EXTERNALLY DEFINABLE SETS AND DEPENDENT PAIRS II

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Abstract. We continue investigating the structure of externally definable sets in NIP theories and preservation of NIP after expanding by new predicates. Most importantly: types over finite sets are uniformly definable; over a model, a family of non-forking instances of a formula (with parameters ranging over a type-definable set) can be covered with finitely many invariant types; we give some criteria for the boundedness of an expansion by a new predicate in a distal theory; naming an arbitrary small indiscernible sequence preserves NIP, while naming a large one doesn’t; there are models of NIP theories over which all 1-types are definable, but not all n-types.

Introduction

A characteristic property of stable theories is the definability of types. Equivalently, every externally definable set is internally definable. In unstable theories this is no longer true. However, as was observed early on by Shelah (e.g. [20]), the class of externally definable sets in NIP theories satisfies some nice properties resembling those in the stable case (e.g. it is closed under projection). In this paper we continue the investigation of externally definable sets in NIP theories started in [8].

As it was established there, every externally definable set $X = \phi(x, b) \cap A$ has an honest definition, which can be seen as the existence of a uniform family of internally definable subsets approximating $X$. Formally, there is $\theta(x, z)$ such that for any finite $A_0 \subseteq X$ there is some $c \in A$ satisfying $A_0 \subseteq \theta(A, c) \subseteq X$. The first section of this paper is devoted to establishing the existence of uniform honest definitions. By uniform we mean that $\theta(x, z)$ can be chosen depending just on $\phi(x, y)$ and not on $A$ or $b$. We achieve this assuming that the whole theory is NIP, combining careful use of compactness with a strong combinatorial result of Alon-Kleitman [2] and Matoušek [17]: the $(p, k)$-theorem. As a consequence we conclude that in an NIP theory types over finite sets are uniformly definable (UDTFS). This confirms a conjecture of Laskowski.

In the next section we consider an implication of the $(p, k)$-theorem for forking in NIP theories. Combined with the results on forking and dividing in NIP theories from [7], we deduce the following: working over a model $M$, let $\{\phi(x, a) : a \models q(y)\}$ be a family of non-forking instances of $\phi(x, y)$, where the parameter $a$ ranges over...
the set of solutions of a partial type \( q \). Then there are finitely many global \( M \)-invariant types such that each \( \phi(x, a) \) from the family belongs to one of them.

In Section 3 we return to the question of naming subsets with a new predicate. In [8] we gave a general condition for the expansion to be NIP: it is enough that the theory of the pair is bounded, i.e., eliminates quantifiers down to the predicate, and the induced structure on the predicate is NIP. Here, we try to complement the picture by providing a general sufficient condition for the boundedness of the pair. In the stable case the situation is quite neatly resolved using the notion of nfcp. However nfcp implies stability, so one has to come up with some generalization of it that is useful in unstable NIP theories. Towards this purpose we introduce dnfcp, i.e., no finite cover property for definable sets of parameters, and its relative version with respect to a set. We also introduce dnfcp' — a weakening of dnfcp with separated variables. Using it, we succeed in the distal, stably embedded, case: if one names a subset of \( M \) which is small, uniformly stably embedded and the induced structure satisfies dnfcp', then the pair is bounded.

In Section 4 we look at the special case of naming an indiscernible sequence. On the one hand, we complement the result in [8] by showing that naming a small indiscernible sequence of arbitrary order type is bounded and preserves NIP. On the other hand, naming a large indiscernible sequence does not.

In the last section we consider models over which all types are definable. While in general even o-minimal theories may not have such models, many interesting NIP theories do (RCF, ACVF, Th(\( \mathbb{Q}_p \)), Presburger arithmetic, etc.). In practice, it is often much easier to check definability of 1-types, as opposed to \( n \)-types, so it is natural to ask whether one implies the other. Unfortunately, this is not true – we give an NIP counterexample. Can anything be said on the positive side? Pillay [18] had established: let \( M \) be NIP and let \( A \subseteq M \) be definable with rosy induced structure. Then if it is 1-stably embedded, it is stably embedded. We observe that Pillay’s results hold when the definable set \( A \) is replaced with a model, assuming that it is uniformly 1-stably embedded. This provides a generalization of the classical theorem of Marker and Steinhorn about definability of types over models in o-minimal theories. We also remark that in NIP theories, there are arbitrary large models with “few” types over them (i.e. such that \( |S(M)| \leq |M|^{|T|} \)).

Preliminaries

0.1. VC-dimension, co-dimension and density. Let \( \mathcal{F} \) be a family of subsets of some set \( X \). Given \( A \subseteq X \), we say that it is shattered by \( \mathcal{F} \) if for every \( A' \subseteq A \) there is some \( S \in \mathcal{F} \) such that \( A \cap S = A' \).

A family \( \mathcal{F} \) is said to have finite VC-dimension if there is some \( n \in \omega \) such that no subset of \( X \) of size \( n \) can be shattered by \( \mathcal{F} \). In this case we let \( VC(\mathcal{F}) \) be the largest integer \( n \) such that some subset of \( X \) of size \( n \) is shattered by it.

The VC co-dimension of \( \mathcal{F} \) is the largest integer \( n \) for which there are \( S_1, \ldots, S_n \in \mathcal{F} \) such that for any \( u \subseteq n \) there is \( b_u \in X \) satisfying \( b_u \in S_i \iff i \in u \). It is well known that \( coVC(\mathcal{F}) < 2^{VC(\mathcal{F})+1} \).

0.2. NIP and alternation. We are working in a monster model \( M \) of a complete first-order theory \( T \).

Recall that a formula \( \phi(x, y) \) is NIP if there are no \( (a_t)_{t \in \omega} \) and \( (b_s)_{s \subseteq \omega} \) such that \( \phi(a_t, b_s) \iff t \in s \). Equivalently, for any indiscernible sequence \( (a_t)_{t \in I} \) and \( b \), there
Lemma 5. Assume that $a_{t_i}$ be an indiscernible sequence and let $E$ be a convex equivalence relation on $I$. If $t = (t_i)_{i < k}$ and $s = (s_i)_{i < k}$ are tuples of elements from $I$, we will write $t \sim_E s$ if $t$ and $s$ have the same quantifier-free order type and $t_iEs_i$ for all $i < k$.

Fact 1. Let $(a_{t_i})_{i \in I}$ be an indiscernible sequence and let $b$ be any finite tuple. Let $\phi(x_0, ..., x_n; y)$ be NIP. Then there is a convex equivalence relation $E$ on $I$ with finitely many classes such that for any $(s_i)_{i \leq n} E (t_i)_{i \leq n}$ from $I$ we have $\phi(a_{t_0}, ..., a_{t_n}; b) \leftrightarrow \phi(a_{t_0}, ..., a_{t_n}; b)$.

Remark 2. In particular, if $I$ is a complete linear order and $\phi(x_0, ..., x_n; y)$ is NIP, then all $\phi$-types over $I$ are definable, possibly after adding finitely many elements extending $I$ on both sides. Why? If $I$ is totally indiscernible, then all $\phi$-types over $I$ are in fact definable using just equality. If it is not, then there is some formula giving the order on the sequence, and by Fact 1 $\phi$-types over $I$ are definable using this order (see [8, Section 3.1]).

In a natural way we define the VC-dimension of a formula in a model $M$ as $\text{VC}(\phi(x, y)) = \text{VC}\{\phi(M, a) : a \in M^n\}$. Notice that this value does not depend on the model, so we’ll talk about VC-dimension of $\phi$ in $T$. Similarly we define VC co-dimension.

It was observed early on by Laskowski that $\phi(x, y)$ is NIP if and only if it has finite VC-dimension and if and only if it has finite VC co-dimension [14]. We also recall an early result of Shelah about counting types over finite sets.

Fact 3 (Shelah/Sauer). The following are equivalent:

1. $\phi(x, y)$ is NIP.
2. There are $k, d \in \omega$ such that for all finite $A$, $|S_\phi(A)| \leq d \cdot |A|^k$.

Then one defines the VC density of $\phi$ to be the infimum of all reals $r$ such that for some $d$, $|S_\phi(A)| \leq d \cdot |A|^r$ for all finite $A$.

0.3. Invariant types. Let $p(x)$ be a global type over a monster model $\mathbb{M}$, invariant over some small submodel $M$. Then one naturally defines $p^{(\omega)}(x) \in S_\omega(\mathbb{M})$, the type of a Morley sequence in it (see [12, Section 2] for details).

Fact 4. Let $T$ be NIP. Assume that $p(x), q(x)$ are global types invariant over a small model $M$. If $p^{(\omega)}|_M = q^{(\omega)}|_M$, then $p = q$.

We will use the following lemma; see [22, Lemma 2.18] for a proof.

Lemma 5. Assume that $T$ is NIP. Let $a$ be given and $q(x) \in S(A')$ be invariant over $C \subseteq A'$. Then there is $D$ of size $\leq |C| + |x| + |a| + |T|$ such that $C \subseteq D \subseteq A'$ and for any $b, b' \in A'$ realizing $q(x)|D$, $\text{tp}(ab/D) = \text{tp}(a'b'/D)$.

0.4. $(p, k)$-theorem. We will need the following theorem from [17].

Fact 6 ($(p, k)$-theorem). Let $F$ be a family of subsets of some set $X$. Assume that the VC co-dimension of $F$ is bounded by $k$. Then, for every $p \geq k$, there is an integer $N$ such that: for every finite subfamily $G \subseteq F$, if $G$ has the $(p, k)$-property meaning that among any $p$ subsets of $G$ some $k$ intersect, then there is an $N$-point set intersecting all members of $G$. 

Remark 7. Although the theorem is stated this way in [17], $N$ depends only on $p$ and $k$ and not on the family $\mathcal{F}$. To see this, assume that, for every $N$, we had a family $\mathcal{F}_N$ on some set $X_N$ of VC co-dimension bounded by $k$ and for which the $(p,k)$-theorem fails for this $N$. Then consider $X$ to be the disjoint union of the sets $X_N$ and $\mathcal{F}$ the union of the families $\mathcal{F}_N$. Then clearly $\mathcal{F}$ has VC co-dimension bounded by $k$ and the theorem fails for it. Also, it follows from the proof.

0.5. **Expansions and stable embeddedness.** Let $A$ be a subset of $M \models T$ and let $L_P = L \cup \{ P(x) \}$, where $P(x)$ is a new unary predicate. We define the structure $(M, A)$ as the expansion of $M$ to an $L_P$-structure where $P(M) = A$. Recall that $\text{Th}(M, A)$ is $P$-bounded if every $L_P$ formula is equivalent to one of the form

$$Q_1 y_1 \in P \ldots Q_n y_n \in P \phi(x, \bar{y}),$$

where $Q_i \in \{\exists, \forall\}$ and $\phi$ is an $L$-formula. We may just say bounded when it creates no confusion.

Given $A \subseteq M \models T$ and a set of formulas $F$, possibly with parameters, we let $A_{\text{ind}(F)}$ be the structure in the language $L(T) \cup \{ D_{\phi(x)}(x) : \phi(x) \in F \}$ with $D_{\phi(x)}(x)$ interpreted as the set $\phi(A)$. When $F = L$, we may omit it. Given $A \subseteq M$ and a tuple $b \in M$, let $A_{F[b]}$ be shorthand for $A_{\text{ind}(F)}$ with $F = \{ \phi(x, b) : \phi \in L \}$.

A set $A \subseteq M$ is called small if for every finite $b \in M$, every finitary type over $Ab$ is realized in $M$. Finally, a set $A \subseteq M$ is stably embedded if for every $\phi(x, y)$ and $c \in M$ there is $\psi(x, z)$ and $b \in A$ such that $\phi(A, c) = \psi(A, b)$. We say that it is uniformly stably embedded if $\psi$ can be chosen depending just on $\phi$, and not on $c$. A definable set is stably embedded if and only if it is uniformly stably embedded, by compactness.

1. **Uniform honest definitions**

1.1. **Uniform honest definitions.** We recall the following result about existence of honest definitions for externally definable sets in NIP theories established in [8].

**Fact 8** (Honest definition). Let $T$ be NIP and let $M$ be a model of $T$ and $A \subseteq M$ any subset. Let $\phi(x, a)$ have parameters in $M$. Then there is an elementary extension $(M', A')$ of the pair $(M, A)$ and a formula $\theta(x, b) \in L(A')$ such that $\phi(A, a) = \theta(A, b)$ and $\theta(A', b) \subseteq \phi(A', a)$.

It can be reformulated as existence of a uniform family of internally definable subsets approximating our externally definable set.

**Corollary 9.** Let $T$ be NIP and let $M$ be a model of $T$ and $A \subseteq M$ any subset. Let $\phi(x, a)$ have parameters in $M$. Then there is $\theta(x, t)$ such that for any finite subset $A_0 \subseteq \phi(A, a)$, there is $b \in A$ such that $A_0 \subseteq \theta(A, b) \subseteq \phi(A, a)$.

**Proof.** Immediately follows from Fact 8 because the extension $(M, A) \prec (M', A')$ is elementary and the condition on $b$ can be stated as a single formula in the theory of the pair. Note that conversely this implies Fact 8 by compactness.

It is natural to ask whether $\theta$ can be chosen in a uniform way depending just on $\phi$, and not on $A$ and $a$ (Question 1.4 from [8]). The aim of this section is to answer this question positively.

First, compactness gives a weak uniformity statement.
Proposition 10. Let $T$ be NIP, and fix a formula $\phi(x,y)$. For every formula $\theta(x,t)$ (in the same variable $x$, but $t$ may vary), fix an integer $n_\theta$. Then there are finitely many formulas $\theta_1(x,t_1), \ldots, \theta_k(x,t_k)$ such that the following holds:

For every $M \models T$ and $A \subseteq M$, for every $a \in M$ there is $i \leq k$ such that, for every subset $A_0 \subseteq \phi(A,a)$ of size at most $n_\theta$, there is $b \in A$ satisfying $A_0 \subseteq \theta_i(A,b) \subseteq \phi(A,a)$.

Proof. Consider the theory $T'$ in the language $L' = L \cup \{P(x), c\}$ saying that if $(M, A) \models T'$ (where $A = P(M)$), then $M \models \theta$ and, for every $\theta \in L$, there is a subset $A_0$ of $\phi(A,c)$ of size at most $n_\theta$ for which there does not exist a $b \in A$ satisfying $A_0 \subseteq \theta(A,b) \subseteq \phi(A,a)$. By Corollary 3 $T'$ is inconsistent. By compactness, we find a finite set of formulas as required.

Combining this with the $(p,k)$-theorem we get the full result.

Theorem 11. Let $T$ be NIP and $\phi(x,y)$ given. Then there is a formula $\chi(x,t)$ such that for every set $A$ of size $\geq 2$, tuple $a$ and finite subset $A_0 \subseteq A$, there is $b \in A$ satisfying:

1. $\phi(A_0,a) = \chi(A_0,b)$,
2. $\chi(A,b) \subseteq \phi(A,a)$.

Proof. By the usual coding tricks, using $|A| \geq 2$, it is enough to find a finite set of formulas $\{\chi_i\}_{i<n}$ such that for every finite set, one of them works.

For every formula $\theta(x,t)$, let $n_\theta$ be its VC-dimension. Proposition 10 gives us a finite set $\{\theta_1, \ldots, \theta_k\}$ of formulas. Using the previous remark, we may assume $k = 1$ and write $\theta(x,t) = \theta_1(x,t)$. Let $N$ be given by Fact 6 taking $p = k = n_\theta$ (using Remark 7).

Let $A_0 \subseteq A \subseteq M \models T$ and $a \in M$ be given, $A_0$ is finite. Let $B \subseteq A^{[k]}$ be the set of tuples $b \in A^{[k]}$ such that $\theta(A,b) \subseteq \phi(A,a)$. Consider the family $\mathcal{F} = \{\theta(d,B) : d \in \phi(A_0,a)\}$ of subsets of $B$. This is a finite family, and by hypothesis the intersection of any $k$ members of it is non-empty. Therefore the $(p,k)$-theorem applies and gives us $N$ tuples $b_1, \ldots, b_N \in B$ such that $\{b_1, \ldots, b_N\}$ intersects any set in $\mathcal{F}$. Unwinding, we see that $\phi(A_0,a) = \bigvee_{i \leq N} \theta(A_0,b_i)$ and $\bigvee_{i \leq N} \theta(A,b_i) \subseteq \phi(A,a)$. So taking $\chi(x,t_1 \ldots t_N) = \bigvee_{i \leq N} \theta(x,t_i)$ works.

1.2. UDTFS. Recall the following classical fact characterizing stability of a formula.

Fact 12. The following are equivalent:

1. $\phi(x,y)$ is stable.
2. There is $\theta(x,z)$ such that, for any $A$ and $a$, there is $b \in A$ satisfying $\phi(A,a) = \theta(A,b)$.
3. There are $m, n \in \omega$ such that $|S_\phi(A)| \leq m \cdot |A|^n$ for any set $A$.

Definition 13. We say that $\phi(x,y)$ has UDTFS (Uniform Definability of Types over Finite Sets) if there is $\theta(x,z)$ such that for every finite $A$ and $a$ there is $b \in A$ such that $\phi(A,a) = \theta(A,b)$. We say that $T$ satisfies UDTFS if every formula does.

Remark 14. If $\phi(x,y)$ has UDTFS, then it is NIP (by Fact 3).

Comparing Fact 12 and Fact 3 naturally leads to the following conjecture of Laskowski: assume that $\phi(x,y)$ is NIP; then it satisfies UDTFS. It was proved for weakly $\omega$-minimal theories in [13] and for $dp$-minimal theories in [11]. An immediate
corollary of Theorem 11 is that if the whole $T$ is NIP, then every formula satisfies UDTFS.

**Theorem 15.** Let $T$ be NIP. Then it satisfies UDTFS.

**Proof.** Follows from Theorem 11 taking $A_0 = A$.

**Remark 16.** This does not fully answer the original question as our argument is using more than just the dependence of $\phi(x,y)$ to conclude UDTFS for $\phi(x,y)$. Looking more closely at the proof of Fact 8 we can say exactly how much NIP is needed. Depending on the VC-dimension of $\phi$, there is a finite set $\Delta_0$ of formulas for which we have to require NIP consisting of formulas of the form $\psi(x_1,\ldots,x_k) = \exists y \bigwedge_i \phi(x_i,y)^{\epsilon(1)}$, where $k$ is at most $\text{VC}(\phi) + 1$.

UDTFS implies that in the statement of the $(p,k)$-theorem for sets inside an NIP theory consistent pieces are uniformly definable.

**Corollary 17.** Let $T$ be NIP. For any $\phi(x,y)$ there is $\psi(y,z)$ and $k \leq N < \omega$ such that: for every finite $A$, if $\{\phi(x,a) : a \in A\}$ is $k$-consistent, then there are $c_0,\ldots,c_{N-1} \in A$ such that $A = \bigcup_{i<N} \psi(A,c_i)$ and $\{\phi(x,a) : a \in \psi(A,c_i)\}$ is consistent for every $i < N$.

1.3. **Strong honest definitions and distal theories.**

**Definition 18.** A theory $T$ is called distal if it satisfies the following property: Let $I + b + J$ be an indiscernible sequence, with $I$ and $J$ finite. For arbitrary $A$, if $I + J$ is indiscernible over $A$, then $I + b + J$ is indiscernible over $A$.

The class of distal theories was introduced in [22], in order to capture the class of dependent theories which do not contain any “stable part”. Examples of distal theories include ordered dp-minimal theories and $\mathbb{Q}_p$.

We will say that $p(x), q(y) \in S(A)$ are orthogonal if $p(x) \cup q(y)$ determines a complete type over $A$.

**Proposition 19 (Strong honest definition).** Let $T$ be distal, $A \subseteq M$ and $a \in M$ arbitrary. Let $(M',A') \succ (M,A)$ be $|M|^+$-saturated. Then for any $\phi(x,y)$ there are $\theta(x,z)$ and $b \in A'$ such that $\models \theta(a,b)$ and $\theta(x,b) \models \text{tp}_\phi(a/A)$.

**Proof.** Let $(M',A') \succ (M,A)$ be $\kappa = |M|^+$-saturated. We show that $p = \text{tp}_L(a/A')$ is orthogonal to any $L$-type $q \in S(A')$ finitely satisfiable in a subset of size $< \kappa$. So take such a $q$, finitely satisfiable in $C \subseteq A'$. By Lemma 5 there is some $D$ of size $< \kappa$, $C \subseteq D \subseteq A'$, such that for any two realizations $I, I' \subseteq A'$ of $q^{(\omega)}|D$, we have $\text{tp}_L(aI/C) = \text{tp}_L(aI'/C)$. Take some $I \models q^{(\omega)}|D$ in $A'$ (exists by saturation of $(M',A')$ and finite satisfiability) and $J \models q^{(\omega)}|M$.

**Claim.** $I + J$ is indiscernible over $aC$.

**Proof.** As $q^{(\omega)}|M$ is finitely satisfiable in $C$, by compactness and saturation of $(M',A')$ there is $J' \models q^{(\omega)}|aDI$ in $A'$.

If $I + J$ is not $aC$-indiscernible, then $I' + J'$ is not $aC$-indiscernible for some finite $I' \subset I$. As both $I' + J'$ and $J'$ realize $q^{(\omega)}|D$ in $A'$, it follows that $J'$ is not indiscernible over $aC$ – a contradiction.

Now, if $b \in M$ is any realization of $q$, then $I + b + J$ is $C$-indiscernible. By the claim and distality, $I + b$ is $aC$-indiscernible. It follows that $\text{tp}(b/Ca)$ is determined
by $\text{tp}(a/A')$. As we can always take a larger $C$, $\text{tp}(b/A'a)$ is determined, so $p$ is orthogonal to $q$ as required.

Consider the set $S^{ls}(A',A)$ of $L$-types over $A'$ finitely satisfiable in $A$. It is a closed subset of $S_L(A')$. By compactness, there is $\theta(x,b) \in p(x)$ such that for any $a' \models \theta(x,b)$ and any $c \models q(y) \in S^{ls}(A',A)$, $\models \phi(a,c) \leftrightarrow \phi(a',c)$. This applies, in particular, to every $c \in A$ and thus $\theta(x,b) \vdash \text{tp}_\phi(a/A)$.

Remark 20. In fact, the argument is only using that every indiscernible sequence in $A'$ is distal.

**Theorem 21.** The following are equivalent:

1. $T$ is distal.
2. For any $\phi(x,y)$ there is $\theta(x,z)$ such that: for any $A$, $a$ and a finite $C \subseteq A$, there is $b \in A$ such that $\models \theta(a,b)$ and $\theta(x,b) \vdash \text{tp}_\phi(a/C)$.

**Proof.** (1)$\Rightarrow$(2): It follows from Proposition 19 that we have: For any finite $C \subseteq A$, there is $b \in A$ such that $\models \theta(a,b)$ and $\theta(x,b) \vdash \text{tp}_\phi(a/C)$. Similarly to the proof of Theorem 11 we can choose $\theta$ depending just on $\phi$.

(2)$\Rightarrow$(1): Let $I+d+J$ be an indiscernible sequence, with $I$ and $J$ infinite. Assume that $I+J$ is indiscernible over $A$, and we show that $I+d+J$ is indiscernible over $A$.

Let $a$ be a finite tuple from $A$ and $\phi(x,y_0...y_n...y_{2n}) \in L$, and let $\theta(x,z)$ be as given for $\phi$ by (2). Without loss of generality $\models \phi(a,b_0...b_n...b_{2n})$ holds for all $b_0 < ... < b_{2n} \in I+J$. Let $I_0 \subseteq I$ be finite. Then for some $b \in I_0$, $\models \theta(a,b)$ and $\theta(x,b) \vdash \text{tp}_\phi(a/I_0)$. If we take $I_0$ to be large enough compared to $|z|$, then there will be some $b_0 < ... < b_n < ... < b_{2n}$ such that $\{b_i\}_{i<n} \cap b = \emptyset$. As we have $\models \forall x \theta(x,b) \rightarrow \phi(x,b_0...b_n...b_{2n})$, by indiscernibility of $I+d+J$ for any $\{b'_i\}_{i<n} \cap b' = \emptyset$ in $I+J$ there is a corresponding $b'$ in $I+J$ such that $\models \forall x \theta(x,b') \rightarrow \phi(x,b_0'...d...b_{2n})$. As $\models \theta(a,b')$ holds by indiscernibility of $I+J$ over $a$, it follows that $\models \phi(a,b_0'...d...b_{2n})$ holds – as wanted.

Remark 22. It follows from this theorem that types over finite sets in distal theories admit uniform definitions of a special “coherent” form as considered in [3, Section 7.1].

2. $(p,k)$-THEOREM AND FORKING

We recall some properties of dividing and forking in NIP theories.

**Fact 23.** Let $T$ be NIP.

1. If $M \models T$, then $\phi(x,a)$ divides over $M \iff$ it forks over $M \iff$ the set $\{\phi(x,a') : a \equiv_M a' \in M\}$ is inconsistent.
2. For any $\phi(x,y)$, the set $\{a : \phi(x,a)$ forks over $M\}$ is type-definable over $M$.
3. If $(a_i)_{i<\omega}$ is indiscernible over $M$ and $\phi(x,a_0)$ does not fork over $M$, then $\{\phi(x,a_i)\}_{i<\omega}$ does not fork over $M$.
4. $\phi(x,a)$ does not fork over $M \iff$ there is a global $M$-invariant type $p$ with $\phi(x,a) \in p$.

**Proof.** (1) and (2) are by [7, Theorem 1.1] and [7, Remark 3.33]; (4) is from [1]. Finally, (3) is well known and follows from (4). Indeed, if $\phi(x,a_0)$ does not fork over $M$, then it is contained in some global type $p(x)$ invariant over $M$. But then by invariance $\{\phi(x,a_i)\}_{i<\omega} \subseteq p(x)$, thus does not fork over $M$. 
Definition 24. Let $M$ be a small model. We say that $(\phi(x, y), q(y))$ (where $\phi \in L(M)$ and $q$ is a partial type over $M$) is a non-forking family over $M$ if, for every $a \models q(y)$, the formula $\phi(x, a)$ does not fork over $M$.

Notice that by Fact 23(2), if $(\phi(x, y), q(y))$ is a non-forking family, then there is some formula $\psi(y) \in q$ such that $(\phi(x, y), \psi(y))$ is a non-forking family.

Proposition 25. Assume that $T$ is NIP, and let $(\phi(x; y), q(y))$ be a non-forking family over $M$. Then there are finitely many global $M$-invariant types $p_1, \ldots, p_{n-1}$ such that for every $a \models q(y)$, there is $i < n$ with $p_i \vdash \phi(x; a)$.

Proof. Let $M < N$ be such that $N$ is $|M|^+$-saturated.

Consider the set $X = \{x \in M : \text{tp}(x/N) \text{ is } M\text{-invariant}\}$; it is type-definable over $N$ by $\{\phi(x; a) \leftrightarrow \phi(x, b) : a, b \in N, a \equiv_M b, \phi \in L\}$. Let $\mathcal{F} \equiv \{Y \subseteq X : Y = X \cap \phi(x, a), a \in q(N)\}$, and notice that the dual VC-dimension of $\mathcal{F}$ is finite, say $k$ (as $\phi(x, y)$ is NIP).

Assume that for any $p < \omega$, $\mathcal{F}$ does not satisfy the $(p, k)$-property. As by Fact 23(2) the set $\{(a_0 \ldots a_{k-1}) : \phi(x, a_0 \ldots a_{k-1}) \text{ forks over } M\}$ is type-definable. By Ramsey, compactness and Fact 23(4) we can find an $M$-indiscernible sequence $(a_i)_{i < \omega} \subseteq q(N)$ such that $\bigwedge_{i < k} \phi(x, a_i)$ forks over $M$, contradicting Fact 23(3) and the assumption on $q$.

Thus $\mathcal{F}$ satisfies the $(p, k)$-property for some $p$. Let $n$ be as given by Fact 6 and define

$$Q(x_0, \ldots, x_{n-1}) \equiv \{x_i \in X\}_{i < n} \cup \left\{ \phi(x_i, a) : a \in q(N) \right\}.$$

As every finite part of $Q$ is consistent by Fact 6 there are $b_0 \ldots b_{n-1}$ realizing it; take $p_i \equiv \text{tp}(b_i/N)$.

Remark 26. If $q(x)$ is a complete type, then this holds with $n = 1$, just by taking some $M$-invariant $p_0(x)$ containing $\phi(x, a)$.

However, we cannot hope to replace invariant $\phi$-types by definable $\phi$-types in the proposition.

Example 27. Let $T$ be the theory of a complete discrete binary tree with a valuation map. Let $M_0$ be the prime model, and take $c$ an element of valuation larger than $\Gamma(M_0)$. Let $d$ be the smallest element in $M_0$. Let $\phi(x; y, z)$ say “if $z = d$, then val($x$) > val($y$), if $z \neq d$, then $x > y$” (where > is the order in the tree). Let $\psi(y, z) = \langle z = d \rangle$. Then $(\phi, \psi)$ is a non-forking family over $M$; however, there is no definable $\phi$-type consistent with $\phi(x; c, d)$.

Remark 28. In [9] it is proved that if $T$ is a VC-minimal theory with unpacking and $M \models T$, then $\phi(x, a)$ does not fork over $M$ if and only if there is a global $M$-definable type $p(x)$ such that $\phi(x, a) \in p$. The previous example shows that the same result cannot hold in a general NIP theory.

Problem 29. Assume $\phi(x, a)$ does not fork over $M$. Is there a formula $\psi(y) \in \text{tp}(a/M)$ such that $\{\phi(x, a) \models \psi(a)\}$ is consistent (and thus does not fork over $M$)?
3. Sufficient conditions for boundedness of $T_P$

In [8] we have demonstrated the following result.

**Fact 30.**

(1) Let $(M, A)$ be bounded. If $M$ is NIP and $A_{\text{ind}}$ is NIP, then $(M, A)$ is NIP.

(2) Let $(M, A)$ be bounded and $A \prec M$. If $M$ is NIP, then $(M, A)$ is NIP.

However, a general sufficient condition for the boundedness of an expansion by a predicate for NIP theories is missing. In the stable case, a satisfactory answer is given in [6]. Recall:

**Definition 31.**

(1) $T$ satisfies nfcp (no finite cover property) if for any $\phi(x, y)$ there is $k < \omega$ such that, for any $A$, if $\{\phi(x, a)\}_{a \in A}$ is $k$-consistent, then it is consistent.

(2) We say that $M \models T$ satisfies nfcp over $A \subseteq M$ if for any $\phi(x, y, z)$ there is $k < \omega$ such that for any $A' \subseteq A$ and $b \in M$, if $\{\phi(x, a, b)\}_{a \in A'}$ is $k$-consistent, then it is consistent.

And then one has:

**Fact 32.** Let $T$ be stable.

(1) [6, Proposition 2.1] Assume that $A \subseteq M \models T$ is small and $M$ has nfcp over $A$. Then $(M, A)$ is bounded.

(2) [6, Proposition 4.6] In fact, “nfcp over $A$” can be relaxed to “$A_{\text{ind}}$ is nfcp”.

In this section we present results towards a possible generalization for unstable NIP theories.

3.1. Dnfcp (nfcp for definable sets of parameters).

**Definition 33.** We say that $M$ satisfies dnfcp over $A \subseteq M$ if for any $\phi(x, y, z)$ there is $k \in \omega$ such that: for any $b \in M$, if $\{\phi(x, a, b)\}_{a \in A}$ is $k$-consistent, then it is consistent.

We remark that dnfcp over $A$ is an elementary property of the pair $(M, A)$.

**Lemma 34.**

(1) nfcp over $A$ $\Rightarrow$ dnfcp over $A$.

(2) If $T$ is stable and $M \models T$, then nfcp $\iff$ nfcp over $M$ $\iff$ dnfcp over $M$.

**Proof.**

(1) Clear.

(2) Assume that $T$ is stable. Then nfcp and nfcp over $M$ are easily seen to be equivalent. Assume that $T$ has fcp; then by Shelah’s nfcp theorem [19, Theorem 4.4] there is a formula $E(x, y, z)$ such that $E(x, y, c)$ is an equivalence relation for every $c$ and for each $k \in \omega$ there is $c_k$ such that $E(x, y, c_k)$ has more than $k$, but finitely many equivalence classes. Taking $\phi(x, y, z) = \neg E(x, y, z)$ and $M$ big enough we see that $\{\phi(x, a, c_k) : a \in M\}$ is $k$-consistent, but inconsistent.

**Lemma 35.** If every formula of the form $\phi(x, y, z)$ with $|x| = 1$ is dnfcp over $A$, then $T$ is dnfcp over $A$.

**Proof.** Assume we have proved that all formulas with $|x| \leq m$ are dnfcp, and we prove it for $|x| = m + 1$. So assume that for every $n < \omega$ we have some $c_n \in M$ such that $\{\phi(x_0...x_m, a, c_n)\}_{a \in A}$ is $n$-consistent, but not consistent. Let $\psi(x_1...x_m, y_1...y_l, z) = \exists x_0 \bigwedge_{i \leq l} \phi(x_0...x_m, y_i, z)$, of course still $\{\psi(x, a, c_n)\}_{a \in A}$ is $[n/l]$-consistent, so consistent for $n$ large enough by the inductive assumption. Let
Remark 39. \(a\) A
\[b \cdots b_m\]
realize it. Then consider \(\Gamma = \{ \theta(x_0, a, c_n b_1 \cdots b_m) \}_{a \in A}\) where \(\theta(x_0, a, c_n b_1 \cdots b_m) = \phi(x_0 b_1 \cdots b_m, a, c_n)\). It is \(l\)-consistent. Again by the inductive assumption, if \(l\) was chosen large enough, there is some \(b_0\) realizing \(\Gamma\), but then
\[b_0 \cdots b_m \models \{ \phi(x_0 \cdots x_m, a, c_n) \}_{a \in A},\]
a contradiction.

Example 36. DLO has dnfcp over models.

The following criterion for boundedness follows from the proof of [6].

Theorem 37. Let \(A \subseteq M\) be small and uniformly stably embedded. Assume that \(M\) has dnfcp over \(A\). Then \((M, A)\) is bounded.

The problem with dnfcp is that it does not seem possible to conclude dnfcp over \(A\) from properties of the induced structure on \(A\). To remedy this, we introduce a weaker variant with separated variables.

Definition 38. We say that \(M\) satisfies dnfcp' over \(A \subseteq M\) if, for any \(\phi(x, y)\) and \(\psi(y, z)\), there is \(k < \omega\) such that for any \(b \in M\), if \(\{ \phi(x, a) : a \in \psi(A, b) \}\) is \(k\)-consistent, then it is consistent. We say that \(T\) has dnfcp' if for any \(M < N\), \(N\) has dnfcp' over \(M\).

Remark 39. Let \((M, A)\) be a pair, and assume that \(A\) is small and \(A_{\text{ind}}\) is saturated. Then if formulas are bounded, \(M\) has dnfcp' over \(A\).

Proof. By assumption \(\exists \gamma \forall a \in P, \psi(a, z) \rightarrow \phi(a; y)\) is equivalent to a bounded formula \(\theta(z)\), for any \(\phi\) and \(\psi\). If dnfcp' does not hold, then there is a consistent bounded type satisfying \(\neg \theta(z)\) and for all \(n, \forall a_1, \ldots, a_n \in P \exists y, \bigwedge \psi(a_i; z) \rightarrow \phi(a_i; y)\). As \(A_{\text{ind}}\) is saturated, it is resplendent, and we can find a type over \(A\) which satisfies this bounded type. By smallness of \(A\) in \(M\), this type is realized by some \(c \in M\). Then again by smallness, there is \(b \in M\) such that \(\psi(a; c) \rightarrow \phi(a; b)\) for all \(a \in A\). This contradicts the hypothesis on \(\theta\).

We can now prove some preservation result.

Lemma 40. Let \(T\) be NIP, \(A \subseteq M \models T\) and assume that \(\text{Th}(A_{\text{ind}}(L_P))\) has dnfcp'. Then \(M\) has dnfcp' over \(A\).

Proof. Let \(\phi(x, y)\) and \(\psi(y, b)\) be given. Let \(\theta_{\phi}(y, s)\) be a uniform honest definition for \(\phi\) and \(\theta_{\psi}(y, t)\) a uniform honest definition for \(\psi\) (by Theorem 14). Let \((M', A') = (M, A)\) be a sufficiently saturated elementary extension; then naturally \(A'_{\text{ind}}(L_P) = A_{\text{ind}}(L_P)\). There is \(c\) such that \(\psi(A, b) = \theta_{\psi}(A, c)\).

Let \(\chi(s)\) be the formula \(\exists d \forall y \in P \theta_{\phi}(y, s) \rightarrow \phi(d, y)\) and let \(k \in \omega\) be as given for \(\theta_{\phi}(y, s) \land \chi(s), \theta_{\psi}(y, t)\) by dnfcp' of \(A_{\text{ind}}(L_P)\) for it. Assume \(\{ \phi(x, a) : a \in \psi(A, b) \}\) is \(k\)-consistent; then \(\{ \theta_{\phi}(a, s) \land \chi(s) : a \in \theta_{\psi}(A, c) \}\) is \(k\)-consistent (let \(d = \{ \phi(x, a_i) \}_{i<k}\), and choose \(c\) such that \(a_i \subseteq \theta_{\phi}(A, c) \subseteq \phi(d, A)\). As \(A_{\text{ind}}(L_P)\) is dnfcp', we conclude that it is consistent. In particular, for any \(n \in \omega\) and \(a_0, \ldots, a_n \in \theta_{\psi}(A, c) = \psi(A, b)\), there is \(c\) such that \(\bigwedge_i \theta_{\phi}(a_i, c) \land \chi(c)\), thus unwinding there is some \(d = \{ \phi(x, a_i) \}_{i<k}\).
3.2. Boundedness of the pair for distal theories. We now aim at giving an analogue of Fact 32 for distal theories and stably embedded predicates.

First, we improve Lemma 40.

**Lemma 41.** Let $T$ be distal, $A \subseteq M \models T$ and assume that $\text{Th}(A_{\text{ind}(L)})$ has dnfcp'. Then $M$ has dnfcp' over $A$.

**Proof.** Follow the proof of Lemma 40, except that we define $\chi(s)$ as $\exists x \forall y \theta_\phi(y, s) \rightarrow \phi(d, y)$, which we can by strong honest definitions (Theorem 21).

Let $A_0$ be a small subset of $M_0$, and take a $|T|^+$-saturated $(M, A) \succ (M_0, A_0)$.

**Lemma 42.** Assume that $T$ is distal and $M$ has dnfcp' over $A$. Let $a \in M, \zeta(x, y) \in L$ and $q(y) \in S(A)$ be an a-definable type. Then the following are equivalent:

1. There is $b \models q$ in $M$ such that $\models \zeta(a, b)$.
2. There is $b \models q$ in $M$ such that $\models \zeta(a, b)$.

**Proof.** By $L_P$-saturation of $(M, A)$ and definability of $q(y)$ over $a$, it is enough to find such a $b$ realizing the $\phi(y, z)$-part of $q(y)$. Assume that it is definable by $d_\phi(z, a)$. Let $\theta(y, t)$ be given by Proposition 19 for $\phi$, and let $d_\theta(t, a)$ define the $\theta$-part of $q$. By dnfcp', the fact that $d_\phi(z, a), d_\theta(t, a)$ define a consistent $\phi, \theta$-type $q_a$ over $P$ is expressible by a bounded formula $\psi_1(a)$ saying:

$$\forall z_1...z_n \in P \forall t_1...t_n \in P \exists y \left( \bigwedge_{i \leq n} \phi(y, z_i) \leftrightarrow d_\phi(z_i, a) \land \bigwedge_{i \leq n} \theta(y, t_i) \leftrightarrow d_\theta(t_i, a) \right),$$

where $n$ is given by dnfcp' for $\phi'(y, z_1z_2t_1t_2) = \phi(y, z_1) \land \neg \phi(y, z_2) \land \theta(y, t_1) \land \neg \theta(y, t_2)$ and $\psi'(z_1z_2t_1t_2, \alpha) = d_\phi(z_1, \alpha) \land \neg d_\phi(z_2, \alpha) \land d_\theta(t_1, \alpha) \land \neg d_\theta(t_2, \alpha)$.

Observe that for any $d \in d_\phi(A, n)$, $M \models \exists b \theta(b, d) \land \zeta(a, b)$ (as $q(y) \land \zeta(a, y)$ is consistent). It can be expressed by a bounded formula $\psi_2(a)$.

Let $a_0 \in M_0$ be such that $(M_0, A_0) \models \psi_1(a_0) \land \psi_2(a_0)$. Assume that there is a finite $C \subseteq A_0$ such that $q_{a_0}(y) \mid_C \land \zeta(a_0, y)$ is inconsistent. Let $d \in d_\phi(A_0, a_0)$ be as given by Theorem 21. Then find some $b \in M_0$ such that $\models \theta(b, d) \land \zeta(a_0, b)$ (by $\psi_2(a_0)$). By the hypothesis on $\theta$, we have $b \models q_{a_0} \mid C$ – a contradiction.

So $q_{a_0}(y) \land \zeta(a_0, y)$ is consistent, and it follows by smallness of $A_0$ in $M_0$ that $(M_0, A_0) \models \forall x \psi_1(x) \land \psi_2(x) \rightarrow \exists b \models q_a(y) \land \zeta(x, y)$. It follows that $(M, A)$ satisfies the same sentence, and unwinding we conclude.

**Theorem 43.** Let $T$ be distal, $A \subseteq M$ is small and (uniformly) stably embedded, and $A_{\text{ind}}$ has dnfcp'. Then $T_P$ is bounded.

**Proof.** By Lemma 41, $M$ has dnfcp' over $A$. Take $(M, A)$ a $|T|^+$-saturated elementary extension of the pair. Let $a, a' \in M$ be such that $A_{[a]} \equiv A_{[a']}$, $A_{[\text{ind}(L)]}$ and $A_{\text{ind}(L)}$ have the same finite height and $A_{[a]} \equiv A_{[a']}$, $A_{\text{ind}(L)}$ and $A_{\text{ind}(L)}$ have the same finite height. Take $b \in M$.

Case 1: $b \in A$. As $A_{[a]} \equiv A_{[a']}$, by $L_P$-saturation we can find $b' \in P$ such that $A_{[ab]} \equiv A_{[a'b']}$.

Case 2: $b \in M \setminus A$. By stable embeddedness and Case 1, we may assume that $\text{tp}(ab/A)$ is a-definable. It is enough to find $b' \in M \setminus A$ such that $\text{tp}(b', a') = \text{tp}(b, a)$ and $\text{tp}(ab'/A)$ is defined over $a'$ the same way $\text{tp}(ab/A)$ is over $a$. Now the previous lemma (and saturation) applies and gives such a $b'$. 

Theorem 43. Let $T$ be distal, $A \subseteq M$ is small and (uniformly) stably embedded, and $A_{\text{ind}}$ has dnfcp'. Then $T_P$ is bounded.
4. Naming indiscernible sequences, again

We recall briefly the story of the question. In [4] Baldwin and Benedikt had established the following.

**Fact 44.** Let $T$ be NIP. Let $I \subset M$ be a small indiscernible sequence indexed by a dense complete linear order. Then $\text{Th}(M, I)$ is bounded and the $L_p$-induced structure on $I$ is just the linear order.

We have demonstrated ([5] Proposition 3.2) that in this case $(M, I)$ is still NIP. In this section we are going to complement the picture by resolving some of the remaining questions: naming a small indiscernible sequence of arbitrary order type preserves NIP, while naming a large indiscernible sequence may create IP.

4.1. Naming an arbitrary small indiscernible sequence.

**Lemma 45.** Let $I$ be small in $M$ and $N \succ M$ such that $I$ is small in $N$. Then $(M, I)$ and $(N, I)$ are elementary equivalent.

**Proof.** We do a back-and-forth starting with the identity mapping from $I$ to $I$, and inductively choosing $A = \{a_1\}_{i<\omega} \subset M$ and $B = \{b_1\}_{i<\omega} \subset N$ such that $\text{tp}_L(AI) = \text{tp}_L(BI)$. Assume we have chosen $\{a_m b_m : m < n\}$ and we pick $a_n \in M$. Consider $p(x, AI) = \text{tp}_L(a_n/AI)$. By the inductive assumption, $p(x, BI)$ is consistent. Let $b_n \in N$ realize it (possibly by smallness). In the end, in particular, $AI \equiv \forall f - L_p BI$.

Assume that $D$ is an $L$-definable set which is uniformly stably embedded in the sense of $T$ (and $T$ eliminates quantifiers in a relational language $L$). Let $P$ name a subset of $D$. Now let $(N, P)$ be a saturated model of the pair.

A formula is $D$-bounded if it is equivalent to one of the form $\psi(x) = Q_1 z_1 \in D \ldots Q_n z_n \in D \bigvee_{i<m} \phi_i(\bar{x}, \bar{z}) \wedge \chi_i(\bar{x}, \bar{z})$, where $\phi_i(\bar{x}, \bar{z})$ is a qf-$L$-formula and $\chi_i(\bar{x}, \bar{z})$ is a qf-$P$-formula (follows from the relationality of $L$).

**Lemma 46.** Let $a, a' \in N$ have the same $D$-bounded type; then $a \equiv L_p a'$.

**Proof.** We do a back-and-forth. Assume that $a \equiv \forall D\text{-bdd} a'$, and let $b \in N$ be arbitrary.

Case 1: $b \in D$. Consider $p(x,a) = \text{tp}_L(ab)$. For any finite $p_0(x,a) \subseteq p(x,a)$ we have $\models \exists x \in D p_0(x,a)$, which is a $D$-bounded formula; thus, $\models \exists x \in D p_0(x,a')$, and by saturation of $N$ there is $b' \in D$ satisfying $ab \equiv \forall D\text{-bdd} a'b'$.

Case 2: $b \notin D$. Possibly adding some points in $D$ using (1), we may assume that $\text{tp}_L(ab/D)$ is $L$-definable over $c = a \cap D$. Take some $b' \in N$ such that $ab \equiv L a'b'$; then $\text{tp}_L(a'b'/D)$ is $L$-definable over $c' = a' \cap D$ using the same formulas.

We want to check that $ab \equiv \forall D\text{-bdd} a'b'$. Let $\psi(\bar{x})$ be a $D$-bounded formula, say $\psi(\bar{x}) = Q_1 z_1 \in D \ldots Q_n z_n \in D \bigvee_{i<m} \phi_i(\bar{x}, \bar{z}) \wedge \chi_i(\bar{x}, \bar{z})$. Then we have: $\models Q_1 x_1 \in D \ldots Q_n x_n \in D \bigvee_{i<m} \phi_i(ab, \bar{x}) \wedge \chi_i(ab, \bar{x}) \iff \models Q\bar{x} \in D \bigvee_{i<m} d\phi_i(c, \bar{x}) \wedge \chi_i'(\bar{x})$ (as we know the truth values of $P(x)$ on $ab$) $\iff \models Q\bar{x} \in D \bigvee_{i<m} d\phi_i(c', \bar{x}) \wedge \chi_i'(\bar{x})$ (as $c \equiv \forall D\text{-bdd} c'$) $\iff \models Q_1 x_1 \ldots Q_n x_n \bigvee_{i<m} \phi_i(a'b', \bar{x}) \wedge \chi_i(a'b', \bar{x})$ (as the truth values of $P(x)$ on $a'b'$ are the same by the choice of $b'$ and assumption on $a'$).

**Lemma 47.** Assume that $\text{Th}(D_{\text{ind}}, P)$ is bounded. Then $\text{Th}(M, P)$ is bounded.

**Proof.** Let $(N, P)$ be saturated. Assume that $P[a] \equiv P[a']$ and let $b$ be given.

If $b \in D$, then we find a $b' \in D$ such that $P[ab] \equiv P[ab']$ by the assumption that $(D, P)$ is bounded and saturation.
If \( b \notin D \), then we take the same \( b' \) as in (2) of the previous lemma and conclude that \( bb' \equiv L_{P}^{qf} aa' \) in the same way (using that \( c \equiv L_{P}^{qf} c' \)), which is sufficient (clearly, if two tuples have the same \( D \)-bounded \( L_{P} \)-type, then they have the same \( P \)-bounded \( L_{P} \)-type).

**Lemma 48.** In the situation as above, if \( T \) is \( NIP \) and \( (D,P) \) with the induced quantifier-free structure is \( NIP \), then \( T_{P} \) is \( NIP \).

**Proof.** As \( D_{\text{ind}}(L_{P}^{qf}) \) is \( NIP \), it follows that \( D_{\text{ind}}(L_{P}^{qf}) \) is \( NIP \). Conclude as in Corollary 2.5 in [8] (and even easier as \( D \) is actually stably embedded).

**Theorem 49.** Let \((M, I) \) be small and \( M \) be \( NIP \). Then \((M, I) \) is \( NIP \).

**Proof.** Let \((M, I) \) be small. By Lemma 45 we may assume that \( M \) is \( (2|I|)^{+} \)-saturated. Let \( I \subseteq J \subseteq M \), where \( J \) is a dense complete indiscernible sequence such that \((M, J) \) is still small. Name \( J \) by \( D \), and let \( T' \) be a Morleyzation of \( T_{D} \). Then by Fact 44, \( T' \) is \( NIP \) and \( D \) is stably embedded. Thus formulas in \( T_{P} \) are \( D \)-bounded by Lemma 46. It is easy to check directly that \((J_{\text{ind}}, I) \) is bounded; thus, \( T_{P} \) is \( P \)-bounded by Lemma 47. Conclude by Fact 30 (as the structure induced on \( I \) is still \( NIP \)).

4.2. Large indiscernible sequence producing IP. Take \( L = \{<, E\} \) and \( T \) saying that \(< \) is DLO and \( E \) is an equivalence relation with infinitely many classes, all of which are dense co-dense with respect to \(< \). It is easy to check by back-and-forth that this theory eliminates quantifiers and that it is \( NIP \). Let \( M/E \) denote the imaginary sort of \( E \)-equivalence classes.

Let \( D \) be an equivalence class, pick some \( x_{0} \in M \) outside of it and take \( P \) to name \( D \cap (\sim \infty, x_{0}) \). Consider the formula

\[
\phi(x) = \exists y \forall s < y \exists t \in P, yEx \land s < t < y \land (\neg \exists u > y, u \in P).
\]

Then \( \phi(x) \) picks out exactly the points equivalent to \( x_{0} \). Easily, that formula is not equivalent to a \( D \)-bounded one (simply because all imaginary elements of equivalence classes different from \( D \) have exactly the same type over \( D \)).

Now consider the following formula:

\[
S(x_{1}, x_{2}) = \exists y_{1}, y_{2}, y_{1}Ex_{1} \land y_{2}Ex_{2} \land L_{0}(y_{1}) \land R_{0}(y_{2}) \land (\forall y_{1} < z < y_{2}, \neg P(z))
\]

where \( L_{0}(y) = \exists t \in P \forall s \in P, t < y \land (s > t \rightarrow y < s) \) and the same for \( R_{0}(y) \), but reversing the inequalities.

**Claim 50.**

1. Let \( D \) be an equivalence class. Then any increasing sequence contained in \( D \) is indiscernible.
2. Let \( G \) be an arbitrary countable graph. Then we can choose \( P \subseteq D \) such that \((M/E, S) \cong G \).

**Proof.** (1) is immediate by the quantifier elimination.

(2) By induction, for every edge \( e_{1}e_{2} \in (M/E)^{2} \) that we want to put, choose a pair of representatives \( a_{1}, a_{2} \in \mathbb{Q} \) such that the interval \((a_{1}, a_{2})\) is disjoint from all the previously chosen intervals. Let \( P \) name the set of points in \( D \) outside of the union of these intervals.

In particular we can choose \( P \) so that \( T_{P} \) interprets the random graph.
Remark 51. We also observe that naming two small indiscernible sequences at once can create IP. This time we name sequences which satisfy \(\neg xEy\) for any two points \(x\) and \(y\) in them. So pick any small \(I_0\). Let \(A = A[I_0] = \{t \in M/E : \exists x \in I_0, xEt\}\). Then \(A\) gets an order \(<_0\) from \(I_0\) induced by \(<\). Fix \(<_1\) any other order on \(A\). Then we can find another sequence \(I_1\) such that \(A[I_1] = A\) and the order induced on \(A\) by \(I_1\) is \(<_1\). With two linear orders, we can code pseudo-finite arithmetic as in [21]. In particular we have IP.

5. Models with definable types

Classically,

Fact 52. \(T\) is stable \(\iff\) for every \(M \models T\), \(|S(M)| \leq |M|^{|T|}\) \(\iff\) for every \(M \models T\), all types over it are definable \(\iff\) there is a saturated \(M \models T\) with all types over it definable.

We start by observing that if \(T\) is NIP, then it has models of arbitrary size with few types over them.

Proposition 53. Let \(T\) be NIP. For any \(\kappa \geq |T|\) there is a model \(M\) with \(|M| = \kappa\) such that \(|S(A)| \leq |A|^{|T|}\) for every \(A \subseteq M\).

Proof. If \(T\) is stable, then every model of size \(\kappa\) works. Otherwise assume \(T\) is unstable and let \(I = (a_\alpha)_{\alpha < \kappa}\) be linearly ordered by \(< (x, y) \in L\). Let \(T^{\text{Sk}}\) be a Skolemization of \(T\), and let \(M = \text{Sk}(I), |M| \leq \kappa + |T|\).

We show that \(S^L(M) \leq \kappa^{|T|}\). Consider

\[ \bar{L} := \{ \phi(x, f(y)) : \phi \in L \text{ and } f \text{ is an } L^{\text{Sk}}\text{-definable function}\}. \]

Notice that every \(\psi(x, y) \in \bar{L}\) is NIP. But then (by Remark 2) for every \(\psi \in \bar{L}\), every \(\psi\)-type over \(I\) is \(<\)-definable; in particular, \(|S^L(I)| \leq |I|^{|T|}\).

Given \(p, q \in S^L(M)\) choose some \(p', q' \in S^L(M)\) with \(p \subseteq p', q \subseteq q'\). It is easy to see that \(p'|_{I} = q'|_{I} \Rightarrow p = q\) (for any \(a \in M\) and \(\phi \in \bar{L}\) we have \(\phi(x, a) \in p \iff \phi(x, f(\bar{b})) \in p'|_{I} \) where \(\bar{b} \subseteq I\) and \(f(\bar{b}) = a\); thus, \(|S^L(M)| \leq |S^L(I)| \leq \kappa^{|T|}\).

Remark 54. Slightly elaborating on the argument, we may construct such an \(M\) which is in addition gross (\(M\) is called gross if every infinite subset definable with parameters from \(M\) is of cardinality \(|M|\); see [15]).

In general one cannot find a model such that all types over it are definable (for example, take RCF and add a new constant for an infinitesimal). However, some interesting NIP theories have models with all types over them definable.

Example 55. (1) \(\mathbb{R}\) as a model of RCF (and this is the only model of RCF with all types definable).

(2) In \(ACVF\) there are arbitrary large models with all types definable (maximally complete fields with \(\mathbb{R}\) as a value group).

(3) \((\mathbb{Z}, +, <)\) is a model of Presburger arithmetic with all types definable (but there are no larger models).

(4) \((\mathbb{Q}_p, +, \times, 0, 1)\) (by [10]).

When looking at a particular example, it is usually much easier to check that 1-types are definable, rather than \(n\)-types, and one can ask if this is actually the same thing.
Definition 56. Let $A$ be a set. We say that it is $(n, m)$-stably embedded if every subset of $A^n$ which can be defined as $\phi(A, a)$ with $|a| \leq m$, can actually be defined as $\psi(A, b)$ with $b \in A$. We say that it is uniformly $(n, m)$-stably embedded if $\psi$ can be chosen depending just on $\phi$ (and not on $a$). A compactness argument shows that for a definable set $A$, it is $(n, m)$-stably embedded if and only if it is uniformly $(n, m)$-stably embedded. Obviously, $(\infty, n)$-stable embeddedness is equivalent to definability of $n$-types over $A$.

Of course, $(n, m)$-stable embeddedness implies $(n', m')$-stable embeddedness for $n' \leq n, m' \leq m$.

Proposition 57. Let $T$ be NIP and assume that $M$ is $(\infty, n)$-stably embedded. Then it is $(n, \infty)$-stably embedded.

Proof. By definability, every type $p \in S_n(M)$ has a unique heir.

Claim 1: If $p \in S_n(M)$ has a unique heir, then it has a unique coheir.

Let $p'(x)$ be the unique global heir of $p$. Let $p_1(x)$ be a global coheir of $p$, and $(a_i)_{i<\omega}$ a Morley sequence in it over $M$. Given $\bar{m} \in M$ and noticing that $tp(a_0/a_1...a_n,M)$ is an heir over $M$ (so is contained in a global heir as $M \models T$) we have that $\models \phi(a_0, ..., a_n, \bar{m})$ if and only if $\phi(x,a_1...a_n\bar{m}) \in p'(x)$. Thus, by Fact 3 $p$ has a unique global coheir.

Claim 2: Every $p \in S_n(A)$ has a unique coheir $\iff A$ is $(n, \infty)$-stably embedded.

$\Rightarrow$: Let $\phi(x,c) \in L(M)$ and consider $p(x) \in S_n(A)$ finitely satisfiable in $\phi(x,c) \cap A$. If it was finitely satisfiable in $\neg\phi(x,c) \cap A$ as well, then $p$ would have two coheirs; thus, there is some $\psi_p(x) \in p(x)$ with $\psi_p(x) \rightarrow A \phi(x,c)$. By compactness we have $\bigvee_{\psi_p(x) \in p(x)} \phi(x,c)$ for finitely many $p_i$'s.

$\Leftarrow$: Let $p_1, p_2$ be two global coheirs of $p \in S_n(A)$, and assume that $\phi(x,a) \in p_1, \neg\phi(x,a) \in p_2$. Let $\psi(x) \in L(A)$ be such that $\psi(A^n) = \phi(A^n, a)$. It follows that $\psi(x) \in p$. But this implies that $p_2$ cannot be a coheir as $\psi(x) \land \neg\phi(x,a)$ is not realized in $A$.

And so it is natural to ask whether $(\infty, 1)$-stable embeddedness of $M$ implies $(\infty, n)$-stable embeddedness. The answer is yes in stable theories, for the obvious reason, and yes in $\alpha$-minimal theories, where by a theorem of Marker and Steinhorn [16], $(1, 1) \to (\infty, \infty)$ for models. However, we show in the next section that this is not true in NIP theories in general. The question remains open for $C$-minimal theories.

5.1. Example of $(\infty, 1) \not\leftrightarrow (\infty, m)$.

5.1.1. General construction. Start with a theory $T$ in a language $L$ containing an equivalence relation $E(x,y)$. Assume $T$ has a model $M_0$ composed of $\omega$-many $E$-equivalence classes, each one finite of increasing sizes, so that any model $M$ of $T$ contains $M_0$ as a sub-model and all the $E$-classes disjoint from $M_0$ are infinite.

We consider the language $L'$ defined as follows:

- For each relation $R(x_1, ..., x_n)$ in $L$, $L'$ then contains a relation $R'(x_1, y_1, ..., x_n, y_n)$.
- Also, $L'$ contains an equivalence relation $\bar{E}(u, v)$, a binary relation $S(u, v)$ and a quaternary relation $U(u_1, v_1, u_2, v_2)$. The relation $S$ will code a graph and $U$ will be an equivalence relation on $S$-edges.
We build an $L'$ structure $N_0$ as follows:

$N_0$ has $\omega$-many $\tilde{E}$-equivalence classes, corresponding to the $E$-equivalence classes of $M_0$. Let $\epsilon$ be an $E$-class, and let $n$ be its size. Then the corresponding $\tilde{E}$ class $\tilde{\epsilon}$ in $N_0$ is a finite regular graph, with $S$ as the edge relation, of degree $n$ (every vertex has degree $n$) and with no cycles of length $\leq n$ (such graphs exist; see e.g. [5, III.1, Theorem 1.4]). The predicate $U$ is interpreted as an equivalence relation between edges so that every vertex is adjacent to exactly one edge from each equivalence class. We fix a bijection $\pi$ between $U$-equivalence classes and elements of the $E$-class $\epsilon$. This being done, for each relation $R(x_1,\ldots,x_n)$ we say that $R'(x_1, y_1,\ldots,x_n, y_n)$ holds in $N_0$ if $\bigwedge_{i\leq n} S(x_i, y_i)$ and if $R(\pi(x_1, y_1),\ldots,\pi(x_n, y_n))$ holds in $M_0$.

Note that any model of $T' = \text{Th}(N_0)$ contains $N_0$ as a submodel and its $\tilde{E}$-classes not in $N_0$ are infinite and composed of disjoint unions of trees with infinite branching. So the graph structure does not interact in any way with the structure coming from the $R'$ relations.

Given a model of $T'$ we can recover a model of $M_0$ by looking at $U$-equivalence classes and we obtain in this way every model of $T$. So there are at least as many 2-types over $N_0$ as there are 1-types over $M_0$. However, the non-realized 1-types over $N_0$ correspond to imaginary types of non-realized $E$-classes over $M_0$. See below.

Assume that $L$ contains a constant for every element of $M_0$. Let $N \models T'$ and denote by $M$ the model of $T$ which we get from $N$. We build a language $L'' \supset L'$:

- We add a constant for every element of $N_0$.
- For every $n \in \omega$, we add a relation $d_n(u,v)$ which holds if and only if $u$ and $v$ are at distance $n$ (in the sense of the shortest path in graph $S(u,v)$).
- For every $\emptyset$-definable set $\phi(x_1,\ldots,x_n,y_1,\ldots,y_m)$ of $M_0$ which is $E$-congruent with respect to the variables $x_i$ (i.e., for $a_i E a'_i$ and $b_i$’s, we have $\phi(a_1,\ldots,a_n,b_1,\ldots,b_m) \leftrightarrow \phi(a'_1,\ldots,a'_n,b_1,\ldots,b_m)$) we add a predicate $W_\phi(x_1,\ldots,x_n,y_1,\ldots,y_m,z_1,\ldots,z_m)$ which we interpret as:
  
  $N \models W_\phi(a_1,\ldots,a_n,b_1,\ldots,b_m)$ if and only if $\bigwedge_{i\leq m} S(b_i,c_i)$ and for some $e_1,\ldots,e_n \in M$ with $e_i$ in the $E$-class corresponding to the $\tilde{E}$-class of $a_i$, we have $M \models \phi(e_1,\ldots,e_n,\pi(b_1,c_1),\ldots,\pi(b_m,c_m))$.

Claim 58. If $T$ eliminates quantifiers in $L$, then $T'$ eliminates quantifiers in $L''$.


Corollary 59. If $T$ is NIP, then $T'$ is NIP.

Corollary 60. Assume that all (imaginary) types of a new $E$ class in $M_0$ are definable; then all 1-types over $N_0$ are definable.

5.1.2. An example of $M_0$ with NIP. Let $L_0 = \{\leq, E\}$. We build an $L_0$-structure $M_0$ as follows:

- The reduct to $\leq$ is a binary tree with a root (every element has exactly two immediate successors, there is a unique element with no predecessor). The tree is of height $\omega$, so every element is at finite distance from the root.
- Two elements are $E$-equivalent if they are at the same distance from the root.
This theory eliminates quantifiers in the language \(L\) obtained from \(L_0\) by adding a constant for every element of \(M_0\), a binary function symbol \(\wedge\) interpreted as \(x \wedge y\) is the maximal element \(z\) such that \(z \leq x\) and \(z \leq y\) and for each \(n\) a predicate \(d_n(x,y)\) saying that the difference between the heights of \(x\) and \(y\) is \(n\). Note that those predicates are \(E\)-congruent.

Clearly, \(M_0\) is NIP, there is a unique imaginary type of a new \(E\)-class over \(M_0\) and this type is definable. However, not all types over \(M_0\) are definable.

So we obtain the required counterexample.

Remark 61. Together with Proposition 57 it follows that also \((1, \infty) \not\to (n, \infty)\) in a general NIP theory. Another example due to Hrushovski witnessing this is presented in Pillay 18 – a proper dense elementary pair of \(ACVF\)'s \(F_1 \prec F_2\) with the same residue field and value group. Then \(F_1\) is \((1, \infty)\)-stably embedded in \(F_2\), but if \(a \in F_2 \setminus F_1\), then the function taking \(x \in F_1\) to \(v(x - a)\) is not \(F_1\)-definable.

5.2. Some positive results. In 18 Pillay had established the following.

Fact 62. Let \(A\) be a definable subset of \(M\). Assume that \(A_{\text{ind}}\) is rosy, \(M\) is NIP over \(A\) and \(A\) is \((1, \infty)\)-stably embedded. Then \(A\) is stably embedded.

In fact, one can replace the definable set \(A\) with a model, at the price of requiring that \((1, \infty)\)-stable embeddedness is uniform. We explain briefly how to modify Pillay’s argument.

Theorem 63. Let \(A \preceq M\). Assume that \(M\) is NIP and rosy, and \(A\) is uniformly \((1, \infty)\)-stably embedded. Then \(A\) is uniformly stably embedded.

Proof. Assume that \(A \preceq M\) is a counterexample to the theorem. We consider \((M, A)\) as a pair with \(P\) naming \(A\). As \(A\) is a model, it follows that \(A_{\text{ind}}\) eliminates quantifiers; thus, every set definable in \(A_{\text{ind}}\) is given by the trace of an \(L\)-formula.

As there are two languages \(L\) and \(L_P\) around, we make a terminology clarification: induced structure is always meant to be with respect to \(L\) formulas, and \((n, m)\)-stable embeddedness always means that sets externally definable by \(L\)-formulas are internally definable by \(L\)-formulas.

Claim. We may assume that \((M, A)\) is saturated (as a pair in the \(L_P\) language).

Proof. Just let \((N, B) \succ (M, A)\) be a saturated extension. Of course, \(A\) is uniformly \((n, \infty)\)-stably embedded in \(M\) if and only if \(B\) is uniformly \((n, \infty)\)-stably embedded in \(N\).

Claim. Let \(f : A \to Z\) be an \(L(M)\)-definable function (namely the trace on \(A\) of an \(L(M)\)-definable relation which happens to define a function on \(A\)), where \(Z\) is some sort in \(A_{\text{ind}}^n\). Then there is an \(L(A)\)-definable relation \(R(x, y)\) and \(k < \omega\) such that \((M, A) \models \forall x \in P \,(R(x, f(x)) \land \exists y \leq k y \in P \, R(x, y))\).

Proof. Let the graph of \(f\) be defined by \(f(x, y, e) \in L(M)\). Let \(\kappa\) be large enough. Working entirely in \(A_{\text{ind}}\), assume that we could choose \((a_i b_i)_{i < \kappa}\) in \(A\) such that \(b_i = f(a_i)\) and \(b_i \notin \text{acl}_L\left((a_j b_j)_{j < i} a_i\right)\) for all \(i\). Following Pillay’s proof of 18 Lemma 3.2 and using saturation of \(A_{\text{ind}}\), we may assume that \((a_i b_i)\) is \(L\)-indiscernible and then find \((b'_i)\) in \(A\) such that \(b'_i = b_i\) if and only if \(i\) is even, and \(tp_L((a_i b_i)_{i < \kappa}) = tp_L((a'_i b'_i)_{i < \kappa})\), so is still \(L\)-indiscernible. But then \((M, A) \models f(a_i, b'_i, e)\) if and only if \(i\) is even – a contradiction to \(M\) being NIP with respect to \(L\)-formulas.
So, by compactness we find some $R(x, y) \in L(A)$ and $k < \omega$ such that $(M, A) \models \forall x \in P \ R(x, f(x)) \land \exists^k y \in P \ R(x, y)$.

Claim. In the previous claim, we can take $k = 1$.

Proof. Pillay’s proof of [18, Lemma 3.3] goes through again, with acl, dcl and forking all considered inside of the $L$-induced structure on $A$ (which is saturated and eliminates quantifiers).

Finally, we conclude by induction on the dimension of the externally definable sets. So let $X = A^{n+1} \cap \phi(x_0, \ldots, x_n, x_{n+1}, c)$ be given, and assume inductively that $A$ is uniformly $(n, \infty)$-stably embedded (the base case given by the assumption).

For any $a \in A$, let $X_a = A^n \cap \phi(x_0, \ldots, x_n, a, c)$. By the inductive assumption, there is some $\psi(x_0, \ldots, x_n, z)$ such that for any $a \in A$, $X_a = A^n \cap \psi(x_0, \ldots, x_n, b)$ for some $b \in A$. By Shelah’s expansion theorem, the function $f : A \rightarrow Z$ sending $a$ to $[b]_{\psi}$ (the canonical parameter of $\psi(x_0, \ldots, x_n, b)$) is externally definable. Thus, by the previous claim, it is actually definable with parameters from $A$. It follows that $X$ is defined by $\psi(x_0, \ldots, x_n, f(x_{n+1}))$.

As an application, we obtain a new proof of a theorem of Marker and Steinhorn [16].

**Corollary 64.** Let $T$ be o-minimal and $M \models T$. Assume that the order on $M$ is complete. Then all types over $M$ are uniformly definable.

**References**


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