TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 367, Number 4, April 2015, Pages 2665–2702
S 0002-9947(2014)06393-9
Article electronically published on December 3, 2014

LIMITS OF POSITIVE FLAT BIVARIATE MOMENT MATRICES

LAWRENCE A. FIALKOW

ABSTRACT. The bivariate moment problem for a sequence $\beta \equiv \beta^{(6)}$ of degree 6 remains unsolved, but we prove that if the associated 10×10 moment matrix $M_3(\beta)$ satisfies $M_3 \succeq 0$ and $rank\ M_3 \leq 6$, then β admits a sequence of *approximate* representing measures, and $\beta^{(5)}$ has a representing measure. More generally, let $\overline{\mathcal{F}_d}$ denote the closure of the positive flat moment matrices of degree $2d$ in n variables. Each matrix in $\overline{\mathcal{F}_d}$ admits computable approximate representing measures, and in 2013, Jiawang Nie and the author began to study concrete conditions for membership in this class. Let $\beta \equiv \beta^{(2d)} = \{\beta_i\}_{i\in\mathbb{Z}_+^n, |i|\leq 2d}$, $\beta_0 > 0$, denote a real n-dimensional sequence of degree $2d$. If the corresponding moment matrix $M_d \equiv M_d(\beta)$ is the limit of a sequence of positive *flat* moment matrices $\{M_d^{(k)}\}$, i.e., $M_d^{(k)} \succeq 0$ and $rank\ M_d^{(k)} = rank\ M_{d-1}^{(k)}$, then i) $M_d \succeq 0$, ii) $rank\ M_d \leq \rho_{d-1} \equiv dim\ \mathbb{R}[x_1,\ldots,x_n]_{d-1}$, and iii) $\beta^{(2d-1)}$ admits a representing measure. We extend our earlier results by proving, conversely, that for $n = 2$, if M_d satisfies certain positivity and rank conditions related to i)-iii), then M_d is the limit of positive flat moment matrices.

1. Introduction

Let $\beta \equiv \beta^{(m)} = \{\beta_i\}_{i\in\mathbb{Z}_+^n, |i|\leq m}$ denote a real n-dimensional sequence of degree m, $\beta_0 > 0$, and let $K \subseteq \mathbb{R}^n$ be a closed set. The Truncated K-Moment Problem (TKMP) concerns conditions for the existence of a positive Borel measure μ, supported in K, such that

$$\beta_i = \int x^i d\mu \ (|i| \leq m) \tag{1.1}$$

(where $i \equiv (i_1,\ldots,i_n) \in \mathbb{Z}_+^n$, $|i| = i_1 + \cdots + i_n$, $x \equiv (x_1,\ldots,x_n) \in \mathbb{R}^n$, and $x^i := x_1^{i_1}\cdots x_n^{i_n}$). β as above is a *truncated moment sequence*, and we refer to μ as a *K-representing measure* for β. In the case $K = \mathbb{R}^n$, we refer to TKMP as the Truncated Moment Problem (TMP) and to μ as a *representing measure*. Although several abstract criteria for the existence of representing measures are known [CF9], [CF7] (cf. Theorems 1.2 and 1.4 below), the only concrete condition available is *flatness* of the moment data, i.e., the moment matrix M_d associated to $\beta^{(2d)}$ satisfies $M_d \succeq 0$ (positive semidefinite) and $rank\ M_d = rank\ M_{d-1}$ [CF7] (cf. Theorem 1.4). With the aid of this condition, TKMP has been solved concretely for K a planar curve of degree 1 or 2 [CF3], [CF5], [CF6], [CF8], and for $y = x^3$ [F]. In particular, for $n = 2$, TMP has been solved concretely for degree 2 ($d = 1$) and degree 4 ($d = 2$) [CF3], [CF5]. Nevertheless, the degree 6 problem ($d = 3$)

Received by the editors February 10, 2013.
2010 *Mathematics Subject Classification.* Primary 47A57, 44A60, 42A70, 30E05; Secondary 15A57, 15-04, 47A20.
Key words and phrases. Truncated moment problem, Riesz functional, moment matrix extension, flat extensions of positive matrices, positive functional.

remains largely unsolved, and similarly for $\beta^{(m)}$ with $m \geq 5$. In view of this difficulty, our focus here is on membership of M_d in $\overline{\mathcal{F}_d}$, the closure of the positive flat moment matrices. For if $M_d(\beta)$ belongs to $\overline{\mathcal{F}_d}$, then the moment problem for β is "almost" solved in two respects: first, there exists a sequence of *approximate* representing measures for β and, second, as described by Theorem 1.2 (cf. [CF9]), $\beta^{(2d-1)}$ has a representing measure. Moreover, as we discuss below, the existence of approximate representing measures provides a criterion for finite convergence in the polynomial optimization program of J.-B. Lasserre [Las]. Our main result, Theorem 1.9, provides a sufficient condition for membership in $\overline{\mathcal{F}_d}$. The main application of Theorem 1.9 is the following simple criterion for approximation by positive flat moment matrices in the degree 6 problem.

Theorem 1.1. *Let $n = 2$ and let $\beta \equiv \beta^{(6)} = \{\beta_{ij}\}_{i,j\geq 0, i+j\leq 6}$. The associated 10×10 moment matrix $M_3 \equiv M_3(\beta)$ belongs to $\overline{\mathcal{F}_3}$ if and only if $M_3 \succeq 0$ and $\rho \equiv rank\ M_3 \leq 6$. In this case, given τ, $\rho \leq \tau \leq 6$, M_3 is in the closure of the rank-τ positive flat moment matrices, and there exists a sequence of (computable) τ-atomic positive measures $\{\mu_k\}$ such that $\beta_{ij} = \lim_{k\to\infty} \int x^i y^j \, d\mu_k$ $(i, j \geq 0,\ i+j \leq 6)$. Moreover, $\beta^{(5)}$ has a representing measure.*

Let $\mathbb{R}[x]_m \equiv \mathbb{R}[x_1, \ldots, x_n]_m := \{p \in \mathbb{R}[x_1, \ldots, x_n] : deg\ p \leq m\}$. We associate to β the *Riesz functional* $L_\beta : \mathbb{R}[x]_m \mapsto \mathbb{R}$ defined by $L_\beta(\sum_{|i|\leq m} a_i x^i) = \sum a_i \beta_i$. If β has a K-representing measure μ, then L_β is *K-positive*, i.e., $p \in \mathbb{R}[x]_m$, $p|K \geq 0 \implies L_\beta(p) \geq 0$; indeed, in this case, $L_\beta(p) = \int_K p \, d\mu \geq 0$. For $K = \mathbb{R}^n$, we say that L_β is *positive*. In the classical Full Multidimensional K-Moment Problem for $\beta \equiv \beta^{(\infty)}$, the Riesz-Haviland Theorem [R], [H] states that β admits a K-representing measure if and only if L_β is K-positive. Such is not the case in TKMP. The proof of Tchakaloff's Theorem [T] shows that if K is *compact*, then K-positivity of L_β does imply that β has a (finitely atomic) K-representing measure. However, for K noncompact, this implication fails; for example, with $n = 1$, $K = \mathbb{R}$, $m = 4$, the sequence 1, 1, 1, 1, 2 has a positive functional but no representing measure (cf. Example 1.5). The following result of [CF9] illustrates the role of K-positivity in TKMP.

Theorem 1.2 ([CF9, Theorem 1.2])**.** *$\beta \equiv \beta^{(2d)}$ (or $\beta \equiv \beta^{(2d+1)}$) admits a K-representing measure if and only if β can be extended to a sequence $\widetilde{\beta} \equiv \widetilde{\beta}^{(2d+2)}$ for which $L_{\widetilde{\beta}}$ is K-positive.*

(The result in [CF9] is stated only for $\beta^{(2d)}$, but it is clear from the proof that it applies as well to $\beta^{(2d+1)}$.)

K-positivity also solves TKMP in the sense of approximation. Let $\mathcal{M}_{n,m}$ denote the set of all real n-dimensional sequences of degree m, viewed as a subset of $\mathbb{R}^\eta$ (endowed with the Euclidean norm), where $\eta := dim\ \mathbb{R}[x]_m$ $(= \binom{n+m}{m})$. Let $\mathcal{R}_{n,m}(K) := \{\beta \in \mathcal{M}_{n,m} : \beta$ *has a K-representing measure*$\}$, and let $\mathcal{P}_{n,m}(K) := \{\beta \in \mathcal{M}_{n,m} :\ L_\beta$ *is K-positive*$\}$. Note that $\mathcal{R}_{n,m}(K) \subset \mathcal{P}_{n,m}(K) \subset \mathcal{M}_{n,m}$ is an inclusion of convex cones, with $\mathcal{M}_{n,m}$ and $\mathcal{P}_{n,m}(K)$ closed; it will become clear in the sequel that in general $\mathcal{R}_{n,m}(K)$ is not closed.

Theorem 1.3 ([FN1, Theorem 2.2])**.** $\mathcal{P}_{n,m}(K) = \overline{\mathcal{R}_{n,m}(K)}$. *If $L_{\beta^{(m)}}$ is K-positive, then there exists a sequence of positive Borel measures $\{\mu_k\}$, each supported in*

K, having finite moments up to degree m, such that for each i, $|i| \le m$, $\beta_i = \lim_{k\to\infty} \int x^i d\mu_k$.

In view of the preceding two results, it would be desirable to have a concrete test for K-positivity, but at present there is no such test that is applicable to a general multisequence. Similarly, in the Full K-Moment Problem, concrete conditions for K-positivity are known only in certain cases, e.g., in Hamburger's solution for $K = \mathbb{R}$ (cf. [A], [KN]) and in K. Schmüdgen's solution for K a basic compact semialgebraic set [S2]. Remarkably, in the compact case treated by Tchakaloff [S2], there is no known concrete test for K-positivity in TKMP, even in the case when K is a basic compact semialgebraic set. In [FN2] we began to study conditions for β to be a limit of "flat" multisequences (as defined below), in which case positivity of L_β is obvious. Moreover, if the flat approximants $\beta^{[k]}$ are known, then corresponding approximate representing measures $\mu^{(k)}$ for β can be explicitly computed (cf. Example 1.5). In [FN2] we obtained concrete necessary and sufficient conditions for flat approximation when $n = 1$, $d \ge 1$ and when $n = d = 2$ (cf. Theorem 1.6 below). In the present note, we extend the results of [FN2] to the bivariate truncated moment problem, i.e., $n = 2$, $d \ge 1$. The conditions that we present in Theorem 1.9 apply in certain cases in which the truncated moment problem is still unsolved, including certain cases of the bivariate degree 6 problem, so we can at least test for approximate representing measures in these cases; Theorem 1.1 provides one such test.

To describe our results in detail, we require some additional terminology. Following [CF7], we associate to $\beta \equiv \beta^{(2d)}$ the *moment matrix* $M_d(\beta)$. For $p \in \mathbb{R}[x]_d$, $p = \sum_{|i|\le d} a_i x^i$, let $\widehat{p} \equiv (a_i)$ denote the vector of coefficients of p with respect to the basis for $\mathbb{R}[x]_d$ consisting of the monomials in degree-lexicographic order. Let $\rho_d = dim\ \mathbb{R}[x]_d$. Then the moment matrix $M_d \equiv M_d(\beta)$ is the $\rho_d \times \rho_d$ matrix defined by

$$\langle M_d(\beta)\widehat{p}, \widehat{q}\rangle = L_\beta(pq)\ (p, q \in \mathbb{R}[x]_d).$$

As noted above, if β has a representing measure, then L_β is positive. Further, if L_β is positive, then $M_d(\beta)$ is positive semidefinite ($M_d(\beta) \succeq 0$), since $\langle M_d(\beta)\widehat{p}, \widehat{p}\rangle = L_\beta(p^2) \ge 0$ ($p \in \mathbb{R}[x]_d$). In general, the preceding implication cannot be reversed, but there is one situation where positivity of $M_d(\beta)$ is equivalent to positivity of L_β. Recall that $p \in \mathbb{R}[x]_{2d}$ is *nonnegative* or *positive semidefinite (psd)* if $p|\mathbb{R}^n \ge 0$, and p is a *sum of squares* (SOS) if there exist $p_1, \dots, p_k \in \mathbb{R}[x]_d$ such that $p = \sum_{j=1}^{k} p_j^2$. Now, if every nonnegative polynomial p in $\mathbb{R}[x]_{2d}$ is SOS , then positive semidefiniteness of $M_d(\beta)$ implies positivity of L_β, since, for p nonnegative, $L_\beta(p) = L_\beta(\sum_{j=1}^{k} p_j^2) = \sum \langle M_d(\beta)\widehat{p}_j, \widehat{p}_j\rangle \ge 0$. A well-known theorem of Hilbert (cf. [Rez]) shows that each psd polynomial is SOS if and only if $n = 1$, $d \ge 1$; $n = d = 2$; or $n \ge 1$, $d = 1$. In each of these cases, checking that L_β is positive reduces to simply verifying that $M_d(\beta)$ is positive semidefinite. Now, for $n = 2$, $d = 3$, there exists $M_3(\beta) \succ 0$ (positive definite) for which L_β is not positive [CF3], [S1]; more generally, [FN2, Proposition 1.6] shows that *except* in the cases of Hilbert's theorem, positive semidefiniteness of a moment matrix does not imply positivity of its Riesz functional.

In the sequel we seek to identify cases of Riesz functional positivity that are beyond the scope of Hilbert's theorem and are not due to the existence of representing measures. The first such examples appear in [EF] for certain bivariate M_3 with *rank* $M_3 = 9$. In the present note we focus on positivity arising from approximation by positive flat moment matrices. The following result of [CF7] illustrates the central role of positive flat moment matrices in TMP.

Theorem 1.4 ([CF7])**.** *Let $\beta \equiv \beta^{(2d)}$. If $M_d(\beta) \succeq 0$ is flat, i.e., rank $M_d =$ rank M_{d-1}, then β admits a unique representing measure which is rank $M_d(\beta)$-atomic. More generally, β has a representing measure if and only if β can be extended to a sequence $\widetilde{\beta} \equiv \widetilde{\beta}^{(2(d+k))}$ (for some k, $0 \le k \le \rho_{2d} -$ rank $M_d(\beta) + 1$) for which $M_{d+k}(\widetilde{\beta})$ is positive semidefinite and flat.*

The result in [CF7] is stated in terms of finitely atomic representing measures, but in [BT] Bayer and Teichmann proved that the existence of a representing measure implies the existence of a finitely atomic representing measure. Theorem 1.4 is the basis for a concrete solution to TKMP when K is the real line $\mathbb{R}$ [CF1], a planar line or curve of degree 2 [CF3], [CF5], [CF6], [CF8], or the planar curve $y = x^3$ [F]. Theorem 1.4 also leads to an algorithmic solution to TMP for the class of *recursively determinate* moment matrices [CF10] and is used by Helton and Nie [HN] in a numerical solution to TKMP based on semidefinite programming (see also [FN3]).

Let $\mathcal{F}_d := \{M_d(\beta) : M_d(\beta) \succeq 0,\ rank\ M_d(\beta) = rank\ M_{d-1}(\beta)\}$, the set of positive flat moment matrices with data of degree $2d$. For $M_d(\beta) \in \mathcal{F}_d$, we refer to β as a *flat* multisequence. Now if $M_d(\beta)$ belongs to $\overline{\mathcal{F}_d}$, the closure of $\mathcal{F}_d$ in any of the (equivalent) norms on the $\rho_d \times \rho_d$ matrices, then, by Theorems 1.3 and 1.4, L_β is positive. Moreover, if $M_d(\beta) = \lim_{k\to\infty} M_d^{(k)}$, where each $M_d^{(k)}$ is positive semidefinite and flat, then a unique representing measure for $M_d^{(k)}$ can be explicitly computed using [CF7] (see Section 2), and this serves as an approximate representing measure for M_d. Although a concrete characterization of positivity for L_β seems unlikely, in view of the ease of detecting flatness (by simply checking the positivity and rank conditions), we are motivated to seek a concrete characterization of membership in $\overline{\mathcal{F}_d}$ (cf. Question 1.7 below). To illustrate membership in $\overline{\mathcal{F}_d}$, consider the example mentioned just before Theorem 1.2.

Example 1.5. We have $n = 1$ and

$$M \equiv M_2(\beta^{(4)}) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

M is positive semidefinite, but not *recursively generated*, so β has no measure (see Section 2). However, with

$$M_2^{(k)} := \begin{pmatrix} 1 & 1 & 1+\frac{1}{k} \\ 1 & 1+\frac{1}{k} & 1+\epsilon_k \\ 1+\frac{1}{k} & 1+\epsilon_k & 2 \end{pmatrix} \ (\epsilon_k = \frac{k^2+\sqrt{k^5-2k^4-k^3}}{k^3}),$$

we see that $M_2^{(k)}$ is positive and flat, with *rank* $M_2^{(k)} =$ *rank* $M_1^{(k)} = 2$, and clearly $\lim_{k\to\infty} M_2^{(k)} = M$, so $M_2(\beta) \in \overline{\mathcal{F}_2}$ and L_β is positive. (Of course, since $n = 1$ in this example, positivity of L_β also follows from $M \succeq 0$ via sums of squares.) Let $\gamma = \sqrt{k^3(k^2 - 2k - 1)}$ and let $\psi = \sqrt{k^2(3k + 2k^2 + k^3 - 4\gamma)}$. Then (as described in Section 2) the unique representing measure for $M_2^{(k)}$, which serves as an approximate representing measure for M, is given by $\mu^{(k)} := \alpha_1^{(k)}\delta_{x_1^{(k)}} + \alpha_2^{(k)}\delta_{x_2^{(k)}}$, where $x_1^{(k)} = \frac{1}{2k^2}(\gamma - \psi) \longrightarrow 1$, $x_2^{(k)} = \frac{1}{2k^2}(\gamma + \psi) \longrightarrow +\infty$, $\alpha_1^{(k)} = \frac{-2k^2+\gamma+\psi}{2\psi} \longrightarrow 1$, $\alpha_2^{(k)} = \frac{-2k^2+\gamma-\psi}{2\psi} \longrightarrow 0$. The fact that $\{x_2^{(k)}\}$ diverges reflects the fact that β has no representing measure.

Now suppose $n \geq 1$ and $M_d(\beta) = \lim_{k\to\infty} M_d(\beta^{[k]})$, where each $\beta^{[k]}$ is a sequence of degree $2d$ with $M_d(\beta^{[k]})$ positive and flat. It follows that

$$M_d(\beta) \succeq 0, \tag{1.2}$$

and lower semicontinuity of rank ([Her, Proposition 1.12(i)]) implies that

$$\textit{rank}\ M_d(\beta) \leq \liminf_{k\to\infty} \textit{rank}\ M_d(\beta^{[k]}) = \liminf_{k\to\infty} \textit{rank}\ M_{d-1}(\beta^{[k]}) \leq \rho_{d-1},$$

whence

$$\textit{rank}\ M_d(\beta) \leq \rho_{d-1}. \tag{1.3}$$

For $n = 1$, $d \geq 1$ or $n = d = 2$, the main result of [FN2], which follows, characterizes membership in $\overline{\mathcal{F}_d}$ in terms of (1.2) and (1.3).

Theorem 1.6 ([FN2]). *Let $n = 1, d \geq 1$ or $n = d = 2$. $M_d \equiv M_d(\beta)$ belongs to $\overline{\mathcal{F}_d}$ if and only if $M_d \succeq 0$ and rank $M_d \leq \rho_{d-1}$. In this case, there exist moment matrices $M_d^{(k)} \equiv M_d(\beta^{[k]})$ $(k \geq 1)$ such that $\lim_{k\to\infty} M_d^{(k)} = M_d$ and for each k, rank $M_d^{(k)} =$ rank $M_{d-1}^{(k)} =$ rank M_d.*

Note also that for $n \geq 1$, $d = 1$, (1.2) and (1.3) imply that $M_1(\beta)$ is flat (with rank 1), so Theorem 1.6 holds in all the cases of Hilbert's theorem; these results motivate the following question.

Question 1.7. Let $n, d \geq 1$. If $M_d(\beta) \succeq 0$ and *rank* $M_d \leq \rho_{d-1}$, does $M_d(\beta)$ belong to $\overline{\mathcal{F}_d}$?

Note that in each case in which Question 1.7 has an affirmative answer, (1.2) and (1.3) provide a simple test for the existence of a representing measure for $\beta^{(2d-1)}$ (via Theorem 1.2). Theorem 1.1 provides a positive answer to Question 1.7 for $n = 2$, $d = 3$; this is perhaps surprising, because several moment theorems which hold within the framework of Hilbert's theorem, e.g., the solution to TMP for $n = d = 2$ [CF5], break down in the bivariate degree 6 case (cf. [F]). To further address membership in $\overline{\mathcal{F}_d}$, observe the necessay condition arising from Theorem 1.2 that positivity of L_β, including membership of $M_d(\beta)$ in $\overline{\mathcal{F}_d}$, entails

$$\beta^{(2d-1)} \in \mathcal{R}_{n,2d-1}. \tag{1.4}$$

Of course, (1.4) is equivalent to TMP for $\beta^{(2n-1)}$ and is therefore difficult to check in general. Nevertheless, the results of this paper provide some positive evidence concerning the following weaker version of Question 1.7.

Question 1.8. Let $n, d \geq 1$. If $M_d \succeq 0$, *rank* $M_d \leq \rho_{d-1}$, and $\beta^{(2d-1)}$ has a representing measure, is M_d a limit of positive flat moment matrices?

Let $M_d \equiv M_d(\beta)$ denote a bivariate moment matrix of degree $2d$, and consider the block matrix decomposition

$$M_d \equiv \begin{pmatrix} M_{d-1} & B_d \\ B_d^T & C_d \end{pmatrix};$$

thus M_{d-1} and B_d together contain the data in $\beta^{(2d-1)}$, and C_d is a Hankel matrix comprised of the data of degree $2d$. As we discuss in Section 2, M_d is positive semidefinite if and only if i) $M_{d-1} \succeq 0$, ii) there exists a matrix W such that $B_d = M_{d-1}W$, and iii) $C_d \succeq C^\flat \equiv C_d^\flat := W^T M_{d-1} W$, i.e., $\Delta^\flat \equiv C_d - C_d^\flat \succeq 0$. Let $\rho_d = dim\ \mathbb{R}[x,y]_d$ $(= \frac{(d+1)(d+2)}{2})$, which coincides with the size of M_d; thus, $r \equiv rank\ M_{d-1} \leq \rho_{d-1}$. Now suppose $\beta^{(2d-1)}$ has a κ-atomic representing measure μ; then $M_d[\mu]$ is of the form

$$M_d[\mu] = \begin{pmatrix} M_{d-1} & B_d \\ B_d^T & C_d[\mu] \end{pmatrix}.$$

Let $\Delta \equiv \Delta[\mu] := C_d - C_d[\mu]$, $s = rank\ \Delta[\mu]$, $\rho[\mu] := s + \kappa$, and

$$M_\Delta := \begin{pmatrix} 0 & 0 \\ 0 & \Delta \end{pmatrix}.$$

We now state our main result.

Theorem 1.9. *Let $n = 2$, $d \geq 1$. Suppose that $M_d(\beta) \succeq 0$ and that $\beta^{(2d-1)}$ has a κ-atomic representing measure μ. If $\Delta \equiv \Delta[\mu] \succeq 0$ and $\rho[\mu] \leq \rho_{d-1}$, then $M_d \in \overline{\mathcal{F}_d}$. Moreover, if $\rho[\mu] \leq \tau \leq \rho_{d-1}$, then there exists a sequence of positive flat moment matrices, $\{M_d^{(k)}\}_{k=1}^\infty$, such that $M_d(\beta) = \lim_{k\to\infty} M_d^{(k)}$ and, for each k, $rank\ M_d^{(k)} = rank\ M_{d-1}^{(k)} = \tau$.*

We note two basic cases where Theorem 1.9 applies. For $M_d \succeq 0$, let

$$M^\flat := \begin{pmatrix} M_{d-1} & B_d \\ B_d^T & C^\flat \end{pmatrix}$$

and

$$M_{\Delta^\flat} := \begin{pmatrix} 0 & 0 \\ 0 & \Delta^\flat \end{pmatrix},$$

so that

$$M_d = M^\flat + M_{\Delta^\flat}. \tag{1.5}$$

Let $r = rank\ M_{d-1}$ and $s^\flat = rank\ \Delta^\flat$. Then $\rho \equiv \text{rank}\ M_d = r + s^\flat$ and $rank\ M^\flat = r$. For the cases covered by Theorem 1.6, we showed in [FN2] that $C^\flat$ is always Hankel; thus, Theorem 1.6 can be derived from the following result.

Corollary 1.10. *The conclusions of Theorem 1.9 hold if $M_d \succeq 0$, $\rho \equiv rank\ M_d \leq \rho_{d-1}$, and $C^\flat$ is Hankel.*

Proof. Suppose $M_d \succeq 0$ and $rank\ M_d \leq \rho_{d-1}$. If $C^\flat$ is a Hankel matrix, then $M^\flat$ is a flat moment matrix extension of M_{d-1} using the data in B_d. Theorem 1.4 thus shows that $M^\flat$ has a unique representing measure μ, which is r-atomic ($r \equiv rank\ M_{d-1}$). Thus, the hypotheses of Theorem 1.9 hold with $\kappa = r$, $\Delta = \Delta^\flat$, $s = s^\flat$, $\rho[\mu] = r + s = \rho \leq \rho_{d-1}$; note from iii) above that $\Delta \succeq 0$ since $M_d \succeq 0$. $\square$

In Example 2.1, (1.2) and (1.3) hold, but $C^\flat$ is not Hankel; however, we show in Section 4 that this example falls within the scope of the next result.

Corollary 1.11. *The conclusions of Theorem* 1.9 *hold if M_d has a κ-atomic representing measure μ with $\kappa \le \rho_{d-1}$; in particular, this applies if $M_d \succeq 0$, rank $M_d \le \rho_{d-1}$, and M_d admits a flat extension M_{d+1}.*

Proof. For μ satisfying the hypothesis, $\Delta[\mu] = 0$, $s = 0$, $\rho[\mu] = \kappa \le \rho_{d-1}$, so Theorem 1.9 applies. In the flat extension case, Theorem 1.4 implies the existence of μ with $\kappa \equiv \mathit{card\ supp}\ \mu = \mathit{rank}\ M_d \le \rho_{d-1}$. □

If we ignore the calculations of ranks in the proof of Theorem 1.9, we arrive at the following criterion for positivity of L_β.

Proposition 1.12. *If $M_d(\beta) \succeq 0$ and $\beta^{(2d-1)}$ has a finitely atomic representing measure μ such that $\Delta[\mu] \succeq 0$, then L_β is positive.*

Section 2 concerns positive moment matrices, including a geometric characteriztion of flatness (Theorem 2.7) that we require in the sequel. Section 3 concerns *determining sequences*, which provide a tool for relating the rank of a moment matrix to the geometry of the support of a representing measure. Using Proposition 3.1 and particular determining sequences that we describe in Section 3, we prove Theorem 1.9 and Proposition 1.12 in Section 4. The proof of Theorem 1.1 is a synthesis of these and other results and appears in Section 5. Theorem 1.1 also yields a new solution to the singular quartic moment problem (Proposition 5.13). The proofs of Theorem 1.1 and of several examples that we present below depend in part on calculations using the computer algebra system in *Mathematica*; in the sequel we refer to these as *symbolic calculations*.

We conclude this section by noting an application of Theorem 1.9 to polynomial optimization on $\mathbb{R}^2$. For $p \in \mathbb{R}[x,y]$, the Optimization Problem entails estimating

$$p_* := \inf_{(x,y)\in\mathbb{R}^2} p(x,y). \tag{1.6}$$

We recall the *moment relaxations* for (1.6) introduced by J.-B. Lasserre [Las]. For $t \ge 1$, we define the t-th Lasserre relaxation by

$$p_t := \mathit{inf}\{L_y(p) : y \equiv y^{(2t)},\ y_{00} = 1,\ M_t(y) \succeq 0\}. \tag{1.7}$$

It is not difficult to check that $p_t \le p_*$ and that for $t' \ge t$, $p_{t'} \ge p_t$; thus, $\{p_t\}$ is convergent, and $p^{mom} \equiv \lim\limits_{t\to\infty} p_t \le p_*$. For fixed t, the infimum in (1.6) is not necessarily attained. Assuming that the infimum *is* attained, at some optimal sequence $y_* \equiv y_*^{(2t)}$, we seek criteria which imply that $L_{y_*}(p) = p_*$, so that we have finite convergence of $\{p_s\}$ to p_* at stage $s = t$. A basic result of [Las] shows that this is the case if $M_t(y_*)$ is flat, and this concrete condition is used as the stopping criterion in the optimization program in [HeLa]. In [FN2, Theorem 1.5] we showed that convergence at stage t occurs, more generally, if L_{y_*} is positive. Thus, Corollary 1.10 provides a broader concrete condition for finite convergence than flatness, namely, $\mathit{rank}\ M_t(y_*) \le \rho_{t-1}$ and $C_t^\flat(y_*)$ Hankel.

2. Positive moment matrices

In this section we present some results concerning positive moment matrices that will be used in the sequel. We begin, more generally, with a real symmetric block matrix of the form

$$M = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}. \tag{2.1}$$

It is well known that M is positive semidefinite if and only if $A \succeq 0$, $B = AW$ for some matrix W, and $C \succeq W^TAW$ $(= B^TW)$ (cf. [CF1]). In this case, let $\Delta \equiv C - W^TAW$ denote the Schur complement, so that $rank\ M = rank\ A + rank\ \Delta$. We have $rank\ M = rank\ A$, and we say that M is a *flat extension* of A if and only if $C = W^TAW$. Flat extensions are uniquely determined by A and B, for if there are matrices W and V such that $AW = B = AV$, then $W^TAW - V^TAV = W^TAV - V^TAV = (AW - AV)^TV = 0$. Given $A \succeq 0$ and $B = AW$ as above, if we set $C^\flat = W^TAW$ and

$$M^\flat = \begin{pmatrix} A & B \\ B^T & C^\flat \end{pmatrix}, \tag{2.2}$$

then clearly $M^\flat$ is a positive, flat extension of A.

Let us denote the positive n-variable moment matrix $M_d \equiv M_d(\beta)$ by

$$M_d(\beta) = \begin{pmatrix} M_{d-1}(\beta) & B_d \\ B_d^T & C_d \end{pmatrix}, \tag{2.3}$$

with $B_d = M_{d-1}W$. In this setting, we sometimes denote $C^\flat$ by $C_d^\flat$. Following [CF2], [CF7], we say that M_d is *flat* if $rank\ M_d = rank\ M_{d-1}$, i.e., $C_d = C_d^\flat$. Theorem 1.4 implies that in this case M_d has a unique representing measure which is $(rank\ M_d)$-atomic. For $M_d \succeq 0$, a rank-preserving extension M_{d+1} is said to be a *flat extension*; in this case, $M_{d+1} \succeq 0$ and M_{d+1} is itself flat. In the sequel, if μ is a positive Borel measure with convergent moments $\beta \equiv \beta^{(2d)}$, we sometimes denote $M_d(\beta)$ by $M_d[\mu]$; moreover, for a moment matrix $M_d(\beta)$, we sometimes refer to a representing measure for β as a representing measure for $M_d(\beta)$.

Let $n = 2$ and suppose $M_d(\beta) \succeq 0$, so that $B_d = M_{d-1}(\beta)W$ for some matrix W (as above). Since $n = 2$, C_d is a *Hankel* matrix; i.e., C_d is constant on cross-diagonals. We note that in general $C^\flat \equiv W^TM_{d-1}W$ need not be Hankel. We illustrate this in the following example of [FN2], which we will analyze further in Section 4.

Example 2.1. Consider the positive moment matrix $M_3(\beta)$ defined by

$$M_3(\beta) = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 4 \end{pmatrix}.$$

Then $B_3 = M_2 W$, where

$$W := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so $C^\flat \equiv W^T M_2 W = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 4 \end{pmatrix}$ and $\Delta \equiv C_3 - C^\flat = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.
In [FN2] we used ad hoc methods to show that $M_3 \in \overline{\mathcal{F}}_3$. In Section 4 we will establish membership in $\overline{\mathcal{F}}_3$ based on Corollary 1.11. □

In the sequel we denote the successive columns of $M_d(\beta)$ by X^i ($|i| \le d$) in degree-lexicographic order. Let $p \in \mathbb{R}[x]_d$, $p = \sum a_i x^i$. We define a corresponding element of $Col\ M_d(\beta)$, the column space of $M_d(\beta)$, by $p(X) := \sum a_i X^i$ ($= M_d \widehat{p}$). Thus, each column dependence relation in $M_d(\beta)$ can be expressed as $p(X) = 0$ for some $p \in \mathbb{R}[x]_d$, $p \not\equiv 0$. Following [CF2], [CF7], we say that $M_d(\beta)$ is *recursively generated* if $p, q, pq \in \mathbb{R}[x]_d$, $p(X) = 0 \Longrightarrow (pq)(X) = 0$. Positivity and recursiveness are necessary conditions for β to have a representing measure [CF7], and for $n = 1$ these conditions are also sufficient [CF1] (cf. Theorem 2.3). In the sequel, to construct examples, we will employ the following result without further reference.

Theorem 2.2 (Structure Theorem [CF2], [CF7]). *If $M_d(\beta)$ is positive semidefinite, then the following properties hold:*
i) $M_{d-1}(\beta)$ is recursively generated.
ii) If $p \in \mathcal{P}_{d-1}$ and $p(X) = 0$ in $Col\ M_{d-1}$, then $p(X) = 0$ in $Col\ M_d$.
iii) If $p \in \mathcal{P}_d$ satisfies $p(X) = 0$ in $Col\ M_d$ and $q \in \mathcal{P}_d$ satisfies $\deg pq \le d$, then $(pq)(X) = 0$ in $Col\ \begin{pmatrix} M_{d-1}(y) & B_d \end{pmatrix}$. Further, if M_d is flat, then $(pq)(X) = 0$ in $Col\ M_d$.

For $n = 1$, we will denote a truncated moment sequence of degree $2d$ by $y^{(2d)} = \{y_0, \ldots, y_{2d}\}$. Then $M_d(y)$ is a Hankel matrix, which we henceforth denote by $H_d \equiv H_d(y) := (y_{a+b})_{0 \le a,b \le d}$. In this case, Theorem 2.2 may be expressed as follows.

Theorem 2.3 ([CF1]). *Suppose $H_d \equiv H_d(y)$ is a positive Hankel matrix, $y_0 > 0$. If $H_d \succ 0$ (positive definite), then H_d admits infinitely many distinct flat extensions and corresponding $(d+1)$-atomic representing measures. If H_d is singular, let $r := min\{s : 1 \le s \le d : \ H_s \text{ is singular}\}$. Let $v = (y_r, \ldots, y_{2r-1})^T$ and let $c \equiv (c_0, \ldots, c_{r-1})^T = M_{r-1}^{-1} v$. Then (a) $y_j = c_0 y_{j-r} + \cdots + c_{r-1} y_{j-1}$ ($r \le j \le 2d-1$) and (b) $y_{2d} \ge c_0 y_{2d-r} + \cdots + c_{r-1} y_{2d-1}$.*

Further, the following are equivalent: i) y has a representing measure; ii) equality holds in (b) (above); iii) H_d is recursively generated; iv) rank $H_d = r$; v) H_d is a flat extension of H_{r-1}. There is strict inequality in (b) (above) $\Longleftrightarrow$ rank $H_d = r + 1$.

Recall the *variety* associated with β and $M_d(\beta)$, defined by

$$\mathcal{V} \equiv \mathcal{V}(\beta) := \bigcap_{p \in \mathbb{R}[x]_d,\, p(X)=0} \mathcal{Z}(p),$$

where $\mathcal{Z}(p) := \{x \in \mathbb{R}^n : p(x) = 0\}$. We will repeatedly use the following result.

Proposition 2.4 ([CF2], [CF7, Prop. 2.10, Cor. 2.12])**.** *If β has a representing measure μ, then supp $\mu \subseteq \mathcal{V}(\beta)$. Furthermore,*
i) rank $M_d \le$ card supp $\mu \le$ card $\mathcal{V}(\beta)$;
ii) for $p \in \mathbb{R}[x]_d$, $p(X) = 0$ in Col M_d (equivalently, $M_d\widehat{p} = 0$) $\Longleftrightarrow$ $p|$supp $\mu \equiv 0$.

The next result shows how the variety can be used to construct the unique representing measure corresponding to a flat extension.

Theorem 2.5 ([CF7, Theorem 1.2])**.** *Suppose $M_d \succeq 0$ admits a flat extension $M_{d+1}(\widetilde{\beta})$, i.e., rank $M_{d+1}(\widetilde{\beta}) =$ rank M_d. Then card $\mathcal{V}(\widetilde{\beta}) = \rho \equiv$ rank M_d, say $\mathcal{V}(\widetilde{\beta}) = \{w_1, \dots, w_\rho\}$. Let $X^{i_1}, \dots, X^{i_\rho}$ denote a basis for the column space of M_d, define $p_j(x) := x^{i_j} \in \mathbb{R}[x]_d$, and let $\omega = (\beta_{i_1}, \dots, \beta_{i_\rho})$. Then the Vandermonde matrix $V \equiv (p_j(w_k))_{1\le j,k\le\rho}$ is invertible, and we define $\alpha \equiv (\alpha_1, \dots, \alpha_\rho)$ by $\alpha^T := V^{-1}\omega^T$. The unique representing measure for $\widetilde{\beta}$ is given by*

$$\mu := \sum_{i=1}^{\rho} \alpha_i \delta_{w_i}. \tag{2.4}$$

We now consider the moment matrix determined by a finitely atomic measure. For $w \equiv (x_1, \dots, x_n) \in \mathbb{R}^n$, let $\zeta \equiv \zeta[w] = (1,\ x_1,\ x_2, \dots,\ x_1^d,\ x_1^{d-1}x_2, \dots,\ x_{n-1}x_n^{d-1},\ x_n^d)$. In the sequel, $\delta_{\{w\}}$ denotes the atomic measure with support $\{w\}$, whose moment sequence $\beta \equiv \beta^{(2d)}$ satisfies $\beta_i = x_1^{i_1} \cdots x_n^{i_n}$ ($i \equiv (i_1, \dots, i_n)$, $|i| \le 2d$). As noted in [Lau], $M_d[\delta_{\{w\}}] \equiv M_d(\beta) = \zeta^T\zeta$, so *rank* $M_d[\delta_{\{w\}}] = 1$. It follows from subadditivity of rank that if μ is a t-atomic measure, $\mu \equiv \sum_{i=1}^{t} \alpha_i \delta_{\{w_i\}}$ with each $\alpha_i > 0$, then

$$\textit{rank } M_d[\mu] = \textit{rank } \sum_{i=1}^{t} \alpha_i M_d[\delta_{\{w_i\}}] \le \sum_{i=1}^{t} \textit{rank } \alpha_i M_d[\delta_{\{w_i\}}] = t. \tag{2.5}$$

In Theorem 2.7, we present a geometric criterion for flatness that we will use in the proof of Theorem 1.9. We begin with a lower estimate for *rank* M_d which complements the upper estimate in Proposition 2.4 i). Given $\mathcal{P} \equiv \{p_1, \dots, p_t\} \subset \mathbb{R}[x]_d$ and points $w \equiv \{w_i\}_{i=1}^t \subset \mathbb{R}^n$, let $V \equiv V[\mathcal{P}; w]$ denote the Vandermonde-type matrix

$$\begin{pmatrix} p_1(w_1) & \cdots & p_t(w_1) \\ \vdots & \vdots & \vdots \\ p_1(w_t) & \cdots & p_t(w_t) \end{pmatrix}. \tag{2.6}$$

Proposition 2.6. *If $w \equiv \{w_1, \dots, w_t\} \subseteq \mathcal{V}(\beta)$, $\mathcal{P} \equiv \{p_1, \dots, p_t\} \subset \mathbb{R}[x]_d$, and $V \equiv V[\mathcal{P}; w]$ is invertible, then rank $M_d(\beta) \ge t$. If M_d has a t-atomic representing measure μ, supp $\mu = w \equiv \{w_1, \dots, w_t\}$, then rank $M_d = t \Longleftrightarrow$ there exist polynomials $\mathcal{P} \equiv \{p_1, \dots, p_t\} \subset \mathbb{R}[x]_d$ such that $V[\mathcal{P}; w]$ is invertible.*

Proof. Let $v := \rho_d$; since V is invertible, $\mathcal{P}$ is independent, so we may extend $\mathcal{P}$ to a basis $\{p_j\}_{j=1}^v$ for $\mathbb{R}[x]_d$. Suppose $p \equiv \sum_{i=1}^{v} c_i p_i \in \mathbb{R}[x]_d$ and $\widehat{p} \in$ *ker* M_d; then $p|\mathcal{V}(\beta) \equiv 0$, so $p(w_j) = 0$ ($1 \le j \le t$).

We first consider the case when $t < v$. Setting

$$\widehat{V} = \begin{pmatrix} p_1(w_1) & \cdots & p_t(w_1) & p_{t+1}(w_1) & \cdots & p_v(w_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_1(w_t) & \cdots & p_t(w_t) & p_{t+1}(w_t) & \cdots & p_v(w_t) \end{pmatrix} \equiv \begin{pmatrix} V & \widetilde{V} \end{pmatrix}$$

and $c \equiv (c_1, \ldots, c_v)^T$, we have

$$\widehat{V}c = 0. \tag{2.7}$$

Now V is invertible, so multiplying (2.7) by V^{-1} on the left, we obtain $Kc = 0$, with K of the form $\begin{pmatrix} I_t & D \end{pmatrix}$, where I_t denotes the $t \times t$ identity matrix and $D := V^{-1}\widetilde{V}$ is of the form $D_{t\times(v-t)} \equiv (d_{ij})_{1\le i\le t, 1\le j\le v-t}$. We thus see that

$$c_i + d_{i1}c_{t+1} + \cdots + d_{i,v-t}c_v = 0 \ (1 \le i \le t). \tag{2.8}$$

Now,

$$\begin{aligned} p &= (-d_{1,1}c_{t+1} - \cdots - d_{1,v-t}c_v)p_1 + \cdots + (-d_{t,1}c_{t+1} - \cdots - d_{t,v-t}c_v)p_t \\ &\quad + c_{t+1}p_{t+1} + \cdots + c_v p_v \\ &= c_{t+1}(-d_{1,1}p_1 - \cdots - d_{t,1}p_t + p_{t+1}) + c_{t+2}(-d_{1,2}p_1 - \cdots - d_{t,2}p_t + p_{t+2}) \\ &\quad + \cdots + c_v(-d_{1,v-t}p_1 - \cdots - d_{t,v-t}p_t + p_v). \end{aligned}$$

The preceding calculation shows that $ker\ M_d$ is spanned by the $v-t$ vectors $\widehat{\eta}_j$ $(1 \le j \le v-t)$, where

$$\eta_j := -d_{1,j}p_1 - \cdots - d_{t,j}p_t + p_{t+j}\ . \tag{2.9}$$

Thus, $dim\ ker\ M_d \le v-t$, and it follows that $rank\ M_d = \rho_d - dim\ ker\ M_d \ge \rho_d - (v-t) = t$.

In the case when $t = v$, we set $\widehat{V} = V$ and conclude from (2.7) that $\widehat{c} = 0$, whence $ker\ M_d = \{0\}$. Thus, $rank\ M_d = \rho_d = v = t$.

Next, suppose μ is a representing measure, $supp\ \mu = w \equiv \{w_i\}_{i=1}^t$. From Proposition 2.4, $w \subseteq \mathcal{V}(\beta)$ and $rank\ M_d \le t$. If there exists $\mathcal{P} \equiv \{p_i\}_{i=1}^t \subset \mathbb{R}[x]_d$ such that $V[\mathcal{P}; w]$ is invertible, then the above argument implies that $rank\ M_d = t$. Conversely, suppose $rank\ M_d = t$, and let $\mathcal{B} \equiv \{X^{i_1}, \ldots, X^{i_t}\}$ denote a basis for $Col\ M_d$. Let $\mathcal{P} = \{x^{i_j}\}_{j=1}^t \subset \mathbb{R}[x]_d$. Suppose $p := \sum_{j=1}^t c_j x^{i_j}$, $p \not\equiv 0$, and $V[\mathcal{P}; w]\widehat{p} = 0$, i.e., $p|supp\ \mu \equiv 0$. Since μ is a representing measure for M_d, Proposition 2.4 ii) implies $p(X) = 0$ in $Col\ M_d$, contradicting the independence of $\mathcal{B}$; thus $V[\mathcal{P}; w]$ is invertible. □

Theorem 2.7. *$M_d \succeq 0$ is flat if and only if there exist a t-atomic representing measure μ for M_d, $supp\ \mu = w \equiv \{w_1, \ldots, w_t\} \subset \mathbb{R}^n$, and polynomials $\mathcal{P} \equiv \{p_1, \ldots, p_t\} \subset \mathbb{R}[x]_{d-1}$, such that $V[\mathcal{P}; w]$ is invertible.*

Proof. The "only if" direction follows immediately from Theorem 2.5 applied to the flat extension M_d of M_{d-1}. For the converse, it follows from Proposition 2.6 (applied to M_{d-1}), and from (2.5), that $t = rank\ M_{d-1} \le rank\ M_d \le t$, so M_d is flat. □

We conclude this section with a change of variables result for $n = 2$ that we will use in Section 5 to simplify certain moment matrices. An analogous change of variables for the truncated complex moment problem appears in [CF5, Proposition

1.7], but here we require a stronger version. Consider the *degree-one map* $\mathcal{T} : \mathbb{R}^2 \mapsto \mathbb{R}^2$ defined by $\mathcal{T}(x, y) = (\widetilde{x}, \widetilde{y}) \equiv (a + \alpha x + \gamma y, b + \delta x + \lambda y)$, where $\alpha\lambda - \gamma\delta \neq 0$ (which insures that $\mathcal{T}$ is a bijection). Given $\beta \equiv \beta^{(2d)}$, let L_β denote its Riesz functional. We define a multisequence $\widetilde{\beta} \equiv \widetilde{\beta}^{(2d)}$ by $\widetilde{\beta}_{ij} = L_\beta(\widetilde{x}^i\widetilde{y}^j)$ $(i, j \geq 0,$ $i + j \leq 2d)$. Let $\widetilde{M_d} \equiv M_d(\widetilde{\beta})$ denote the moment matrix for $\widetilde{\beta}$ and let $L_{\widetilde{\beta}}$ denote its Riesz functional. For $p \in \mathbb{R}[x, y]_{2d}$, $p(x, y) \equiv \sum a_{ij}x^iy^j$, we have $L_{\widetilde{\beta}}(p) = L_\beta(p(\widetilde{x}, \widetilde{y})) = L_\beta(\sum a_{ij}(a + \alpha x + \gamma y)^i(b + \delta x + \lambda y)^j)$. Let $\mathcal{P}_d$ denote the space of polynomial vectors of degree at most d, i.e., $\mathcal{P}_d = \{\widehat{p} : p \in \mathbb{R}[x, y]_d\}$, and define a matrix $J \equiv J_d : \mathcal{P}_d \mapsto \mathcal{P}_d$ by $J\widehat{p} := \widehat{p(\widetilde{x}, \widetilde{y})}$ $(p \in \mathbb{R}[x, y]_d)$; for example, $J\widehat{x} \equiv J(0, 1, 0, 0, ..., 0)^T = (a, \alpha, \gamma, 0, \ldots, 0)^T$.

Proposition 2.8. *i)* $\widetilde{M_d} = J^T M_d J$;

ii) J is invertible;

iii) $\widetilde{M_d} \succeq 0 \Longleftrightarrow M_d \succeq 0$;

iv) rank $\widetilde{M_d} =$ *rank* M_d.

v) The formula $\mu = \widetilde{\mu} \circ \mathcal{T}$ is a one-to-one correspondence between representing measures for β and $\widetilde{\beta}$, which preserves measure class and cardinality of support; moreover, $\mathcal{T}($supp $\mu) =$ supp $\widetilde{\mu}$.

vi) $\widetilde{M_d}$ $(\succeq 0)$ admits a flat extension if and only if M_d $(\succeq 0)$ admits a flat extension.

vii) For $p \in \mathbb{R}[x, y]_d$, define $q(\widetilde{x}, \widetilde{y}) := p \circ \mathcal{T}^{-1}(\widetilde{x}, \widetilde{y}) \in \mathbb{R}[\widetilde{x}, \widetilde{y}]_d$. Then in Col $\widetilde{M_d}$, $q(\widetilde{X}, \widetilde{Y}) = J^T p(X, Y)$. In particular, $p(X, Y) = 0$ if and only if $q(\widetilde{X}, \widetilde{Y}) = 0$, and $\mathcal{V}(\widetilde{\beta}) = \mathcal{T}(\mathcal{V}(\beta))$; further, $\widetilde{M_d}$ is recursively generated if and only if M_d is recursively generated.

viii) $\widetilde{M_d} \in \mathcal{F}_d$ if and only if $M_d \in \mathcal{F}_d$.

ix) Let τ satisfy $1 \leq \tau \leq \rho_{d-1}$ and let $\mathcal{F}_{d,\tau} = \{M_d \in \mathcal{F}_d :$ rank $M_d = \tau\}$. Then $\widetilde{M_d} \in \overline{\mathcal{F}_{d,\tau}}$ if and only if $M_d \in \overline{\mathcal{F}_{d,\tau}}$.

x) For $M_d \succeq 0$, $\widetilde{M_d^\flat} \equiv J^T M_d^\flat J = (\widetilde{M_d})^\flat$; in particular, $C_d^\flat$ is Hankel in $M_d^\flat$ if and only if the $C_d^\flat$ block of $(\widetilde{M_d})^\flat$ is Hankel.

Proof. The proofs of i)-vi) are direct analogues of those of the corresponding parts of [CF5, Proposition 1.7], so we omit the details. For vii), we have $q(\widetilde{X}, \widetilde{Y}) = \widetilde{M_d}\widehat{q} = \widetilde{M_d}\widehat{p \circ \mathcal{T}^{-1}} = J^T M_d((J\widehat{p \circ \mathcal{T}^{-1}})) = J^T M_d(\widehat{p \circ \mathcal{T}^{-1} \circ \mathcal{T}}) = J^T M_d\widehat{p} = J^T p(X, Y)$. Since J is invertible, it follows that $q(\widetilde{X}, \widetilde{Y}) = 0 \Longleftrightarrow p(X, Y) = 0$; the other conclusions follow similarly. For viii), first note that, relative to $\widetilde{\beta}$, for $1 \leq k \leq d$, we also have the subsequence $\widetilde{\beta}^{(2k)}$ and its moment matrix $\widetilde{M_k}$ (a submatrix of $\widetilde{M_d}$). For $p \in \mathbb{R}[x, y]_d$, *deg* $p(x, y) =$ *deg* $p(\widetilde{x}, \widetilde{y})$, so J maps $\mathcal{P}_k$ onto itself and the map J_k associated to $\widetilde{\beta}^{(2k)}$ satisfies $J_k = J|\mathcal{P}_k$. Each J_k is invertible, and $\widetilde{M_k} = J_k^T M_k J_k$. In particular, *rank* $\widetilde{M_{d-1}} =$ *rank* M_{d-1}. It now follows from iii) and iv) that $\widetilde{M_d}$ is positive and flat if M_d is positive and flat. The converse follows by symmetry (using the transform $\mathcal{T}^{-1}$). For ix), suppose $M_d = \lim_{k\to\infty} M_d^{[k]}$, where each $M_d^{[k]}$, corresponding to $\beta^{[k]}$ (a sequence of degree $2d$), is positive and flat, with *rank* $M_d^{[k]} = \tau$. Then $\widetilde{M_d} \equiv J^T M_d J = \lim_{k\to\infty} J^T M_d^{[k]} J$, and iii)-iv) and viii) imply that each $J^T M_d^{[k]} J$ is positive and flat, with *rank* $J^T M_d^{[k]} J = \tau$. Thus

$M_d \in \overline{\mathcal{F}_{d,\tau}} \Longrightarrow \widetilde{M}_d \in \overline{\mathcal{F}_{d,\tau}}$, and the converse follows by symmetry (again using the transform $\mathcal{T}^{-1}$).

For x), since $M_d \succeq 0$, there is a matrix W such that

$$M_d = \begin{pmatrix} M_{d-1} & M_{d-1}W \\ W^T M_{d-1} & C_d \end{pmatrix},\ C_d \succeq C_d^\flat \equiv W^T M_{d-1} W,$$

and

$$M_d^\flat = \begin{pmatrix} M_{d-1} & M_{d-1}W \\ W^T M_{d-1} & W^T M_{d-1} W \end{pmatrix}.$$

Denoting

$$J \equiv J_d = \begin{pmatrix} J_{d-1} & K \\ 0 & L \end{pmatrix},$$

we have

$$\widetilde{M}_d \equiv J^T M_d J = \begin{pmatrix} \widetilde{M}_{d-1} & \widetilde{B}_d \\ \widetilde{B}_d^T & \widetilde{C}_d \end{pmatrix},$$

where $\widetilde{M}_{d-1} = J_{d-1}^T M_{d-1} J_{d-1}$ and $\widetilde{B}_d = \widetilde{M}_{d-1}\widetilde{W}$, with $\widetilde{W} = J_{d-1}^{-1}(K+WL)$. Thus

$$(\widetilde{M}_d)^\flat = \begin{pmatrix} \widetilde{M}_{d-1} & \widetilde{B}_d \\ \widetilde{B}_d^T & \widetilde{C}_d^\flat \end{pmatrix},$$

with $\widetilde{C}_d^\flat \equiv \widetilde{W}^T \widetilde{M}_{d-1} \widetilde{W} = (K^T + L^T W^T)(J_{d-1}^{-1})^T (J_{d-1}^T M_{d-1} J_{d-1}) J_{d-1}^{-1}(K+WL) = (K+WL)^T M_{d-1}(K+WL)$. A block matrix calculation now shows that the last expression coincides with the C-block of $J^T M_d^\flat J$, so $(\widetilde{M}_d)^\flat$ coincides with $\widetilde{M_d^\flat} \equiv J^T M_d^\flat J$. Now, if $C_d^\flat$ is Hankel in $M_d^\flat$, then $M_d^\flat$ is a moment matrix, and therefore so also is $\widetilde{M_d^\flat}$. Thus $(\widetilde{M}_d)^\flat$ $(= \widetilde{M_d^\flat})$ is a moment matrix, whence its $C_d^\flat$ block is Hankel. The converse follows by symmetry. $\square$

3. Determining sequences for bivariate polynomials

In this section we examine certain determining sequences for bivariate polynomials that will be used in proving Theorem 1.9. By a *determining sequence* for $\mathbb{R}[x,y]_d$ we mean a sequence of distinct points $\Gamma \equiv \{(x_k, y_k)\}_{k=1}^\infty \subset \mathbb{R}^2$ with the following property: for $p \in \mathbb{R}[x,y]_d$, if there exists $k_p \in \mathbb{N}$ such that $p(x_k, y_k) = 0$ for every $k \geq k_p$, then $p \equiv 0$. In particular, Γ is determining if each nonzero polynomial in $\mathbb{R}[x,y]_d$ has at most a finite number of zeros in Γ.

The following result will prove useful in relating the geometry of the support of a representing measure to the rank of a moment matrix.

Proposition 3.1. *Suppose* $\mathcal{P} \equiv \{p_1(x,y), \ldots, p_t(x,y)\}$ *is a set of independent polynomials in* $\mathbb{R}[x,y]_d$. *If* $w^{(j)} \equiv \{w_i^{(j)}\}_{i=1}^\infty$ *is a determining sequence for* $\mathbb{R}[x,y]_d$ $(1 \leq j \leq t)$, *then given* $k > 0$, *there exist integers* $k_1, \ldots, k_t \geq k$ *such that* $V[\mathcal{P}; w_{k_1}^{(1)}, \ldots, w_{k_t}^{(t)}]$ *is invertible.*

Note that we are not assuming that the determining sequences in Proposition 3.1 are distinct; however, in the proof of Theorem 1.9 in Section 4, the sequences will be distinct. Furthermore, the integers $k_1, \ldots, k_t$ need not be distinct, but clearly the points $w_{k_1}^{(1)}, \ldots, w_{k_t}^{(t)}$ are necessarily distinct. We also note that although we have formulated Proposition 3.1 only for 2 variables, the concept of determining sequence and the proof that we present below are valid as well for n variables.

Proof of Proposition 3.1. The proof is by induction on $t \geq 1$. Since $p_1(x,y) \not\equiv 0$ and $w^{(1)}$ is a determining sequence, there exists $k_1 \geq k$ such that $p_1(w_{k_1}^{(1)}) = \frac{1}{k_1^m} Q_d^{(m)}(w_1) \neq 0$, so the result holds for $t = 1$. Assume by induction that the result holds up to $t-1$. Since $\mathcal{P}_{t-1} \equiv \{p_1, \ldots, p_{t-1}\}$ is independent, there exist $k_1, \ldots, k_{t-1} \geq k$ such that $V_{t-1} \equiv V[\mathcal{P}_{t-1}; w_{k_1}^{(1)}, \ldots, w_{k_{t-1}}^{(t-1)}]$ is invertible. Let $V(x,y)$ be defined by

$$V(x,y) := \begin{pmatrix} p_1(w_{k_1}^{(1)}) & \cdots & p_{t-1}(w_{k_1}^{(1)}) & p_t(w_{k_1}^{(1)}) \\ \vdots & \vdots & \vdots & \vdots \\ p_1(w_{k_{t-1}}^{(t-1)}) & \cdots & p_{t-1}(w_{k_{t-1}}^{(t-1)}) & p_t(w_{k_{t-1}}^{(t-1)}) \\ p_1(x,y) & \cdots & p_{t-1}(x,y) & p_t(x,y) \end{pmatrix}, \tag{3.1}$$

and set $p(x,y) := det\ V(x,y)\ (\in \mathbb{R}[x]_d)$. Expanding $det\ V(x,y)$ with respect to the bottom row, we have

$$p(x,y) = D_1 p_1(x,y) - D_2 p_2(x,y) + \cdots + (-1)^{t+1} D_t p_t(x,y),$$

where each D_i is a minor of $p(x,y)$ and $D_t = det\ V_{t-1}$ ($\neq 0$, by induction). Since $\{p_1, \ldots, p_t\}$ is independent in $\mathbb{R}[x,y]_d$ and $D_t \neq 0$, then $p(x,y) \not\equiv 0$ in $\mathbb{R}[x,y]_d$. Now, since $\{w_i^{(t)}\}_{i=1}^\infty$ is a determining sequence, there exists $k_t \geq k$ such that $p(w_{k_t}^{(t)}) \neq 0$, whence $V[\mathcal{P}; w_{k_1}^{(1)}, \ldots, w_{k_t}^{(t)}]$ is invertible. $\square$

We require a determining sequence for $\mathbb{R}[x,y]_d$ that is compatible with a moment matrix construction in the proof of Theorem 1.9, and to this end we will focus on the sequence $\Gamma \equiv \{(k + \frac{1}{k^m}, \alpha k)\}_{k=1}^\infty$, where $m \in \mathbb{N}$ satisfies $m \geq d$ and $\alpha \in \mathbb{R}$ is nonzero.

Proposition 3.2. *Γ is a determining sequence for $\mathbb{R}[x,y]_d$. If $p \in \mathbb{R}[x,y]_d$ and $p \not\equiv 0$, then p has at most $md + d$ zeros of the form $(k + \frac{1}{k^m}, \alpha k)$, where $k \in \mathbb{N}$.*

Let $p \in \mathbb{R}[x,y]_d$, $p(x,y) = \sum_{i,j \geq 0, i+j \leq d} a_{ij} x^i y^j$. For $\alpha \neq 0$ and $m \in \mathbb{N}$, $m \geq d$, let $q_d^{(m)}(x) := p(x + \frac{1}{x^m}, \alpha x)$, and define $Q_d^{(m)} \in \mathbb{R}[x]_{md+d}$ by

$$Q_d^{(m)}(x) := x^{md} q_d^{(m)}(x) = x^{md} \sum_{i,j \geq 0, i+j \leq d} a_{ij} (x + \frac{1}{x^m})^i (\alpha x)^j. \tag{3.2}$$

It follows from (3.2) that to prove Proposition 3.2, it suffices to show that $Q_d^{(m)}(x) \not\equiv 0$; i.e., $Q_d^{(m)}$ has some nonzero coefficient. The coefficients of $Q_d^{(m)}$ as a polynomial in $\mathbb{R}[x]_{md+d}$ are not the coefficients a_{ij} of $p(x,y)$, but, rather, certain linear combinations of the a_{ij}. To show that at least one such coefficient is nonzero, we require some auxiliary results. In the sequel we say that a power monomial x^k appearing in $Q_d^{(m)}$ is *affiliated* with coefficient a_{ij} of $p(x,y)$ if the formal expansion of (3.2) into a sum of monomials contains one or more terms of the form $\gamma a_{ij} x^k$, say $\gamma_r a_{ij} x^k$ $(1 \leq r \leq s)$ for some $s \geq 1$, where $\gamma_r \equiv \alpha^j \delta_r$ is a nonzero absolute constant (i.e., independent of all a_{uv}); we sometimes write this as $x^k \rightsquigarrow a_{ij}$. It is clear from (3.2) that for fixed i, j, k, $\sum_{r=1}^s \gamma_r a_{ij} x^k$ is a term of the form $\Gamma a_{ij} x^k$, where $\Gamma = \alpha^j \beta$ and β is some binomial coefficient. For our purposes, it is necessary to identify exactly which monomial powers x^k appear in $Q_d^{(m)}$ and with which coefficients of $p(x,y)$ each monomial power is affiliated. For example, if $p(x,y) = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2$, then $Q_2^{(m)}(x) = a_{00}x^{2m} + a_{10}x^{2m+1} + a_{10}x^m + a_{01}\alpha x^{2m+1} + a_{20}x^{2m+2} + 2a_{20}x^{2m+1} + a_{20} + a_{11}\alpha x^{2m+2} + a_{11}\alpha x^{m+1} + a_{02}\alpha^2 x^{2m+2}$, so the sequences of coefficients with which each power monomial of $Q_2^{(m)}$ is affiliated are as follows:

$x^0 \rightsquigarrow a_{20}$; $x^m \rightsquigarrow a_{10}$; $x^{m+1} \rightsquigarrow a_{20}, a_{11}$; $x^{2m} \rightsquigarrow a_{00}$; $x^{2m+1} \rightsquigarrow a_{10}, a_{01}$; $x^{2m+2} \rightsquigarrow a_{20}, a_{11}, a_{02}$.

To identify the powers and affiliations in $Q_d^{(m)}$, consider the index set for the monomials in $p(x,y)$, $\mathbb{Z}_d^2 := \{(i,j) | i,j \geq 0,\ i+j \leq d\}$, and for $m \geq d$, let $\mathcal{S}_{m,d} = \{rm + s : r, m \in \mathbb{Z},\ 0 \leq r \leq d,\ 0 \leq s \leq r\}$. The requirement $m \geq d$ is necessary to insure that $\mathcal{S}_{m,d}$ contains $\frac{(d+1)(d+2)}{2}$ distinct elements. It is now easy to check that the map $\phi \equiv \phi_d : \mathbb{Z}_d^2 \longrightarrow \mathcal{S}_{m,d}$, defined by $\phi(i,j) = (d-i)m + j$ is a bijection, with inverse $\psi : \mathcal{S}_{m,d} \longrightarrow \mathbb{Z}_d^2$ given by $\psi(rm+s) = (d-r, s)$. (Indeed, if $(i,j) \in \mathbb{Z}_d^2$, then $r \equiv d-i$ and $s \equiv j$ satisfy $0 \leq r \leq d$ and $s = j \leq d-i = r$, so ϕ is well-defined, and similarly for ψ.) Note also that

$$\phi_{d-1}(i,j) + m = \phi_d(i,j). \tag{3.3}$$

Now let $r_d(x) = \sum_{i,j\geq 0, i+j \leq d-1} a_{ij}(x + \frac{1}{x^m})^i(\alpha x)^j$ (which we can interpret as $q_{d-1}^{(m)}(x)$ relative to $\sum_{i,j\geq 0, i+j\leq d-1} a_{ij}x^i y^j$), and set $s_d(x) = \sum_{i,j\geq 0, i+j=d} a_{ij}(x + \frac{1}{x^m})^i(\alpha x)^j$, so that $q_d^{(m)}(x) = r_d(x) + s_d(x)$. Thus, $Q_d^{(m)}(x) = x^m(x^{(d-1)m}r_d(x)) + x^{md}s_d(x)$, or

$$Q_d^{(m)}(x) = x^m Q_{d-1}^{(m)}(x) + x^{md}s_d(x). \tag{3.4}$$

Lemma 3.3. *Let $d \geq 1$, $m \geq d$. For $(i,j) \in \mathbb{Z}_d^2$, the power monomials x^k in $Q_d^{(m)}$ that are affiliated with coefficient a_{ij} of $p(x,y)$ are $x^{\phi(i,j)}$, $x^{\phi(i,j)+m+1}, \ldots, x^{\phi(i,j)+i(m+1)}$.*

Proof. The proof is by induction on $d \geq 1$. For $d = 1$, $m \geq 1$, $Q_1^{(m)}(x) = a_{00}x^m + a_{10}x^{m+1} + a_{10}x^0 + a_{01}\alpha x^{m+1}$. The unique power affiliated with a_{00} is x^m, and $\phi(0,0) = m$, $i = 0$. The powers affiliated with a_{10} are x^0 and x^{m+1}, and $\phi(1,0) = 0$, $i = 1$. The unique power affiliated with a_{01} is x^{m+1}, and $\phi(0,1) = m+1$, $i = 0$.

Assume the result is true up to $d-1$ and consider $Q_d^{(m)}$. For $i,j \geq 0$ with $i + j \leq d-1$, it follows by induction that the powers affiliated with a_{ij} in $Q_{d-1}^{(m)}$ are precisely $x^{\phi_{d-1}(i,j)}$, $x^{\phi_{d-1}(i,j)+m+1}, \ldots, x^{\phi_{d-1}(i,j)+i(m+1)}$. Thus, (3.4) implies that the powers affiliated with a_{ij} in $Q_d^{(m)}$ are precisely $x^{\phi_{d-1}(i,j)+m}$, $x^{\phi_{d-1}(i,j)+2m+1}$, $\ldots, x^{\phi_{d-1}(i,j)+(i+1)m+i}$, and (3.3) implies that these powers coincide with $x^{\phi_d(i,j)}$, $x^{\phi_d(i,j)+m+1}, \ldots, x^{\phi_d(i,j)+i(m+1)}$.

Now consider the case $i,j \geq 0$, $i + j = d$. It follows from (3.4) that the powers affiliated with a_{ij} in $Q_d^{(m)}$ are those affiliated with a_{ij} in the formal expansion of $x^{md}s_d(x)$, i.e., those powers which appear in the complete expansion of $x^{md}(x + \frac{1}{x^m})^i(\alpha x)^{d-i}$. The monomial powers in this expansion are $x^{md}x^i x^{d-i}$, $x^{md}x^{i-1}x^{-m}x^{d-i}$, $x^{md}x^{i-2}x^{-2m}x^{d-i}, \ldots, x^{md}x^{-im}x^{d-i}$, which simplify to x^{md+d}, $x^{md-m+d-1}$, $x^{md-2m+d-2}$, $\ldots$, $x^{md-mi+d-i}$, and this sequence coincides with $x^{\phi(i,d-i)+i(m+1)}$, $x^{\phi(i,d-i)+(i-1)(m+1)}, \ldots, x^{\phi(i,d-i)}$. $\square$

Corollary 3.4. *For $(i,j) \in \mathbb{Z}_d^2$, the powers affiliated with a_{ij} in $Q_d^{(m)}$ are $x^{\phi(i,j)}$, $x^{\phi(i-1,j+1)}, \ldots, x^{\phi(0,j+i)}$.*

Proof. For $0 \le a \le i$, $\phi(i,j)+a(m+1) = (d-i)m+j+am+a = (d-i+a)m+j+a = (d-(i-a))m+j+a = \phi(i-a, j+a)$. The result now follows from Lemma 3.3. $\square$

Note that every power of x in the expansion of the right hand side of (3.2) is affiliated with some a_{ij}. The preceding result shows that each such power of x is of the form $x^{\phi(i',j')}$ for some $(i',j') \in \mathbb{Z}_d^2$, and it also shows that every such power is affiliated with at least one coefficient in $Q_d^{(m)}$, namely, $a_{i'j'}$. Since $\phi : \mathbb{Z}_d^2 \longrightarrow \mathcal{S}_{m,d}$ is a bijection, it also follows that the powers in $Q_d^{(m)}$ can be expressed as x^{rm+s} with $0 \le r \le d$, $0 \le s \le r$.

Proof of Proposition 3.2. It suffices to show that $Q_d^{(m)}$ has at most $md+m$ distinct real roots. Suppose to the contrary that $Q_d^{(m)}$ has more than $md+d$ roots. Since *deg* $Q_d^{(m)} \le md+d$, then $Q_d^{(m)} \equiv 0$ in $\mathbb{R}[x]$, and we will show that this implies $p(x,y) \equiv 0$ in $\mathbb{R}[x,y]$, contradicting the hypothesis. We may list the powers corresponding to $\mathcal{S}_{m,d}$ by increasing degree as

$$x^0,\ x^m,\ x^{m+1},\ x^{2m},\ x^{2m+1},\ x^{2m+2}, \ldots, x^{dm},\ x^{dm+1}, \ldots,\ x^{dm+d}. \tag{3.5}$$

We will prove by induction on the position, τ, of a power in this list ($1 \le \tau \le \frac{(d+1)(d+2)}{2}$) that for the τ-th power x^{rm+s} ($0 \le r \le d$, $0 \le s \le r$), every coefficient a_{ij} of $p(x,y)$ with which x^{rm+s} is affiliated satisfies $a_{ij} = 0$.

Note from Corollary 3.4 that if x^0 is affiliated with coefficient a_{ij} of $p(x,y)$, then $0 = \phi(i-\rho, j+\rho)$ for some ρ, $0 \le \rho \le i$. Now $0 = (d-(i-\rho))m + j + \rho = (d-i+\rho)m + j + \rho$, and thus $\rho = 0$, $d = i$, $j = 0$. Conversely, $\phi(d,0) = 0$, so x^0 is affiliated only with $a_{d,0}$; since $Q_d^{(m)} \equiv 0$, it follows that $a_{d,0} = 0$.

Now assume by induction that for the first $\tau - 1$ powers in (3.5), each coefficient a_{ij} with which any of these powers is affiliated satisfies $a_{ij} = 0$. Let x^{rm+s} denote the τ-th power (for some r, s with $0 \le r \le d$, $0 \le s \le r$). Now $rm+s = \phi(i,j)$ for a *unique* $(i,j) \in \mathbb{Z}_d^2$, and Corollary 3.4 shows that x^{rm+s} is affiliated with a_{ij}. Next, suppose x^{rm+s} is also affiliated with some other coefficient $a_{i',j'}$, $(i,j) \ne (i',j')$. Now $\phi(i',j') = r'm + s'$ for some r', s' with $0 \le r' \le d$, $0 \le s' \le r'$. Lemma 3.3 shows that $x^{\phi(i',j')}$ comes first in the sequence of powers affiliated with $a_{i',j'}$, and since this sequence is ordered by strictly increasing degree of the powers, we have $r'm + s' < rm + s$. By induction, since $x^{r'm+s'}$ is a lower power than x^{rm+s}, it follows that $a_{i',j'} = 0$. Thus, whether or not x^{rm+s} is affiliated only with a_{ij}, we can now represent the total coefficient of x^{rm+s} in $Q_d^{(m)}$ as γa_{ij} for some nonzero absolute constant γ. Since $Q_d^{(m)} \equiv 0$, it now follows that $a_{ij} = 0$, so *every* coefficient with which x^{rm+s} is affiliated equals 0. By induction, we conclude that for every power x^{rm+s} ($0 \le r \le d$, $0 \le s \le r$), each coefficient with which x^{rm+s} is affiliated equals 0. Since, given a_{ij}, $x^{\phi(i,j)}$ is affiliated with a_{ij} (Corollary 3.4), it follows that each $a_{ij} = 0$. Since $p(x,y) \not\equiv 0$ by hypothesis, this contradiction implies that $Q_d^{(m)}$ has at most $md+d$ real roots, which completes the proof. $\square$

The following example shows that if $m < d$, then we may have $p(x,y) \not\equiv 0$, but $Q_d^{(m)} \equiv 0$. Let $d = 2$, $m = 1$, and for $\alpha \ne 0$ and $a_{11} \ne 0$, set $a_{00} = -\alpha a_{11}$, $a_{02} = -\frac{a_{11}}{\alpha}$, $a_{01} = a_{10} = a_{20} = 0$. Then $p(x,y) = -a_{11}(\alpha - xy + \frac{1}{\alpha}y^2) \not\equiv 0$, but

$q_2^{(1)}(x) \equiv p(x+\frac{1}{x}, \alpha x) = -a_{11}(\alpha - (\alpha x^2+\alpha) + \frac{1}{\alpha}(\alpha^2 x^2)) = 0$. Thus $Q_2^{(1)}(x) \equiv x q_2^{(1)}(x) = 0$.

We continue this section by identifying another determining sequence that we will use in proving Theorem 1.9.

Proposition 3.5. *Let $m \in \mathbb{N}$, $m \geq d$. If $p \in \mathbb{R}[x,y]_d$ and $p \not\equiv 0$, then p has at most $md+d$ zeros of the form $(\frac{1}{k^m}, k)$, where $k \in \mathbb{N}$.*

Proof. For $p(x,y) \equiv \sum_{i,j\geq 0, i+j\leq d} a_{ij}x^i y^j$, let $r_d^{(m)}(x) := p(\frac{1}{x^m}, x)$, and define $R_d^{(m)} \in \mathbb{R}[x]_{md+d}$ by

$$R_d^{(m)}(x) := x^{md} r_d^{(m)}(x) = x^{md} \sum_{i,j\geq 0, i+j\leq d} a_{ij}(\frac{1}{x^m})^i x^j. \tag{3.6}$$

We will prove that

$$R \equiv R_d(x) = \sum_{i=0}^{d}\sum_{j=0}^{i} a_{d-i,j} x^{im+j}. \tag{3.7}$$

If p has more than $md+m$ zeros of the form $(\frac{1}{k^m}, k)$, then R has more than $md+m$ roots, so $R \equiv 0$ in $\mathbb{R}[x]$. Since $m \geq d$, the powers x^{im+j} $(0 \leq i \leq d$, $0 \leq j \leq i)$ are distinct; moreover, each coefficient of p is of the form $a_{d-i,j}$ for some i,j with $0 \leq i \leq d$, $0 \leq j \leq i$. It thus follows from (3.7) that each coefficient of p equals 0, so $p \equiv 0$, a contradiction. It remains to establish (3.7). Write $p(x,y) = u(x,y) + v(x,y)$, where $u(x,y) = \sum_{i,j\geq 0, i+j\leq d-1} a_{ij}x^i y^j$ and $v(x,y) = \sum_{i,j\geq 0, i+j=d} a_{ij}x^i y^j$. Then $R_d(x) = x^m x^{(d-1)m} u(\frac{1}{x^m}, x) + x^{md} v(\frac{1}{x^m}, x)$, and thus

$$R_d(x) = x^m R_{d-1}(x) + x^{md} v(\frac{1}{x^m}, x). \tag{3.8}$$

Now (3.7) follows from (3.8) by a straightforward induction on $d \geq 1$; we omit the details. □

We conclude this section by describing a determining sequence, corresponding to a prescribed point $(x_0, y_0) \in \mathbb{R}^2$, that we will use in Section 5. For $r, s \in \mathbb{N}$, with $s > 0$ and $r > (d-1)s$, consider the sequence $\Lambda := \{(x_0 + \frac{1}{k^r}, y_0 + \frac{1}{k^{r+s}})\}_{k=1}^{\infty}$.

Proposition 3.6. *Λ is a determining sequence for $\mathbb{R}[x,y]_d$. If $p \in \mathbb{R}[x,y]_d$ and $p \not\equiv 0$, then p has at most $(r+s)d$ zeros of the form $(x_0 + \frac{1}{k^r}, y_0 + \frac{1}{k^{r+s}})$, where $k \in \mathbb{N}$.*

The hypothesis $s > 0$ is necessary: for $d = 1$, if $a, b, c \in \mathbb{R}$ are nonzero, with $b = -c$, if $a + bx_0 + cy_0 = 0$, and if we take $r = 1$ and $s = 0$, then $p(x,y) := a + bx + cy$ vanishes on Λ. We will use notation similar to that in the proof of Proposition 3.2. For $p \in \mathbb{R}[x,y]_d$, $p(x,y) = \sum_{i,j\geq 0, i+j\leq d} a_{ij}x^i y^j$, let $q(x) = p(x_0 + \frac{1}{x^r}, y_0 + \frac{1}{x^{r+s}})$, and define $Q_d \in \mathbb{R}[x]_{(r+s)d}$ by

$$Q_d(x) := x^{(r+s)d} q(x) = x^{(r+s)d} \sum_{i,j\geq 0, i+j\leq d} a_{ij}(x_0 + \frac{1}{x^r})^i (y_0 + \frac{1}{x^{r+s}})^j. \tag{3.9}$$

It follows readily that

$$Q_d = x^{r+s}Q_{d-1} + x^{(r+s)d}\sum_{i=0}^{d} a_{i,d-i}(x_0 + \frac{1}{x^r})^i(y_0 + \frac{1}{x^{r+s}})^{d-i}. \tag{3.10}$$

Lemma 3.7. *The powers that appear in* Q_d *are* x^{ir+js} *with* $0 \le i \le j \le d$, *and*

$$x^{ir+js} \rightsquigarrow a_{j-i+a,d-j+b} \ (a,b \ge 0,\ a+b \le i). \tag{3.11}$$

Moreover, the complete expansion of (3.10) *includes the term* $a_{j-i,d-j}x^{ir+js}$ (*corresponding to* $a = b = 0$ *in* (3.11)).

We have already seen that $s > 0$ is necessary for Proposition 3.6. In order for the exponents $ir + js$ $(0 \le i \le j \le d)$ to be distinct, it is necessary and sufficient that $r > (d-1)s$, which is the requirement in the definition of Λ.

Proof of Lemma 3.7. The proof is by induction on $d \ge 1$. For $d = 1$, $p(x,y) = a_{00}+a_{10}x+a_{01}y$, so $Q_1(x) = a_{00}x^{r+s}+a_{10}x_0x^{r+s}+a_{10}x^s+a_{01}y_0x^{r+s}+a_{01}$. Thus x^0 is affiliated only with a_{01} (in agreement with $i = j = 0$, $a = b = 0$ in (3.11)), and x^s is affiliated only with a_{10} (in agreement with $i = 0$, $j = 1$, $a = b = 0$). For x^{r+s}, we have $i = j = 1$, and this power is affiliated with a_{00} $(a = b = 0)$, a_{01} $(a = 0, b = 1)$, and a_{10} $(a = 1, b = 0)$. By inspection, we also see that in $Q_1(x)$ we have the required terms $a_{01}x^0$, $a_{10}x^s$, and $a_{00}x^{r+s}$.

Now assume the result holds for $d-1$. Thus, in Q_{d-1}, for $0 \le i \le j \le d-1$, the power x^{ir+js} is affiliated precisely with the coefficients $a_{j-i+a,(d-1)-j+b}$, where $a,b \ge 0$, $a+b \le i$; moreover, the expansion of (3.10) (with $d-1$ taking the place of d) contains the term $a_{j-i,(d-1)-j}x^{ir+js}$. It follows from (3.10) that in Q_d the power $x^{(i+1)r+(j+1)s}$ is also affiliated with these coefficients. Setting $i' := i+1$, $j' := j+1$, in Q_d, for $1 \le i' \le j' \le d$, the power $x^{i'r+j's}$ is affiliated with

$$a_{j'-i'+a,d-j'+b} \ (a,b \ge 0,\ a+b \le i'-1);$$

moreover, the expansion of (3.10) for Q_d contains the terms

$$a_{j-i,(d-1)-j}x^{(i+1)r+(j+1)s} = a_{j'-i',d-j'}x^{i'r+j's}.$$

To complete the proof of (3.11) (but with i and j in (3.11) replaced by i' and j'), it is necessary to show that i) for the case $i' = 0$, $0 \le j' \le d$, $x^{j's} \rightsquigarrow a_{j',d-j'}$ and Q_d contains the term $a_{j',d-j'}x^{j's}$; and ii) $x^{i'r+j's}$ is affiliated with $a_{j'-i'+a,d-j'+b}$ when $0 \le i' \le j' \le d$ and $a+b = i'$.

Note that in (3.10), the expansion of $x^{(r+s)d}\sum_{i=0}^{d} a_{i,d-i}(x_0 + \frac{1}{x^r})^i(y_0 + \frac{1}{x^{r+s}})^{d-i}$ is of the form

$$x^{(r+s)d}\sum_{i=0}^{d} a_{i,d-i}(\sum_{u=0}^{i}\binom{i}{u}x_0^u x^{-r(i-u)})(\sum_{v=0}^{d-i}\binom{d-i}{v}y_0^v x^{-(r+s)(d-i-v)}). \tag{3.12}$$

Upon further expansion, this shows that

$$x^{(u+v)r+(v+i)s} \rightsquigarrow a_{i,d-i} \ (0 \le i \le d,\ 0 \le u \le i,\ 0 \le v \le d-i). \tag{3.13}$$

To show i), given $0 \le j' \le d$, in (3.13) we let $i = j'$, $u = v = 0$ and see that x^{is} is affiliated with $a_{i,d-i}$. Further, with $u = v = 0$, it is clear from (3.12) that Q_d contains the term $a_{i,d-i}x^{is}$. For ii), we have $1 \le i' \le j' \le d$, $a+b = i'$, and we seek to show that $x^{i'r+j's} \rightsquigarrow a_{j'-i'+a,d-j'+b}$. Set $\tilde{i} \equiv j' - b$ so that $0 \le \tilde{i} \le d$ and

$a_{j'-i'+a,d-j'+b} = a_{\tilde{i},d-\tilde{i}}$. Let $u := i' - b$ $(0 \le u \le d)$ and $v := b$ $(0 \le v \le d - \tilde{i})$. It now follows from (3.13) that $x^{i'r+j's} = x^{(u+v)r+(v+\tilde{i})s} \rightsquigarrow a_{\tilde{i},d-\tilde{i}}$, which proves ii) and completes the induction. $\square$

Proof of Proposition 3.6. If $p(x, y)$ has more than $(r + s)d$ zeros in Λ, then Q_d has more than $(r + s)d$ distinct real roots, so $Q_d \equiv 0$. Under this assumption, we will prove that each coefficient of $p(x, y)$ equals 0, in contradiction to $p \not\equiv 0$. From Lemma 3.7, with $i = 0$, $0 \le j \le d$, $a = b = 0$, we see that in Q_d, x^{js} is affiliated solely with $a_{j,d-j}$, so $a_{j,d-j} = 0$ $(0 \le j \le d)$. Let us assume by induction that for some i, $1 \le i \le d$, we have $a_{uv} = 0$ for $u, v \ge 0$, $d \ge u+v \ge d-i+1$. We will show that each coefficient of degree $d-i$ equals 0. Lemma 3.7 implies that for $i \le j \le d$, the coefficient of x^{ir+js} in Q_d is of the form

$$\alpha_{00}a_{j-i,d-j} + \alpha_{10}a_{j-i+1,d-j} + \alpha_{01}a_{j-i,d-j+1} + \cdots + \alpha_{i0}a_{j,d-j} + \cdots + \alpha_{0i}a_{j-i,d-j+i}$$

for certain absolute constants α_{uv} with $\alpha_{00} = 1$. Except for $a_{j-i,d-j}$, each coefficient of $p(x, y)$ that appears in the preceding expression has degree at least $d - i + 1$, so by induction each such coefficient equals 0. It follows that the coefficient of x^{ir+js} in Q_d reduces to $\alpha_{00}a_{j-i,d-j}$, whence $a_{j-i,d-j} = 0$. Thus, each coefficient of p of degree $d - i$ equals 0; the result now follows by induction. $\square$

4. Limits of positive flat moment matrices

In this section we prove Theorem 1.9. Throughout this section $M_d \equiv M_d(\beta)$ (as in (2.3)) denotes a bivariate moment matrix of degree $2d$. Now suppose $\beta^{(2d-1)}$ has a κ-atomic representing measure μ, so that $M_d[\mu]$ is of the form

$$M_d[\mu] := \begin{pmatrix} M_{d-1} & B_d \\ B_d^T & C_d[\mu] \end{pmatrix}.$$

Let $\Delta \equiv \Delta[\mu] := C_d - C_d[\mu]$, $s \equiv rank\ \Delta[\mu]$, $\rho[\mu] \equiv s + \kappa$, and

$$M_\Delta := \begin{pmatrix} 0 & 0 \\ 0 & \Delta \end{pmatrix}.$$

We now re-state Theorem 1.9 for ease of reference.

Theorem 4.1. *Let $n = 2$, $d \ge 1$. Suppose that $M_d(\beta) \succeq 0$ and that $\beta^{(2d-1)}$ has a κ-atomic representing measure μ. If $\Delta \equiv \Delta[\mu] \succeq 0$ and $\rho[\mu] \le \rho_{d-1}$, then $M_d \in \overline{\mathcal{F}_d}$. Moreover, if $\rho[\mu] \le \tau \le \rho_{d-1}$, then there exists a sequence of positive flat moment matrices, $\{M_d^{(k)}\}$, such that $M_d(\beta) = \lim_{k\to\infty} M_d^{(k)}$ and for each k, rank $M_d^{(k)} =$ rank $M_{d-1}^{(k)} = \tau$.*

Proof. Denote μ as $\mu \equiv \sum_{i=1}^{\kappa} \alpha_i\delta_{w_i}$, where the points w_i are distinct and each $\alpha_i > 0$. If $\rho[\mu] < \rho_{d-1}$ and $\rho[\mu] < \tau \le \rho_{d-1}$, choose additional points $w_{\rho[\mu]+1}, \ldots, w_\tau$ so that all of the $\tau - s$ points are distinct. (If $\tau = \rho[\mu]$, omit all reference to these additional points in the sequel.) Let $R, S \in \mathbb{N}$, with $S > 0$ and $R > (d-1)S$. With $w_j \equiv (x_j, y_j)$ $(1 \le j \le \kappa$ and $\kappa + s + 1 \le j \le \tau)$, we apply Proposition 3.6 to define the determining sequences $w^{(j)} \equiv \{w_k^{(j)}\}_{k=1}^\infty$ $(1 \le j \le \kappa$ and $\kappa + s + 1 \le j \le \tau)$, where $w_k^{(j)} := (x_j + \frac{1}{k^R}, y_j + \frac{1}{k^{R+S}})$.

Next, note that Δ is a Hankel matrix, say $\Delta = (u_{i+j})_{0\le i,j\le d}$. We first consider the case when Δ is a moment matrix, i.e., $u_0 > 0$. If $\Delta \succ 0$ (positive definite), then $s = d+1$ and Δ has an s-atomic representing measure ν by Theorem 2.3 (cf. [CF1]). Otherwise, Δ is positive and singular, i.e., $s \le d$, so [FN2, Theorem 3.1] implies that there is a sequence, $\{H_d(y^{[k]})\}$, of positive Hankel matrices such that $\Delta = \lim_{k\to\infty} H_d(y^{[k]})$ and $rank\ H_d(y^{[k]}) = rank\ H_{d-1}(y^{[k]}) = s$. It follows from Theorem 1.4 that $H_d(y^{[k]})$ admits an s-atomic representing measure ν_k. Thus, for the remainder of the proof of this case, replacing Δ by $H_d(y^{[k]})$ and C_d by $C_d[\mu] + H_d(y^{[k]})$ if necessary, we may assume that $rank\ \Delta = s$ and that Δ admits an s-atomic representing measure $\nu \equiv \sum_{i=\kappa+1}^{\kappa+s} a_i \delta_{\{\alpha_i\}}$ (where each $a_i > 0$ and the α_i are distinct). Further, for the purposes of approximation, in the sequel we may assume that $\alpha_i \neq 0$ ($\kappa+1 \le i \le \kappa+s$). For if a (unique) $\alpha_{i'}$ satisfies $\alpha_{i'} = 0$, then consider the measures $\nu^{(l)}$ ($l \ge 1$) obtained from ν by replacing $\alpha_{i'} = 0$ by $\alpha_{i'} = \frac{1}{l}$ (with l large enough so that the α_i remain distinct). Clearly, $\Delta = \lim_{l\to\infty} H_d[\nu^{(l)}]$, and we claim that $rank\ H_d[\nu^{(l)}] = s$ for all sufficiently large l. Since $\nu^{(l)}$ is s-atomic, then (2.5) implies that $rank\ H_d[\nu^{(l)}] \le s$. The reverse inequality follows from lower semicontinuity of rank (cf. Section 1), since $\Delta = \lim_{l\to\infty} H_d[\nu^{(l)}]$ and $rank\ \Delta = s$. In the sequel, by replacing Δ by $H_d[\nu^{(l)}]$ (for sufficiently large l), we may thus assume that $rank\ \Delta = s$ and that Δ has an s-atomic measure ν (as above) in which the α_i are distinct and *nonzero*. Now, for $\kappa+1 \le j \le \kappa+s$, choose an integer $m_j \ge d-1$. Applying Proposition 3.2 with d replaced by $d-1$, we define the determining sequences $w^{(j)} \equiv \{w_k^{(j)}\}_{k=1}^\infty$ ($\kappa+1 \le j \le \kappa+s$), where $w_k^{(j)} := (k + \frac{1}{k^{m_j}}, \alpha_j k)$.

Let $v \equiv \rho_{d-1}$ and $t \equiv \tau$. Let $p_1(x,y), \ldots, p_v(x,y)$ denote the basis for $\mathbb{R}[x,y]_{d-1}$ consisting of all of the monomials, and let $\mathcal{P} := \{p_1, \ldots, p_t\}$. We next apply Proposition 3.1 to $\mathcal{P}$ and the determining sequences $w^{(j)}$ ($1 \le j \le t$). It follows that given $k > 0$, there exist $k_1, \ldots, k_t \ge k$ such that $V \equiv V[\mathcal{P}, w_{k_1}^{(1)}, \ldots, w_{k_t}^{(t)}]$ (as defined in Section 2) is invertible; in particular, the points $w_j \equiv w_{k_j}^{(j)}$ ($1 \le j \le t$) are distinct.

We define three atomic measures that will be used in constructing the flat approximants to M_d. Let

$$\mu^{(k)} := \sum_{j=1}^{\kappa} \alpha_j \delta_{w_j} = \sum_{j=1}^{\kappa} \alpha_j \delta_{(x_j + \frac{1}{k_j^R}, y_j + \frac{1}{k_j^{R+S}})},$$

$$\nu^{(k)} := \sum_{j=\kappa+1}^{\kappa+s} \frac{a_j}{k_j^{2d}} \delta_{w_j} \equiv \sum_{j=\kappa+1}^{\kappa+s} \frac{a_j}{k_j^{2d}} \delta_{\{(k_j + \frac{1}{k_j^{m_j}}, \alpha_j k_j)\}},$$

$$\sigma^{(k)} := \sum_{j=\kappa+s+1}^{t} \frac{1}{k} \delta_{w_j} \equiv \sum_{j=\kappa+s+1}^{t} \frac{1}{k} \delta_{(x_j + \frac{1}{k_j^R}, y_j + \frac{1}{k_j^{R+S}})}.$$

Now let $\omega^{(k)} := \mu^{(k)} + \nu^{(k)} + \sigma^{(k)}$ and set

$$M_d^{(k)} \equiv M_d[\omega^{(k)}] = M_d[\mu^{(k)}] + M_d[\nu^{(k)}] + M_d[\sigma^{(k)}].$$

Straightforward calculations show that $\lim_{k\to\infty} M_d[\mu^{(k)}] = M_d[\mu]$, $\lim_{k\to\infty} M_d[\nu^{(k)}] = M_\Delta$, and $\lim_{k\to\infty} M_d[\sigma^{(k)}] = 0$. Thus, $\lim_{k\to\infty} M_d^{(k)} = M_d[\mu] + M_\Delta = M_d(\beta)$.

To complete the proof of this case, it remains to show that $M_d^{(k)}$ is flat. Since $\omega^{(k)}$ is a representing measure for $M_d^{(k)}$, *supp* $\omega^{(k)} = \{w_1, \ldots, w_t\}$, $\mathcal{P} \subset \mathbb{R}[x,y]_{d-1}$, and $V[\mathcal{P}; w_1, \ldots, w_t]$ is invertible, the conclusion that $M_d^{(k)}$ is flat follows directly from Theorem 2.7. This completes the proof when $u_0 > 0$.

We next consider the case when $u_0 = 0$. In this case, it follows from Theorem 2.3 that $u_j = 0$ $(0 \le j \le 2d-1)$ and $\gamma := u_{2d} \ge 0$. If $\gamma = 0$, then $\Delta = 0$ and $s = 0$; in this case, we may proceed as above except that we skip the step involving Δ and $\nu^{(k)}$ and, in particular, we do not define the sequences $w^{(j)}$ $(\kappa + 1 \le j \le \kappa + s)$. Assuming that $\gamma > 0$, we have $s = 1$, and for fixed $m \ge d-1$, we use Proposition 3.5 to define the determining sequence $w^{(\kappa+1)} \equiv \{w_k^{(\kappa+1)}\}_{k=1}^{\infty}$, where $w_k^{(\kappa+1)} = (\frac{1}{k^m}, k)$. We next apply Proposition 3.1 exactly as before to produce the points $w_1, \ldots, w_t$ with an invertible Vandermonde V. We define $\mu^{(k)}$ and $\sigma^{(k)}$ as before, but we now define

$$\nu^{(k)} := \frac{\gamma}{k_j^{2d}} \delta_{w_{k_j}^{(j)}},$$

where $j = \kappa + 1$. The proof now continues exactly as in the case $u_0 > 0$, beginning with the definition of $\omega^{(k)} := \mu^{(k)} + \nu^{(k)} + \sigma^{(k)}$. $\square$

We illustrate Theorem 4.1, and particularly Corollary 1.11, with a continuation of Example 2.1.

Example 4.2. In Example 2.1 we have $M_3 \succeq 0$ and *rank* $M_3 = 6$, with column relations $X^2 = 1$, $X^3 = X$, $X^2Y = Y$, and $Y^3 = 2Y$. It is straightforward to check that by propagating these column relations forward, i.e., by defining $X^4 := X^2$, $X^3Y := XY$, $X^2Y^2 := Y^2$, $XY^3 := 2XY$, $Y^4 := 2Y^2$, we construct a positive flat moment matrix extension M_4. Thus β has a 6 atomic representing measure μ which may be explicitly computed as described in Section 2 (cf. [CF7]). The support of μ is the variety of M_4, which consists of the common solutions to $x^2 = 1$ and $y^3 = 2y$, namely $(x_1, y_1) \equiv (-1, 0)$, $(x_2, y_2) \equiv (-1, -\sqrt{2})$, $(x_3, y_3) \equiv (-1, \sqrt{2})$, $(x_4, y_4) \equiv (1, 0)$, $(x_5, y_5) \equiv (1, -\sqrt{2})$, $(x_6, y_6) \equiv (1, \sqrt{2})$. Note that $\mathcal{B} \equiv \{1, X, Y, XY, Y^2, XY^2\}$ is a basis for the column space of M_3 and define corresponding polynomials $p_1(x,y) := 1$, $p_2(x,y) = x$, $p_3(x,y) := y$, $p_4(x,y) = xy$, $p_5(x,y) := y^2$, $p_6(x,y) = xy^2$. The Vandermonde $V \equiv (p_i(x_j, y_j))_{1 \le i,j \le 6}$ is invertible. Let v be the vector of moments corresponding to the p_i, i.e., $v := (\beta_{00}, \beta_{10}, \beta_{01}, \beta_{11}, \beta_{02}, \beta_{12})^T = (1, 0, 0, 0, 1, 0)^T$. We compute $\alpha \equiv (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)^T := V^{-1}v = (\frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8})^T$. Then $\mu := \sum_{i=1}^{6} \alpha_i \delta_{(x_i, y_i)}$ is a representing measure for M_3.

We now use the method of Theorem 4.1 to approximate M_3 with positive flat moment matrices. Using Proposition 3.6 with $d = 3$, $r = 3$, $s = 1$ (r and s here have the same meaning as in Proposition 3.6), we perturb the support of μ as follows. For $k > 0$, $1 \le j \le 6$, let $x_k^{(j)} = x_j + \frac{1}{(jk)^3}$ and $y_k^{(j)} = y_j + \frac{1}{(jk)^4}$. We now define

$$\omega^{(k)} \equiv \mu^{(k)} := \sum_{j=1}^{6} \alpha_j \delta_{(x_k^{(j)}, y_k^{(j)})}.$$

Since $\Delta[\mu] = 0$, we set $\nu^{(k)} = 0$, and since $\rho[\mu] = 6 = \rho_2$, we set $\sigma^{(k)} = 0$. With $M_3^{(k)} := M_3[\mu^{(k)}]$, it is clear that $\lim_{k \to \infty} M_3^{(k)} = M_3$. We performed a numerical test using *Mathematica*, where all calculations are performed exactly in $\mathbb{Q}[\sqrt{2}]$. In particular, calculations of the ranks of the approximants are exact because there are no roundoff errors in the matrices. Let $\beta^{[k]}$ denote the moment sequence corresponding to $M_3^{(k)}$ and let $\Delta_k = \max_{a,b\geq 0, a+b\leq 6} |\beta_{ab} - \beta_{ab}^{[k]}|$. We found $\Delta_1 \approx 0.4$, $\Delta_2 \approx 0.09$, $\Delta_3 \approx 0.05$, $\Delta_4 \approx 0.02$, $\Delta_5 \approx 0.01$, $\Delta_{10} \approx 0.002$, $\Delta_{20} \approx 0.0002$, $\Delta_{50} \approx 10^{-5}$. Perhaps surprisingly, every $M_3^{(k)}$ that we tested was flat, even for $k = 1$, where the error is large. Note also that in implementing Proposition 3.1, for each $k > 0$ we used $k_j = jk$ $(1 \leq j \leq 6)$. Experiments with each $k_j = k$ did not yield flat approximants $M_3^{(k)}$.

We next illustrate Theorem 4.1, particularly Corollary 1.10, in a case where $C^\flat$ is Hankel. In such a case, let μ denote the unique, $(rank\ M_{d-1})$-atomic measure associated with $M^\flat \equiv M_d[\mu]$, let $\Delta \equiv \Delta[\mu] = \Delta^\flat$, and let $\kappa = r \equiv rank\ M_{d-1}$. In any example of this type, we may apply the method of Theorem 4.1 directly or we may apply a slightly different and simpler approach that we next describe. In this approach, we do not define the determining sequences $w^{(1)}, \ldots, w^{(r)}$, and we replace $\mu^{(k)}$ with μ. Further, if $r + s < \tau$, we choose α_j $(r + s + 1 \leq j \leq t)$ so that $\alpha_1, \ldots, \alpha_t$ are distinct and nonzero. For $m \geq d - 1$, we use Proposition 3.2 to define determining sequences $w^{(j)} \equiv \{(k + \frac{1}{k^m}, \alpha_j k)\}_{k=1}^{\infty}$ $(r + s + 1 \leq j \leq t)$. Given $k \geq 1$, we then apply Proposition 3.1 to $p_{r+1}, \ldots, p_t$ and the sequences $w^{(r+1)}, \ldots, w^{(t)}$ to produce points $w_{r+1}, \ldots, w_t$ (dependent on k) leading to an invertible $(\tau - r) \times (\tau - r)$ Vandermonde V. As before, we use $w_{r+1}, \ldots, w_{r+s}$ to define $\nu^{(k)}$ and $w_{r+s+1}, \ldots, w_t$ to define $\sigma^{(k)}$. Setting $\omega^{(k)} := \mu + \nu^{(k)} + \sigma^{(k)}$, the proof then proceeds along the lines of the proof of Theorem 4.1; we omit the details.

Example 4.3. To define

$$M_4 \equiv M_4(\beta^{(8)}) = \begin{pmatrix} M_3 & B_4 \\ B_4^T & C_4 \end{pmatrix},$$

we begin with

$$M_3 = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 2 & 0 & 2 & 0 & 5 \\ 0 & 1 & 0 & 0 & 2 & 0 & 2 & 0 & 5 & 0 \\ 1 & 0 & 2 & 2 & 0 & 5 & 0 & 5 & 0 & 14 \\ 1 & 0 & 2 & 2 & 0 & 5 & 0 & 5 & 0 & 14 \\ 0 & 2 & 0 & 0 & 5 & 0 & 5 & 0 & 14 & 0 \\ 2 & 0 & 5 & 5 & 0 & 14 & 0 & 14 & 0 & 42 \\ 0 & 2 & 0 & 0 & 5 & 0 & 5 & 0 & 14 & 0 \\ 2 & 0 & 5 & 5 & 0 & 14 & 0 & 14 & 0 & 42 \\ 0 & 5 & 0 & 0 & 14 & 0 & 14 & 0 & 42 & 0 \\ 5 & 0 & 14 & 14 & 0 & 42 & 0 & 42 & 0 & 132 \end{pmatrix}.$$

M_3 is positive semidefinite, with $rank\ M_3 = 7$, and column dependence relations $X^2 = Y$, $X^3 = XY$, and $X^2Y = Y^2$. From Theorem 2.2, a positive M_4 requires that in $\begin{pmatrix} M_3 & B_4 \end{pmatrix}$ we have $X^4 = X^2Y$, $X^3Y = XY^2$, $X^2Y^2 = Y^3$, so B_4 must be of the form

$$B_4 = \begin{pmatrix} 2 & 0 & 5 & 0 & 14 \\ 0 & 5 & 0 & 14 & 0 \\ 5 & 0 & 14 & 0 & 42 \\ 5 & 0 & 14 & 0 & 42 \\ 0 & 14 & 0 & 42 & 0 \\ 14 & 0 & 42 & 0 & 132 \\ 0 & 14 & 0 & 42 & 0 \\ 14 & 0 & 42 & 0 & 132 \\ 0 & 42 & 0 & 132 & x \\ 42 & 0 & 132 & x & y \end{pmatrix}.$$

A calculation shows that there exists W satisfying $B_4 = M_3 W$ and

$$C^\flat = \begin{pmatrix} 14 & 0 & 42 & 0 & 132 \\ 0 & 14 & 0 & 132 & x \\ 42 & 0 & 132 & x & y \\ 0 & 132 & x & x^2+428 & x(y-416) \\ 132 & x & y & x(y-416) & 179464 + x^2 - 844y + y^2 \end{pmatrix}.$$

In the sequel, to satisfy the requirement that $C^\flat$ be Hankel, we set $y = x^2 + 428$. We next define C_4 by

$$C_4 = \begin{pmatrix} 15 & 0 & 43 & 0 & 133 \\ 0 & 43 & 0 & 133 & x \\ 43 & 0 & 133 & x & x^2+429 \\ 0 & 133 & x & x^2+429 & x(x^2+12) \\ 133 & x & x^2+429 & x(x^2+12) & x^4+13x^2+1417+\epsilon \end{pmatrix} \quad (\epsilon \geq 0),$$

so that

$$\Delta \equiv C_4 - C^\flat = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1+\epsilon \end{pmatrix}.$$

Since $\Delta \succeq 0$, then $M_4 \succeq 0$. However, since $X^4 \neq X^2 Y$ in the column space, M_4 is not recursively generated and thus has no representing measure. Since $M_4 \succeq 0$, we have $\mathit{rank}\ M_4 = \mathit{rank}\ M_3 + \mathit{rank}\ \Delta$. Thus, $\mathit{rank}\ M_4 = 9$ if $\epsilon = 0$ and $\mathit{rank}\ M_4 = 10$ if $\epsilon > 0$, so in either case M_4 satisfies the hypotheses of Theorem 4.1, and we will illustrate how to approximate M_4 with positive flat moment matrices.

We begin with the case when $\epsilon = 0$ and $\mathit{rank}\ M_4 = 9$, and we let μ denote the unique (7-atomic) representing measure for $M^\flat$ (which can be explicitly computed from [CF7]; cf. Section 2). We first choose $\tau = 9$, i.e., we will approximate M_4 by flat moment matrices $M_4^{(k)}$ with $\mathit{rank}\ M_4^{(k)} = 9$. Since $\epsilon = 0$, Δ has the unique representing measure $\nu := \frac{1}{2}\delta_{\{1\}} + \frac{1}{2}\delta_{\{-1\}}$. Following the proof of Theorem 4.1 and the modified approach sketched above (in particular, applying Propositions 3.1 and 3.2), we choose $m = 3$, and for $k \geq 1$, we set $k_1 = k_2 = k$, $a_1 = a_2 = \frac{1}{2}$, and $\alpha_1 = 1$, $\alpha_2 = -1$. Now we define

$$\nu^{(k)} := \frac{1}{2k^8}\delta_{(k+\frac{1}{k^3},k)} + \frac{1}{2k^8}\delta_{(k+\frac{1}{k^3},-k)}.$$

Let $\omega^{(k)} := \mu+\nu^{(k)}$ and set $M_4^{(k)} = M_4[\omega^{(k)}]$. Clearly, $\lim_{k\to\infty} M_4^{(k)} = M^\flat + M_\Delta = M_4$, and a symbolic calculation shows that $rank\ M_4^{(k)} = rank\ M_3^{(k)} = 9$.

We next choose $\tau = 10$. Note that for $k \geq 1$, Propositions 3.1 and 3.2 guarantee the existence of $k_1, k_2, k_3 \geq k$ corresponding to an invertible Vandermonde, but there is no explicit formula for the k_i values. In this case, if we choose $k_1 = k_2 = k_3 = k$, then we find $rank\ M_3^{(k)} = 9$, so $M_4^{(k)}$ (which has rank 10) is not flat. Instead, we define $\nu^{(k)}$ as above, except that we now try $k_1 = k$ and $k_2 = 2k$. Further, following the method discussed above, we try $k_3 = 3k$, $\alpha_3 = 2$, and we define $\sigma^{(k)} := \frac{1}{k^{10}}\delta_{\{(3k+\frac{1}{(3k)^3}, 6k)\}}$. Setting $\omega^{(k)} := \mu + \nu^{(k)} + \sigma^{(k)}$ and $M_4^{(k)} := M_4[\omega^{(k)}]$, we see that $\lim_{k\to\infty} M_4^{(k)} = M^\flat + M_\Delta + 0 = M_4$, and a symbolic calculation shows that $rank\ M_4^{(k)} = rank\ M_3^{(k)} = 10$.

We next consider the case when $\epsilon > 0$, i.e., $rank\ \Delta = 3$. In this case, Δ is not recursively generated and so has no representing measure. Nevertheless, [FN2] implies that Δ may be approximated by rank 3 positive flat Hankel matrices. Indeed, for $k > 0$, if we define

$$\nu_k := \frac{1}{2}\delta_{\{1\}} + \frac{1}{2}\delta_{\{-1\}} + \frac{\epsilon}{k^8}\delta_{\{k\}},$$

then $\lim_{k\to\infty} H_4[\nu_k] = \Delta$, and $rank\ H_4[\nu_k] = rank\ H_3[\nu_k] = 3$. We may therefore assume that $\Delta = \Delta^{(k)} := H_4[\nu_k]$, and we proceed as outlined above, with $r = 7$ and $\rho = \tau = \rho_3 = 10$. Let $\alpha_1 = 1$, $\alpha_2 = -1$, $\alpha_3 = k$, and $a_1 = a_2 = \frac{1}{2}$, $a_3 = \frac{\epsilon}{k^8}$. For fixed $j \geq k$, we choose $k_1 = j$, $k_2 = 2j$, $k_3 = 3j$ (based on Proposition 3.1 and experimentation), and define

$$\nu^{(k)} = \frac{1}{2k_1^8}\delta_{\{(k_1+\frac{1}{k_1^3}, k_1)\}} + \frac{1}{2k_2^8}\delta_{\{(k_2+\frac{1}{k_2^3}, -k_2)\}} + \frac{\epsilon}{k^8}\frac{1}{k_3^8}\delta_{\{(k_3+\frac{1}{k_3^3}, k\cdot k_3)\}}.$$

We now set $\omega^{(k)} := \mu + \nu^{(k)}$ and $M_4^{(k)} := M_4[\omega^{(k)}]$. Clearly, $\lim_{k\to\infty} M_4^{(k)} = M^\flat + \lim_{k\to\infty} M_{\Delta^{(k)}} = M^\flat + M_\Delta = M_4$, and a symbolic calculation shows that $rank\ M_4^{(k)} = rank\ M_3^{(k)} = 10$.

To conclude the example we re-define C_4 as

$$C_4 = \begin{pmatrix} 14 & 0 & 42 & 0 & 132 \\ 0 & 42 & 0 & 132 & x \\ 42 & 0 & 132 & x & x^2+428 \\ 0 & 132 & x & x^2+428 & x(x^2+12) \\ 132 & x & x^2+428 & x(x^2+12) & x^4+13x^2+1416+\epsilon \end{pmatrix} \quad (\epsilon > 0),$$

so that the Hankel matrix Δ is no longer a moment matrix, i.e., $\Delta_j = 0$ $(0 \leq j \leq 7)$ and $\Delta_8 = \epsilon > 0$. We have $r = 7$, $\rho = 8$, and we choose $\tau = 9$. Using Propositions 3.1 and 3.5 (with $k_1 = k$ and $k_2 = 2k$ in Proposition 3.1), let $\nu^{(k)} := \frac{\epsilon}{k^8}\delta_{\{(\frac{1}{k^3}, k)\}}$ and $\sigma^{(k)} := \frac{1}{k^{10}}\delta_{\{(\frac{1}{(2k)^3}, 2k)\}}$. Setting $\omega^{(k)} := \mu + \nu^{(k)} + \sigma^{(k)}$ and $M_4^{(k)} := M_4[\omega^{(k)}]$, we see that $\lim_{k\to\infty} M_4^{(k)} = M^\flat + M_\Delta + 0 = M_4$, and a symbolic calculation shows that $rank\ M_4^{(k)} = rank\ M_3^{(k)} = 9$.

We conclude this section with a proof of Proposition 1.12.

Proof of Proposition 1.12. The proof consists of repeating the proof of Theorem 4.1 up to the point where we define $\omega^{(k)} := \mu^{(k)} + \nu^{(k)} + \sigma^{(k)}$, set

$$M_d^{(k)} \equiv M_d[\omega^{(k)}] = M_d[\mu^{(k)}] + M_d[\nu^{(k)}] + M_d[\sigma^{(k)}],$$

and show that $\lim_{k\to\infty} M_d^{(k)} = M_d[\mu] + M_\Delta = M_d(\beta)$. This shows that M_d is in the closure of moment matrices with measures, so L_β is positive. Since we are not interested in the ranks of the approximating moment matrices, it is also possible to re-work the proof in a simpler way, without using determining sequences. $\square$

5. Flat approximation in the degree 6 moment problem

Our main application is Theorem 1.1, which is included in the following result.

Theorem 5.1. *$M_3 \equiv M_3(\beta^{(6)})$ belongs to $\overline{\mathcal{F}_3}$ if and only if $M_3 \succeq 0$ and $\rho \equiv$ rank $M_3 \le 6$. In this case, given τ, $\rho \le \tau \le 6$, M_3 is in the closure of the rank-τ positive flat moment matrices, and there exists a sequence of (computable) τ-atomic positive measures $\{\mu_k\}$ such that $\beta_{ij} = \lim_{k\to\infty} \int x^i y^j \, d\mu_k$ $(i,j \ge 0,\ i+j \le 6)$. Further, either $M_3^\flat$ is a moment matrix or M_3 admits a flat extension M_4, whence $\beta^{(5)}$ has a representing measure.*

Let $M_d = \begin{pmatrix} M_{d-1} & B_d \\ B_d^T & C_d \end{pmatrix}$ denote a positive semidefinite moment matrix. We recall from Section 2 two basic properties that we will use repeatedly. It follows from [CF2, Proposition 3.9] that if $M_d \succeq 0$, then

$$p \in \mathbb{R}[x]_{d-1},\ p(X) = 0 \text{ in } Col\ M_{d-1} \Longrightarrow p(X) = 0 \text{ in } Col\ M_d. \tag{5.1}$$

Further, Theorem 2.2 iii) implies that $\begin{pmatrix} M_{d-1} & B_d \end{pmatrix}$ is recursively generated, i.e.,

$$p, q, pq \in \mathbb{R}[x]_d,\ p(X)=0 \text{ in } Col \begin{pmatrix} M_{d-1} & B_d \end{pmatrix} \Longrightarrow (pq)(X)=0 \text{ in } Col \begin{pmatrix} M_{d-1} & B_d \end{pmatrix}. \tag{5.2}$$

If $M_d \succeq 0$, then $Ran\ B_d \subseteq Ran\ M_{d-1}$, so there exists W such that $B_d = M_{d-1}W$. Let $C_d^\flat := B_d^T W = W^T M_{d-1} W$; note that $C_d^\flat = C_d^\flat W^T$ and that $C_d^\flat$ is independent of W satisfying $B_d = M_{d-1}W$. Let $M_d^\flat := \begin{pmatrix} M_{d-1} & B_d \\ B_d^T & C_d^\flat \end{pmatrix}$. The following property is useful in computing $M_d^\flat$:

$$p \in \mathbb{R}[x]_d,\ p(X) = 0 \text{ in } Col \begin{pmatrix} M_{d-1} & B_d \end{pmatrix} \Longrightarrow p(X) = 0 \text{ in } Col\ M_d^\flat. \tag{5.3}$$

As discussed in Section 2, *rank* $M_d^\flat =$ *rank* M_{d-1}. Thus, if $C_d^\flat$ is Hankel, then $M_d^\flat$ is a flat moment matrix extension of M_{d-1} (using the data in B_d) and thus has a representing measure (cf. Theorem 1.4). Note that in the case when $d = 3$, since $C_3^\flat \equiv (c_{ij})_{1\le i,j\le 4}$ is real symmetric, $C_3^\flat$ is Hankel if and only if $c_{31} = c_{22}$, $c_{41} = c_{32}$, and $c_{42} = c_{33}$.

We begin the proof of Theorem 5.1 with a series of preliminary results based on the value of $r \equiv$ *rank* M_2.

Proposition 5.2. *If $M_3 \succeq 0$, rank $M_3 \le 6$ and $r \equiv$ rank $M_2 \le 3$, then $C_3^\flat$ is Hankel.*

Proof. Suppose $r=1$. Then in $Col\, M_3$, $X=\alpha 1$, $Y=\beta 1$ (for certain scalars α, β), so (5.2) and (5.3) imply that in $Col\, M_3^\flat$, $X^iY^j=\alpha^i\beta^j 1$. Then for $i+j=k+l=3$ with $1\le i,l\le 3$, $0\le j,k\le 2$, $\langle X^iY^j, X^kY^l\rangle = \alpha^{i+k}\beta^{j+l} = \langle X^{i-1}Y^{j+1}, X^{k+1}Y^{l-1}\rangle$, so $C_3^\flat$ is Hankel.

Next, let $r=2$. If $rank\ M_1=1$, then by recursiveness in M_2, it follows that $rank\ M_2=1$, a contradiction. Thus $rank\ M_1 = rank\ M_2 = 2$, so M_2 is flat. Thus M_2 has a unique flat extension $\widetilde{M}_3$. Since $\widetilde{M}_3$ is recursively generated, it follows readily from (5.2) that $B_3=\widetilde{B}_3$. Now (5.3) implies that $M_3^\flat = \widetilde{M}_3$, so $C_3^\flat = \widetilde{C}_3$, whence $C_3^\flat$ is Hankel.

Now suppose $r=3$. As above, if $rank\ M_1=1$, then by recursiveness of M_2, $rank\ M_2=1$, a contradiction. Further, if $rank\ M_1=3$, then M_2 is flat, so we may proceed exactly as in the $r=2$ case (above) to conclude that $C_3^\flat$ is Hankel. We may thus assume that $rank\ M_1=2$, and we first consider the case when $\{1,X\}$ is a column basis, with $Y=\alpha 1+\beta X$ for some $\alpha,\beta\in\mathbb{R}$. By recursiveness in M_2, $XY=\alpha X+\beta X^2$ and $Y^2=\alpha Y+\beta XY$, so $\{1,X,X^2\}$ is a basis for $Col\ M_2$. From (5.2), in $Col\ B_3$ we have

$$X^2Y=\alpha X^2+\beta X^3, \tag{5.4}$$

$$XY^2=\alpha XY+\beta X^2Y, \tag{5.5}$$

$$Y^3=\alpha Y^2+\beta XY^2. \tag{5.6}$$

These relations and the value of β_{50} completely determine the other moments of degree 5; thus from (5.4), $\beta_{41}=\alpha\beta_{40}+\beta\beta_{50}$, $\beta_{32}=\alpha\beta_{31}+\beta\beta_{41}$, $\beta_{23}=\alpha\beta_{22}+\beta\beta_{32}$; from (5.5), $\beta_{14}=\alpha\beta_{13}+\beta\beta_{23}$; and from (5.6), $\beta_{05}=\alpha\beta_{04}+\beta\beta_{14}$. With these values, a calculation shows that there exists W such that $B_3=M_2W$, and a further calculation shows that $C_3^\flat\equiv W^TM_2W$ is Hankel. In the remaining case, a basis is $\{1,Y\}$, and the proof is entirely analogous. □

We next analyze cases of Theorem 5.1 with $r=4$ or $r=5$. Since $M_3\succeq 0$, then M_2 is recursively generated, so in these cases we must have $\{1,X,Y\}$ independent in $Col\ M_2$. Thus, in $Col\ M_2$ there is a dependence relation of the form $p(X,Y)=0$, with $deg\ p(x,y)=2$. Given a degree-one map $\mathcal{T}$, let $\widetilde{M}_2$ denote the moment matrix corresponding to M_2 under $\mathcal{T}$ (cf. Proposition 2.8), and let $q(\widetilde{x},\widetilde{y})=p\circ\mathcal{T}^{-1}(\widetilde{x},\widetilde{y})$ $\in\mathbb{R}[\widetilde{x},\widetilde{y}]_d$; clearly $q(\widetilde{x},\widetilde{y})=0$ if and only if $p(x,y)=0$. It is well known that corresponding to $p(x,y)$ there is a degree-one map $\mathcal{T}$ such that the variety $q(\widetilde{x},\widetilde{y})=0$ is one of the following: $\widetilde{x}\widetilde{y}=0$, $\widetilde{x}\widetilde{y}=1$, $\widetilde{y}=\widetilde{x}^2$, $\widetilde{x}^2+\widetilde{y}^2=1$, $\widetilde{x}^2=1$, $\widetilde{x}^2=0$, $\widetilde{x}^2=-1$, $\widetilde{x}^2+\widetilde{y}^2=0$, $\widetilde{x}^2+\widetilde{y}^2=-1$ (cf. [SH, p. 405]). We note that column dependence relations corresponding to any of the last four cases cannot occur in $\widetilde{M}_2$ if $\widetilde{M}_2\succeq 0$ and $rank\ \widetilde{M}_2\ge 4$. To see this, we may scale $\widetilde{\beta}$ so that $\widetilde{\beta}_{00}=1$, and we denote $\widetilde{M}_2$ as

$$\widetilde{M}_2\equiv\begin{pmatrix} 1 & a & b & c & d & e\\ a & c & d & f & g & h\\ b & d & e & g & h & k\\ c & f & g & p & q & w\\ d & g & h & q & w & s\\ e & h & k & w & s & t\end{pmatrix}. \tag{5.7}$$

If $\widetilde{X}^2 = 0$, then $c = f = g = p = q = w = 0$; since $c = 0$, positivity of $\widetilde{M_2}$ implies $a = d = h = 0$, and, similarly, $w = 0$ implies $s = 0$. Thus, *rank* $\widetilde{M_2} \le 3$, a contradiction. Similarly, if $\widetilde{X}^2 + \widetilde{Y}^2 = 0$, then $c + e = 0$ and $p + w = 0$ imply $c = e = p = w = 0$, and it follows as above (via positivity) that *rank* $\widetilde{M_2} \le 2$. If $\widetilde{X}^2 = -1$, then $c = -1$, which violates positivity in $\widetilde{M_2}$; similarly, if $\widetilde{X}^2 + \widetilde{Y}^2 = -1$, then $c + e = -1$, which is impossible. Motivated by the preceding discussion, in the following sequence of results we assume that M_2 admits a column dependence relation corresponding to one of the following varieties: $xy = 0$, $xy = 1$, $x^2 = y$, $x^2 = 1$, or $x^2 + y^2 = 1$. In each case we will show that under the hypotheses of Theorem 5.1, M_3 satisfies the following property:

$$C_3^\flat \text{ is Hankel or } M_3 \text{ admits a flat extension } M_4. \tag{5.8}$$

In view of Theorem 1.4 and Corollaries 1.10 and 1.11, it is clear that (5.8) implies the conclusions of Theorem 5.1, including the existence of a representing measure for $\beta^{(5)}$.

In the sequel, to simplify certain symbolic calculations, we always scale β so that $\beta_{00} = 1$ (without loss of generality). If $\mathcal{B}$ is a collection of columns of M_d which forms a basis of the column space of M_d, we will denote by $[M_d]_{\mathcal{B}}$ the compression of M_d to the rows and columns indexed by the elements of $\mathcal{B}$. Thus, if $M_d \succeq 0$, then $[M_d]_{\mathcal{B}} \succ 0$.

Proposition 5.3. *Suppose $M_3 \succeq 0$, with $\rho \le 6$ and $r = 4$. If M_2 has a column relation $XY = 0$, then $C_3^\flat$ is Hankel.*

Proof. As noted above, $\{1, X, Y\}$ is independent. Suppose first that *Col* M_2 has the basis $\mathcal{B} \equiv \{1, X, Y, X^2\}$. Writing $Y^2 = c_1 1 + c_2 X + c_3 Y + c_4 X^2$, in $\begin{pmatrix} M_2 & B_3 \end{pmatrix}$ we have $X^2Y = 0$, $0 = XY^2 = c_1 X + c_2 X^2 + c_4 X^3$ and $Y^3 = c_1 Y + c_3 Y^2$. Further, M_2 is of the form

$$M_2 = \begin{pmatrix} 1 & a & b & c & 0 & d \\ a & c & 0 & e & 0 & 0 \\ b & 0 & d & 0 & 0 & f \\ c & e & 0 & g & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ d & 0 & f & 0 & 0 & h \end{pmatrix}, \tag{5.9}$$

with

$$J \equiv [M_2]_{\mathcal{B}} = \begin{pmatrix} 1 & a & b & c \\ a & c & 0 & e \\ b & 0 & d & 0 \\ c & e & 0 & g \end{pmatrix} \succ 0,$$

$$\begin{pmatrix} c_1 & c_2 & c_3 & c_4 \end{pmatrix}^T = J^{-1} \begin{pmatrix} d & 0 & f & 0 \end{pmatrix}^T, \ h = \begin{pmatrix} d & 0 & f & 0 \end{pmatrix} \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \end{pmatrix}^T.$$

From the above column relations, B_3 must be of the form

$$B_3 = \begin{pmatrix} e & 0 & 0 & f \\ g & 0 & 0 & 0 \\ 0 & 0 & 0 & h \\ t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u \end{pmatrix} \tag{5.10}$$

for certain t and u. Since X^3 in B_3 belongs to $Ran\ M_2$, it follows that $X^3 = d_1 1 + d_2 X + d_3 Y + d_4 X^2$, with $(\ d_1\ d_2\ d_3\ d_4\ ) = J^{-1}(\ e,\ g,\ 0,\ t\ )^T$. In particular, $\gamma := \langle X^3, Y^2 \rangle = 0$, and a symbolic calculation shows that

$$\gamma = (\ d,\ 0,\ f,\ 0\ )(\ d_1\ d_2\ d_3\ d_4\ )^T = \frac{(bf - d^2)(e^3 - 2ceg + ag^2 + (c^2 - ac)t)}{det(J)}.$$

Further, since $Y^3 = c_1 Y + c_3 Y^2$, it follows that $u = c_1 f + c_3 h$. Now let

$$W = \begin{pmatrix} d_1 & 0 & 0 & 0 \\ d_2 & 0 & 0 & 0 \\ d_3 & 0 & 0 & c_1 \\ d_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_2 \end{pmatrix}.$$

Then $M_2 W = B_3$, and in $C_3^\flat \equiv W^T M_2 \equiv (c_{ij})_{1 \le i,j \le 4}$, we have $c_{31} = 0 = c_{22}$ and $c_{42} = 0 = c_{32}$. Thus $C_3^\flat$ is Hankel if and only if $c_{41} = 0\ (= c_{32})$. A symbolic calculation shows that

$$c_{41} = \frac{-\gamma\omega}{det(J)}\ (\omega \equiv -bde^2 + (c^3 - 2ace + e^2)f + (bcd + a^2 f - cf)g),$$

and since $\gamma = 0$, then $c_{41} = 0$. Thus $C_3^\flat$ is Hankel, which completes the proof in this case.

Suppose next that $XY = 0$ and that $B \equiv \{1, X, Y, Y^2\}$ is a basis for $Col\ M_2$. In view of the previous case, we may assume that $X^2 \in \langle 1, X, Y \rangle$. Thus, $X^2 = c_1 1 + c_2 X + c_3 Y$, with $(\ c_1\ c_2\ c_3\ ) = M_1^{-1}(\ c\ e\ 0)^T$. In particular, $g = (\ c\ e\ 0\ )(c_1\ c_2\ c_3\ )^T$ and $0 = \langle X^2, Y^2 \rangle = (\ d\ 0\ f\ )(\ c_1\ c_2\ c_3\ )^T$ (which entails $c^2 - ae)(d^2 - bf) = 0$). Denoting M_2 and B_3 as in (5.9) and (5.10), t in X^3 is uniquely determined via recursiveness in $(\ M_2\ B_3\ )$ by

$$X^3 = c_1 X + c_2 X^2, \tag{5.11}$$

i.e., $t = c_1 e + c_2 g$.

Let $K = [M_2]_{\mathcal{B}}\ (\succ 0)$. Since $M_3 \succeq 0$, Y^3 in B_3 is in $Ran\ M_2$, so $Y^3 = k_1 1 + k_2 X + k_3 Y + k_4 Y^2$, where $(\ k_1\ k_2\ k_3\ k_4\ ) = K^{-1}(\ f,\ 0,\ h,\ u\ )^T$. This relation and (5.10) imply that $0 = \langle Y^3, X^2 \rangle = k_1 c + k_2 e \equiv \lambda$. A symbolic calculation now shows that

$$\lambda = \frac{(ae - c^2)(f^3 - 2dfh + bh^2 + (d^2 - bf)u)}{det(K)}.$$

Setting

$$W = \begin{pmatrix} 0 & 0 & 0 & k_1 \\ c_1 & 0 & 0 & k_2 \\ 0 & 0 & 0 & k_3 \\ c_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_4 \end{pmatrix},$$

we have $M_2 W = B_3$, and a further symbolic calculation shows that

$$C_3^\flat \equiv B_3^T W = \begin{pmatrix} * & 0 & 0 & c_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * \end{pmatrix},$$

with

$$c_{14} \equiv \langle Y^3, X^3 \rangle = k_1 e + k_2 g = \frac{-\lambda(acd + b^2 e - de)}{det(K)}.$$

Since $\lambda = 0$, it follows that $C_3^\flat$ is Hankel, whence the result follows. □

Proposition 5.4. *Suppose $M_3 \succeq 0$, with $\rho \leq 6$ and $r = 4$. If M_2 has a column relation $XY = 1$, then $C_3^\flat$ is Hankel.*

Proof. Since $XY = 1$, we may denote M_2 as

$$M_2 = \begin{pmatrix} 1 & a & b & c & 1 & d \\ a & c & 1 & e & a & b \\ b & 1 & d & a & b & f \\ c & e & a & g & c & 1 \\ 1 & a & b & c & 1 & d \\ d & b & f & 1 & d & h \end{pmatrix}. \tag{5.12}$$

Since $r = 4$, $\{1, X, Y\}$ is independent, and we first consider the case when $\mathcal{B} \equiv \{1, X, Y, X^2\}$ is a column basis. Let

$$J \equiv [M_2]_{\mathcal{B}} = \begin{pmatrix} 1 & a & b & c \\ a & c & 1 & e \\ b & 1 & d & a \\ c & e & a & g \end{pmatrix} \succ 0,$$

and set $(\ c_1\ c_2\ c_3\ c_4\ )^T := J^{-1}(\ d\ b\ f\ 1\ )^T$. Then

$$h = (\ d\ b\ f\ 1\ )(\ c_1\ c_2\ c_3\ c_4\ )^T \tag{5.13}$$

and $Y^2 = c_1 1 + c_2 X + c_3 Y + c_4 X^2$. By recursiveness, in $Col\ (\ M_2\ B_3\ )$ we have $X^2 Y = X$, $XY^2 = Y$, $Y^3 = c_1 Y + c_2 XY + c_3 Y^2 + c_4 X^2 Y$. Now B_3 is of the form

$$B_3 = \begin{pmatrix} e & a & b & f \\ g & c & 1 & d \\ c & 1 & d & h \\ t & e & a & b \\ e & a & b & f \\ a & b & f & u \end{pmatrix}$$

for some $t \equiv \beta_{50}$ and $u \equiv \beta_{05}$. Setting $(\ k_1\ k_2\ k_3\ k_4\ )^T := J^{-1}(\ e\ g\ c\ t\ )^T$, the condition $X^3 \in Ran\ M_2$, i.e., $X^3 = k_1 1 + k_2 X + k_3 Y + k_4 X^2$, is equivalent to the condition that $\lambda := k_1 d + k_2 b + k_3 f + k_4 1 - a$ satisfies $\lambda = 0$. Setting

$$W \equiv \begin{pmatrix} k_1 & 0 & 0 & 0 \\ k_2 & 1 & 0 & c_4 \\ k_3 & 0 & 1 & c_1 \\ k_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_2 \\ 0 & 0 & 0 & c_3 \end{pmatrix},$$

we have $M_2 W = B_3$, so that $C_3^\flat = B_3^T W$. Denoting $C_3^\flat \equiv (c_{ij})_{1 \leq i,j \leq 4}$, a symbolic calculation shows that $c_{31} = c_{22}$, $c_{42} = c_{33}$, and $c_{32} = 1$. Thus , $C_3^\flat$ is Hankel if and only if $c_{41} \equiv k_1 f + k_2 d + k_3 h + k_4 b$ satisfies $c_{41} = 1$ $(= c_{32})$. Now, using (5.13), a symbolic calculation shows that

$$c_{41} - 1 = \frac{\lambda\gamma}{det(J)},$$

where γ is a polynomial in the moments of M_2. Since, from above, $\lambda = 0$, we have $c_{41} = 1$, whence $C_3^\flat$ is Hankel.

In the remaining case with $XY = 1$, we may assume that M_2 has the column basis $\mathcal{B} \equiv \{1, X, Y, Y^2\}$. In view of the previous case, we may also assume that $X^2 = c_1 1 + c_2 X + c_3 Y$, where $(\ c_1\ c_2\ c_3\ )^T = M_1^{-1}(\ c\ e\ a\ )^T$, so, in particular, we have

$$c_1 c + c_2 e + c_3 a = g,\ c_1 d + c_2 b + c_3 f = 1. \tag{5.14}$$

In $(\ M_2\ B_3\ )$, by recursiveness, we have $X^3 = c_1 X + c_2 X^2 + c_3 1$, $X^2 Y = X$, $XY^2 = Y$. Further, since $Y^3 \in Ran\ M_2$, in $Col\ (\ M_2\ B_3\ )$ we have

$$Y^3 = k_1 1 + k_2 X + k_3 Y + k_4 Y^2, \tag{5.15}$$

where $(\ k_1\ k_2\ k_3\ k_4\ )^T = L^{-1}(\ f\ d\ h\ u\ )^T$ for

$$L \equiv [M_2]_{\mathcal{B}} = \begin{pmatrix} 1 & a & b & d \\ a & c & 1 & b \\ b & 1 & d & f \\ d & b & f & h \end{pmatrix} (\succ 0).$$

In particular, (5.15) entails

$$b \equiv \langle Y^3, X^2 \rangle = k_1 c + k_2 e + k_3 a + k_4. \tag{5.16}$$

Define W by

$$W = \begin{pmatrix} c_3 & 0 & 0 & k_1 \\ c_1 & 1 & 0 & k_2 \\ 0 & 0 & 1 & k_3 \\ c_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_4 \end{pmatrix},$$

so that $M_2 W = B_3$. A symbolic calculation shows that $C_3^\flat := W^T M_2 W \equiv (c_{ij})_{1 \le i,j \le 4}$ is Hankel if and only if $1 = c_{41} \equiv \langle X^3, Y^3 \rangle = c_1 d + c_2 b + c_3 f$ and $c_{41} = c_{14}$; the former condition follows from (5.14) and the latter from the fact that $C_3^\flat$ is real symmetric, so the proof is complete. □

Proposition 5.5. *Suppose $M_3 \succeq 0$, with $\rho \le 6$ and $r = 4$. If M_2 has a column relation $Y = X^2$, then $C_3^\flat$ is Hankel.*

Proof. Since $r = 4$ and M_2 is recursively generated, then $\{1, X, Y\}$ is independent, and we first consider the case when $\{1, X, Y, Y^2\}$ is a column basis for M_2. Thus, there is a column relation of the form $XY = c_1 1 + c_2 X + c_3 Y + c_4 Y^2$. The corresponding curve $xy = c_1 + c_2 x + c_3 y + c_4 y^2$ has no x^2 term, so its discriminant is positive, and it thus represents a (possibly degenerate) hyperbola. Thus, by applying an appropriate degree-one map and Proposition 2.8 iii), iv), vii), M_2 may be transformed into a rank-4 positive $\widetilde{M_2}$ with a column relation of the form $\widetilde{X}\widetilde{Y} = 0$ or $\widetilde{X}\widetilde{Y} = \widetilde{1}$. It thus follows from Propositions 5.3 and 5.4 that the C block of $(\widetilde{M_d})^\flat$ is Hankel, whence Proposition 2.8 x) implies that $C_3^\flat$ is Hankel.

We may now assume that $\{1, X, Y, XY\}$ is a column basis, with a column relation of the form $Y^2 = k_1 1 + k_2 X + k_3 Y + k_4 XY$. Denoting M_2 as M_d with $d = 2$, we see that M_d is *recursively determinate* in the sense of [CF10]; i.e., there is a degree-reducing relation $X^n = p(X, Y)$ $(n = 2,\ deg\ p = 1 < n)$ and a degree-preserving relation $Y^m = q(X, Y)$ $(m = 2,\ deg\ q = 2 = m)$. Since M_d is positive, recursively generated, and recursively determinate with $n + m - 2 = d$, it follows from [CF10, Corollary 2.4] that M_2 admits a unique flat extension $\widetilde{M_3}$. In this extension, $\widetilde{B}_3$ is completely determined from M_2 by recursiveness, i.e., $\widetilde{X^3} = \widetilde{XY}$, $\widetilde{X^2Y} = \widetilde{Y^2}$, $\widetilde{XY^2} = (xq)(\widetilde{X}, \widetilde{Y})$, $\widetilde{Y^3} = (yq)(\widetilde{X}, \widetilde{Y})$. Since $M_2 = \widetilde{M_2}$ and $\begin{pmatrix} M_2 & B_3 \end{pmatrix}$ is recursively generated, the same column relations as just above determine B_3 so that $B_3 = \widetilde{B}_3$. It now follows immediately from (5.3) that $C_3^\flat = \widetilde{C}_3$, and since $\widetilde{C}_3$ is Hankel, the result follows. ☐

Proposition 5.6. *Suppose $M_3 \succeq 0$, with $\rho \le 6$ and $r = 4$. If M_2 has a column relation $X^2 = 1$, then $C_3^\flat$ is Hankel.*

Proof. The proof is essentially the same as that of the preceding result. If $\{1, X, Y, Y^2\}$ is a column basis for M_2, then there is a hyperbola relation $XY = c_1 1 + c_2 X + c_3 Y + c_4 Y^2$, so it follows as in the proof of Proposition 5.5 that $C_3^\flat$ is Hankel. In the remaining case, $\{1, X, Y, XY\}$ is a column basis, and it follows exactly as in the proof of Proposition 5.5 that $M_3^\flat$ is a moment matrix, whence $C_3^\flat$ is Hankel. ☐

Proposition 5.7. *Suppose $M_3 \succeq 0$, with $\rho \le 6$ and $r = 4$. If M_2 has a column relation $X^2 + Y^2 = 1$, then $C_3^\flat$ is Hankel.*

Proof. As in the previous proof, $\{1, X, Y\}$ is independent. We consider first the case when $\{1, X, Y, X^2\}$ is a column basis for M_2. It follows that XY is a linear combination of the basis columns, but since this relation has no Y^2 term, the relation represents a hyperbola, so the result follows from the hyperbola cases above (exactly as in the proof of Proposition 5.5, via Proposition 2.8 x)). In the remaining case, $\{1, X, Y\}$ is independent and $\{1, X, Y, X^2\}$ is dependent, so there is a degree-reducing column relation of the form $X^2 = c_1 1 + c_2 X + c_3 Y$. Since we also have the degree-preserving relation $Y^2 = 1 - X^2$, M_2 is recursively determinate, and the proof proceeds exactly as in the conclusion of the proof of Proposition 5.5. ☐

We next present cases where $r = 5$. Since there is nothing to prove in Theorem 5.1 if M_3 is flat, and (5.8) clearly holds as well in this case, we may assume $rank\ M_3 = 6$ in these cases.

Proposition 5.8. *Suppose $M_3 \succeq 0$, with $\rho = 6$ and $r = 5$. If M_2 has a column relation $XY = 0$, then M_3 satisfies* (5.8).

Proof. We have M_2 as in (5.9), and $\begin{pmatrix} M_2 & B_3 \end{pmatrix}$ is recursively generated, so B_3 is as in (5.10). Since $X^2Y = XY^2 = 0$ in B_3, (5.3) implies that the same column relations hold in $C_3^\flat$, so $C_3^\flat$ has the form

$$C_3^\flat = \begin{pmatrix} \kappa & 0 & 0 & \tau \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \tau & 0 & 0 & \delta \end{pmatrix}. \tag{5.17}$$

We may suppose $\tau \neq 0$, for otherwise $C_3^\flat$ is Hankel and the result follows. A symbolic calculation shows that $\tau = \tau_1\tau_2$ where

$$\tau_1 = (e^3 - 2ceg + ag^2 + (c^2 - ae)t),\ \tau_2 = (f^3 - 2dfh + bh^2 + (d^2 - bf)u).$$

Let $J = [M_2]_{\mathcal{B}}$, where $\mathcal{B}$ denotes the column basis $\{1, X, Y, X^2, Y^2\}$. Now $J \succ 0$, so $\kappa = [X^3]J^{-1}[X^3]^T$, where $[X^3] = (\ e\ g\ 0\ t\ 0\ )$; note that since $\tau \neq 0$ and $C_3^\flat \succeq 0$, then $\kappa > 0$. Consider $\Delta := C_3 - C_3^\flat$ and write $\Delta \equiv (\Delta_{ij})_{1\le i,j\le 4}$.

We claim that $\Delta_{11} \neq 0$. Let us denote $C_3^\flat$ as $(c_{ij})_{1\le i,j\le 4}$, so that $c_{31} = c_{22} = c_{13} = 0$. Since C_3 $(= C_3^\flat + \Delta)$ is Hankel, it follows that $\Delta_{31} = \Delta_{22} = \Delta_{13}$. If $\Delta_{11} = 0$, then, since $\Delta \succeq 0$, it follows that $\Delta_{21} = \Delta_{31} = \Delta_{41} = 0$, whence $\Delta_{22} = 0$. Positivity of Δ now implies $\Delta_{32} = \Delta_{42} = 0$. Since $C_3 \equiv C_3^\flat + \Delta$ is Hankel, it follows that $\Delta_{32} = \tau$, which contradicts $\tau \neq 0$.

Now $M_3 \succeq 0$ and *rank* $M_3 = 6$, so

$$\Delta \succeq 0,\ rank\ \Delta = 1. \tag{5.18}$$

Since $C_3^\flat + \Delta = C_3$ and C_3 is Hankel, (5.17) and (5.18) imply that for $p \equiv \Delta_{11} > 0$ and certain scalars q and ϵ, Δ has the form

$$\Delta = \begin{pmatrix} p & qp & q^2p & \epsilon \\ qp & q^2p & q^3p & q\epsilon \\ q^2p & q^3p & q\epsilon & q^2\epsilon \\ \epsilon & q\epsilon & q^2\epsilon & k \end{pmatrix},\ \epsilon \neq q^3p.$$

Since *rank* $\Delta = 1$, $XY^2 = q^2X^3$ in *Col* Δ, whence $q^4p = q\epsilon$. Now, if $q \neq 0$, then $\epsilon = q^3p$, a contradiction. Thus $q = 0$, so the Hankel matrix C_3 is of the form

$$C_3 = \begin{pmatrix} v & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & w \end{pmatrix}.$$

We now have $X^2Y = XY^2 = 0$ in M_3, and we next construct a flat extension M_4. Since $(\ M_3\ B_4\ )$ must be recursively generated in any flat extension M_4, in B_4 we must define $X^3Y = X^2Y^2 = XY^3 = 0$. From this structure, it follows that in B_4, X^4 must be of the form $X^4 = (\ g\ t\ 0\ v\ 0\ 0\ x\ 0\ 0\ 0\ )^T$ (for some x), and Y^4 must have the form $Y^4 = (\ h\ 0\ u\ 0\ 0\ w\ 0\ 0\ 0\ y\ )^T$ (for some y). Since $p > 0$, M_3 has a column basis $\{1, X, Y, X^2, Y^2, X^3\}$. Let $\widetilde{J}$ denote the compression of M_3 to these rows and columns, and let $\widetilde{X}^4 \equiv (\ g\ t\ 0\ v\ 0\ x)^T$. For each x, let $w_x = \widetilde{J}^{-1}\widetilde{X}^4$. For M_4 to be a positive extension of M_3 it is necessary that $X^4 \in Ran\ M_3$, and this is equivalent to the condition that $\psi \equiv \langle \widetilde{Y}^3, w_x\rangle = \langle X^4, Y^3\rangle = 0$, where $\widetilde{Y}^3 \equiv (\ f\ 0\ h\ 0\ u\ 0)^T$. A symbolic calculation shows that

$$\psi = \frac{\tau_2(Q - \tau_1 x)}{det(\widetilde{J})}$$

(where Q is some polynomial in the moments of M_3). Since $\tau_1 \neq 0$, it follows that there exists a unique x $(= \frac{Q}{\tau_1})$ such that $X^4 \in Ran\ M_3$. Further, setting $\widetilde{Y}^4 = (\ h\ 0\ u\ 0\ w\ 0\ )^T$ and $w_y = \widetilde{J}^{-1}\widetilde{Y}^4$, if we define $y := \langle \widetilde{Y}^3, w_y\rangle$, then the resulting Y^4 satisfies $Y^4 \in Ran\ M_3$, as required for positivity of M_4. With these values for the moments of B_4, we now compute $C_4^\flat \equiv (c_{ij})_{1\le i,j\le 5}$ via (5.3). Since $X^3Y = X^2Y^2 = XY^2 = 0$ in $C_4^\flat$, it is easy to see that $C_4^\flat$ is Hankel if and only if

$c_{51} = 0$. A symbolic calculation of c_{51} using (5.3) shows that $c_{51} = \psi$, and by our choice of x, $\psi = 0$, so the existence of a flat extension M_4 is established. $\square$

Proposition 5.9. *Suppose $M_3 \succeq 0$, with $\rho = 6$ and $r = 5$. If M_2 has a column relation $XY = 1$, then M_3 satisfies* (5.8).

Proof. The proof is very similar to that of the previous result, so we omit many of the details. Symbolic calculation shows that either $C_3^\flat$ is Hankel, in which case the result follows, or $C_3^\flat$ is not Hankel because $\tau \equiv (C_3^\flat)_{41} - 1 \neq 0$. In the latter case we proceed to construct a flat extension M_4 as follows. In $\begin{pmatrix} M_2 & B_3 \end{pmatrix}$ we have $X^2Y = X$ and $XY^2 = Y$. As in the previous proof, by examining the structures of $C_3^\flat$ and Δ in detail, we see that $X^2Y = XY^2 = 0$ in Δ, whence $X^2Y = X$ and $XY^2 = Y$ hold in M_3. In M_4 we must then have $X^3Y = X^2$, $X^2Y^2 = 1$, $XY^3 = Y^2$ in $\begin{pmatrix} M_3 & B_4 \end{pmatrix}$ (due to recursiveness), so in B_4 it remains to define $x \equiv \beta_{70}$ and $y \equiv \beta_{07}$. A symbolic calculation shows that there exists x such that $X^4 \in Ran\ M_3$ if and only if an equation of the form $Q - \tau x = 0$ has a solution (for a particular polynomial expression Q in the moments of M_3). Since $\tau \neq 0$, there is a unique solution x. Next, β_{07} can be defined so that $Y^4 \in Ran\ M_3$ exactly as in the proof of the preceding result. Now (5.3) implies that in $Col\ C_4^\flat$ we have $X^3Y = X^2$, $X^2Y^2 = 1$, and $XY^3 = Y^2$, so $C_4^\flat$ is Hankel if and only if $(C_4^\flat)_{51} = 1$. A symbolic calculation now shows that $(C_4^\flat)_{51} - 1 = (Q - \tau x)R$ (for some well-defined rational expression R in the moment data). Since $Q - \tau x = 0$, a flat extension M_4 is established, so the result follows. $\square$

Proposition 5.10. *Suppose $M_3 \succeq 0$, with $\rho = 6$ and $r = 5$. If M_2 has a column relation $Y = X^2$, then M_3 satisfies* (5.8).

Proof. By recursiveness, we have $X^3 = XY$ and $X^2Y = Y^2$ in $\begin{pmatrix} M_2 & B_3 \end{pmatrix}$, and so, by (5.3), these relations also hold in $C_3^\flat$. Thus, X^3 and X^2Y in $C_3^\flat$ are Hankel with respect to each other, and it follows that $C_3^\flat \equiv (c_{ij})_{1 \le i,j \le 4}$ is Hankel if and only if $c_{42} = c_{33}$, in which case we are done. Assuming this is not the case, we will establish a flat extension M_4. Consider $\Delta := C_3 - C_3^\flat$, $rank\ \Delta = 1$, and note that the leftmost two columns of Δ must be Hankel with respect to each other. If $p \equiv \Delta_{11} \neq 0$, then from the form of C_3 and $C_3^\flat$, it follows that Δ is of the form

$$\Delta = \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \delta & w \\ 0 & 0 & w & s \end{pmatrix}, \ \delta > 0,$$

or

$$\Delta = \begin{pmatrix} p & qp & q^2p & q^3p \\ qp & q^2p & q^3p & q^4p \\ q^2p & q^3p & \delta & q^5p \\ q^3p & q^4p & q^5p & q^6p \end{pmatrix}, \ \delta \neq q^4p, \ q \neq 0.$$

In the former case, $rank\ \Delta \geq 2$, a contradiction. In the latter case, since $p \neq 0$ and $rank\ \Delta = 1$, then $Col\ 3 = q^2 Col\ 1$, which implies $\delta = q^4p$, a contradiction. Thus, $p = 0$, so the positivity of Δ and the Hankel property of its leftmost two columns imply that

$$\Delta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \delta & w \\ 0 & 0 & w & s \end{pmatrix}, \ \delta > 0, \ \delta s = w^2.$$

It follows that in $C_3 \equiv C_3^\flat + \Delta$, we have $X^3 = XY$ and $X^2Y = Y^2$. Thus M_3 is positive and recursively generated. Since $\delta > 0$, it follows that a column basis for M_3 is given by $\{1, X, Y, XY, Y^2, XY^2\}$. We thus have a column relation of the form $Y^3 = k_1 1 + k_2 X + k_3 Y + k_4 XY + k_5 Y^2 + k_6 XY^2$. Denoting M_3 as M_d with $d = 3$, we see that M_d is recursively determinate, i.e., $X^n = p(X, Y)$ $(n = 2,\ deg\ p = 1)$ and $Y^m = q(X, Y)$ $(m = 3,\ deg\ q = 3)$. Since M_d is positive, recursively generated, and recursively determinate with $n + m - 2 = d$, it follows from [CF10, Corollary 2.4] that M_3 admits a unique flat extension M_4, whence the result follows. ☐

Proposition 5.11. *Suppose $M_3 \succeq 0$, with $\rho = 6$ and $r = 5$. If M_2 has a column relation $X^2 = 1$, then M_3 satisfies* (5.8).

Proof. The proof is essentially the same as that of the preceding result; replacing $X^2 = Y$ by $X^2 = 1$ has no effect on the argument there that either $C_3^\flat$ is Hankel or M_3 is recursively determinate with $n + m - 2 = d$ (in the sense of [CF10, Corollary 2.4]), whence M_3 admits a flat extension M_4. ☐

Proposition 5.12. *Suppose $M_3 \succeq 0$, with $\rho = 6$ and $r = 5$. If M_2 has a column relation $X^2 + Y^2 = 1$, then M_3 admits a flat extension M_4.*

Proof. Since $\{1, X, Y\}$ is independent in $Col\ M_3$, it follows from [CF5, Proposition 1.12] that $M_3(\beta)$ corresponds to a complex moment matrix $\mathcal{M}_3(\gamma)$ with columns $1, Z, \bar{Z}$ independent and $\bar{Z}Z = 1$ in the column space. [CF5, Theorem 1.1] now implies that $\mathcal{M}_3(\gamma)$ has a flat extension $\mathcal{M}_4$. Under the inverse correspondence of [CF5, Proposition 1.12], $\mathcal{M}_4$ corresponds to a real moment matrix $\widetilde{M}_4$ that is a flat extension of M_3. ☐

We are now prepared to prove Theorem 5.1.

Proof of Theorem 5.1. The "only if" direction is clear from the discussion in Section 1. For the converse, let $r = rank\ M_2$. If $r = 6$, then M_3 is flat, so there is nothing to prove. For $1 \le r \le 3$, the result follows from Proposition 5.2 and Corollary 1.10. Suppose $4 \le r \le 5$. As noted earlier, in these cases recursiveness in M_2 implies that $\{1, X, Y\}$ is independent in $Col\ M_2$. Since M_2 is singular, there is a column dependence relation of the form $AX^2 + BXY + CY^2 + DX + EY + F1 = 0$, with $A^2 + B^2 + C^2 \neq 0$. Let $p(x, y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$ and consider the planar variety $p(x, y) = 0$. As discussed earlier in this section (following Proposition 5.2), there is a degree-one map $\mathcal{T}$ $(\mathcal{T}(x, y) \equiv (\widetilde{x}, \widetilde{y}))$, such that the variety corresponding to $q(\widetilde{x}, \widetilde{y}) := (p \circ \mathcal{T}^{-1})(\widetilde{x}, \widetilde{y})$ is one of the following: $\widetilde{x}\widetilde{y} = 0$, $\widetilde{x}\widetilde{y} = 1$, $\widetilde{y} = \widetilde{x}^2$, $\widetilde{x}^2 = 1$, $\widetilde{x}^2 + \widetilde{y}^2 = 1$. Applying $\mathcal{T}$ to M_3 via Proposition 2.8, it follows from Proposition 2.8 iii), iv), vii) that the resulting moment matrix $\widetilde{M}_3$ is positive, with $rank\ \widetilde{M}_3 = rank\ M_3$, $\{\widetilde{1}, \widetilde{X}, \widetilde{Y}\}$ independent in $Col\ \widetilde{M}_2$, and with one of the following column dependence relations: $\widetilde{XY} = \widetilde{0}$, $\widetilde{XY} = \widetilde{1}$, $\widetilde{Y} = \widetilde{X^2}$, $\widetilde{X^2} = \widetilde{1}$, $\widetilde{X^2} + \widetilde{Y^2} = \widetilde{1}$. It now follows from Propositions 5.3-5.12 that $\widetilde{M}_3$ satisfies (5.8). Proposition 2.8 vi) and x) thus imply that M_3 satisfies (5.8), so the proof is completed by applications of Corollaries 1.10 and 1.11. ☐

The bivariate singular quartic problem was solved in [CF5] (with one exception that we note below). We conclude with a new formulation based on Theorem 5.1.

Proposition 5.13. *For $n = 2$, let $\beta \equiv \beta^{(4)}$ and suppose $M_2(\beta)$ is singular. The following are equivalent:*
i) β admits a representing measure;
ii) M_2 is positive and recursively generated, and rank $M_2 \le$ card $\mathcal{V}(\beta)$;
iii) M_2 admits a positive extension $M_3 \equiv M_3(\widetilde{\beta})$ satisfying rank $M_3 \le 6$.

The equivalence of i) and ii) is proved in [CF5], except that [CF5] neglects to treat the case when M_2 has a column relation equivalent to $X^2 = 1$ under a degree-one map; we treat the $X^2 = 1$ case now.

Lemma 5.14. *Suppose $X^2 = 1$ in Col M_2. Then $\beta \equiv \beta^{(4)}$ has a representing measure if and only if M_2 is positive, recursively generated, and satisfies rank $M_2 \le$ card $\mathcal{V}(\beta)$.*

Proof. The necessity of the conditions is clear. For the converse, if M_1 is singular, the existence of a flat extension follows from [CF3] and from the equivalence of the real and complex truncated moment problems [CF7]. We may thus assume that $M_1 \succ 0$, and we consider next the case when $\{1, X, Y, XY\}$ is a basis for *Col* M_2. Then $Y^2 = q(X, Y)$ for some $q \in \mathbb{R}[x, y]_2$, and since also $X^2 = 1$, it follows that M_2 is recursively determinate with $n + m - 2 = d$ (in the sense of [CF10, Corollary 2.4]), so [CF10] implies the existence of a flat extension M_3. If $\{1, X, Y, Y^2\}$ is a column basis, then there is a hyperbola relation $XY = a1 + bX + cY + dY^2$, so it follows from [CF8] that either M_2 has a flat extension M_3 or M_2 admits a positive, recursively generated extension of rank 5, which in turn has a flat extension M_4.

In the remaining case, we have a column basis $\mathcal{B} \equiv \{1, X, Y, XY, Y^2\}$, and we will construct a flat extension $M_3(\widetilde{\beta})$. Since $X^2 = 1$, we may denote M_2 by

$$M_2 = \begin{pmatrix} 1 & a & b & 1 & c & d \\ a & 1 & c & a & b & f \\ b & c & d & b & f & g \\ 1 & a & b & 1 & c & d \\ c & b & f & c & d & p \\ d & f & g & d & p & q \end{pmatrix}. \tag{5.19}$$

In *Col* $(\,M_2 \;\; B_3\,)$ we must have $X^3 = X$ and $X^2Y = Y$, so B_3 has the form

$$B_3 = \begin{pmatrix} a & b & f & g \\ 1 & c & d & p \\ c & d & p & q \\ a & b & f & g \\ b & f & g & w \\ f & g & w & s \end{pmatrix} \tag{5.20}$$

for certain w, s. Let $J = [M_2]_{\mathcal{B}}$. Since $J \succ 0$, it is clear that $B_3 = M_2 W$ for some W. Let $C_3^\flat \equiv W^T M_2 W$. Since $X^3 = X$ and $X^2Y = Y$ in *Col* $(\,B_3^T \;\; C_3^\flat\,)$, then $C_3^\flat \equiv (c_{ij})_{1 \le i,j \le 4}$ is Hankel if and only if $c_{42} = c_{33}$. A symbolic calculation shows that $\tau \equiv c_{42} - c_{33}$ is of the form $\tau = \frac{\tau_1 \tau_2}{Det\ J}$, where τ_1 and τ_2 are polynomials in the moments of M_2. Further, τ_1 can be expressed as $\tau_1 = \sigma + \rho w$, where σ and ρ are also polynomials in the moments of M_2. Let K denote the compression of M_2 to rows and columns indexed by $1, X, Y, XY$. Then $K \succ 0$, and a symbolic calculation shows that $Det\ K = \rho\rho'$ for some moment polynomial ρ'. Thus, $\rho \neq 0$, and it follows that there is a unique $w \equiv \widetilde{w}$ such that $\tau_1 = 0$. Setting $\widetilde{\beta}_{14} = \widetilde{w}$, then $\tau = 0$ and $C_3^\flat$ is Hankel, so the existence of a flat extension $M_3(\widetilde{\beta})$ is established. □

Proof of Proposition 5.13. We first show that ii) implies iii). If ii) holds and M_1 is singular, then [CF3] and the equivalence of the real and complex truncated moment problems [CF7] together imply that M_2 has a flat extension, so iii) follows in this case. We now assume that $M_1 \succ 0$, so that M_2 has a degree 2 column dependence relation. Note from Proposition 2.8 that all of the properties in i)-iii) (i.e., positivity, rank, variety, existence of measureness, recursiveness, existence of positive extensions with prescribed rank) are invariant under degree-one maps. From Proposition 2.8 and the discussion following Proposition 5.2, we may thus assume that M_2 has a column dependence corresponding to one of the five varieties considered in the proof of Theorem 5.1. In each of these cases, except for $x^2 = 1$, the results of [CF5] show that when ii) holds, then either M_2 admits a flat extension M_3 or M_2 has a positive extension M_3 with $rank\ M_3 \leq 1 + rank\ M_2 \leq 6$, and M_3 admits a flat extension M_4. The proof of Lemma 5.14 establishes the same conclusions in the case when $x^2 = 1$. Thus, ii) implies iii). If iii) holds, then Theorem 5.1 implies that $L_{\widetilde{\beta}}$ is positive, so i) follows from Theorem 1.2. Since i) always implies ii) (cf. Section 2), the proof is complete. □

Acknowledgment

The author wishes to thank Professor Jiawang Nie for an invitation to visit him at the University of California, San Diego, where this work commenced in January 2012. This paper is a sequel to [FN2], and the author benefited from helpful conversations with Professor Nie.

Added in proof

A very general and concrete sufficient condition for positivity of L_β was recently discovered by Grigoriy Blekherman [Bl]. Blekherman's results imply that if $n \geq 1$, $d \geq 3$, and $rank\ M_d \leq 3d - 3$, then L_β is positive. They also imply that Theorem 1.1 cannot be extended to moment problems of higher degree.

References

[A] N. I. Akhiezer, *The classical moment problem and some related questions in analysis*, translated by N. Kemmer, Hafner Publishing Co., New York, 1965. MR0184042 (32 #1518)

[BT] Christian Bayer and Josef Teichmann, *The proof of Tchakaloff's theorem*, Proc. Amer. Math. Soc. **134** (2006), no. 10, 3035–3040 (electronic), DOI 10.1090/S0002-9939-06-08249-9. MR2231629 (2007d:44004)

[B] Sterling K. Berberian, *Lectures in functional analysis and operator theory*, Graduate Texts in Mathematics, No. 15, Springer-Verlag, New York, 1974. MR0417727 (54 #5775)

[Bl] Grigoriy Blekherman, *Positive Gorenstein ideals*, Proc. Amer. Math. Soc. **143** (2015), no. 1, 69–86, DOI 10.1090/S0002-9939-2014-12253-2. MR3272733

[Co] John B. Conway, *A course in functional analysis*, 2nd ed., Graduate Texts in Mathematics, vol. 96, Springer-Verlag, New York, 1990. MR1070713 (91e:46001)

[CF1] Raúl E. Curto and Lawrence A. Fialkow, *Recursiveness, positivity, and truncated moment problems*, Houston J. Math. **17** (1991), no. 4, 603–635. MR1147276 (93a:47016)

[CF2] Raúl E. Curto and Lawrence A. Fialkow, *Solution of the truncated complex moment problem for flat data*, Mem. Amer. Math. Soc. **119** (1996), no. 568, x+52, DOI 10.1090/memo/0568. MR1303090 (96g:47009)

[CF3] Raúl E. Curto and Lawrence A. Fialkow, *Flat extensions of positive moment matrices: relations in analytic or conjugate terms*, Nonselfadjoint operator algebras, operator theory, and related topics, Oper. Theory Adv. Appl., vol. 104, Birkhäuser, Basel, 1998, pp. 59–82. MR1639649 (99i:47026)

[CF4] Raúl E. Curto and Lawrence A. Fialkow, *Flat extensions of positive moment matrices: recursively generated relations*, Mem. Amer. Math. Soc. **136** (1998), no. 648, x+56, DOI 10.1090/memo/0648. MR1445490 (99d:47015)

[CF5] Raúl E. Curto and Lawrence A. Fialkow, *Solution of the singular quartic moment problem*, J. Operator Theory **48** (2002), no. 2, 315–354. MR1938799 (2003j:47017)

[CF6] Raúl E. Curto and Lawrence A. Fialkow, *Solution of the truncated parabolic moment problem*, Integral Equations Operator Theory **50** (2004), no. 2, 169–196, DOI 10.1007/s00020-003-1275-3. MR2099788 (2005f:47041)

[CF7] Raúl E. Curto and Lawrence A. Fialkow, *Truncated K-moment problems in several variables*, J. Operator Theory **54** (2005), no. 1, 189–226. MR2168867 (2006e:47032)

[CF8] Raúl E. Curto and Lawrence A. Fialkow, *Solution of the truncated hyperbolic moment problem*, Integral Equations Operator Theory **52** (2005), no. 2, 181–218, DOI 10.1007/s00020-004-1340-6. MR2216081 (2007a:47017)

[CF9] Raúl E. Curto and Lawrence A. Fialkow, *An analogue of the Riesz-Haviland theorem for the truncated moment problem*, J. Funct. Anal. **255** (2008), no. 10, 2709–2731, DOI 10.1016/j.jfa.2008.09.003. MR2464189 (2009i:47039)

[CF10] Raúl E. Curto and Lawrence A. Fialkow, *Recursively determined representing measures for bivariate truncated moment sequences*, J. Operator Theory **70** (2013), no. 2, 401–436, DOI 10.7900/jot.2011sep06.1943. MR3138363

[EF] Chirakkal Easwaran and Lawrence Fialkow, *Positive linear functionals without representing measures*, Oper. Matrices **5** (2011), no. 3, 425–434, DOI 10.7153/oam-05-30. MR2858497 (2012m:47033)

[F] Lawrence A. Fialkow, *Solution of the truncated moment problem with variety $y = x^3$*, Trans. Amer. Math. Soc. **363** (2011), no. 6, 3133–3165, DOI 10.1090/S0002-9947-2011-05262-1. MR2775801 (2012c:47044)

[FN1] Lawrence Fialkow and Jiawang Nie, *Positivity of Riesz functionals and solutions of quadratic and quartic moment problems*, J. Funct. Anal. **258** (2010), no. 1, 328–356, DOI 10.1016/j.jfa.2009.09.015. MR2557966 (2010j:47017)

[FN2] Lawrence Fialkow and Jiawang Nie, *On the closure of positive flat moment matrices*, J. Operator Theory **69** (2013), no. 1, 257–277, DOI 10.7900/jot.2010may11.1890. MR3029497

[FN3] Lawrence Fialkow and Jiawang Nie, *The truncated moment problem via homogenization and flat extensions*, J. Funct. Anal. **263** (2012), no. 6, 1682–1700, DOI 10.1016/j.jfa.2012.06.004. MR2948227

[H] E. K. Haviland, *On the momentum problem for distribution functions in more than one dimension. II*, Amer. J. Math. **58** (1936), no. 1, 164–168, DOI 10.2307/2371063. MR1507139

[KN] M. G. Kreĭn and A. A. Nudel′man, *The Markov moment problem and extremal problems.* Ideas and problems of P. L. Čebyšev and A. A. Markov and their further development; Translated from the Russian by D. Louvish; Translations of Mathematical Monographs, Vol. 50, American Mathematical Society, Providence, R.I., 1977. MR0458081 (56 #16284)

[HN] J. William Helton and Jiawang Nie, *A semidefinite approach for truncated K-moment problems*, Found. Comput. Math. **12** (2012), no. 6, 851–881, DOI 10.1007/s10208-012-9132-x. MR2989475

[HeLa] Didier Henrion and Jean-Bernard Lasserre, *GloptiPoly: global optimization over polynomials with Matlab and SeDuMi*, ACM Trans. Math. Software **29** (2003), no. 2, 165–194, DOI 10.1145/779359.779363. MR2000881 (2004g:90084)

[Her] Domingo A. Herrero, *Approximation of Hilbert space operators. Vol. 1*, 2nd ed., Pitman Research Notes in Mathematics Series, vol. 224, Longman Scientific & Technical, Harlow, 1989. MR1088255 (91k:47002)

[Las] Jean B. Lasserre, *Global optimization with polynomials and the problem of moments*, SIAM J. Optim. **11** (2000/01), no. 3, 796–817, DOI 10.1137/S1052623400366802. MR1814045 (2002b:90054)

[Lau] Monique Laurent, *Sums of squares, moment matrices and optimization over polynomials*, Emerging applications of algebraic geometry, IMA Vol. Math. Appl., vol. 149, Springer, New York, 2009, pp. 157–270, DOI 10.1007/978-0-387-09686-5_7. MR2500468 (2010j:13054)

[Rez] Bruce Reznick, *Some concrete aspects of Hilbert's 17th Problem*, Real algebraic geometry and ordered structures (Baton Rouge, LA, 1996), Contemp. Math., vol. 253, Amer. Math. Soc., Providence, RI, 2000, pp. 251–272, DOI 10.1090/conm/253/03936. MR1747589 (2001i:11042)

[R] M. Riesz, Sur le problème des moments, Troisième Note, *Arkiv für Matematik, Astronomi och Fysik* **17** (1923), 1-52.

[SH] S. L. Salas and E. Hille, *Calculus: One and several variables*, Fourth Edition, John Wiley and Sons, 1982.

[S1] Konrad Schmüdgen, *An example of a positive polynomial which is not a sum of squares of polynomials. A positive, but not strongly positive functional*, Math. Nachr. **88** (1979), 385–390, DOI 10.1002/mana.19790880130. MR543417 (81b:12024)

[S2] Konrad Schmüdgen, *The K-moment problem for compact semi-algebraic sets*, Math. Ann. **289** (1991), no. 2, 203–206, DOI 10.1007/BF01446568. MR1092173 (92b:44011)

[T] Vladimir Tchakaloff, *Formules de cubatures mécaniques à coefficients non négatifs* (French), Bull. Sci. Math. (2) **81** (1957), 123–134. MR0094632 (20 #1145)

Department of Computer Science, State University of New York, New Paltz, New York 12561

E-mail address: fialkowl@newpaltz.edu