CORRECTION TO
“COMBINATORICS AND GEOMETRY OF POWER IDEALS”:
TWO COUNTEREXAMPLES FOR POWER IDEALS
OF HYPERPLANE ARRANGEMENTS

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Abstract. We disprove Holtz and Ron’s conjecture that the power ideal $C_{A,-2}$ of a hyperplane arrangement $A$ (also called the internal zonotopal space) is generated by $A$-monomials. We also show that, in contrast with the case $k \geq -2$, the Hilbert series of $C_{A,k}$ is not determined by the matroid of $A$ for $k \leq -6$.

Remark. This note is a corrigendum to our article [1], and we follow the notation of that paper.

1. Introduction

Let $A = \{H_1, \ldots, H_n\}$ be a hyperplane arrangement in a vector space $V$; say $H_i = \{x \mid l_i(x) = 0\}$ for some linear functions $l_i \in V^*$. Call a product of (possibly repeated) $l_i$’s an $A$-monomial in the symmetric algebra $\mathbb{C}[V^*]$. Let Lines($A$) be the set of lines of intersection of the hyperplanes in $A$. For each $h \in V$ with $h \neq 0$, let $\rho_A(h)$ be the number of hyperplanes in $A$ not containing $h$. Let $\rho = \rho(A) = \min_{h \in V} (\rho_A(h))$. For all integers $k \geq -(\rho + 1)$, consider the power ideals

$$I_{A,k} := \left\{ h^{\rho_A(h)+k+1} \mid h \in V, h \neq 0 \right\}, \quad I'_{A,k} := \left\{ h^{\rho_A(h)+k+1} \mid h \in \text{Lines}(A) \right\}$$

in the symmetric algebra $\mathbb{C}[V]$. It is convenient to regard the polynomials in $I_{A,k}$ as differential operators, and to consider the space of solutions to the resulting system of differential equations:

$$C_{A,k} = I^\perp_{A,k} := \left\{ f(x) \in \mathbb{C}[V^*] \mid h \left( \frac{\partial}{\partial x} \right)^{\rho_A(h)+k+1} f(x) = 0 \text{ for all } h \neq 0 \right\}$$

which is known as the inverse system of $I_{A,k}$. Define $C'_{A,k}$ similarly. These objects arise naturally in numerical analysis, algebra, geometry, and combinatorics. For references, see [1,3].

One important question is to compute the Hilbert series of these spaces of polynomials, graded by degree, as a function of combinatorial invariants of $A$. Frequently, the answer is expressed in terms of the Tutte polynomial of $A$. This has been done

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successfully in many cases. One strategy used independently by different authors has been to prove the following:

(i) There is a spanning set of $A$-monomials for $C_{A,k}$.
(ii) There is an exact sequence $0 \to C_{A \setminus H,k}(-1) \to C_{A,k} \to C_{A/H,k} \to 0$ of graded vector spaces.
(iii) Therefore, the Hilbert series of $C_{A,k}$ is an evaluation of the Tutte polynomial of $A$.

Here $A \setminus H$ and $A/H$ are the deletion and contraction of $H$, respectively.

For $k \geq -1$, this method works very nicely. Dahmen and Micchelli [2] were the first ones to do this for $C'_{A,-1}$. Postnikov, Shapiro, and Shapiro [3] did it for $C_{A,0}$, while Holtz and Ron [3] did it for $C_{A,0}'$. In [1] we did it for $C_{A,k}$ for all $k \geq -1$, and showed that $C_{A,0}' = C_{A,0}$ and $C_{A,-1}' = C_{A,-1}$.

For $k \leq -3$ this approach does not work in full generality. In [1] we showed that (i) is false in general for $C_{A,k}$, and left (ii) and (iii) open, suggesting the problem of measuring $C_{A,k}$. For $k \leq -6$, (ii) and (iii) are false, as we will show in Propositions 4 and 5, respectively. In fact, we will see that the Hilbert series of $C_{A,k}$ is not even determined by the matroid of $A$.

The intermediate cases are interesting and subtle, and deserve further study; notably the case $k = -2$, which Holtz and Ron call the internal zonotopal space. In [3] they proved (ii) and (iii) and conjectured (i) for $C'_{A,-2}$. In [1, Proposition 4.5.3] – a restatement of Holtz and Ron’s Conjecture 6.1 in [3] – we put forward an incorrect proof of this conjecture; the last sentence of our argument is false. In fact their conjecture is false, as we will see in Proposition 2.

2. The case $k = -2$: Internal zonotopal spaces

Before showing why Holtz and Ron’s conjecture is false, let us point out that the remaining statements about $C_{A,-2}$ that we made in [1] are true. The easiest way to derive them is to prove that $C_{A,-2} = C'_{A,-2}$, and simply note that Holtz and Ron already proved those statements for $C'_{A,-2}$:

Lemma 1. We have $C_{A,k} = C'_{A,k}$ for any $k$ with $-(\rho + 1) \leq k \leq 0$.

Proof. By [1, Theorem 4.17] we have $I_{A,0} = I'_{A,0}$, so it suffices to show that $I_{A,j} = I'_{A,j}$ implies that $I_{A,j-1} = I'_{A,j-1}$ as long as these ideals are defined. If $I_{A,j} = I'_{A,j}$, then for any $h \in V \setminus \{0\}$ we have $h^{\rho_A(h)+j+1} = \sum f_i h_i^{\rho_A(h_i)+j+1}$ for some polynomials $f_i$, where the $h_i$’s are the lines of the arrangement. As long as the exponents are positive, taking partial derivatives in the direction of $h$ gives $h^{\rho_A(h)+j} = \sum g_i h_i^{\rho_A(h_i)+j}$ for some polynomials $g_i$. □

The following result shows that (i) does not hold for $C_{A,-2}$.

Proposition 2. [3, Conjecture 6.1] is false: The “internal zonotopal space” $C_{A,-2}$ is not necessarily spanned by $A$-monomials.

Proof. Let $H$ be the hyperplane arrangement in $\mathbb{C}^4$ determined by the linear forms $y_1, y_2, y_3, y_1 - y_4, y_2 - y_4, y_3 - y_4$. We have

$$I'_{H,-2} = \langle x_1^1, x_2^1, x_3^1, (\epsilon_1 x_1 + \epsilon_2 x_2 + \epsilon_3 x_4 + x_4)^2 \rangle = \langle x_1, x_2, x_3, x_4^2 \rangle$$
as $\epsilon_1, \epsilon_2, \epsilon_3$ range over $\{0,1\}$. The other generators of $I_{H,-2}$ are of degree at least 3, and are therefore in $I_{H,-2}'$ already, so

$$I_{H,-2} = \langle x_1, x_2, x_3, x_4^2 \rangle, \quad C_{H,-2} = \text{span}(1, y_4).$$

Therefore $C_{H,-2}$ is not spanned by $H$-monomials. \hfill \Box

As Holtz and Ron pointed out, if [3, Conjecture 6.1] had been true, it would have implied [3, Conjecture 1.8], an interesting spline-theoretic interpretation of

for this condition is trivial unless \{coplanar, and 0 otherwise. However, the matroid of \A h when \A is unimodular. The arrangement above is unimodular, but it does not provide a counterexample to [3, Conjecture 1.8]. In fact, Matthias Lenz [4] has recently put forward a proof of this weaker conjecture.

3. The case \(k \leq -6\)

In this section we show that when \(k \leq -6\), the Hilbert series of \(C_{\A,k}\) is not a function of the Tutte polynomial of \A. In fact, it is not even determined by the matroid of \A. Recall that $\rho = \rho(\A) := \min_{h \in V} (\rho_\A(h))$. Say $h \in V$ is large if it is on the maximum number of hyperplanes, so $\rho_\A(h) = \rho$.

**Lemma 3.** The degree 1 component of $C_{\A,-\rho}$ is

$$(C_{\A,-\rho})_1 = (\text{span}\{h \in V : h \text{ is large}\})^\perp$$
in $V^*$.

**Proof.** An element $f$ of $C_{\A,-\rho}$ needs to satisfy the differential equations

$$h(\partial/\partial x)^{\rho_\A(h)-\rho+1} f(x) = 0$$
for all non-zero vectors $h \in V$. If $f$ is linear, then this condition is trivial unless $h$ is large; and in that case it says that $f \perp h$. \hfill \Box

**Proposition 4.** For $k \leq -6$, the Hilbert series of $C_{\A,k}$ is not determined by the matroid of $\A$.

**Proof.** First assume $k = -2m$. Let $L_1, L_2, L_3$ be three different lines through 0 in $\mathbb{C}^3$ and consider an arrangement $\A$ of 3m (hyper)planes consisting of $m$ generically chosen planes $H_{i1}, \ldots, H_{im}$ passing through $L_i$ for $i = 1, 2, 3$. Then $\rho = 2m$ and the only large lines are $L_1, L_2, \text{ and } L_3$. Therefore $\dim(C_{\A,-2m})_1$ equals 1 if $L_1, L_2, L_3$ are coplanar, and 0 otherwise. However, the matroid of $\A$ does not know whether $L_1, L_2, L_3$ are coplanar.

More precisely, consider two versions $\A_1$ and $\A_2$ of the above construction; in $\A_1$ the lines $L_1, L_2, L_3$ are coplanar, and in $\A_2$ they are not. Notice that $\A_1$ and $\A_2$ have the same matroid: the rank 3 matroid whose non-bases are the triples $\{H_{ia}, H_{ib}, H_{ic}\}$ for $1 \leq i \leq 3$ and $1 \leq a < b < c \leq m$. However, $\dim(C_{\A_1,-2m})_1 \neq \dim(C_{\A_2,-2m})_1$.

The case $k = -2m-1$ is similar. It suffices to add a generic plane to the previous arrangements. \hfill \Box

**Proposition 5.** For $k \leq -6$, the sequence of graded vector spaces

$$0 \to C_{\A\setminus H,k}(-1) \to C_{\A,k} \to C_{\A/H,k} \to 0$$
of [1, Proposition 4.4.1] is not necessarily exact, even if $H$ is neither a loop nor a coloop.
Proof. We will not need to recall the maps that define this sequence; we will simply show an example where right exactness is impossible because \( \dim(C_{A,k})_1 = 0 \) and \( \dim(C_{A/k,k})_1 = 1 \). We do this in the case \( k = -2m \); the other one is similar.

Consider the arrangement \( A = A_2 \) of the proof of Proposition 4 and the plane \( H = H_{11} \). We have \( \dim(C_{A,-2m})_1 = 0 \). In the contraction \( A/H \), the planes \( H_{12}, \ldots, H_{1m} \) become the same line \( L_1 \) in \( H \), while the other \( 2m \) planes of \( A \) become generic lines in \( H \). Therefore \( \rho(A/H) = 2m \) and \( (C_{A/H,-2m})_1 = L_1^\perp \) in \( H^* \), which is one-dimensional. \( \square \)

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References