B(\gamma_i, \delta+\delta') \cap Q_i - H_i)$*, and*
- *for* $-\widetilde{d}_{\gamma_i}$ *and* $-\widetilde{d_\gamma \circ \pi'_i}$ *on* $N(Q_i - H_i - U(\gamma_i, 2\delta - \delta'))$*.*

Proof. Let us take gradient-like smooth vector fields $\tilde{V}$, $\tilde{V}'$ and $\tilde{W}$ for $\widetilde{d}_{p_i}$, $\widetilde{d}_{p'_i}$ and $\widetilde{d}_{\gamma_i}$ on $N(Q_i - H_i)$. We prepare a decomposition of $Q_i - \operatorname{int} H_i$ as follows:

$$Q_i - \operatorname{int} H_i = \bigcup_{\alpha=1}^{8} A_\alpha. \tag{5.20}$$

See Figure 2. Here, we define

$$\begin{aligned}
A_1 &:= (Q_i - \operatorname{int} H_i) \cap (B(\gamma_i, \delta) - U(\{p_i, p'_i\}, r+\delta'))\,, \\
A_2 &:= (Q_i - \operatorname{int} H_i) - (U(\gamma_i, 2\delta-\delta') \cup U(\{p_i, p'_i\}, r+\delta'))\,, \\
A_3^* &:= B(p_i, r+\delta') \cap B(\gamma_i, \delta+\delta'), \\
A_4^* &:= B(p_i, r+\delta') \cap A(\gamma_i; \delta+\delta', 2\delta-\delta'), \\
A_5^* &:= B(p_i, r+\delta') - U(\gamma_i, 2\delta-\delta').
\end{aligned}$$

Similarly, we put

$$\begin{aligned}
A_6^* &:= B(p'_i, r+\delta') \cap B(\gamma_i, \delta+\delta'), \\
A_7^* &:= B(p'_i, r+\delta') \cap A(\gamma_i; \delta+\delta', 2\delta-\delta'), \\
A_8^* &:= B(p'_i, r+\delta') - U(\gamma_i, 2\delta-\delta').
\end{aligned}$$

We define $A_3, A_4, \cdots, A_8$ by

$$A_\alpha := A_\alpha^* \cap Q_i - \operatorname{int} H_i \text{ for } \alpha = 3, 4, \ldots, 8.$$

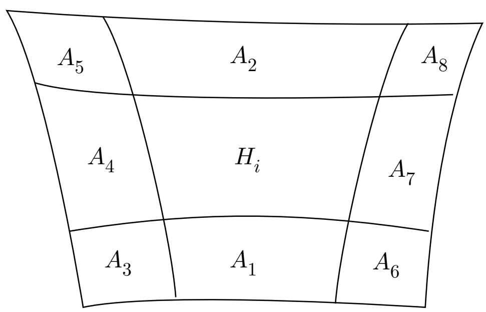

FIGURE 2. The decomposition of Q_i

We take smooth functions h_α ($\alpha = 1, \ldots, 8$) on $N(Q_i - H_i)$ such that

$$\begin{aligned}
&0 \le h_\alpha \le 1, \\
&h_\alpha \equiv 1 \text{ on } \tilde{A}_\alpha, \\
&\operatorname{supp}(h_\alpha) \subset B(\tilde{A}_\alpha, \delta'/100).
\end{aligned}$$

We define a vector field $\tilde{X}$ as

$$\begin{aligned}
\tilde{X} := h_1\tilde{W} &- h_2\tilde{W} \\
&+ h_3(\tilde{V}+\tilde{W}) + h_4\tilde{V} + h_5(\tilde{V}-\tilde{W}) \\
&+ h_6(\tilde{V}'+\tilde{W}) + h_7\tilde{V}' + h_8(\tilde{V}'-\tilde{W}).
\end{aligned}$$

Then, we can show that $\tilde{X}$ satisfies the conclusion of Lemma 5.7 as follows. We will prove it only on $N(A_3)$.

We consider the integral flow $\tilde{\Phi}$ of $\tilde{X}$ and the pull-back $\Phi_t := f_i^{-1} \circ \tilde{\Phi}_t \circ f_i$. It suffices to show that

$$\Phi \pitchfork \mathrm{dist}_{p_i} \tag{5.21}$$

$$\Phi \pitchfork \mathrm{dist}_{\gamma_i} \tag{5.22}$$

$$\Phi \pitchfork \mathrm{dist}_{p} \circ \pi_i' \tag{5.23}$$

$$\Phi \pitchfork \mathrm{dist}_{\gamma} \circ \pi_i' \tag{5.24}$$

on $N(A_3)$. We can write

$$\tilde{X} = \alpha \tilde{V} + \beta \tilde{W}$$

for smooth functions $\alpha, \beta \geq 0$ with $1 \leq \alpha + \beta \leq 3$ on $N(A_3)$. By a direct calculus, we have

$$|\tilde{X}| \geq \sqrt{2} - \theta(\varepsilon),$$
$$\angle(\tilde{X}, \tilde{V}) < \gamma + \theta(\varepsilon),\ \angle(\tilde{X}, \tilde{W}) < \gamma + \theta(\varepsilon)$$

on $N(A_3)$. Here, $\cos \gamma = 1/\sqrt{10}$.

Let us set

$$X := \left.\frac{d}{dt}\right|_{t=0+} \Phi(x,t) \in T_x M_i.$$

Then we have

$$X(x) = df_i^{-1}(\tilde{X}(\tilde{x})).$$

We set

$$V := df_i^{-1}(\tilde{V}),\ W := df_i^{-1}(\tilde{W}).$$

Then we obtain

$$V \doteqdot \nabla d_{p_i},\ W \doteqdot \nabla d_{\gamma_i}.$$

Here, $A \doteqdot A'$ means $|A, A'| < \theta(\varepsilon)$.

Since f_i is a $\theta(\varepsilon)$-almost isometry, we have

$$|X| \doteqdot |\tilde{X}|,\ |V, X| \doteqdot |\tilde{V}, \tilde{X}|,\ \angle(V, X) \doteqdot \angle(\tilde{V}, \tilde{X}).$$

Hence, we obtain

$$|X| \geq \sqrt{2} - \theta(\varepsilon),$$
$$\angle(p_i', X) \geq \angle(p_i', V) - \angle(V, X) > \pi - \gamma - \theta(\varepsilon).$$

Therefore, we have

$$(d_{p_i})'(X) = -|X| \cos \angle(p_i', X) \geq 1/\sqrt{5} - \theta(\varepsilon).$$

This implies (5.21).

For any fixed scale (o), we set

$$d\pi_i' := (\exp^{(o)}_{\pi_i'(x)})^{-1} \circ (\pi_i')^{(o)}_x \circ \exp^{(o)}_x.$$

By Proposition 3.10, we have

$$\angle(p', d\pi_i'(X)) < \gamma + \theta(\varepsilon),\ |d\pi_i'(X)| > \sqrt{2} - \theta(\varepsilon).$$

Therefore, we obtain

$$(\mathrm{dist}_p \circ \pi_i')_x^{(o)}(\exp_x^{(o)}(X)) = -|d\pi_i'(X)| \cos \angle_{\pi_i'(x)}(p', d\pi_i'(X)) \\ > 1/\sqrt{5} - \theta(\varepsilon).$$

Thus, we obtain (5.23).

In a similar way to the above, we can prove (5.22) and (5.24). □

By Lemma 5.7, we obtain an isotopy φ based on Φ satisfying (5.17) through (5.19).

Therefore, we conclude that if $F_i \approx D^2$, then $D_i \approx D^3$.

5.4. **The case that F_i is a Mobius band.** We consider the case that $F_i \approx$ Mö. We prove that

Lemma 5.8. *D_i is $(3, \theta(\varepsilon))$-strained.*

Proof. We first define a domain L_i similar to (5.14):

$$L_i := A(p_i; r/2, |pp'| - r/2) \cap B(\gamma_i, 3\delta).$$

To prove Lemma 5.8, it suffices to show that

(5.25) $$L_i \text{ is } (3, \theta(\varepsilon))\text{-regular.}$$

By Theorem 2.33 and Lemma 2.40, we have $L_i \approx$ Mö$\times I$. Let $\hat{L}_i$ be the orientable double cover which is homeomorphic to $(S^1 \times I) \times I$. Since $\hat{L}_i$ is a covering space of L_i, $\hat{L}_i$ has the metric of Alexandrov space with $L_i \equiv \hat{L}_i/\langle\sigma\rangle$ for an isometric involution σ on $\hat{L}_i$.

Since the projection $\hat{L}_i \to L_i$ is a local isometry, to prove (5.25), it suffices to show that

(5.26) $$\hat{L}_i \text{ is } (3, \theta(\varepsilon))\text{-regular.}$$

L_i converges to the following closed domain L_∞ in X:

$$L_\infty = A(p; r/2, |pp'| - r/2) \cap B(\gamma, 3\delta).$$

We may assume that $\hat{L}_i$ converges to some two-dimensional space Y^2. Note that L_∞ is 1-strained, and hence L_i and $\hat{L}_i$ are also 1-strained. Therefore,

(5.27) $$Y \text{ is 1-strained.}$$

From the form of L_∞, we have that Y^2 is a two-disk having no ε-singular points. Indeed, if Y has a boundary-point in the sense of Alexandrov space, then from an argument similar to the proof of Assertion 5.5, $\hat{L}_i$ contains a domain homeomorphic to $D^2 \times I$ or Mö $\times I$. This is a contradiction, and hence Y has no boundary. By this and (5.27), Y is 2-strained. Therefore, $\hat{L}_i$ is 3-strained; this is the assertion (5.26). This implies (5.25) and completes the proof of Lemma 5.8. □

By Lemma 5.8 and Theorem 3.2, we have a Lipschitz flow Φ which is gradient-like for dist_{p_i} near D_i. We divide D_i into H_i and K_i as follows:

$$K_i := \text{the union of flow curves of } \Phi \\ \text{starting from } F_i \text{ in } B(\gamma_i, 2\delta) - \mathrm{int}\, B_i'.$$
$$H_i := D_i - \mathrm{int}\, K_i.$$

Note that the union of flow curves of Φ starting from ∂F_i is contained in $A(\gamma_i; \delta - \delta'', \delta + \delta'')$ for some small $\delta'' > 0$. By the construction, $K_i \approx \text{Mö} \times I$.

We will prove that

Assertion 5.9. *H_i is homeomorphic to $S^1 \times D^2$ and the circle fiber structure on H_i induced by one on $S^1 \times D^2$ is compatible to π_i'.*

Proof. Let Q_i be a closed neighborhood of H_i obtained in a way similar to the construction of Q_i in subsection 5.3. We actually define

$$Q := A(\gamma; \delta - \delta'', 2\delta + \delta') - U(\{p, p'\}, r - 2\delta'),$$
$$Q_i := \pi_i'^{-1}(Q).$$

We prepare a decomposition of $Q_i - \operatorname{int} H_i = \bigcup_{\alpha=1}^{8} A_\alpha$ in a way similar to (5.20) in Lemma 5.7. Actually, we define A_5, A_2, A_8 as in Lemma 5.7, and other A_α's are defined by

$$A_1 := (Q_i - \operatorname{int} H_i) \cap (B(\gamma_i, \delta + \delta'') - U(\{p_i, p_i'\}, r + \delta'))\,,$$
$$A_3 := (Q_i - \operatorname{int} H_i) \cap B(p_i, r + \delta') \cap B(\gamma_i, \delta + \delta''),$$
$$A_6 := (Q_i - \operatorname{int} H_i) \cap B(p_i', r + \delta') \cap B(\gamma_i, \delta + \delta''),$$
$$A_4 := (Q_i - \operatorname{int}(H_i \cup A_3 \cup A_5)) \cap B(p_i, r + \delta'),$$
$$A_7 := (Q_i - \operatorname{int}(H_i \cup A_6 \cup A_8)) \cap B(p_i', r + \delta').$$

Since $\nabla \mathrm{dist}_{p_i}$ and $\nabla \mathrm{dist}_{\gamma_i}$ are almost perpendicular to each other on $Q_i - \operatorname{int} H_i$, we can obtain a flow Φ which has nice transversality as in Lemma 5.7. We can also construct an isotopy from the identity to some homeomorphism which deforms Q_i to H_i inside Q_i. Therefore, we obtain a circle fibration of H_i over Q which is compatible to the generalized Seifert fibration π_i'. This completes the proof of Assertion 5.9. □

Therefore, we conclude that if $F_i \approx \text{Mö}$, then $D_i \approx \text{Mö} \times I$.

Proof of Theorem 1.5. It remain to show that each component $M''_{i,C}$ of M''_i has the structure of a generalized solid torus or generalized solid Klein bottles. This is clear from Sections 5.3 and 5.4. □

5.5. **Proof of Corollary 1.6.** To prove Corollary 1.6, we show elementary lemmas. We define the mapping class group $\mathrm{MCG}(F)$ of a topological space F to be the set of all isotopy classes of homeomorphisms of F.

Lemma 5.10. *Let F be a topological space. For any element γ of the mapping class group* $\mathrm{MCG}(F)$*, we fix a homeomorphism $\varphi_\gamma : F \to F$ such that $\varphi_\gamma \in \gamma$. Let us set $B = F \times [0,1]$ and $\pi : B \to [0,1]$ a projection. For any homeomorphisms $f_i : F \to \pi^{-1}(i)$, for $i = 0, 1$, there exist $\gamma \in$* $\mathrm{MCG}(F)$ *and a homeomorphism $h : F \times [0,1] \to B$ respecting π such that, for every $x \in F$, $h(x, 0) = f_0(x)$ and $h(x, 1) = f_1 \circ \varphi_\gamma(x)$.*

Proof. Let us set $F_t = \pi^{-1}(t) = F \times \{t\}$. Let us define the translation $\chi_t : F_0 \to F_t$ by $\chi_t(x, 0) = (x, t)$, and set a homeomorphism $\tilde{f}_t = \chi_t \circ f_0 : F \to F_t$. Note that $\tilde{f}_0 = f_0$. Let us take an element $\gamma \in \mathrm{MCG}(F)$ represented by a homeomorphism $f_1^{-1} \circ \tilde{f}_1$ of F. Then, there is a homeomorphism $g_t : F \to F$, for $0 \le t \le 1$, such that

$$g_0 = \mathrm{id} \text{ and } \tilde{f}_1 \circ g_1 = f_1 \circ \varphi_\gamma.$$

Therefore, setting $h_t = \tilde{f}_t \circ g_t : F \to F_t$, we obtain

$$h_0 = f_0 \text{ and } h_1 = f_1 \circ \varphi_\gamma.$$

Hence, defining $h : F \times [0,1] \to B$ by $h(x,t) = h_t(x)$, h satisfies the desired condition. □

Lemma 5.11. *Let Y be a generalized solid torus or a generalized solid Klein bottle. Let $\pi : Y \to S^1$ be a projection as in* (1.2)*. Then, there is a continuous surjection*

$$\eta : Y \to [0,1]$$

such that $\eta^{-1}(1) = \partial Y$ and, setting

$$\Phi = (\pi, \eta) : Y \to S^1 \times [0,1],$$

Φ is an S^1-bundle over $S^1 \times (0,1]$. Further, for every $x \in S^1$, $\Phi^{-1}(x,0)$ is a one point set or a circle, and the homeomorphic type of the fiber $\Phi^{-1}(x,0)$ changes if and only if that of $\pi^{-1}(x)$ changes.

Proof. Let us take ordered points $t_1, t_2, \dots, t_{2N-1}, t_{2N} \in S^1$ changing the fiber of π. Then, for a small $\varepsilon > 0$, setting $I_k = [t_k - \varepsilon, t_k + \varepsilon] \subset S^1$, $\pi^{-1}(I_k)$ is homeomorphic to $K_1(P^2)$.

We regard $K_1(P^2) = \bigcup_{t \in [-1,1]} D(t)$ as in Definition 1.4. Let us define a continuous surjection $\theta : K_1(P^2) \to [0,1]$ by

$$\theta(x,y,z) = \begin{cases} z^2 & \text{if } t > 0, \\ x^2 + y^2 & \text{if } t \le 0. \end{cases}$$

This is well defined. (θ is like the square of the distance function from the center of each surface $D(t)$. If $t \le 0$, then the center means a point $D(t) \cap \{x^2 + y^2 = 0\}$ of disk $D(t)$, and if $t > 0$, then the center means a centric circle $D(t) \cap \{z = 0\}$ of a Mobius band $D(t)$.) Let us fix a homeomorphism $\varphi_k : K_1(P^2) \to \pi^{-1}(I_k)$ respecting π. We define a continuous surjection

$$\eta_k = \theta \circ \varphi_k^{-1} : \pi^{-1}(I_k) \to [0,1].$$

Thus, a continuous surjection from the disjoint union of $\pi^{-1}(I_k)$'s to $[0,1]$ is defined and satisfies the desired property.

It remains to show that the domain of the η_k's can extend to the whole Y, satisfying the desired property. Let $J_k := [t_k + \varepsilon, t_{k+1} - \varepsilon] \subset S^1$ be the interval between I_k and I_{k+1}. Let us set $F_k = \pi^{-1}(t_k + \varepsilon)$ which is homeomorphic to D^2 or Mö. Let $G_k = \pi^{-1}(t_{k+1} - \varepsilon)$ which is homeomorphic to F_k.

Suppose that $F_k \approx D^2$. We recall that $D(-1) \subset \partial K_1(P^2)$ is defined as

$$\{(x,y,z) \in \mathbb{R}^3 \,|\, x^2 + y^2 - z^2 = -1, x^2 + y^2 \le 1\}/(x,y,z) \sim -(x,y,z).$$

We identify this as $D^2 = \{(x,y) \,|\, x^2 + y^2 \le 1\}$ by a map

$$D(-1) \ni [x,y,z] \mapsto (x,y) \in D^2.$$

Then, via φ_k, the map $\eta_k : F_k \to [0,1]$ can be identified as the map

$$\theta' : D^2 \to [0,1];\ (x,y) \mapsto x^2 + y^2,$$

namely, $\eta_k = \theta' \circ \varphi_k^{-1}$. Similarly, $\eta_{k+1} = \theta' \circ \varphi_{k+1}^{-1}$. Here, φ_k and φ_{k+1} are restricted on $D^2 \subset \partial K_1(P^2)$. Let $r : D^2 \to D^2; (x,y) \mapsto (x,-y)$ be the reflection with respect to the x-axis. We note that $\theta' \circ r = \theta'$ and r represents a unique non-trivial element of the mapping class group $\mathrm{MCG}(D^2)$ ($\cong \mathbb{Z}_2$) of D^2. By using Lemma 5.10, we obtain a homeomorphism

$$\varphi_k' : D^2 \times J_k \to \pi^{-1}(J_k),$$

respecting projections π and $D^2 \times J_k \to J_k$ such that $\varphi_k' = \varphi_k$ on F_k and either

$$\varphi_k' = \varphi_{k+1} \qquad \text{on } G_k, \text{ or}$$
$$\varphi_k' = \varphi_{k+1} \circ r \text{ on } G_k.$$

Hence,

$$\eta_k' = \theta' \circ (\varphi_k')^{-1} : \pi^{-1}(J_k) \to [0,1]$$

satisfies

$$\eta_k' = \eta_k \text{ on } F_k \text{ and } \eta_k' = \eta_{k+1} \text{ on } G_k.$$

Therefore, if the fiber of π on J_k is a disk, then η_k and η_{k+1} extend to the map η_k' on $\pi^{-1}(J_k)$, satisfying the desired property.

Next, we assume that $F_k \approx$ Mö. We recall that $D(1)$ is

$$\{(x,y,z) \mid x^2 + y^2 - x^2 = 1, |z| \leq 1\}/(x,y,z) \sim -(x,y,z).$$

Let us identify $D(1) \subset \partial K_1(P^2)$ as Mö defined by

$$\text{Mö} = S^1 \times [-1,1]/(x,s) \sim (-x,-s)$$

via a map

$$D(1) \ni [x,y,z] \mapsto \left[\frac{(x,y)}{\sqrt{x^2+y^2}}, z\right] \in \text{Mö}.$$

Then, η_k is identified as a projection

$$\theta'' : \text{Mö} \ni [x,s] \mapsto s^2 \in [0,1],$$

via φ_k. Namely, $\eta_k = \theta'' \circ \varphi_k^{-1}$ on F_k. We can see that $\eta_{k+1} = \theta'' \circ \varphi_{k+1}^{-1}$ on G_k. Let us fix a homeomorphism $r : \text{Mö} \to \text{Mö}$ defined by $r[x,s] = [\bar{x}, s]$, where $\bar{x}$ is the complex conjugate of x in $S^1 \subset \mathbb{C}$. Then, r reverses the orientation of ∂Mö. Hence, r represents a unique non-trivial element of the mapping class group MCG(Mö) $\cong \mathbb{Z}_2$, and we note that $\theta'' \circ r = \theta''$. By Lemma 5.10, there exists a homeomorphism

$$\varphi_k'' : \text{Mö} \times J_k \to \pi^{-1}(J_k),$$

respecting projections π and Mö $\times J_k \to J_k$, such that $\varphi_k'' = \varphi_k$ on F_k and either

$$\varphi_k'' = \varphi_{k+1} \qquad \text{on } G_k, \text{ or}$$
$$\varphi_k'' = \varphi_{k+1} \circ r \text{ on } G_k.$$

Since $\theta'' = \theta'' \circ r$, we obtain a continuous surjection

$$\eta_k'' = \theta'' \circ (\varphi_k'')^{-1} : \pi^{-1}(J_k) \to [0,1]$$

satisfying

$$\eta_k'' = \eta_k \text{ on } F_k \text{ and } \eta_k'' = \eta_{k+1} \text{ on } G_k.$$

By summarizing the above, we obtain a continuous surjection

$$\eta : Y \to [0,1]$$

satisfying the desired condition. $\square$

Proof of Corollary 1.6. We may assume that X has only one boundary component ∂X. By Theorem 1.5, there are decompositions

$$M_i = M_i' \cup M_i'' \text{ and } X = X' \cup X''$$

satisfying the following:

(1) X'' is a collar neighborhood of ∂X. We fix a homeomorphism $\varphi : \partial X \times [0,1] \to X''$ such that $\varphi(\partial X \times \{0\}) = \partial X$ and $\varphi(\partial X \times \{1\}) = \partial X'$.
(2) M_i' is a generalized Seifert fiber space over $X' \approx X$. We fix a fibration $f_i' : M_i' \to X'$ of it.
(3) M_i'' is a generalized solid torus or a generalized solid Klein bottle. We fix a projection $\pi_i : M_i'' \to \partial X \approx S^1$ as (1.2) in Definition 1.4.
(4) The maps f_i', π_i and φ are compatible in the following sense. For any $x \in \partial X$,
$$\pi_i^{-1}(x) \cap \partial M_i'' = (f_i')^{-1}(\varphi(x,1))$$
holds.

By Lemma 5.11, we obtain a continuous surjection

$$\eta_i : M_i'' \to [0,1]$$

such that

(5) $\eta_i^{-1}(1) = \partial M_i''$.
(6) Setting $g_i = (\pi_i, \eta_i) : M_i'' \to \partial X \times [0,1]$, the restriction of g_i on
$$g_i^{-1}(\partial X \times (0,1])$$
is an S^1-bundle.
(7) For every $x \in \partial X$, $g_i^{-1}(x,0)$ is one point set or a circle. The fiber of g_i changes at $x \in \partial X$ if and only if the fiber of π_i changes at x.

Then, the map

$$f_i'' = \varphi \circ g_i : M_i'' \to X''$$

satisfies

$$f_i' = f_i'' \text{ on } M_i' \cap M_i''.$$

Therefore, the gluing $f_i : M_i \to X$ of maps f_i' and f_i'' defined by

$$f_i = \begin{cases} f_i' \text{ on } M_i' \\ f_i'' \text{ on } M_i'' \end{cases}$$

is well-defined. The map f_i satisfies the topological condition desired in Corollary 1.6.

From the proof of Theorem 1.5 and the construction of X'', for any $\varepsilon > 0$ and large i, we can take $\pi_i : M_i'' \to \partial X$ as an ε-approximation and $\varphi : \partial X \times [0,1] \to X''$ satisfying

$$\big| |\varphi(x,t), \varphi(x',t')| - |x,x'| \big| < \varepsilon$$

for any $x, x' \in \partial X$ and $t, t' \in [0,1]$. Then, one can show that f_i is an approximation. □

6. The case that X is a circle

Let $\{M_i^3\}$ be a sequence of closed three-dimensional Alexandrov spaces with curvature ≥ -1 and uniformly bounded diameter. Suppose that M_i converges to a circle X. We will prove Theorem 1.7.

Proof of Theorem 1.7. We first show

Lemma 6.1. *For large i, $\Sigma_x \approx S^2$ for all $x \in M_i$. In particular, M_i is a topological manifold.*

Proof. Indeed, by Proposition 2.70, we may assume that $\operatorname{diam} \Sigma_{x_i}$ is almost π for each $x_i \in M_i$. It follows from Theorem 2.35 and $\partial M_i = \emptyset$ that Σ_{x_i} is homeomorphic to the suspension over a circle, which is 2-sphere. Therefore, by Theorem 2.34, M_i is a topological manifold. $\square$

By taking a rescaling, we may assume that M_i converges to the unit circle $X = S^1 = \{e^{i\theta} \in \mathbb{C} \mid \theta \in [0, 2\pi]\}$. We take points $p^+ := 1$ and $p^- := -1 \in S^1$, and prepare points p_i^+ and $p_i^- \in M_i$ converging to p^+ and p^-, respectively. Let us set $q^+ := \sqrt{-1}$ and $q^- := -\sqrt{-1} \in S^1$, and take $q_i^+, q_i^- \in M_i$ such that $q_i^\pm \to q^\pm$.

Let us take δ_i the diameter of a part of $\partial B(p_i, \pi/2)$ which is GH-close to $q^+ \in S^1$. We consider metric balls

$$B_i^+ := B(p_i^+, \ell_i - \delta_i) \text{ and } B_i^- := B(p_i^-, \ell_i - \delta_i).$$

Here, $\ell_i = |p_i^+, p_i^-|/2$. By the construction, $B_i^+ \cap B_i^- = \emptyset$. We prove the next

Lemma 6.2. *$B_i^\pm$ is homeomorphic to $F_i^\pm \times [0,1]$. Here, $F_i^\pm$ is homeomorphic to S^2, P^2, T^2 or K^2.*

Proof. We will prove this assertion only for B_i^+. Let us set $B_i := B_i^+$ and $p_i := p_i^+$.

By Lemma 6.1, M_i is a manifold. We will implicitly use this fact throughout the following argument.

Remark that

$$\partial B_i \text{ is disconnected.} \tag{6.1}$$

If B_i does not satisfy Assumption 2.28, then there exists a sequence $\hat{p}_i \in M_i$ where we may assume that $\hat{p}_i = p_i$ and $\partial B_i \approx \Sigma_{p_i} \approx S^2$. Hence ∂B_i is connected. This is a contradiction.

Therefore, B_i must satisfy Assumption 2.28. Then, by Theorem 2.27, there exist $\varepsilon_i \to 0$ and points $\hat{p}_i \in M_i$, where we may assume that $\hat{p}_i = p_i$, such that a limit $(Y, y_0) := \lim_{i\to\infty}(\frac{1}{\varepsilon_i} B_i, p_i)$ exists and has dimension ≥ 2. We remark that Y has a line, because $\tilde{\angle} q_i^+ p_i q_i^- \to \pi$. It follows from Theorem 2.19 that Y is isometric to $S \times \mathbb{R}$ for some non-negatively cured Alexandrov space S of dimension at least one.

If $\dim S = 2$, then by Theorem 2.34, S has no boundary and the topology of B_i can be determined. By the remark (6.1), S is compact and, hence, S is either homeomorphic to S^2 or P^2 or is isometric to a flat torus or a flat Klein bottle. Again, by using Theorem 2.34, we conclude that $B_i \approx S \times I$.

If $\dim S = 1$, then by Theorems 1.3 and 1.5, the topology of B_i can be determined. It follows from (6.1) that S is compact. Hence S is isometric to a circle or an interval. If S is a circle, then Y has no singular point. Then we can use Theorem 2.25, and therefore we conclude that B_i is homeomorphic to $T^2 \times I$ or

$K^2 \times I$. If S is an interval, then by Theorem 1.5, B_i is homeomorphic to $S^2 \times I$, $P^2 \times I$ or $K^2 \times I$.

This completes the proof of Lemma 6.2. □

Recall that $q_i^\pm$ are points in M_i converging to $q^\pm = \pm\sqrt{-1} \in S^1$. Let us consider

$$D_i^\pm := B(q_i^\pm, \pi/2) - \operatorname{int}(B_i^+ \cup B_i^-).$$

Let us set

$$S_i^\pm := B_i^\pm \cap D_i^+.$$

By Lemma 6.2, $S_i^\pm \approx F_i^\pm$.

Lemma 6.3. *There is a homeomorphism $\phi_i : F_i^+ \times [0,1] \to D_i^+$ such that $\phi_i(F_i^+ \times \{0\}) = S_i^+$ and $\phi_i(F_i^+ \times \{1\}) = S_i^-$.*

Proof. Let W_i be the component of $S(p_i, \ell_i)$ converging to $q = \sqrt{-1} \in S^1$. Recall that $\delta_i = \operatorname{diam} W_i$. Then $\delta_i \to 0$.

Let us take $q_i \in W_i$ and consider any limit Y of a rescaling sequence:

$$(\frac{1}{\delta_i} M_i, q_i) \to (Y, q_\infty). \tag{6.2}$$

Let $\gamma_\infty^\pm$ be rays starting at q_∞ which are limits of geodesics $q_i p_i^\pm$. Since $\tilde{\angle} p_i^+ q_i z_i^- \to \pi$, $\gamma_\infty := \gamma_\infty^+ \cup \gamma_\infty^-$ is a line in Y.

Let W_∞ be the limit of W_i under the convergence (6.2). By the choice of δ_i, $\operatorname{diam} W_\infty = 1$. We will prove that

Assertion 6.4. *Y is isometric to $W_\infty \times \mathbb{R}$. In particular,* $\dim Y \geq 2$.

Proof of Assertion 6.4. Let us consider functions

$$f_i^\pm(\cdot) := \tilde{d}_i(p_i^\pm, \cdot) - \tilde{d}_i(p_i^\pm, q_i),$$
$$b^\pm(\cdot) := \lim_{t\to\infty} d(\gamma_\infty^\pm(t), \cdot) - t.$$

Here, $\tilde{d}_i$ is the original metric of M_i multiplied by $1/\delta_i$. The functions $b^\pm$ are the Busemann functions of the rays $\gamma_\infty^\pm$. Then, we can show that $f_i^\pm$ converges to $b^\pm$. Therefore, we obtain $W_\infty = (b^+)^{-1}(0)$. This completes the proof of Assertion 6.4. □

By Assertion 6.4, $\dim W_\infty = 1$ or 2. If $\dim W_\infty = 2$, then by Theorem 2.34, we have a homeomorphism

$$\phi_i : D_i^+ \approx W_\infty \times [-1, 1]$$

with respect to functions $f_i^\pm$ and $b^\pm$. Namely,

$$\phi_i((f_i^\pm)^{-1}(t)) = (b^\pm)^{-1}(t)$$

whenever t is near $\{-1, 1\}$. In particular,

$$S_i^+ = (f_i^+)^{-1}(1) \approx (b^\pm)^{-1}(0) = W_\infty \approx (f_i^-)^{-1}(1) = S_i^-.$$

In this case, $W_\infty \approx S^2$, P^2, T^2 or K^2.

If $\dim W_\infty = 1$, then W_∞ is a circle or an interval. If W_∞ is a circle, then by Theorem 2.25 and some flow argument, there is a circle fiber bundle

$$\pi_i : D_i^+ \to W_\infty \times [-1, 1]$$

such that $\pi_i^{-1}(W_\infty \times \{\pm 1\}) = S_i^\pm$. In this case, $S_i^\pm \approx T^2$ or K^2.

If W_∞ is an interval, then by using Theorem 1.5 and some flow argument, we have a homeomorphism

$$\phi_i : D_i^+ \to S_i^+ \times [-1,1]$$

such that $\phi_i(S_i^\pm) = S_i^+ \times \{\pm 1\}$. In this case, $S_i^\pm \approx S^2$, P^2 or K^2.

This completes the proof of Lemma 6.3. □

Let F_i be a topological space homeomorphic to $F_i^\pm \approx S_i^\pm$. By Lemmas 6.2 and 6.3, we obtain homeomorphisms

$$\varphi_i^\pm : F_i \times [0,1] \to B_i^\pm,$$
$$\psi_i^\pm : F_i \times [0,1] \to D_i^\pm$$

such that they send the boundaries to the boundaries. Therefore, $M_i = B_i^+ \cup B_i^- \cup D_i^+ \cup D_i^-$ is an F_i-bundle over S^1. □

7. The case that X is an interval

Let $\{M_i\}$ be a sequence of three-dimensional closed Alexandrov spaces of curvature ≥ -1 with $\operatorname{diam} M_i \leq D$. Suppose that M_i converges to an interval I. Let $\partial I = \{p, p'\}$, and let $p_i, p_i' \in M_i$ converge to p, p', respectively. We divide M_i into $M_i = B_i \cup D_i \cup B_i'$, where $B_i = B(p_i, r)$, $B_i' = B(p_i', r)$ for small $r > 0$, and $D_i := M_i - \operatorname{int}(B_i \cup B_i')$.

Proof of Theorem 1.8. In a way similar to the proof of Lemma 6.3, we can prove that there exists a homeomorphism $\phi_i : F_i \times I \to D_i$ such that $\phi_i(F_i \times 0) = \partial B_i$ and $\phi_i(F_i \times 1) = \partial B_i'$, where F_i is homeomorphic to one of S^2, P^2, T^2 and K^2.

Next, we will find the topologies of B_i (and B_i'). If B_i does not satisfy Assumption 2.28, then B_i is homeomorphic to D^3 or $K_1(P^2)$. Hence, we may assume that there exist sequences $\delta_i \to 0$ and $\hat{p}_i$ such that a limit $(Y, y_0) = \lim_{i\to\infty} \frac{1}{\delta_i}(B_i, \hat{p}_i)$ exists, where we may assume that $\hat{p}_i = p_i$ and Y is a non-compact non-negatively curved Alexandrov space of $\dim Y \geq 2$.

If $\dim Y = 3$ with a soul $S \subset Y$, then Theorem 2.59 implies B_i is homeomorphic to one of the following:

- D^3, $K_1(P^2)$ and $B(\mathrm{pt})$ if $\dim S = 0$,
- $S^1 \times D^2$ and $S^1 \tilde{\times} D^2$ if $\dim S = 1$, and
- $B(N(S))$ and $B(S_2)$ and $B(S_4)$ if $\dim S = 2$.

Here, $N(S)$ is a non-trivial line bundle over a closed surface S of non-negative curvature and $B(N(S))$ is a metric ball around S in $N(S)$, and $B(S_i)$ is a metric ball around S_i in $L_i = L(S_i)$ for $i = 2, 4$ (see 2.8.1). $B(N(S))$ is homeomorphic to one of the non-trivial twisted I-bundles over a closed surface S with connected boundary. We determine the topology of $B(N(S))$ as follows: If $S \approx S^2$, $N(S)$ is isometric to $S \times \mathbb{R}$, which is a contradiction. If $S \approx P^2$, we have the line bundle $N(\hat{S})$ induced by the double covering $\pi : \hat{S} \to S$. Since $N(\hat{S}) = \hat{S} \times \mathbb{R}$, we find that $N(S) = \hat{S} \times \mathbb{R}/(x,t) \sim (\sigma(x), -t)$, where σ is the involution on $\hat{S}$ with $\hat{S}/\sigma = S$. Thus $B(N(S))$ is a twisted I-bundle over P^2, which is homeomorphic to $P^3 - \operatorname{int} D^3$. If S is homeomorphic to either T^2 or K^2, then $N(S)$ is a complete flat three-manifold. By [W, Theorem 3.5.1] we obtain that $B(N(S))$ is a twisted I-bundle over T^2, which is homeomorphic to Mö$\times S^1$, an orientable I-bundle $K^2 \tilde{\times} I$ over K^2, and a non-trivial non-orientable I-bundle $K^2 \hat{\times} I$ over K^2.

If $\dim Y = 2$ and $\partial Y = \emptyset$, then Y is either homeomorphic to $\mathbb{R}^2$ or isometric to a flat cylinder or a flat Mobius strip.

Suppose that $Y \approx \mathbb{R}^2$. Let us denote by m the number of essential singular points in Y. Then $m \le 2$. When $m \le 1$, Theorem 1.3 together with Lemma 4.5 implies that $B_i \approx S^1 \times D^2$ or $B(\mathrm{pt})$. If $m = 2$, then Y is isometric to the envelope $\mathrm{dbl}\,(\mathbb{R}_+ \times [0,\ell])$ for some $\ell > 0$. Let B be a closed ball around $\{0\} \times [0,\ell]$ in Y. By Theorem 1.3, B_i is a generalized Seifert fiber space over B and its boundary ∂B_i is homeomorphic to T^2 or K^2. We may assume that B_i has actually two singular orbits over two singular points $(0,0)$ and $(0,\ell)$ in Y. Here, a singular orbit is either a $(2,1)$-type fiber corresponding to the core of $U_{2,1}$ or the interval fiber of M_{pt} in this case. The topology of B_i is determined as follows: When two singular orbits are both $(2,1)$-type, $\mathrm{int}\,B_i$ is homeomorphic to $U'_{2,1} \cup_\partial U'_{2,1}$. Since $U'_{2,1}$ is an $\mathbb{R}$-bundle over Mö, $\mathrm{int}\,B_i$ is an $\mathbb{R}$-bundle over K^2. By the boundary condition, B_i is homeomorphic to $K^2 \tilde{\times} I$ if $\partial B_i \approx T^2$ or $K^2 \hat{\times} I$ if $\partial B_i \approx K^2$. When singular fibers of B_i are $(2,1)$-type and an interval, $\mathrm{int}\,B_i$ is homeomorphic to $U'_{2,1} \cup_\partial M'_{\mathrm{pt}}$. Then B_i is homeomorphic to one of $B(S_2) \subset L_{2,1}$ with $S_2 \approx P^2$. When B_i has two singular interval fibers, $\mathrm{int}B_i$ is homeomorphic to $M'_{\mathrm{pt}} \cup_\partial M'_{\mathrm{pt}}$, which is L_4. Then B_i is homeomorphic to $B(S_4)$.

If Y is a flat cylinder, then ∂B_i is not connected, and hence this case cannot happen.

If Y is isometric to a flat Mobius strip, then B_i is an S^1-bundle over Mö. Therefore, we have $B_i \approx \text{Mö} \times S^1$ or $K^2 \tilde{\times} I$.

If $\dim Y = 2$ and $\partial Y \neq \emptyset$, then Y is either isometric to a flat half cylinder $S^1(\ell) \times [0,\infty)$ or $[0,\ell] \times \mathbb{R}$ or homeomorphic to an upper half plane $\mathbb{R}^2_+ = \mathbb{R} \times [0,\infty)$.

If Y is a flat half cylinder, then ∂Y has no essential singular point. Therefore, B_i is a fiber bundle over S^1 with the fiber homeomorphic to D^2 or Mö. In other words, this is a generalized solid torus of type 0 or a generalized solid Klein bottle of type 0.

If $Y \equiv [0,\ell] \times \mathbb{R}$, then ∂B_i is not connected, and hence this case cannot happen.

Suppose that Y is homeomorphic to $\mathbb{R}^2_+$. Let us set $m := \sharp\mathrm{Ess}\,(\mathrm{int}\,Y)$ and $n := \sharp\mathrm{Ess}\,(\partial Y)$. Then $m \le 1$ and $n \le 2$.

If $m = 0$ and $n \le 1$, then by Lemma 5.4, B_i is homeomorphic to one of D^3, $\text{Mö} \times I$ or $K_1(P^2)$.

If $m = 0$ and $n = 2$, then Y is isometric to $\mathbb{R}_+ \times [0,\ell]$ for some $\ell > 0$. Let $B := [0,c] \times [0,\ell]$ for some $c > 0$. By Corollary 1.6, there is a continuous surjective map

$$\pi : B_i \to B.$$

We may assume that B_i has two topologically singular points converging to the corners $(0,0)$ and $(0,\ell)$ of Y. We divide B into two domains,

$$A_j = [0,c] \times \{y \in [0,\ell] \,|\, (-1)^j (y - \ell/2) \ge 0\} \subset B,$$

for $j = 1,2$. Since B_i has two topologically singular points, $\pi^{-1}(A_j) \approx K_1(P^2)$. Then, B_i is homeomorphic to $K_1(P^2) \cup_{D^2} K_1(P^2)$ if $\pi^{-1}(A_1 \cap A_2) \approx D^2$ or $K_1(P^2) \cup_{\text{Mö}} K_1(P^2)$ if $\pi^{-1}(A_1 \cap A_2) \approx \text{Mö}$. By Lemma 2.61 and Remark 2.62, B_i is homeomorphic to $B(\mathrm{pt})$ or $B(S_2) \subset L_{2,2}$ with $S_2 \approx S^2$.

If $m = 1$, then $n = 0$ and Y is isometric to a cut envelope $\mathbb{R} \times [0,h]/(x,y) \sim (-x,y)$ for some $h > 0$. Let $B := Y \cap \{(x,y) \,|\, x \le r\}$ which is homeomorphic to D^2. By Theorem 1.5, there is a generalized Seifert fibration $\pi_i : W_i \to B$ such that

B_i is homeomorphic to a gluing of W_i and $F_i \times [-r, r]$ via a homeomorphism

$$\partial F_i \times [-r, r] \supset \partial F_i \times \{x\} \mapsto \pi_i^{-1}(x) \subset \pi_i^{-1}(\{(x, h) \in B \mid x \in [-r, r]\})$$

for all $x \in [-r, r]$. Here, F_i is D^2 or Mö. We may assume that W_i contains a singular orbit over the singular point $(0, 0) \in \mathrm{int}B$. If the singular orbit is a circle, then W_i is isomorphic to a Seifert solid torus $V_{2,1}$ of $(2, 1)$-type. Remark that W_i can be regarded as an I-bundle over Mö, which corresponds to the preimage of the Seifert fibration over $\{0\} \times [0, h] \subset B$. Then, B_i is isomorphic to an I-bundle over Mö $\cup_\partial F_i$. Therefore, it is $P^2 \tilde{\times} I \approx P^3 - \mathrm{int}D^3$ if $F_i \approx D^2$ or $K^2 \hat{\times} I$ if $F_i \approx$ Mö. If the singular orbit is an interval, then Theorem 1.3 shows that W_i is isomorphic to M'_{pt}. Recall that B_i is homeomorphic to the union $W_i \cup F_i \times I$. Therefore, B_i is homeomorphic to $B(S_2) \subset L_{2,2}$ with $S_2 \approx S^2$ if $F_i \approx D^2$ or $B(S_2) \subset L_{2,3}$ with $S_2 \approx P^2$ if $F_i \approx$ Mö.

This completes the proof of Theorem 1.8. □

8. The case that X is a single-point set

Lemma 8.1. *If M is a three-dimensional non-negatively curved closed Alexandrov space, then a finite covering of M is T^3, $S^1 \times S^2$ or simply-connected.*

Proof. We may assume that $|\pi_1(M)| = \infty$. Then a universal covering $\tilde{M}$ of M has a line. Thus, $\tilde{M}$ is isometric to the product $\mathbb{R}^k \times X_0$, where $1 \leq k \leq 3$ and X_0 is a $(3 - k)$-dimensional non-negatively curved closed Alexandrov space.

- If $k = 3$, then $\tilde{M}$ is the Euclidean space. Then a finite covering of M is T^3.
- If $k = 2$, then X_0 is a circle. Then $\tilde{M}$ is not simply-connected. This is a contradiction.
- If $k = 1$, then X_0 is homeomorphic to S^2. Then a finite covering of M is homeomorphic to $S^1 \times S^2$.

□

Proof of Corollary 1.9. Let $\{M_i\}$ be a sequence of three-dimensional closed Alexandrov spaces of curvature ≥ -1 with $\operatorname{diam} M_i \leq D$, which converges to a point $\{*\}$. Let $\delta_i := \operatorname{diam} M_i$. Then the rescaled space $\frac{1}{\delta_i} M_i$ is an Alexandrov space with curvature $\geq -\delta_i^2$ having diameter one. Then, the limit Y of the rescaled sequence $\frac{1}{\delta_i} M_i$ is a non-negatively curved Alexandrov space of dimension ≥ 1. If $\dim Y = 1$, then M_i is homeomorphic to a space in the conclusion of Theorems 1.7 and 1.8. If $\dim Y = 2$ and $\partial Y = \emptyset$, then M_i is homeomorphic to a generalized Seifert fiber space having at most 4 singular fibers. If $\dim Y = 2$ and $\partial Y \neq \emptyset$, then M_i is homeomorphic to a space in the conclusion of Theorem 1.5 with at most 4 topologically singular points. If $\dim Y = 3$, then by the Stability Theorem, M_i is homeomorphic to Y. In this case, the topology of Y is already obtained in Lemma 8.1. □

9. Appendix: ε-regular covering of the boundary of an Alexandrov surface

Let X be an Alexandrov surface with non-empty compact boundary ∂X. Let us denote C by a component of ∂X. The purpose of this section is to prove Lemma 9.9, which states the existence of an ε-regular covering of C, used in Section 5.

We will first prepare a division of C by consecutive arcs $\gamma_1, \gamma_2, \ldots, \gamma_n$ with $\partial \gamma_\alpha = \{p_\alpha, p_{\alpha+1}\}$ and $p_{n+1} = p_1$. We next prove that this division makes the desired regular covering $\{B_\alpha, D_\alpha\}_{\alpha=1,2,\ldots,n}$ of C.

For $\varepsilon > 0$, we define

$$S_\varepsilon(\partial X) := \{p \in \partial X \mid L(\Sigma_p) \le \pi - \varepsilon\},$$

where $L(\Sigma_p)$ is the length of Σ_p. Note that $S_\varepsilon(\partial X)$ is a finite set. We set

$$R_\varepsilon(\partial X) := \partial X - S_\varepsilon(\partial X)$$

and

$$S_\varepsilon(C) := S_\varepsilon(\partial X) \cap C \text{ and } R_\varepsilon(C) := R_\varepsilon(\partial X) \cap C.$$

We review fundamental properties.

Lemma 9.1. *For $\varepsilon > 0$ and $p \in R_\varepsilon(\partial X)$, there exists $\delta > 0$ such that for every $x \in B(p, \delta) - \partial X$, we have*

$$|\nabla d_{\partial X}|(x) > \cos \varepsilon.$$

Proof. Suppose the contrary. Then, there are a sequence $\delta_i \to 0$ and $x_i \in B(p, \delta_i) - \partial X$ such that $|\nabla d_{\partial X}|(x_i) \le \cos \varepsilon$. Taking a subsequence, we consider the limit $x_\infty \in B(o_p, 1) \subset T_p X$ of x_i under the convergence $(\frac{1}{\delta_i} X, p) \to (T_p X, o_p)$.

If $|\partial T_p X, x_\infty| > 0$, then

$$|\nabla d_{\partial T_p X}|(x_\infty) > -\cos\left(\pi - \frac{\varepsilon}{2}\right) = \cos\left(\frac{\varepsilon}{2}\right).$$

By the lower-semicontinuity of angles,

$$\liminf_{i \to \infty} |\nabla d_{\partial X}|(x_i) \ge |\nabla d_{\partial T_p X}|(x_\infty).$$

This implies a contradiction.

When $|\partial T_p X, x_\infty| = 0$, we take $y_\infty \in B(o_p, 1) - U(\partial T_p X, 1/2)$ such that

$$\frac{|\partial T_p X, y_\infty|}{|x_\infty, y_\infty|} = \cos \angle x_\infty y_\infty \partial T_p X > \cos\left(\frac{\varepsilon}{2}\right).$$

We take a sequence $y_i \in B(p, \frac{3\delta_i}{2}) - U(\partial X, \frac{\delta_i}{4})$ such that $y_i \to y_\infty$ under the convergence $(\frac{1}{\delta_i} X, p) \to (T_p X, o_p)$. Since the distance function $d_{\partial X}$ is λ-concave for some λ on $\mathrm{int} X$,

$$\begin{aligned} \frac{|\partial X, y_i| - |\partial X, x_i|}{|x_i, y_i|} &\le \frac{\lambda}{2} |x_i y_i| + (d_{\partial X})'_{x_i}(\uparrow_{x_i}^{y_i}) \\ &\le \frac{\lambda}{2} |x_i y_i| + |\nabla d_{\partial X}|(x_i). \end{aligned}$$

Remark that $x_i y_i \subset \mathrm{int}\, X$ (Remark 9.2, later). It is obvious that

$$\frac{|\partial X, y_i| - |\partial X, x_i|}{|x_i y_i|} \to \frac{|\partial T_p X, y_\infty|}{|x_\infty, y_\infty|} \text{ (as } i \to \infty).$$

Therefore, we conclude that

$$\cos(\varepsilon/2) \le \cos \varepsilon.$$

This is a contradiction. Therefore, we have the conclusion of Lemma 9.1. $\square$

Remark 9.2. The interior of an Alexandrov space is strictly convex. In fact, let $p, q \in \mathrm{int}\, M$. For every $x, y \in \mathrm{int}\,(pq)$ (the relative interior), $\Sigma_x \equiv \Sigma_y$ ([Pet Para]). If x is near p, then $x \in \mathrm{int}\, M$, and hence $\partial \Sigma_x = \emptyset$. Then $\partial \Sigma_y = \emptyset$. Therefore, $pq \subset \mathrm{int}\, M$.

Corollary 9.3. *For any $\varepsilon, s > 0$, there is $\delta_1 > 0$ such that*

$$|\nabla d_{\partial X}| > \cos\varepsilon$$

on $B(\partial X, \delta_1) - (\partial X \cup U(S_\varepsilon(\partial X), s))$.

Proof. The proof is provided by Lemma 9.1 and the Lebesgue covering lemma. $\square$

Lemma 9.4. *For any $\varepsilon > 0$, there is $\delta_2 > 0$ such that*

$$B(\partial X, \delta_2) - \partial X$$

is $(2, \varepsilon)$-strained.

Proof. For any $p \in \partial X$, there is $\delta_p > 0$ such that

$$B(p, \delta_p) - \{p\}$$

has no ε'-critical point for d_p, where, $\varepsilon' \ll \varepsilon$. Therefore, $B(p, \delta_p) - \partial X$ is $(1, \varepsilon')$-strained, and hence this is $(2, \varepsilon)$-strained. Since ∂X is compact, there is $\delta > 0$ such that, for any $p \in \partial X$, there exists $q \in \partial X$ with $B(p, \delta) \subset B(q, \delta_q)$. Therefore, $B(\partial X, \delta) - \partial X$ is $(2, \varepsilon)$-strained. $\square$

From now on, we use the notation $\tilde\angle(A; B, C)$ defined as follows. Let A, B and C be positive numbers satisfying a part of the triangle inequality: $B + C \geq A$ and $A + C \geq B$. If $A + B \geq C$, then taking a geodesic triangle $\triangle abc$ in the hyperbolic plane $\mathbb{H}^2$ with side lengths $|ab| = C$, $|bc| = A$ and $|ca| = B$, we set $\tilde\angle(A; B, C) := \angle bac$. Otherwise, $\tilde\angle(A; B, C) := 0$.

Let us start to construct a division of $C \subset \partial X$ to construct an ε-regular covering. Let us fix a small positive number $\varepsilon > 0$.

Lemma 9.5. *For any $p \in \partial X$, there is $s > 0$ such that for any $q \in B(p, s) \cap \partial X - \{p\}$ and $x \in \widehat{pq} - (\{q\} \cup U(p, |pq|/2))$, we have*

$$\tilde\angle(|pq|; |px|, L(\widehat{xq})) > \pi - \varepsilon.$$

Here, $\widehat{pq}$ is an arc joining p and q in ∂X. In particular,

$$\tilde\angle pxq > \pi - \varepsilon.$$

Proof. Suppose the contrary. Then, there are $p \in \partial X$, $s_i \to 0$, $q_i \in S(p, s_i) \cap \partial X$ and $x_i \in \widehat{pq_i} - (\{q_i\} \cup U(p, |pq_i|/2))$ such that

$$\tilde\angle(|pq_i|; |px_i|, L(\widehat{x_iq_i})) \leq \pi - \varepsilon.$$

Taking a subsequence, we may assume that q_i, x_i converges to q_∞, x_∞, respectively, under the convergence $(\frac{1}{s_i}X, p) \to (T_pX, o_p)$. Then, $q_\infty \in \partial T_pX$, $|o_p, q_\infty| = 1$ and $x_\infty \in o_pq_\infty$.

If $x_\infty \neq q_\infty$, then

$$\lim_{i\to\infty} \tilde\angle(|pq_i|; |px_i|, L(\widehat{x_iq_i})) = \tilde\angle o_px_\infty q_\infty = \pi.$$

This is a contradiction.

Otherwise, $x_\infty = q_\infty$. We take $r_\infty \in \partial T_pX$ such that

$$q_\infty \in o_pr_\infty,\ |o_p, r_\infty| > 3/2.$$

We choose $r_i \in \partial X$ such that $r_i \to r_\infty$ as $i \to \infty$ under the convergence $(\frac{1}{s_i}X, p) \to (T_pX, o_p)$. Since $\widehat{x_ir_i}$ is a quasigeodesic containing q_i, by the comparison theorem for quasigeodesics [PP QG], we have

$$\tilde\angle(|pq_i|; |px_i|, L(\widehat{x_iq_i})) \geq \tilde\angle(|pr_i|; |px_i|, L(\widehat{x_ir_i})).$$

Since $L(\widehat{x_ir_i})/s_i \to |x_\infty r_\infty|$ ([PP QG]), we obtain

$$\tilde{\angle}(|pr_i|; |px_i|, L(\widehat{x_ir_i})) \to \tilde{\angle} o_p x_\infty r_\infty = \pi.$$

This is a contradiction. $\square$

Lemma 9.6. *For $p \in R_\varepsilon(\partial X)$, there is $s > 0$ such that for any $q \in B(p,s) \cap \partial X - \{p\}$ and $x \in \widehat{pq} - \{p,q\}$, we have*

$$\tilde{\angle}(|pq|; |px|, L(\widehat{xq})) > \pi - \varepsilon \text{ or } \tilde{\angle}(|pq|; L(\widehat{px}), |xq|)) > \pi - \varepsilon.$$

In particular, $\tilde{\angle} pxq > \pi - \varepsilon$.

Proof. Suppose the contrary. Then there are $p \in \partial X$, $s_i \to 0$, $q_i \in S(p, s_i) \cap \partial X$ and $x_i \in \widehat{pq_i} - \{p, q_i\}$ such that

$$\tilde{\angle}(|pq_i|; |px_i|, L(\widehat{x_iq_i})) \le \pi - \varepsilon \text{ and} \tag{9.1}$$

$$\tilde{\angle}(|pq_i|; L(\widehat{px_i}), |x_iq_i|)) \le \pi - \varepsilon. \tag{9.2}$$

We may assume that q_i and x_i converge to q_∞ and x_∞, respectively, under the convergence $(\frac{1}{s_i}X, p) \to (T_pX, o_p)$. Then, $q_\infty \in \partial T_pX$, $|o_pq_\infty| = 1$ and $x_\infty \in o_pq_\infty$.

If $q_\infty \ne o_p$, then by the same argument as in the proof of Lemma 9.5, we have

$$\tilde{\angle}(|pq_i|; |px_i|, L(\widehat{x_iq_i})) \to \pi.$$

This is a contradiction to (9.1).

Otherwise, $q_\infty = o_p$. We take $r_\infty \in \partial T_pX \cap S(o_p, 1) - \{q_\infty\}$ and $r_i \in \partial X$ such that $r_i \to r_\infty$. Since $\widehat{x_ir_i}$ is a quasigeodesic containing p, by the comparison theorem for quasigeodesics, we have

$$\tilde{\angle}(|pq_i|; L(\widehat{px_i}), |x_iq_i|)) \ge \tilde{\angle}(|r_iq_i|; L(\widehat{r_ix_i}), |x_iq_i|)).$$

Since $L(\widehat{r_ix_i})/s_i \to |r_\infty o_p|$, we obtain

$$\tilde{\angle}(|r_iq_i|; L(\widehat{r_ix_i}), |x_iq_i|)) \to \tilde{\angle} q_\infty o_p r_\infty > \pi - \varepsilon.$$

This is a contradiction to (9.2). $\square$

Definition 9.7. Let $\gamma = \widehat{pq}$ be an arc joining p and q in ∂X. We say that γ is *strictly ε-strained by $\partial\gamma = \{p, q\}$* if

$$\tilde{\angle} pxq > \pi - \varepsilon \text{ for all } x \in \operatorname{int}\gamma, \tag{9.3}$$

and if setting ξ and η as the directions of quasigeodesics $\widehat{xp}$ and $\widehat{xq}$ at x, respectively, we have

$$\angle(\xi, \uparrow_x^p) < \varepsilon \text{ and } \angle(\eta, \uparrow_x^q) < \varepsilon. \tag{9.4}$$

Remark that an arc $\widehat{pq}$ in Lemma 9.6 is strictly ε-strained by $\{p, q\}$. Indeed, we assume that $\tilde{\angle}(|pq|; |px|, L(\widehat{xq})) > \pi - \varepsilon$ for some $x \in \operatorname{int}\widehat{pq}$. We obtain $\tilde{\angle} pxq \ge \tilde{\angle}(|pq|; |px|, L(\widehat{xq})) > \pi - \varepsilon$. Let ξ and η be the directions of $\widehat{xp}$ and $\widehat{xq}$ at x, respectively. Since $\dim X = 2$, ξ and η attain the diameter of Σ_x, i.e.

$$\angle(\xi, \eta) = L(\Sigma_x).$$

Hence, we have

$$\begin{aligned}\angle(\xi, \eta) &= \angle(\xi, \uparrow_x^p) + \angle(\uparrow_x^p, \eta) \\ &= \angle(\xi, \uparrow_x^p) + \angle(\uparrow_x^p, \uparrow_x^q) + \angle(\uparrow_x^q, \eta) \\ &\ge \angle(\uparrow_x^p, \uparrow_x^q) \ge \tilde{\angle} pxq > \pi - \varepsilon.\end{aligned}$$

Since $L(\Sigma_p) \le \pi$, we obtain

$$\angle(\xi, \uparrow_x^p) + \angle(\uparrow_x^q, \eta) < \varepsilon.$$

In particular, (9.4) holds.

Let us fix a component C of ∂X. By Lemma 9.5 and $\sharp S_\varepsilon(C) < \infty$, there is $s > 0$ such that for every $p \in S_\varepsilon(C)$, taking $q^+, q^- \in S(p,s) \cap C$, we have

$$\tilde\angle(|pq^\pm|; |px|, L(\widehat{xq^\pm})) > \pi - \varepsilon$$

for all $x \in \beta_p^\pm - (U(p, s/2) \cup \{q^\pm\})$, where $\beta_p^\pm := \widehat{pq^\pm}$.

Let us consider the set

$$C - U(S_\varepsilon(C), s) = C - \bigcup_{p \in S_\varepsilon(C)} \mathrm{int}\,(\beta_p^+ \cup \beta_p^-). \tag{9.5}$$

This consists of finitely many arcs. We prove that each component K of it is divided into finitely many strictly ε-strained arcs.

Lemma 9.8. *Let K be an arc in $R_\varepsilon(C)$ with $\partial K = \{p, q\}$. There are consecutive arcs $\gamma_\alpha = \widehat{p_\alpha p_{\alpha+1}}$, $\alpha = 1, 2, \dots, n$ with $p_1 = p$ and $p_{n+1} = q$ such that each γ_α is strictly ε-strained by $\{p_\alpha, p_{\alpha+1}\}$.*

Proof. By repeatedly using Lemma 9.6, we have a set Φ of consecutive arcs starting from p contained in K,

$$\Phi = \{\gamma_1, \gamma_2, \dots, \gamma_n\}$$

such that each γ_α is strictly ε-strained by $\partial\gamma_\alpha$. Here, "consecutive arcs starting from p" means that each γ_α forms $\gamma_\alpha = \widehat{p_\alpha p_{\alpha+1}} \subset K$ and $p_1 = p$.

In what follows, Φ denotes any such finite sequence of arcs as above. Let us set

$$L(\Phi) := \sum_{\alpha=1}^{n} L(\gamma_\alpha).$$

We consider the value $\ell := \sup_\Phi L(\Phi)$. Since γ_α are consecutive and contained in K, we have $\ell \le L(K)$. To prove the lemma, we show that there exists Φ with $L(\Phi) = L(K)$.

If $\ell = L(K)$, then there is $\Phi = \{\gamma_\alpha\}_{1 \le \alpha \le n}$ such that p_n is arbitrarily close to q. If there is Φ with $L(\Phi) = L(K)$, then the proof is done. Otherwise, by using Lemma 9.6 for q, we can take Φ such that $\gamma_{n+1} := \widehat{p_n q}$ is strictly ε-strained by $\partial\gamma_{n+1}$. Then we obtain an extension

$$\tilde\Phi := \Phi \cup \{\gamma_{n+1}\}$$

of Φ with $L(\tilde\Phi) = L(K)$. This is a contradiction. Therefore, if $\ell = L(K)$, then there is Φ attaining $L(\Phi) = L(K)$.

We assume that $\ell < L(K)$. By a similar argument as above, we have $\Phi = \{\gamma_\alpha\}_{1 \le \alpha \le n}$ such that $L(\Phi) = \ell$. Again, by a similar argument as above, we have an extension $\tilde\Phi$ of Φ. Hence $L(\tilde\Phi) > \ell$. This is a contradiction. $\square$

By Lemma 9.8 and the decomposition (9.5), we obtain a division of C:

$$C = \left(\bigcup_{p \in S_\varepsilon(C)} \beta_p^+ \cup \beta_p^- \right) \cup \left(\bigcup_K \bigcup_{i=1}^{n_K} \gamma_\alpha^K \right), \tag{9.6}$$

where $\beta_p^{\pm} := \widehat{pq^{\pm}}$ and K denotes any arc component of $C - U(S_\varepsilon(C), s)$. For each K, γ_α^K $(1 \le \alpha \le n_K)$ expresses a strictly ε-strained arc by $\partial\gamma_\alpha^K$, obtained in Lemma 9.8.

By using a division (9.6) of C, we prove the existence of an ε-regular covering of C.

Lemma 9.9. *There is an ε-regular covering of C.*

Proof. Let us fix a division of C obtained as (9.6). Fixing a component K, we write $n = n_K$, $\gamma_\alpha = \gamma_\alpha^K$. Each γ_α forms $\gamma_\alpha = \widehat{p_\alpha p_{\alpha+1}}$. We take a small positive number r such that

$$|\nabla d_{p_\alpha}| > 1 - \varepsilon \text{ on } B(p_\alpha, 2r) - \{p_\alpha\} \text{ for all } \alpha, \tag{9.7}$$

$$B_\alpha \cap B_{\alpha'} = \emptyset \text{ for all } \alpha \neq \alpha', \tag{9.8}$$

where $B_\alpha := B(p_\alpha, r)$.

By the condition (9.3), there is a small positive number δ with $\delta \ll r$ such that, setting

$$D_\alpha := B(\gamma_\alpha, \delta) - \operatorname{int}(B_\alpha \cup B_{\alpha+1}),$$

we have

$$\tilde{\angle} p_\alpha x p_{\alpha+1} > \pi - \varepsilon$$

for all $x \in D_\alpha$. Further, by (9.3) and (9.4), δ can be chosen that for every $x \in D_\alpha$ and $y \in C$ with $|xC| = |xy|$, we have

$$|\angle p_\alpha x y - \pi/2| < 2\varepsilon \quad \text{and} \quad |\angle p_{\alpha+1} x y - \pi/2| < 2\varepsilon.$$

To use later, we set

$$\Phi_K := \{B_\alpha\}_{1 \le \alpha \le n} \cup \{D_\alpha\}_{1 \le \alpha \le n-1}.$$

For $p \in S_\varepsilon(C)$, there are unique components K^+ and K^- of $C - U(S_\varepsilon(C, s))$ with $\beta_p^{\pm} \cap K^{\pm} \neq \emptyset$. We take unique elements $q^{\pm} \in \beta_p^{\pm} \cap K^{\pm}$. Recall that $s > 0$ is a small positive number satisfying the conclusion of Lemma 9.5 for p, and $|\nabla d_p| > 1 - \varepsilon$ on $B(p, s) - \{p\}$. For $q^{\pm} \in K^{\pm}$, we provided numbers $r^{\pm}$ satisfying (9.7) and (9.8), above. Let us set

$$B_p := B(p, s/2) \text{ and } D_p^{\pm} := B(\beta_p^{\pm}, \delta) - \operatorname{int}(B_p \cup B(q^{\pm}, r^{\pm})).$$

If we retake δ small enough, we have

$$|\angle p x q^{\pm} - \pi/2| < \varepsilon$$

for all $x \in D_p^{\pm}$.

Thus, we obtain an ε-regular covering

$$\{B_p, D_p^{\pm}\}_{p \in S_\varepsilon(C)} \cup \bigcup_K \Phi_K$$

of C. $\square$

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