

ON THE STABLE MOMENT GRAPH OF AN AFFINE KAC–MOODY ALGEBRA

MARTINA LANINI

ABSTRACT. In 1980 Lusztig proved a stabilisation property of the affine Kazhdan-Lusztig polynomials. In this paper we give a categorical version of such a result using the theory of sheaves on moment graphs. This leads us to associate with any Kac-Moody algebra its stable moment graph.

1. INTRODUCTION

Finding the characters of the irreducible modular representations of semisimple algebraic groups is a fundamental problem in the representation theory of finite groups. In [Lu80a] Lusztig stated a conjecture which related these characters to the affine Kazhdan-Lusztig polynomials and which was similar to a conjecture by Kazhdan and Lusztig (cf. [KL79]) on the composition series of Verma modules for semisimple complex Lie algebras. The Kazhdan-Lusztig conjecture has been solved using geometric methods and the theory of \mathcal{D} -modules (cf. [BB], [BK]), while Lusztig's conjecture has not yet been completely settled. Indeed, by [AJS], [KL93], [KT] it is possible to recover it as a limit from the characteristic zero case, but this means that Lusztig's conjecture is not yet known to be true for all the characteristics it is supposed to hold.

The attempt of finding a new approach to these character conjectures brought Fiebig to apply moment graph techniques in representation theory. In particular, he associated a certain moment graph with any complex symmetrisable Kac-Moody algebra \mathfrak{g} : its Bruhat graph, that is, an oriented graph with labeled edges and whose set of vertices is given by the elements of the Weyl group W of \mathfrak{g} . A central role in this theory is played by indecomposable Braden-MacPherson sheaves on the Bruhat graph of \mathfrak{g} , which correspond to indecomposable projective objects admitting a Verma flag in an equivariant version of the representation category \mathcal{O} of \mathfrak{g} (cf. [Fie08a]). Moreover, moment graphs and Braden-MacPherson sheaves can also be constructed in the positive characteristic setting, in which case the moment graph is attached to a semisimple reductive algebraic group G . In this way, Fiebig could relate Kazhdan-Lusztig and Lusztig conjectures to a question on the graded rank of the stalks of indecomposable Braden-MacPherson sheaves. In particular, he stated a conjectural formula connecting this rank explicitly to Kazhdan-Lusztig polynomials. Such a formula is known to hold in characteristic zero and, in this case, to be equivalent to the character formula conjectured by Kazhdan and Lusztig (cf. [Fie08a]), while in positive characteristic it is expected to be true for characteristics bigger than the Coxeter number of G .

Received by the editors January 28, 2013 and, in revised form, April 30, 2013.
2010 *Mathematics Subject Classification*. Primary 05E10, 17B67, 20C08.

©2015 American Mathematical Society
Reverts to public domain 28 years from publication

Fiebig's conjectural formula motivated our paper [L11], where we lifted at the categorical level of sheaves on moment graphs certain properties of Kazhdan-Lusztig polynomials and parabolic Kazhdan-Lusztig polynomials. The methods we developed in [L11] work in any characteristic under certain technical assumptions, even in cases in which the formula has been proven to fail. In [FW] Fiebig and Williamson related indecomposable Braden-MacPherson sheaves on a Bruhat graph to parity sheaves on the corresponding flag variety, a modular counterpart of intersection cohomology complexes introduced by Juteau, Mautner and Williamson (cf. [JMW]). Once this connection is established, by [FW, Theorem 9.2] the failure of the formula corresponds to the failure of a parity sheaf to be perverse. In this setting, our results proved that even in the cases in which parity sheaves are not perverse, they still satisfy certain elementary properties that intersection cohomology complexes have in characteristic zero. Anyway, giving a categorical version of properties of Kazhdan-Lusztig polynomials is not at all trivial and is rather interesting, and also in the case in which the connection between Braden-MacPherson sheaves and these polynomials is known. Indeed, the lifting at a categorical level provides us with extra structure which may help in the understanding of objects related to these ubiquitous polynomials.

While in [L11] we considered rather basic equalities involving Kazhdan-Lusztig polynomials, in this paper the result we want to deal with is much more complex. It concerns a stabilisation property of the affine Kazhdan-Lusztig polynomials, which has been proven by Lusztig in [Lu80b]. More precisely, for any affine Weyl group $\widehat{\mathcal{W}}$, Lusztig defined in [Lu80b] its *periodic module* \mathbf{M} , that is, a free $\mathbb{Z}[q^{\pm\frac{1}{2}}]$ -module with a standard basis indexed by the set \mathcal{A} of alcoves of $\widehat{\mathcal{W}}$ and also equipped with a structure of a module over the Hecke algebra \mathbf{H} of $\widehat{\mathcal{W}}$. By extending methods of [KL79], he found a nice basis of \mathbf{M} and showed that the change of basis matrix from the standard basis to this one is given by certain polynomials in $\mathbb{Z}[q]$: the generic polynomials, which we will denote by $\{Q_{A,B}\}_{A,B \in \mathcal{A}}$. A fundamental property of these polynomials is that they are invariant under finite coweight translation, that is, $Q_{A,B} = Q_{A+\lambda, B+\lambda}$ for any finite coweight λ . On the other hand, we have another family of polynomials which are attached to the affine Hecke algebra: the Kazhdan-Lusztig polynomials $\{P_{x,y}\}_{x,y \in \widehat{\mathcal{W}}}$. Since it is possible to identify the Weyl group $\widehat{\mathcal{W}}$ with its set of alcoves, we may index any affine Kazhdan-Lusztig polynomial by a pair of alcoves and ask whether $P_{A,B}$ and $Q_{A,B}$ are related. This is precisely the point of the theorem by Lusztig we want to lift at the level of sheaves on moment graphs. Indeed, what he proved is that if the alcoves A, B are *deep enough* in the fundamental chamber \mathcal{C}^+ , the generic polynomial $Q_{A,B}$ coincides with the corresponding affine Kazhdan-Lusztig $P_{A,B}$. The goal of this paper is hence to describe how the stalks of Braden-MacPherson sheaves on certain finite intervals of the Bruhat graph behave and in particular to show this stabilisation property. A crucial point in the proof of such a property is the introduction of a new object: the *stable moment graph* $\widehat{\mathcal{G}}^{\text{stab}}$.

Before outlining the structure of the paper and illustrating in more detail in what way we obtain this categorical analogue of Lusztig's theorem, we want to motivate our interest in such a result. By linking the affine Hecke algebra and its periodic module, the moment graph version of the stabilisation property we mentioned above should allow us to investigate both \mathbf{H} and \mathbf{M} from a categorical point of view. Let $\widehat{\mathfrak{g}}$ be an affine Kac-Moody algebra and $\widehat{\mathcal{W}}$ the corresponding

(affine) Weyl group. The affine Hecke algebra of $\widehat{\mathcal{W}}$ controls the representation theory of $\widehat{\mathfrak{g}}$ at a non-critical level and in this case moment graph techniques have already been applied (cf. [Fie08a]), while the periodic module, according to a conjecture by Feigin and Frenkel (cf. [FF92]), governs the representation theory of $\widehat{\mathfrak{g}}$ at a critical level, where a moment graph picture has been missing so far. We believe that the *stable moment graph* we associate with $\widehat{\mathfrak{g}}$ in this paper will be relevant in this direction. From the geometric point of view, \mathbf{M} is related to the semi-infinite flags (cf. [FF90]) –also referred to as periodic affine Schubert varieties by Lusztig (cf. [Lu90])– and we expect $\widehat{\mathcal{G}}^{\text{stab}}$ to be connected to these objects too. It would be very interesting to understand the link between the stable moment graph and this geometric counterpart. Finally, we want to mention that the definition of $\widehat{\mathcal{G}}^{\text{stab}}$ arises from rather intriguing combinatorics, which have been our starting point and which we are going to address in this work.

The paper is organised as follows.

The aim of the second section is to recollect the basics of the theory of affine Kac-Moody algebras, in order to fix the notation.

In Section 3 we recall the definition of the category of k -moment graphs on a lattice we introduced in [L14], where k stands for a local ring with $2 \in k^\times$. Let us fix once and for all an affine (untwisted) Kac-Moody algebra $\widehat{\mathfrak{g}}$, a Borel subalgebra $\widehat{\mathfrak{b}}$ and a Cartan subalgebra $\widehat{\mathfrak{h}}$. Let us moreover denote by \mathcal{W} the Weyl group of $\widehat{\mathfrak{g}}$ and $\widehat{\mathcal{S}}$ the set of simple reflections corresponding to the data we fixed. For any quadruple $(\widehat{\mathfrak{g}} \supseteq \widehat{\mathfrak{b}} \supseteq \widehat{\mathfrak{h}}, J)$, where $J \subseteq \widehat{\mathcal{S}}$, we define the Bruhat moment graph $\widehat{\mathcal{G}}^J$ (see §3.2). Moreover, we associate with $\widehat{\mathfrak{g}} \supseteq \widehat{\mathfrak{b}} \supseteq \widehat{\mathfrak{h}}$ another k -moment graph: the periodic one, that we denote by $\widehat{\mathcal{G}}^{\text{per}}$ and which coincides with $\widehat{\mathcal{G}}^\emptyset$ up to the orientation of the edges (see §3.3).

In Section 4 we focus on the parabolic Bruhat graph $\widehat{\mathcal{G}}^{\text{par}}$, that is, the one corresponding to the choice of J being \mathcal{S} , the set of finite simple reflections. The vertices of this graph are given by the alcoves in the fundamental chamber, and the goal of this section is to study the behaviour of intervals $\widehat{\mathcal{G}}^{\text{par}}_{[[A,B]]}$, with A and B *deep enough* in C^+ . Our first hope was that the stabilisation phenomenon described by Lusztig would have been visible also at the moment graph level. This is unluckily not completely true. Indeed, for any fixed pair of alcoves A, B in the fundamental chamber it is possible to translate them in such a way that $\widehat{\mathcal{G}}^{\text{par}}$ restricted to the corresponding interval and considered as an *unlabeled* graph is invariant under translation by positive multiples of ρ , and the coweight is defined as half the sum of the finite positive coroots (see Lemma 4.7). Since the label function is a fundamental data that a moment graph comes with, the next step is to discuss the labeling of the edges of such intervals. It turns out that the set of edges is in this case bipartite in *stable* and *non-stable* edges, whose labels have a good and a bad behaviour, respectively (see Corollary 4.10 and Lemma 4.12). This gives rise to the definition of a new k -moment graph attached to $\widehat{\mathfrak{g}}$: the *stable* moment graph $\widehat{\mathcal{G}}^{\text{stab}}$ (see §4.3); that is, a subgraph of $\widehat{\mathcal{G}}^{\text{par}}$ having the stabilisation property we were looking for. It is the main character of this paper.

Section 5 deals with the category of sheaves on a k -moment graph. After recalling definitions and basic results concerning these objects, we introduce the new notion of a pushforward functor and we prove that it is right adjoint to the pullback functor we introduced in [L11]. Braden-MacPherson sheaves make their appearance in the

last part of this section. We recall a lemma of [L11] telling us that the pullback of an isomorphism preserves indecomposable Braden Mac-Pherson sheaves (see Lemma 5.14). This result will be an important tool in the proof of our main theorem and we will refer to it as “property of the pullback”.

In the following section we are finally able to state the moment graph analogue of Lusztig’s theorem, that is, the stalks of indecomposable Braden-MacPherson sheaves on finite intervals of $\widehat{\mathcal{G}}^{\text{par}}$ deep enough in \mathcal{C}^+ stabilise too. More precisely, let $\mathcal{I} = [A, B]$ be an interval deep enough in the fundamental chamber. Then

Theorem 1.1. *Under some assumptions on the base ring, for any alcove $C \in [A, B]$ and for all $\mu \in \mathbb{Z}_{\geq 0}\rho$,*

$$\mathcal{B}(B)^C \cong \mathcal{B}(B + \mu)^{C+\mu},$$

where $\mathcal{B}(B)$, resp. $\mathcal{B}(B + \mu)$, is an indecomposable BMP-sheaves on $\widehat{\mathcal{G}}^{\text{par}}_{|[A, B]}$, resp. $\widehat{\mathcal{G}}^{\text{par}}_{|[A+\mu, B+\mu]}$.

Here the stable moment graph comes into play. Because A and B are deep enough in the fundamental chamber, by results of Section 4, the two k -moment graphs $\widehat{\mathcal{G}}^{\text{par}}_{|[A, B]}$ and $\widehat{\mathcal{G}}^{\text{par}}_{|[A+\mu, B+\mu]}$ are in general not isomorphic, while there is always an isomorphism of k -moment graphs between $\widehat{\mathcal{G}}^{\text{stab}}_{|[A, B]}$ and $\widehat{\mathcal{G}}^{\text{stab}}_{|[A+\mu, B+\mu]}$, for $\mu \in \mathbb{Z}_{\geq 0}\rho$. Since the stable moment graph is a subgraph of $\widehat{\mathcal{G}}^{\text{par}}$, there is a morphism $\widehat{\mathcal{G}}^{\text{stab}} \hookrightarrow \widehat{\mathcal{G}}^{\text{par}}$. The following diagram summarises this situation:

$$\begin{array}{ccc} \widehat{\mathcal{G}}^{\text{par}}_{|[A, B]} & & \widehat{\mathcal{G}}^{\text{par}}_{|[A+\mu, B+\mu]} \\ \uparrow g & & \uparrow g_\mu \\ \widehat{\mathcal{G}}^{\text{stab}}_{|[A, B]} & \longrightarrow & \widehat{\mathcal{G}}^{\text{stab}}_{|[A+\mu, B+\mu]} \end{array}$$

We then get a functor $\cdot^{\text{stab}} := g^* : \mathbf{Sh}(\widehat{\mathcal{G}}^{\text{par}}_{|[A, B]}) \rightarrow \mathbf{Sh}(\widehat{\mathcal{G}}^{\text{stab}}_{|[A, B]})$. The main theorem of the last two sections is the following one.

Theorem 1.2. *The functor $\cdot^{\text{stab}} : \mathbf{Sh}_k(\widehat{\mathcal{G}}^{\text{par}}_{|[A, B]}) \rightarrow \mathbf{Sh}_k(\widehat{\mathcal{G}}^{\text{stab}}_{|[A, B]})$ preserves indecomposable Braden-MacPherson sheaves.*

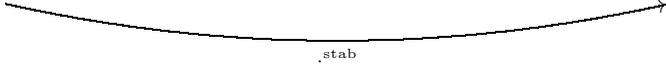
Once this result is proven, the stabilisation property for stalks of indecomposable Braden-MacPherson sheaves follows by applying the property of the pullback we mentioned before.

We conclude the section by demonstrating the theorem above in the subgeneric case, that is, the one corresponding to $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_2$. In particular, we show that for any finite interval of $\widehat{\mathcal{G}}^{\text{stab}}$, the associated indecomposable Braden-MacPherson sheaf is isomorphic to its structure sheaf \mathcal{L} (see §5.2.2 for the definition of a structure sheaf). This is done by proving the equivalent claim (see Proposition 5.13) that \mathcal{L} is flabby. On the other hand, by Fiebig’s multiplicity one result (cf. [Fie10a]), it is known that indecomposable Braden-MacPherson sheaves on finite intervals of $\widehat{\mathcal{G}}^{\text{par}}$ are isomorphic to the corresponding structure sheaves (in Appendix A we give an alternative proof of this fact). This finishes the proof of the subgeneric case.

In Section 7 we address the general case. Easily, we reduce (see Lemma 7.1) to proving that the image under \cdot^{stab} of an indecomposable Braden-MacPherson sheaf is flabby and indecomposable. In order to show this, we use a local-global approach,

developed by Fiebig in [Fie08a], which enables us to translate our claim in terms of certain modules over the structure algebra \mathcal{Z} , that is, the space of global sections of the structure sheaf. More precisely, we express the functor \cdot^{stab} as a composition of five functors:

$$\mathbf{Sh}(\widehat{\mathcal{G}}_{|\mathcal{Z}}^{\text{par}}) \xrightarrow{i_*} \mathbf{Sh}(\widehat{\mathcal{G}}^{\text{par}}) \xrightarrow{p_{\text{par}}^*} \mathbf{Sh}(\widehat{\mathcal{G}}) \xrightarrow{\cdot^{\text{per}}} \mathbf{Sh}(\widehat{\mathcal{G}}^{\text{per}}) \xrightarrow{j^*} \mathbf{Sh}(\widehat{\mathcal{G}}_{|\mathcal{Z}}^{\text{per}}) \xrightarrow{\cdot^{\text{opp}}} \mathbf{Sh}(\widehat{\mathcal{G}}_{|\mathcal{Z}}^{\text{stab}})$$



 \cdot^{stab}

The functors i_* , p_{par}^* , j^* and \cdot^{opp} preserve indecomposable Braden-MacPherson sheaves (and hence flabbiness), by results of [L11]. Therefore it is only left to prove that the functor \cdot^{per} maps Braden-MacPherson sheaves to flabby sheaves. To get such a result, translation functor techniques are needed. Once we recall some definitions from [Fie08a] and [L14], we consider any partial order \trianglelefteq on $\widehat{\mathcal{W}}$ satisfying certain properties (see §7.1.3), we define the notion of \trianglelefteq -flabbiness (see Definition 7.6) and, generalising the result of [Fie08a], we prove that translation functors preserve \trianglelefteq -flabby objects (see Theorem 7.17). The desired property of the functor \cdot^{per} now follows by a rather easy inductive procedure (see Proposition 7.20).

Finally, we obtain the indecomposability by considering localisation at certain ideals of *special* \mathcal{Z} -modules (see §7.2.1), which allows us to use the subgeneric case –proven in Section 6– and hence to conclude (see Proposition 7.25).

Appendix A deals with the subgeneric case. In particular, we prove in a very explicit way that indecomposable Braden-MacPherson sheaves on any finite interval of $\widehat{\mathcal{G}}^{\text{par}}$, satisfying the GKM-condition (cf. Definition 3.3), are isomorphic to the corresponding structure sheaves. As mentioned before, this result could be obtained by applying [Fie10a, Theorem 6.4], or –for $k = \mathbb{Q}$ – it could also be derived from topological facts, such as the rational smoothness of the underlying varieties together with [BMP, Theorem 1.6]. The advantage of our proof is that it only uses the definition of Braden-MacPherson sheaves, together with the explicit description of finite intervals of $\widehat{\mathcal{G}}^{\text{par}}$ given in Section 3.

2. AFFINE KAC-MOODY ALGEBRAS

We want to fix some notation relative to affine Kac-Moody algebras. The main reference is [Kac, §6].

2.1. Basic notation. Let us consider a finite-dimensional simple complex Lie algebra \mathfrak{g} and $\mathfrak{b} \supseteq \mathfrak{h}$ a Borel and a Cartan subalgebra. This data uniquely determines a set of simple roots $\Pi = \Pi(\mathfrak{b}, \mathfrak{h})$ and a root system $\Delta = \Delta_+ \sqcup \Delta_-$, where Δ_+ , resp. Δ_- , denotes the set of positive, resp. negative, roots.

We now want to consider the (untwisted) affine Kac-Moody algebra $\widehat{\mathfrak{g}}$. As a vector space, it is defined as

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t^{\pm 1}] \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

If $\kappa : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$ denotes the Killing form of \mathfrak{g} , then the commutation relations are as follows:

$$\begin{aligned} [c, \widehat{\mathfrak{g}}] &= 0, \\ [D, x \otimes t^n] &= nx \otimes t^n, \\ [x \otimes t^n, y \otimes t^m] &= [x, y] \otimes t^{m+n} + n\delta_{m,-n}\kappa(x, y)c, \end{aligned}$$

for $x, y \in \mathfrak{g}, n, m \in \mathbb{Z}$.

Moreover, there is a Cartan subalgebra of $\widehat{\mathfrak{g}}$, which corresponds to \mathfrak{h} , namely

$$\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}D,$$

and we also have

$$\widehat{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\delta \oplus \mathbb{C}\Lambda_0,$$

where, if $\langle \cdot, \cdot \rangle : \widehat{\mathfrak{h}} \times \widehat{\mathfrak{h}}^* \rightarrow \mathbb{C}$ is the canonical pairing, it holds that

$$\begin{aligned} \langle \delta, \mathfrak{h} \oplus \mathbb{C}c \rangle &= \{0\}, \\ \langle \delta, D \rangle &= 1, \\ \langle \Lambda_0, \mathfrak{h} \oplus \mathbb{C}D \rangle &= \{0\}, \\ \langle \Lambda_0, c \rangle &= 1. \end{aligned}$$

Thus the set of real roots $\widehat{\Delta}^{\text{re}}$ of $\widehat{\mathfrak{g}}$ has a nice description in terms of the root system of \mathfrak{g} . Indeed, we have (cf. [Kac, Proposition 6.3])

$$\widehat{\Delta}^{\text{re}} = \{\alpha + n\delta \mid \alpha \in \Delta, n \in \mathbb{Z}\},$$

while the set of positive real roots is given by

$$\widehat{\Delta}_+^{\text{re}} = \{\alpha + n\delta \mid \alpha \in \Delta, n \in \mathbb{Z}_{>0}\} \cup \Delta_+.$$

Finally, if $\theta \in \Delta_+$ is the highest root, then

$$\widehat{\Pi} = \Pi \cup \{-\theta + \delta\}$$

is the set of (affine) simple roots.

2.2. The affine Weyl group and its set of alcoves. For any $\alpha \in \widehat{\Delta}^{\text{re}}$ let us denote by $s_\alpha \in GL(\widehat{\mathfrak{h}}^*)$ the reflection whose action on $v \in \widehat{\mathfrak{h}}^*$ is defined by

$$s_\alpha(v) = v - \langle v, \alpha^\vee \rangle \alpha.$$

The affine Weyl group $\widehat{\mathcal{W}}$ is then generated by the reflections s_α with $\alpha \in \widehat{\Delta}_+^{\text{re}}$, while we may identify the finite Weyl group \mathcal{W} with the subgroup of $\widehat{\mathcal{W}}$ generated by reflections which are indexed by finite positive real roots.

Let us denote by

$$\widehat{\mathcal{S}} = \{s_\alpha \mid \alpha \in \widehat{\Pi}\}$$

the set of *simple reflections* of $\widehat{\mathcal{W}}$. Then $(\widehat{\mathcal{W}}, \widehat{\mathcal{S}})$ is a Coxeter system and the set $\widehat{\mathcal{T}}$ of *reflections* of $\widehat{\mathcal{W}}$ can be also obtained by conjugating $\widehat{\mathcal{S}}$, i.e.

$$\widehat{\mathcal{T}} = \{s_\alpha \mid \alpha \in \widehat{\Delta}_+^{\text{re}}\} = \{ws w^{-1} \mid w \in \widehat{\mathcal{W}}, s \in \widehat{\mathcal{S}}\}.$$

Moreover, we denote by $\ell : \widehat{\mathcal{W}} \rightarrow \mathbb{Z}_{\geq 0}$ the length function and by \leq the Bruhat order on $\widehat{\mathcal{W}}$.

Let $\widehat{\mathfrak{h}}_{\mathbb{R}}^*$ and $\mathfrak{h}_{\mathbb{R}}^*$ be the \mathbb{R} -span of $\widehat{\Pi}$ and of Π , respectively. We shall now recall another realisation of $\widehat{\mathcal{W}}$, as a group of affine transformations of $\mathfrak{h}_{\mathbb{R}}^*$. This is obtained by identifying $\mathfrak{h}_{\mathbb{R}}^*$ with the affine space $\widehat{\mathfrak{h}}_1^* \bmod \mathbb{R}\delta$, where

$$\widehat{\mathfrak{h}}_1^* := \{ \lambda \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle \lambda, c \rangle = 1 \}.$$

Because \mathfrak{g} is symmetrisable, by [Kac, Lemma 2.1], there is a bilinear form $(\cdot, \cdot) : \widehat{\mathfrak{g}} \times \widehat{\mathfrak{g}} \rightarrow \mathbb{C}$ that induces an isomorphism $\nu : \mathfrak{h} \rightarrow \mathfrak{h}^*$ such that, if α is a finite root, we may identify α^\vee and $\frac{2\alpha}{(\alpha, \alpha)}$. Then, the action of $\widehat{\mathcal{W}}$ on $\lambda \in \mathfrak{h}^*$ is given by

$$(1) \quad s_{\alpha+n\delta}(\lambda) = \lambda - \left(\langle \lambda, \alpha^\vee \rangle + \frac{2n}{(\alpha, \alpha)} \right) \alpha = s_\alpha(\lambda) - n\alpha^\vee.$$

Denote by Q^\vee the coroot lattice of \mathfrak{g} and by T_μ the translation by $\mu \in Q^\vee$, that is, the transformation defined as $T_\mu(\lambda) = \lambda + \mu$ for any $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$. This is an element of the affine Weyl group, since $T_{-n\alpha^\vee} = s_{\alpha+n\delta}s_\alpha$. It is easy to check that for any $w \in \widehat{\mathcal{W}}$ and for any $\mu \in Q^\vee$ we have $wT_\mu w^{-1} = T_{w(\mu)}$, so the group of translations by an element of the coroot lattice turns out to be a normal subgroup. A well-known fact is that $\widehat{\mathcal{W}} = \mathcal{W} \ltimes Q^\vee$ (cf. [H, Proposition 4.2]).

Denote by

$$H_{\alpha,n} := \left\{ \lambda \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle \lambda, \alpha^\vee \rangle = -\frac{2n}{(\alpha, \alpha)} \right\} = \{ \lambda \in \mathfrak{h}_{\mathbb{R}}^* \mid (\lambda, \alpha) = -n \}$$

and observe that the affine reflection $s_{\alpha+n\delta}$ fixes pointwise such a hyperplane. We call *alcoves* the connected components of

$$\mathfrak{h}_{\mathbb{R}}^* \setminus \bigcup_{\alpha+n\delta \in \widehat{\Delta}_+^{\text{re}}} H_{\alpha,n}$$

and write \mathcal{A} for the set of all alcoves.

The *fundamental (Weyl) chamber* is

$$\mathcal{C}^+ := \{ \lambda \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle \lambda, \alpha^\vee \rangle > 0 \ \forall \alpha \in \Pi \}$$

and an element $\lambda \in \mathcal{C}^+$ is called *dominant weight*. We denote by \mathcal{A}^+ the set of all alcoves contained in \mathcal{C}^+ .

We now state a 1-1 correspondence between $\widehat{\mathcal{W}}$ and \mathcal{A} (cf. [H, Theorem 4.8]). In order to do that, we fix an alcove A^+ , that is, the unique alcove in \mathcal{A}^+ which contains the null vector in its closure. A^+ is usually called *the fundamental alcove* and it has the property that every element $\lambda \in A^+$ is such that $0 < (\lambda, \alpha) < 1$ for all $\alpha \in \Delta_+$ (cf. [H, §4.3]).

The affine Weyl group $\widehat{\mathcal{W}}$ acts on the left (by (1)) simply transitively on \mathcal{A} (cf. [H, §4.5]), and so we obtain

$$(2) \quad \begin{array}{ccc} \widehat{\mathcal{W}} & \xrightarrow{1-1} & \mathcal{A} \\ w & \mapsto & wA^+. \end{array}$$

Let us observe that each wall of A^+ is fixed by exactly one reflection $s \in \widehat{\mathcal{S}}$. We say that such a wall is the *s-wall* of A^+ . In general every $A \in \mathcal{A}$ has one and only one wall in the $\widehat{\mathcal{W}}$ -orbit of the *s-wall* of A^+ . This is called the *s-wall* of A .

The affine Weyl group acts on itself by right multiplication, so it makes sense to define a right action of $\widehat{\mathcal{W}}$ on \mathcal{A} . It is of course enough to define such an action for the generators of the group. Thus for each alcove A let As be the unique alcove having in common with A the s -wall.

3. MOMENT GRAPHS

Let us fix once and for all a local ring k such that 2 is an invertible element. In the first part of this section we recall the definition of the category of k -moment graphs on a lattice, while in the second part we focus on certain moment graphs, which are relevant for the representation theory of affine Kac-Moody algebras.

3.1. The category of k -moment graphs on a lattice.

Definition 3.1 (cf. [Fie10b]). Let Y be a lattice of finite rank. A *moment graph on the lattice Y* is given by $(\mathcal{V}, \mathcal{E}, \preceq, l)$, where:

- (MG1) $(\mathcal{V}, \mathcal{E})$ is a directed graph without directed cycles or multiple edges;
- (MG2) \preceq is a partial order on \mathcal{V} such that if $x, y \in \mathcal{V}$ and $E : x \rightarrow y \in \mathcal{E}$, then $x \preceq y$;
- (MG3) $l : \mathcal{E} \rightarrow Y \setminus \{0\}$ is a map called the *label function*.

Following Fiebig’s notation (cf. [Fie10b]), we will write $x \text{---} y$ if we are forgetting about the orientation of the edge.

For any lattice Y of finite rank, let us denote the extended lattice by $Y_k = Y \otimes_{\mathbb{Z}} k$.

Definition 3.2 (cf. [L11]). Let \mathcal{G} be a moment graph on the lattice Y . We say that \mathcal{G} is a *k -moment graph* on Y if all labels are non-zero in Y_k , that is, the image $\overline{l(E)} := l(E) \otimes 1$ is a non-zero element of Y_k .

By abuse of notation, we will also often denote by $l(E)$ its image in Y_k .

Definition 3.3 (cf. [FW]). The pair (\mathcal{G}, k) is called a *GKM-pair* if all pairs E_1, E_2 of distinct edges containing a common vertex are such that

$$k \cdot l(E_1) \cap k \cdot l(E_2) = \{0\}.$$

This property is very important and, in the next sections, will give a restriction on the ring k .

3.1.1. Morphisms of k -moment graphs.

Definition 3.4 (cf. [L11]). A morphism between two k -moment graphs

$$f : (\mathcal{V}, \mathcal{E}, \preceq, l) \rightarrow (\mathcal{V}', \mathcal{E}', \preceq', l')$$

is given by $(f_{\mathcal{V}}, \{f_{l,x}\}_{x \in \mathcal{V}})$, where

- (MORPH1) $f_{\mathcal{V}} : \mathcal{V} \rightarrow \mathcal{V}'$ is any map of posets such that, if $x \text{---} y \in \mathcal{E}$, then either $f_{\mathcal{V}}(x) \text{---} f_{\mathcal{V}}(y) \in \mathcal{E}'$, or $f_{\mathcal{V}}(x) = f_{\mathcal{V}}(y)$.

For an edge $E : x \rightarrow y \in \mathcal{E}$ such that $f_{\mathcal{V}}(x) \neq f_{\mathcal{V}}(y)$, we will denote

$$f_{\mathcal{E}}(E) := f_{\mathcal{V}}(x) \text{---} f_{\mathcal{V}}(y).$$

- (MORPH2) For all $x \in \mathcal{V}$, $f_{l,x} : Y_k \rightarrow Y_k \in \text{Aut}_k(Y_k)$ is such that, if $E : x \rightarrow y \in \mathcal{E}$ and $f_{\mathcal{V}}(x) \neq f_{\mathcal{V}}(y)$, the following two conditions are verified:
- (MORPH2a) $f_{l,x}(l(E)) = h \cdot l'(f_{\mathcal{E}}(E))$, for some $h \in k^\times$;
 - (MORPH2b) $\pi \circ f_{l,x} = \pi \circ f_{l,y}$, where π is the canonical quotient map $\pi : Y_k \rightarrow Y_k/l'(f_{\mathcal{E}}(E))Y_k$.

Given two morphisms of k -moment graphs

$$f : \mathcal{G} = (\mathcal{V}, \mathcal{E}, \leq, l) \rightarrow \mathcal{G}' = (\mathcal{V}', \mathcal{E}', \leq', l') \quad \text{and} \quad g : \mathcal{G}' \rightarrow \mathcal{G}'' = (\mathcal{V}'', \mathcal{E}'', \leq'', l''),$$

we may set

$$g \circ f := (g_{\mathcal{V}'} \circ f_{\mathcal{V}}, \{g_{l', f_{\mathcal{V}}(x)} \circ f_{l,x}\}_{x \in \mathcal{V}}).$$

It is an easy check that this composition is well-defined and associative.

Definition 3.5. We denote by $\mathbf{MG}(Y_k)$ the category whose objects are the k -moment graphs on Y and corresponding morphisms.

3.1.2. *Isomorphisms of k -moment graphs.* In [L11], we gave an abstract definition of isomorphisms but we did not notice that the notion we were considering actually coincided with the one of invertible morphisms. The following lemma proves this fact.

Lemma 3.6. *Let us consider $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \leq, l), \mathcal{G}' = (\mathcal{V}', \mathcal{E}', \leq', l') \in \mathbf{MG}(Y_k)$, and a morphism between them $f = (f_{\mathcal{V}}, \{f_{l,x}\}_{x \in \mathcal{V}}) \in \text{Hom}_{\mathbf{MG}(Y_k)}(\mathcal{G}, \mathcal{G}')$. f is an isomorphism in the categorical sense if and only if the following two conditions hold:*

- (ISO1) $f_{\mathcal{V}}$ is bijective;
- (ISO2) for all $u \rightarrow w \in \mathcal{E}'$, there exists exactly one $x \rightarrow y \in \mathcal{E}$ such that $f_{\mathcal{V}}(x) = u$ and $f_{\mathcal{V}}(y) = w$.

Proof. To begin with we show that a morphism satisfying (ISO1) and (ISO2) is invertible. Denote by $f^{-1} := (f'_{\mathcal{V}'}, \{f'_{l',u}\}_{u \in \mathcal{V}'})$, where we set $f'_{\mathcal{V}'} := f_{\mathcal{V}}^{-1}$ and $f'_{l',u} := f_{l, f_{\mathcal{V}}^{-1}(u)}$. We have to verify that f^{-1} is well-defined, that is, we have to check conditions (MORPH2a) and (MORPH2b). Suppose there exists an edge $F : u \rightarrow w \in \mathcal{E}'$; then, by (ISO2), there is an edge $E : x \rightarrow y \in \mathcal{E}$ such that $f_{\mathcal{V}}(x) = u$ and $f_{\mathcal{V}}(y) = w$. Since f satisfies (MORPH2a), $f_{l,x}(l(E)) = h \cdot l'(F)$ for $h \in k^\times$, and we get

$$f'_{l',u}(l'(F)) = f_{l, f_{\mathcal{V}}^{-1}(u)}^{-1}(l'(F)) = f_{l,x}^{-1}(l'(F)) = h^{-1} \cdot l(E).$$

Now, let $\mu \in Y_K$ and take $\lambda := f_{l,y}^{-1}(\mu)$. By (MORPH2a), $\mu = f_{l,y}(\lambda) = f_{l,x}(\lambda) + r \cdot l'(F)$ for some $r \in k$. It follows that

$$\begin{aligned} f'_{l',u}(\mu) &= f_{l,x}^{-1}(\mu) \\ &= f_{l,x}^{-1}(f_{l,x}(\lambda) + r l'(F)) \\ &= \lambda + r \cdot f_{l,x}^{-1}(l'(F)) \\ &= f_{l,y}^{-1}(\mu) + r \cdot h^{-1} \cdot l(E) \\ &= f'_{l',w}(\mu) + r' \cdot l(E). \end{aligned}$$

Suppose f is an isomorphism. If (ISO1) is not satisfied, then $f_{\mathcal{V}}$, and hence f , is not invertible. Moreover, (ISO1) implies that for all $u \rightarrow v \in \mathcal{E}'$, there exists at most one $x \rightarrow y \in \mathcal{E}$ such that $f_{\mathcal{V}}(x) = u$ and $f_{\mathcal{V}}(y) = v$ (otherwise $f_{\mathcal{V}}$ would not

be injective). Now, let f be the following homomorphism (we do not care about the $f_{l,x}$'s):



Condition (ISO1) holds, but f is not invertible, since $f_V^{-1}(u) \neq f_V^{-1}(w)$ but $f_V^{-1}(u) - f_V^{-1}(w) \notin \mathcal{E}$ (this contradicts (MORPH1)). \square

Isomorphisms between k -moment graphs were an important tool in [L11] in order to obtain a categorical analogue of certain equalities between Kazhdan-Lusztig polynomials.

3.2. Bruhat moment graphs. From now on we fix $\widehat{\mathfrak{g}} \supseteq \widehat{\mathfrak{b}} \supseteq \widehat{\mathfrak{h}}$ and keep the notation of Section 2. For any subset $J \subset \widehat{\mathcal{S}}$ we define the (affine) Bruhat moment graph

$$\widehat{\mathcal{G}}^J = \widehat{\mathcal{G}}(\widehat{\mathfrak{g}} \supseteq \widehat{\mathfrak{b}} \supseteq \widehat{\mathfrak{h}}, J) = (\mathcal{V}, \mathcal{E}, \leq, l)$$

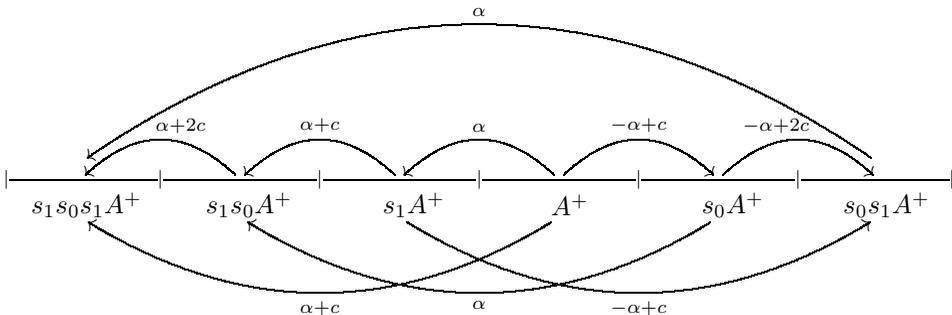
associated to the data $(\widehat{\mathfrak{g}} \supseteq \widehat{\mathfrak{b}} \supseteq \widehat{\mathfrak{h}}, J)$ as follows.

Definition 3.7. $\widehat{\mathcal{G}}^J$ is the moment graph on the affine coroot lattice \widehat{Q}^\vee which is given by

- (i) $\mathcal{V} = \widehat{\mathcal{W}}^J$, where $\widehat{\mathcal{W}}^J$ stands for the set of minimal representatives of the equivalence classes of $\widehat{\mathcal{W}}/\langle J \rangle$;
- (ii) $\mathcal{E} = \left\{ x \rightarrow y \mid x < y, \exists \alpha \in \widehat{\Delta}_+^{\text{re}}, \exists w \in \langle J \rangle \text{ such that } ywx^{-1} = s_\alpha \right\}$;
- (iii) $l(x \rightarrow s_\alpha xw^{-1}) := \alpha^\vee$.

If $J = \emptyset$, we will simply write $\widehat{\mathcal{G}}$ instead of $\widehat{\mathcal{G}}^\emptyset$ and call it the *regular Bruhat graph* of $\widehat{\mathfrak{g}}$, while, if $J = \mathcal{S} := \{s_\alpha \mid \alpha \in \Pi\}$, we will denote $\widehat{\mathcal{G}}^{\text{par}} := \widehat{\mathcal{G}}^\mathcal{S}$ and call it the *parabolic Bruhat graph*.

Example 3.8. Let $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_2$. In this case, $\widehat{\Delta}_+^{\text{re}} = \{\pm\alpha + n\delta \mid n \in \mathbb{Z}_{>0}\} \cup \{\alpha\}$, where α is the (unique) positive root of \mathfrak{sl}_2 and $(\alpha, \alpha) = 2$. The corresponding regular Bruhat graph $\widehat{\mathcal{G}}$ is an infinite graph whose vertices are given by the words in two letters ($s_1 := s_\alpha$ and s_0) without repetition. Two elements are connected if and only if the difference between their lengths is odd and in this case the edge is oriented from the shorter to the longer one. Thanks to the correspondence (2), we may identify the set of vertices with the set of alcoves of $\widehat{\mathfrak{g}}$. If we restrict $\widehat{\mathcal{G}}$ to the interval $[A^+, s_1s_0s_1A^+]$, we get the following graph:



3.3. The periodic moment graph. We now want to associate another moment graph with $\widehat{\mathfrak{g}}$. In order to do this, we need to recall the notion of generic order on the set of alcoves.

3.3.1. *Two partial orders on the set of alcoves.* Following [Lu80b], we provide the set of alcoves with two partial orders.

First of all, the Bruhat order on $\widehat{\mathcal{W}}$ induces a partial order on \mathcal{A} . Indeed, for all alcoves $A, B \in \mathcal{A}$ with $A = xA^+, B = yA^+, x, y \in \widehat{\mathcal{W}}$ we may set

$$A \leq B \iff x \leq y.$$

We still call it the *Bruhat order*.

We observe that in general if we look at two alcoves it is not obvious at all if they are comparable with respect to the Bruhat order without knowing the corresponding elements in $\widehat{\mathcal{W}}$.

Next, we recall Lusztig’s definition of a nicer partial order \preceq on \mathcal{A} , in the sense that for all pairs of alcoves we will be able to say whether they are comparable and, if it is the case, to establish which one is the bigger one.

Each $H \in \bigcup_{\alpha+n\delta \in \widehat{\Delta}_+^{\text{re}}} H_{\alpha,n}$ divides $\mathfrak{h}_{\mathbb{R}}^*$ in two half spaces: $\mathfrak{h}_{\mathbb{R}}^* = H^+ \sqcup H \sqcup H^-$, where H^+ is the half space that intersects every translate of \mathcal{C}^+ . Let $A \in \mathcal{A}$. If H is the reflecting hyperplane between A and $As, s \in \widehat{\mathcal{S}}$, we consider the partial order generated by

$$A \preceq As \text{ if } A \in H^-.$$

Notice that it is not clear in general how \leq and \preceq are related. Let us denote by X^\vee the *lattice of (finite) integral coweights*, that is (again under the identification of $\mathfrak{h}_{\mathbb{R}}$ and $\mathfrak{h}_{\mathbb{R}}^*$),

$$(3) \quad X^\vee := \{ \lambda \in \mathfrak{h}_{\mathbb{R}}^* \mid (\lambda, \alpha) \in \mathbb{Z} \ \forall \alpha \in \Delta \}.$$

Proposition 3.9 ([Soe97], Claim 4.4). *Deep enough in \mathcal{C}^+ , \leq and \preceq coincide; that is, for all $\lambda \in X^\vee \cap \mathcal{C}^+, A, B \in \mathcal{A}$ the following are equivalent:*

- (1) $A \preceq B$;
- (2) $n\lambda + A \leq n\lambda + B$ for $n \gg 0$.

Because of this result Lusztig called \preceq the *generic Bruhat order*. We remark that \preceq is invariant under translation by finite coweights.

Definition 3.10. The *periodic moment graph*

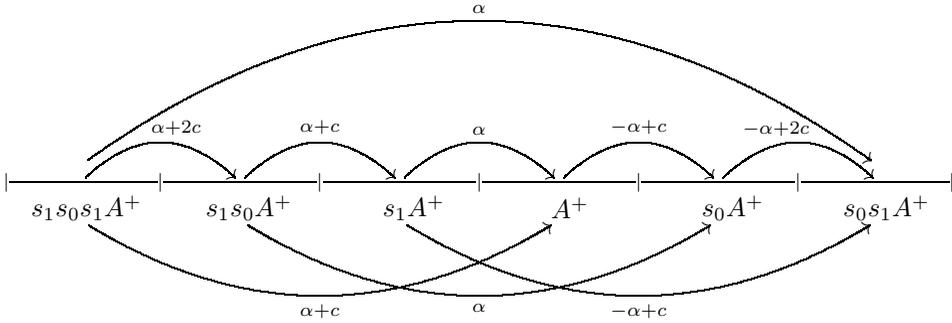
$$\widehat{\mathcal{G}}^{\text{per}} = \widehat{\mathcal{G}}^{\text{per}}(\widehat{\mathfrak{g}} \supseteq \widehat{\mathfrak{b}} \supseteq \widehat{\mathfrak{h}}) = (\mathcal{V}, \mathcal{E}, \preceq, l)$$

is a moment graph on \widehat{Q}^\vee and is given by

- (i) $\mathcal{V} = \mathcal{A}$, the set of alcoves of $\widehat{\mathcal{W}}$;
- (ii) $\mathcal{E} = \{ xA^+ \rightarrow yA^+ \mid xA^+ \preceq yA^+, \exists \alpha \in \widehat{\Delta}_+^{\text{re}} \text{ such that } y = s_\alpha x \}$;
- (iii) $l(xA^+ \rightarrow s_\alpha xA^+) := \alpha^\vee$.

Remark 3.11. Let us observe that we identified $\widehat{\mathcal{W}}$ and \mathcal{A} by (2) and therefore $\widehat{\mathcal{G}}$ and $\widehat{\mathcal{G}}^{\text{per}}$ coincide as labeled *unoriented* graphs, as we can see by comparing Example 3.8 with the following one.

Example 3.12. Let $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_2$. If we restrict the corresponding periodic moment graph to the interval $[s_1s_0s_1A^+, s_0s_1A^+]$, we get the following moment graph:



4. FINITE INTERVALS OF $\widehat{\mathcal{G}}^{\text{PAR}}$ DEEP ENOUGH IN THE FUNDAMENTAL CHAMBER

This section is devoted to the study of certain intervals of the parabolic Bruhat graph $\widehat{\mathcal{G}}^{\text{par}}$, that is, the Bruhat graph corresponding to the data $(\widehat{\mathfrak{g}} \supseteq \widehat{\mathfrak{b}} \supseteq \widehat{\mathfrak{h}}, \mathcal{S})$.

4.1. Two descriptions. There are actually two descriptions of the graph $\widehat{\mathcal{G}}^{\text{par}}$: one identifies the set of vertices with the finite coroot lattice Q^\vee , while the other identifies the set of vertices with \mathcal{A}^+ , the set of alcoves in the fundamental chamber.

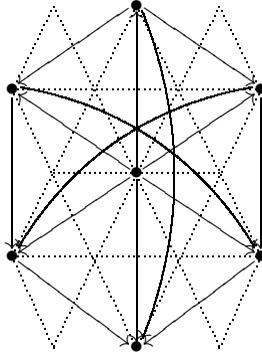
As the affine Weyl group acts on $\mathfrak{h}_{\mathbb{R}}^*$, we can consider the $\widehat{\mathcal{W}}$ -orbit of 0, that is, the finite coroot lattice Q^\vee . Moreover $\text{Stab}_{\widehat{\mathcal{W}}}(0) = \mathcal{W} = \langle \mathcal{S} \rangle$, the finite Weyl group, and hence $\widehat{\mathcal{W}}/\mathcal{W}$ is in bijection with the coroot lattice via the mapping $w \rightarrow w(0)$. Clearly, for any pair of minimal length representatives $x, y \in \widehat{\mathcal{W}}/\mathcal{W}$, there exist $w \in \mathcal{W}$ and $\alpha + n\delta \in \widehat{\Delta}_+^{\text{re}}$ such that $y = s_{\alpha+n\delta}xw$ if and only if $y(0) = s_{\alpha+n\delta}x(0)$, that is, $y(0) = x(0) - n\alpha^\vee$.

On the other hand, $\mathcal{W} \setminus \widehat{\mathcal{W}}$ is clearly in bijection with $\widehat{\mathcal{W}}/\mathcal{W}$ via the mapping $x \mapsto x^{-1}$. The set of minimal representatives for the equivalence classes, under the correspondence (2), is given by the set \mathcal{A}^+ of the alcoves in the fundamental chamber. Moreover, we will connect $xA^+, yA^+ \in \mathcal{A}^+$ if and only if there exist an element of the finite Weyl group $w \in \mathcal{W}$ and an affine positive root $\alpha \in \widehat{\Delta}_+^{\text{re}}$ such that $x = wys_\alpha$, that is, $x^{-1} = s_\alpha y^{-1}w^{-1}$.

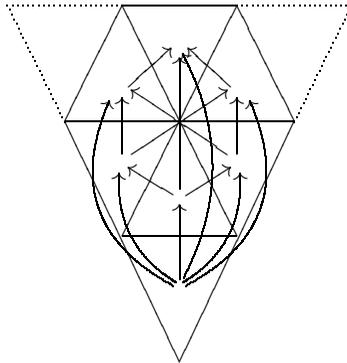
Example 4.1. Let $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_3$ and let \mathcal{W}^{par} be the set of minimal length representatives of $\widehat{\mathcal{W}}/\mathcal{W}$. Let us consider the interval $[e, s_\beta s_\alpha s_\beta s_0] \subset \mathcal{W}^{\text{par}}$; then the two

descriptions of $\widehat{\mathcal{G}}^{\text{par}}$ are as follows (we omit the labels):

(i) Description via the finite coroot lattice



(ii) Description via \mathcal{A}^+ the set of alcoves in \mathcal{C}^+



As we can see in the previous example, in the description of $\widehat{\mathcal{G}}^{\text{par}}$ via the alcoves in the fundamental chamber, the set of edges seems to have a very complex structure, while in the other one the order on the set of vertices is hard to understand. Since we are interested in the study of intervals, the description via the finite coroot lattice turns out to be not very useful for our purposes, unless $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_2$.

4.1.1. *The $\widehat{\mathfrak{sl}}_2$ case.* If $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_2$, it is actually possible to give a very explicit description of $\widehat{\mathcal{G}}^{\text{par}}$. In this case we may identify the finite root lattice with the finite coroot lattice and then the set of vertices is $\mathcal{V} = \mathbb{Z}\alpha$. For any pair $n, m \in \mathbb{Z}$, it is immediate to check that

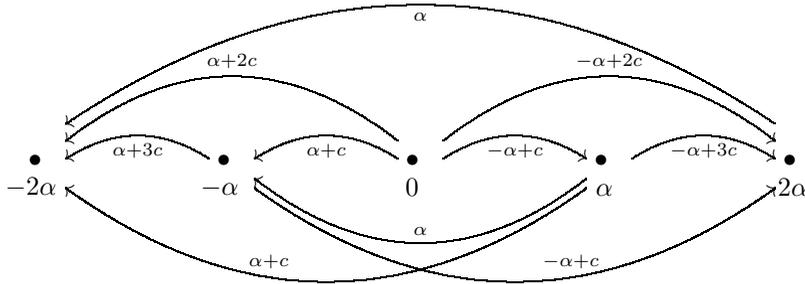
$$(4) \quad s_{\alpha-(n+m)\delta}(n\alpha) = m\alpha;$$

then $\widehat{\mathcal{G}}^{\text{par}}$ is a fully connected graph. Notice that, even if (4) holds for any pair of integers n and m , we do not allow loops, so $n \neq m$ always. It follows that

$$(5) \quad l(n\alpha \text{ --- } m\alpha) = \begin{cases} -\alpha + (n+m)c & \text{if } n+m \geq 0, \\ \alpha - (n+m)c & \text{if } n+m < 0. \end{cases}$$

Finally, observe that $\alpha = s_0(0)$ and $-\alpha = s_\alpha s_0(0)$; so, for any pair of $n \neq m \in \mathbb{Z}$, $n\alpha < m\alpha$ if and only if either $|n| < |m|$ or $n = -m > 0$.

Example 4.2. The interval $[0, -2\alpha]$ of $\widehat{\mathcal{G}}^{\text{par}}$ looks as in the following picture:



The second part of this section is devoted to showing that finite intervals of $\widehat{\mathcal{G}}^{\text{par}}$ “deep enough” in \mathcal{C}^+ surprisingly have a very regular structure.

4.2. Nice behaviour of finite intervals of $\widehat{\mathcal{G}}^{\text{par}}$. In this paragraph, we will consider only the description of $\widehat{\mathcal{G}}^{\text{par}}$ in which the set of vertices coincides with \mathcal{A}^+ .

Definition 4.3. Let $\lambda, \mu \in \mathcal{C}^+$. We say that

- (i) λ is *strongly linked* to μ if $\lambda = \mu + x\alpha$, for some $x \in \mathbb{R}$ and $\alpha \in \Delta_+$;
- (ii) λ is *linked* to μ if $\lambda = w(\mu + n\alpha)$, for some $n \in \mathbb{R}$, $\alpha \in \Delta_+$ and $w \in \mathcal{W}$.

Remark that the fundamental chamber \mathcal{C}^+ is a fundamental domain with respect to the left action of the finite Weyl group (cf. [H, §1.12]), so the element in point (ii) is unique.

Proposition 4.4. *There exists a $K > 0$, depending only on the root system Δ , such that if $\lambda \in \mathcal{C}^+$ and d_λ is the minimum of distances from λ to the borders of \mathcal{C}^+ , then all $\mu \in \mathcal{C}^+$ linked to λ and such that $|\lambda - \mu| < K \cdot d_\lambda$ are strongly linked to λ .*

Proof. For any $\lambda \in \mathcal{C}^+$ and any finite positive root $\alpha \in \Delta_+$ we denote by $r_{\lambda,\alpha}$ the line $\{\lambda + \alpha x \mid x \in \mathbb{R}\} \subseteq \mathfrak{h}_{\mathbb{R}}^+$. It is clear that the set of finite dominant weights strongly linked to λ corresponds to $(\bigcup_{\alpha \in \Delta_+} r_{\lambda,\alpha}) \cap \mathcal{C}^+$. On the other hand, we may describe the set of $\mu \in \mathcal{C}^+$ linked to λ as follows. Fix $\alpha \in \Delta_+$ and consider the line $r_{\lambda,\alpha}$. Each time that such a line hits a wall of \mathcal{C}^+ , it reflects off the wall and goes on this way. Let us denote by $\tilde{r}_{\lambda,\alpha}$ the piecewise linear path inside of \mathcal{C}^+ so obtained. Now $\bigcup_{\alpha \in \Delta_+} \tilde{r}_{\lambda,\alpha}$ is the set of finite dominant weights linked to λ .

Thus it is enough to show that there exists a $K > 0$ such that if $\mu \in \tilde{r}_{\lambda,\alpha}$ and $|\lambda - \mu| < K \cdot d_\lambda$, then $\mu \in r_{\lambda,\alpha}$. Notice that the finite Weyl group acts on $\mathfrak{h}_{\mathbb{R}}^*$ as a group of orthogonal transformations; hence we may reduce to show that for all $w \in \mathcal{W} \setminus \{e, s_\alpha\}$, the distance of the weight $w(\lambda)$ from the line $r_{\lambda,\alpha}$ is not less than $K \cdot d_\lambda$. Moreover, one may think of this reduction as an “unfolding” back of $\tilde{r}_{\lambda,\alpha}$ to $r_{\lambda,\alpha}$ and consider the conjugates of λ instead of λ .

Since the distance of $w(\lambda)$ from the line $r_{\lambda,\alpha}$ is the minimum of the distances of $w(\lambda)$ from $\lambda + x\alpha$ for $x \in \mathbb{R}$, we have to show that $|\lambda - x\alpha - w(\lambda)|^2 \geq K^2 d_\lambda^2$ for all $x \in \mathbb{R}$. Computing the square norm, and denoting $\lambda^w := \lambda - w(\lambda)$, we have:

$$|\alpha|^2 x^2 + 2(\lambda^w, \alpha)x + |\lambda^w|^2 - K^2 d_\lambda^2 \geq 0 \quad \forall x \in \mathbb{R}.$$

Hence this is equivalent to showing that the discriminant $D^w = (\lambda^w, \alpha)^2 - |\alpha|^2 |\lambda^w|^2 + |\alpha|^2 K^2 d_\lambda^2 \leq 0$.

To start with, let us notice that $D^{s_\alpha w} = D^w$, since $\lambda^{s_\alpha w} = \lambda - w(\lambda) + \langle w(\lambda), \alpha^\vee \rangle \alpha = \lambda^w + \langle w(\lambda), \alpha^\vee \rangle \alpha$; hence:

$$\begin{aligned} D^{s_\alpha w} &= (\lambda^{s_\alpha w}, \alpha)^2 - |\alpha|^2 |\lambda^{s_\alpha w}|^2 + |\alpha|^2 K^2 d_\lambda^2 \\ &= (\lambda^w + \langle w(\lambda), \alpha^\vee \rangle \alpha, \alpha)^2 - |\alpha|^2 (\lambda^w + \langle w(\lambda), \alpha^\vee \rangle \alpha, \lambda^w + \langle w(\lambda), \alpha^\vee \rangle \alpha) \\ &\quad + |\alpha|^2 K^2 d_\lambda^2 \\ &= (\lambda^w, \alpha)^2 + 2\langle w(\lambda), \alpha^\vee \rangle |\alpha|^2 (\lambda^w, \alpha) + \langle w(\lambda), \alpha^\vee \rangle^2 |\alpha|^4 \\ &\quad - |\alpha|^2 |\lambda^w|^2 - 2|\alpha|^2 (\langle w(\lambda), \alpha^\vee \rangle \alpha, \lambda^w) - \langle w(\lambda), \alpha^\vee \rangle^2 |\alpha|^4 + |\alpha|^2 K^2 d_\lambda^2 \\ &= (\lambda^w, \alpha)^2 - |\alpha|^2 |\lambda^w|^2 + |\alpha|^2 K^2 d_\lambda^2 = D^w. \end{aligned}$$

Now if $w^{-1}(\alpha)$ is a finite negative root, then clearly $(s_\alpha w)^{-1}(\alpha) \in \Delta_+$, and hence, using the invariance property just proved, in what follows we may assume that $w \in \mathcal{W} \setminus \{e, s_\alpha\}$ is such that $w^{-1}(\alpha) \in \Delta_+$.

Now denote by Δ_+^w the set of positive roots sent to negative roots by w^{-1} , let C^w be the (closed convex rational) cone $\langle \Delta_+^w \rangle_{\mathbb{R}^+}$ generated by the elements of Δ_+^w and notice that α is not in $\pm C^w$. Indeed, α is not in C^w since all elements of this cone are sent to non-negative linear combinations of finite negative roots by w^{-1} and, on the other hand, α is a finite positive root while all elements in $-C^w$ are non-negative linear combinations of finite negative roots.

Let L^w be the set of weights λ^w , where λ runs in \mathcal{C}^+ , and fix a reduced expression $s_{i_1} \dots s_{i_r}$, with $s_{i_j} := s_{\alpha_{i_j}}$, for $\alpha_j \in \Pi$. Then we have $w(\lambda) = \lambda - (a_1 \beta_{i_1} \dots a_r \beta_{i_r})$, where $\beta_j = s_{i_1} \dots s_{i_{j-1}}(\alpha_{i_j})$ for $j = 1, \dots, r$. Notice that $a_{i_j} \geq 0$ for all j since $\lambda \in \mathcal{C}^+$ and, moreover, $\{\beta_{i_1}, \dots, \beta_{i_r}\} = \Delta_+^w$. This shows that $L^w \subseteq C^w$.

Let $\pi : \mathfrak{h}_{\mathbb{R}}^* \setminus \{0\} \rightarrow \mathbb{P}(\mathfrak{h}_{\mathbb{R}}^*)$ be the quotient map to the projective space of $\mathfrak{h}_{\mathbb{R}}^*$. Given two non-zero vectors $u, v \in \mathfrak{h}_{\mathbb{R}}^*$ we denote by $[u, v]$ the angle between them; clearly this symbol depends only on the lines generated by u and v up to sign to change and up to supplementary angles. In particular, the map $\mathbb{P}(\mathfrak{h}_{\mathbb{R}}^*)^2 \rightarrow \mathbb{R}$ defined by $(\pi[u], \pi[v]) \mapsto \cos^2[u, v]$ is well-defined.

Since C^w is a closed convex rational cone we have that $\pi(C^w \setminus \{0\})$ is closed in $\mathbb{P}(\mathfrak{h}_{\mathbb{R}}^*)$. Hence the map $\pi(C^w \setminus \{0\}) \rightarrow \mathbb{R}$ sending $\pi(\mu) \mapsto \cos^2[\mu, \alpha]$ achieves a maximal value $M^{\alpha, w}$, and this maximal value is less than 1 since $\pi(\alpha) \notin \pi(C^w \setminus \{0\})$. In particular we have $\cos^2[\lambda^w, \alpha] \leq M^{\alpha, w} < 1$ for all $\lambda \in \mathcal{C}^+ \setminus \{0\}$ since $L^w \subseteq C^w$.

Finally, since there are only a finite number of pairs (α, w) , we have $M := \max M^{\alpha, w} < 1$. Now notice that $w(\lambda) \notin \mathcal{C}^+$, because $w \neq e$, so $|\lambda^w| \geq d_\lambda$, as the segment from λ to $w(\lambda)$ must cross a wall of \mathcal{C}^+ .

We have to show $D^w \leq 0$. Since

$$(\lambda^w, \alpha) = |\lambda^w| |\alpha| \cos[\lambda^w, \alpha],$$

our inequality becomes $\cos^2[\lambda^w, \alpha] \leq 1 - K^2 d_\lambda^2 / |\lambda^w|^2$. But we have $\cos^2[\lambda^w, \alpha] \leq M < 1$ and $1 - K^2 d_\lambda^2 / |\lambda^w|^2 > 1 - K^2$. Hence it is enough to choose K such that $M \leq 1 - K^2$. This finishes the proof. \square

Let ρ be half the sum of the finite positive coroots, that is, $\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha^\vee$. Moreover, for any alcove $A \in \mathcal{A}$, let us denote by c_A its centroid.

By using Proposition 4.4, together with the identification $\alpha^\vee = 2\alpha / (\alpha, \alpha)$ for all $\alpha \in \Delta$, we get the following characterisation of finite intervals of $\widehat{\mathcal{G}}^{\text{par}}$ which are deep enough from the walls of the dominant chamber.

Lemma 4.5. *Let $A, B \in \mathcal{A}^+$. Then there exists an integer $m_0 = m_0(A, B)$ such that for any $\lambda \in X \cap m\rho + \mathcal{C}^+$, with $m \geq m_0$, for any pair $D, E \in [A + \lambda, B + \lambda]$ there is an edge $D \text{ --- } E$ in $\widehat{\mathcal{G}}^{\text{par}}$ if and only if*

- (i) *either $E = Ds_\alpha$ for some $\alpha \in \widehat{\Delta}_+^{\text{re}}$;*
- (ii) *or $E = D + a\alpha$ for some $a \in \mathbb{Z} \setminus \{0\}$ and $\alpha \in \Delta_+$.*

Proof. First of all, let us set $m_1 = \max_{E, D \in [A, B]} n_0(E, D)$, where n_0 is the one in Proposition 3.9. It follows that for any $m > m_1$ there is an isomorphism of posets $[A + m\rho, B + m\rho] \cong [A + m_1\rho, B + m_1\rho]$.

By the previous proposition, we know that in $\widehat{\mathcal{G}}_{[A+m\rho, B+m\rho]}^{\text{par}}$ edges adjacent to a vertex corresponding to a given alcove D are of the desired type if for any $E \in [A + m\rho, B + m\rho]$ it holds that

$$|c_D - c_E| < K \cdot d_{c_D}.$$

Observe that for any $n \geq 0$ and for any $F \in [A + m_1\rho, B + m_1\rho]$ we have $c_{F+n\rho} = c_F + n\rho$, and hence, for all $G \in [A + m_1\rho, B + m_1\rho]$, $|c_{F+n\rho} - c_{G+n\rho}| = |c_F - c_G|$. Moreover, if $n > 0$, $d_{c_{F+n\rho}} > d_{c_F}$, and therefore it makes sense to consider

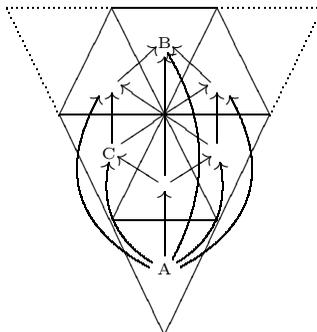
$$n_F := \min \{n \mid K \cdot d_{c_{F+n\rho}} > |c_F - c_G| \text{ for all } G \in [A + m_1\rho, B + m_1\rho]\}.$$

Finally, we may set

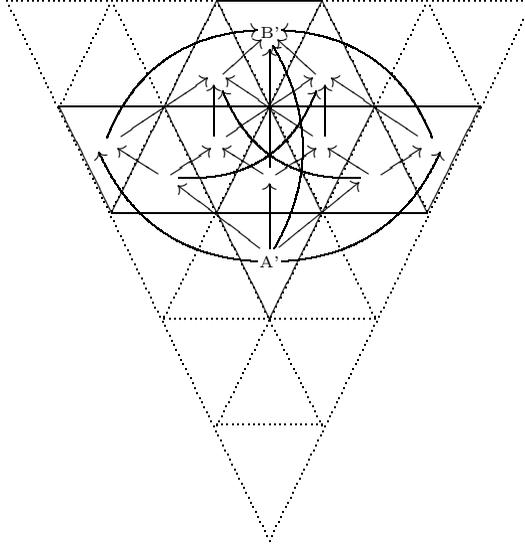
$$m_0 := m_1 + \max \{n_F \mid F \in [A + m_1\rho, B + m_1\rho]\}. \quad \square$$

We say that the edges of type (i), which are given by reflections, are *stable*, while the ones of type (ii), that are given by translations, are *non-stable*. We denote the corresponding sets \mathcal{E}_S , resp. \mathcal{E}_{NS} .

Example 4.6. Let $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_3$ and $A = A^+$, $B = s_0s_1s_2s_1A^+$. Then in the interval $[A, B]$ of $\widehat{\mathcal{G}}^{\text{par}}$ there are edges that are neither stable nor non-stable, as the one between $A = A^+$ and $C = s_0s_1A^+$.



It is enough to translate the interval of $(\alpha + \beta)^\vee = \alpha + \beta$ to get the structure described in Lemma 4.5.



Lemma 4.7. *For any pair $A, B \in \mathcal{A}^+$, $B \leq A$ and for any pair $\lambda_1 = m_1\rho$, $\lambda_2 = m_2\rho \in X^\vee \cap \mathcal{C}^+$ ($m_1, m_2 \geq m_0(A, B)$), then $\widehat{\mathcal{G}}_{[A+\lambda_1, B+\lambda_1]}^{\text{par}}$ and $\widehat{\mathcal{G}}_{[A+\lambda_2, B+\lambda_2]}^{\text{par}}$ are isomorphic as oriented graphs.*

Proof. Set $\mu := \lambda_2 - \lambda_1$. The isomorphism we are looking for is given by $C \mapsto C + \mu$. Observe that, by Proposition 3.9, the Bruhat order coincides in the fundamental chamber with the generic one and so it is invariant by finite coweight translation; then the map we have just defined is an isomorphism of posets. Moreover $C = xA^+$ is connected to $D = yA^+$ in $\widehat{\mathcal{G}}_{[A+\lambda_1, B+\lambda_1]}^{\text{par}}$ if and only if $C + \mu$ is connected to $D + \mu$ in $\widehat{\mathcal{G}}_{[A+\lambda_2, B+\lambda_2]}^{\text{par}}$; indeed:

- (i) we have $D = Cs_\alpha$ for some $\alpha \in \widehat{\Delta}_+^{\text{re}}$ if and only if $yx^{-1} = s_\alpha$ if and only if $T_\mu yx^{-1}T_{-\mu} = T_\mu s_\alpha T_{-\mu}$, that is, $T_\mu yx^{-1}T_{-\mu} \in \widehat{\mathcal{T}}$, and this is equivalent to $T_\mu yx^{-1}T_{-\mu} = s_\beta$ for some $\beta \in \widehat{\Delta}_+^{\text{re}}$, that is, $D + \mu = (C + \mu)s_\beta$,
- (ii) while $D = C + a\alpha$ if and only if $D + \mu = C + a\alpha + \mu = (C + \mu) + a\alpha$. \square

Remark 4.8. We want to stress the fact that in Lemma 4.7 we are *not* proving the existence of an isomorphism of moment graphs, but only between the underlying oriented graphs, that is, we are not considering labels. Our first hope was that we could find a collection $\{f_{l,C}\}_{C \in [A+\lambda_1, B+\lambda_1]}$ satisfying condition (MORPh2a) and (MORPh2b). In the next two paragraphs, we will see that this is not the case. In particular, it turns out that the labels of stable edges are invariant by finite coroot translation (cf. Lemma 4.10), while the ones of non-stable edges are not (cf. Lemma 4.12).

From now on we will denote by $w \in \widehat{\mathcal{W}}$ the corresponding alcove $wA^+ \in \mathcal{A}$, thanks to the identification (2) of the affine Weyl group with its set of alcoves. In particular, if wA^+ is contained in the fundamental chamber, we will write $w \in \mathcal{A}^+$.

4.2.1. *Stable edges.* Let $|\widehat{\mathcal{S}}| = n$ and fix a numbering of the simple reflections. We define the permutation $\sigma_{A,\mu} \in S_n$, for $A \in \mathcal{A}$ and $\mu \in X^\vee$, in the following way: $\sigma_{A,\mu}(i) = j$ if the image under the translation by μ of the s_i -th wall of A is the s_j -th wall of $A + \mu$ (cf. §2.2). Let $\widetilde{\mathcal{W}}$ be the extended affine Weyl group, that is, $\widetilde{\mathcal{W}} = \widehat{\mathcal{W}} \ltimes \Omega$, where $\Omega := X^\vee/Q^\vee$ (cf. [Lu83]).

Lemma 4.9. *For any $\mu \in X^\vee$ the permutation defined above is independent on $A \in \mathcal{A}$, i.e. there exists $\sigma_\mu \in S_n$ such that $\sigma_{A,\mu} = \sigma_\mu$ for any alcove A .*

Proof. We know that $T_{k\alpha^\vee} = s_{-\alpha+k\delta}s_\alpha$ for $k \in \mathbb{Z}_{\geq 0}$ and $\alpha \in \Delta$. Since we are reflecting twice in the same direction (orthogonal to α), the walls of $A + k\alpha^\vee$ have the same numbering as the ones of A .

Thus for any $\mu \in X^\vee$ there exists an element $\omega \in \Omega$ and roots $\alpha_1, \dots, \alpha_r \in \Delta$ such that $T_\mu = \omega s_{\alpha_1, k_1} s_{\alpha_1} \dots s_{\alpha_r, k_r} s_{\alpha_r}$ and the numbering of the walls of $A + \mu$ only depends on ω . □

Let us denote (by abuse of notation) also by $\sigma_\mu : \widehat{Q}^\vee \rightarrow \widehat{Q}^\vee$ the automorphism of the affine coroot lattice induced by the map $\alpha_i \mapsto \alpha_{\sigma_\mu(i)}$ for α_i corresponding to the simple reflection $s_i \in \widehat{\mathcal{S}}$. Let us observe that σ_μ preserves the positive cone $\langle \widehat{\Pi} \rangle_{\mathbb{R}_{\geq 0}}$ by definition, and the following result is straightforward.

Corollary 4.10. *Let $x \in \mathcal{W}$, $t \in \widehat{\mathcal{T}}$, $\mu \in X^\vee$ be such that $x, xt, T_\mu x, T_\mu xt \in \mathcal{A}^+$. Then,*

$$l(T_\mu x \text{ --- } T_\mu xt) = \sigma_\mu(l(x \text{ --- } xt)).$$

4.2.2. *Non-stable edges.* Now we describe how labels of non-stable edges change. In order to do so, we need to recall the following result.

Proposition 4.11 ([H], Proposition 4.1). *Let $z = T_{z(0)}v$, where $z(0) \in Q^\vee$ and $v \in \mathcal{W}$. Then, for any $\alpha + n\delta \in \widehat{\Delta}_+^{re}$,*

$$(6) \quad z s_{\alpha+n\delta} z^{-1} = s_{v(\alpha)+r\delta} \quad \text{with } r = n - (v(\alpha), z(0)).$$

In the proof of Lemma 4.12 we will also need the following equality:

$$(7) \quad (\alpha + n\delta)^\vee = \alpha^\vee + \frac{2n}{(\alpha, \alpha)} c.$$

Lemma 4.12. *Let $x \in \mathcal{A}^+$ and $x = T_{x(0)}w$, with $x(0) \in Q^\vee$ and $w \in \mathcal{W}$.*

(i) *If $\alpha \in \Delta$, $n \in \mathbb{Z}_{>0}$ and $T_{n\alpha^\vee}x \in \mathcal{A}^+$, then*

$$(8) \quad l(x \text{ --- } T_{n\alpha^\vee}x) = \pm \left(w^{-1}(\alpha^\vee) + \frac{2}{(\alpha, \alpha)} ((\alpha, x(0)) + n) c \right) \in \widehat{\Delta}_+^{re}.$$

(ii) *Let $y = T_{a\alpha^\vee}x$, for some $a \in \mathbb{Z}$ and $\alpha \in \Delta_+$. Let moreover $\mu \in X^\vee$, $\omega \in \Omega$ and $\gamma \in Q^\vee$ be such that $T_\mu = \omega T_\gamma$. Then, if $y, T_\mu x, T_\mu y \in \mathcal{A}^+$,*

$$(9) \quad l(T_\mu x \text{ --- } T_\mu y) = \sigma_\mu(l(x \text{ --- } y)) + \langle \gamma, \alpha^\vee \rangle \sigma_\mu(c).$$

Proof.

(i) We have to determine $u \in \mathcal{W}$ and $\gamma \in \widehat{\Delta}_+^{re}$ such that $x^{-1}uT_{n\alpha^\vee}x = s_\gamma$. Because $T_{n\alpha^\vee} = s_{-\alpha+n\delta}s_\alpha$, we have

$$x^{-1}uT_{n\alpha^\vee}x = x^{-1}us_{-\alpha+n\delta}s_\alpha x.$$

Clearly, to get a reflection, we have to choose $u = s_\alpha$.

We may now apply the proposition above twice. First consider it with $z = s_\alpha$ and $n = 0$ and we obtain $s_\alpha s_{-\alpha+a\delta} s_\alpha = s_{\alpha+a\delta}$. Next, by applying Proposition 4.11 with $z = x^{-1}$, i.e. $z(0) = -w^{-1}x(0)$ and $v = w^{-1}$, we obtain

$$x^{-1}uT_{n\alpha^\vee}x = s_{w^{-1}(\alpha)+k\delta},$$

with $k = n - (w^{-1}(\alpha), -w^{-1}x(0)) = n + (\alpha, x(0))$, since the bilinear form is \mathcal{W} -invariant. The result then follows from (7) and the fact that $w(\alpha)^\vee = w(\alpha^\vee)$ for all $\alpha \in \Delta$ and $w \in \mathcal{W}$.

(ii) Let us observe that $T_\mu x - T_\mu y = T_{a\alpha^\vee}(T_\mu x)$. If $x = T_{x(0)}w$, then $T_\mu x = T_{\mu+x(0)}w = \omega T_{\gamma+x(0)}w$ (this expression is unique), and we may apply Lemma 4.9 and point (i) of this lemma with $T_\gamma x$ instead of x to get

$$\begin{aligned} l(T_\mu x - T_\mu y) &= \sigma_\mu(l(T_\gamma x - T_\gamma y)) \\ &= \pm\sigma_\mu\left(w^{-1}(\alpha^\vee) + \frac{2}{(\alpha, \alpha)}((\alpha, \gamma + x(0)) + n)c\right) \in \widehat{\Delta}_+^{\text{re}} \\ &= \sigma_\mu(l(x - y)) + \langle \gamma, \alpha^\vee \rangle \sigma_\mu(c). \end{aligned} \quad \square$$

4.3. The stable moment graph. We are now ready to define the main character of this paper: the *stable moment graph* $\widehat{\mathcal{G}}^{\text{stab}}$. This is the moment graph having as a set of vertices the alcoves in the fundamental chamber (that we identify with the corresponding elements of the Weyl group), equipped with the Bruhat order (that here coincides with the generic one); we connect two vertices if and only if there exists a real positive root $\alpha \in \widehat{\Delta}_+^{\text{re}}$ such that $y = xs_\alpha$, and in this case we set $l(x - xs_\alpha) := \alpha^\vee$.

Then we have:

Lemma 4.13. *For any interval $[y, w]$ and for any $\mu \in X^\vee$ there exists an isomorphism of k -moment graphs $\widehat{\mathcal{G}}_{[y, w]}^{\text{stab}} \rightarrow \widehat{\mathcal{G}}_{[y+\mu, w+\mu]}^{\text{stab}}$ for all k .*

Proof. Since the order on the set of vertices of $\widehat{\mathcal{G}}^{\text{stab}}$ is invariant by finite coweight translation, we have an isomorphism of posets given by the map $f_\nu : x \mapsto x + \mu$. This map also induces a bijection between set of edges, as we have already seen in the proof of Lemma 4.7.

For any $x \in [y, w]$ we set $f_{l, x} = \sigma_\mu$ (see Lemma 4.9), and the data $(f_\nu, \{f_{l, x}\})$ gives us an isomorphism of k -moment graphs for any k . □

5. SHEAVES ON MOMENT GRAPHS

The notion of sheaf on a moment graph is due to Braden and MacPherson (cf. [BMP]) and has been used by Fiebig in several papers (cf. [Fie08b], [Fie08a], [Fie11], [Fie10b], [Fie10a]). In the first part of this section, we recall the definition of a category of sheaves on a k -moment graph and we present two important examples, namely, the structure sheaf and the canonical –or BMP– sheaf. In the second part, for any homomorphism of k -moment graphs f , we define the *pullback* functor f^* and the *pushforward* functor f_* . These two functors turn out to have the same adjointness property as in classical sheaf theory (see Proposition 5.9).

5.1. The category of sheaves on a k -moment graph. As in the previous sections, let Y be a lattice of finite rank and k a local ring (with $2 \in k^*$). Let us denote the symmetric algebra of Y by $S = \text{Sym}(Y)$ and set $S_k := S \otimes_{\mathbb{Z}} k$ as its extension. As a polynomial ring, S_k has a natural \mathbb{Z} -grading, but we keep the convention

(coming from geometry) of doubling it, that is, we set $(S_k)_{\{2\}} = Y_k$. From now on, all the S_k -modules will be finitely generated and equipped with this \mathbb{Z} -grading. Moreover, we will consider only degree zero morphisms between them. Finally, for $j \in \mathbb{Z}$ and M a graded S_k -module we denote by $M\langle j \rangle$ the graded S_k -module obtained from M by shifting the grading by j , that is, $M\langle j \rangle_{\{i\}} = M_{\{j+i\}}$.

Definition 5.1 (cf. [BMP]). Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \triangleleft, l) \in \mathbf{MG}(Y_k)$; then a *sheaf* \mathcal{F} on \mathcal{G} is given by the following data $(\{\mathcal{F}^x\}, \{\mathcal{F}^E\}, \{\rho_{x,E}\})$:

- (SH1) for all $x \in \mathcal{V}$, \mathcal{F}^x is an S_k -module;
- (SH2) for all $E \in \mathcal{E}$, \mathcal{F}^E is an S_k -module such that $l(E) \cdot \mathcal{F}^E = \{0\}$;
- (SH3) for $x \in \mathcal{V}$, $E \in \mathcal{E}$, $\rho_{x,E} : \mathcal{F}^x \rightarrow \mathcal{F}^E$ is a homomorphism of S_k -modules defined if x is incident to the edge E .

Definition 5.2 ([FieNotes]). Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \triangleleft, l) \in \mathbf{MG}(Y_k)$ and let $\mathcal{F} = (\{\mathcal{F}^x\}, \{\mathcal{F}^E\}, \{\rho_{x,E}\})$, $\mathcal{F}' = (\{\mathcal{F}'^x\}, \{\mathcal{F}'^E\}, \{\rho'_{x,E}\})$ be two sheaves on it. A *morphism* $\varphi : \mathcal{F} \rightarrow \mathcal{F}'$ is given by the following data:

- (i) for all $x \in \mathcal{V}$, $\varphi^x : \mathcal{F}^x \rightarrow \mathcal{F}'^x$ is a homomorphism of S_k -modules;
- (ii) for all $E \in \mathcal{E}$, $\varphi^E : \mathcal{F}^E \rightarrow \mathcal{F}'^E$ is a homomorphism of S_k -modules

such that, for any $x \in \mathcal{V}$ on the border of $E \in \mathcal{E}$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}^x & \xrightarrow{\rho_{x,E}} & \mathcal{F}^E \\ \downarrow \varphi^x & & \downarrow \varphi^E \\ \mathcal{F}'^x & \xrightarrow{\rho'_{x,E}} & \mathcal{F}'^E \end{array}$$

Definition 5.3. Let $\mathcal{G} \in \mathbf{MG}(Y_k)$. We denote by $\mathbf{Sh}_k(\mathcal{G})$ the category whose objects are the sheaves on \mathcal{G} and whose morphisms are as in Definition 5.2.

5.2. Combinatorial sheaf theory. In this paragraph we recall and introduce notions such as space of sections, flabbiness, pullback and pushforward functors, which mimic the corresponding ones in classical sheaf theory.

5.2.1. Sections of sheaves. Let us consider a sheaf $\mathcal{F} \in \mathbf{Sh}_k(\mathcal{G})$ and let \mathcal{I} be a subset of the set of vertices \mathcal{V} of \mathcal{G} . The the set of *local sections* of \mathcal{F} over \mathcal{I} is defined as follows:

$$\Gamma(\mathcal{I}, \mathcal{F}) := \left\{ (m_x) \in \prod_{x \in \mathcal{I}} \mathcal{F}^x \mid \begin{array}{l} \rho_{x,E}(m_x) = \rho_{y,E}(m_y) \\ \text{for all } E : x \text{ --- } y \in \mathcal{E}, x, y \in \mathcal{I} \end{array} \right\}.$$

We write $\Gamma(\mathcal{F})$ for $\Gamma(\mathcal{V}, \mathcal{F})$ and call it the set of *global sections* of \mathcal{F} .

5.2.2. The structure sheaf and the structure algebra. With any k -moment graph, it is possible to associate its structure sheaf \mathcal{Z} , that is, the sheaf on \mathcal{G} given by $\mathcal{Z}^x = S_k$ for all $x \in \mathcal{V}$, $\mathcal{Z}^E = S_k/l(E) \cdot S_k$ for all $E \in \mathcal{E}$ and $\rho_{x,E} : \mathcal{Z}^x = S_k \rightarrow \mathcal{Z}^E = S_k/l(E) \cdot S_k$ the canonical quotient map, for any vertex $x \in \mathcal{V}$ on the edge $E \in \mathcal{E}$. Then the *structure algebra* \mathcal{Z} of \mathcal{G} is the set of global sections of \mathcal{Z} , namely

$$\mathcal{Z} := \Gamma(\mathcal{Z}) = \left\{ (z_x)_{x \in \mathcal{V}} \in \prod_{x \in \mathcal{V}} S_k \mid \begin{array}{l} z_x - z_y \in l(E) \cdot S_k \\ \text{for all } E : x \text{ --- } y \in \mathcal{E} \end{array} \right\}.$$

The symmetric algebra S_k acts on the structure algebra via diagonal action, and it is easy to check that \mathcal{Z} is actually an algebra under componentwise addition and multiplication.

5.2.3. *Flabby sheaves.* Once we have obtained the analogue of the spaces of sections, we would like to define the concept of flabby sheaves. Clearly, in order to do so, the notion of an open set is needed. We declare $\mathcal{I} \subseteq \mathcal{V}$ to be *open* if it is upwardly closed, that is, if and only if whenever $x \in \mathcal{I}$ and $y \geq x$, then also $y \in \mathcal{I}$.

Definition 5.4 (cf. [Fie08a]). A sheaf \mathcal{F} on \mathcal{G} is *flabby* if the map $\Gamma(\mathcal{F}) \rightarrow \Gamma(\mathcal{I}, \mathcal{F})$, given by the projection on the \mathcal{I} -components, is surjective for any $\mathcal{I} \subseteq \mathcal{V}$ open.

5.3. **Pullback and pushforward of sheaves.** Let $f = (f_{\mathcal{V}}, \{f_{l,x}\}) : \mathcal{G} = (\mathcal{V}, \mathcal{E}, \leq, l) \rightarrow \mathcal{G}' = (\mathcal{V}', \mathcal{E}', \leq', l')$ be a morphism of k -moment graphs. We want to define two functors:

$$\begin{array}{ccc} & f^* & \\ \text{Sh}_k(\mathcal{G}) & \xleftarrow{\quad} & \text{Sh}_k(\mathcal{G}') \\ & f_* & \end{array}$$

From now on, for any $\varphi \in \text{Aut}_k(Y_k)$, we will also denote by φ the automorphism of S_k that it induces.

We need a lemma in order to make consistent the definitions we are going to give.

Lemma 5.5. *Let $s \in S_k$, $f \in \text{Hom}_{\text{MG}(Y_k)}(\mathcal{G}, \mathcal{G}')$, $\mathcal{F} \in \text{Sh}_k(\mathcal{G})$ and $\mathcal{H} \in \text{Sh}_k(\mathcal{G}')$. Let $E : x \text{ --- } y \in \mathcal{E}$ and $F : f_{\mathcal{V}}(x) \text{ --- } f_{\mathcal{V}}(y) \in \mathcal{E}'$; then*

- (i) *the twisted actions of S_k on \mathcal{F}^E defined via $s \cdot m_E := f_{l,x}^{-1}(s) \cdot m_E$ and $s \cdot m_E := f_{l,y}^{-1}(s) \cdot m_E$ coincide on $\mathcal{F}^E / l'(F) \cdot \mathcal{F}^E$ (\cdot denotes the action of S_k on \mathcal{F}^E before the twist); moreover, $l'(F) \cdot \mathcal{F}^E = \{0\}$ in both cases;*
- (ii) *the twisted actions of S_k on \mathcal{H}^F defined via $s \cdot n_F := f_{l,x}(s) \cdot n_F$ and $s \cdot n_F := f_{l,y}(s) \cdot n_F$ coincide on $\mathcal{H}^F / l(E) \cdot \mathcal{H}^F$ (\cdot denotes the action of S_k on \mathcal{H}^F before the twist); moreover, $l(E) \cdot \mathcal{H}^F = \{0\}$ in both cases.*

Proof. It is enough to prove the claim for $s \in (S_k)_{\{2\}} = Y_k$, since S_k is a k -algebra generated by Y_k .

- (i) The statement follows from (MORPH2a), (MORPH2b) and the computations we made in the proof of Lemma 3.6.
- (ii) It is an immediate consequence of conditions (MORPH2a), (MORPH2b). □

If φ is an automorphism of S_k , for any S_k -module M , we will denote $\text{Tw}_{\varphi} : M \rightarrow M$ as the map sending M to M and twisting the action of S_k on M by φ .

5.3.1. *Definitions.*

Definition 5.6. Let $\mathcal{F} \in \text{Sh}_k(\mathcal{G})$; then $f_*\mathcal{F} \in \text{Sh}_k(\mathcal{G}')$ is defined as follows:

(PUSH1) for any $u \in \mathcal{V}'$,

$$(f_*\mathcal{F})^u := \Gamma(f_{\mathcal{V}}^{-1}(u), \mathcal{F}),$$

and the structure of S_k -module is given by $s \cdot (m_x)_{x \in f_{\mathcal{V}}^{-1}(u)} := (s \cdot m_x)_{x \in f_{\mathcal{V}}^{-1}(u)}$;

(PUSH2) for any $u \in \mathcal{V}'$,

$$(f_*\mathcal{F})^F := \bigoplus_{E: f_{\mathcal{E}}(E)=F} \mathcal{F}^E,$$

and the action of S_k is twisted in the following way: $s \cdot (m_E)_{E: f_{\mathcal{E}}(E)=F} := (f_{l,x}^{-1}(s) \cdot m_E)_{E: f_{\mathcal{E}}(E)=F}$, where x is incident to the edge E ;

(PUSH3) for all $u \in \mathcal{V}'$ and $F \in \mathcal{E}'$, such that u is in the border of the edge F , $(f_*\rho)_{u,F}$ is defined as the composition of the following maps:

$$\Gamma(f_{\mathcal{V}'}^{-1}(u), \mathcal{F}) \hookrightarrow \bigoplus_{x: f_{\mathcal{V}}(x)=u} \mathcal{F}^x \xrightarrow{\oplus \rho_{x,E}} \bigoplus_{E: f_{\mathcal{V}}(E)=F} \mathcal{F}^E \xrightarrow{\text{Tw}} \bigoplus_{E: f_{\mathcal{V}}(E)=F} \mathcal{F}^E,$$

where $\text{Tw} = \bigoplus \text{Tw}_{f_{l,x}^{-1}}$.

We call f_* a *direct image* or a *pushforward* functor.

Definition 5.7. Let $\mathcal{H} \in \mathbf{Sh}_k(\mathcal{G}')$; then $f^*\mathcal{H} \in \mathbf{Sh}_k(\mathcal{G})$ is defined as follows:

(PULL1) for all $x \in \mathcal{V}$, $(f^*\mathcal{H})^x := \mathcal{H}^{f_{\mathcal{V}}(x)}$ and the action of S_k is twisted by $f_{l,x}$;

(PULL2) for all $E : x \text{ --- } y \in \mathcal{E}$,

$$(f^*\mathcal{H})^E = \begin{cases} \mathcal{H}^{f_{\mathcal{V}}(x)}/l(E)\mathcal{H}^{f_{\mathcal{V}}(x)} & \text{if } f_{\mathcal{V}}(x) = f_{\mathcal{V}}(y), \\ \mathcal{H}^{f_{\mathcal{E}}(E)} & \text{otherwise,} \end{cases}$$

and of $s \in S_k$ acts on $(f^*\mathcal{H})^E$ via $f_{l,x}(s)$;

(PULL3) for all $x \in \mathcal{V}$ and $E \in \mathcal{E}$, such that x is in the border of the edge E ,

$$(f^*\rho)_{x,E} = \begin{cases} \text{canonical quotient map} & \text{if } f_{\mathcal{V}}(x) = f_{\mathcal{V}}(y), \\ \text{Tw}_{f_{l,x}} \circ \rho_{f_{\mathcal{V}}(x), f_{\mathcal{E}}(E)} \circ \text{Tw}_{f_{l,x}^{-1}} & \text{otherwise.} \end{cases}$$

We call f^* an *inverse image* or a *pullback* functor.

Example 5.8. Let $\mathcal{G} \in \mathbf{MG}(Y_k)$ and let $p : \mathcal{G} \rightarrow \{\text{pt}\}$ be the homomorphism of k -moment graphs having $p_{l,x} = \text{id}_{Y_k}$ for all x , the vertex of \mathcal{G} . Then, for any $\mathcal{F} \in \mathbf{Sh}_k(\mathcal{G})$ $p_*(\mathcal{F}) = \Gamma(\mathcal{F})$. Moreover $p^*(S_k) = \mathcal{L}$, the structure sheaf of \mathcal{G} .

5.3.2. *Adjunction formula.* Although the following result will not be used in the rest of the paper, we include it for completeness.

Proposition 5.9. Let $f \in \text{Hom}_{\mathbf{MG}(Y_k)}(\mathcal{G}, \mathcal{G}')$; then f^* is left adjoint to f_* . That is, for all pair of sheaves $\mathcal{F} \in \mathbf{Sh}_k(\mathcal{G})$ and $\mathcal{H} \in \mathbf{Sh}_k(\mathcal{G}')$ the following equality holds:

$$(10) \quad \text{Hom}_{\mathbf{Sh}_k(\mathcal{G})}(f^*\mathcal{H}, \mathcal{F}) = \text{Hom}_{\mathbf{Sh}_k(\mathcal{G}')}(\mathcal{H}, f_*\mathcal{F}).$$

Proof. Let us take $\varphi \in \text{Hom}_{\mathbf{Sh}_k(\mathcal{G})}(f^*\mathcal{H}, \mathcal{F})$, that is, $\varphi = (\{\varphi^x\}_{x \in \mathcal{V}}, \{\varphi^E\}_{E \in \mathcal{E}})$ such that for all $x \in \mathcal{V}$ and $E \in \mathcal{E}$, with x on the border of E , the following diagram commutes:

$$(11) \quad \begin{array}{ccc} (f^*\mathcal{H})^x & \xrightarrow{\varphi^x} & \mathcal{F}^x \\ \downarrow (f^*\rho)_{x,E} & & \downarrow \rho_{x,E} \\ (f^*\mathcal{H})^E & \xrightarrow{\varphi^E} & \mathcal{F}^E \end{array}$$

We want to show that there is a bijective map $\gamma : \text{Hom}_{\mathbf{Sh}_k(\mathcal{G})}(f^*\mathcal{H}, \mathcal{F}) \rightarrow \text{Hom}_{\mathbf{Sh}_k(\mathcal{G}')}(\mathcal{H}, f_*\mathcal{F})$ and it is given by $\varphi = (\{\varphi^x\}_{x \in \mathcal{V}}, \{\varphi^E\}_{E \in \mathcal{E}}) \mapsto \psi = (\{\psi^u\}_{u \in \mathcal{V}'}, \{\psi^F\}_{F \in \mathcal{E}'})$, where

$$\psi^u := \bigoplus_{x \in f_{\mathcal{V}}^{-1}(u)} \varphi^x \quad \text{and} \quad \psi^F := \bigoplus_{E \in f_{\mathcal{E}}^{-1}(F)} \varphi^E.$$

We start by verifying that this map is well-defined. We have to show that for any $h \in \mathcal{H}^u$, $\psi^u(h) \in (f_*\mathcal{F})^u = \Gamma(f_{\mathcal{V}}^{-1}(u), \mathcal{F})$, that is, for any $x, y \in f_{\mathcal{V}}^{-1}(u)$ such that $E : x \rightarrow y \in \mathcal{E}$, $\rho_{x,E}(\varphi^x(h)) = \rho_{y,E}(\varphi^y(h))$.

From diagram (11), we get the following commutative diagram:

$$(12) \quad \begin{array}{ccc} (f^*\mathcal{H})^x = \mathcal{H}^{f_{\mathcal{V}}(x)} = \mathcal{H}^u & \xrightarrow{\varphi^x} & \mathcal{F}^x \\ \downarrow (f^*\rho')_{x,E} & & \downarrow \rho_{x,E} \\ (f^*\mathcal{H})^E = \mathcal{H}^u / l(E)\mathcal{H}^u & \xrightarrow{\varphi^E} & \mathcal{F}^E \\ \uparrow (f^*\rho')_{y,E} & & \uparrow \rho_{x,E} \\ (f^*\mathcal{H})^y = \mathcal{H}^{f_{\mathcal{V}}(y)} = \mathcal{H}^u & \xrightarrow{\varphi^y} & \mathcal{F}^y \end{array}$$

But $(f^*\rho')_{y,E} = (f^*\rho')_{x,E}$ by definition (they are both the canonical projection) and we obtain

$$\rho_{x,E} \circ \varphi^x = \varphi^E \circ (f^*\rho')_{x,E} = \varphi^E \circ (f^*\rho')_{y,E} = \rho_{y,E} \circ \varphi^y.$$

It is clear that the map $\gamma : \text{Hom}_{\mathbf{Sh}_k(\mathcal{G})}(f^*\mathcal{H}, \mathcal{F}) \rightarrow \text{Hom}_{\mathbf{Sh}_k(\mathcal{G}')}(\mathcal{H}, f_*\mathcal{F})$ defined is injective. To conclude our proof, we have to show the surjectivity of γ .

Suppose $\psi = (\{\psi^u\}_{u \in \mathcal{V}'}, \{\psi^F\}_{F \in \mathcal{E}'}) \in \text{Hom}_{\mathbf{Sh}_k(\mathcal{G}')}(\mathcal{H}, f_*\mathcal{F})$, where, for all $u \in \mathcal{V}'$ and $F \in \mathcal{E}'$ such that u is on the border of F , the following diagram commutes:

$$(13) \quad \begin{array}{ccc} \mathcal{H}^u & \xrightarrow{\psi^u} & \Gamma(f_{\mathcal{V}}^{-1}(u), \mathcal{F}) \\ \downarrow \rho_{u,F} & & \downarrow \oplus (\text{Tw}_{f_{L,x}} \circ \rho_{x,E}) \\ (f^*\mathcal{H})^F & \xrightarrow{\psi^F} & \bigoplus_{E \in f_{\mathcal{E}}^{-1}(F)} \mathcal{F}^E \end{array}$$

We claim that there exist $\varphi = (\{\varphi^x\}) \in \text{Hom}_{\mathbf{Sh}_k(\mathcal{G})}(f^*\mathcal{H}, \mathcal{F})$ such that $\gamma(\varphi) = \psi$.

For any $x \in \mathcal{V}$, let us consider $u := f_{\mathcal{V}}(x)$ and define φ^x as the composition of the following maps:

$$\mathcal{H}^u \xrightarrow{\psi^u} \Gamma(f_{\mathcal{V}}^{-1}(u), \mathcal{F}) \hookrightarrow \bigoplus_{y \in f_{\mathcal{V}}^{-1}(u)} \mathcal{F}^y \twoheadrightarrow \mathcal{F}^x$$

φ^x

For any $E : x \rightarrow y \in \mathcal{E}$ such that $f_{\mathcal{V}}(x) \neq f_{\mathcal{V}}(y)$, that is, there exists an edge $F \in \mathcal{E}'$ such that $f_{\mathcal{E}}(E) = F$, we define φ^E as the composition of the following maps:

$$\mathcal{H}^F \xrightarrow{\psi^F} \bigoplus_{L \in f_{\mathcal{E}}^{-1}(F)} \mathcal{F}^L \xrightarrow{\text{Tw}_{f_{L,y}}} \bigoplus_{L \in f_{\mathcal{E}}^{-1}(F)} \mathcal{F}^L \twoheadrightarrow \mathcal{F}^E$$

φ^E

Now it is clear that $\gamma(\varphi) = \psi$. Indeed, if $u \notin f_{\mathcal{V}}(\mathcal{V})$, then $\psi^u = 0$ and the claim is trivial. Otherwise, $u \in f_{\mathcal{V}}(\mathcal{V})$ and we get the following diagram, where all the squares are commutative:

$$\begin{array}{ccccccc}
 & & & \varphi^x & & & \\
 & & & \curvearrowright & & & \\
 \mathcal{H}^u & \xrightarrow{\psi^u} & \Gamma(f_{\mathcal{V}}^{-1}(u), \mathcal{F}) \hookrightarrow & \bigoplus_{y \in f_{\mathcal{V}}^{-1}(u)} \mathcal{F}^y & \twoheadrightarrow & \mathcal{F}^x & \\
 \downarrow \rho_{u,F} & & \downarrow (f_*\rho)_{y,F} & \downarrow \bigoplus \rho_{z,L} & & \downarrow \rho_{x,E} & \\
 \mathcal{H}^F & \xrightarrow{\psi^F} & \bigoplus_{L \in f_{\mathcal{E}}^{-1}(F)} \mathcal{F}^L & \xrightarrow{\text{Tw}f_{l,y}} & \bigoplus_{L \in f_{\mathcal{E}}^{-1}(F)} \mathcal{F}^L & \twoheadrightarrow & \mathcal{F}^E \\
 & & & \curvearrowleft & & & \\
 & & & \varphi^E & & &
 \end{array}$$

□

As an application of the previous proposition, we get the following corollary.

Corollary 5.10. *Let $\mathcal{G} \in \mathbf{MG}(Y_k)$ and let \mathcal{Z} , resp. \mathcal{Z} , be its structure sheaf, resp. its structure algebra. Then the functors $\Gamma(-), \text{Hom}_{\mathbf{Sh}_k(\mathcal{G})}(\mathcal{Z}, -) : \mathbf{Sh}_k(\mathcal{G}) \rightarrow \mathcal{Z}$ -modules are naturally equivalent. In particular, we get the following isomorphism of S_k -modules:*

$$\mathcal{Z} \cong \text{End}_{\mathbf{Sh}_k(\mathcal{G})}(\mathcal{Z}).$$

Proof. Consider the homomorphism $p : \mathcal{G} \rightarrow \{\text{pt}\}$, where we set $p_{l,x} = \text{id}_{Y_k}$ for all x , the vertex of \mathcal{G} . The structure sheaf of $\{\text{pt}\}$ is just a copy of S_k and, for all $\mathcal{F} \in \mathbf{Sh}_k(\mathcal{G})$, by Proposition 5.9, we get

$$\text{Hom}_{\mathbf{Sh}_k(\mathcal{G})}(p^*S_k, \mathcal{F}) = \text{Hom}_{\mathbf{Sh}_k(\{\text{pt}\})}(S_k, p_*\mathcal{F}).$$

But we have already noticed in Example 5.8 that $p^*S_k \cong \mathcal{Z}$ and $p_*\mathcal{F} = \Gamma(\mathcal{F})$. Finally, as $\text{Hom}_{S_k}(S_k, \mathcal{Z}) \cong \mathcal{Z}$, we get the claim. □

5.4. BMP-sheaves. The following definition, due to Fiebig and Williamson, generalises the one of canonical sheaves given by Braden and MacPherson in [BMP]. These sheaves will play a fundamental role in the rest of this paper.

Definition 5.11 (cf. [FW]). Let $\mathcal{G} \in \mathbf{MG}(Y_k)$ and let \mathcal{B} be a sheaf on it. We say that \mathcal{B} is a *Braden-MacPherson sheaf* if it satisfies the following properties:

- (BMP1) for any $x \in \mathcal{V}$, $\mathcal{B}^x \in S$ -mod is free;
- (BMP2) for any $E : x \rightarrow y \in \mathcal{E}$, $\rho_{y,E} : \mathcal{B}^y \rightarrow \mathcal{B}^E$ is surjective with kernel $l(E) \cdot \mathcal{B}^y$;
- (BMP3) \mathcal{B} is flabby;
- (BMP4) for any $x \in \mathcal{V}$, the map $\Gamma(\mathcal{B}) \rightarrow \mathcal{B}^x$ is surjective.

Hereafter, Braden-MacPherson sheaves will be referred to also as *BMP-sheaves* or *canonical sheaves*.

An important theorem, characterising Braden-MacPherson sheaves, is the following one.

Theorem 5.12 (cf. [J, Proposition 3.5(d), Proposition 3.12]; [FW, Theorem 6.3]).
 Let $\mathcal{G} \in \mathbf{MG}(Y_k)$. Then

- (i) For any $w \in \mathcal{V}$, there is a unique (up to isomorphism) Braden-MacPherson sheaf $\mathcal{B}(w) \in \mathbf{Sh}_k(\mathcal{G})$ with the following properties:

$$\begin{aligned} &(\text{BMP0}) \mathcal{B}(w) \text{ is indecomposable in } \mathbf{Sh}_k(\mathcal{G}); \\ &(\text{BMP1a}) \mathcal{B}(w)^w \cong S_k \text{ and } \mathcal{B}(w)^x = 0, \text{ unless } x \leq w. \end{aligned}$$

- (ii) Let \mathcal{B} be a Braden-MacPherson sheaf such that the set $\{v \in \mathcal{V} \mid \mathcal{B}^v \neq 0\}$ is bounded above. Then, there are $w_1, \dots, w_r \in \mathcal{V}$ and $l_1 \dots l_r \in \mathbb{Z}$ such that

$$\mathcal{B} \cong \mathcal{B}(w_1)\langle l_r \rangle \oplus \dots \oplus \mathcal{B}(w_r)\langle l_r \rangle.$$

We also want to quote a result by Fiebig, that will be used later, which tells us that the structure sheaf is not in general flabby, but that if it is the case, then it is isomorphic to an indecomposable BMP-sheaf.

Proposition 5.13 ([Fie10a], Proposition 4.2). *Let $\mathcal{G} \in \mathbf{MG}(Y_k)$ be such that it has a highest vertex w . Then $\mathcal{B}(w) \cong \mathcal{L}$ if and only if \mathcal{L} is flabby.*

5.4.1. *Pullback of BMP-sheaves.* We conclude this section by recalling a result from our paper [L11].

Lemma 5.14 (cf. [L11]). *Let \mathcal{G} and \mathcal{G}' be two k -moment graphs on Y , both with a unique maximal vertex, w resp. w' , and let $f : \mathcal{G} \rightarrow \mathcal{G}'$ be an isomorphism. If $\mathcal{B}(w)$ and $\mathcal{B}'(w')$ are the corresponding indecomposable Braden-MacPherson sheaves, then $\mathcal{B}(w) \cong f^* \mathcal{B}'(w')$ as k -sheaves on \mathcal{G} .*

6. STATEMENT OF THE MAIN RESULT AND PROOF OF THE SUBGENERIC CASE

In order to provide a new approach to Kazhdan-Lusztig and Lusztig conjectures, Fiebig applied the theory of sheaves on moment graphs to the representation theory of complex symmetrisable Kac-Moody algebras and of semisimple reductive algebraic groups over a field of positive characteristic, and stated a conjectural formula relating Kazhdan-Lusztig polynomials to the stalks of indecomposable Braden-MacPherson sheaves (cf. [Fie08a], [Fie10b], [Fie11]). More precisely, if $\widehat{\mathcal{G}}^J$ is the Bruhat graph we defined in §3.2, then for any $w \in \widehat{\mathcal{W}}^J$ we can consider the subgraph $\widehat{\mathcal{G}}_w^J := \widehat{\mathcal{G}}_{\{\leq w\}}^J$. It is a finite k -moment graph (for any k) with highest vertex w , hence we may build the corresponding indecomposable Braden-MacPherson sheaf $\mathcal{B}^J(w) \in \mathbf{Sh}_k(\widehat{\mathcal{G}}_w^J)$. For any free and finitely generated S_k -module $M = \bigoplus_{i=1}^n S\langle l_i \rangle$, its graded rank is

$$\text{rk} M = \sum_{i=1}^n q^{-l_i} \in \mathbb{Z}[q^{\pm 1}].$$

Finally, let us denote by $P_{y,w}^{J,-1}$ Deodhar’s analogue of Kazhdan-Lusztig polynomials at the parameter $u = -1$ (cf. [Deo]).

Question 6.1 (cf. [Fie10b], Conjecture 4.4). *Under which assumptions on the characteristic of the base field do we have that if $y \leq w$, then $\text{rk}(\mathcal{B}^J(w))^y = P_{y,w}^{J,-1}(q)$?*

This question motivated our paper [L11], where we were able to give the moment graph analogue of several properties of Kazhdan-Lusztig polynomials. The interpretation of certain equalities of these polynomials at a categorical level has the

advantage of furnishing a deeper understanding of several phenomena, since they are lifted to a setting where there is extra structure. The problem we address in this paper concerns a stabilisation property of the affine Kazhdan-Lusztig polynomials, proved by Lusztig in [Lu80b], while he was trying to find support for his conjecture on modular representations.

Let \mathcal{W} denote a finite Weyl group, $\widehat{\mathcal{W}}$ its affinisation and \mathcal{S} the set of *finite* simple reflections. Then, as we have already noticed, the set of minimal representatives of $\mathcal{W} \setminus \widehat{\mathcal{W}}$ is in bijection with the set of alcoves in the fundamental chamber and the corresponding parabolic Kazhdan-Lusztig polynomials can be indexed by pairs of alcoves $A, B \in \mathcal{A}^+$. The result that Lusztig proved can be reformulated in terms of these polynomials. In particular, quoting Soergel’s reformulation (cf. [Soe97, Theorem 6.1]), the parabolic Kazhdan-Lusztig polynomials $P_{A,B}^{J,-1}$ indexed by pairs of alcoves deep enough in the fundamental chamber stabilise, in the sense that, for any pair of alcoves A, B , there exists a polynomial $Q_{A,B}$ with integer coefficients such that

$$\lim_{\mu \in \mathcal{C}^+} P_{A+\mu, B+\mu}^{J,-1} = Q_{A,B}.$$

The polynomials $Q_{A,B}$ are called *generic polynomials* and turn out to have a realisation very similar to the one of the regular Kazhdan-Lusztig polynomials. Indeed, Lusztig in [Lu80b] associated to every affine Weyl group $\widehat{\mathcal{W}}$ its periodic module \mathbf{M} , that is, the free $\mathbb{Z}[q^{\pm \frac{1}{2}}]$ -module with a set of generators –or standard basis– indexed by the set of all alcoves \mathcal{A} . It is possible to define an involution and to prove that there exists a self-dual basis of \mathbf{M} : the *canonical basis*. In this setting, the generic polynomials are the coefficients of the change basis matrix. Our interest in the periodic module is motivated by the fact that \mathbf{M} governs the representation theory of the affine Kac-Moody algebra, whose Weyl group is $\widehat{\mathcal{W}}$, at the critical level.

The aim of the rest of this paper is to study the behaviour of indecomposable Braden-MacPherson sheaves on finite intervals of the parabolic Bruhat graph $\widehat{\mathcal{G}}^{\text{par}}$ deep enough in \mathcal{C}^+ (see §4). More precisely, let $\mathcal{I} = [A, B]$ be an interval deep enough in the fundamental chamber. Inspired by [Lu80b, Proposition 11.15], we claim that, under some assumptions on the base ring, for any $C \in [A, B]$ and for all $\mu \in \mathbb{Z}_{\geq 0}\rho$,

$$(14) \quad \mathcal{B}(B)^C \cong \mathcal{B}(B + \mu)^{C+\mu},$$

where $\mathcal{B}(B)$, resp. $\mathcal{B}(B + \mu)$, is an indecomposable BMP-sheaf on $\widehat{\mathcal{G}}_{[A,B]}^{\text{par}}$, resp. $\widehat{\mathcal{G}}_{[A+\mu, B+\mu]}^{\text{par}}$.

We showed in §4 that $\widehat{\mathcal{G}}_{[A,B]}^{\text{par}}$ is in general not isomorphic to $\widehat{\mathcal{G}}_{[A+\mu, B+\mu]}^{\text{par}}$ as a k -moment graph, so unluckily we cannot use Lemma 5.14 to get the isomorphism of S_k -modules above. On the other hand, we proved in Lemma 4.13 that, for all $\mu \in X^\vee$, there is an isomorphism of k -moment graphs,

$$\tau_\mu : \widehat{\mathcal{G}}_{[A,B]}^{\text{stab}} \rightarrow \widehat{\mathcal{G}}_{[A+\mu, B+\mu]}^{\text{stab}}.$$

Thus, by Lemma 5.14, we get an isomorphism between the indecomposable canonical sheaf $\mathcal{B}(B)$ on $\widehat{\mathcal{G}}_{[A,B]}^{\text{stab}}$ and $\tau_\mu^* \mathcal{B}(B + \mu)$, the pullback of the indecomposable Braden-MacPherson sheaf $\mathcal{B}(B + \mu)$ on $\widehat{\mathcal{G}}_{[A+\mu, B+\mu]}^{\text{stab}}$.

For any finite interval \mathcal{I} deep enough in the fundamental chamber, consider the monomorphism $g_{\mathcal{I}} : \widehat{\mathcal{G}}_{\mathcal{I}}^{\text{stab}} \hookrightarrow \widehat{\mathcal{G}}_{\mathcal{I}}^{\text{par}}$, given by $g_{\mathcal{I}, \nu} = \text{id}_\nu$ and $g_{\mathcal{I}, l, x} = \text{id}$ for all $x \in \mathcal{I}$.

We obtain the functor

$$\cdot^{\text{stab}} : \mathbf{Sh}(\widehat{\mathcal{G}}_{|\mathcal{I}}^{\text{par}}) \rightarrow \mathbf{Sh}(\widehat{\mathcal{G}}_{|\mathcal{I}}^{\text{stab}}),$$

defined by the setting $\mathcal{F} \mapsto \mathcal{F}^{\text{stab}} := g_{\mathcal{I}}^*(\mathcal{F})$. The goal of the rest of this paper is to prove the following result.

Theorem 6.2. *Let \mathcal{I} be a finite interval deep enough in the fundamental chamber and let k be such that $(\widehat{\mathcal{G}}_{|\mathcal{I}}^{\text{par}}, k)$ is a GKM-pair. Then the functor $\cdot^{\text{stab}} : \mathbf{Sh}_k(\mathcal{G}_{|\mathcal{I}}^{\text{par}}) \rightarrow \mathbf{Sh}_k(\mathcal{G}_{|\mathcal{I}}^{\text{stab}})$ preserves indecomposable Braden-MacPherson sheaves.*

We will prove this theorem via explicit calculations in the $\widehat{\mathfrak{sl}}_2$ -case, while for the general case we will need results and methods developed by Fiebig in [Fie11].

Once Theorem 6.2 is proven, we will obtain the isomorphism in (14) by applying Lemma 4.13.

Theorem 6.3. *Let $\mathcal{I} = [A, B]$ be deep enough in the fundamental chamber and let k be such that $(\widehat{\mathcal{G}}_{|\mathcal{I}}^{\text{par}}, k)$ is a GKM-pair. For any alcove $C \in [A, B]$ and for all $\mu \in \mathbb{Z}_{\geq 0}\rho$,*

$$\mathcal{B}(B)^C \cong \mathcal{B}(B + \mu)^{C+\mu}.$$

6.1. The subgeneric case. Using the same terminology as Fiebig in [Fie13], we denominate the *subgeneric* the case in which $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_2$.

In the subgeneric case, for any finite interval \mathcal{I} and any k such that $(\widehat{\mathcal{G}}_{|\mathcal{I}}^{\text{par}}, k)$ is a GKM-pair, the corresponding indecomposable BMP-sheaf is isomorphic to the structure sheaf \mathcal{A} of $\widehat{\mathcal{G}}_{|\mathcal{I}}^{\text{par}}$. There are several ways to prove this fact.

The first one comes from the topology of the underlying variety. We try to give a rough idea of this argument. To start with, let us suppose $k = \mathbb{Q}$. In the case of a finite interval \mathcal{I} of a Bruhat moment graph, its structure algebra and the space of global sections of the indecomposable BMP-sheaf on \mathcal{I} having maximal support describe the T -equivariant cohomology and intersection cohomology, respectively, of the corresponding Richardson variety. Since all the Richardson varieties are in the subgeneric case rationally smooth, the intersection cohomology coincides with the usual cohomology, and then the structure sheaf with the canonical sheaf. It is possible to perform the same argument, using parity sheaves and results from [FW] in the positive characteristic setting.

An alternative proof comes from Fiebig’s multiplicity one result. Indeed, using the inductive formula (2.2.c) of [KL79], it is easy to show that $P_{A,B}^{J^-,1} = 1$ for all $A, B \in \mathcal{A}^+$ with $A \leq B$. We then have to apply the following theorem

Theorem 6.4 (cf. [Fie10a], Theorem 6.4). *Suppose that k and w are such that (\mathcal{G}_w, k) is a GKM-pair. For all $x \leq w$ we have*

$$\mathcal{B}(w)^x \cong S_k \quad \text{if and only if} \quad P_{x,w} = 1.$$

But $\mathcal{B}(w)^x \cong S_k$ if and only if $\mathcal{B}(w)^y \cong S_k$ for all $y \in [x, w]$ (see, e.g. [Fie10a, Theorem 5.1]), which, together with (BMP2), implies that the indecomposable BMP-sheaf $\mathcal{B}(w)$ on $\mathcal{G}_{[x,w]}$ is isomorphic to the structure sheaf of such a graph.

A third proof is presented in the appendix of this paper and makes use only of some combinatorial arguments together with intrinsic properties of indecomposable BMP-sheaves.

In this paragraph, $\widehat{\mathcal{G}}^{\text{par}}$, resp. $\widehat{\mathcal{G}}^{\text{stab}}$, denotes the parabolic moment graph, resp. the stable moment graph, for $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_2$.

We have seen above that for any two vertices x, w with $x \leq w$ the stalk of the Braden-MacPherson sheaf on $\widehat{\mathcal{G}}_{\leq w}^{\text{par}}$ is $\mathcal{B}(w)^x \cong S_k$. This fact, by Proposition 5.13, is equivalent to the flabbiness of the structure sheaf on $\widehat{\mathcal{G}}_{\leq w}^{\text{par}}$. Therefore in order to show that the functor \cdot^{stab} preserves indecomposable canonical sheaves, again by Proposition 5.13, it is enough to verify that, for any vertex w , the structure sheaf \mathcal{A} on $\widehat{\mathcal{G}}_{\leq w}^{\text{stab}}$ is still flabby.

Recall that the set of vertices of $\widehat{\mathcal{G}}^{\text{par}}$ (and so of $\widehat{\mathcal{G}}^{\text{stab}}$) can be identified with the finite (co)root lattice, that is, $\mathbb{Z}\alpha$, where $\alpha = \alpha^\vee$ is the positive (co)root of A_1 . Moreover, $\widehat{\mathcal{G}}^{\text{par}}$ is a complete graph and the label function is given, up to a sign, by $l(h\alpha - k\alpha) = -\alpha + (h+k)c$. By definition, we get $\widehat{\mathcal{G}}^{\text{stab}}$ from $\widehat{\mathcal{G}}^{\text{par}}$ by deleting the non-stable edges; then $h\alpha - k\alpha \in \mathcal{E}^{\text{stab}}$ if and only if $\text{sgn}(h) = -\text{sgn}(k)$ (where, by convention, we set $\text{sgn}(0) = -$).

Lemma 6.5. *Let $r \in \mathbb{Z}_{>0}$. If $n \in \mathbb{Z}$, set, for any $h \in \mathbb{Z}$, with $h\alpha \leq n\alpha$,*

$${}^e z_{n\alpha, h\alpha}^r := \begin{cases} 0 & \text{if } |h| \in [|n| - r + 1, |n|], \\ \prod_{i=0}^{r-1} [(\alpha + (|n| - h - i)c)(|n| - h - i)] & \text{if } h \in (0, |n| - r], \\ \prod_{i=0}^{r-1} [(-\alpha + (|n| + h - i)c)(|n| + h - i)] & \text{if } h \in [r - |n|, 0]. \end{cases}$$

If $(\widehat{\mathcal{G}}_{\leq n\alpha}^{\text{stab}}, k)$ is a GKM-pair, then ${}^e z_{n\alpha}^r = ({}^e z_{n\alpha, h\alpha}^r) \in \Gamma(\{\leq n\alpha\}, \mathcal{A})_{\{r\}}$.

Proof. We verify that, for any $h, k \in \mathbb{Z}$ such that $h\alpha, k\alpha \leq n\alpha$, if $h\alpha - k\alpha$ is an edge, then

$$(15) \quad {}^e z_{n\alpha, h\alpha}^r - {}^e z_{n\alpha, k\alpha}^r \equiv 0 \pmod{-\alpha + (h+k)c}.$$

We may clearly suppose $h > 0$ and $k \leq 0$.

Let us at first consider $h \in [|n| - r + 1, n]$. If $-k \in [|n| - r + 1, n]$, then ${}^e z_{n\alpha, h\alpha}^r = {}^e z_{n\alpha, k\alpha}^r = 0$ and there is nothing to prove. Otherwise, $k \in [r - |n|, 0]$ and

$$(16) \quad {}^e z_{n\alpha, h\alpha}^r - {}^e z_{n\alpha, k\alpha}^r = 0 - \prod_{i=0}^{r-1} [(-\alpha + (|n| + k - i)c)(|n| + k - i)].$$

Observe that $-\alpha + (h+k)c$ divides $\prod_{i=0}^{r-1} [(-\alpha + (|n| + k - i)c)(|n| + k - i)]$ if and only if there exists an $i \in [0, r - 1]$ such that $|n| - i = h$, i.e. $h - |n| = -i$. But we supposed $h \in [|n| - r + 1, n]$, that is, precisely, $h - |n| \in [-r + 1, 0]$.

Let consider the case $h \in (0, |n| - r]$. If $-k \in [|n| - r + 1, n]$, then

$$(17) \quad {}^e z_{n\alpha, h\alpha}^r - {}^e z_{n\alpha, k\alpha}^r = \prod_{i=0}^{r-1} [(\alpha + (|n| - h - i)c)(|n| - h - i)] - 0.$$

Notice that $-\alpha + (h+k)c$ divides $\prod_{i=0}^{r-1} [(\alpha + (|n| - h - i)c)(|n| - h - i)]$ if and only if there exists an $i \in [0, r - 1]$ such that $|n| - i = -k$, i.e. $-k - |n| = -i$. But we assumed $-k \in [|n| - r + 1, |n|]$, that is, precisely, $-k - |n| \in [-r + 1, 0]$.

Otherwise, $k \in [r - |n|, 0]$ and

$$\begin{aligned}
 {}^e z_{n\alpha, h\alpha}^r - {}^e z_{n\alpha, k\alpha}^r &= \prod_{i=0}^{r-1} [(\alpha + (|n| - h - i)c)(|n| - h - i)] \\
 &\quad - \prod_{i=0}^{r-1} [(-\alpha + (k + |n| - i)c)(k + |n| - i)] \\
 &\equiv \prod_{i=0}^{r-1} [(k + h + |n| - h - i)(|n| - h - i)c] \\
 &\quad - \prod_{i=0}^{r-1} [(-k - h + k + |n| - i)(|n| + k - i)c] \pmod{-\alpha + (h+k)c} \\
 &= c^r \prod_{i=0}^{r-1} [(k + |n| - i)(|n| - h - i) - (-h + |n| - i)(|n| + k - i)] \\
 &= 0. \tag*{\square}
 \end{aligned}$$

Lemma 6.6. *Let $r \in \mathbb{Z}_{>0}$. If $n \in \mathbb{Z}$, for any $h \in \mathbb{Z}$, such that $h\alpha \leq n\alpha$, we set*

$$\begin{aligned}
 &{}^o z_{n\alpha, h\alpha}^r \\
 &:= \begin{cases} 0 & \text{if } |h| \in [|n| - r + 2, |n|], \\ \prod_{i=0}^{r-1} [(\alpha + (|n| - h - i)c)(|n| - h - i + 1)] & \text{if } h \in (0, |n| - r + 1], \\ \prod_{i=0}^{r-1} [(-\alpha + (|n| + h - i + 1)c)(|n| + h - i)] & \text{if } h \in [r + n - 1, 0]. \end{cases}
 \end{aligned}$$

If $(\widehat{\mathcal{G}}_{\leq n\alpha}^{stab}, k)$ is a GKM-pair, then ${}^o z_{n\alpha}^r = ({}^o z_{n\alpha, h\alpha}^r) \in \Gamma(\{\leq n\alpha\}, \mathcal{A})_{\{r\}}$.

Proof. The proof is very similar to the one of the previous lemma and therefore we omit it. \square

Define ${}^e z_{n\alpha}^0 := (1)_{h\alpha \leq n\alpha}$.

Lemma 6.7. *Let $r \in \mathbb{Z}_{\geq 0}$ and $m, n \in \mathbb{Z}$ be such that $m\alpha \leq n\alpha$. If $(\widehat{\mathcal{G}}_{\leq n\alpha}^{stab}, k)$ is a GKM-pair, for all $z \in \Gamma([m\alpha, n\alpha], \mathcal{A})_{\{r\}}$, there exist ${}^o s_k^i, {}^e s_k^j \in (S_k)_{\{i\}}$, with $i \in [0, r], j \in (0, r]$ and p such that $p\alpha \in [m\alpha, n\alpha]$, such that*

$$(18) \quad z = \sum_{j=1}^r {}^e s_p^j ({}^e z_{p\alpha}^{r-j})_{p\alpha \in [m\alpha, n\alpha]} + \sum_{i=0}^r {}^o s_p^i ({}^o z_{p\alpha}^r)_{p\alpha \in [m\alpha, n\alpha]}.$$

Proof. Let $h\alpha$ be the maximal vertex in $[m\alpha, n\alpha]$ such that $z_{h\alpha} \neq 0$. We prove the statement by induction on $l = \#[m\alpha, h\alpha]$.

If such a vertex does not exist, that is, $l = 0$, then $z = (0)$ and there is nothing to prove.

We should consider four cases: $n > 0$ and $l > 0$; $n > 0$ and $l \leq 0$; $n \leq 0$ and $l > 0$; $n \leq 0$ and $l \leq 0$. Actually, we will verify only the first case, since the others can be proven in a very similar way.

Let $n > 0$ and $h > 0$. If $h = n$, then we set $z' = z - z_{n\alpha}^e z_{n\alpha}^0$ and the result follows from the inductive hypothesis. Otherwise, $h < n$ and then $\prod_{i=0}^{n-h-1} (\alpha + (n-h-i)c)$ divides $z_{h\alpha}$ in S_k , and we may set

$$(19) \quad {}^e s_h^{r-n+h} := \prod_{i=0}^{n-h-1} [(\alpha + (n-h+i)c)(n-h-i)]^{-1} \cdot z_{h\alpha} \in (S_k)_{\{r-n+h\}}.$$

Now $z' := z - e s_h^{r-n+h+1} \cdot (e z_{p\alpha}^{n-h})_{p\alpha \in [m\alpha, n\alpha]} \in \Gamma([m\alpha, n\alpha], \mathcal{A})_{\{r\}}$ has the property that $z'_{p\alpha} = 0$ for all $p \in [h\alpha, n\alpha]$, and we obtain the statement from the inductive hypothesis. \square

Corollary 6.8. *For any $n \in \mathbb{Z}$ and any field k such that $(\widehat{\mathcal{G}}_{\leq n\alpha}^{stab}, k)$ is a GKM-pair, the structure sheaf \mathcal{A} on $\widehat{\mathcal{G}}_{\leq n\alpha}^{stab}$ is flabby.*

Proof. We have to show that every local section $z \in \Gamma(\mathcal{I}, \mathcal{A})$ with \mathcal{I} open can be extended to a global section $\tilde{z} \in \Gamma(\widehat{\mathcal{G}}_{\leq n\alpha}^{stab}, \mathcal{A})$. Since the set of vertices of $\widehat{\mathcal{G}}^{stab}$ is totally ordered, then any open set of $\widehat{\mathcal{G}}_{\leq n\alpha}^{stab}$ is actually an interval, that is, there exists an $m \in \mathbb{Z}$ such that $\mathcal{I} = [m\alpha, n\alpha]$.

Suppose $z \in \Gamma(\mathcal{I}, \mathcal{A})_{\{r\}}$; then by Lemma 6.7, we can write

$$(20) \quad z = \sum_{j=1}^r e s_k^j (e z_{k\alpha}^{r-j})_{k\alpha \in [m\alpha, n\alpha]} + \sum_{i=0}^r o s_k^i (o z_{k\alpha}^r)_{k\alpha \in [m\alpha, n\alpha]}.$$

By Lemma 6.5 and Lemma 6.6, z is a sum of extensible sections, and so it is extensible as well. \square

Finally, we get the following theorem.

Theorem 6.9. *Let $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_2$, let \mathcal{I} be a finite interval deep enough in the fundamental chamber and let k be such that $(\widehat{\mathcal{G}}_{|\mathcal{I}}^{par}, k)$ is a GKM-pair. Then for every finite interval \mathcal{I} , the functor \cdot^{stab} preserves indecomposable canonical sheaves.*

7. THE GENERAL CASE

In order to prove Theorem 6.2, we have to show that, for any interval \mathcal{I} deep enough in the fundamental chamber, if \mathcal{B} is an indecomposable Braden-MacPherson sheaf on $\widehat{\mathcal{G}}_{|\mathcal{I}}^{par}$, then \mathcal{B}^{stab} is indecomposable and satisfies properties (BMP1), (BMP2), (BMP3), (BMP4).

Lemma 7.1. *If $\mathcal{F} \in \mathbf{Sh}(\widehat{\mathcal{G}}_{|\mathcal{I}}^{par})$ satisfies (BMP1), (BMP2), (BMP4), then also \mathcal{F}^{stab} has the same properties.*

Proof. To begin with, let us observe that (BMP1) and (BMP2) follow immediately from the definition of the pullback functor, as $\cdot^{stab} = i_{\mathcal{I}}^*$.

Moreover $\widehat{\mathcal{G}}_{|\mathcal{I}}^{stab}$ is obtained from $\widehat{\mathcal{G}}_{|\mathcal{I}}^{par}$ by deleting the non-stable edges, and hence there is an inclusion

$$\begin{aligned} \Gamma(\mathcal{I}, \mathcal{F}) &= \left\{ (m_x) \in \bigoplus_{x \in \mathcal{I}} \mathcal{F}^x \mid \begin{array}{l} \rho_{x,E}(m_x) = \rho_{y,E}(m_y) \\ \forall E : x \longrightarrow y \in \mathcal{E}, x, y \in \mathcal{I} \end{array} \right\} \\ &\subseteq \left\{ (m_x) \in \bigoplus_{x \in \mathcal{I}} \mathcal{F}^x = \bigoplus_{x \in \mathcal{I}} (\mathcal{F}^{stab})^x \mid \begin{array}{l} \rho_{x,E}(m_x) = \rho_{y,E}(m_y) \\ \forall E : x \longrightarrow y \in \mathcal{E}_S, x, y \in \mathcal{I} \end{array} \right\} \\ &= \Gamma(\mathcal{I}, \mathcal{F}^{stab}). \end{aligned}$$

By assumption there is a surjective map $\Gamma(\mathcal{I}, \mathcal{F}) \rightarrow \mathcal{F}^x = (\mathcal{F}^{stab})^x$, and, from the inclusion above, it follows that the map $\Gamma(\mathcal{I}, \mathcal{F}^{stab}) \rightarrow (\mathcal{F}^{stab})^x$ is surjective too. \square

Therefore it is only left to show that \mathcal{B}^{stab} is a flabby indecomposable sheaf on $\widehat{\mathcal{G}}_{|\mathcal{I}}^{stab}$.

7.1. Flabbiness. Let $\widehat{\mathcal{G}}$ denote the regular Bruhat graph of $\widehat{\mathfrak{g}}$, while we let $\widehat{\mathcal{G}}^{\text{per}}$ be the periodic graph of §3.3.

It is possible to define a functor $\cdot^{\text{per}} : \mathbf{Sh}(\widehat{\mathcal{G}}) \rightarrow \mathbf{Sh}(\widehat{\mathcal{G}}^{\text{per}})$ in the following way. Let $\mathcal{F} = (\{\mathcal{F}^x\}, \{\mathcal{F}^E\}, \{\rho_{x,E}\})$; then we set $(\mathcal{F}^{\text{per}})^x = \mathcal{F}^x$ for any $x \in \mathcal{V}$, $(\mathcal{F}^{\text{per}})^E = \mathcal{F}^E$ for any $E \in \mathcal{E}$ and $\rho_{x,E}^{\text{per}} = \rho_{x,E}$.

A fundamental step in the proof of the flabbiness of $\mathcal{B}^{\text{stab}}$ consists in showing that \cdot^{per} maps canonical sheaves to flabby sheaves.

7.1.1. Translation functors. Here we recall the definition of (left and right) translation functors and of the corresponding categories of special modules.

Let us denote by \mathcal{Z} the structure algebra of the (affine regular) Bruhat graph $\widehat{\mathcal{G}}$. To any simple reflection $s \in \widehat{\mathcal{S}}$, we associate the two following automorphisms of the structure algebra.

First, let σ_s be defined as

$$\sigma_s : \begin{array}{ccc} \mathcal{Z} & \longrightarrow & \mathcal{Z} \\ (z_x)_{x \in \widehat{\mathcal{W}}} & \longmapsto & (z'_x)_{x \in \widehat{\mathcal{W}}} \end{array} \quad \text{with } z'_x = z_{xs}.$$

Secondly, if τ_s is the automorphism of the symmetric algebra S_k induced by the mapping $\lambda \mapsto s(\lambda)$ for all $\lambda \in \widehat{\mathfrak{h}}^*$, we set:

$$s\sigma : \begin{array}{ccc} \mathcal{Z} & \longrightarrow & \mathcal{Z} \\ (z_x)_{x \in \widehat{\mathcal{W}}} & \longmapsto & (z'_x)_{x \in \widehat{\mathcal{W}}} \end{array} \quad \text{with } z'_x = \tau_s(z_{sx}).$$

Let $\mathcal{Z} - \text{mod}^f$ be the category of \mathcal{Z} -modules which are finitely generated and free as S_k -modules. Moreover, let us denote by \mathcal{Z}^s and ${}^s\mathcal{Z}$ the S_k -submodules of \mathcal{Z} consisting of σ_s - and ${}_s\sigma$ -invariants, respectively. To start with, let us observe that \mathcal{Z} is a \mathcal{Z}^s - and ${}^s\mathcal{Z}$ -module under pointwise multiplication. Let $c^s = (c^s_x)_{x \in \widehat{\mathcal{W}}}$ be the element of \mathcal{Z} given by $c^s_x = x(\alpha_s)$, while $\overline{\alpha}_s \in \mathcal{Z}$ denotes the constant section whose components are all equal to α_s . Thus the following lemma tells us that \mathcal{Z} decomposes in a nice way as a \mathcal{Z}^s -module, resp. a ${}^s\mathcal{Z}$ -module

- Lemma 7.2.** (i) *There is a decomposition $\mathcal{Z} = \mathcal{Z}^s \oplus c^s \mathcal{Z}^s$ of \mathcal{Z}^s -modules (cf. [Fie08b], Lemma 5.1).*
 (ii) *There is a decomposition $\mathcal{Z} = {}^s\mathcal{Z} \oplus \overline{\alpha}_s {}^s\mathcal{Z}$ of ${}^s\mathcal{Z}$ -modules (cf. [L14], Lemma 4.1).*

- Definition 7.3.** (i) – The (right) translation on the wall is the functor $\theta^{s,\text{on}} : \mathcal{Z} - \text{mod}^f \rightarrow \mathcal{Z}^s - \text{mod}^f$ given by $M \mapsto \text{Res}_{\mathcal{Z}}^{\mathcal{Z}^s} M$.
 – The (right) translation out of the wall is the functor $\theta^{s,\text{out}} : {}^s\mathcal{Z} - \text{mod}^f \rightarrow \mathcal{Z} - \text{mod}^f$ defined by the $N \mapsto \text{Ind}_{\mathcal{Z}}^{\mathcal{Z}^s} N$.
 – By composition, we get a functor $\theta^s := \theta^{s,\text{out}} \circ \theta^{s,\text{on}} : \mathcal{Z} - \text{mod}^f \rightarrow \mathcal{Z} - \text{mod}^f$ that we call a (right) translation functor (cf. [Fie08b]).
 (ii) – The (left) translation on the wall is the functor ${}^{s,\text{on}}\theta : \mathcal{Z} - \text{mod}^f \rightarrow {}^s\mathcal{Z} - \text{mod}^f$ given by $M \mapsto \text{Res}_{\mathcal{Z}}^{{}^s\mathcal{Z}} M$.
 – The (left) translation out of the wall is the functor ${}^{s,\text{out}}\theta : {}^s\mathcal{Z} - \text{mod}^f \rightarrow \mathcal{Z} - \text{mod}^f$ defined by the $N \mapsto \text{Ind}_{\mathcal{Z}}^{{}^s\mathcal{Z}} N$. Again, this functor is well-defined thanks to Lemma 7.2.
 – By composition, we get a functor ${}^s\theta := {}^{s,\text{out}}\theta \circ {}^{s,\text{on}}\theta : \mathcal{Z} - \text{mod}^f \rightarrow \mathcal{Z} - \text{mod}^f$ that we call a (left) translation functor (cf. [L14]).

Observe that the functors $\theta^{s,\text{out}}$ and ${}^{s,\text{out}}\theta$ are well-defined due to Lemma 7.2.

Remark 7.4. The advantage of defining left translation functors is that they can also be applied in the parabolic setting. Indeed, if \mathcal{Z}^J is the structure algebra of the Bruhat graph $\widehat{\mathcal{G}}^J = \widehat{\mathcal{G}}(\widehat{\mathfrak{g}} \supseteq \widehat{\mathfrak{h}} \supseteq \widehat{\mathfrak{h}}, J)$, then the automorphism ${}_s\sigma^J : \mathcal{Z}^J \rightarrow \mathcal{Z}^J$ given by $(z_x) \mapsto (z'_x)$ with $z'_x = \tau_s(z_{sx})$ is still well-defined for any finitary $J \subseteq \widehat{\mathcal{S}}$ (cf. [L14], where the definition of left translation functors leads to a categorification of a parabolic Hecke module). If $J = \mathcal{S}$, the set of finite simple reflections, we denote the corresponding translation functor by ${}^s\theta^{\text{par}}$.

Remark 7.5. In the case of a finite Weyl group, left translation functors have been studied in [Str04] in the setting of Harish-Chandra bimodules.

7.1.2. *Flabby \mathcal{Z} -modules.* For all k -moment graphs \mathcal{K} , Fiebig considered the localisation functor

$$\mathcal{L} : \mathcal{Z}(\mathcal{K}) - \text{mod} \rightarrow \mathbf{Sh}_k(\mathcal{K}),$$

whose definition we are going to recall.

First of all we need to set some notation. Let Q be the quotient field of S_k and, for any $M \in \mathcal{Z} - \text{mod}^f$, let us write $M_Q := M \otimes_{S_k} Q$. If \mathcal{K} is a Bruhat graph, then (see §3.1 of [Fie08a]) there is a decomposition $M_Q := M \cap \bigoplus_{x \in \mathcal{V}} M_Q^x$ and so a canonical inclusion $M \subseteq \bigoplus_{x \in \mathcal{V}} M_Q^x$. For all subsets of the set of vertices $\Omega \subseteq \mathcal{V}$, we may define

$$M_\Omega := M \cap \bigoplus_{x \in \Omega} M_Q^x,$$

$$M^\Omega := M/M_{\mathcal{V} \setminus \Omega} = \text{im} \left(M \rightarrow M_Q \rightarrow \bigoplus_{x \in \Omega} M_Q^x \right).$$

For any vertex $x \in \mathcal{V}$, we set

$$\mathcal{L}(M)^x = M^x.$$

For any edge $E : x \text{ --- } y$, let us consider $\mathcal{Z}(E) = \{(z_x, z_x) \in S_k \oplus S_k \mid z_x - z_y \in l(E) \cdot S_k\}$ and $M(E) := \mathcal{Z}(E) \cdot M^{x,y}$. For $m = (m_x, m_y) \in M(E)$, let us set $\pi_x((m)) = m_x$, $\pi_y((m)) = m_y$ and define $\mathcal{L}(M)^E$ as the pushout in the following diagram of S_k -modules:

$$\begin{array}{ccc} M(E) & \xrightarrow{\pi_x} & M^x \\ \pi_y \downarrow & & \rho_{x,E} \downarrow \\ M^y & \xrightarrow{\rho_{y,E}} & \mathcal{L}(M)^E \end{array}$$

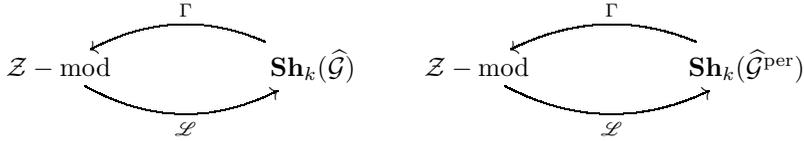
In this way, we also get the maps $\rho_{x,E}$ and $\rho_{y,E}$.

Finally, from [Fie08a, Theorem 3.5], the functor \mathcal{L} is left adjoint to the global section functor

$$\Gamma : \mathbf{Sh}_k(\mathcal{K}) \rightarrow \mathcal{Z}(\mathcal{K}) - \text{mod}.$$

From now on, we will deal again with Bruhat moment graphs and we will denote once again by \mathcal{Z} the structure algebra of the regular affine Bruhat graph $\widehat{\mathcal{G}}$. To start with, let us notice that, being the structure algebra of a moment graph independent from the partial order on the set of vertices, \mathcal{Z} coincides with the structure algebra of the periodic moment graph $\widehat{\mathcal{G}}^{\text{per}}$. Therefore we may summarise this setting as

follows:



Since for a BMP-sheaf \mathcal{F} on $\widehat{\mathcal{G}}$ it holds that $\mathcal{L} \circ \Gamma(\mathcal{F}) \cong \mathcal{F}$ (cf. [Fie08a]), our claim is equivalent to the fact that $\mathcal{L}(\Gamma(\mathcal{F}^{\text{per}})) = \mathcal{L}(\Gamma(\mathcal{F}))^{\text{per}}$ is a flabby sheaf on $\widehat{\mathcal{G}}^{\text{per}}$ if \mathcal{F} is a Braden MacPherson sheaf. We will prove it using translation functors.

7.1.3. *Three properties of the Bruhat order.* Let us consider the following properties of the Bruhat order:

- (PO1) The elements w and tw are comparable for all $w \in \widehat{\mathcal{W}}$ and $t \in \widehat{\mathcal{T}}$; the relations between all such pairs w, tw generate the partial order.
- (PO2) We have $[w, ws] = \{w, ws\}$ for all $w \in \widehat{\mathcal{W}}$ and $s \in \widehat{\mathcal{S}}$ such that $w < ws$.
- (PO3) (i) For $x, y \in \widehat{\mathcal{W}}$ such that $x < xs$ and $y \leq xs$, we have $ys \leq xs$;
 (ii) For $x, y \in \widehat{\mathcal{W}}$ such that $xs < x$ and $xs \leq y$, we have $xs \leq ys$.

From now on, \trianglelefteq will denote a partial order on $\widehat{\mathcal{W}}$ having the same properties as above and $\widehat{\mathcal{G}}_{\trianglelefteq}$ the moment graph on $\widehat{\mathcal{Q}}^{\vee}$ obtained from the regular Bruhat graph $\widehat{\mathcal{G}}$ by changing the orientation of the edges according to this partial order. This graph still satisfies the property of not having oriented cycles, thanks to (PO1) above.

Definition 7.6. An object $M \in \mathcal{Z} - \text{mod}$ is \trianglelefteq -flabby if the corresponding sheaf $\mathcal{L}(M)$ is a flabby sheaf on $\widehat{\mathcal{G}}_{\trianglelefteq}$.

In [Fie08b] Fiebig proved that translation functors preserve a \leq -flabby object, while he was proving them to preserve a stronger property of objects in $\mathcal{Z} - \text{mod}^f$. In what follows, we are going to recall the main steps of his argument and to point out that, whenever a proof depends on the Bruhat order on the set of vertices, it actually depends only on the properties which have been listed above.

7.1.4. *The moment graph $\widehat{\mathcal{G}}_{\trianglelefteq}^s$.* A first fundamental observation made by Fiebig is that it is possible to identify the set of invariants \mathcal{Z}^s with the structure algebra of the Bruhat moment graph $\widehat{\mathcal{G}}^s = \widehat{\mathcal{G}}(\widehat{\mathfrak{g}} \supseteq \widehat{\mathfrak{b}} \supseteq \widehat{\mathfrak{h}}, \{s\})$ (cf. [Fie08b, §5.1]). We were able to define Bruhat moment graphs, since the Bruhat order is preserved by the quotient map (see [BjBr, Proposition 2.5.1]). We now want to show that it makes sense to define a quotient graph of $\widehat{\mathcal{G}}_{\trianglelefteq}$ too, if J consists of only one simple reflection. Let \trianglelefteq be a partial order having the above properties; then for any $s \in \widehat{\mathcal{S}}$, we will denote $\widehat{\mathcal{W}}_{\trianglelefteq}^s$ the set of minimal representatives wrt \trianglelefteq of the equivalence classes of $\widehat{\mathcal{W}}/\{e, s\}$. Moreover, for any $x \in \widehat{\mathcal{W}}$ let us set $\bar{x}_{\trianglelefteq} := \min\{x, xs\}$, where the minimum is taken with respect to \trianglelefteq .

Lemma 7.7. *The map $\widehat{\mathcal{W}} \rightarrow \widehat{\mathcal{W}}_{\trianglelefteq}^s$, given by $u \mapsto \bar{u}_{\trianglelefteq}$, is order preserving.*

Proof. Let us suppose that $u \trianglelefteq v$ in $\widehat{\mathcal{W}}$. We want to show that $\bar{u}_{\trianglelefteq} \trianglelefteq \bar{v}_{\trianglelefteq}$. First of all, since $\bar{u}_{\trianglelefteq} \trianglelefteq u \trianglelefteq v$, if $v = \bar{v}_{\trianglelefteq}$, there is nothing to prove. Otherwise, $\bar{v}_{\trianglelefteq} = vs \triangleleft v$ and again either $\bar{u}_{\trianglelefteq} = u \triangleleft us$ or $\bar{u}_{\trianglelefteq} = us \triangleleft u$. In the first case, we may apply (PO3) (ii) with $x = us$ and $y = v$ and obtain $\bar{u}_{\trianglelefteq} \trianglelefteq \bar{v}_{\trianglelefteq}$. On the other hand, if $\bar{u}_{\trianglelefteq} = us \triangleleft u$, by (PO3) (ii) with $x = u$ and $y = v$, it follows that $\bar{u}_{\trianglelefteq} \trianglelefteq \bar{v}_{\trianglelefteq}$. \square

This leads us to the definition of the moment graph $\widehat{\mathcal{G}}_{\triangleleft}^s$ on \widehat{Q}^{\vee} , having as a set of vertices $\widehat{\mathcal{W}}_{\triangleleft}^s$, equipped with the partial order \triangleleft , edges $x \rightarrow y$ if either $x \text{ --- } y$ or $x \text{ --- } ys$ was an edge of $\widehat{\mathcal{G}}_{\triangleleft}$, in which case the label is the same. It is clear that \mathcal{Z}^s coincides with the structure algebra of this graph \mathcal{Z}' . Again, we say that $N \in \mathcal{Z}'\text{-mod}^f$ is \triangleleft -flabby if $\mathcal{L}(N)$ is a flabby sheaf on $\widehat{\mathcal{G}}_{\triangleleft}^s$.

For all $\mathcal{I} \subseteq \widehat{\mathcal{W}}$, let us denote by $\overline{\mathcal{I}}_{\triangleleft}$ the image of \mathcal{I} in $\widehat{\mathcal{W}}_{\triangleleft}^s$ under the map $u \mapsto \overline{u}_{\triangleleft}$. The following two results do not depend on the partial order.

Proposition 7.8 (cf. [Fie08b], Proposition 5.3). *Let $M \in \mathcal{Z}\text{-mod}^f$ and $N \in \mathcal{Z}'\text{-mod}^f$. Then for $\mathcal{I} \subseteq \widehat{\mathcal{W}}$ such that $s \cdot \mathcal{I} = \mathcal{I}$ we have*

$$(\theta^{s, on} M)_{\overline{\mathcal{I}}_{\triangleleft}} = \theta^{s, on}(M^{\mathcal{I}}), \quad (\theta^{s, out} N)^{\mathcal{I}} = \theta^{s, out}(N^{\overline{\mathcal{I}}_{\triangleleft}})$$

and

$$(\theta^{s, on} M)_{\overline{\mathcal{I}}_{\triangleleft}} = \theta^{s, on}(M_{\mathcal{I}}), \quad (\theta^{s, out} N)_{\mathcal{I}} = \theta^{s, out}(N_{\overline{\mathcal{I}}_{\triangleleft}}).$$

Lemma 7.9 (cf. [Fie08a], Lemma 3.3). *For any $M \in \mathcal{Z}\text{-mod}^f$, resp. $N \in \mathcal{Z}'\text{-mod}^f$, and any $\mathcal{I} \in \widehat{\mathcal{W}}$, resp. $\mathcal{J} \subseteq \widehat{\mathcal{W}}_{\triangleleft}^s$, it holds that $\Gamma(\mathcal{I}, \mathcal{L}(M)) = \Gamma(\mathcal{L}(M^{\mathcal{I}}))$, resp. $\Gamma(\mathcal{J}, \mathcal{L}(N)) = \Gamma(\mathcal{L}(N^{\mathcal{J}}))$.*

Therefore we get

Lemma 7.10. *Let $M \in \mathcal{Z}\text{-mod}^f$ be such that the adjunction morphism $M \mapsto \Gamma(\mathcal{L}(M))$ is an isomorphism. If M is \triangleleft -flabby, then also $\theta^{s, on}(M) \in \mathcal{Z}'\text{-mod}^f$ is \triangleleft -flabby.*

Proof. By Lemma 7.7, if $\overline{\mathcal{I}}_{\triangleleft} \subseteq \mathcal{W}_{\triangleleft}^s$ is an upwardly closed subset (wrt to \triangleleft), then also its preimage in $\widehat{\mathcal{W}}$ is upwardly closed wrt \triangleleft .

We want to show that the canonical map

$$\Gamma(\mathcal{L}(\theta^{s, on}(M))) \rightarrow \Gamma(\overline{\mathcal{I}}_{\triangleleft}, \mathcal{L}(\theta^{s, on}(M)))$$

is surjective.

By combining Lemma 7.9 and Proposition 7.8, we get

$$\Gamma(\overline{\mathcal{I}}_{\triangleleft}, \mathcal{L}(\theta^{s, on}(M))) = \Gamma(\mathcal{L}(\theta^{s, on}(M^{\mathcal{I}}))).$$

Because $(\Gamma \circ \mathcal{L}) \circ \theta^{s, on} = \theta^{s, on} \circ (\Gamma \circ \mathcal{L})$, our claim is equivalent to the surjectivity of the map

$$\theta^{s, on}(\Gamma \circ \mathcal{L}(M)) \rightarrow \theta^{s, on}(\Gamma \circ \mathcal{L}(M^{\mathcal{I}})).$$

By applying once again Lemma 7.9,

$$\theta^{s, on}(\Gamma \circ \mathcal{L}(M^{\mathcal{I}})) = \theta^{s, on}(\Gamma(\mathcal{I}, \mathcal{L}(M))),$$

and hence the statement follows from the assumption of M being \triangleleft -flabby, since $\theta^{s, on}$ is right-exact. \square

7.1.5. *Local definition of translation out of the wall.* In order to also prove that $\theta^{s, out}$ preserves \triangleleft -flabby objects, we have to give a realisation of it in terms of sheaves. Let us consider $N \in \mathcal{Z}'\text{-mod}^f$ such that $\theta^{s, out}(N)$ is isomorphic to $\Gamma(\mathcal{L}(\theta^{s, out}(N)))$. Let us set

$$\mathcal{M} := \mathcal{L}(\theta^{s, out}(N)) \in \mathbf{Sh}_k(\mathcal{G}_{\triangleleft}) \quad \text{and} \quad \mathcal{N} := \mathcal{L}(N) \in \mathbf{Sh}_k(\mathcal{G}_{\triangleleft}^s).$$

Let us denote by $p_s : \widehat{\mathcal{G}}_{\triangleleft} \rightarrow \widehat{\mathcal{G}}_{\triangleleft}^s$ the morphism of k -moment graphs defined as $p_{s, \nu} : x \mapsto \overline{x}_{\triangleleft}$ and $p_{s, l, x} = \text{id}$ for all $x \in \widehat{\mathcal{W}}$.

We may now reformulate Lemma 5.4 of [Fie08b] in this notation. The proof only needs (PO1), (PO2), (PO3) of \trianglelefteq .

Lemma 7.11 (cf. [Fie08b], Lemma 5.4). *It holds that $\mathcal{M} = p_s^*(\mathcal{N})$.*

Recall that a poset P is *directed* if for any pair of elements $u, v \in P$ there exists a $z \in P$ such that $u, v \trianglelefteq z$. The following result is a generalisation of [BjBr, Proposition 2.2.9].

Lemma 7.12. *The affine Weyl group \widehat{W} equipped with the partial order \trianglelefteq is a directed poset.*

Proof. By induction on $\ell(u) + \ell(v)$. The base step is $u = v = e$, and in this case we can take $z = e$. Now let $u = s_1 s_2 \cdots s_r$ with s_1, s_2, \dots, s_r simple reflections and $r \geq 1$. Let us set $u' := us_r$; since $\ell(u') < \ell(u)$ there exists z' such that $u', v \trianglelefteq z'$. By (PO2), we know that u and u' are comparable, so we have two cases.

If $u \triangleleft u'$, then we take $z = z'$ and our claim follows from the inductive hypothesis. Otherwise, $u' \triangleleft u$ and we have again to consider two cases for $z', z's_r$. Suppose first that $z' \triangleleft z's_r$. Let $x = z'$ and $y = xs_r$ and notice that $x = z' < z's_r = xs_r$ and $y = us_r \trianglelefteq z' = zs_r$, so we may apply (PO3) (i) and find $u = ys_r \trianglelefteq xs_r = z's_r$. Hence in this case we take $z = z's_r$ since we also have $v \trianglelefteq z' < z's_r = z$.

Finally suppose $z's_r \triangleleft z'$. Let $x = z's_r$ and $y = us_r$ and notice that $x \triangleleft xs_r$ and $y = us_r \trianglelefteq z' = xs_r$, so we may again apply (PO3) (i) and find $u = ys_r \trianglelefteq xs_r = z'$. So we take $z = z'$ and our claim is proven. \square

For $u, v \in \widehat{W}$ we will write $u \dot{\triangleleft} v$ if $u \trianglelefteq v$ and $[u, v] = \{u, v\}$. The following lemma generalises [BjBr, Corollary 2.2.8].

Lemma 7.13. *Let $s \in \widehat{\mathcal{S}}$ and $t \in \widehat{\mathcal{T}}$ be such that $s \neq t$. Let moreover $w \in \widehat{W}$ be such that $w \dot{\triangleleft} ws, wt$. Then $ws, wt \dot{\triangleleft} wts$.*

Proof. The claim follows by repeatedly applying (PO3).

To start with, by (PO3) (ii) with $x = ws$ and $y = wt$, we get $w \trianglerighteq wts$. Since $s \neq t$, it holds that $wts \neq w$ and hence $wts \triangleleft wt$. Indeed, if it were not the case, $wts \in [w, wt] \setminus \{w, wt\}$, but by assumption $[w, wt,] = \{w, wt\}$. From (PO2) we deduce that $wts \dot{\triangleleft} wt$.

Now we apply (PO3) (i) with $x = wt$ and $y = w$ and it follows that $ws \trianglerighteq wts$. It is only left to show that $[ws, wts] = \{ws, wts\}$.

Suppose there were a $z \in [ws, wts] \setminus \{ws, wts\}$. By (PO1), either $z \triangleleft zs$ or $zs \triangleleft z$. In the first case, by twice applying (PO3) (i) (once with $y = ws$ and $x = z$ and once with $y = z$ and $x = wts$) we obtain $zs \in [w, wt] = \{w, wt\}$, which is a contradiction, since we supposed $zs \neq ws, wts$. On the other hand, if $zs \triangleleft z$, we repeat the same argument as before, once z has been substituted by zs , and we get $z \in [w, wt]$. It follows that either $z = w$ or $z = wt$, which is not possible because $w \triangleleft ws$ and $wt \triangleleft wts$, while we assumed $zs \triangleleft z$. \square

For $z \trianglelefteq v \in \widehat{W}$, a *maximal chain* from z to v of length k is a sequence $z = z_0 \dot{\triangleleft} z_1 \dot{\triangleleft} \dots \dot{\triangleleft} z_{k-1} \dot{\triangleleft} z_k = v$, where $z_i \in \widehat{W}$.

Lemma 7.14. *Let $u, v \in \widehat{W}$ be such that $u \triangleleft v$ and let $s \in \widehat{\mathcal{S}}$ be such that $vs \triangleleft v$ and $us \triangleleft u$. Then for any $z \in (us, v] \setminus \{\trianglerighteq u\}$ which is minimal, any maximal chain from z to v of length k induces a maximal chain from us to v of the same length.*

Proof. By the minimality assumption, $us \triangleleft z = z_0$, and, by (PO1), there exists a $t_0 \in \widehat{\mathcal{T}}$ such that $z_0 = ust_0$. From Lemma 7.13 with $w = us$ and $t = t_0$, we get $z_0, u \triangleleft z_0s$. If $z_1 = z_0s$, we obtain a maximal chain

$$us \triangleleft z_0s \triangleleft \dots \triangleleft z_{k-1} \triangleleft z_k = v.$$

Otherwise, we again apply Lemma 7.13 with $z_0 = w$ and $t = t_1$. Once again, if $z_2 = z_1s$, we are done, and if not, we proceed as before. In this way we get a chain from us to v of length k .

Observe that all the $z_i s$ are still in the interval, since $z_i s \triangleright v$, by (PO3) (i) with $x = vs$ and $y = z_i$. □

For any $u, v \in \widehat{\mathcal{W}}$ with $w \triangleleft v$ let us denote by $d(u, v)$ the supremum of the lengths of all maximal chains from u to v .

Corollary 7.15. *Let $u, v \in \widehat{\mathcal{W}}$ be such that $u \triangleleft v$ and let $s \in \widehat{\mathcal{S}}$ be such that $vs \triangleleft v$ and $us \triangleleft u$. Let $z \in (us, v] \setminus \{\triangleright u\}$ not be minimal; then $d(z, v) < d(u, v)$.*

Proposition 7.16. *Let $N \in \mathcal{Z}'\text{-mod}^f$ be such that the adjunction morphism $N \mapsto \Gamma(\mathcal{L}(N))$ is an isomorphism. If N is \triangleleft -flabby, then also $\theta^{s,\text{out}}(N) \in \mathcal{Z}\text{-mod}^f$ is \triangleleft -flabby.*

Proof. Let us keep the same notation as before. We claim that, for any set $\mathcal{I} \subseteq \widehat{\mathcal{W}}$ upwardly closed wrt \triangleleft the canonical map $\Gamma(\mathcal{M}) \rightarrow \Gamma(\mathcal{I}_{\triangleleft}, \mathcal{M})$ is surjective.

If $\mathcal{I}s = \mathcal{I}$, the proof is very similar to the one of Lemma 7.10; therefore we omit it.

Let us set $\{\triangleright u\} = \{v \in \widehat{\mathcal{W}} \mid v \triangleright u\}$, and analogously for $\{\triangleright u\}$, by [Fie08a, Proposition 4.2] it is enough to demonstrate that for all $u \in \widehat{\mathcal{W}}$ the following map is surjective:

$$\Gamma(\{\triangleright u\}, \mathcal{M}) \rightarrow \Gamma(\{\triangleright u\}, \mathcal{M}).$$

Let us consider $m \in \Gamma(\{\triangleright u\}, \mathcal{M})$.

We want to show that it is possible to extend it to a section $\tilde{m} \in \Gamma(\{\triangleright u\}, \mathcal{M})$.

If $us \triangleright u$, then, from (PO3) (ii) with $x = us$ and $y = v$, if $v \triangleright u$ it holds $vs \triangleright u$ too. It follows that $\{\triangleright u\} \setminus \{u, us\}$ is s -invariant. Moreover it is upwardly closed because the interval $[u, us] = \{u, us\}$ by (PO2). Hence we can apply what we have already proven and get that if we restrict m to $\{\triangleright u\} \setminus \{u, us\}$ it extends to a global section. It is then enough to consider the case in which m is supported at most on the vertex us , that is, $m_x = 0$ for all $x \in \{\triangleright u\} \setminus \{us\}$. By definition of the pullback functor and Lemma 7.11, $\mathcal{M}^u = \mathcal{M}^{us}$, and so we can extend m to the section $\tilde{m} \in \{\triangleright u\}$, by the setting $\tilde{m}_u := m_{us}$.

Let us now assume $us \triangleleft u$ and consider the support of M , that is,

$$\text{supp}(\mathcal{M}) = \{v \in \widehat{\mathcal{W}} \mid \mathcal{M}^v \neq 0\}.$$

Observe that $\text{supp}(\mathcal{M})$ is finite, since otherwise $\Gamma(\mathcal{M})$ would not be a finitely generated S_k -module. Therefore, by Lemma 7.12, we find a $v \in \widehat{\mathcal{W}}$ such that $\text{supp}(\mathcal{M}) \subseteq \{\triangleleft v\}$. Let us observe that we may assume $vs \triangleright v$ without loss of generality.

Let us now proceed by induction on $d(u, v)$. The base step is $u = v$ and it is trivial. Otherwise, by Lemma 7.14, we may suppose by induction that m extends to any vertex $z \in (us, v] \setminus \{us, u\}$ which is not minimal. We get in this way a section m' .

Next, let us assume $z \triangleleft us$ and $z \neq u$. In this case $z \triangleleft zs$. Indeed, suppose it were not the case. By (PO1) z and zs are comparable and by (PO3) (ii) with $x = u$ and $y = z$, it holds that $us \trianglelefteq zs$. This contradicts the assumptions $z \triangleleft us$ and $z \neq u$. We have already proven that it is possible to extend $m'_{|\triangleright z}$ to the vertex z . Since the minimal vertices are not comparable, there are not edges between them, and hence we extended m' to a section $m'' \in \Gamma(\{\triangleright us\} \setminus \{u, us\}, \mathcal{M})$. But $\{\triangleright us\} \setminus \{u, us\}$ is now an s -stable set, and therefore m'' extends to a global section and in particular to the vertex u . \square

Finally, as $\theta^s = \theta^{s,out} \circ \theta^{s,on}$ and since Lusztig proved that the generic order \preceq also has properties (PO1), (PO2) and (PO3) (cf. [Lu80b]), we get the following theorem.

Theorem 7.17. $\theta^s : \mathcal{Z} - \text{mod}^f \rightarrow \mathcal{Z} - \text{mod}^f$ preserves both \leq -flabby and \preceq -flabby objects.

7.1.6. *Special modules.* Let $B_e \in \mathcal{Z} - \text{mod}^f$ be the free S_k -module of rank one on which $z = (z_x)_{x \in \widehat{W}}$ acts via multiplication by z_e .

Definition 7.18.

- (i) The category of (right) *special modules* is the full subcategory \mathcal{H} of $\mathcal{Z} - \text{mod}^f$ whose objects are isomorphic to a direct summand of a direct sum of modules of the form $\theta^{s_{i_1}} \circ \dots \circ \theta^{s_{i_r}}(B_e^J)\langle n \rangle$, where $s_{i_1}, \dots, s_{i_r} \in \widehat{S}$ and $n \in \mathbb{Z}$;
- (ii) the category of *parabolic special modules* is the full subcategory \mathcal{H}^{par} of $\mathcal{Z}^{\text{par}} - \text{mod}^f$ whose objects are isomorphic to a direct summand of a direct sum of modules of the form $\theta^{s_{i_1}} \circ \dots \circ \theta^{s_{i_r}}(B_e^J)\langle n \rangle$, where $s_{i_1}, \dots, s_{i_r} \in \widehat{S}$ and $n \in \mathbb{Z}$.

A fundamental characterisation of special modules is the following one.

Theorem 7.19 (cf. [Fie08b]). *Let $M \in \mathcal{Z} - \text{mod}^f$. Then $M \in \mathcal{H}$ if and only if it is isomorphic to the space of global sections of a Braden-MacPherson sheaf on \widehat{G} .*

Thanks to the above result, we are now able to prove the following, which will make clear why we are dealing with such objects.

Proposition 7.20. *Let \mathcal{F} be a Braden-MacPherson sheaf on \widehat{G} . Then \mathcal{F}^{per} is a flabby sheaf on \widehat{G}^{per} .*

Proof. We want to show that $F = \Gamma(\mathcal{F})$ is a flabby object in $\mathcal{Z}^{\text{per}} - \text{mod}^f$. By Theorem 7.19, we know that $F \in \mathcal{H}$, so we may prove our result by induction. If $F = B_e$, there is nothing to prove. We have to show now that, if the claim is true for $M \in \mathcal{H}$, then it holds also for $\theta_s(M) \in \mathcal{H}$, that, again by Theorem 7.19, is still isomorphic to the global sections of a Braden-MacPherson sheaf on \widehat{G} . But now by the inductive hypothesis we get that M is a flabby object in $\mathcal{Z}^{\text{per}} - \text{mod}^f$, and so, by applying Theorem 7.17, $\theta_s(M) = \theta_s^{\text{per}}(M)$ is also a flabby object in $\mathcal{Z}^{\text{per}} - \text{mod}^f$. \square

7.1.7. *Decomposition of the functor \cdot^{stab} .* The functor \cdot^{stab} may be obtained by composing the following five functors:

$$\mathbf{Sh}(\widehat{\mathcal{G}}_{|\mathcal{I}}^{\text{par}}) \xrightarrow{i_*} \mathbf{Sh}(\widehat{\mathcal{G}}^{\text{par}}) \xrightarrow{p_{\text{par}}^*} \mathbf{Sh}(\widehat{\mathcal{G}}) \xrightarrow{\cdot^{\text{per}}} \mathbf{Sh}(\widehat{\mathcal{G}}^{\text{per}}) \xrightarrow{j^*} \mathbf{Sh}(\widehat{\mathcal{G}}_{|\mathcal{I}}^{\text{per}}) \xrightarrow{\cdot^{\text{opp}}} \mathbf{Sh}(\widehat{\mathcal{G}}_{|\mathcal{I}}^{\text{stab}})$$

$\xrightarrow{\cdot^{stab}}$

Here

- $i : \widehat{\mathcal{G}}_{|\mathcal{I}}^{\text{par}} \hookrightarrow \widehat{\mathcal{G}}^{\text{par}}$ and $j : \widehat{\mathcal{G}}_{|\mathcal{I}}^{\text{per}} \hookrightarrow \widehat{\mathcal{G}}^{\text{per}}$ are the morphisms of moment graphs induced by the corresponding inclusions of subgraphs;
- $p_{\text{par}} : \mathcal{G} \rightarrow \widehat{\mathcal{G}}^{\text{par}}$ is the quotient morphism defined by $p_{\text{par},\nu} : x \mapsto x^{\text{par}}$, the minimal representative of $x\mathcal{W}$, and $p_{\text{par},l,x} = \text{id}_{\widehat{Q}^\vee}$ for all x ;
- \cdot^{opp} is the pullback of the isomorphism of moment graphs $f : \widehat{\mathcal{G}}_{|\mathcal{I}}^{\text{stab}} \rightarrow \widehat{\mathcal{G}}_{|\mathcal{I}}^{\text{per}}$ defined as $f_\nu = \text{id}_{\widehat{Q}^\vee}$ and $f_{l,x}(\lambda) = x^{-1}(\lambda)$ for all $x \in \mathcal{I}$ and $\lambda \in Q^\vee$.

Some properties of the above functors are needed.

Proposition 7.21. *The functors p_{par}^* and \cdot^{opp} preserve indecomposable BMP-sheaves.*

Proof. The fact that p_{par}^* maps indecomposable BMP-sheaves to indecomposable BMP-sheaves is just a particular case of Theorem 6.2 of [L11], in the reformulation we gave in [L14, Theorem 6.1].

In Lemma 5.2 of [L11], we proved that f defined as above is an isomorphism, and hence, by Lemma 5.14, the pullback \cdot^{opp} preserves Braden-MacPherson sheaves. \square

Theorem 7.22. *Let $\mathcal{F} \in \mathbf{Sh}(\widehat{\mathcal{G}}_{|\mathcal{I}}^{\text{par}})$ be a Braden-MacPherson sheaf; then $\mathcal{F}^{\text{stab}} \in \mathbf{Sh}(\widehat{\mathcal{G}}_{|\mathcal{I}}^{\text{stab}})$ is a flabby sheaf.*

Proof. The statement follows by combining results of this section, together with the fact that the functors i_* and j^* clearly map flabby sheaves to flabby sheaves. \square

7.2. Indecomposability. Here we demonstrate the only step missing in the proof of Theorem 6.2. From now on, we will denote by $\widehat{\mathcal{W}}^{\text{par}}$ the set of minimal representatives for the equivalence classes of $\widehat{\mathcal{W}}/\mathcal{W}$.

7.2.1. *Localisation of special \mathcal{Z}^{par} -modules.* Let $\beta \in \Delta_+$; we consider the following localisation of the symmetric algebra S_k :

$$(21) \quad S_k^\beta := S_k[(\nu\alpha + n\epsilon)^{-1} \mid \alpha \in \Delta_+ \setminus \{\beta\}, n \in \mathbb{Z}].$$

Fiebig used this localisation in [Fie11], in order to relate the category of regular special modules to a category introduced by Andersen, Jantzen and Soergel in [AJS].

Let us denote by $\widehat{\mathcal{W}}_\beta$ the subgroup of $\widehat{\mathcal{W}}$ generated by the affine reflections $s_{\beta,n}$, for $n \in \mathbb{Z}$, and by $\widehat{\mathcal{W}}^\beta$ the set of orbits for the left action of $\widehat{\mathcal{W}}_\beta$ on $\widehat{\mathcal{W}}^{\text{par}}$. Remark that the group $\widehat{\mathcal{W}}_\beta$ is isomorphic to the Weyl group of $\widehat{\mathfrak{sl}}_2$. For any subset $\Omega \subseteq \widehat{\mathcal{W}}^{\text{par}}$, let us write moreover $\mathcal{Z}^{\text{par},\beta}(\Omega) := \mathcal{Z}^{\text{par}}(\Omega) \otimes_{S_k} S_{k,\beta}$.

Lemma 7.23 (cf. [Fie11], Lemma 3.1). *Let $\Omega \subset \widehat{\mathcal{W}}^{par}$ be finite; then*

$$\begin{aligned} \mathcal{Z}^{par,\beta}(\Omega) &= \left\{ (z_x) \in \bigoplus_{x \in \Omega} S_k^\beta \mid \begin{array}{l} z_x \equiv z_y \pmod{(\beta + n\delta)^\vee} \\ \text{if } \exists w \in \mathcal{W}, n \in \mathbb{Z} \text{ s.t. } ywx^{-1} = s_{\beta+n\delta} \end{array} \right\} \\ &= \bigoplus_{\Theta \in \widehat{\mathcal{W}}^\beta} \mathcal{Z}^{par,\beta}(\Omega \cap \Theta). \end{aligned}$$

Proof. Omitted, since Fiebig’s proof of [Fie11, Lemma 3.1] works exactly the same in this parabolic setting too. □

For $M \in \mathcal{H}^{par}$, we set $M^\beta := M \otimes_{S_k} S_{k,\beta}$. Because any special module is a module on $\mathcal{Z}(\Omega)$ for some $\Omega \subset \widehat{\mathcal{W}}^{par}$ finite (see [L14, Lemma 4.3]), the decomposition of the previous lemma induces the following decomposition:

$$(22) \quad M^\beta = \bigoplus_{\Theta \in \widehat{\mathcal{W}}^\beta} M^{\beta,\Theta}.$$

Next we are going to show that this localisation procedure preserves special modules. In particular, we prove that, under the localisation, a special module having support on a finite interval deep enough in the fundamental chamber splits in a direct sum of special modules for the subgeneric parabolic structure algebra, that is, the one corresponding to the case $\widehat{\mathfrak{g}} = \mathfrak{sl}_2$.

Lemma 7.24. *Let $M \in \mathcal{H}^{par}$ such that \mathcal{Z}^{par} acts on it via $\mathcal{Z}^{par}(\mathcal{I})$, for \mathcal{I} a finite interval deep enough in \mathcal{C}^+ and $M^\beta = \bigoplus_{\Theta \in \widehat{\mathcal{W}}^\beta} M^{\beta,\Theta}$. Then, for any $\Theta \in \mathcal{W}^\beta$, $M^{\beta,\Theta}$ is isomorphic to a $\mathcal{Z}^{par}(\widehat{\mathfrak{sl}}_2)$ -special module.*

Proof. We prove by induction that any $M^{\beta,\Theta}$ is a special module for the structure algebra of $\widehat{\mathcal{G}}_{|\Theta}^{par}$. If $M = B_e$, there is nothing to prove. Suppose the lemma holds for $M \in \mathcal{H}^{par}$; we have to show that it is also true for ${}^s\theta^{par}(M) = \bigoplus_{\Theta \in \widehat{\mathcal{W}}^\beta} {}^s\theta^{par}(M)^{\beta,\Theta}$. Thus it is enough to demonstrate it for one module $M^{\beta,\Theta}$.

In order to do this, we follow the proof of [Fie11, Lemma 3.5]. If $\Theta = \Theta_s$, then ${}^s\theta^{par}(M)^{\beta,\Theta} = M^{\beta,\Theta} \otimes_{s\mathcal{Z}^{par,\beta}(\Theta)} \mathcal{Z}^{par,\beta}(\Theta)$, since, by [L14, Lemma 4.3], the inclusion ${}^s\mathcal{Z}^{par,\beta}(\Omega) \subset \mathcal{Z}^{par,\beta}(\Omega)$ contains ${}^s\mathcal{Z}^{par,\beta}(\Theta) \subset \mathcal{Z}^{par,\beta}(\Theta)$ as a direct summand. Otherwise, $\Theta \neq \Theta_s$ and the inclusion ${}^s\mathcal{Z}^{par,\beta}(\Theta \cup \Theta_s) \subset \mathcal{Z}^{par,\beta}(\Theta) \oplus \mathcal{Z}^{par,\beta}(\Theta_s)$ is an isomorphism on each direct summand. Therefore we get the equality ${}^s\theta^{par}(M)^{\beta,\Theta} = M^{\beta,\Theta} \oplus M^{\beta,\Theta_s}$. In both cases, we obtain the claim by applying the inductive hypothesis because $\mathcal{Z}^{par,\beta}$ acts on $M^{\beta,\Theta}$ via $\mathcal{Z}^{par,\beta}(\mathcal{I} \cap \Theta)$ and clearly $\mathcal{Z}^{par,\beta}(\mathcal{I} \cap \Theta) = \mathcal{Z}^{par}(\mathcal{I} \cap \Theta)$.

Now the statement follows because, by Lemma 4.5, for any finite interval \mathcal{I} deep enough in the fundamental chamber and any $\Theta \in \widehat{\mathcal{W}}^\beta$, the interval $\mathcal{I} \cap \Theta$ is isomorphic (as a moment graph) to a finite parabolic interval of the subgeneric moment graph. □

7.2.2. A property of indecomposable BMP-sheaves. Before proving our last result, we first have to recall an intrinsic property of indecomposable canonical sheaves, which has already been used by Braden and MacPherson in [BMP] to show the existence of these objects. We follow a reformulation due to Fiebig.

For any k -moment graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \triangleleft, l) \in \mathbf{MG}(Y_k)$ and for any $x \in \mathcal{V}$, let us consider the following sets (cf. [Fie08a, §4.2]):

$$\mathcal{E}_{\delta_x} := \{E \in \mathcal{E} \mid E : x \rightarrow y\}$$

and

$$\mathcal{V}_{\delta_x} := \{y \in \mathcal{V} \mid \exists E \in \mathcal{E}_{\delta_x} \text{ such that } E : x \rightarrow y\}.$$

Consider $\mathcal{F} \in \mathbf{Sh}_k(\mathcal{G})$ and define \mathcal{F}^{δ_x} to be the image of $\Gamma(\{\triangleright x\}, \mathcal{F})$ under the composition u_x of the following maps:

$$(23) \quad \Gamma(\{\triangleright x\}, \mathcal{F}) \hookrightarrow \bigoplus_{y \triangleright x} \mathcal{F}^y \longrightarrow \bigoplus_{y \in \mathcal{V}_{\delta_x}} \mathcal{F}^y \xrightarrow{\oplus \rho_{y,E}} \bigoplus_{E \in \mathcal{E}_{\delta_x}} \mathcal{F}^E$$

$\underbrace{\hspace{15em}}_{u_x}$

Moreover, let us set

$$d_x := (\rho_{x,E})_{E \in \mathcal{E}_{\delta_x}}^T : \mathcal{F}^x \longrightarrow \bigoplus_{E \in \mathcal{E}_{\delta_x}} \mathcal{F}^E.$$

If \mathcal{B} is an indecomposable *BMP*-sheaf, that is, $\mathcal{B} = \mathcal{B}(w)$ for some $w \in \mathcal{V}$, then conditions (BMP3) and (BMP4) of Definition 5.11 may be replaced by the following condition (cf. [BMP, Theorem 1.4]):

(BMP3') for all $x \in \mathcal{V}$, with $x \triangleleft w$, $d_x : \mathcal{B}(w)^x \rightarrow \mathcal{B}(w)^{\delta_x}$ is a projective cover in the category of graded S_k -modules.

Since a (graded) finitely generated S_k -module is projective if and only if it is free, $\mathcal{B}(w)^x$ is isomorphic to the graded free S_k -module with a minimal number of generators which map surjectively to $\mathcal{B}(w)^{\delta_x}$.

Proposition 7.25. *Let \mathcal{I} be a finite interval of $\widehat{\mathcal{G}}^{par}$ deep enough in \mathcal{C}^+ and let $\mathcal{B} \in \mathbf{Sh}(\widehat{\mathcal{G}}_{\mathcal{I}}^{par})$ be an indecomposable Braden-MacPherson sheaf. Then \mathcal{B}^{stab} is also indecomposable as a sheaf on $\widehat{\mathcal{G}}_{\mathcal{I}}^{stab}$.*

Proof. Since \mathcal{B} is indecomposable, by Theorem 5.12, $\mathcal{B} = \mathcal{B}(w)$ for some $w \in \mathcal{I}$, that implies $\mathcal{B}(w)^x = 0 = \mathcal{B}^{stab,x}$ for all $x > w$ ($x \in \mathcal{I}$) and $\mathcal{B}(w)^w \cong S_k \cong \mathcal{B}^{stab,w}$. Suppose that $\mathcal{B}^{stab} = \mathcal{C} \oplus \mathcal{D}$; then for what we have just observed, we may take \mathcal{C} and \mathcal{D} such that $\mathcal{C}^x = \mathcal{D}^x = 0$ for all $x > w$, $\mathcal{C}^w \cong S_k$ and $\mathcal{D}^w = 0$. Let $y \in \mathcal{I}$ be a maximal vertex such that $\mathcal{D}^y \neq 0$. For any $E : y \rightarrow z \in \mathcal{E}_{\delta_y}$, by the definition of the Braden-MacPherson sheaf, $\rho_{z,E} : \mathcal{B}(w)^z = \mathcal{B}^{stab,z} = \mathcal{C}^z \rightarrow \mathcal{B}^E = \mathcal{B}^{stab,E}$ is surjective with kernel $l(E) \cdot \mathcal{B}^z = l(E) \cdot \mathcal{C}^z$, and this implies $\mathcal{D}^E = 0$.

We now localise $\Gamma(\mathcal{B})$ at a finite simple root β . Remark that, since we are representing the parabolic Bruhat graph using alcoves, we are taking the quotient of \mathcal{G}^{opp} instead of \mathcal{G} . It means that we have to twist the action of S_k on any vertex x by x^{-1} . However, once the action of the symmetric algebra is twisted, the two previous results still work in the same way. By combining Theorem 7.19 and Lemma 7.24 we know that $\mathcal{L}(\Gamma(\mathcal{B})^\beta)$ is a direct sum of Braden-MacPherson sheaves on certain moment graphs, and each one of them is isomorphic to a finite interval of the subgeneric parabolic Bruhat graph. From the definition of \mathcal{L} , it follows that $\mathcal{L}(\Gamma(\mathcal{B}^{stab})^\beta) = (\mathcal{L}(\Gamma(\mathcal{B})^\beta))^{stab}$.

We have already proven that $\rho_{y,E}(\mathcal{D}^y) = 0$ for any $E \in \mathcal{E}_{\delta_y} \cap \mathcal{E}_S$, and we want to show that $\rho_{y,E}(\mathcal{D}^y) = 0$ for any $E \in \mathcal{E}_{\delta_y}$. If it were not the case, there would be a non-stable edge $F \in \mathcal{E}_{\delta_y} \cap \mathcal{E}_{NS}$ such that $\rho_{y,E}(\mathcal{D}^y) \neq 0$. Let $\beta \in \Delta_+$ be such that $l(F) = (\beta + n\delta)^\vee$ for some $n \in \mathbb{Z}$. Localising at β , we would get $\rho_{y,F}^\beta(\mathcal{D}^{y,\beta}) \neq 0$, and from the $\widehat{\mathfrak{sl}}_2$ -case, it follows that $\rho_{y,E}^\beta(\mathcal{D}^{y,\beta}) \neq 0$ for all $E \in \mathcal{E}_{\delta_y}$ in the β -direction, but we proved that this is not the case.

We are now ready to conclude. From what we showed, it follows that $u_y(\mathcal{C}^y) = \mathcal{B}^{\delta y}$, and this implies $\mathcal{D}^y = 0$, since (\mathcal{B}^y, u_y) is a projective cover of $\mathcal{B}^{\delta y}$. \square

APPENDIX A. FINITE PARABOLIC INTERVALS IN THE SUBGENERIC CASE

In this appendix we want to study the behaviour of indecomposable BMP-sheaves on finite intervals of $\widehat{\mathcal{G}}^{\text{par}}$ in the subgeneric case, that is, $\widehat{\mathfrak{g}} = \mathfrak{sl}_2$, for k a UFD. In particular, we want to interpret in the moment graph setting the fact that the corresponding parabolic Kazhdan-Lusztig polynomials are all trivial. More precisely, we want to show that the structure sheaf \mathcal{A} and the canonical sheaf \mathcal{B}^{par} are in this case isomorphic. Hence to prove our claim, it will be enough to define a surjective map $S_k \rightarrow \widehat{\mathcal{B}}^{\text{par}}(w)^{\delta x}$ for any pair of vertices $x < w$ in $\widehat{\mathcal{G}}^{\text{par}}$.

Recall that the set of vertices is in this case totally ordered, so we may enumerate the vertices as follows, once we have identified the finite root α with the corresponding coroot α^\vee : $v_0 = 0, v_1 = \alpha, v_2 = -\alpha, \dots, v_h = (-1)^{h+1}[\frac{h+1}{2}]\alpha, \dots$

From now on, we denote the edges as $E_{h,k} : v_h \text{ --- } v_k$ and the labels as $l_{h,k} := l(E_{h,k})$; we write moreover $l_{h,k} = \alpha + n_{h,k}c$. Actually, the label of an edge $E_{h,k}$ is by definition $\pm l_{h,k}$; however, there exists an isomorphic k -moment graph with the same sets of vertices and edges, but with this other label function and, by Lemma 5.14, the corresponding indecomposable canonical sheaves are isomorphic.

We will prove in several steps that, if $v_j \leq v_i$ and $(\widehat{\mathcal{G}}^{\text{par}}_{[[v_j, v_i]], k})$ is a GKM-pair, then $(\mathcal{B}^{\text{par}}(v_i))^{v_j} \cong S_k$ by induction on $i - j$.

Let us fix once and for all $\mathcal{I} = \{v_i, v_{i-1}, \dots, v_{j+1}\}$.

Lemma A.1. *Let $r \in \mathbb{N}$ be such that $r < i - j$. If $(\widehat{\mathcal{G}}^{\text{par}}, k)$ is a GKM-pair, and $z \in \Gamma(\mathcal{I}, \mathcal{B}^{\text{par}}(v_i))_{\{r\}}$, then z is uniquely determined by its first $r + 1$ components, that is, the restriction map*

$$\Gamma(\mathcal{I}, \mathcal{B}^{\text{par}}(v_i))_{\{r\}} \longrightarrow \Gamma(\{v_i, v_{i-1}, \dots, v_{i-r}\}, \mathcal{B}^{\text{par}}(v_i))_{\{r\}}$$

is injective.

Proof. Let $z \in \Gamma(\mathcal{I}, \mathcal{B}^{\text{par}}(v_i))_{\{r\}}$ such that $z_{v_i} = z_{v_{i-1}} = \dots = z_{v_{i-r}} = 0$. Observe that for any $j + 1 \leq h < i - r \leq k \leq i$ one has $z_{v_h} \equiv z_{v_k} \equiv 0 \pmod{-\alpha + n_{h,k}c}$.

From the GKM-property it follows that $GCD(-\alpha + n_{h,k}c, -\alpha + n_{h,l}c) = 1$ for any $i - r \leq k \neq l \leq i$. Since S_k is a UFD, the entry z_{v_h} has to be divisible by $(-\alpha + n_{h,i-r}c)(-\alpha + n_{h,i-r+1}c) \dots (-\alpha + n_{h,i}c)$. This is a polynomial of degree $r + 1$ while z_{v_h} was a polynomial of degree r , so $z_{v_h} = 0$. \square

Lemma A.2. *Let $r \in \mathbb{N}$ be such that $r < i - j$. We have $\dim_k \Gamma(\mathcal{I}, \mathcal{B}^{\text{par}}(v_i))_{\{r\}} = \binom{r+2}{2}$.*

Proof. By Lemma A.1,

$$\dim_k \Gamma(\mathcal{I}, \mathcal{B}^{\text{par}}(v_i))_{\{r\}} = \dim_k \Gamma(\{v_i, v_{i-1}, \dots, v_{i-r}\}, \mathcal{B}^{\text{par}}(v_i))_{\{r\}}.$$

Clearly, $\Gamma(\{v_i, v_{i-1}, \dots, v_{i-r}\}, \mathcal{B}^{\text{par}}(v_i))_{\{r\}} \subseteq \bigoplus_0^r (S_k)_{\{r\}}$ and $\dim_k \bigoplus_0^r (S_k)_{\{r\}} = (r + 1)^2$.

By definition an element $m \in \bigoplus_0^r (S_k)_{\{r\}}$ is in $\Gamma(\{v_i, v_{i-1}, \dots, v_{i-r}\}, \mathcal{B}^{\text{par}}(v_i))_{\{r\}}$ if it satisfies some (linear) conditions given by the labels of the edges. If we prove that such conditions are linearly independent, then we know that

$$\dim_k \Gamma(\{v_i, v_{i-1}, \dots, v_{i-r}\}, \mathcal{B}^{\text{par}}(v_i))_{\{r\}} = \dim_k \bigoplus_0^r (S_k)_{\{r\}} - \# \text{ edges.}$$

We noticed in §4.1.1 that in the $\widehat{\mathfrak{sl}}_2$ -case all the vertices are connected, so the number of edges is equal to the number of pairs of different elements in a set with $r + 1$ elements, that is, $\binom{r+1}{2}$. Then,

$$\dim_k \Gamma(\{v_i, v_{i-1}, \dots, v_{i-r}\}, \mathcal{B}^{\text{par}}(v_i))_{\{r\}} = (r + 1)^2 - \binom{r + 1}{2} = \binom{r + 2}{2}.$$

It is left to show that the conditions are linearly independent.

Let $i - r \leq h < k \leq i$ and define the element $(m^{(h,k)}) \in \bigoplus_0^r (S_k)_{\{r\}}$ in the following way:

$$m_{v_l}^{(h,k)} := \begin{cases} c \prod_{m \in \{i, i-1, \dots, i-r\} \setminus \{h,k\}} (-\alpha + n_{h,m} c) & \text{if } l = h, \\ 0 & \text{otherwise.} \end{cases}$$

Now $m_{v_l}^{(h,k)} = m_{v_m}^{(h,k)}$ for any $l, m \neq h$ and $c \prod (-\alpha + n_{h,m} c) \equiv 0 \pmod{\alpha + n_{h,m} c}$. By the GKM-property, $l_{h,k}$ does not divide $m_{v_h}^{(h,k)}$, while $m_{v_k}^{(h,k)} = 0$.

So for any condition coming from the edge $E_{l,m}$ we built an $r + 1$ -tuple which verifies all conditions except the $E_{l,m}$ -th. It follows that all conditions are linearly independent. \square

Let us denote by $m_\alpha, m_c \in \Gamma(\mathcal{I}, \mathcal{B}^{\text{par}}(v_i))_{\{1\}}$ the constant sections $m_{\alpha,v} = \alpha$, $m_{c,v} = c$ for all $v \in \mathcal{I}$. Denote moreover by $u_{v_j} := \bigoplus \rho_{v_h, E_{h,j}}$, where $\rho_{v_h, E_{h,j}} : S_k \rightarrow S_k / (E_{h,j} \cdot S_k)$ are the canonical quotient maps.

Lemma A.3. *Let $r \in \mathbb{N}$ and let $(\widehat{\mathcal{G}}^{\text{par}}, k)$ be a GKM-pair. The vector subspace of $(\mathcal{B}_{v_i}^{\text{par}})^{v_j}$ generated by*

$$u_{v_j}(m_\alpha^r), u_{v_j}(m_\alpha^{r-1} m_c) \dots u_{v_j}(m_\alpha m_c^{r-1}), u_{v_j}(m_c^r)$$

has dimension equal to $r + 1$ if $r < i - j$ or dimension equal to $i - j$ otherwise.

Proof. At first notice that $(\mathcal{B}^{\text{par}}(v_i))^{E_{j,k}} = S_k / (l_{j,k} \cdot S_k) \cong k[c]$ by the mapping $\alpha \mapsto n_{j,k} c$. Then

$$u_{v_j}(m_\alpha^k m_c^{t-k}) = (n_{j,i}^k, n_{j,i-1}^k, \dots, n_{j,j+1}^k) c.$$

We obtain the following matrix:

$$N = \begin{pmatrix} 1 & 1 & \dots & 1 \\ n_{j,i}^k & n_{j,i-1}^k & \dots & n_{j,j+1}^k \\ n_{j,i}^{2k} & n_{j,i-1}^{2k} & \dots & n_{j,j+1}^{2k} \\ \vdots & \vdots & & \vdots \\ n_{j,i}^t & n_{j,i-1}^t & \dots & n_{j,j+1}^t \end{pmatrix}.$$

By the GKM-property it follows that $n_{j,k} \neq n_{j,h}$ for all pair $j + 1 \leq k \neq h \leq i$ and N is a Vandermonde matrix. In particular, such a matrix is not singular and so it has maximal rank, that is, $rk(N) = t + 1$ if $t < i - j$ and $rk(N) = i - j$ otherwise. \square

Lemma A.4. *There exists a section $m_0 \in \Gamma(\mathcal{I}, \mathcal{B}^{\text{par}}(v_i))_{\{1\}}$ such that $u_{v_j}(m_0) = 0$ and $m_{0,v} \neq 0$ for all $v \in \mathcal{I}$.*

Proof. Let $v_j = r\alpha$. Define $m_{0,v_h} := (r - s)l_{j,h} = (r - s)(-\alpha + (r + s)c)$ if $v_h = s\alpha$.

Notice that $(m_0) \in \Gamma(\mathcal{I}, \mathcal{B}^{\text{par}}(v_i))$; indeed for any pair of vertices $v_h = s\alpha$, $v_k = t\alpha$, one has $l_{h,k} = \alpha + (s+t)c$ and

$$\begin{aligned} m_{0,v_h} - m_{0,v_k} &= (r-s)(-\alpha + (r+s)c) - (r-t)(-\alpha + (r+t)c) \\ &= s\alpha - s^2c - t\alpha + t^2c \\ &= \alpha(s-t) + c(t^2 - s^2) \\ &= (s-t)(-\alpha + (s+t)c) \\ &\equiv 0 \pmod{-\alpha + (s+t)c}. \end{aligned}$$

Finally, by definition, $m_{0,v_h} \neq 0$ for any $v_h \in \mathcal{I}$ and $u_{v_j}((m_0)) = 0$. □

Lemma A.5. *Let $r \in \mathbb{N}$ be such that $r < i - j$. The collection of monomials $\{m_\alpha^l m_c^h m_0^k \mid l, h, k \geq 0, l + h + k = r\}$ is a basis of $\Gamma(\mathcal{I}, \mathcal{B}^{\text{par}}(v_i))_{\{r\}}$ as a k -vector space.*

Proof. Since the number of monomials in three variables of degree r is $\binom{r+2}{2}$ and by Lemma A.2 $\dim_k \Gamma(\mathcal{I}, \mathcal{B}^{\text{par}}(v_i)) = \binom{r+2}{2}$ as well, it is enough to prove that all monomials in m_α, m_c, m_0 are linearly independent. We prove the claim by induction on r .

Let $r = 1$. If $xm_\alpha + ym_c + zm_0 = 0$, then clearly $0 = u_{v_j}(xm_\alpha + ym_c + zm_0) = xu_{v_j}(m_\alpha) + yu_{v_j}(m_c) + zu_{v_j}(m_0)$. By Lemma A.4 $u_{v_j}(m_0) = 0$, so $xu_{v_j}(m_\alpha) + yu_{v_j}(m_c) = 0$. But by Lemma A.3 $u_{v_j}(m_\alpha)$ and $u_{v_j}(m_c)$ generate a vector space of dimension 2; then $x = y = 0$. Finally, from $zm_0 = 0$ and Lemma A.4 it follows that $z = 0$.

Now let $r > 1$. Let $z = \sum_{l+m+n=r} x_{l,m,n} m_\alpha^l m_c^m m_0^n = 0$. We can write $z = z_1 + z_0 m_0$, where z_1 is such that m_0 does not appear. Then by Lemma A.3 $u_{v_j}(z) = u_{v_j}(z_1) + u_{v_j}(z_0)u_{v_j}(m_0) = u_{v_j}(z_1) = 0$. From Lemma A.3 we know that all $u_{v_j}(m_\alpha^l m_c^{r-l})$ are linearly independent, and so the chain of equalities

$$0 = u_{v_j}(z_1) = u_{v_j}\left(\sum_{l+m=r} x_{l,m,0} m_\alpha^l m_c^m\right) = \sum_{l+m=r} x_{l,m,0} u_{v_j}(m_\alpha^l m_c^m)$$

implies $x_{l,m,0} = 0$ for all pair l, m , that is, $c_1 = 0$. Thus we obtain $c_0 m_0 = 0$ and we conclude by Lemma A.4 that $c_0 = 0$. Finally,

$$0 = c_0 = \sum_{l+m+n=r-1} x_{l,m,n+1} m_\alpha^l m_c^m m_0^n$$

is a linear combination of monomials in m_α, m_c, m_0 of degree $r - 1$, and so by the inductive hypothesis we have $x_{l,m,n+1} = 0$ for all l, m, n . □

Theorem A.6. *If $v_j \leq v_i$ and $(\widehat{\mathcal{G}}_{[v_j, v_i]}^{\text{par}}, k)$ is a GKM-pair, then $(\mathcal{B}^{\text{par}}(v_i))^{v_j} \cong S_k$.*

Proof. We prove that $(\mathcal{B}^{\text{par}}(v_i))^{\delta v_i}$ coincides with the u_{v_i} image of the ring generated by m_α and m_c . If $r < i - j$, by Lemma A.5, $\Gamma(\mathcal{I}, \mathcal{B}^{\text{par}}(v_i))_{\{r\}}$ is generated by $\{m_\alpha^l m_c^h m_0^k \mid l, h, k \geq 0, l + h + k = r\}$. From Lemma A.4 it follows that $(\mathcal{B}^{\text{par}}(v_i))^{\delta v_i} = u_{v_i}(\Gamma(\mathcal{I}, \mathcal{B}^{\text{par}}(v_i))_{\{r\}})$ is contained in the ring generated by $u_{v_i}((m_\alpha))$ and $u_{v_i}((m_c))$.

Otherwise, $r \geq i - j$ and $\bigoplus_{E_{i,k} \in \mathcal{E}_{\delta v_i}} (\mathcal{B}^{\text{par}}(v_i))^{E_{i,k}} \cong k[c]^{i-j}$, having dimension $i - j$. Then by Lemma A.3 $u_{v_i}(m_\alpha)$ and $u_{v_i}(m_c)$ generate $(\mathcal{B}^{\text{par}}(v_i))^{\delta v_i}$.

Thus we have a surjective map $S_k \rightarrow \bigoplus_{E_{i,k} \in \mathcal{E}_{\delta v_i}} (\mathcal{B}^{\text{par}}(v_i))^{E_{i,k}}$ given by the mapping $\alpha \mapsto m_\alpha$, and we conclude that $c \mapsto m_c$ and $(\mathcal{B}^{\text{par}}(v_i))^{v_j} \cong S_k$. □

ACKNOWLEDGEMENTS

The author wishes to thank Peter Fiebig for encouraging her to look at the stable moment graph and for sharing his ideas. The author would like to acknowledge Rocco Chirivì, whose suggestions were always very useful. The author also owes many thanks to Vladimir Shchigolev for his careful reading of a very preliminary version of this paper.

REFERENCES

- [AJS] H. H. Andersen, J. C. Jantzen, and W. Soergel, *Representations of quantum groups at a p th root of unity and of semisimple groups in characteristic p : independence of p* (English, with English and French summaries), *Astérisque* **220** (1994), 321. MR1272539 (95j:20036)
- [AF08] Tomoyuki Arakawa and Peter Fiebig, *On the restricted Verma modules at the critical level*, *Trans. Amer. Math. Soc.* **364** (2012), no. 9, 4683–4712, DOI 10.1090/S0002-9947-2012-05467-5. MR2922606
- [AF09] Tomoyuki Arakawa and Peter Fiebig, *The linkage principle for restricted critical level representations of affine Kac-Moody algebras*, *Compos. Math.* **148** (2012), no. 6, 1787–1810, DOI 10.1112/S0010437X12000395. MR2999304
- [BB] Alexandre Beilinson and Joseph Bernstein, *Localisation de g -modules* (French, with English summary), *C. R. Acad. Sci. Paris Sér. I Math.* **292** (1981), no. 1, 15–18. MR610137 (82k:14015)
- [BjBr] Anders Björner and Francesco Brenti, *Combinatorics of Coxeter groups*, Graduate Texts in Mathematics, vol. 231, Springer, New York, 2005. MR2133266 (2006d:05001)
- [Bou] N. Bourbaki, *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines* (French), *Actualités Scientifiques et Industrielles*, No. 1337, Hermann, Paris, 1968. MR0240238 (39 #1590)
- [BMP] Tom Braden and Robert MacPherson, *From moment graphs to intersection cohomology*, *Math. Ann.* **321** (2001), no. 3, 533–551, DOI 10.1007/s002080100232. MR1871967 (2003g:14030)
- [BK] Jean-Luc Brylinski and Masaki Kashiwara, *Démonstration de la conjecture de Kazhdan-Lusztig sur les modules de Verma* (French, with English summary), *C. R. Acad. Sci. Paris Sér. A-B* **291** (1980), no. 6, A373–A376. MR596075 (81k:17004)
- [Deo] Vinay V. Deodhar, *On some geometric aspects of Bruhat orderings. II. The parabolic analogue of Kazhdan-Lusztig polynomials*, *J. Algebra* **111** (1987), no. 2, 483–506, DOI 10.1016/0021-8693(87)90232-8. MR916182 (89a:20054)
- [Dy91] Matthew Dyer, *On the “Bruhat graph” of a Coxeter system*, *Compositio Math.* **78** (1991), no. 2, 185–191. MR1104786 (92c:20076)
- [Dy93] Matthew Dyer, *Hecke algebras and shellings of Bruhat intervals*, *Compositio Math.* **89** (1993), no. 1, 91–115. MR1248893 (95c:20053)
- [FF90] Boris Feigin and Edward V. Frenkel, *Affine Kac-Moody algebras and semi-infinite flag manifolds*, *Comm. Math. Phys.* **128** (1990), no. 1, 161–189. MR1042449 (92f:17026)
- [FF92] Boris Feigin and Edward Frenkel, *Affine Kac-Moody algebras at the critical level and Gelfand-Dikiĭ algebras*, *Infinite analysis, Part A, B* (Kyoto, 1991), *Adv. Ser. Math. Phys.*, vol. 16, World Sci. Publ., River Edge, NJ, 1992, pp. 197–215. MR1187549 (93j:17049)
- [Fie08a] Peter Fiebig, *Sheaves on moment graphs and a localization of Verma flags*, *Adv. Math.* **217** (2008), no. 2, 683–712, DOI 10.1016/j.aim.2007.08.008. MR2370278 (2008m:17044)
- [Fie08b] Peter Fiebig, *The combinatorics of Coxeter categories*, *Trans. Amer. Math. Soc.* **360** (2008), no. 8, 4211–4233, DOI 10.1090/S0002-9947-08-04376-6. MR2395170 (2009g:20087)
- [FieNotes] Peter Fiebig, *Moment graphs in representation theory and geometry*, *Notes* (2009), 32 pp.
- [Fie10a] Peter Fiebig, *The multiplicity one case of Lusztig’s conjecture*, *Duke Math. J.* **153** (2010), no. 3, 551–571, DOI 10.1215/00127094-2010-031. MR2667425 (2011g:20069)

- [Fie10b] Peter Fiebig, *Lusztig’s conjecture as a moment graph problem*, Bull. Lond. Math. Soc. **42** (2010), no. 6, 957–972, DOI 10.1112/blms/bdq058. MR2740015 (2012c:20120)
- [Fie11] Peter Fiebig, *Sheaves on affine Schubert varieties, modular representations, and Lusztig’s conjecture*, J. Amer. Math. Soc. **24** (2011), no. 1, 133–181, DOI 10.1090/S0894-0347-2010-00679-0. MR2726602 (2012a:20072)
- [Fie13] Peter Fiebig, *On the subgeneric restricted blocks of affine category \mathcal{O} at the critical level*, Symmetries, integrable systems and representations, Springer Proc. Math. Stat., vol. 40, Springer, Heidelberg, 2013, pp. 65–84, DOI 10.1007/978-1-4471-4863-0-4. MR3077681
- [FW] Peter Fiebig, Geordie Williamson, *Parity sheaves, moment graphs and the p -smooth locus of Schubert varieties*, to appear in Ann. Inst. Four.
- [GKM] Mark Goresky, Robert Kottwitz, and Robert MacPherson, *Equivariant cohomology, Koszul duality, and the localization theorem*, Invent. Math. **131** (1998), no. 1, 25–83, DOI 10.1007/s002220050197. MR1489894 (99c:55009)
- [H] James E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge University Press, Cambridge, 1990. MR1066460 (92h:20002)
- [IM] N. Iwahori and H. Matsumoto, *On some Bruhat decomposition and the structure of the Hecke rings of p -adic Chevalley groups*, Inst. Hautes Études Sci. Publ. Math. **25** (1965), 5–48. MR0185016 (32 #2486)
- [J] Jens Carsten Jantzen, *Moment graphs and representations* (English, with English and French summaries), Geometric methods in representation theory. I, Sémin. Congr., vol. 24, Soc. Math. France, Paris, 2012, pp. 249–341. MR3202705
- [JMW] Daniel Juteau, Carl Mautner, and Geordie Williamson, *Parity sheaves*, J. Amer. Math. Soc. **27** (2014), no. 4, 1169–1212, DOI 10.1090/S0894-0347-2014-00804-3. MR3230821
- [Kac] Victor G. Kac, *Infinite-dimensional Lie algebras*, Progress in Mathematics, vol. 44, Birkhäuser Boston Inc., Boston, MA, 1983. An introduction. MR739850 (86h:17015)
- [KT] Masaki Kashiwara and Toshiyuki Tanisaki, *Kazhdan-Lusztig conjecture for affine Lie algebras with negative level*, Duke Math. J. **77** (1995), no. 1, 21–62, DOI 10.1215/S0012-7094-95-07702-3. MR1317626 (96j:17016)
- [Ka] Shin-ichi Kato, *On the Kazhdan-Lusztig polynomials for affine Weyl groups*, Adv. in Math. **55** (1985), no. 2, 103–130, DOI 10.1016/0001-8708(85)90017-9. MR772611 (86d:20048)
- [KL79] David Kazhdan and George Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), no. 2, 165–184, DOI 10.1007/BF01390031. MR560412 (81j:20066)
- [KL80] David Kazhdan and George Lusztig, *Schubert varieties and Poincaré duality*, Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979), Proc. Sympos. Pure Math., XXXVI, Amer. Math. Soc., Providence, R.I., 1980, pp. 185–203. MR573434 (84g:14054)
- [KL93] David Kazhdan, George Lusztig, Tensor structures arising from affine Lie algebras. I–IV, *J. Amer. Math. Soc.* **6** (1993), 905–947; *J. Amer. Math. Soc.* **6** (1993), 949–1011; *J. Amer. Math. Soc.* **7** (1994), 335–381; *J. Amer. Math. Soc.* **7** (1994), 383–453. MR1186962; MR1239506; MR1239507
- [Ku] Shrawan Kumar, *Kac-Moody groups, their flag varieties and representation theory*, Progress in Mathematics, vol. 204, Birkhäuser Boston Inc., Boston, MA, 2002. MR1923198 (2003k:22022)
- [L11] Martina Lanini, *Kazhdan-Lusztig combinatorics in the moment graph setting*, J. Algebra **370** (2012), 152–170, DOI 10.1016/j.jalgebra.2012.07.023. MR2966832
- [L14] Martina Lanini, *Categorification of a parabolic Hecke module via sheaves on moment graphs*, Pacific J. Math. **271** (2014), no. 2, 415–444, DOI 10.2140/pjm.2014.271.415. MR3267535
- [Lu80a] George Lusztig, *Some problems in the representation theory of finite Chevalley groups*, The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979), Proc. Sympos. Pure Math., vol. 37, Amer. Math. Soc., Providence, R.I., 1980, pp. 313–317. MR604598 (82i:20014)

- [Lu80b] George Lusztig, *Hecke algebras and Jantzen's generic decomposition patterns*, Adv. in Math. **37** (1980), no. 2, 121–164, DOI 10.1016/0001-8708(80)90031-6. MR591724 (82b:20059)
- [Lu83] George Lusztig, *Singularities, character formulas, and a q -analog of weight multiplicities*, Analysis and topology on singular spaces, II, III (Luminy, 1981), Astérisque, vol. 101, Soc. Math. France, Paris, 1983, pp. 208–229. MR737932 (85m:17005)
- [Lu90] George Lusztig, *Intersection cohomology methods in representation theory*, II (Kyoto, 1990), Math. Soc. Japan, Tokyo, 1991, pp. 155–174. MR1159211 (93e:20059)
- [Soe97] Wolfgang Soergel, *Kazhdan-Lusztig polynomials and a combinatoric[s] for tilting modules*, Represent. Theory **1** (1997), 83–114 (electronic), DOI 10.1090/S1088-4165-97-00021-6. MR1444322 (98d:17026)
- [Soe07] Wolfgang Soergel, *Kazhdan-Lusztig-Polynome und unzerlegbare Bimoduln über Polynomringen* (German, with English and German summaries), J. Inst. Math. Jussieu **6** (2007), no. 3, 501–525, DOI 10.1017/S1474748007000023. MR2329762 (2009c:20009)
- [Str04] Catharina Stroppel, *A structure theorem for Harish-Chandra bimodules via coinvariants and Golod rings*, J. Algebra **282** (2004), no. 1, 349–367, DOI 10.1016/j.jalgebra.2004.07.037. MR2097586 (2005g:17020)

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF MELBOURNE, PARKVILLE, VICTORIA 3010, AUSTRALIA

E-mail address: `martina.lanini@unimelb.edu.au`