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WELL-POSEDNESS OF GENERAL BOUNDARY-VALUE PROBLEMS FOR SCALAR CONSERVATION LAWS

BORIS ANDREIANOV AND KARIMA SBIHI

Abstract. In this paper we investigate well-posedness for the problem $u_t + \operatorname{div}\varphi(u) = f$ on $(0,T)\times\Omega$, $\Omega \subset \mathbb{R}^N$, with initial condition $u(0,\cdot) = u_0$ on Ω and with general dissipative boundary conditions $\varphi(u)\cdot\nu \in \beta_{(t,x)}(u)$ on $(0,T)\times\partial\Omega$. Here for a.e. $(t,x) \in (0,T)\times\partial\Omega$, $\beta_{(t,x)}(\cdot)$ is a maximal monotone graph on $\mathbb{R}$. This includes, as particular cases, Dirichlet, Neumann, Robin, obstacle boundary conditions and their piecewise combinations.

As for the well-studied case of the Dirichlet condition, one has to interpret the *formal boundary condition* given by β by replacing it with the adequate *effective boundary condition*. Such effective condition can be obtained through a study of the boundary layer appearing in approximation processes such as the vanishing viscosity approximation. We claim that the formal boundary condition given by β should be interpreted as the effective boundary condition given by another monotone graph $\tilde\beta$, which is defined from β by the projection procedure we describe. We give several equivalent definitions of entropy solutions associated with $\tilde\beta$ (and thus also with β).

For the notion of solution defined in this way, we prove existence, uniqueness and L^1 contraction, monotone and continuous dependence on the graph β. Convergence of approximation procedures and stability of the notion of entropy solution are illustrated by several results.

1. Introduction

While there exists extensive literature on the Cauchy and Cauchy-Dirichlet problems for scalar conservation law $u_t + \operatorname{div}\varphi(u) = 0$, other initial-boundary value problems have received very little attention. The purpose of this paper is to define a notion of entropy solution for a wide class of boundary conditions that we call dissipative boundary conditions; to justify this definition through convergence of natural approximation procedures; and to establish well-posedness results for the so defined entropy solutions.

1.1. Dissipative boundary conditions for conservation laws.

Let Ω be an open domain in $\mathbb{R}^N$ with Lipschitz boundary, $N \geq 1$, and $T > 0$. We consider the following initial-boundary value problem for a scalar conservation law:

$$(H_{\varphi,\beta}(u_0,f)) \qquad \begin{cases} u_t + \operatorname{div}\varphi(u) = f & \text{in } Q_T := (0,T)\times\Omega, \\ u|_{t=0} = u_0 & \text{in } \Omega, \\ \varphi_{\nu(x)}(u) := \varphi(u)\cdot\nu(x) \in \beta_{(t,x)}(u) & \text{on } \Sigma := (0,T)\times\partial\Omega. \end{cases}$$

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Here $\varphi : \mathbb{R} \longrightarrow \mathbb{R}^N$ is a continuous function (for the sake of simplicity, the reader may assume that φ is Lipschitz continuous, although most of our results hold without this assumption);[1] $u_0 \in L^\infty(\Omega)$; and f is a measurable function on Q_T with $\int_0^T \|f(t,\cdot)\|_{L^\infty(\Omega)} < \infty$.

Further, in $(H_{\varphi,\beta}(u_0, f))$, the unit outward normal vector on $\partial\Omega$ is denoted by ν, and the boundary condition is prescribed (formally) in terms of β that is a map from Σ to the set $\mathbb{B}$ of all *maximal monotone graphs* on $\mathbb{R}$. Clearly, some measurability assumption is needed on the map $\beta : (t,x) \in \Sigma \mapsto \beta_{(t,x)} \in \mathbb{B}$. In the sequel, we always extend $\beta_{(t,x)}$ to a maximal monotone graph from $\overline{\mathbb{R}}$ to $\overline{\mathbb{R}}$ and require the following:

$$\begin{array}{l}\text{for all } k \in \mathbb{R},\ (t,x) \mapsto \inf \beta_{(t,x)}(k) \text{ and } (t,x) \mapsto \sup \beta_{(t,x)}(k)\\ \text{are measurable } \overline{\mathbb{R}}\text{-valued functions w.r.t. the Hausdorff measure on } \Sigma.\end{array} \tag{1.1}$$

This encompasses different classical boundary conditions. For instance, the graph $\beta_{(t,x)} = \{u^D(t,x)\} \times \mathbb{R}$ prescribes the Dirichlet boundary condition "$u = u^D$ on Σ"; the graph $\beta_{(t,x)} := \mathbb{R} \times \{-g(t,x)\}$ prescribes the condition "$-\varphi(u) \cdot \nu(x) = g$" that we will call the *Neumann condition*, by analogy with the Neumann boundary conditions for the general convection-diffusion problems of the kind $u_t - \operatorname{div} a(u,\nabla u) = f$. It is also easy to include the more general conditions of the kind $\lambda u + (1-\lambda)(-\varphi(u) \cdot \nu) = g$, $\lambda \in (0,1)$, conditions that interpolate between the Dirichlet and the Neumann ones (these are known as *Robin conditions* in the convection-diffusion context). To give one more example, the (bilateral) *obstacle boundary conditions* "$u^m \le u \le u^M$ on Σ" correspond to the graph

$$\beta_{(t,x)} = \Big(\{u^m(t,x)\} \times \mathbb{R}^-\Big) \cup \Big([u^m(t,x), u^M(t,x)] \times \{0\}\Big) \cup \Big(\{u^M(t,x)\} \times \mathbb{R}^+\Big).$$

For the sake of simplicity, the reader may consider

$$\begin{array}{l}\beta_{(t,x)}(r) = \beta^0_{(t,x)}(r - u^D(t,x)) - g(t,x) \text{ with } u^D \in L^\infty(\Sigma),\ g \in L^\infty(\Sigma)\\ \text{and with a maximal monotone graph } \beta^0_{(t,x)} \text{ such that } \beta^0_{(t,x)}(0) \ni 0;\end{array} \tag{1.2}$$

this contains the aforementioned cases and, e.g., the case of *mixed Dirichlet-Neumann boundary conditions*.

In the context of parabolic problems $u_t - \operatorname{div} a(u,\nabla u) = f$, it is well known that the boundary conditions of the kind $\beta_{(t,x)}(u) + a(u,\nabla u) \cdot \nu(x) \ni 0$ lead to the L^1 *contraction property* (see, e.g., [38] for a study of the associated stationary elliptic problem; see also [3]); that's why we call these conditions *dissipative boundary conditions*. It is customary to interpret the physically admissible weak solutions (called *entropy solutions* since the founding work [21] of Kruzhkov) of a scalar conservation law as limits of the *vanishing viscosity approximation* that, in our case, would take the form

$$\left\{\begin{array}{l} u^\varepsilon_t - \operatorname{div}(-\varphi(u^\varepsilon) + \varepsilon\nabla u^\varepsilon) = f, \quad u^\varepsilon|_{t=0} = u_0,\\ \Big(\beta_{(t,x)}(u^\varepsilon) + (-\varphi(u^\varepsilon) + \varepsilon\nabla u^\varepsilon) \cdot \nu(x)\Big)|_{(t,x)\in\Sigma} \ni 0. \end{array}\right. \tag{1.3}$$

Then it is clear that the boundary condition in $(H_{\varphi,\beta}(u_0, f))$ is the *formal* limit of the dissipative boundary condition $\beta_{(t,x)}(u^\varepsilon) + (-\varphi(u^\varepsilon) + \varepsilon\nabla u^\varepsilon) \cdot \nu(x) \ni 0$ in (1.3)

[1] Note that the the results of the present paper can be easily extended to the case of x-dependent φ (and x-dependent β) in one space dimension, using the non-linear semigroup theory. We refer to [5] for this extension and for a brief summary of the present paper, with ideas and results presented in a technically simplified setting.

(here we should assume some regularity of $\beta_{(t,x)}$ in (t,x) in order for a solution u^ε to exist; for instance, for the Dirichlet BC case we need $u^D \in L^2(0,T;H^{-1/2}(\partial\Omega))$). Moreover let u^ε, $\hat{u}^\varepsilon$ be solutions of problem (1.3) with the same dissipative boundary condition and with data u_0, f and $\hat{u}_0, \hat{f}$, respectively. The L^1 contraction property holds under rather weak restrictions on Ω and φ (see, e.g., [6, 26]):

$$\|u^\varepsilon(t,\cdot) - \hat{u}^\varepsilon(t,\cdot)\|_{L^1(\Omega)} \le \|u_0 - \hat{u}_0\|_{L^1(\Omega)} + \|f - \hat{f}\|_{L^1(\Omega)}.$$

Provided the $L^1(Q_T)$ compactness of the sequences $(u^\varepsilon)_\varepsilon$, $(\hat{u}^\varepsilon)_\varepsilon$ with $\varepsilon \to 0$ is known, it is inherited at the limit $\varepsilon \to 0$. Therefore we expect that the boundary condition satisfied at the limit is also a dissipative one.

But what is this limit boundary condition as $\varepsilon \to 0$ in (1.3)? The compactness of $(u^\varepsilon)_\varepsilon$ in $L^1(Q_T)$ gives no information on convergence of u^ε on the boundary. The term $\varepsilon\nabla u^\varepsilon \cdot \nu(x)$ on the boundary becomes singular as $\varepsilon \to 0$; therefore passage to the limit in boundary conditions is by no means straightforward. As a matter of fact, in general,

the boundary condition "$\varphi(u)\cdot\nu(x) \in \beta_{(t,x)}(u)$" is not the correct limit obtained from the boundary conditions $\beta_{(t,x)}(u^\varepsilon) + (-\varphi(u^\varepsilon) + \varepsilon\nabla u^\varepsilon)\cdot\nu(x) \ni 0$.

The Dirichlet condition case discussed below is a well-known illustration of this fact.

1.2. **Classical results on the Dirichlet case.** Within the whole variety of dissipative boundary conditions, only the Dirichlet case received much attention in the framework of conservation laws. The celebrated result of Bardos, LeRoux and Nédélec [10] states that the Dirichlet condition "$u = u^D$ on Σ" should be seen as a *formal condition* and that *it must be interpreted* by stating that the trace $(\gamma u)(t,x)$ of u at a point $(t,x) \in \Sigma$ belongs to the subset $I(t,x)$ of $\mathbb{R}$ defined in terms of $u^D(t,x)$ and of the function $r \mapsto \varphi_{\nu(x)}(r) = \varphi(r)\cdot\nu(x)$ as follows:

(1.4)
$$I(t,x) = \Big\{ z \in \mathbb{R} \;\Big|\; \operatorname{sign}(z - u^D(t,x))(\varphi_{\nu(x)}(z) - \varphi_{\nu(x)}(k)) \ge 0 \quad \forall k \in [u^D(t,x) \wedge z, u^D(t,x) \vee z] \Big\}.$$

Here and in the sequel, $\wedge$ (respectively, $\vee$) denotes the min (resp., the max) operation. We denote by $\mathcal{H}^N$ the $N-$dimensional Hausdorff measure on Σ.

The *effective boundary condition*

$$(\gamma u)(t,x) \in I(t,x) \;\; \mathcal{H}^N\text{-a.e. on } \Sigma \tag{1.5}$$

is known as the *BLN condition*; in this paper, we will use the reformulation of the BLN condition in terms of a *maximal monotone (sub)graph.* Such graph interpretation was first made explicit, for the Dirichlet case, by Dubois and LeFloch in [18] (see in particular [18, Fig. 1.1]). Another useful interpretation of the BLN condition going back to [18] is the following:

$$I(t,x) = \Big\{ z \in \mathbb{R} \,\Big|\, \varphi_{\nu(x)}(z) = \mathrm{God}[\varphi_{\nu(x)}](z, u^D(t,x)) \Big\},$$

where $\mathrm{God}[\psi] : \mathbb{R}^2 \to \mathbb{R}$ is the Godunov numerical flux associated to a given scalar flux $\psi : \mathbb{R} \to \mathbb{R}$. Recall that the Godunov flux is given by the expression

$$\mathrm{God}[\psi](a,b) = \begin{cases} \min_{z\in[a,b]} \psi(z), & \text{if } a \le b, \\ \max_{z\in[b,a]} \psi(z), & \text{if } b \le a. \end{cases} \tag{1.6}$$

The functional framework of the paper [10] is the space $L^\infty(0,T;BV(\Omega))$ (actually, the solutions belong to the space $BV(Q_T)$). There are two good reasons for that. Firstly, the BV in space regularity of u guarantees the existence of a trace γu of u on Σ, necessary in order to give sense to the BLN condition. Secondly, uniform in ε BV estimates on the solutions of the approximating problems (1.3) are available, for BV data u_0 and u^D and for Lipschitz continuous flux function φ. Bardos, LeRoux and Nédélec show that for the above data and flux, there exists a unique $L^\infty(0,T;BV(\Omega))$ *entropy solution* of the conservation law satisfying (pointwise on Σ) the BLN boundary condition and that this solution is the limit of the vanishing viscosity approximation.

More recently, Otto in [27, 28] (see also [25]) provided a formulation suitable for merely L^∞ data u_0 and u^D; Porretta, Vovelle [35] and Ammar, Carrillo and Wittbold [2] extended the definition and results to the framework of L^1 data (see the papers for the precise assumptions on u^D) and merely continuous flux function φ, in a bounded domain Ω. The L^1 framework requires an appropriate notion of solution; in [2, 35] the notion of *renormalized solution* from [11] was used. In the Otto formulation, existence of a *(strong) boundary trace* γu of u on Σ is not assumed; a BLN kind of condition is reformulated in terms of *weak normal boundary traces* of $\varphi(u)$ and of the associated *boundary entropy fluxes* $\mathcal{F}(u;u^D,k)$ (the existence of the weak traces is a relatively simple consequence of the fact that u is a Kruzhkov entropy solution of the scalar conservation law inside $(0,T)\times\Omega$). We refer to [25, 27, 28] and to [35, 39, 41] for details and results related to the approach of Otto.

1.3. **Strong traces of entropy solutions on the boundary.** Although the definition of [27, 28] and the aforementioned generalizations were a remarkable step forward in the study of boundary value problems for conservation laws, it was possible to bypass the use of weak traces and the associated boundary entropies' techniques of [27, 28]. Indeed, for the sake of simplicity let us start with the following flux non-degeneracy assumption:

$$\forall \xi\in\mathbb{R}^N\setminus\{0\}\ \forall c\in\mathbb{R}\ \ \text{the Lebesque measure of the set } \{z\,|\,\xi\cdot\varphi(z)=c\}\ \text{ is zero.} \tag{1.7}$$

Using the approach of *kinetic solutions* (see [24, 34]), Vasseur in [40] has shown that for φ regular enough,

$$\begin{aligned}&\text{under (1.7), any } L^\infty \text{ Kruzhkov entropy solution in } Q_T\\ &\text{admits a \textit{strong trace} } \gamma u \text{ on } \Sigma.\end{aligned} \tag{1.8}$$

The *non-degeneracy assumption* (1.7) on φ is typical for the "compactification properties" in the theory of kinetic solutions; see Perthame [34] and references therein. As pointed out by Vasseur, (1.8) gives sense to the pointwise BLN condition (1.5) for *general* L^∞ entropy solutions, and not only for solutions corresponding to BV data; thus the weak trace technique of Otto [27, 28] is bypassed (yet for general (t,x)-dependent flux φ, the approach of [27, 28] remains the most powerful; see in particular the results of Vallet [39]). Further results in the spirit of (1.8) were obtained by Kwon and Vasseur [23] for the case $N=1$ (see also [7, 37], where we treat the case of a flat boundary using a hint due to Panov). To the authors' knowledge, the strongest generalization of (1.8) is the result of Panov [32] obtained using the technique of *parametrized families of H-measures* (see also [30,33]); Panov drops all regularity assumptions on φ, and, in a sense, he also drops non-degeneracy assumptions of the kind (1.7). Because of its importance for our paper, we should make the latter statement more precise:

- (upon rotating axes and localizing around a point x^* of the boundary) the boundary $\partial\Omega$ is represented by the graph of a Lipschitz[2] function g on W, i.e.,
$$\partial\Omega \cap U = \{(g(x'), x') \mid x' \in W\},$$
$$\Omega \cap U = \{(x_0, x') \mid x_0 = y + g(x'),\ x' \in W, y \in (0, h)\}$$
for some neighbourhood U of x^*, some neighbourhood V of zero in R^{N-1}, and some $h > 0$; further, the unit exterior normal field $\Big(\nu(g(x'), x')\Big)_{x' \in W}$ is lifted inside $\Omega \cap U$ by the formula $\nu(x_0, x') = \frac{1}{\sqrt{1+|\nabla g(x')|^2}}\Big(-1, \nabla g(x')\Big)$ (the field is constant in $x_0 \in [0, h)$);
- for $x \in \partial\Omega \cap U$, consider the *singular mapping* $V\varphi_{\nu(x)} : r \mapsto \int_0^r |\varphi'(s) \cdot \nu(x)|\, ds$ on $\mathbb{R}$ (notice that the mapping is independent of x_0, and it depends on x' continuously);
- then for any $u \in L^\infty(Q_T)$ that is a Kruzhkov entropy solution in Q_T, there exists

(1.9)
$$\operatorname{ess\,lim}_{y \downarrow 0} V\varphi_{\nu(x)}\Big(u(t, y + g(x'), x')\Big) =: \Big(\gamma V\varphi_{\nu(x)}(u)\Big)(t, x) \quad \text{in } L^1((0,T) \times W),$$
where $x := (g(x'), x')$ is a generic point of $U \cap \partial\Omega$; recall that $\nu(y + g(x'), x') \equiv \nu(g(x'), x')$.

Statement (1.9) is actually a reinterpretation of the localization property that appears in the proof [32, p. 571] of Panov; we use it to give a sense to pointwise formulations of boundary conditions, in the same vein as Vasseur in [40]. If for all $\xi \in \mathbb{R}^N \setminus \{0\}$ the function $r \mapsto \varphi(r) \cdot \xi$ is non-constant on any interval (this is a weaker version of (1.7) typical for the technique of parametrized H-measures; see [30, 32, 33]), then $V\varphi_{\nu(x_0)}$ is an invertible function (which means that strong trace γu exists). If φ is not a BV function, one can use another singular mapping instead of the map $r \mapsto \int_0^r |\varphi'(z)\cdot\nu(x_0)|\, dz$ (which is not well defined), e.g.,

$$V\varphi_{\nu(x)}(r) = \int_0^r \mathbb{1}_F(s)\, ds, \qquad \begin{array}{l} F \text{ being the union of all the intervals} \\ \text{where the map } s \mapsto \varphi(s) \cdot \nu(x) \text{ does not vary.} \end{array}$$

Remark 1.1. By the definition of the singular mapping, $V\varphi_{\nu(x)}(\cdot)$ has the properties of being monotone non-decreasing and of being constant on the same intervals where $\varphi_{\nu(x)}(\cdot)$ is constant. Therefore $\varphi(r) \cdot \nu(x) = \Phi_{\nu(x)} \circ V\varphi_{\nu(x)}$ with some continuous function $\Phi_{\nu(x)} : \mathbb{R} \to \mathbb{R}$. As a consequence of (1.9), there exists the strong trace $\gamma\varphi(u) \cdot \nu(x)$ (with the same meaning as in (1.9)) which is equal to $\Phi_{\nu(x)}\Big(\gamma V\varphi_{\nu(x)}(u)\Big)$.

In the same way, one can represent the projections on the direction $\nu(x)$ of the semi-Kruzhkov entropy fluxes

$$q^{\pm}(u, k) := \operatorname{sign}^{\pm}(u - k)\Big(\varphi(u) - \varphi(k)\Big) \tag{1.10}$$

[2]While the setting of Panov [32] is C^1 regular domains, the author indicates that the generalization to Lipschitz and, more generally, Lipschitz deformable boundaries in the sense of [17] is straightforward.

with the help of continuous functions $Q^{\pm}_{\nu(x)}(\cdot,\cdot)$ of two variables:

$$q^{\pm}(u,k)\cdot\nu(x) = Q^{\pm}_{\nu(x)}\Big(V\varphi_{\nu(x)}(u)\,,\, V\varphi_{\nu(x)}(k)\Big). \tag{1.11}$$

Hence for a couple $u, \hat{u}$ of entropy solutions, it follows that a strong trace of $q^{\pm}(u,\hat{u})\cdot\nu(x)$ exists and can be represented as $Q^{\pm}_{\nu(x)}\Big(\gamma V\varphi_{\nu(x)}(u)\,,\,\gamma V\varphi_{\nu(x)}(\hat{u})\Big)$. The same is true for the Kruzhkov fluxes:

$$\begin{array}{l} q(u,k)\cdot\nu(x) = Q_{\nu(x)}\Big(V\varphi_{\nu(x)}(u)\,,\, V\varphi_{\nu(x)}(k)\Big) \\ \text{with } \ q := q^+ + q^-,\ \ Q_{\nu(x)} := Q^+_{\nu(x)} + Q^-_{\nu(x)}. \end{array} \tag{1.12}$$

1.4. **Interpretation of a general boundary condition.** The Bardos-LeRoux-Nédélec condition (1.4), (1.5) is generally recognized as the correct *interpretation* of the Dirichlet boundary condition; this is justified in particular by convergence of vanishing viscosity or numerical approximations of the boundary value problem (see Vovelle [41]), considered as quite natural approximations. Observations of *viscous or numerical boundary layers* explain how the formal boundary condition $u = u^D$ on Σ transforms into the effective boundary condition (1.4), (1.5).

The strong trace result of [40] was used by Bürger, Frid and Karlsen in [13] in order to give sense to the formal *zero-flux boundary condition* (in our terminology, this is the Neumann boundary condition with $g \equiv 0$) in the particular but important case $\varphi(0) = 0 = \varphi(1)$. Under this assumption and for $[0,1]$-valued initial data, the zero-flux boundary condition for $u_t + \operatorname{div}\varphi(u) = 0$ can be understood *literally* (see [13]) (in the sense that the problem is well-posed and solutions are limits of the vanishing viscosity approximation).

Let us stress that in general, also for the zero-flux boundary condition "$\varphi(u)\cdot\nu = 0$" a boundary layer would form in approximate solutions, and this *formal zero-flux boundary condition* would transform into some different effective boundary condition. For a simple example, consider the zero-flux problem for the transport equation $u_t + u_x = 0$ on $[0,1]$; as in [10], arguing along characteristics one sees that the zero-flux condition (that reads "$u = 0$" because $\varphi = Id$) at the right boundary $x = 1$ must be merely dropped.

It is the purpose of this paper to provide a natural interpretation for a general dissipative boundary condition (formally given by a family β of maximal monotone graphs $\beta_{(t,x)}(\cdot)$) under the form of an effective boundary condition. Most generally, this effective boundary condition can be written under the form

$$\begin{array}{l} \mathcal{H}^N\text{-a.e. on } \Sigma, \text{ the couple } \Big(\gamma V\varphi_\nu u, \varphi(\gamma V\varphi_\nu u)\cdot\nu\Big) \\ \text{lies in the graph } \tilde{\beta}_{(t,x)}(\cdot)\circ\Big(V\varphi_\nu\Big)^{-1}, \end{array} \tag{1.13}$$

with $\tilde{\beta}$ to be defined and with the notation $\gamma V\varphi_\nu u := \Big(\gamma V\varphi_{\nu(x)}(u)\Big)(t,x)$.

To clarify the essence of the condition (1.13), consider the case where $V\varphi_{\nu(x)} = Id$ can be taken (recall that this is the case if (1.7) holds). Then (1.13) means that $(\gamma u)(t,x) \in Dom\,\tilde{\beta}_{(t,x)}(\cdot)$, and from the definition of $\tilde{\beta}$ in Section 2 we will see that this automatically includes the equality $\tilde{\beta}_{(t,x)}(\gamma u(t,x)) = \varphi(\gamma u(t,x))\cdot\nu$. Thus the condition "$\varphi(u)\cdot\nu(x) = \tilde{\beta}_{(t,x)}(u)$ on Σ" can be understood *literally* as a pointwise equality; this is why we call it an effective boundary condition. Notice that condition (1.13) takes the form "$(\gamma V\varphi_\nu u)(t,x) \in V\varphi_\nu(I(t,x))$ a.e. on Σ"; i.e., it prescribes some set $I(t,x)$ of possible trace values of u on the boundary. Recall that the BLN condition (1.4) has the same form.

The effective BC graph $\tilde\beta_{(t,x)}$ featured in (1.13) will be characterized in Section 2 as:

$(A_{\tilde\beta})$ *$\tilde\beta_{(t,x)}$ is the "closest" to the $\beta_{(t,x)}$ maximal monotone subgraph of the graph $\Big\{(r,\varphi(r)\cdot\nu(x))\,|\,r\in\mathbb{R}\Big\}$ that contains all the points of crossing of $\beta_{(t,x)}$ with the graph of the function $\varphi_{\nu(x)}=\varphi\cdot\nu(x)$.*

For simplicity, let us look at the case where φ is a C^1 function; then the monotonicity of $\tilde\beta_{(t,x)}$ means that the domain of the graph contains either isolated points $r\in\mathbb{R}$ such that $\varphi(r)\cdot\nu(x)\in\beta_{(t,x)}(r)$ or intervals where $\varphi'(\cdot)\cdot\nu(x)\ge 0$. Therefore heuristically, (1.13) can be understood as follows (assume for simplicity that γu exists):

$(B_{\tilde\beta})$

Fix a point $(t,x)\in\Sigma$; denote $\tilde u:=(\gamma u)(t,x)$, $\varphi_\nu:=\varphi_{\nu(x)}$, and $\beta_{(t,x)}=\beta$. Then

- *either the boundary condition is satisfied literally in the sense that $(\tilde u,\varphi_\nu(\tilde u))\in\beta$;*
- *or $\varphi'_\nu(\tilde u)\ge 0$; i.e., the characteristics at the point (t,x) associated with the expression $u_t+\operatorname{div}\varphi(u)$ exits the domain (in which case it is natural to ignore the boundary condition). In the latter case, the flux $\varphi_\nu(\tilde u)$ is as close to $\beta(\tilde u)$ as possible.*

We also point out in Remark 2.7 a useful characterization of the effective BC graph $\tilde\beta_{(t,x)}$ in terms of $\beta_{(t,x)}$ and of the Godunov numerical flux (1.6) associated with the scalar flux function $\varphi_{\nu(x)}$.

In view of the description $(B_{\tilde\beta})$ of $\tilde\beta$, the interpretation of the formal BC "$\varphi_\nu(\tilde u)\in\beta(\tilde u)$" as "$\varphi_\nu(\tilde u)=\tilde\beta(\tilde u)$" can appear as a rather natural one. Yet the only convincing justification we can think of would be in terms of approximation. Namely, we should use the formal boundary condition given by β on one of the approximation schemes that are well-established in the context of conservation laws (such as the vanishing viscosity approximation or approximation with a monotone consistent finite volume scheme), pass to the limit in the sequence of the approximate solutions, and identify the boundary condition satisfied at the limit. If this can be achieved only for some restricted class of "regular" data u_0,f, graphs β or fluxes φ, then a further justification can be provided by a passage to the limit from the "regular" problem (where the correct BC is already identified) to the general problem.

1.5. **Former results and a summary of the paper.** Beyond the Cauchy-Dirichlet problem described in Section 1.2 and the "simple case" of the zero-flux problem treated in [13], we are not aware of works on initial-boundary value problems for conservation laws.

The present paper develops the approach initiated in the thesis [37] of K. Sbihi; see [7,8]. The graph $\tilde\beta$ (in a different but equivalent representation; see Section 2) was introduced in [7,37]. The passage from β to $\tilde\beta$ was justified in [7,37] in the case of a flat boundary of non-degenerate, in the sense of (1.7), flux φ and for quickly growing at infinity (t,x)-independent graph β. A combination of the vanishing viscosity method and non-linear semigroup methods was used in this argument.

Notice that the technique of [7, 37] is rather restrictive because it is based upon a strong compactness on the boundary of the sequence of approximate solutions. In [8], the definition of $\tilde{\beta}$ was further supported through an argument of monotone dependence on β; a notion of *measure-valued* (or *entropy-process*) solution was introduced in order to simplify the convergence analysis for different approximation methods.

Let us give an outline of the paper. Proposition 3.3 and Theorems 4.1 and 5.2 are its main results.

In Section 2, we discuss in detail the properties and different characterizations of the projected graph $\tilde{\beta}$; this long section can be omitted by a reader convinced by the heuristic arguments of Section 1.4 and not interested in details of some proofs. In Section 3 we provide several equivalent definitions of entropy solutions, sub- and super-solutions for the formal problem $(H_{\varphi,\beta}(u_0, f))$. These definitions lead, in a rather straightforward way, to uniqueness, comparison and continuous dependence results proved in Section 4 under minimal restrictions on β and φ.

In the existence part of the paper, several restrictions on the behaviour of φ and β are needed for ensuring boundedness and compactness of sequences of approximate solutions. In Section 5, we give a short but somewhat artificial proof of existence of entropy solutions (namely, we use not β but the projected graphs $\tilde{\beta}$ to construct approximate solutions). In Section 6, we discuss in length the pertinence of the use of $\tilde{\beta}$. First, we justify the appearance of the effective boundary condition using the vanishing viscosity parabolic approximation recalled in the Appendix. Second, we give several stability results for entropy solutions of the hyperbolic problem $(H_{\varphi,\beta}(u_0, f))$, with a focus on stability with respect to different approximations of the BC graphs β. In Section 7, first we improve the existence results in the one-dimensional case, dropping most of the assumptions on φ and β with the help of the BV_{loc} estimates due to Bürger, Karlsen, García and Towers [14, 15]. Second, following Eymard, Gallouët and Herbin [20] we present a notion of *entropy-process solution* that is useful in order to prove convergence of approximations with only weak compactness properties; it can be exploited under the additional, quite restrictive, assumption that an entropy solution exists already.

2. The effective BC graph

Throughout the section, we fix a point $(t, x) \in \Sigma$. We are given a maximal monotone graph $\beta_{(t,x)}$ on $\mathbb{R}$ and a continuous function $\varphi_{\nu(x)}$ on $\mathbb{R}$; the associated *"semi-Kruzhkov" entropy fluxes* (more precisely, their normal components) are defined as

$$q^{\pm}_{\nu(x)}(z, k) := \operatorname{sign}^{\pm}(z - k)\Big(\varphi_{\nu(x)}(z) - \varphi_{\nu(x)}(k)\Big). \tag{2.1}$$

2.1. **Preliminaries: Undershoot and overshoot sets, increasing envelopes.**

Let us start with a series of definitions and notation.

Definition 2.1 (See Figure 2.1 for an illustration)**.**

- For a closed sub-interval I of $\mathbb{R}$, introduce the *upper increasing envelope*[3] $\varphi^+_x(I; \cdot)$ and the *lower increasing envelope* $\varphi^-_x(I; \cdot)$ of $\varphi_{\nu(x)}$ on I by setting,

[3] It is easily seen that $\varphi^+_x(I; \cdot)$, respectively $\varphi^-_x(I; \cdot)$, is a non-decreasing function that is continuous and whose graph lies above (respectively, below) the graph of $\varphi_{\nu(x)}|_I$.

for $r \in I$,

$$\varphi_x^+(I;r) := \inf\Big\{\psi(r) \mid \psi \geq \varphi_{\nu(x)} \text{ and } \psi \text{ is non-decreasing on } I\Big\}, \tag{2.2}$$

$$\varphi_x^-(I;r) := \sup\Big\{\psi(r) \mid \psi \leq \varphi_{\nu(x)} \text{ and } \psi \text{ is non-decreasing on } I\Big\}. \tag{2.3}$$

- Define the *overshoot set* $D^+_{(t,x)} \subset \mathbb{R}$ and the *undershoot set* $D^-_{(t,x)} \subset \mathbb{R}$ by[4]

$$\begin{aligned} D^+_{(t,x)} &:= \Big\{ z \in \mathbb{R} \mid \sup \beta_{(t,x)}(z) \geq \varphi_{\nu(x)}(z) \Big\}, \\ D^-_{(t,x)} &:= \Big\{ z \in \mathbb{R} \mid \inf \beta_{(t,x)}(z) \leq \varphi_{\nu(x)}(z) \Big\}; \end{aligned} \tag{2.4}$$

  also introduce the *crossing set*[5] $D^0_{(t,x)} := \Big\{ r \in \mathbb{R} \mid \varphi_{\nu(x)}(r) \in \beta_{(t,x)}(r) \Big\} \equiv D^+_{(t,x)} \cap D^-_{(t,x)}$.
- *Subgraphs* of the graph of $\varphi_{\nu(x)}$ are defined as the graphs of restrictions of $\varphi_{\nu(x)}|_E$ on different subsets E of $\mathbb{R}$. Among these, we distinguish *monotone subgraphs* characterized by the property $\varphi_{\nu(x)}(a) \leq \varphi_{\nu(x)}(b)$ for all $a, b \in E$ with $a \leq b$. Finally, those monotone subgraphs that do not possess a non-trivial extension (within the class of monotone subgraphs) are called *maximal monotone subgraphs* of the graph of $\varphi_{\nu(x)}$.
- Denote by $\mathbb{B}_x$ the set of all maximal monotone subgraphs of the graph of $\varphi_{\nu(x)}$. Denote by $\mathbb{B}^0_{(t,x)}$ the set of all elements of $\mathbb{B}_x$ whose domain contains $D^0_{(t,x)}$.[6]
- Denote by $\overline{\mathbb{B}}_x$ (respectively, by $\overline{\mathbb{B}^0_{(t,x)}}$) the set of all maximal monotone graphs on $\mathbb{R}$ obtained as extensions of elements of $\mathbb{B}_x$ (respectively, of $\mathbb{B}^0_{(t,x)}$).[7]
- Define the monotone function $\widetilde{\mathcal{B}}_{(t,x)}$ on $\mathbb{R}$ as the closest to the $\beta_{(t,x)}$ element of $\overline{\mathbb{B}^0_{(t,x)}}$.

The notion of "the closest" in the last definition should be made precise: indeed, we now show that the definition of $\widetilde{\mathcal{B}}_{(t,x)}$ is correct, interpreted as the extremality property (2.5).

Proposition 2.2. *The function $\widetilde{\mathcal{B}}_{(t,x)}$ is correctly defined, in the sense that*

(2.5) *there exists $\widetilde{\mathcal{B}}_{(t,x)} \in \overline{\mathbb{B}^0_{(t,x)}}$ that realizes,* simultaneously *for all $z \in \mathbb{R}$, the minimum over all $\overline{\mu} \in \overline{\mathbb{B}^0_{(t,x)}}$ of the distance dist$\Big(\,\overline{\mu}(z)\,,\, \beta_{(t,x)}(z)\,\Big)$.*

---

[4] In definition (2.4), we actually extend $\beta_{(t,x)}$ to a maximal monotone graph from $\overline{\mathbb{R}}$ to $\overline{\mathbb{R}}$ so that $\beta_{(t,x)}(z)$ is never empty, but it may de reduced to $\{+\infty\}$ or to $\{-\infty\}$. With this convention, $\mathbb{R} = D^+_{(t,x)} \cup D^-_{(t,x)}$.

[5] Indeed, $D^0_{(t,x)}$ is the set of crossing points of $\beta_{(t,x)}$ with the graph of $\varphi_{\nu(x)}$.

[6] It follows that for any $\mu \in \mathbb{B}^0_{(t,x)}$, $\Big\{(z, \varphi_{\nu(x)}(z)) \mid z \in D^0_{(t,x)}\Big\} \subset \mu \subset \Big\{(z, \varphi_{\nu(x)}(z)) \mid z \in \mathbb{R}\Big\}$.

[7] First, it easily follows from the continuity of $\varphi_{\nu(x)}$ and the intermediate value theorem that for each $\mu \in \mathbb{B}_x$ there exists a unique extension $\overline{\mu} \in \overline{\mathbb{B}}_x$. The graph $\overline{\mu}$ is actually the graph of a single-valued continuous function on $\mathbb{R}$; moreover, on every connected component (a, b) of the set $\{z \in \mathbb{R} \mid \overline{\mu}(z) \neq \varphi_{\nu(x)}(z)\}$ the function $\overline{\mu}$ takes the constant value equal to the value of $\varphi_{\nu(x)}$ on $\{a, b\} \cap \mathbb{R}$.

Furthermore, $\widetilde{\mathcal{B}}_{(t,x)}$ *can be expressed in terms of the upper (respectively, lower) increasing envelopes of the graph of* $\varphi_{\nu(x)}$ *on the connected components*[8] I *of* $D^+_{(t,x)}$ *(respectively, of* $D^-_{(t,x)}$*):*

$$\text{(2.6)}\quad \widetilde{\mathcal{B}}_{(t,x)} := \left(\bigcup\nolimits_I \left\{ \left(z, \varphi_x^-(I;z)\right) \,\middle|\, I \textit{ is a connected component of } D^-_{(t,x)} \right\} \right) \cup \left(\bigcup\nolimits_I \left\{ \left(z, \varphi_x^+(I;z)\right) \,\middle|\, I \textit{ is a connected component of } D^+_{(t,x)} \right\} \right).$$

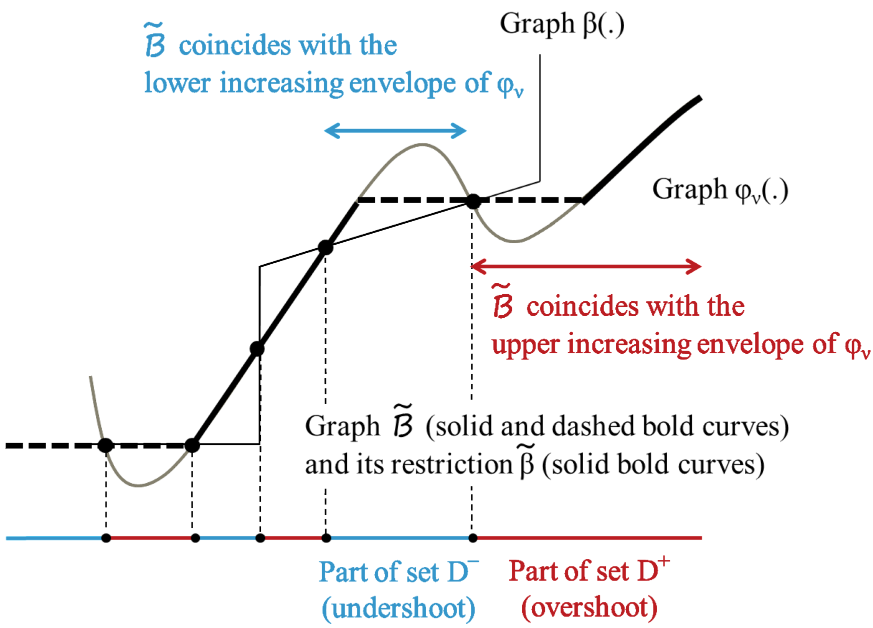

FIGURE 1. Example of construction of the projected graph $\widetilde{\beta}$.

Proof. By definition of the class $\overline{\mathbb{B}^0_{(t,x)}}$, $\widetilde{\mathcal{B}}_{(t,x)}|_{D^0_{(t,x)}}$ coincides with $\varphi_{\nu(x)}|_{D^0_{(t,x)}}$. Let I be a connected component of $D^+_{(t,x)}$ or of $D^-_{(t,x)}$; the endpoints of I are either infinite or belong to $D^0_{(t,x)}$. Therefore, we only need to make explicit the definition of $\widetilde{\mathcal{B}}_{(t,x)}$ on the interior of I, and for a proof of (2.5) we can consider $z \in I$ separately for every connected component I of $D^+_{(t,x)}$ or of $D^-_{(t,x)}$. To be specific, consider $I \subset D^+_{(t,x)}$. From (2.2), one easily sees that

$$\text{(2.7)}\qquad \varphi_{\nu(x)} \le \varphi_x^+(I;\cdot) \le \beta \ \text{ on } \ I.$$

Every function $\overline{\mu} \in \overline{\mathbb{B}^0_{(t,x)}}$ is constant on each interval where it does not coincide with $\varphi_{\nu(x)}$, while $\varphi_x^+(I;\cdot)$ and β are monotone on these intervals. Thus from (2.7)

[8] Let us recall that I is a *connected component* of $K \subset \mathbb{R}$ if I is an interval, and moreover, for all intervals J such that $I \subset J \subset K$, one has $J = I$.

it follows that

$$\forall \overline{\mu} \in \overline{\mathbb{B}^0_{(t,x)}} \quad \overline{\mu} \le \varphi^+_x(I;\cdot) \le \beta \ \text{ on } \ I. \tag{2.8}$$

Now, one easily checks that the family $\{\overline{\mu}|_I \mid \ \overline{\mu} \in \overline{\mathbb{B}^0_{(t,x)}}\}$ is stable by the sup operation; therefore it possesses a greatest element that we call ψ. This element is the restriction of $\widetilde{\mathcal{B}}_{(t,x)}$ on I. Indeed, from (2.8), for all $\overline{\mu} \in \overline{\mathbb{B}^0_{(t,x)}}$ one has in particular $\overline{\mu} \le \psi \le \beta$. Therefore we can set $\widetilde{\mathcal{B}}_{(t,x)}|_I := \psi$, and (2.5) gets verified for all $z \in I$.

We have seen in (2.8) that $\widetilde{\mathcal{B}}_{(t,x)}|_I = \psi \le \varphi^+_x(I;\cdot)$. The function ψ is nondecreasing and its graph lies above $\varphi_{\nu(x)}|_I$ by (2.7); thus according to (2.2), $\psi \ge \varphi^+_x(I;\cdot)$. Therefore $\widetilde{\mathcal{B}}_{(t,x)}|_I$ coincides with $\varphi^+_x(I;\cdot)$, for every connected component I of $D^+_{(t,x)}$ or of $D^-_{(t,x)}$. This yields (2.6). □

2.2. Definition and equivalent characterizations of $\tilde{\beta}$.

Definition 2.3. The graph $\tilde{\beta}_{(t,x)}$ is the part of $\widetilde{\mathcal{B}}_{(t,x)}$ contained within the graph of $\varphi_{\nu(x)}$.

It is clear from the above definition that $\tilde{\beta}_{(t,x)} \in \mathbb{B}^0_{(t,x)}$. Namely, $\tilde{\beta}_{(t,x)}$ is a maximal monotone subgraph of $\varphi_{\nu(x)}$ containing the crossing points with $\beta_{(t,x)}$. Moreover, the unique extension $\widetilde{\mathcal{B}}_{(t,x)}$ of $\tilde{\beta}_{(t,x)}$ to a maximal monotone graph on $\mathbb{R}$ satisfies (2.5). Thus Definition 2.3 is a precise expression of $(A_{\tilde{\beta}})$, in view of the extremality property (2.5).

Notice that, according to (2.6), $\beta_{(t,x)}$ intervenes in the construction of $\tilde{\beta}_{(t,x)}$ uniquely through the sets $D^\pm_{(t,x)}$ that gather the points $z \in \mathbb{R}$ such that $\pm(\beta_{(t,x)}(z) - \varphi_{\nu(x)}(z)) \cap \mathbb{R}^+ \neq \emptyset$.

Remark 2.4. The operation $\widetilde{\mathcal{P}}_x$ that transforms the maximal monotone graph $\beta_{(t,x)}$ into the maximal monotone graph $\widetilde{\mathcal{B}}_{(t,x)}$ is a projection on $\mathbb{B}_x$. Indeed, we have $\widetilde{\mathcal{P}}_x^2 = \widetilde{\mathcal{P}}_x$ and $\widetilde{\mathcal{P}}_x|_{\mathbb{B}_x} = Id$. With a slight abuse of notation (the graph $\tilde{\beta}_{(t,x)} = \widetilde{\mathcal{B}}_{(t,x)}|_{\mathrm{Dom}\,\tilde{\beta}_{(t,x)}}$ being monotone but not necessarily maximal),[9] we will say that the operation $\tilde{\ }: \ \beta_{(t,x)} \mapsto \tilde{\beta}_{(t,x)}$ is a projection.

Let us give alternative characterizations of $\tilde{\beta}_{(t,x)}$. Recall that $\tilde{\beta}_{(t,x)}$ is a subgraph of the graph of $\varphi_{\nu(x)}$; thus it is fully characterized by its domain.

Proposition 2.5. *The domain of the graph $\tilde{\beta}_{(t,x)}$ given by Definition* 2.3 *can be equivalently defined by any of the following properties:*

(i) *In terms of the semi-Kruzhkov entropy fluxes* (2.1), *one has*

$$Dom\,\tilde{\beta}_{(t,x)} = \Big\{ z \in \mathbb{R} \Big| \ \Big(\forall k \in D^-_{(t,x)} \ \ q^-_\nu(z,k) \ge 0\Big) \& \Big(\forall k \in D^+_{(t,x)} \ \ q^+_\nu(z,k) \ge 0\Big)\Big\}.$$

(ii) *For $z \in \mathbb{R}$, denote $\beta^{-1}_{(t,x)}(\varphi_{\nu(x)}(z)) =: \Big[m_{(t,x)}(z), M_{(t,x)}(z)\Big]$; this is a non-empty*[10] *closed interval of $\overline{\mathbb{R}}$. Notice that $z < m_{(t,x)}(z)$ (resp., $z > M_{(t,x)}(z)$)*

[9] Actually, uniqueness of solutions to the boundary value problems with dissipative boundary condition encoded by a graph β stems from the monotonicity of β only. Existence may depend on how wide the domain of β is. In the sequel we will see that the monotone graph $\tilde{\beta}$, while it is not maximal, leads to existence and uniqueness for the problem in hand.

[10] We mean that $\tilde{\beta}_{(t,x)}$ is extended to a maximal monotone graph on $\overline{\mathbb{R}}$.

for $z\in D^-_{(t,x)}\backslash D^0_{(t,x)}$ (resp., for $z\in D^-_{(t,x)}\backslash D^0_{(t,x)}$). With this notation, we have

$$\begin{array}{rcl} Dom\,\tilde\beta_{(t,x)} &=& D^0_{(t,x)}\\ && \bigcup\ \left\{ z\in D^-_{(t,x)}\backslash D^0_{(t,x)}\ \Big|\ \varphi_{\nu(x)}(k)\geq\varphi_{\nu(x)}(z)\ \forall k\in\Big[\,z\,,\,m_{(t,x)}(z)\Big]\ \right\}\\ && \bigcup\ \left\{ z\in D^+_{(t,x)}\backslash D^0_{(t,x)}\ \Big|\ \varphi_{\nu(x)}(k)\leq\varphi_{\nu(x)}(z)\ \forall k\in\Big[\,M_{(t,x)}(z)\,,\,z\Big]\ \right\}. \end{array}$$

Remark 2.6. Characterization (i), in its spirit, goes back to the idea of Carrillo [16], further developed by Ammar, Carrillo and Wittbold [2], for the case of the Dirichlet problem. In [16], $u^D=0$ and thus $\beta_{(t,x)}=\{0\}\times\mathbb{R}$; therefore $D^\pm_{(t,x)}=\mathbb{R}^\pm$ in this case. In [2], $\beta_{(t,x)}=\{u^D(t,x)\}\times\mathbb{R}$ and thus $D^-_{(t,x)}=(-\infty,u^D(t,x)]$, $D^+_{(t,x)}=[u^D(t,x),+\infty)$. Further, notice that for the Dirichlet boundary condition, $\inf\beta^{-1}_{(t,x)}(\varphi_{\nu(x)}(z))=u^D(t,x)=\sup\beta^{-1}_{(t,x)}(\varphi_{\nu(x)}(z))$. Thus we see that characterization (ii) is precisely the Bardos-LeRoux-Nédélec set (1.4). Representation (ii) of $\tilde\beta_{(t,x)}$ is therefore a generalization of the BLN condition; it appeared in the previous works [37] and [7,8] of the authors (see in particular [7, formula (4)]).

Proof. Throughout the proof, we write $\widetilde D^0_{(t,x)}:=\mathrm{Dom}\,\tilde\beta_{(t,x)}=\Big\{z\in\mathbb{R}\,|\,\widetilde{\mathcal{B}}_{(t,x)}(z)=\varphi_{\nu(x)}(z)\Big\}$.

(i) Let us assume that $z\in\widetilde D^0_{(t,x)}$. Consider, e.g., $k\in D^-_{(t,x)}$: according to (2.5) in this case we have $\varphi_{\nu(x)}(k)\geq\widetilde{\mathcal{B}}_{(t,x)}(k)$. Then

$$q^-_\nu(z,k)=sign^-(z-k)(\varphi_{\nu(x)}(z)-\varphi_{\nu(x)}(k))\geq sign^-(z-k)(\widetilde{\mathcal{B}}_{(t,x)}(z)-\widetilde{\mathcal{B}}_{(t,x)}(k))\geq 0$$

by the monotonicity of $\widetilde{\mathcal{B}}_{(t,x)}$. The case $k\in D^-_{(t,x)}$ is analogous.

Reciprocally, assume that for all $k\in D^\pm_{(t,x)}$ one has $q^\pm_\nu(z,k)\geq 0$. For the sake of being definite, assume that z belongs to a connected component I of $D^+_{(t,x)}$. Let $k\in I$, $k<z$; by assumption we have $\varphi_{\nu(x)}(k)\leq\varphi_{\nu(x)}(z)$ for all such k. Keeping in mind the characterization (2.6) in Proposition 2.2, we see that $\varphi^+_x(k)=\varphi_{\nu(x)}(k)$ if and only if $k\in\widetilde D^0_{(t,x)}$. It follows that $\tilde u$ verifies $\varphi^+_x(k)\leq\varphi_{\nu(x)}(z)$ for all $k\in I$, $k<z$. By the definition of the upper increasing envelope φ^+_x, this exactly means that $\varphi_{\nu(x)}(z)=\varphi^+_x(z)$. Hence $z\in\widetilde D^0_{(t,x)}$.

(ii) The case $z\in D^0_{(t,x)}$ is trivial. Let I be the connected component of the complement of $D^0_{(t,x)}$ that contains z; for the sake of being definite, assume that $I\subset D^+_{(t,x)}$. Then from the monotonicity of $\beta_{(t,x)}$ we have

$$\varphi_{\nu(x)}(k)\leq\sup\beta_{(t,x)}(k)\leq\beta_{(t,x)}(M_{(t,x)}(z))\ni\varphi_{\nu(x)}(z)\tag{2.9}$$

for $k\in I\cap(-\infty,M_{(t,x)}(z))$. By the characterization (2.6) of $\widetilde{\mathcal{B}}_{(t,x)}$ on I, $z\in\widetilde D^0_{(t,x)}\cap I$ if and only if $\varphi_{\nu(x)}(k)\leq\varphi_{\nu(x)}(z)$ for all $k\in I$, $k\leq z$. Taking into account (2.9), we can reformulate this as follows: $z\in\widetilde D^0_{(t,x)}\cap I$ if and only if, firstly, $[M_{(t,x)}(z),z]\subset I$ and secondly, $\varphi_{\nu(x)}(k)\leq\varphi_{\nu(x)}(z)$ for all $k\in[M_{(t,x)}(z),z]$. This justifies the statement of (ii). ∎

Remark 2.7. One more convenient description of $\tilde\beta_{(t,x)}$ can be given in terms of the Godunov numerical fluxes (1.6):

(2.10)

$$\tilde\beta_{(t,x)} = \Big\{ (z,F) \in \mathbb{R}^2 \,\Big|\, \exists (r,F) \in \beta_{(t,x)} \text{ such that } F = \varphi_{\nu(x)}(z) = \mathrm{God}[\varphi_{\nu(x)}](z,r) \Big\}.$$

This description is easily inferred from (1.6), (2.6) and Definition 2.3.

2.3. **Order and metric structure on $\mathbb{B}_x$.** Fix $x \in \partial\Omega$. Recall that $\mathbb{B}_x$ is the set of all maximal monotone subgraphs of $\varphi_{\nu(x)}$, $\mathbb{B}$ is the set of all maximal monotone graphs of $\overline{\mathbb{R}}$, and $\overline{\mathbb{B}}_x$ is the subset of $\mathbb{B}$ obtained by extension (which is unique) from $\mathrm{Dom}\tilde\beta$ to $\mathbb{R}$ of elements $\tilde\beta \in \mathbb{B}_x$.

Let us define an order relation and a distance for maximal monotone graphs under study. They are most naturally defined on $\overline{\mathbb{B}}_x$.

Definition 2.8. For $\widetilde{\mathcal{B}}^1, \widetilde{\mathcal{B}}^2 \in \overline{\mathbb{B}}_x$, define the uniform distance

$$d_x(\widetilde{\mathcal{B}}^1, \widetilde{\mathcal{B}}^2) := \|\widetilde{\mathcal{B}}^1 - \widetilde{\mathcal{B}}^2\|_\infty = \sup_{\mathbb{R}} |\widetilde{\mathcal{B}}^1 - \widetilde{\mathcal{B}}^2|.$$

Define the order relation " $\succeq_x$ " on $\overline{\mathbb{B}}_x$ by

$$\widetilde{\mathcal{B}}^1 \succeq_x \widetilde{\mathcal{B}}^2 \quad \text{if} \quad \widetilde{\mathcal{B}}^1 \geq \widetilde{\mathcal{B}}^2 \ \text{ pointwise on } \mathbb{R}.$$

Since every $\tilde\beta \in \mathbb{B}_x$ possesses a unique extension $\widetilde{\mathcal{B}}^2 \in \overline{\mathbb{B}}_x$, we can define d_x and $\succeq_x$ on $\mathbb{B}_x$ by writing, e.g., $d_x(\tilde\beta^1, \tilde\beta^2) := d_x(\widetilde{\mathcal{B}}^1, \widetilde{\mathcal{B}}^2)$. Further, every $\beta \in \mathbb{B}$ gives rise to the projection $\widetilde{\mathcal{B}} := \widetilde{\mathcal{P}}_x\beta$ on $\mathbb{B}_x$. Thus we can extend d_x to a semi-distance on $\beta \in \mathbb{B}$, and we can extend $\succeq_x$ to a binary relation on $\mathbb{B}$ by assigning $\beta^1 \succeq_x \beta^2$ whenever $\widetilde{\mathcal{P}}_x\beta \succeq_x \widetilde{\mathcal{P}}_x\beta$.

In Sections 4, 6.2.2, 6.2.3 we will use these definitions in combination with the following lemma.

Lemma 2.9. *One can represent the distance $d_x(\cdot,\cdot)$ by the formulas*

(2.11)

$$\begin{aligned} d_x(\tilde\beta^1, \tilde\beta^2) &= \sup_{a,b\in\mathbb{R}} \mathrm{sign}\,(b-a)\big(\widetilde{\mathcal{B}}^1(a) - \widetilde{\mathcal{B}}^2(b)\big) \\ &= \sup\Big\{ \mathrm{sign}\,(b-a)\big(\varphi_\nu(a) - \varphi_\nu(b)\big) \mid a \in \mathrm{Dom}(\tilde\beta^1),\ b \in \mathrm{Dom}(\tilde\beta^2) \Big\}. \end{aligned}$$

One can express the relation $\beta^1 \succeq_x \beta^2$ through the formula $d^-_x(\tilde\beta^1, \tilde\beta^2) = 0$, where

(2.12)

$$\begin{aligned} d^-_x(\tilde\beta^1, \tilde\beta^2) &:= \sup_{a,b\in\mathbb{R}} \mathrm{sign}^-(b-a)(\widetilde{\mathcal{B}}^1(a) - \widetilde{\mathcal{B}}^2(b)) \\ &= \sup\Big\{ \big(\widetilde{\mathcal{B}}^1(a) - \widetilde{\mathcal{B}}^2(b)\big)^- \mid a > b \Big\} \\ &= \sup\Big\{ \big(\varphi_\nu(a) - \varphi_\nu(b)\big)^- \mid a \in \mathrm{Dom}(\tilde\beta^1),\ b \in \mathrm{Dom}(\tilde\beta^2),\ a > b \Big\} \\ &= \sup\Big\{ \Big(Q^+_{\nu(x)}(\tilde a, \tilde b)\Big)^- \mid \tilde a \in V\varphi_{\nu(x)}\,\mathrm{Dom}(\tilde\beta^1),\ \tilde b \in V\varphi_{\nu(x)}\,\mathrm{Dom}(\tilde\beta^2) \Big\}, \end{aligned}$$

where we have used the singular mapping $V\varphi_{\nu(x)}$ and the notation of Section 1.3.

Proof. On the one hand, from the monotonicity of $\widetilde{\mathcal{B}}^2$, we have

$$\begin{aligned} &\sup_{a,b\in\mathbb{R}} \mathrm{sign}\,(b-a)\big(\widetilde{\mathcal{B}}^1(a) - \widetilde{\mathcal{B}}^2(b)\big) \\ &\quad = \sup_{a,b\in\mathbb{R}} \mathrm{sign}\,(b-a)\big(\widetilde{\mathcal{B}}^1(a) - \widetilde{\mathcal{B}}^2(a) + \widetilde{\mathcal{B}}^2(a) - \widetilde{\mathcal{B}}^2(b)\big) \\ &\quad \leq |\widetilde{\mathcal{B}}^1(a) - \widetilde{\mathcal{B}}^2(a)| \leq \|\widetilde{\mathcal{B}}^1 - \widetilde{\mathcal{B}}^2\|_\infty = d_x(\tilde\beta^1, \tilde\beta^2). \end{aligned}$$

On the other hand, consider $a = k$ and $b_n = k+\frac{1}{n}$; then $b_n = k-\frac{1}{n}$ in the left-hand side of the above expression, with $n \to \infty$. Using the continuity of $\widetilde{\mathcal{B}}^2$ we get for all $k \in \mathbb{R}$,

$$\big|\widetilde{\mathcal{B}}^1(k) - \widetilde{\mathcal{B}}^2(k)\big| \le \sup_{a,b\in\mathbb{R}} \operatorname{sign}(b-a)\big(\widetilde{\mathcal{B}}^1(a) - \widetilde{\mathcal{B}}^2(b)\big).$$

Hence we derive the first equality in (2.11). Further, recall that $\widetilde{\mathcal{B}}^i$ is constant on each connected component of the complement of $\operatorname{Dom}\tilde\beta^i$, while $\widetilde{\mathcal{B}}^i|_{\operatorname{Dom}\tilde\beta^i} = \varphi_\nu|_{\operatorname{Dom}\tilde\beta^i}$; this implies the second equality in (2.11).

In the same way, we justify the first three equalities in (2.12). The last equality in (2.12) is evident from the definitions of $V\varphi_{\nu(x)}$ and $Q^+_{\nu(x)}$. $\square$

3. Notion of solution

Let us start with the following notation. Given $\beta_{(t,x)} \in \mathbb{B}$, in the previous section we have constructed its projection $\widetilde{\mathcal{B}}_{(t,x)} \in \overline{\mathbb{B}}_x$. Then we write

$$\widetilde{D}^-_{(t,x)} := \Big\{k \in \mathbb{R} \,|\, \widetilde{\mathcal{B}}_{(t,x)}(k) \le \varphi_{\nu(x)}(k)\Big\} \equiv D^-_{(t,x)} \cup \operatorname{Dom}\tilde\beta_{(t,x)};$$
$$\widetilde{D}^+_{(t,x)} := \Big\{k \in \mathbb{R} \,|\, \widetilde{\mathcal{B}}_{(t,x)}(k) \ge \varphi_{\nu(x)}(k)\Big\} \equiv D^+_{(t,x)} \cup \operatorname{Dom}\tilde\beta_{(t,x)};$$
$$\widetilde{D}^0_{(t,x)} := \operatorname{Dom}\tilde\beta_{(t,x)} \equiv \widetilde{D}^-_{(t,x)} \cap \widetilde{D}^+_{(t,x)}.$$

Recall that $D^-_{(t,x)}, D^+_{(t,x)}$ and $D^0_{(t,x)}$ are the undershoot, the overshoot and the crossing sets for the graph $\beta_{(t,x)}$ given the normal flux $\varphi_{\nu(x)}$. Similarly, $\widetilde{D}^-_{(t,x)}, \widetilde{D}^+_{(t,x)}$ and $\widetilde{D}^0_{(t,x)}$ are the undershoot, the overshoot and the crossing sets for the projected graph $\widetilde{\mathcal{B}}_{(t,x)}$. These sets appear as sets of boundary traces of entropy sub-solutions, super-solutions and solutions, respectively, according to the definitions we now give.

Note the following localized version of the celebrated definition of the entropy solution due to Kruzhkov [21]. Recall that $q^\pm(\cdot,\cdot)$ are the semi-Kruzhkov entropy fluxes defined by (1.10).

Definition 3.1. Let $\widehat{Q_T}$ be an open sub-domain of $Q_T = (0,T)\times\Omega$. A function $u \in L^\infty(\widehat{Q_T})$ is called the entropy solution of problem $u_t + \operatorname{div}\varphi(u) = f$, $u|_{t=0} = u_0$ if for all $k \in \mathbb{R}$ and for all $\xi \in \mathcal{D}\Big(\widehat{Q_T} \cup (\{0\}\times\Omega)\Big)^+$,

$$\int_0^T\!\!\int_\Omega \Big(-(u-k)^\pm\xi_t - q^\pm(u,k)\cdot\nabla\xi\Big) - \int_\Omega (u_0-k)^\pm\xi(0,\cdot) \le \int_0^T\!\!\int_\Omega \operatorname{sign}^\pm(u-k) f\,\xi. \tag{3.1}$$

If only the sign "plus" (respectively, "minus") is chosen in (3.1), then u is an entropy sub-solution (respectively, an entropy super-solution) in $\widehat{Q_T}$.

Remark 3.2. Notice that entropy solutions, sub- and super-solutions are quasi-solutions in the sense of Panov (see [32]). This implies that the boundary traces in the sense of Section 1.3, used in the definitions of the next section, do exist. It should also be noted that, according to the result of [31] (see also [40]), entropy solutions in the whole cylinder Q_T actually belong to $C(0,T;L^1_{loc}(\Omega))$; in particular the initial datum u_0 is assumed in the sense of strong L^1_{loc} trace.

3.1. **Equivalent definitions of entropy solutions, sub-solutions, and super-solutions.** Now we include the boundary condition in the definition. We need one more notation:

$$\Sigma^{\pm}(k) := \{(t,x) \in \Sigma \mid k \in D^{\pm}_{(t,x)}\}. \tag{3.2}$$

In order to describe simultaneously the key features of entropy solutions, we gather a series of equivalent definitions in the following definition and proposition.

Proposition 3.3 (Definition of an entropy solution)**.** *Let $u \in L^\infty(Q_T)$.*

If any of the below items (i)-(iv) are satisfied, u is called an entropy solution of problem $(H_{\varphi,\beta}(u_0, f))$. Indeed, the assertions (i)-(iv) are equivalent:

(i) *The function u verifies the entropy inequalities* (3.1) *with $\xi \in \mathcal{D}([0,T)\times\Omega)^+$; moreover, for $\mathcal{H}^N$-a.e. $(t,x) \in \Sigma$, the strong trace $\gamma V\varphi_{\nu(x)} u$ belongs to the set $V\varphi_{\nu(x)} \widetilde{D}^0_{(t,x)}$.*

(ii) *The function u verifies the entropy inequalities* (3.1) *with $\xi \in \mathcal{D}([0,T)\times\Omega)^+$; moreover, for $\mathcal{H}^N$-a.e. $(t,x) \in \Sigma$, the strong trace $\gamma V\varphi_{\nu(x)} u$ verifies*

$$\forall k \in \widetilde{D}^0_{(t,x)} \quad Q_{\nu(x)}\Big(\gamma V\varphi_{\nu(x)} u\,,\, V\varphi_{\nu(x)} k\Big) \geq 0. \tag{3.3}$$

Here, according to (1.12)*, $Q_{\nu(x)}$ represents the normal component of the Kruzhkov entropy flux $q(u,k) = \mathrm{sign}\,(u-k)(\varphi(u) - \varphi(k))$.*

(iii) *The function u verifies the entropy inequalities* (3.1) *with $\xi \in \mathcal{D}([0,T)\times\Omega)^+$; moreover, for $\mathcal{H}^N$-a.e. $(t,x) \in \Sigma$, the strong trace $\gamma V\varphi_{\nu(x)} u$ verifies*

$$\forall k \in D^{\pm}_{(t,x)} \quad Q^{\pm}_{\nu(x)}(\gamma V\varphi_{\nu(x)} u, V\varphi_{\nu(x)} k) \geq 0. \tag{3.4}$$

Here, $Q^{\pm}_{\nu(x)}$ are defined by (1.11)*.*

(iv) *The function u verifies the up-to-the-boundary entropy inequalities with remainder term:*

$$\begin{aligned} &\forall k \in \mathbb{R}\ \ \forall \xi \in \mathcal{D}([0,T) \times \overline{\Omega})^+, \\ &\quad \int_0^T\!\!\int_\Omega \Big(-(u-k)^{\pm}\xi_t - q^{\pm}(u,k)\cdot\nabla\xi\Big) - \int_\Omega (u_0 - k)^{\pm}\xi(0,\cdot) \\ &\qquad \leq \int_0^T\!\!\int_\Omega \mathrm{sign}^{\pm}(u-k) f\,\xi + \iint_\Sigma C_k \wedge \Big(\beta_{(t,x)}(k) - \varphi_{\nu(x)}(k)\Big)^{\mp} \xi(t,x). \end{aligned} \tag{3.5}$$

Here, C_k is a constant[11] *that depends on $\|u\|_\infty$ and on k.*

Moreover, if the sets $\Sigma^{\pm}(k)$ in (3.2) *are regular enough in the sense that*[12]

$$\begin{aligned} &\textit{for a countable dense set of values of } k, \\ &\textit{the space } \mathcal{D}(\Sigma^{\pm}(k)) \textit{ is dense in } L^1(\Sigma^{\pm}(k)), \end{aligned} \tag{3.6}$$

[11]Truncation by C_k is needed in order that the right-hand side be finite. Indeed, recall that we have extended $\beta_{(t,x)}$ to an $\overline{\mathbb{R}}$-valued graph.

[12]This is a kind of separation property for $\Sigma^{\pm}$ and the complementary sets $\Sigma\setminus\Sigma^{\pm}$; it is satisfied in many practical situations, but it fails, e.g., in $\beta_{(t,x)} = \{u^D(t,x)\} \times \mathbb{R}$ (the Dirichlet case) with u^D that is the characteristic function of a Cantor set of positive measure.

then (i)-(iv) are also equivalent to

(v) *The function u verifies the following up-to-the-boundary entropy inequalities:*

$$\begin{aligned}&\forall k \in \mathbb{R}\ \ \forall \xi \in \mathcal{D}([0,T)\times\overline{\Omega})^+ \ \ \textit{such that}\ \xi|_{\Sigma\setminus\Sigma^\pm(k)} = 0,\\ &\int_0^T\!\!\int_\Omega \Big(-(u-k)^\pm \xi_t - q^\pm(u,k)\cdot\nabla\xi\Big) - \int_\Omega (u_0-k)^\pm \xi(0,\cdot)\\ &\quad \le \int_0^T\!\!\int_\Omega \operatorname{sign}^\pm(u-k) f\,\xi.\end{aligned} \tag{3.7}$$

Remark 3.4. Let us provide a few comments on the different items of Proposition 3.3 and their use for establishing well-posedness for problem $(H_{\varphi,\beta}(u_0,f))$ in the setting of entropy solutions.

- Inequalities (3.5) are multi-valued, but, approximating k from below and from above, it is enough to require that (3.5) hold in its less restrictive version, i.e., with $(\inf \beta_{(t,x)}(k) - \varphi_{\nu(x)}(k))^-$ and with $(\sup \beta_{(t,x)}(k) - \varphi_{\nu(x)}(k))^+$, respectively, in the right-hand side.
- Definition (i) is a straightforward interpretation of the *formal BC*, encoded by $\beta_{(t,x)}$, as the *effective BC* given by its projection $\tilde{\beta}_{(t,x)}$: recall that $\widetilde{D}^0_{(t,x)}$ is the domain of $\tilde{\beta}_{(t,x)}$.
- Definition (ii) of entropy solutions encrypts, in a rather direct way, the dissipative nature of the boundary condition expressed by $\tilde{\beta}_{(t,x)}$. Combination of items (i) and (ii) leads to an immediate proof of uniqueness, comparison and L^1 contraction for entropy solutions.
- Explicit use of boundary traces in definitions (i), (ii) makes it delicate to establish existence. Indeed, one of the important features of a definition should be the stability of the notion of entropy solution under L^1_{loc} convergence in Q_T. Existence arguments for definitions (i), (ii) were devised in [7, 37], but they are quite restrictive (namely, they require strong compactness of boundary traces on Σ, which is not implied by a mere $L^1_{loc}(Q_T)$ convergence).
- "Traceless" definitions (iv) and (v) by global entropy inequalities (cf. [8] for a different version of definition (iv)) are clearly stable under L^1_{loc} convergence.
- Definition (v) is reminiscent of those of Carrillo [16], and Ammar, Carrillo and Wittbold [2]. Yet in full generality, (v) cannot be used, e.g., when $\Sigma^\pm$ have a fractal nature. Definition (iv) is a way to bypass the subtlety of the simultaneous choice of k and ξ imposed in [16]; the idea is to incorporate a remainder term that vanishes, on parts of the boundary, for particular choices of k. An approach similar to (iv) was used by Vovelle [41], with a simpler choice of the remainder term suitable for an inhomogeneous Dirichlet boundary condition.
- Finally, definition (iii) provides a link between (i)-(ii) and (iv)-(v): it uses both traces and the "$D^\pm$ vocabulary". This definition can be put in correspondence with the pointwise interpretation by Rouvre and Gagneux [36] of the Carrillo boundary condition.

The following proposition defines entropy sub- and super-solutions of problem $(H_{\varphi,\beta}(u_0,f))$.

Proposition 3.5 (Definition of entropy sub- and super-solutions). *Let $u \in L^\infty(Q_T)$. If any of the below items (i)-(iv) are satisfied, u is called an entropy sub-solution of problem $(H_{\varphi,\beta}(u_0,f))$. Indeed, the assertions (i)-(iv) are equivalent:*

(i) *The function u verifies the entropy inequalities* (3.1) *with the sign "plus" and $\xi \in \mathcal{D}([0,T)\times\Omega)$, $\xi \geq 0$; moreover, for $\mathcal{H}^N$-a.e. $(t,x)\in\Sigma$, the strong trace $\gamma V\varphi_{\nu(x)}u$ lies in $V\varphi_{\nu(x)}\widetilde{D}^-_{(t,x)}$.*

(ii) *The function u verifies the entropy inequalities* (3.1) *with the sign "plus" with $\xi \in \mathcal{D}([0,T)\times\Omega)$, $\xi \geq 0$; moreover, for $\mathcal{H}^N$-a.e. $(t,x)\in\Sigma$, the strong trace $\gamma V\varphi_{\nu(x)}u$ verifies*

$$\forall k \in \widetilde{D}^+_{(t,x)};\quad Q^+_{\nu(x)}\Big(\gamma V\varphi_{\nu(x)}u\,,\,V\varphi_{\nu(x)}k\Big) \geq 0. \tag{3.8}$$

(iii) *Item (ii) holds with $k\in D^+_{(t,x)}$ (in the place of $k\in\widetilde{D}^+_{(t,x)}$) in* (3.8).

(vi) *The function u verifies the up-to-the-boundary entropy inequalities with remainder term:*

$$\begin{gathered}\forall k\in\mathbb{R}\ \ \forall\xi\in\mathcal{D}([0,T)\times\overline{\Omega}),\ \xi\geq 0,\\ \int_0^T\!\!\int_\Omega\Big(-(u-k)^+\xi_t - q^+(u,k)\cdot\nabla\xi\Big) - \int_\Omega (u_0-k)^+\xi(0,\cdot)\\ \leq \int_0^T\!\!\int_\Omega \operatorname{sign}^+(u-k)f\,\xi + \iint_\Sigma C_k\wedge\Big(\beta_{(t,x)}(k)-\varphi_{\nu(x)}(k)\Big)^-\,\xi(t,x).\end{gathered} \tag{3.9}$$

Here, C_k is a constant that depends on $\|u\|_\infty$ and on k.

Further, exchange the signs "plus" and "minus" in the above properties: they remain equivalent, and if any of them are satisfied, u is called an entropy super-solution of problem $(H_{\varphi,\beta}(u_0,f))$.

The proof of Proposition 3.5 uses the same tools as the proof of Proposition 3.3 given below; we omit the details.

Remark 3.6. A function u is an entropy solution of problem $(H_{\varphi,\beta}(u_0,f))$ if and only if it is both an entropy sub- and super-solution of the problem.

3.2. **Proof of the equivalence of different definitions.** Before turning to the proof, let us state the key technical lemma that allows for use of strong traces defined in the way of Section 1.3.

Lemma 3.7. *There exists a sequence $(\xi_n)_n$ of Lipschitz functions on $\overline{\Omega}$ such that $0\leq\xi_n\leq 1$, $\xi_n|_{\partial\Omega}=1$, $\xi_n\to 0$ on Ω as $n\to\infty$, and for all $\xi\in\mathcal{D}([0,T)\times\overline{\Omega})$, for all $k\in\mathbb{R}$ there hold*

$$\begin{gathered}\lim_{n\to\infty}\left(\int_0^T\!\!\int_\Omega\Big(-(u-k)^\pm(\xi\xi_n)_t - q^\pm(u,k)\cdot\nabla(\xi\xi_n)\Big) - \int_\Omega (u_0-k)^\pm(\xi\xi_n)(0,\cdot)\right)\\ = -\lim_{n\to\infty}\int_0^T\!\!\int_\Omega \xi\, q^\pm(u,k)\cdot\nabla\xi_n = -\iint_\Sigma \xi\, Q^\pm_{\nu(x)}\Big(\gamma V_{\varphi_{\nu(x)}}u\,,\,V_{\varphi_{\nu(x)}}k\Big)\end{gathered} \tag{3.10}$$

and

$$\begin{gathered}\lim_{n\to\infty}\int_0^T\!\!\int_\Omega q^\pm(u,k)\cdot\nabla(\xi(1-\xi_n))\\ = \int_0^T\!\!\int_\Omega q^\pm(u,k)\cdot\nabla\xi - \iint_\Sigma \xi\, Q^\pm_{\nu(x)}\Big(\gamma V_{\varphi_{\nu(x)}}u\,,\,V_{\varphi_{\nu(x)}}k\Big).\end{gathered} \tag{3.11}$$

Proof. For $n \in \mathbb{N}$, the function ξ_n is defined almost explicitly. Firstly, a partition of unity $(\chi^i)_{i=0}^M$ on $\overline{\Omega}$ is used such that $\operatorname{supp}\chi^0 \subset \Omega$ and for $i = 1, \ldots, M$, $\operatorname{supp}\chi^i \subset U^i$ where U^i is an open set of the kind considered in Section 1.3. Then for each $i = 1, \ldots, M$, in the local coordinates of U_i as described in Section 1.3 we take the function

$$\pi_n^i(x_0, x') := n\Big(\tfrac{1}{n} - (x_0 - g^i(x'))\Big)^+$$

(the function g^i being associated with the neighbourhood U^i). Then we assign

$$\xi_n := \sum\nolimits_{i=1}^M \chi^i\, \pi_n^i.$$

Clearly, it only remains to justify (3.10) and (3.11).

Notice that $\nabla \xi_n = \sum_{i=1}^M \nabla\chi^i\, \pi_n^i + \sum_{i=1}^M \chi^i\, \nabla\pi_n^i$ and the first term in the right-hand side vanishes as $n \to \infty$, while the second one permits us to make appeal to the strong normal traces of $V_{\varphi_{\nu(x)}} u$. Indeed, by construction, $\nabla\pi_n^i(\cdot)$ is aligned with the field of normals $\nu(\cdot)$ lifted inside U^i; it is supported on $\{0 < y = x_0 - g^i(x') < \frac{1}{n}\}$ and its absolute value is $n\sqrt{1 + |\nabla g^i(x')|^2}$. The limit of the expression

$$\begin{aligned}&\int_0^T\!\!\int_{U^i} \xi\chi^i\, q^\pm(u,k)\cdot\nabla\pi_n^i\\ &\equiv n\int_0^{\frac{1}{n}}\Big(\int_0^T\!\!\int_{W^i} \xi\chi^i\, q^\pm(u(t, y{+}g^i(x'), x;), k)\cdot\nu(x_0,x')\Big(\sqrt{1+|\nabla g^i(x')|^2}dtdx\Big)\Big)\,dy\end{aligned}$$

(here W^i is a boundary neighbourhood corresponding to U^i; see Section 1.3) exists and equals

$$\iint_\Sigma \xi\chi^i Q^\pm_{\nu(x)}\Big(\gamma V_{\varphi_{\nu(x)}}u\,,\, V_{\varphi_{\nu(x)}}k\Big)$$

according to Section 1.3 and because $\Big(\sqrt{1+|\nabla g^i(x')|^2}dtdx\Big)$ is precisely the surface measure on the boundary Σ. Then by a straightforward passage to the limit, both (3.10) and (3.11) hold. $\square$

Proof of Proposition 3.3. Throughout the proof, we use the following notation. If a point $(t,x) \in \Sigma$ is fixed, we set $\tilde{V} := \gamma V\varphi_{\nu(x)}u$, then pick (arbitrarily) $\tilde{u} \in [V\varphi_{\nu(x)}]^{-1}(\tilde{V})$. We also use the sequence $(\xi_n)_n$ of Lemma 3.7.

Notice that all the definitions contain entropy inequalities (3.1). We concentrate on the equivalence of the complementary properties related to the boundary condition.

<u>(i) ⇒ (ii)</u> The claim is straightforward by the definition (1.11) of $Q^\pm_{\nu(x)}$ and the monotonicity of the graph of $\varphi_{\nu(x)}|_{\widetilde{D}^0_{(t,x)}}$.

<u>(ii) ⇒ (i)</u> This implication is a consequence of the maximality of the graph $\widetilde{\beta}_{(t,x)}$ as a monotone subgraph of $\varphi_{\nu(x)}$. Thanks to inequality (3.3), we have $\operatorname{sign}(\tilde{u}-k)(\varphi_{\nu(x)}(\tilde{u}) - \varphi_{\nu(x)}(k)) \geq 0$ for all $k \in \widetilde{D}^0_{(t,x)}$; thus, $\varphi_{\nu(x)}$ is monotone not only on $\widetilde{D}^0_{(t,x)}$ but also on $\tilde{u} \cup \widetilde{D}^0_{(t,x)}$. Thus $\tilde{u} \in \widetilde{D}^0_{(t,x)}$ and $\tilde{V} \in V\varphi_{\nu(x)}\widetilde{D}^0_{(t,x)}$, which proves (i).

<u>(i) ⇔ (iii)</u> This equivalence follows from Proposition 2.5(i).

(i) $\Rightarrow$ (iv) As a preliminary step, we assess the following property (see [7, 37]): for all $k \in \mathbb{R}$, for all $\xi \in \mathcal{D}([0,T)\times\overline{\Omega})^+$,

$$\begin{aligned}(3.12)\qquad \int_0^T\!\!\int_\Omega \Big(-(u-k)^\pm \xi_t - q^\pm(u,k)\cdot\nabla\xi\Big) - \int_\Omega (u_0-k)^\pm \xi(0,\cdot) \\ \le\ -\iint_\Sigma Q^\pm_{\nu(x)}\Big(\gamma V\varphi_{\nu(x)}u\,,\, V\varphi_{\nu(x)}k\Big)\,\xi(t,x).\end{aligned}$$

Indeed, taking (by approximation) for the test function in (3.1) a non-negative function $\xi \in \mathcal{D}([0,T)\times\overline{\Omega})$ multiplied by the truncation $(1-\xi_n)$, we get (3.12) from (3.11) of Lemma 3.7.

It remains to justify using the information that $\tilde u \in \widetilde{D}^0_{(t,x)}$, that

$$(3.13)\quad -Q^\pm_{\nu(x)}\Big(\gamma V_{\varphi_{\nu(x)}}u\,,\, V_{\varphi_{\nu(x)}}k\Big) \equiv -q^\pm_{\nu(x)}(\tilde u,k)\ \le\ C_k \wedge \Big(\beta_{(t,x)}(k) - \varphi_{\nu(x)}(k)\Big)^\mp.$$

The upper bound of the left-hand side of (3.13) by $C_k := 2\max\big\{|\varphi(z)|\ \big|\ |z| \le k + \|u\|_\infty\big\}$ is evident. Further, if $k \in D^\pm_{(t,x)}$, then we already know that (i) implies (3.4), which gives

$$(3.14)\qquad -Q^\pm_{\nu(x)}\Big(\gamma V_{\varphi_{\nu(x)}}u\,,\, V_{\varphi_{\nu(x)}}k\Big)\ \le\ 0\ \le\ \Big(\beta_{(t,x)}(k) - \varphi_{\nu(x)}(k)\Big)^\mp,$$

proving (3.13) for this case. Let us study the remaining values of k.

For the sake of being definite, let us consider $k \in D^-_{(t,x)}$, $k < \tilde u$; then the goal is to estimate $-q^+_{\nu(x)}(\tilde u,k)$ from above by $R_k := \Big(\beta_{(t,x)}(k) - \varphi_{\nu(x)}(k)\Big)^-$. Consider the four possible cases:

- If $\varphi_{\nu(x)}(\tilde u) \ge \varphi_{\nu(x)}(k)$, there is nothing to prove because $-q^+_{\nu(x)}(\tilde u,k) \le 0 \le R_k$.
- If $\beta_{(t,x)}(k) \le \varphi_{\nu(x)}(\tilde u) < \varphi_{\nu(x)}(k)$, then

$$\begin{aligned} -q^+_{\nu(x)}(\tilde u,k) &\equiv \varphi_{\nu(x)}(k) - \varphi_{\nu(x)}(\tilde u) \le \varphi_{\nu(x)}(k) - \beta_{(t,x)}(k) \\ &= \Big(\beta_{(t,x)}(k) - \varphi_{\nu(x)}(k)\Big)^- = R_k.\end{aligned}$$

- If $\tilde u \in D^-_{(t,x)}$, then $\varphi_{\nu(x)}(\tilde u) \ge \beta_{(t,x)}(\tilde u)$ and from the monotonicity of $\beta_{(t,x)}$, we have

$$-q^+_{\nu(x)}(\tilde u,k) \equiv \varphi_{\nu(x)}(k) - \varphi_{\nu(x)}(\tilde u) \le \varphi_{\nu(x)}(k) - \beta_{(t,x)}(\tilde u) \le \varphi_{\nu(x)}(k) - \beta_{(t,x)}(k) = R_k.$$

- The case $\tilde u \in D^+_{(t,x)}$, $k \in D^-_{(t,x)}$, and $\varphi_{\nu(x)}(\tilde u) < \beta_{(t,x)}(k) < \varphi_{\nu(x)}(k)$ remains; let us show that this is impossible. Indeed, in this case there exists $k' \in (k,\tilde u]$ that belongs to $D^0_{(t,x)}$. Then $k' \in \widetilde{D}^0_{(t,x)}$ according to the definition of this graph. Yet also $\tilde u \in \widetilde{D}^0_{(t,x)}$; by the definition of $D^0_{(t,x)}$ and the monotonicity of $\varphi_{\nu(x)}|_{\widetilde{D}^0_{(t,x)}}$, we infer

$$\beta_{(t,x)}(k') \ni \varphi_{\nu(x)}(k') \le \varphi_{\nu(x)}(\tilde u) < \beta_{(t,x)}(k).$$

This contradicts the monotonicity of $\beta_{(t,x)}$ because $k < k'$.

(iv) $\Rightarrow$ (iii) It is enough to justify inequalities (3.4). We work with mollifying sequences $(\xi^\alpha)_\alpha$ on Σ (extended smoothly inside Ω) that are supported in an α-neighbourhood of some $\sigma \in \Sigma$; as $\alpha \to 0$, ξ^α concentrates to the Dirac measure supported at σ.

Fix $k \in \mathbb{R}$ and consider, e.g., $\sigma = (t,x) \in \Sigma^+(k)$. Almost every point of $\Sigma^+(k)$ is its point of density (see, e.g., [19]), which means in particular that for $\mathcal{H}^N$-a.e. $\sigma \in \Sigma^+(k)$,

$$\lim_{\alpha\to 0} \iint_\Sigma C_k \wedge \Big(\beta_{(t,x)}(k) - \varphi_{\nu(x)}(k)\Big)^- \xi^\alpha(t,x) = 0. \tag{3.15}$$

Indeed, the integrand in the right-hand side is bounded by C_k and by the definition (3.2); it is zero for $(t,x) \in \Sigma^+(k)$.

Now we generate inequalities (3.4) by taking the test functions $\xi^\alpha \xi_n$ (with $(\xi_n)_n$ constructed in Lemma 3.7). Consequently using (3.10), (3.5) and (3.15), we infer that

$$\begin{aligned}
- Q^+_{\nu(x)}\Big(\gamma V_{\varphi_{\nu(x)}} u\,,\, V_{\varphi_{\nu(x)}} k\Big)|_{(t,x)=\sigma} &= -\lim_{\alpha\to 0} \iint_\Sigma \xi^\alpha\, Q^+_{\nu(x)}\Big(\gamma V_{\varphi_{\nu(x)}} u\,,\, V_{\varphi_{\nu(x)}} k\Big) \\
&= \lim_{\alpha\to 0} \lim_{n\to\infty} \Bigg(\int_0^T \int_\Omega \Big(-(u-k)^\pm (\xi^\alpha \xi_n)_t - q^+(u,k) \cdot \nabla(\xi^\alpha \xi_n) \Big) \\
&\qquad\qquad - \int_\Omega (u_0 - k)^\pm (\xi^\alpha \xi_n)(0,\cdot) \Bigg) \\
&\le \lim_{\alpha\to 0} \iint_\Sigma C_k \wedge \Big(\beta_{(t,x)}(k) - \varphi_{\nu(x)}(k)\Big)^- \xi^\alpha(t,x) = 0.
\end{aligned}$$

Similarly, the case $\sigma \in \Sigma^-(k)$ leads to the inequality

$$-Q^-_{\nu(x)}\Big(\gamma V_{\varphi_{\nu(x)}} u\,,\, V_{\varphi_{\nu(x)}} k\Big)|_{(t,x)=\sigma} \le 0.$$

<u>(iv) $\Rightarrow$ (v)</u> Inequalities (3.7) are immediate from (3.5).

<u>(v) & (3.6) $\Rightarrow$ (i)</u> Actually, we'd rather prove (iii). Under the assumption that there exists a mollifying sequence $(\xi^\alpha)_\alpha$ on Σ that concentrates at $\sigma \in \Sigma^\pm(k)$ and, moreover, that is identically zero on $\Sigma \setminus \Sigma^\pm(k)$, we can repeat the proof of the above implication "(iv) $\Rightarrow$ (iii)". A sufficient condition is the density of $\mathcal{D}(\Sigma^\pm(k))$ in $L^1(\Sigma^\pm(k))$. Moreover, this assumption is needed only for a countable dense set of values of k: indeed, the proof of the implication "(iii) $\Rightarrow$ (i)" can be rewritten so that it uses only a dense subset of values of k satisfying (3.4). Thus, (3.6) is enough to derive the trace condition in (i). $\square$

4. Uniqueness, comparison, continuous dependence

Following the ideas of [6, 12, 22, 26], we introduce the "uniqueness condition":

$$\text{either } \Omega \text{ is bounded or } N = 1 \text{ or } \varphi \text{ is locally Hölder continuous of order } 1 - \tfrac{1}{N}. \tag{4.1}$$

In the classical case of a locally Lipschitz continuous flux φ this assumption holds automatically.

The following result contains uniqueness of an entropy solution for problem $(H_{\varphi,\beta}(u_0, f))$, L^1 contraction and the comparison property with respect to the initial datum u_0 and the source term f, and a comparison and stability property with respect to the choice of $\beta_{(t,x)}(\cdot)$.

Theorem 4.1. *Assume* (4.1). *Let* u^1 *be an entropy sub-solution for problem* $(H_{\varphi,\beta^1}(u_0^1, f^1))$*; let* u^2 *be an entropy super-solution for problem* $(H_{\varphi,\beta^2}(u_0^2, f^2))$*. Then for a.e.* $s \in (0,T)$*,*

$$\int_\Omega (u^1 - u^2)^+(s) \le \int_\Omega (u_0^1 - u_0^2)^+ + \int_0^s \int_\Omega sign^+(u^1 - u^2)(f^1 - f^2) + \int_0^t \int_{\partial\Omega} d_x^-(\tilde\beta^1_{(t,x)}, \tilde\beta^2_{(t,x)}). \tag{4.2}$$

In particular, if $u_0^1 \le u_0^2$ *a.e. on* Ω*,* $f^1 \le f^2$ *a.e. on* Q_T*; and if* $\beta^1_{(t,x)} \succeq_x \beta^2_{(t,x)}$ $\mathcal{H}^N$*-a.e. on* Σ*, then* $u^1 \le u^2$ *a.e. on* Q_T*. In particular, there exists at most one entropy solution to* $(H_{\varphi,\beta}(u_0, f))$*.*

Note that whenever φ is locally Lipschitz continuous, we can localize the contraction property using the finite speed of propagation, following Kruzhkov [21].

Proof. Consider the case of a bounded domain Ω. We apply the Kruzhkov doubling of variables argument inside the domain to deduce the Kato inequality: for a.e. $s \in (0,T)$, for all $\xi \in \mathcal{D}(\Omega)$,

$$\int_\Omega (u^1 - u^2)^+(s)\,\xi - \int_\Omega (u_0^1 - u_0^2)^+ \xi(0,\cdot) \le \int_0^s \int_\Omega \Big(q^+(u^1,u^2)\cdot\nabla\xi + \mathrm{sign}^+(u^1-u^2)(f^1-f^2)\,\xi \Big). \tag{4.3}$$

Now we take $\xi = 1 - \xi_n$ with $(\xi_n)_n$ constructed in Lemma 3.7, and let $n \to \infty$. Since there exists a strong normal boundary trace of $q^+(u_1,u_2)$ expressed in the way of Remark 1.1 in Section 1.3, we find the inequality

$$\int_\Omega (u^1-u^2)^+(s) \le \int_\Omega (u_0^1-u_0^2)^+ + \int_0^s\int_\Omega \mathrm{sign}^+(u^1-u^2)(f^1-f^2) - \int_0^s\int_\Omega Q^+_{\nu(x)}(\gamma V\varphi_{\nu(x)}(u^1), \gamma V\varphi_{\nu(x)}(u^2)).$$

It remains to show that $-Q^+_{\nu(x)}(\gamma V\varphi_{\nu(x)}(u^1), \gamma V\varphi_{\nu(x)}(u^2)) \le d_x^-(\tilde\beta^1_{(t,x)}, \tilde\beta^2_{(t,x)})$ pointwise on $(0,T)\times\partial\Omega$. This is true because whenever the term on the left is non-zero, we have

$$-Q^+_{\nu(x)}(\gamma V\varphi_{\nu(x)}(u^1), \gamma V\varphi_{\nu(x)}(u^2)) = -(\varphi_{\nu(x)}(\tilde u^1) - \varphi_{\nu(x)}(\tilde u^2)) \tag{4.4}$$

with some $\tilde u^1 > \tilde u^2$ such that

$$V\varphi_{\nu(x)}(\tilde u^1) = \gamma V\varphi_{\nu(x)}(u^1) \in V\varphi_{\nu(x)}\widetilde{D}^-(t,x)$$

and

$$V\varphi_{\nu(x)}(\tilde u^2) = \gamma V\varphi_{\nu(x)}(u^2) \in V\varphi_{\nu(x)}\widetilde{D}^+(t,x)$$

(here we have used the trace properties of entropy sub- and super-solutions; see Proposition 3.5(i)). Thus $\varphi_{\nu(x)}(\tilde u^1) \ge \widetilde{\mathcal{B}^1}_{(t,x)}(\tilde u^1)$, $\varphi_{\nu(x)}(\tilde u^2) \le \widetilde{\mathcal{B}^2}_{(t,x)}(\tilde u^2)$, so that the right-hand side of (4.4) fulfills

$$-(\varphi_{\nu(x)}(\tilde u^1) - \varphi_{\nu(x)}(\tilde u^2)) \le -(\widetilde{\mathcal{B}^1}_{(t,x)}(\tilde u^1) - \widetilde{\mathcal{B}^2}_{(t,x)}(\tilde u^2)) \le d_x^-(\widetilde{\mathcal{B}^1}_{(t,x)}, \widetilde{\mathcal{B}^2}_{(t,x)}),$$

where we have used the definition of d_x^- and the fact that $\tilde u^1 > \tilde u^2$.

For the case when Ω is unbounded, we proceed in the same way we got the up-to-the boundary Kato inequality, i.e., inequality (4.3) with a test function $\xi \in \mathcal{D}(\overline{\Omega})$.

Assuming either that $N=1$, or that $N\geq 2$ and φ is locally Hölder continuous of order $1-\frac{1}{N}$, we use the techniques known for scalar conservation laws with infinite speed of propagation (see, e.g., [6, 12, 22]) and eventually deduce (4.2). $\square$

5. Existence: A formal proof

In this section, we establish existence on an entropy solution, but we take for granted that the formal BC, encrypted by the graphs $\beta_{(t,x)}$, should be replaced by the boundary condition expressed with the help of their projections $\widetilde{\mathcal{B}}_{(t,x)}=\widetilde{\mathcal{P}}_x\beta_{(t,x)}$. Section 6 contains a longer but more convincing discussion of the problem of existence and convergence of approximations.

For general graphs β satisfying the measurability assumption (1.1), we cannot hope for existence of a bounded solution (it is enough to consider, e.g., the situation where unbounded Dirichlet data are imposed: in this case, one needs the notion of a renormalized solution, as used by Porretta, Vovelle [35] and by Ammar, Carrillo and Wittbold [2]). We control the L^∞ norm of solutions or approximate solutions by assuming existence of a rich enough family of simple (constant in space) sub- and super-solutions to the problem. Namely, we require that one of the two following assumptions be fulfilled: either

$$\begin{aligned}&f=0,\text{ and there exist }(A^-_m)_{m\in\mathbb{N}},(A^+_m)_{m\in\mathbb{N}}\subset\mathbb{R}^\pm\text{ such that }A^\pm_m\to\pm\infty\\&\text{as }m\to\infty\quad\text{and}\quad\text{for }\mathcal{H}^N\text{-a.e. }(t,x)\in\Sigma,\text{ for all }m\in\mathbb{N},\ A^\pm_m\in\widetilde{D}^\pm_{(t,x)}\end{aligned}\tag{5.1}$$

or

$$\begin{aligned}&\text{the measures of the sets }\mathcal{A}^\pm:=\Big\{k\in\mathbb{R}^\pm\,|\,k\in\widetilde{D}^\pm_{(t,x)}\ \text{ for }\mathcal{H}^N\text{-a.e. }(t,x)\in\Sigma\Big\}\\&\text{are infinite.}\end{aligned}\tag{5.2}$$

Note that (5.2) is ensured by the following:

$$\exists A\text{ for }\mathcal{H}^N\text{-a.e.}(t,x)\in\Sigma,\quad(-\infty,-A]\subset\widetilde{D}^-_{(t,x)}\ \text{ and }\ [A,+\infty)\subset\widetilde{D}^+_{(t,x)}.\tag{5.3}$$

Remark 5.1. Given a formal BC graph β, it is not immediate to check whether (5.3), (5.1), or (5.2) hold. Let us give sufficient conditions.

Firstly, by definition we have $D^\pm_{(t,x)}\subset\widetilde{D}^\pm_{(t,x)}$, where $D^+_{(t,x)}$, resp. $D^-_{(t,x)}$, is the overshoot (resp., the undershoot) set defined in Section 2; $D^\pm_{(t,x)}$ are computed directly from the relative positions of the graphs $\beta_{(t,x)}$ and $\varphi_{\nu(x)}$. Thus replacing $\widetilde{D}^\pm_{(t,x)}$ by $D^\pm_{(t,x)}$ in each of the assumptions (5.3), (5.1), or (5.2), we get stronger but easier-to-check restrictions (cf. [7, 8]).

Secondly, if there exists $C>0$ such that for all $x\in\partial\Omega$ each of the functions $\varphi_{\nu(x)}|_{(-\infty,C]}$ and $\varphi_{\nu(x)}|_{[C,+\infty)}$ is either non-decreasing or non-increasing, then it is easily checked that assumption (5.3) (and thus also (5.2)) holds.

Assume in addition that the limit flux φ is genuinely non-linear in the sense

$$\begin{aligned}&\forall\,\Xi\in\mathbb{R}^{N+1}\backslash\{0\}\ \forall c\in\mathbb{R}\ \text{ the Lebesque measure of the set}\\&\{z\,|\,\Xi\cdot(z,\varphi(z))=c\}\ \text{ is zero.}\end{aligned}\tag{5.4}$$

Notice that the latter assumption implies (1.7); in particular the singular mapping $V\varphi_{\nu(x)}$ can be taken to be Id in this case.

The main result is the following theorem.

Theorem 5.2. *Assume that φ satisfies* (4.1) *and* (5.4). *Let $u_0 \in L^\infty(\Omega)$ and $\int_0^T \|f(t,\cdot)\|_\infty \, dt < \infty$. Assume that β satisfies* (1.1) *and any of the assumptions* (5.1), (5.2). *Then there exists a unique entropy solution of problem $(H_{\varphi,\beta}(u_0, f))$.*

Proof. Uniqueness is contained in Theorem 4.1. For proving existence, we exploit the vanishing viscosity method in which we use directly the projected graphs $\widetilde{\mathcal{B}}_{(t,x)} = \widetilde{\mathcal{P}}_x \beta_{(t,x)}$. We apply two results that are justified in the sequel. Firstly, we construct approximate solutions u^ε by the vanishing viscosity method, using Proposition 7.2 (see also Remark 7.2) of the Appendix. Indeed, $k \mapsto \widetilde{\mathcal{B}}_{(t,x)}(k) =: b(t,x;k)$ being a continuous function for fixed $(t,x) \in \Sigma$, from (1.1) we deduce that the map b on $\Sigma \times \mathbb{R}$ is Carathéodory. Because Proposition 7.2 requires that b be bounded, we pick some value $M > 0$ depending on $\|u_0\|_\infty + \int_0^T \|f(t,\cdot)\|_\infty \, dt$ and on $(A_m^\pm)_m$ or $\mathcal{A}^\pm$ in the assumptions (M is chosen as *a priori* a bound of $\|u\|_\infty$, to be justified later). We proceed by truncating φ and $\widetilde{\mathcal{B}}_{(t,x)}$ as follows: e.g., under assumption (5.1) we take m such that $[A_m^-, A_m^+] \supset [-M, M]$ with $M = \|u_0\|_\infty$ and take the convention that

(5.5)

$$\begin{aligned} &\varphi \text{ is constant on } (-\infty, A_m^-] \text{ and on } [A_m^+, +\infty) \text{ (equal to } \varphi(A_m^\pm)\text{, resp.)}, \\ &\widetilde{\mathcal{B}}_{(t,x)} \text{ is constant on } (-\infty, A_m^-] \text{ and on } [A_m^+, +\infty) \text{ (equal to } \widetilde{\mathcal{B}}_{(t,x)}(A_m^\pm)\text{, resp.)}. \end{aligned}$$

Therefore we get existence of vanishing viscosity approximations $(u^\varepsilon)_\varepsilon$ corresponding to the truncated graphs.

Let us stress the fact that, because $A_m^\pm \in \widetilde{D}_{(t,x)}^\pm$, truncation (5.5) does not change the fact that $\widetilde{\mathcal{B}}_{(t,x)}$ is a maximal monotone subgraph of $\varphi_{\nu(x)}$. For the same reason, the truncated graphs φ and $\widetilde{\mathcal{B}}$ fulfill assumption (5.1) with the same sequences $(A_m^\pm)_m$; hence by Proposition 7.2(ii) the solutions obey an L^∞ estimate that does not depend on the truncation level chosen in (5.5).

Now we can exploit Theorem 6.2 stated and proved in Section 6.1. Its assumptions (6.1)–(6.3) are fulfilled: indeed, notice that we have required the genuine non-linearity property (5.4) that implies compactness (see, e.g., [30, 33]) and that $\widetilde{\mathcal{B}}$ is bounded by $\max_{[A_m^-, A_m^+]} |\varphi|$ due to the truncation convention (5.5). We deduce existence of an entropy solution u to the truncated problem $(H_{\varphi,\beta}(u_0, f))$. Yet we have also ensured that $\|u\|_\infty \le M$; therefore the constructed solution u also solves the original problem $(H_{\varphi,\beta}(u_0, f))$ (before the truncation (5.5)). This ends the proof. □

6. Justification of the effective boundary condition

The goal of this section is to provide evidence in favor of the interpretation (1.13), $(A_{\tilde{\beta}})$ of the effective boundary condition. As was already stated in the introduction, a natural way to justify a notion of solution is to see problem $(H_{\varphi,\beta}(u_0, f))$ as the limit of a family of problems for which the notion of solution is unambiguous: one derives the solution notion from passage-to-the-limit arguments. In this section, we do it in two complementary ways, following the general idea of our previous works ([7] and [8], respectively).

Firstly, in Section 6.1 we rely on the classical notion of a weak solution to parabolic problems with additional viscosity term, vanishing at the limit. The entropy formulation of $(H_{\varphi,\beta}(u_0, f))$ is obtained as a *singular limit* formulation: indeed, the limit problem loses its parabolic character. Unfortunately, for a practical application of this technique we will need several restrictions on the behaviour of β and φ. To separate the technical details from the key idea of the proof, we assume, without comment, that approximate solutions possess uniform bounds and a strong compactification property. Notice that the techniques of Section 6.1 are very different from the ones of the preceding works [7, 37], where we also needed the difficult-to-ensure compactification assumptions on the sequence of approximate solutions on the boundary.

Remark 6.1. Although the arguments of [7,37] are less general, they have the advantage of showing quite explicitly how the projected graph $\tilde{\beta}$ (in its characterization [7, formula (4)], equivalent to the characterization of Proposition 2.5(ii)) appears from β.

In a sense, with [7, formula (4)] one can "observe" the formation of the boundary layer (see [37] for details). The arguments we use in this paper are more indirect; they lead to the characterization of Proposition 2.5(i) via the formulation (3.5).

Secondly, in Section 6.2 we consider approximations of $(H_{\varphi,\beta}(u_0, f))$ by purely hyperbolic problems of the same type (but with possibly different data and non-linearities) and exploit the stability and comparison principle of Theorem 4.1 in order to extend the entropy formulation "by heredity". This allows us, e.g., to concentrate on the case of smooth and/or compactly supported initial data that may be useful in the context of a locally Lipschitz flux φ (cf. Section 7.1). For the Dirichlet or obstacle condition, we can approximate the boundary data either pointwise or using the Lusin theorem. For a less evident application, one can approximate a general graph β by a bi-monotone sequence of graphs $\beta^{\delta,\lambda}$ satisfying the assumptions of the previous section (by bi-monotonicity, we mean that $\beta^{\delta,\lambda}$ decreases as $\delta \downarrow 0$ and increases as $\lambda \downarrow 0$). In this way we can justify the use of the projected graph $\widetilde{\beta}$ for the homogeneous Neumann boundary condition (whereas the justification in the way of Section 6.1 does not work in this case); see [8].

6.1. **Convergence of the vanishing viscosity approximation.** Let us provide a basic convergence argument for the vanishing viscosity approximation (without any additional regularization or approximation of data and non-linearities).

We make the following *a priori* assumptions on data and non-linearities of problem (1.3):

(6.1)
$$\text{For all } \varepsilon \in (0,1) \text{ there exists a weak solution}^{13}\ u^\varepsilon \in L^2(0,T; H^1_{loc}(\Omega)) \text{ of (1.3);}\\ \text{moreover, the family } (\sqrt{\varepsilon}\nabla u^\varepsilon)_\varepsilon \text{ is bounded in } L^2_{loc}([0,T]\times\Omega).$$

(6.2)
$$\text{There exists } u \in L^\infty(Q_T) \text{ and a sequence } \varepsilon_m \text{ decreasing to zero as } m \to \infty\\ \text{such that } u^{\varepsilon_m} \to u \text{ in } L^1_{loc}([0,T]\times\Omega) \text{ as } m \to \infty.$$

$$\begin{array}{l} \text{There exists } G \in L^1_{loc}([0,T]\times\partial\Omega) \text{ such that} \\ |b^\varepsilon(t,x)| \le G(t,x) \text{ for } \mathcal{H}^N\text{-a.e. } (t,x) \in \Sigma, \text{ uniformly in } \varepsilon \in (0,1), \end{array} \tag{6.3}$$

where $b^\varepsilon(t,x) \in \beta_{(t,x)}(u^\varepsilon(t,x))$ is the value realized in the multi-valued boundary condition of (1.3) (namely, $b^\varepsilon(t,x) := \gamma_w(\varphi(u^\varepsilon) - \varepsilon\nabla u^\varepsilon)\cdot\nu(x)$, the right-hand side

[13]See the Appendix for a precise definition of a weak solution.

having the meaning of the weak normal boundary trace of the divergence-measure field $(\varphi(u^\varepsilon) - \varepsilon\nabla u^\varepsilon)$; see [17]). Writing $\beta_{(t,x)}(u^\varepsilon(t,x))$, we use without further mention the restriction $u^\varepsilon|_\Sigma$ of u^ε on the boundary, understood in the sense of traces of Sobolev functions.

Theorem 6.2. *Assume that* u_0, f *and* φ, β *are such that* (6.1), (6.2) *and* (6.3) *hold. Then* u *is an entropy solution of problem* $(H_{\varphi,\beta}(u_0, f))$.

Remark 6.3. In practice, (6.2) can be fulfilled as a compactness property. In this case, let us suppose that the uniqueness condition (4.1) of Theorem 4.1 holds. Then from the uniqueness of the accumulation point u we deduce that the whole family u^ε converges, as $\varepsilon \to 0$, to the entropy solution of $(H_{\varphi,\beta}(u_0, f))$.

Proof. It is classical (see, e.g., Carrillo [16]) to deduce from the weak formulation of (1.3) the Kruzhkov entropy inequalities (3.1) with $\mathcal{D}([0,T) \times \Omega)$ test functions (i.e., entropy formulation *inside the domain*). One readily passes to the limit in this entropy formulation using the property (6.2) and the uniform L^2_{loc} bound on $\sqrt{\varepsilon}\nabla u^\varepsilon$ contained in assumption (6.1). In our case, the delicate issue is to pass to the limit in the *up-to-the-boundary* entropy formulation of (1.3). Our goal is to deduce the characterization (3.5) of entropy solution.

To this end, we reproduce the arguments of [16], but we now take $\xi \in \mathcal{D}([0,T)\times\overline{\Omega})$, $\xi \geq 0$. We multiply (1.3) by the test function $H_\alpha(u^\varepsilon - k)\xi$, where H_α is a Lipschitz regularization of sign $^+$ (the case of sign $^-$ is similar) such that $H'_\alpha(r) = \frac{1}{\alpha}\mathbb{1}_{(0,\alpha)}(r)$. In addition, we substitute the term $\varphi(u^\varepsilon)$ in (1.3) by $\varphi(u^\varepsilon) - \varphi(k)$, which results in the "new" boundary condition

$$(\varphi(u^\varepsilon) - \varphi(k) - \varepsilon\nabla u^\varepsilon) \cdot \nu(x) \in \beta_{(t,x)}(u^\varepsilon) - \varphi_{\nu(x)}(k).$$

Using the chain rule in time (see, e.g., [1, 29]), using, in addition, [16, Lemma 1] to make the term $\lim_{\alpha\to 0^+} \int_\Omega (\varphi(u) - \varphi(k))H'_\alpha(u^\varepsilon - k)\xi$ disappear, dropping the positive term $\varepsilon H'_\alpha(u^\varepsilon - k)|\nabla u^\varepsilon|^2$ at the limit $\alpha \to 0^+$, we derive the "parabolic up-to-the-boundary entropy equality"

$$\begin{aligned}(6.4)\qquad &\int_0^T\int_\Omega \Big(-(u^\varepsilon - k)^+\xi_t - q^+(u^\varepsilon, k)\cdot\nabla\xi\Big) - \int_\Omega (u_0 - k)^+\xi(0,\cdot)\\ &\leq -\int_\Sigma \operatorname{sign}^+(u^\varepsilon - k)(b^\varepsilon(t,x) - \varphi_{\nu(x)}(k))\xi - \varepsilon\int_0^T\int_\Omega sign^+(u^\varepsilon - k)\nabla u^\varepsilon\cdot\nabla\xi\end{aligned}$$

with some $b^\varepsilon(t,x) \in \beta_{(t,x)}(u^\varepsilon)$ satisfying (6.3). Recall that we have assumed that $f = 0$, the general case being similar. In the right-hand side of (6.4), by the monotonicity of $\beta_{(t,x)}$ we have, pointwise on Σ, the multi-valued inequality

$$(6.5)\qquad -\operatorname{sign}^+(u^\varepsilon - k)(b^\varepsilon(t,x) - \varphi_{\nu(x)}(k)) \leq (\beta_{(t,x)}(k) - \varphi_{\nu(x)}(k))^-.$$

Here, the quantity in the right-hand side can be infinite, which makes problematic the localization arguments. Under assumption (6.3) (see Remarks 6.6–6.9 for comments and generalizations) the left-hand side of (6.5) is upper bounded by the $L^1_{loc}([0,T] \times \partial\Omega)$ function defined by $G_k(t,x) := G(t,x) + |\varphi(k)|$. Letting $\varepsilon \to 0^+$, from the L^1_{loc} convergence assumption (6.2) we deduce

$$\begin{aligned}(6.6)\qquad &\int_0^T\int_\Omega \Big(-(u - k)^+\xi_t - q^+(u, k)\cdot\nabla\xi\Big) - \int_\Omega (u_0 - k)^+\xi(0,\cdot)\\ &\qquad\leq \int_0^T\int_{\partial\Omega} G_k(t,x) \wedge (\beta_{(t,x)}(k) - \varphi_{\nu(x)}(k))^-\,\xi.\end{aligned}$$

Now, since u is an entropy solution inside the domain, we can use the strong normal boundary trace $\gamma q^+_{\nu(x)}(u,k)$ of $q^+(u,k)$ and generate it with the help of the sequence $(\xi_n)_n$ of Lemma 3.7. The positive test function ξ being arbitrary, we deduce

$$\gamma q^+_{\nu(x)}(u,k) \ \le\ G_k(t,x)\wedge(\beta_{(t,x)}(k)-\varphi_{\nu(x)}(k))^- \ \mathcal{H}^N\text{-a.e. on } \Sigma$$

(the inequality holds at the Lebesgue points of the left- and right-hand sides). Now notice that we can provide a more precise upper bound for the left-hand side: taking

$$C_k := 2\max\big\{|\varphi(z)| \ \big|\ |z|\le k+\|u\|_\infty\big\} \ge \|q^+(u,k)\|_\infty,$$

we have $|q^+_{\nu(x)}(u,k)|\le C_k$ pointwise, so that

$$\gamma q^+_{\nu(x)}(u,k) \ \le\ C_k\wedge(\beta_{(t,x)}(k)-\varphi_{\nu(x)}(k))^-. \tag{6.7}$$

Combining the entropy inequalities inside the domain (namely, (6.6) with ξ replaced by $\xi(1-\xi_n)$, with boundary cut-off functions $(\xi_n)_n$ constructed in Lemma 3.7) with (3.11) and (6.7), we finally deduce (3.5). ☐

The simplest example combining Proposition 7.2 and Theorem 6.2 is the following:

Example 6.4. Assume that φ satisfies (4.1) and (5.4). Assume that β fulfills the analogues of assumptions (5.2) or (5.1) with $\widetilde{D}^{\pm}_{(t,x)}$ replaced by $D^{\pm}_{(t,x)}$ (this makes the assumptions stronger; see Remark 5.1). Assume that the graphs $\beta_{(t,x)}$ are single-valued uniformly bounded on $\mathbb{R}$ functions.

Then for all viscosity parameters $\varepsilon>0$, solutions u^ε of the parabolic problem (1.3) exist; moreover, u^ε converge, as $\varepsilon\to 0$, to the unique entropy solution of $(H_{\varphi,\beta}(u_0,f))$.

The justification of this example is contained in the proof of Theorem 5.2.

Several comments are in order; indeed, we need to discuss generalizations and further applications of Theorem 6.2. First, consider the existence and compactification assumptions (6.1) and (6.2).

Remark 6.5.

(i) As we show in the Appendix, the existence assumption (6.1) is verified, e.g., in the case where $\beta_{(t,x)}$ are monotone continuous functions having $\mathbb{R}$ for their domain. But this assumption is not a necessary one. For example, existence for the Dirichlet problem for (1.3) is well known for regular enough bounded Dirichlet data u^D. If the lack of regularity (in (t,x)) of the family $(\beta_{(t,x)})_{(t,x)\in\Sigma}$ does not allow for existence of a solution u^ε, replacing β with a regularized graph β^ε (e.g., the Yosida regularization can be used, pointwise in (t,x)), one can easily generalize the convergence result of Theorem 6.2.

(ii) Property (6.2) is ensured in the case where, firstly, the flux φ is genuinely non-linear in the sense (5.4) and, secondly, a uniform L^∞ estimate on the family $(u^\varepsilon)_\varepsilon$ is available.

(iii) According to Proposition 7.2(ii) (see also Remark 7.2), uniform L^∞ estimates on u^ε are available in the case (A.3) or (A.4) hold. These assumptions exclude important cases. Indeed, for (e.g., homogeneous) Neumann and Robin boundary conditions it is easy to get existence of u^ε, but uniform L^∞ bounds may require additional restrictions on φ; see, e.g., the work of Bürger, Frid and Karlsen [13] on the Neumann BC case.

(iv) Without L^∞ estimates, the issue of convergence of vanishing viscosity approximations becomes quite delicate. For example, in the case of homogeneous Neumann boundary conditions, the family $(u^\varepsilon)_\varepsilon$ may be unbounded and nevertheless converge pointwise to a limit $u \in L^\infty(Q_T)$.

In the present paper, we limit our investigation to the case where uniform L^∞ bounds (coming from constant sub- and super-solutions) are available. We leave the study of the more delicate situations to a future work.

Further, assumption (6.3) is made in order to simplify the proof of Theorem 6.2 and also because it is enough for the existence result of Theorem 5.2. Assumption (6.3) is of a technical nature; unfortunately, it cannot be completely bypassed. We make several comments on (6.3).

Remark 6.6. Assumption (6.3) is trivially satisfied whenever the graphs $\beta_{(t,x)}$ are uniformly bounded. It also holds if u^ε are uniformly bounded and for all $M > 0$, the sets $\beta_{(t,x)}([-M, M])$ are bounded uniformly in $(t, x) \in \Sigma$. A different situation where (6.3) holds is when the sequence $(b^\varepsilon)_\varepsilon$ is convergent in $L^1(\Sigma)$ (or even in $L^1_{loc}([0, T] \times \partial\Omega)$). This was actually the case under the restrictions imposed in our previous works (see [7, 37]).

Remark 6.7. Assumption (6.3) can be replaced by the equi-integrability assumption on $(b^\varepsilon)_\varepsilon$. Indeed, setting $G^\varepsilon_k(t, x) := |b^\varepsilon(t, x)| + |\varphi(k)|$, we get (6.6) with G^ε_k in the place of G_k. The equi-integrability assumption implies that the family of functions $\Big(G^\varepsilon_k(t, x) \wedge (\beta_{(t,x)}(k) - \varphi_{\nu(x)}(k))^-\Big)_\varepsilon$ weakly converges to an $L^1_{loc}([0, T] \times \partial\Omega)$ function that we denote B_k. The proof of Theorem 6.2 leads to the inequalities

$$\int_0^T \int_{\partial\Omega} \xi \gamma q^+_{\nu(x)}(u, k) \le \int_0^T \int_{\partial\Omega} \xi B_k \tag{6.8}$$

for non-negative continuous compactly supported functions ξ on Σ. Moreover, $\gamma q^+_{\nu(x)}(u, k) \le C_k$; therefore $\gamma q^+_{\nu(x)}(u, k) \le C_k \wedge B_k \le C_k \wedge (\beta_{(t,x)}(k) - \varphi_{\nu(x)}(k))^-$ pointwise on Σ.

Remark 6.8. In view of practical application of Theorem 6.2 (which is a conditional result) it would be very useful to replace the dominating assumption (6.3) on $(b^\varepsilon)_\varepsilon$ by the mere L^1_{loc} boundedness assumption

$$\int_0^T \int_{\partial\Omega \cap K} |b^\varepsilon(t, x)| \le const_K \text{ uniformly in } \varepsilon \in (0, 1), \text{ for all compact } K \subset \partial\Omega. \tag{6.9}$$

For example, for the case where $\varphi(0) = 0$, $\beta_{(t,x)}(0) \ni 0$ and 0 is in the interior of $\operatorname{Dom} \beta_{(t,x)}$, the bound (6.9) is satisfied automatically. Indeed, under these restrictions, along with the existence result for u^ε (see Proposition 7.2 and Remark 7.2 in the Appendix) there comes a uniform estimate

$$\int_0^T \int_{\partial\Omega \cap K} b^\varepsilon(t, x) u^\varepsilon(t, x) \le const_K.$$

Due to the monotonicity of $\beta_{(t,x)}$, (6.9) follows readily, while (6.3) is not guaranteed.

With (6.9) in hand the approach of the preceding remark can be applied, but with a locally finite measure replacing the L^1_{loc} function B_k. Unfortunately, starting from (6.8) with B_k a measure the localization argument cannot be continued (see [9, Example 2]).

Remark 6.9. Yet in many important cases, (6.3) is not needed; it can be bypassed whenever the set of finite values of $\beta_{(t,x)}(k)$ is regular enough.

Indeed, introduce $S_k^\pm := \{(t,x) \in \Sigma \mid \sup(\pm\beta_{(t,x)}) < +\infty\}$. For instance, assume that for a dense set of values of k,

$$\begin{aligned} &S_k^\pm = E_{k,\infty}^\pm \bigcup \Big(\cup_{M\in\mathbb{N}} E_{k,M}^\pm\Big) \text{ where } \mathcal{H}^N(E_{k,\infty}^\pm) = 0,\\ &\text{the sets } E_{k,M}^\pm \text{ are open in } \Sigma \text{ and } (\beta_{(t,x)}(k) - \varphi_{\nu(x)}(k))^\pm \le M \text{ on } E_{k,M}^\pm. \end{aligned} \tag{6.10}$$

Under this assumption, we can simply localize inequalities (6.4) using test functions ξ such that $\xi|_\Sigma$ is supported in $E_{k,M}^-$ and then apply (6.5) in the situation where its right-hand side is finite. Then we directly get

$$\begin{aligned} &\gamma q_{\nu(x)}^+(u,k) \le (\beta_{(t,x)}(k) - \varphi_{\nu(x)}(k))^-\\ &\mathcal{H}^N\text{-a.e. on } S_k^- \equiv \{(t,x) \,|\, (\beta_{(t,x)}(k) - \varphi_{\nu(x)}(k))^- \ne +\infty\}, \end{aligned}$$

it being understood that we have $q_{\nu(x)}^+(u,k) \le C_k$ on the complement of this set. This establishes (6.7) and complements the proof of Theorem 6.2 with assumption (6.3) replaced by (6.10).

Such modification of Theorem 6.2 can be applied, e.g., to Dirichlet or obstacle problems under mild regularity assumptions on the boundary data. Indeed, the existence result for the Dirichlet problem is well known, as well as the uniform L^∞ bound on u^ε. Assumption (6.3) of Theorem 6.2 is circumvented in the way of Remark 6.9. To be specific, for the Dirichlet graphs $\beta_{(t,x)} = \{u^D(t,x)\} \times \mathbb{R}$ property (6.10) is fulfilled for continuous and even for piecewise continuous u^D (yet it is not fulfilled in the case of "fractal" data u^D).

Example 6.10 (cf. Bardos, LeRoux and Nédélec [10] and Vasseur [40])**.** Assume that Ω is bounded, φ satisfies (5.4), and $u_0 \in L^\infty(\Omega)$. Assume that β is the Dirichlet graph corresponding to a piecewise continuous datum $u^D \in L^2(0,T; H^{1/2}(\partial\Omega)) \cap L^\infty(\Sigma)$. Then for all viscosity parameters $\varepsilon > 0$, solutions u^ε of the parabolic problem (1.3) exist; then u^ε converge, as $\varepsilon \to 0$, to the unique entropy solution of $(H_{\varphi,\beta}(u_0, f))$.

Analogous results hold for the case of obstacle boundary conditions with piecewise continuous data $u^m, u^M \in \bigcap L^2(0,T; H^{1/2}(\partial\Omega)) \cap L^\infty(\Sigma)$.

6.2. **Stability of the notion of entropy solution.** Let us consider a sequence of problems of the kind $(H_{\varphi,\beta}(u_0, f))$ associated with data u_0^δ, f^δ and non-linearities $\varphi^\delta, \beta^\delta$ (here δ is a parameter; for the sake of being definite assume that δ is positive and converges to zero). We assume that there exist associated entropy solutions u^δ; we want to deduce an entropy formulation for an accumulation point u of u^δ, assuming *ad hoc* convergence of $u_0^\delta, f^\delta, \varphi^\delta, \beta^\delta$ to some limits $u_0^\delta, f^\delta, \varphi^\delta, \beta^\delta$.

In the three subsections 6.2.1–6.2.3 below, we will demonstrate three different kinds of heredity for the notion of entropy solution: the one coming from compactification of $(u^\delta)_\delta$ (due to a genuine non-linearity assumption on the flux φ), the one coming from monotone approximation procedures, and the one where the L^1 contraction property of Theorem 4.1 makes $(u^\delta)_\delta$ a Cauchy sequence. Because we now treat a hyperbolic problem, the boundary condition can be encoded either by the formal BC graphs $\beta_{(t,x)}$ (via formulation (3.5)) or by the projected graphs $\tilde{\mathcal{B}}_{(t,x)} = \widetilde{\mathcal{P}}_x \beta_{(t,x)}$ that directly describe the effective BC. We will exploit the two possibilities.

In §6.2.1, we will need a notion of convergence of maximal monotone graphs. Let us take the following:

(6.11)

$$\beta^\delta_{(t,x)} \to \beta_{(t,x)} \text{ as } \delta \to 0 \quad \text{if} \quad \liminf_{\delta\to 0} \beta^\delta_{(t,x)}(k) \le \beta_{(t,x)}(k) \le \limsup_{\delta\to 0} \beta^\delta_{(t,x)}(k)$$

at every point k of continuity of $\beta_{(t,x)}(k)$.

This assumption is satisfied, e.g., if $\beta^\delta_{(t,x)}$ are the Yosida approximations of $\beta_{(t,x)}$ (Yosida approximation is a classical way for regularizing maximal monotone graphs; see, e.g., [38] and §6.2.1). A different notion of convergence of β^δ can be given in terms of the distance d_x between the projected graphs $\widetilde{\mathcal{P}}_x\beta^\delta$; it is used in §6.2.3, and the corresponding order relation is exploited in §6.2.2.

Throughout the section, we assume that

$$\exists M > 0 \ \ \|u^\delta\|_\infty \le M \ \text{ uniformly in } \delta \tag{6.12}$$

(in particular, $\|u_0^\delta\|_\infty$ and $\int_0^T \|f^\delta(t,\cdot)\|_\infty\,dt$ should obey uniform in δ bounds). In order to enforce the L^∞ estimate (6.12), we actually assume that either (5.1) or (5.2) is fulfilled (with δ-dependent f and the sets $\widetilde{D}^\pm_{(t,x)}$).

Lemma 6.11. *Assume* (4.1).

(i) Assume $\|u_0^\delta\|_\infty$ and $\int_0^T \|f^\delta(t,\cdot)\|_\infty\,dt$ are bounded uniformly in δ and assumption (5.2) *holds with $\mathcal{A}^\pm$ independent of δ. Then* (6.12) *holds.*

(ii) Assume that $\|u_0^\delta\|_\infty$ is bounded uniformly in δ and assumption (5.1) *holds with $(A^\pm_m)_m$ independent*[14] *of δ. Then* (6.12) *holds.*

The lemma is shown by using the comparison principle of Theorem 4.1 and the appropriate sub- and super-solutions of problems $(H_{\varphi^\delta,\beta^\delta}(u_0^\delta, f^\delta))$ that stem either from assumption (5.1) (constants $A^\pm_m$ are used) or from assumption (5.2) (in this case, the construction described in Remark 7.2 is used).

6.2.1. *Heredity by compactness.* In this subsection, let us assume the following properties:

$$\begin{array}{l} u_0^\delta \ \text{ converge to } u_0 \text{ in } L^1_{loc}(\Omega), \quad f^\delta \ \text{ converge to } f \text{ in } L^1_{loc}(Q_T), \\ \varphi^\delta \text{ converge to } \varphi \text{ uniformly on every compact interval of } \mathbb{R}, \\ \text{and for } \mathcal{H}^N\text{-a.e. } (t,x)\in\Sigma, \quad \beta^\delta_{(t,x)} \to \beta_{(t,x)} \text{ in the sense of (6.11).} \end{array} \tag{6.13}$$

(Note that it is enough to assume relative compactness of $(u^\delta)_\delta$ and of $(f^\delta)_\delta$.)

Proposition 6.12. *Assume the data $u_0^\delta, f^\delta, \varphi^\delta, \beta^\delta$ converge in the sense of* (6.13)*. Assume* (5.3) *or* (5.1) *holds with A or $(A^\pm_m)_m$ that are suitable for φ^δ, β^δ simultaneously for all $\delta > 0$. Assume φ is genuinely non-linear in the sense of* (5.4)*, and assume that* (4.1) *holds.*

Assume $\|u_0^\delta\|_\infty$ is uniformly bounded; in the case (5.3) *assume $\int_0^T \|f^\delta(t,\cdot)\|_\infty\,dt$ is uniformly bounded. Consider a family $(u^\delta)_\delta$ of entropy solutions of $(H_{\varphi^\delta,\beta^\delta}(u_0^\delta, f^\delta))$. Then there exists an accumulation point u of $(u^\delta)_\delta$, as $\delta \to 0$, and u is an entropy solution of the limit problem $(H_{\varphi,\beta}(u_0, f))$.*

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[14] This assumption can be generalized; e.g., it is enough that $c_m \le \pm A^{\pm,\delta}_m \le C_m$ with $c_m \to \infty$ as $m \to \infty$.

Proof. First of all, the uniform L^∞ bound (6.12) follows by Lemma 6.11. Then L^1_{loc} relative compactness of $(u^\delta)_\delta$ is a consequence of the convergence of φ^δ to φ and of assumption (5.4) (see, e.g., [30, 33] and [24]). It remains to pass to the limit in the entropy formulation for $H_{\varphi^\delta,\beta^\delta}(u_0^\delta, f^\delta)$; to do this, we pick the up-to-the-boundary entropy inequalities (3.5) written for u^δ. Let us focus on the case of the entropies $(\cdot - k)^+$; the case of $(\cdot - k)^-$ is analogous. Passage to the limit in the left-hand side is straightforward, using (6.13). Thus we only have to establish that, for $\xi \in \mathcal{D}(\Sigma)$,

$$\begin{aligned}\liminf_{\delta\to 0} \iint_\Sigma C_k \wedge \Big(\inf \beta^\delta_{(t,x)}(k) - \varphi^\delta_{\nu(x)}(k)\Big)^- \xi \\ \le \iint_\Sigma C_k \wedge \Big(\inf \beta_{(t,x)}(k) - \varphi_{\nu(x)}(k)\Big)^- \xi\end{aligned} \tag{6.14}$$

(see the first point of Remark 3.4). Recall that C_k may depend only on k, $\|u^\delta\|_\infty$ and a local bound of $|\varphi^\delta|$; thus we can take C_k independent of δ. Consequently, the dominated convergence theorem for (6.14) can be used.

According to Lemma 6.13 below, convergence (6.11) of β^δ does imply that for a.e. $k \in \mathbb{R}$, there holds $\inf \beta^\delta_{(t,x)}(k) \to \inf \beta_{(t,x)}(k)$ for $\mathcal{H}^N$-a.e. $(t,x) \in \Sigma$ as $\delta \to 0$ (the convergence takes place in $\overline{\mathbb{R}}$). Since $\varphi^\delta(k)$ tends to $\varphi(k)$, the left-hand side integrand in (6.14) converges $\mathcal{H}^N$-a.e. to the integrand of the right-hand side. This justifies (6.14) for all $k \in \mathbb{R}$ except, maybe, for a set of measure zero. Because the left-hand side of (3.5) is continuous in k, one readily extends (3.5) to all $k \in \mathbb{R}$. This ends the proof. $\square$

Lemma 6.13. *Under assumption* (1.1)*, a.e. point $k \in \mathbb{R}$ is a continuity point of $\beta_{(t,x)}$ simultaneously for $\mathcal{H}^N$-a.e. $(t,x) \in \Sigma$.*

Proof. Consider the $[0,+\infty]$-valued map $\theta(t,x;k) := \sup \beta_{(t,x)}(k) - \inf \beta_{(t,x)}(k)$; it is measurable on $\Sigma \times \mathbb{R}$ due to (1.1). Indeed, by (1.1), for all $\ell, c \in \mathbb{R}$, $m_\ell(c) := \{(t,x) \in \Sigma \mid \inf \beta_{(t,x)}(\ell) > c\}$ is an $\mathcal{H}^N$-measurable subset of Σ; thus $m_\ell(c) \times [\ell, +\infty]$ is measurable on $\Sigma \times \overline{\mathbb{R}}$ w.r.t. to the measure $\mathcal{H}^N \otimes \mu$ where μ is given, for instance, by $\mu([a,b)) := \arctan b - \arctan a$. Now, due to the lower semicontinuity of the map $\ell \mapsto \inf \beta_{(t,x)}(\ell)$, the sets $\{k \mid \inf \beta_{(t,x)}(k) > c\}$ are open. Therefore

$$\Big\{(t,x;k) \mid \inf \beta_{(t,x)}(k) > c\Big\} = \bigcup\nolimits_{\ell \in \mathbb{Q}} m_\ell(c) \times [\ell, +\infty],$$

which is a countable union of measurable sets. Hence it is measurable on $\Sigma \times \mathbb{R}$ w.r.t. $\mathcal{H}^N \otimes \mu$; thus $(t,x;k) \mapsto \inf \beta_{(t,x)}(k)$ is measurable.

Now, for all $\sigma = (t,x) \in \Sigma$, $\theta(t,x;\cdot)$ is zero μ-a.e on $\mathbb{R}$ due to the monotonicity of $\beta_{(t,x)}$. Applying the Fubini-Tonnelli theorem, we see that

$$\int_{\mathbb{R}} \Big(\int_\Sigma \theta(\sigma;k)\, d\mathcal{H}^N(\sigma) \Big)\, d\mu(k) = 0.$$

Thus for a.e. $k \in \mathbb{R}$, the function $\theta(\cdot;k)$ is zero $\mathcal{H}^N$-a.e. on Σ, which was to be proved. $\square$

Now let us give an application of Proposition 6.12 to a Yosida-like approximation of β.

Example 6.14. Assume that φ satisfies (4.1) and (5.4). Assume that β fulfills the analogues of assumptions (5.2) or (5.1) with $\widetilde{D}^\pm_{(t,x)}$ replaced by $D^\pm_{(t,x)}$ (this makes the assumptions stronger). Then the entropy solution u of $(H_{\varphi,\beta}(u_0, f))$ is the limit of u^δ, where u^δ are limits of the vanishing viscosity method applied to problems $(H_{\varphi,\beta^\delta}(u_0,f))$ with β^δ the approximation (6.15) of β that we describe below.

Indeed, consider, e.g., the case where (5.1) holds with $D^{\pm}_{(t,x)}$ in the place of $\widetilde{D}^{\pm}_{(t,x)}$, and pick m such that u_0 takes values in $[A^-_m, A^+_m]$. Then there exist $b^{\pm}_m(t,x) \in \beta_{(t,x)}(A^{\pm}_m)$ such that $b^+_m(t,x) \geq \varphi_{\nu(x)}(A^+_m)$ and $b^-_m(t,x) \leq \varphi_{\nu(x)}(A^-_m)$. Without loss of generality, we may assume that $\pm b^+_m(t,x) > 0$ (otherwise we can modify β without changing the effective BC graph $\tilde{\beta}$ in the interval $[A^-_m, A^+_m]$, as in the case of truncations (5.5)). We can use the result of Theorem 6.2 for the case of single-valued continuous graphs β^δ_m, $\delta > 0$, defined as follows:

$$\beta^\delta_m = \Big\{ (z,b) \mid z + \delta\, b^+_m \frac{z^+}{A^+_m} - \delta\, b^-_m \frac{z^-}{A^-_m} \in \beta^{-1}(b) + \delta b \Big\} \tag{6.15}$$

(here we have skipped the parameters $(t,x) \in \Sigma$). Approximation (6.15) ensures the convergence property (6.11). It is inspired by the Yosida approximation $\beta^\delta = (\beta^{-1}+\delta I)^{-1}$, but by construction, it has the additional property that $\beta^\delta_m(A^{\pm}_m) = b^{\pm}_m$. Therefore $A^{\pm}_m \in D^{\pm,\delta}_{(t,x)}$ for all $(t,x) \in \Sigma$; hence we can use the truncation convention (5.5) simultaneously for all δ. Applying Proposition 6.12, we deduce that solutions u^δ of $(H_{\varphi,\beta^\delta}(u_0,f))$ (u^δ being obtained via Theorem 6.2) converge to the unique entropy solution of $(H_{\varphi,\beta}(u_0,f))$.

6.2.2. *Heredity by monotonicity.* In this subsection, let us assume the following properties:

$$\begin{aligned} &\varphi^\delta = \varphi \text{ for all } \delta, \ \ u^\delta_0 \downarrow_{\delta\to 0} u_0, \quad f^\delta \downarrow_{\delta\to 0} f, \\ &\text{and for } \mathcal{H}^N\text{-a.e. } (t,x) \in \Sigma, \quad \beta^\delta_{(t,x)} \succeq_x \beta^\alpha_{(t,x)} \text{ if } 0 < \delta \leq \alpha. \end{aligned} \tag{6.16}$$

The case where $u^\delta_0 \uparrow_{\delta\to 0} u_0$, $f^\delta \uparrow_{\delta\to 0} f$ and $\beta^\delta_{(t,x)} \downarrow_{\delta\to 0}$ (in the sense $\succeq_x$) can be considered in the same way. In this subsection, we will work with projected graphs $\widetilde{\mathcal{B}}^\delta_{(t,x)}$ in the place of $\beta^\delta_{(t,x)}$.

By Definition 2.8, we have $\widetilde{\mathcal{B}}^\delta_{(t,x)} \geq \widetilde{\mathcal{B}}^\alpha_{(t,x)}$ pointwise on $\mathbb{R}$ if $0 < \delta < \alpha$. Therefore it is automatic that, for a.e. (t,x), $\widetilde{\mathcal{B}}^\delta_{(t,x)} \uparrow \Psi_{(t,x)}$ as $\delta \to 0$, with some non-decreasing function $\Psi_{(t,x)}$ which could possibly take infinite values. Under the assumptions we take, we can truncate φ, β (which means that, e.g., (5.5) is assumed) without changing the solutions u^δ. It follows that $\Psi_{(t,x)}$ is finite everywhere because it is bounded by $\|\varphi\|_\infty < \infty$. Finally, recall that for all $\delta > 0$, $\widetilde{\mathcal{B}}^\delta_{(t,x)}$ is a continuous monotone function that is constant on every connected component of the set $\Big\{ k \in \mathbb{R} \,|\, \widetilde{\mathcal{B}}^\delta_{(t,x)}(k) \neq \varphi_{\nu(x)}(k) \Big\}$. It is easy to see that this structure is inherited by passage to a monotone limit; therefore in the sequel we will write $\widetilde{\mathcal{B}}_{(t,x)}$ in the place of Ψ.

The compactness of $(u^\delta)_\delta$ is automatic from its monotonicity, ensured by the comparison principle of Theorem 4.1. We have

Proposition 6.15. *Assume the data $u^\delta_0, f^\delta, \beta^\delta$ converge monotonically, in the sense of* (6.16). *Assume* (5.1) *or* (5.2) *holds, with $(A^{\pm}_m)_m$ or $\mathcal{A}^{\pm}$ that are suitable for φ^δ, β^δ simultaneously for all $\delta > 0$. Assume that* (4.1) *holds.*

Assume $\|u^\delta_0\|_\infty$ is uniformly bounded; in (5.2) *assume $\int_0^T \|f^\delta(t,\cdot)\|_\infty\, dt$ is uniformly bounded. Consider a family $(u^\delta)_\delta$ of entropy solutions of $(H_{\varphi,\beta^\delta}(u^\delta_0, f^\delta))$.*

Then there exists a limit u of u^δ, as $\delta \to 0$, and u is an entropy solution of the limit problem $(H_{\varphi,\beta}(u_0, f))$ with the graph β given by $\beta_{(t,x)} := \widetilde{\mathcal{B}}_{(t,x)} = \lim_{\delta\to 0} \widetilde{\mathcal{B}}^\delta_{(t,x)}$.

Proof. The uniform L^∞ bound (6.12) follows by Lemma 6.11. By Theorem 4.1 and due to the monotone convergence assumption (6.16) on the data, we deduce that $u^\delta \le u^\alpha$ a.e. on Q_T for $0 < \delta \le \alpha$. Thus $u := \lim_{\delta\to 0} u^\delta$ is well defined a.e. on Q_T (one can start by considering a sequence $(\delta_n)_n$ decreasing to zero; at the very end, we find that u is an entropy solution of $(H_{\varphi,\beta}(u_0, f))$, which ensures the uniqueness of an accumulation point of $(u^\delta)_\delta$).

As in Proposition 6.12, we readily pass to the limit in the left-hand side of the entropy formulation (3.5) written for u^δ. In the right-hand side, we can choose to write

$$(\widetilde{\mathcal{B}}^\delta_{(t,x)}(k) - \varphi_{\nu(x)}(k))^\mp \quad \text{in the place of} \quad (\beta^\delta_{(t,x)}(k) - \varphi_{\nu(x)}(k))^\mp.$$

Indeed, u^δ, being the entropy solution corresponding to $\beta_{(t,x)}$, is also the entropy solution corresponding to the graph $\widetilde{\mathcal{B}}^\delta_{(t,x)}$ (both graphs lead to the same admissible traces set $\widetilde{D}^0_{(t,x)}$). Then by the monotone convergence theorem we readily pass to the limit in the right-hand side of inequalities (3.5) written for u^δ. The proof is complete. ☐

Remark 6.16. It is easy to check that for all $x \in \partial\Omega$ the projection $\widetilde{\mathcal{P}}_x$ on $\mathbb{B}$ is an order-preserving operator. Therefore the monotonicity property of $\beta^\delta_{(t,x)}$ in the sense of the relation $\succeq_x$ is implied by its monotonicity in δ in the pointwise (multi-valued) sense. In this case the limit u of Proposition 6.15 is also an entropy solution of $(H_{\varphi,\beta}(u_0, f))$ with the graph $\beta_{(t,x)}$ obtained as $\lim_{\delta\to 0} \beta^\delta_{(t,x)}$ (the limit here is in the sense of (6.11)).

The following example complements Example 6.14. The corresponding existence claim eventually attains the same generality as the result of Theorem 5.2.

Example 6.17. Assume that φ satisfies (4.1) and (5.4). Assume that β fulfills (5.2) or (5.1).

There exists a family of bi-monotone graphs $\left(\beta^{\delta,\lambda}\right)_{\delta,\lambda>0}$ which satisfies the assumptions of Example 6.14. The entropy solutions $u^{\delta,\lambda}$ (that can be constructed, e.g., in the way of Example 6.14) of the associated problem $(H_{\varphi,\beta^{\delta,\lambda}}(u_0, f))$ converge to an entropy solution u of $(H_{\varphi,\beta}(u_0, f))$ as δ, λ tend to zero.

Indeed, assume, e.g., that β satisfies (5.1). We approximate β by $\beta^{\delta,\lambda} := \beta + \partial I_{[-1/\delta, 1/\lambda]}$, where $\partial I_{[a,b]}$ (the subdifferential of the indicator function of $[a, b]$) is the obstacle graph corresponding to the interval $[a, b]$. This ensures that

$$\beta^{\delta,\lambda}(k) = +\infty \ge \varphi_{\nu(x)}(k) \text{ for } k > 1/\lambda,$$
$$\beta^{\delta,\lambda}(k) = -\infty \le \varphi_{\nu(x)}(k) \text{ for } k < -1/\delta,$$

so that $(-\infty, -1/\delta] \subset D^{-,\delta,\lambda}_{(t,x)}$ and $[1/\lambda, +\infty) \subset D^{+,\delta,\lambda}_{(t,x)}$ for all $(t, x) \in \Sigma$. Thus $\beta^{\delta,\lambda}$ fulfills the assumptions of Example 6.14.

Furthermore, whenever

$$-1/\delta \in D^-_{(t,x)} \text{ and } 1/\lambda \in D^+_{(t,x)}, \tag{6.17}$$

as in the localization procedure in the proof of Theorem 5.2, we see that $\widetilde{\mathcal{B}}^{\delta,\lambda}_{(t,x)} := \widetilde{\mathcal{P}}_x\beta^{\delta,\lambda}_{(t,x)}$ coincides with $\widetilde{\mathcal{B}}_{(t,x)}$ in the interval $[-1/\delta, 1/\lambda]$. Due to assumption (5.1), we can construct sequences of parameters δ and λ going to zero and satisfying (6.17). Moreover, due to (5.1) solutions $u^{\delta,\lambda}$ constructed in Example 6.14 take their values within some fixed interval $[-M, M]$.

By construction, $(\beta^{\delta,\lambda}_{(t,x)})_{\delta>0}$ decreases as $\delta \downarrow 0$ for every $\lambda > 0$, and $(\beta^{\delta,\lambda}_{(t,x)})_{\lambda>0}$ increases as $\lambda \downarrow 0$ for every $\delta > 0$. As $\delta \to 0$, we can use Proposition 6.15 to infer that $u^{\delta,\lambda} \uparrow_{\delta\to 0^+} u^{0,\lambda}$ and $u^{0,\lambda}$ is the entropy solution associated with the graph $\beta^{0,\lambda} := \beta + \partial I_{(-\infty,1/\lambda]}$. As $\lambda \to 0$, using the analogue of Proposition 6.15 we deduce that $u^{0,\lambda} \downarrow_{\lambda\to 0^+} u$ and u is the unique entropy solution of problem $(H_{\varphi,\beta}(u_0, f))$. Exchanging the order of passage to the limit, we also get $u^{\delta,\lambda} \downarrow_{\lambda\to 0^+} u^{\delta,0} \uparrow_{\delta\to 0^+} u$. By the squeeze lemma, we infer that $u^{\delta,\lambda} \to u$ as $(\delta, \lambda) \to (0, 0)$.

6.2.3. *Heredity by L^1 contraction.* In this subsection, let us assume the following properties:

$$\begin{aligned}&\varphi^\delta = \varphi \text{ for all } \delta,\ \ u_0^\delta - u_0 \to 0 \quad \text{in } L^1(\Omega), \quad f^\delta - f \to 0 \ \text{ in } L^1(Q_T),\\ &\text{and for } \mathcal{H}^N\text{-a.e. } (t,x) \in \Sigma, \quad d_x(\beta^\delta_{(t,x)}, \beta_{(t,x)}) \to 0 \text{ as } \delta \to 0\end{aligned} \tag{6.18}$$

with $d_x(\beta^\delta_{(t,x)}, \beta_{(t,x)}) \equiv d_x(\tilde\beta^\delta_{(t,x)}, \tilde\beta_{(t,x)})$ given by Definition 2.8. The practical interpretation of the above convergence is therefore

$$\widetilde{\mathcal{B}}^\delta_{(t,x)} = \widetilde{\mathcal{P}}_x\beta^\delta_{(t,x)} \longrightarrow \widetilde{\mathcal{B}}_{(t,x)} = \widetilde{\mathcal{P}}_x\beta_{(t,x)} \ \text{ uniformly on } \mathbb{R}. \tag{6.19}$$

The convergence of $(u^\delta)_\delta$ follows by the L^1 contraction principle of Theorem 4.1. We have

Proposition 6.18. *Assume the data $u_0^\delta, f^\delta, \beta^\delta$ converge in the sense of* (6.18)*. Assume that* (5.1) *or* (5.2) *holds, with $(A^\pm_m)_m$ or $\mathcal{A}^\pm$ that are suitable for φ^δ, β^δ simultaneously for all $\delta > 0$. Assume that* (4.1) *holds.*

Assume $\|u_0^\delta\|_\infty$ is uniformly bounded; in the case (5.2) *assume $\int_0^T \|f(t,\cdot)\|_\infty\, dt$ is uniformly bounded. Consider a family $(u^\delta)_\delta$ of entropy solutions of $(H_{\varphi,\beta^\delta}(u_0^\delta, f^\delta))$. Then there exists a limit u of u^δ, as $\delta \to 0$, and u is an entropy solution of the limit problem $(H_{\varphi,\beta}(u_0, f))$.*

Proof. As in Propositions 6.12 and 6.15, the L^∞ bound (6.12) is immediate. To continue, from inequalities (4.2) of Theorem 4.1 we deduce

$$\int_\Omega |u^\delta - u^\alpha|(t) \le \int_\Omega |u_0^\delta - u_0^\alpha| + \int_0^t\int_\Omega |f^\delta - f^\alpha| + \int_0^t\int_{\partial\Omega} d_x(\beta^\delta_{(t,x)}, \beta^\alpha_{(t,x)}). \tag{6.20}$$

By the triangular inequality (recall that d_x, when used on $\overline{\mathbb{B}}_x$, is a distance) and the convergence properties (6.18) we see that the right-hand side of (6.20) tends to zero as $\max\{\delta, \alpha\} \to 0$. Thus by the Cauchy criterion, $(u_\delta)_\delta$ converges in $L^\infty(0, T; L^1(\Omega))$, as $\delta \to 0$, to some limit u. Then u fulfills (3.5). Indeed, the passage to the limit in up-to-the-boundary entropy inequalities (3.5) written for u^δ is straightforward. In particular, in the right-hand side we can substitute $\beta^\delta(k)$ by $\widetilde{\mathcal{B}}^\delta(k)$; the latter expression converges to $\widetilde{\mathcal{B}}(k)$, due to (6.19). We conclude using the dominated convergence theorem. $\square$

A trivial application of (6.18) is for approximation of the initial data u_0 and source data f. Let us give another application which is suitable, e.g., for approximation in the sense of the Lusin theorem of merely measurable Dirichlet or obstacle boundary data by continuous in (t,x) data:

Example 6.19. Assume that (5.1) or (5.2) holds, with $(A^{\pm}_m)_m$ or $\mathcal{A}^{\pm}$ that is suitable for φ^δ, β^δ simultaneously for all $\delta>0$. Assume that (4.1) holds.

Assume that $\beta^\delta \to \beta$ in the following sense:

(6.21)

$$\text{the } \mathcal{H}^N \text{ measure of the set } R_\delta := \Big\{(t,x)\in\Sigma \,|\, \beta^\delta_{(t,x)} \neq \beta_{(t,x)}\Big\} \text{ vanishes as } \delta\to 0.$$

Then solution u^δ of problem $(H_{\varphi,\beta^\delta}(u_0,f))$ tends, as $\delta\to 0$, to a limit u that solves problem $(H_{\varphi,\beta}(u_0,f))$.

This result follows readily from Proposition 6.18: indeed, u_0,f being fixed, (6.21) gives (6.18).

The next application, which complements Example 6.10, uses pointwise approximation of the obstacle problem (the case $u^m=u^M$ of the obstacle problem yields the Dirichlet problem).

Example 6.20. In the setting of Example 6.19, in the place of (6.21) assume that β^δ is the obstacle graph

$$\beta^\delta_{(t,x)} = \Big(\{u^m_\delta(t,x)\}\times\mathbb{R}^-\Big) \,\cup\, \Big([u^m_\delta(t,x),u^M_\delta(t,x)]\times\{0\}\Big) \,\cup\, \Big(\{u^M_\delta(t,x)\}\times\mathbb{R}^+\Big).$$

Assume that u^m_δ and u^M_δ obey uniform L^∞ bounds and converge $\mathcal{H}^N$-a.e. on Σ to limits u^m and u^M, respectively. Then solutions u^δ of problem $(H_{\varphi,\beta^\delta}(u_0,f))$ converge to a limit u that solves problem $(H_{\varphi,\beta}(u_0,f))$ with the obstacle graph β corresponding to u^m,u^M.

The proof is straightforward, taking into account the following lemma:

Lemma 6.21. *Assume β^δ, β are obstacle graphs corresponding to u^m_δ, u^m and u^M_δ, u^M that take values in some compact subset I of $\mathbb{R}$. Let $\omega_\varphi:\mathbb{R}^+\mapsto\mathbb{R}^+$ be the modulus of continuity of φ on I. Then for all $(t,x)\in\Sigma$ there holds $d_x(\beta^\delta,\beta)\le \omega_\varphi\Big(\max\{|u^m_\delta-u^m|,|u^M_\delta-u^M|\}\Big)$.*

The proof relies on the fact that $\widetilde{\mathcal{B}}^\delta_{(t,x)}(z)$ and $\widetilde{\mathcal{B}}_{(t,x)}(z)$ coincide except when their values fall within one of the two strips

$$S^m := \varphi_{\nu(x)}([\min\{u^m,u^m_\delta\},\max\{u^m,u^m_\delta\}),$$
$$S^M := \varphi_{\nu(x)}([\min\{u^M,u^M_\delta\},\max\{u^M,u^M_\delta\}]);$$

then $d_x(\beta^\delta,\beta)=\|\widetilde{\mathcal{B}}^\delta_{(t,x)}-\widetilde{\mathcal{B}}_{(t,x)}\|_\infty$ is less than or equal to the width of the strips, which does not exceed $\omega_\varphi\Big(\max\{|u^m_\delta-u^m|,|u^M_\delta-u^M|\}\Big)$.

7. Further existence and convergence results

Here we explore convergence of approximations in two complementary directions. In Section 7.1 we discard the genuine non-linearity assumption (5.4) and exploit BV_{loc} estimates for proving compactness. This technique is limited to the one-dimensional case (with a simple generalization to half-space or strip domains). In

Section 7.2 we set up a framework for studying measure-valued (entropy-process) solutions, so that we can replace the strong compactness in L^1_{loc} of sequences of approximate solutions by their weak-* compactness in L^∞.

7.1. **The one-dimensional case: Existence via BV_{loc} estimate.** The technique of this section relies upon translation arguments for proving localized BV estimates. It goes back to Bürger et al. [14, 15], where the idea was introduced in the context of finite volume numerical approximations.

Theorem 7.1. *Assume that $\Omega = [0, +\infty)$ and φ is locally Lipschitz continuous. Let β be a maximal monotone graph on $\mathbb{R}$, independent of $t \in (0, T)$. Then for all $u_0 \in L^\infty((0, +\infty))$ there exists an entropy solution of $(H_{\varphi,\beta}(u_0, f))$ with $f = 0$.*

Remark 7.2. In the case of a single boundary point x and of a t-independent graph β, assumption (5.1) is automatically fulfilled. Indeed, in this case we can drop the subscripts (t, x); the points $\pm\infty$ are accumulation points of the sets $\widetilde{D}^{\pm}$ because otherwise we have, e.g., $\varphi_\nu > \widetilde{\mathcal{B}}$ on $[M, +\infty)$, which contradicts the maximality of $\widetilde{\mathcal{B}}$.

Proof. According to Proposition 6.18, it is enough to prove the theorem for a dense subset of data in L^1. In order to recover existence for general L^∞ data, we can use Proposition 6.15 applied to a bi-monotone data approximation. Uniform L^∞ bounds are ensured by (5.1) and the assumption $f = 0$, due to Remark 7.2. Substituting β by $\widetilde{\mathcal{B}}$ as in Section 5, we may assume that β is bounded.

Thus we pick $u_0 \in C^\infty(\Omega)$ with compact in $\mathbb{R}^+$ support and such that $u_0 \equiv k_0 = const$ on some interval $(0, \eta)$. Existence of a solution u^ε to the parabolic regularized problem (1.3) follows by the results of [37, 38]; we can also apply Proposition 7.2 from the Appendix. Therefore assumptions (6.1) and (6.3) of Theorem 6.2 hold, and it remains to guarantee (6.2) in order to apply Theorem 6.2 and conclude the proof.

To this end, we extend u^ε continuously by u_0 for $t \le 0$; notice that for $t < 0$, the extended function u^ε satisfies $u^\varepsilon_t + \varphi(u^\varepsilon)_x = \varepsilon u^\varepsilon_{xx} + r(x)$ where

$$r : x \mapsto \varphi(u_0)_x - \varepsilon(u_0)_{xx} \tag{7.1}$$

is an $L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ function, by the assumptions on φ and u_0. Moreover, we can choose $k_0 \in \widetilde{D}^0$, which means that $\varphi_\nu(k_0) = \widetilde{\mathcal{B}}(k_0)$. Therefore the extended function u^ε is an entire solution (i.e., a solution defined for $t \in \mathbb{R}$) of problem

$$\begin{cases} u^\varepsilon_t - \operatorname{div}\left(-\varphi(u^\varepsilon) + \varepsilon\, u^\varepsilon_x\right) = r(x)\, \mathbb{1}_{t<0}, \\ \Big(\widetilde{\mathcal{B}}(u^\varepsilon) + (-\varphi(u^\varepsilon) + \varepsilon\, u^\varepsilon_x) \cdot (-1)\Big)|_{x=0} = 0. \end{cases} \tag{7.2}$$

Now, the key fact is that we can control the L^1 time translates of u^ε by a linear modulus of continuity, because solutions of (7.2) verify the L^1 contraction principle that can be shown, e.g., as in [29] or as in [16] (we only have to take into account an original boundary condition):

(7.3)

$$\int_{\mathbb{R}} |\tilde{u}^\varepsilon(t) - \tilde{u}^\varepsilon(t-\tau)| \le \int_{\mathbb{R}} |\tilde{u}^\varepsilon(0) - \tilde{u}^\varepsilon(-\tau)| + \int_0^t \int_{\mathbb{R}} |r\, \mathbb{1}_{s<0} - r\, \mathbb{1}_{s-\tau<0}|\, ds = \tau\, \|r\|_{L^1}.$$

Therefore $u^\varepsilon \in BV(0, T; L^1(0, +\infty))$, with a uniform in ε bound. Then we can use the idea of [14, Lemma 4.2] and [15, Lemma 5.4]: for $a > 0$, using the mean-value theorem for each $\varepsilon > 0$ we can find a contour $(0,T) \times \{c^\varepsilon\}$ with $0 < c^\varepsilon < a$ such that $\mathrm{TotVar}\, u^\varepsilon$ along these contours is uniformly bounded by $\frac{C}{a}$. The variation of u_0 is also bounded; therefore using the classical estimate of Bardos, LeRoux and Nédélec [10] for the Dirichlet problem for viscous conservation law (with boundary datum given by the values of u^ε on our contour), we get the bound

$$\mathrm{TotVar}\, u^\varepsilon|_{\{(t,x)\,|\,t\in(0,T),\ x\geq a\}} \leq \frac{C}{a}, \tag{7.4}$$

with C that depends only on u_0 and on the Lipschitz constant of φ. With the Cantor diagonal argument, we deduce compactness of $(u^\varepsilon)_\varepsilon$ in $L^1_{loc}((0,T)\times(0,+\infty))$. Combined with the aforementioned uniform L^∞ bound on u^ε, this finally proves (6.2). □

7.2. **Entropy-process solutions.** As soon as existence of an entropy solution is established[15] and the uniqueness assumption (4.1) is fulfilled, we can prove convergence of, e.g., vanishing viscosity approximations without the genuine non-linearity assumption (5.4) (though we still need a uniform L^∞ estimate). To do so, it is enough to adapt the device of measure-valued solutions; here, we use the version called the entropy-process solution due to Gallouët et al. [20].

Definition 7.3. Let $\mu \in L^\infty(Q_T \times (0,1))$. Then μ is called an entropy-process solution of problem $(H_{\varphi,\beta}(u_0,0))$ if μ verifies the following up-to-the-boundary entropy inequalities with remainder term (which is, in general, multi-valued):
(7.5)

$$\forall k\in\mathbb{R}\ \ \forall \xi \in \mathcal{D}([0,T)\times\overline{\Omega})^+,\quad \int_0^1\!\!\int_0^T\!\!\int_\Omega \Big(-(\mu(\alpha)-k)^\pm \xi_t - q^\pm(\mu(\alpha),k)\cdot\nabla\xi\Big) \\ -\int_\Omega (u_0-k)^\pm \xi(0,\cdot) \leq \iint_\Sigma C_k \wedge \Big(\beta_{(t,x)}(k) - \varphi_{\nu(x)}(k)\Big)^\mp \xi(t,x).$$

Here, C_k is a constant that depends on $\|\mu\|_\infty$ and on k.

Proposition 7.4.

(i) Let μ be an entropy-process solution of $(H_{\varphi,\beta}(u_0,0))$. Then it verifies the entropy-process inequalities
(7.6)

$$\forall k\in\mathbb{R},\quad \int_0^1\!\!\int_0^T\!\!\int_\Omega \Big(-(\mu(\alpha)-k)^\pm\xi_t - q^\pm(\mu(\alpha),k)\cdot\nabla\xi\Big) - \int_\Omega (u_0-k)^\pm\xi(0,\cdot)\leq 0$$

with $\xi \in \mathcal{D}([0,T)\times\Omega)^+$. Moreover, for $\mathcal{H}^N$-a.e. $(t,x)\in\Sigma$, the weak *normal boundary trace of the flux verifies*

$$\forall k \in D^\pm_{(t,x)},\quad \gamma_w\Big(\int_0^1 q^\pm(\mu(\cdot;\alpha),k)\,d\alpha\cdot\nu(\cdot)\Big)(t,x)\geq 0. \tag{7.7}$$

(ii) Let $\mu\in L^\infty(Q_T\times(0,1))$ such that μ satisfies (7.6), (7.7). Then for $\mathcal{H}^N$-a.e. $(t,x)\in\Sigma$, the weak *normal boundary trace of the flux verifies*

$$\forall k \in \widetilde{D}^0_{(t,x)} \equiv \mathrm{Dom}\,\tilde\beta_{(t,x)},\quad \gamma_w\Big(\int_0^1 q(\mu(\cdot;\alpha),k)\,d\alpha\cdot\nu(\cdot)\Big)(t,x)\geq 0. \tag{7.8}$$

[15]Let us stress that neither for conservation laws in the whole space nor for the Dirichlet problem (see, e.g., Vovelle [41]) is this assumption needed.

(iii) Let $\mu \in L^\infty(Q_T \times (0,1))$ such that μ satisfies (7.6), (7.8). *If, in addition,* (4.1) *holds, then μ coincides with the entropy solution u in the sense $\mu(\cdot;\alpha) = u(\cdot)$ a.e. on $Q_T \times (0,1)$.*

Notice that, although we do not prove directly the equivalence of Definition 7.3 and any of the formulations (7.6), (7.7) and (7.6), (7.8), such equivalence holds whenever an entropy solution exists and it is unique.

Proof.

(i) Inequalities (7.6) are immediate from the definition of an entropy-process solution. In order to deduce (7.7), one proceeds as in the proof of the claim "(iv) $\Rightarrow$ (iii)" in Proposition 3.3. The only difference is that, while using the analogue of (3.10), one replaces the (strong) trace $Q^{\pm}_{\nu(x)}\Big(\gamma V_{\varphi_{\nu(x)}} u\,,\, V_{\varphi_{\nu(x)}} k\Big)$ by the (weak) trace $\gamma_w\left(\int_0^1 q^{\pm}(\mu(\cdot;\alpha),k)\,d\alpha\cdot\nu(\cdot)\right)(t,x)$.

(ii) Assume, for instance, that $k \in D^+_{(t,x)}$. Let us show that (7.8) holds for this value of k. It is enough to prove (7.8) separately with q^+ and q^- in the place of q; moreover, the first of the two inequalities is already contained in (7.7) since $k \in D^+_{(t,x)}$. Set $k_0 := \sup\Big\{\kappa \le k \,|\, \kappa \in D^-_{(t,x)}\Big\}$; note that k_0 may take the value $-\infty$. In order to prove the statement, it is enough to get

$$\begin{aligned}
&\gamma_w\left(\int_0^1 q^-(\mu(\cdot;\alpha),k)\,d\alpha\cdot\nu(\cdot)\right)\\
&\qquad \ge \gamma_w\left(\int_0^1 q^-(\mu(\cdot;\alpha),k_0)\,d\alpha\cdot\nu(\cdot)\right) \text{ at the point } (t,x)
\end{aligned} \tag{7.9}$$

(indeed, the latter quantity is non-negative by (7.7) because $k_0 \in D^-_{(t,x)}$: recall that $D^-_{(t,x)}$ is a closed set). Because $k \in \operatorname{Dom}\tilde\beta_{(t,x)}$, by Proposition 2.5(ii) we have

$$\varphi_{\nu(x)}(\kappa) \le \varphi_{\nu(x)}(k) \text{ for all } \kappa \in (k_0,k). \tag{7.10}$$

The idea of the proof is the following: we have $q^-_\nu(\mu(\alpha),k) = -\varphi_{\nu(x)}(\mu(\alpha)) + \varphi_{\nu(x)}(k) \ge 0$ whenever $\mu(\alpha) \in (k_0,k)$, and therefore

$$\begin{aligned}
\int_0^1 q^-(\mu(\alpha),k)\,d\alpha\cdot\nu &= \int_{[\mu(\alpha)\le k_0]} q^-_\nu(\mu(\alpha),k)\,d\alpha + \int_{[k_0<\mu(\alpha)<k]} q^-_\nu(\mu(\alpha),k)\,d\alpha + 0\\
&\ge \int_{[\mu(\alpha)\le k_0]} q^-_\nu(\mu(\alpha),k_0)\,d\alpha + 0 = \int_0^1 q^-(\mu(\alpha),k_0)\,d\alpha\cdot\nu.
\end{aligned} \tag{7.11}$$

Here we have used (7.10), which holds at the point (t,x) but not necessarily at every point. We want to write an inequality of the kind (7.11) in a neighbourhood $B_\delta \cap \Omega$ of (t,x), and then take weak traces at the point (t,x). In order to do so, we use an ε-approximate inequality of the kind (7.11) for $(s,y) \in \Omega \cap B_\delta$, with B_δ a δ-sized neighbourhood of (t,x) and with ε vanishing as δ vanishes. This is possible due to the continuity arguments. Indeed, a generic point of Σ is a point of approximate continuity of the normal field; thus we can write (in the place of (7.10)) that $\varphi_{\nu(y)}(\kappa) \le \varphi_{\nu(y)}(k) + \varepsilon$ for $\kappa \in (k_0,k)$ and for a set of points $(s,y) \in B_\delta \setminus C_\delta$ such that $\operatorname{meas}(C_\delta)/\operatorname{meas}(B_\delta) \to 0$ as $\delta \to 0$. Taking the weak trace, we get (7.9) with the additional term $(-\varepsilon)$ on the right-hand side. Then, letting δ go to zero, we see that (7.9) is justified and the proof is complete.

(iii) The proof is analogous to the one of Theorem 4.1, except that it is based on the doubling of variables (inside the domain) for entropy-process solutions. As in [20, 41], for entropy-process solution μ and an entropy solution u corresponding to the same data, we get the Kato inequality analogous to (4.3):

$$\forall\,\xi\in\mathcal{D}(\Omega),\quad \int_0^1\!\int_\Omega |\mu(\alpha)-u|(t)\,\xi\,d\alpha \le -\int_0^1\!\int_0^t\!\int_\Omega q(\mu(\alpha),u)\cdot\nabla\xi\,d\alpha. \tag{7.12}$$

Assume for simplicity that Ω is bounded (other cases will exploit $\xi\in\mathcal{D}(\overline{\Omega})$ that is then sent to the limit 1, as in [6, 12, 22]). We simply take $\xi=1-\xi_n$ with the construction of Lemma 3.7; as in the proof of (3.11), we deduce at the limit $n\to\infty$,

$$\int_0^1\!\int_\Omega |\mu(\alpha)-u|(t)\,d\alpha \le -\int_0^t\!\int_\Omega \gamma_w \int_0^1 q(\mu(\alpha),u)\cdot\nu(x)\,d\alpha. \tag{7.13}$$

Transforming the right-hand side of (7.13) and using the existence of *strong* trace $\gamma V_{\varphi_{\nu(x)}}u$ we get

$$\begin{aligned}\int_0^1\!\int_\Omega |\mu(\alpha)-u|(t)\,d\alpha &\le -\int_0^t\!\int_\Omega \gamma_w \int_0^1 \;Q_{\nu(x)}(V_{\varphi_{\nu(x)}}\mu(\alpha),V_{\varphi_{\nu(x)}}u)\,d\alpha\\ &\equiv -\int_0^t\!\int_\Omega \gamma_w \int_0^1 \;Q_{\nu(x)}(V_{\varphi_{\nu(x)}}\mu(\alpha),\gamma V_{\varphi_{\nu(x)}}u)\,d\alpha.\end{aligned} \tag{7.14}$$

Yet, according to the characterization Proposition 3.3(i) of u, $\gamma V_{\varphi_{\nu(x)}}u\in\mathrm{Dom}\tilde{\beta}_{(t,x)}$ in a generic point of Σ; thus using (7.8) (notice that one can replace $q(\mu(\alpha),k)\cdot\nu(x)$ in (7.8) by the expression $Q_{\nu(x)}(V_{\varphi_{\nu(x)}}\mu(\alpha),V_{\varphi_{\nu(x)}}k)$) we readily find that the right-hand side of (7.14) is non-positive. It follows that $\mu(\alpha)-u$ is zero a.e., which ends the proof. □

Corollary 7.5. *In the assumptions of Theorem* 6.2*, drop the genuine non-linearity assumption* (5.4)*, but suppose that there exists an entropy solution of problem* $(H_{\varphi,\beta}(u_0,f))$*. Then the conclusion of the theorem still holds.*

For the proof, it is enough to use the device of non-linear weak-$*$ convergence, following [20, 41], to derive the entropy-process formulation (7.5) along the lines of the proof of Theorem 6.2. One concludes using Proposition 7.4(i)-(iii).

Conclusion

We investigated the issue of definition, justification and uniqueness of entropy solutions to scalar conservation laws with non-linear dissipative boundary conditions. Although existence of entropy solutions and convergence of approximations are addressed in much generality, technical restrictions we had to impose leave place for a future work, e.g., exploiting the notions of renormalized entropy solutions ([2, 11, 35]) and of *weak* boundary traces and boundary entropy-flux pairs ([28]), as was done for the Dirichlet problem.

Appendix: Existence for the viscosity regularized problem

In this paper, we establish existence of entropy solutions of $(H_{\varphi,\beta}(u_0,f))$ via construction of approximate solutions (in most cases, we need a multi-step approximation). Therefore we need some basic existence result to produce approximate solutions; this is the purpose of the present Appendix. Existence results of this kind were already established by the second author and Wittbold in [38] (see also [37]), for the case of t-independent graph β such that $0 \in \beta(0)$. Other results can be found in [4].

Here we follow a different strategy (in the place of the convex analysis and non-linear semigroup methods of [37, 38], we use the Galerkin scheme and time compactness arguments) in the context that better suits our needs. Consider the following parabolic problem (for simplicity, we set $f \equiv 0$):

$$\text{(A.1)} \qquad \begin{cases} u_t - \operatorname{div}(-\varphi(u)+\varepsilon\nabla u) = 0, \quad u|_{t=0} = u_0, \\ b(t,x;u) + (-\varphi(u)+\varepsilon\nabla u)\cdot\nu(x) = 0, \end{cases}$$

where b is a Carathéodory function (single-valued $b(t,x;\cdot)$ that replaces the maximal monotone graph $\beta_{(t,x)}$); more precisely

for all $z \in \mathbb{R}$ $\;b(\cdot,\cdot;u)$ is measurable,
and for a.e. $(t,x) \in \Sigma$, $b(t,x;\cdot)$ is a continuous strictly increasing function.

Moreover, we assume that b is bounded:

$$\sup_{(t,x)\in\Sigma,\ z\in\mathbb{R}} |b(t,x;z)| < +\infty.$$

The parameter ε in (A.1) could be removed by a scaling argument, but we keep it in order to state an ε-independent L^∞ estimate on u^ε that is needed in order to generate a limit of the sequence $(u^\varepsilon)_\varepsilon$, as $\varepsilon \to 0$.

Proposition A.1.
(i) Under the above assumptions, suppose in addition that φ is bounded on $\mathbb{R}$. Then there exists a solution u^ε to problem (A.1)*: namely, $u^\varepsilon \in L^2(0,T;H^1_{loc}(\Omega))$ and for all $\xi \in \mathcal{D}([0,T)\times\Omega)$,*

$$\text{(A.2)} \quad \int_0^T\!\!\int_\Omega -u\xi_t - \int_\Omega u_0\xi(0,\cdot) + \int_0^T\!\!\int_\Omega \Big(-\varphi(u)+\varepsilon\nabla u\Big)\cdot\nabla\xi + \int_0^T\!\!\int_{\partial\Omega} b(\cdot;u)\,\xi = 0.$$

Moreover, $\sqrt{\varepsilon}\nabla u^\varepsilon$ is bounded in $L^2(0,T;L^2_{loc}(\Omega))$ uniformly in $\varepsilon \in (0,1)$.
(ii) Under the assumption that, upon a modification of b on a subset of Σ of zero $\mathcal{H}^N$ measure,

$$\text{(A.3)} \qquad \begin{array}{l} \text{there exist } (A^-_m)_{m\in\mathbb{N}}, (A^+_m)_{m\in\mathbb{N}} \subset \mathbb{R}^\pm \text{ such that } A^\pm_m \to \pm\infty \text{ as } m\to\infty \\ \text{and for all } (t,x)\in\Sigma, \text{ for all } m\in\mathbb{N},\ \pm b(t,x;A^\pm_m) \geq \pm\varphi_{\nu(x)}(A^\pm_m), \end{array}$$

we have, uniformly in $\varepsilon > 0$, the estimate $\|u^\varepsilon\|_\infty \leq M$ with M that depends on $\|u_0\|_\infty$ and on $(A^\pm_m)_m$ in assumption (A.3).

Remark A.2. For non-zero f, existence is shown in the same way. In property (ii), hypothesis

$$\text{(A.4)} \quad \mathcal{A}^\pm := \Big\{ k \in \mathbb{R}^\pm \mid \pm b(t,x;k) \geq \pm\varphi_{\nu(x)}(k) \text{ for all } (t,x)\in\Sigma \Big\} \text{ are of infinite measure}$$

can be assumed in the place of (A.3), and the bound M would depend on $\|u_0\|_\infty$, $\int_0^T \|f(t,\cdot)\|_\infty \, dt$ and on the sets $\mathcal{A}^\pm$ in assumption (A.4).

In the place of a constant in t and x sub- and super-solutions, in this case we construct super-solutions of the kind $M^\pm(t)$ taking values in $\mathcal{A}^\pm$ and such that $\pm M^\pm(\cdot)$ are non-decreasing, with $\pm M^\pm(0) \geq \|u_0\|_\infty$ and with the absolutely continuous part of the derivative $\pm(M^\pm)'(t)$ that is greater than or equal to $\|f(t,\cdot)\|_\infty$ on $(0,T)$.

Proof. For the proof, it is enough to use the Galerkin method, which we expose briefly in order to focus on the difficulties induced by the non-linearities φ and b.

In the case Ω is bounded, picking an orthonormal basis $(v_i)_{i\in\mathbb{N}}$ in $H^1(\Omega)$, we construct $u_n \in C^1([0,T];\mathrm{span}\{v_1,\dots,v_n\})$ as a solution to the ODE system obtained from (A.2) by substituting u by the unknown function $u_n(t,x) = \sum_{i=1}^n c_i(t)v_i(x)$, substituting u_0 by its projection $u_{0,n}$ on $\mathrm{span}\{v_1,\dots,v_n\}$, and testing it with $\xi(t,x) = v_i(x)\mu(t)$, $i = 1,\dots,n$, $\mu \in \mathcal{D}([0,T))$. Local existence of a solution follows from the Cauchy-Peano theorem. Taking u_n itself for the test function, with $\mu(t)$ approximating $\mathbb{1}_{[0,s)}(t)$ we find

$$\begin{aligned}\frac{1}{2}\int_\Omega u_n^2(s,\cdot) + \int_0^s\!\!\int_\Omega \Big(\varepsilon|\nabla u_n|^2 - \varphi(u_n)\cdot\nabla u_n\Big) + \int_0^s\!\!\int_{\partial\Omega} b(\cdot;u_n)u_n \\ = \frac{1}{2}\int_\Omega u_{0,n}^2 \leq \frac{1}{2}\int_\Omega u_0^2.\end{aligned} \tag{A.5}$$

Thanks to trace inequalities and the boundedness assumptions on φ and b together with the L^∞ bound on u_0, we get an $L^2(Q_T)$ estimate on ∇u_n. Such an estimate precludes the blow-up and guarantees the global in time existence of u_n. For the case of unbounded domain, the mere L^∞ bounds on u_0, φ, b are not sufficient: thus we have to localize the estimate taking, e.g., the weight $\eta(x) = \exp(-c|x - x_0|)$ for some $x_0 \notin \overline{\Omega}$. In this case, we work in the weighted H^1 space and use weighted trace inequalities; as an outcome, we get an $L^2(0,T;L^2_{loc}(\Omega))$ bound on u_n.

Thus we have, in addition, the uniform in n estimate of ∇u_n in $L^2(0,T;H^1_{loc}(\Omega))$. We extract a subsequence weakly convergent to a limit u^ε and pass to the limit in the formulation. To this end, the a.e. convergence of u_n to u^ε is needed in order to pass to the limit in the non-linearity $\varphi(u_n)$. It is obtained by translation techniques in time, following [1]. Indeed, assume for simplicity that Ω is bounded (otherwise we use exponentially decreasing in x weights, as above). We "integrate" the weak formulation (A.2) from t to $t+\delta t$, then test it with $\xi = u_n(t+\delta t) - u_n(t)$ (this corresponds to taking well-chosen test functions in the formulation (A.2) written for u_n and with test functions $\xi \in \mathcal{D}([0,T);\mathrm{span}\{v_1,\dots,v_n\})$). Using the Fubini theorem, the aforementioned $L^2(0,T;H^1(\Omega))$ bound on u_n, the trace inequality and the L^∞ bound on $\varphi(u_n)$ and $b(\cdot;u_n)$, we deduce that

$$\int_0^{T-\delta t} \Big| u_n(t+\delta t) - u_n(t)\Big|^2 \leq const\,\delta t.$$

The estimate of the space translates being trivial due to the $L^2(Q_T)$ bound on ∇u_n, by the Fréchet-Kolmogorov compactness criterion we conclude the $L^1(Q_T)$ convergence of u_n (if Ω is unbounded, we use weights η and get $L^1((0,T);L^1_{loc}(\Omega))$ convergence). The limit being unique in $\mathcal{D}'$, it is identified with u_ε; extracting a further subsequence, we may assume the a.e. convergence in Q_T. Finally, $(b(\cdot;u_n))_n$ being bounded, it converges weakly-* in $L^\infty(\Sigma)$ to some limit that we denote by

b^ε. Now for all i, we can take v_i as a test function and pass to the limit in the Galerkin formulation; we find that

$$\text{(A.6)}\quad \int_0^T\!\!\int_\Omega -u^\varepsilon \xi_t - \int_\Omega u_0\xi(0,\cdot) + \int_0^T\!\!\int_\Omega \Big(-\varphi(u^\varepsilon)+\varepsilon\nabla u^\varepsilon\Big)\cdot\nabla\xi + \int_0^T\!\!\int_{\partial\Omega} b^\varepsilon\,\xi = 0$$

for all $\xi \in \mathcal{D}'([0,T)\times\Omega)$ (this is obtained by density). It remains to identify b^ε with $b(\cdot;u^\varepsilon)$, which is done using the monotonicity of $b(t,x;\cdot)$ and the classical Minty argument.

To do so, for the sake of simplicity assume that Ω is bounded (for the general case, one has to replace the test functions u^ε, u_n below by truncated test functions $u^\varepsilon\eta$, $u_n\eta$ with an exponentially decaying weight η). Comparing the Galerkin formulation for u_n (with test function u_n) and the weak formulation (A.6) for u^ε (with test function u^ε, taken by density), using the Fatou lemma to ensure that $\|u^\varepsilon(T,\cdot)\|^2_{L^2(\Omega)} \le \liminf_{n\to\infty}\|u_n(T,\cdot)\|^2_{L^2(\Omega)}$, we can eventually write

(A.7)
$$\int_0^T\!\!\int_{\partial\Omega} b^\varepsilon u^\varepsilon + \varepsilon\int_0^T\!\!\int_\Omega |\nabla u^\varepsilon|^2 \ge \limsup_{n\to\infty}\Big(\int_0^T\!\!\int_{\partial\Omega} b(\cdot;u_n)u_n + \varepsilon\int_0^T\!\!\int_\Omega |\nabla u_n|^2\Big).$$

Due to the weak lower semicontinuity of the L^2 norm, we infer

$$\text{(A.8)}\qquad \int_0^T\!\!\int_{\partial\Omega} b^\varepsilon u^\varepsilon \ge \limsup_{n\to\infty}\int_0^T\!\!\int_{\partial\Omega} b(\cdot;u_n)u_n.$$

Here we are in the following setting: $u_n|_\Sigma$ converges to $u^\varepsilon|_\Sigma$ in $L^2(\Sigma)$ weakly (this is due to the trace inequalities), and $b(\cdot;u_n)$ converges to b^ε in $L^\infty(\Sigma)$ weakly-* (and thus, in $L^2(\Sigma)$ weakly, because we have assumed that $\partial\Omega$ is bounded); moreover, $z\mapsto b(\cdot;z)$ is monotone and inequality (A.8) holds. In this setting, the Minty argument applies (see, e.g., [1, 29, 38]), which allows us to conclude that $b^\varepsilon = b(\cdot;u^\varepsilon)$ a.e. on Σ. Thus (A.6) becomes (A.2), and the proof of existence is complete.

(ii) Now using (A.3), take $A_m^\pm$ satisfying $A_m^- \le -\|u_0\|_\infty$ and $\|u_0\|_\infty \le A_m^+$; due to (A.3), the constants A_m^- and A_m^+ are sub- and super-solutions to problem (A.1), respectively. The result stems from the comparison principle for weak solutions, sub-solutions and super-solutions of (A.1) using, e.g., the technique of [16]. It consists of taking $H_\alpha(u^\varepsilon - A_m^+)\xi$ (with H_α the Lipschitz regularization of sign$^+$ function as used in the proof of Theorem 6.2) with $\xi\in\mathcal{D}([0,T)\times\overline{\Omega})^+$; the factor ξ can be dropped if Ω is bounded. As in the proof of Theorem 6.2, we deduce the Kato inequality

$$-\int_0^T\!\!\int_\Omega (u^\varepsilon - A_m^+)^+\,\xi_t + \int_0^T\!\!\int_\Omega \operatorname{sign}^+(u^\varepsilon - A_m^+)\Big(-\varphi(u^\varepsilon)+\varphi(A_m^+)+\varepsilon\nabla u^\varepsilon\Big)\cdot\nabla\xi \le 0.$$

We let ξ converge to e^{-t} and prove that $(u^\varepsilon - A_m^+)^+\le 0$ a.e. (if Ω is unbounded, we use (4.1) as in [6, 26]). A uniform upper bound for u^ε is proved; the lower bound by A_m^- is analogous.

Notice that the technique we've used exploits assumption (4.1), yet it is possible to bypass this assumption. Indeed, by approximation one can always construct solutions satisfying the above L^∞ bound. To this end, one can, e.g., substitute the original problem by the problem set up in $(0,T)\times(\Omega\cap B_R)$ where B_R is the ball of radius R centered at the origin; the part $\Sigma'_R := (0,T)\times\partial B_R\cap\Omega$ of the boundary can be supplemented with the homogeneous Dirichlet boundary condition. Then existence of solutions u_R in the space $L^2(0,T;H^1_{0,R}(\Omega))$ of functions that are zero in $(0,T)\times(\Omega\setminus B_R)$ is proved by the same Galerkin method. Notice that the constants $A^{\pm}_m$ are still sub- and super-solutions of this modified problem; B_R being bounded, assumption (4.1) is automatically satisfied and the L^∞ bound on u_R is valid. Finally, convergence of u_R to a limit u^ε is established with the same tools as in the proof of (i). □

Remark A.3. While estimating ∇u^ε in L^2_{loc}, for the sake of simplicity we have assumed that b is bounded, and thus we have not exploited the monotonicity of b in these estimates. Actually, it is enough to assume, e.g., that $b(t,x;0)$ is bounded; in addition, estimate (A.5) brings an $L^1_{loc}([0,T]\times\partial\Omega)$ estimate of the product $(b(\cdot;u_n)-b(t,x;0))\,u_n\geq 0$, which is inherited at the limit $u_n\to u^\varepsilon$. Similarly, instead of the uniform bound on φ we could assume inequalities of the kind $\left|\int_0^z\varphi(s)\,ds\right|\leq C+\operatorname{sign} z\,(b(t,x'z)-b(t,x;0))$.

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Laboratoire de Mathématiques CNRS UMR 6623, Université de Franche-Comté, 16 route de Gray, 25030 Besançon Cedex, France

E-mail address: `boris.andreianov@univ-fcomte.fr`

Laboratoire de Mathématiques CNRS UMR 6623, Université de Franche-Comté, 16 route de Gray, 25030 Besançon Cedex, France

E-mail address: `sbihi_k@yahoo.fr`