STABILITIES OF HOMOTHETICALLY SHRINKING YANG-MILLS SOLITONS

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Abstract. In this paper we introduce entropy-stability and F-stability for homothetically shrinking Yang-Mills solitons, employing entropy and the second variation of the $F$-functional respectively. For a homothetically shrinking soliton which does not descend, we prove that entropy-stability implies F-stability. These stabilities have connections with the study of Type-I singularities of the Yang-Mills flow. Two byproducts are also included: We show that the Yang-Mills flow in dimension four cannot develop a Type-I singularity, and we obtain a gap theorem for homothetically shrinking solitons.

1. Introduction

In this paper we introduce entropy-stability and F-stability for homothetically shrinking (Yang-Mills) solitons. Let $E$ be a trivial $G$-vector bundle over $\mathbb{R}^n$ and of rank $r$. Here the gauge group $G$ is a Lie subgroup of $SO(r)$. A homothetically shrinking soliton, centered at the space-time point $(x_0 = 0, t_0 = 1)$, is a connection $A(x)$ on $E$ such that

$$(d^\nabla)^*F + \frac{1}{2}i_x F = 0,$$

where $F$ is the curvature of $A(x)$, $(d^\nabla)^*$ denotes the formal adjoint of the covariant exterior differentiation $d^\nabla$, and $i_x$ stands for the interior product by the position vector $x$.

A homothetically shrinking soliton $A(x)$ gives rise to a special solution of the Yang-Mills flow. In fact in the exponential gauge of $A(x)$,

$$A(x, t) = A_j(x, t)dx^j := (1 - t)^{-\frac{1}{2}}A_j((1 - t)^{-\frac{1}{2}}x)dx^j$$

is a solution to the Yang-Mills flow. On the other hand, homothetically shrinking solitons are closely related to Type-I singularities of the Yang-Mills flow. Weinkove [22] proved that Type-I singularities of the Yang-Mills flow are modelled by homothetically shrinking solitons whose curvatures do not vanish identically. Examples of homothetically shrinking solitons have been found in [10,22]. In this paper, we restrict ourselves to homothetically shrinking solitons which have uniform bounds on $|\nabla^k A(x)|$ for each $k \geq 1$. In fact, Weinkove showed in [22] that Type-I singularities of the Yang-Mills flow can be modelled by such solitons.

Recently, Colding and Minicozzi [8] discovered two functionals for immersed surfaces in Euclidean space, i.e. the $F$-functional and the entropy. Critical points of
both functionals are self-shrinkers of the mean curvature flow. Colding and Minicozzi introduced entropy-stability and F-stability for self-shrinkers. Inspired by their work, in this paper we aim to introduce corresponding stabilities for homothetically shrinking Yang-Mills solitons. In fact there are many aspects in common concerning the entropy-stability and F-stability for self-similar solutions to various geometric flows, which includes mean curvature flow, Ricci flow, harmonic map heat flow, and Yang-Mills flow. For the entropy-stability and linearly stability of geometric flows, which includes mean curvature flow, Ricci flow, harmonic map heat flow, and Yang-Mills flow. For the entropy-stability and F-stability for self-similar solutions to various geometric flows, which includes mean curvature flow, Ricci flow, harmonic map heat flow, and Yang-Mills flow.

We begin with the definition of an $F$-functional. Let $x_0$ be a point in $\mathbb{R}^n$ and $t_0$ a positive number. The $F$-functional with respect to $(x_0, t_0)$, defined on the space of connections on $E$, is given by

\begin{equation}
F_{x_0, t_0}(A) = t_0^2 \int_{\mathbb{R}^n} |F|^2 (4\pi t_0)^{-\frac{n}{2}} e^{-\frac{|x-x_0|^2}{4t_0}} \, dx.
\end{equation}

The functional $F_{x_0, t_0}$ can trace back to the monotonicity formula of the Yang-Mills flow. For the monotonicity formula see [7, 12, 18]. Let $A(x, t)$ be a solution to the Yang-Mills flow on $E$ and let

\begin{equation}
\Phi_{x_0, t_0}(A(x, t)) = (t_0 - t)^2 \int_{\mathbb{R}^n} |F|^2 [4\pi (t_0 - t)]^{-\frac{n}{2}} e^{-\frac{|x-x_0|^2}{4(t_0 - t)}} \, dx.
\end{equation}

Along the Yang-Mills flow, $\Phi_{x_0, t_0}$ is non-increasing in $t$. Moreover $\Phi_{x_0, t_0}$ is preserved if and only if $A(x, 0)$ is a homothetically shrinking soliton centered at $(x_0, t_0)$. Here a homothetically shrinking soliton centered at $(x_0, t_0)$ is a connection on $E$ satisfying the equation

\begin{equation}
(d^\nabla)^* F + \frac{1}{2t_0} i_{x-x_0} F = 0.
\end{equation}

The $F$-functional leads to another characterization of homothetically shrinking solitons: Critical points of $F_{x_0, t_0}$ are exactly homothetically shrinking solitons centered at $(x_0, t_0)$; moreover, $(x_0, t_0, A_0)$ is a critical point of the function $(x, t, A) \mapsto F_{x, t}(A)$ if and only if $A_0$ is a homothetically shrinking soliton centered at $(x_0, t_0)$.

The $\lambda$-entropy of a connection $A(x)$ on the bundle $E$ is defined by

\begin{equation}
\lambda(A) = \sup_{x_0 \in \mathbb{R}^n, t_0 > 0} F_{x_0, t_0}(A).
\end{equation}

A crucial fact is the following

**Proposition 1.1.** Let $A(x, t)$ be a solution to the Yang-Mills flow on the bundle $E$. Then the entropy $\lambda(A(x, t))$ is non-increasing in $t$.

The entropy is a rescaling invariant. More precisely, let $A(x)$ be a connection on $E$ and $A^c$ a rescaling of $A(x)$ given by $A^c_i(x) = c^{-1} A_i(c^{-1} x)$, $c > 0$. Then $F_{cx_0, ct_0}(A^c) = F_{x_0, t_0}(A)$ and hence $\lambda(A^c) = \lambda(A)$. In particular the entropy of each time-slice of the homothetically shrinking Yang-Mills flow, induced from a homothetically shrinking soliton, is preserved. The entropy is also invariant under translations of a connection. Let $A(x)$ be a connection on $E$, $x_1 \in \mathbb{R}^n$ a given point, and $A_i(x) = A_i(x + x_1)$. Then we have $F_{x_0 - x_1, t_0}(A) = F_{x_0, t_0}(A)$ and hence $\lambda(A) = \lambda(A)$. 
In general the entropy $\lambda(A)$ of a connection $A(x)$ is not attained by any $F_{x_0,t_0}(A)$. However if $A(x)$ is a homothetically shrinking soliton centered at $(x_0,t_0)$, then $\lambda(A) = F_{x_0,t_0}(A)$. In fact we prove the following

**Proposition 1.2.** Let $A(x)$ be a homothetically shrinking soliton centered at $(0,1)$ such that $i_VF \neq 0$ for any non-zero $V \in \mathbb{R}^n$. Then the function $(x_0,t_0) \mapsto F_{x_0,t_0}(A)$ attains its strict maximum at $(0,1)$.

Note that if $i_VF = 0$ for some non-zero vector $V$, then $A(x)$ can be viewed as a connection on a $G$-vector bundle over any hyperplane perpendicular to $V$ and we say $A(x)$ descends (to $V^\perp$).

Entropy-stability and F-stability are defined for homothetically shrinking solitons.

**Definition 1.1.** A homothetically shrinking soliton $A(x)$ is called entropy-stable if it is a local minimum of the entropy, among all perturbations $\tilde{A}(x)$, such that $||\tilde{A} - A||_{C^1}$ is sufficiently small.

Entropy-stability of homothetically shrinking solitons has direct connections with Type-I singularities of the Yang-Mills flow. For example, given an entropy-unstable homothetically shrinking soliton $A(x)$, by definition we can find a perturbation $\tilde{A}(x)$ of $A(x)$ such that $||\tilde{A} - A||_{C^1}$ is arbitrarily small and has less entropy. Then by comparing the entropy, the Yang-Mills flow starting from $\tilde{A}$ cannot converge back to a rescaling of $A(x)$. Moreover, the Yang-Mills flow cannot develop a Type-I singularity modelled by $A(x)$, due to the fact that the entropy is a rescaling invariant.

Let $A_0(x)$ be a homothetically shrinking soliton centered at $(x_0,t_0)$. For a 1-parameter family of deformations $(x_s,t_s,A_s)$ of $(x_0,t_0,A_0)$, let $V = \frac{dx_s}{ds}|_{s=0}, q = \frac{dt_s}{ds}|_{s=0}, \theta = \frac{dA_s}{ds}|_{s=0}$.

**Definition 1.2.** $A_0(x)$ is called F-stable if for any compactly supported $\theta$, there exist a real number $q$ and a vector $V$ such that

$$F''_{x_0,t_0}(q,V,\theta) := \frac{d^2}{ds^2}|_{s=0}F_{x_s,t_s}(A_s) \geq 0.$$

Entropy-stability has an apparent connection with the singular behavior of the Yang-Mills flow; however the F-stability is more practical when we are trying to do classification. The classification of entropy-stable homothetically shrinking solitons can be relied on the classification of F-stable ones. In fact we have the following relation for entropy-stability and F-stability.

**Theorem 1.3.** Let $A(x)$ be a homothetically shrinking soliton such that $i_VF \neq 0$ for any non-zero $V \in \mathbb{R}^n$. If $A(x)$ is entropy-stable, then it is F-stable.

Let $A_0(x)$ be a homothetically shrinking soliton centered at $(0,1)$. Denote

$$L\theta = -[(d\nabla)^*d\nabla \theta + \mathcal{R}(\theta) + i_\xi d\nabla \theta],$$

where $\mathcal{R}(\theta)(\partial_j) := [F_{ij}, \theta]$. For the homothetically shrinking soliton $A_0(x)$, we have

$$L(d\nabla)^*F = (d\nabla)^*F$$

and

$$Li_VF = \frac{1}{2}i_VF, \quad \forall V \in \mathbb{R}^n.$$
The second variation of the $F$-functional and at $A_0$ is given by
\begin{equation}
\frac{1}{2} F_{0,1}''(q, V, \theta) = \int_{\mathbb{R}^n} \left( -L\theta + 2q(d\nabla)^* F - iVF, \theta \right) G dx
- \int_{\mathbb{R}^n} \left( q^2 |(d\nabla)^* F|^2 + \frac{1}{2} |iVF|^2 \right) G dx,
\end{equation}
where $G(x) = (4\pi)^{-\frac{n}{2}} \exp(-\frac{|x|^2}{4})$. Denote the space of $\theta$ satisfying $L\theta = -\lambda\theta$ by $E_\lambda$. We have the following characterization for $F$-stability.

**Theorem 1.4.** $A_0(x)$ is $F$-stable if and only if the following properties are satisfied:
\begin{itemize}
  \item $E_{-1} = \{ c(d\nabla)^* F, \ c \in \mathbb{R} \}$;
  \item $E_{-\frac{1}{2}} = \{ iVF, \ V \in \mathbb{R}^n \}$;
  \item $E_\lambda = \{ 0 \}$, for any $\lambda < 0$ and $\lambda \neq -1, -\frac{1}{2}$.
\end{itemize}

Theorem 1.4 amounts to saying that $A_0(x)$ is $F$-stable if and only if $-L$ is non-negative definite modulo the vector space spanned by $(d\nabla)^* F$ and $iVF$. This is actually the reflection of the invariance property of the $F$-functional and the entropy under rescalings and translations. Since Colding-Minicozzi’s work [8], classification problem of $F$-stable self-shrinkers of the mean curvature flow has drawn much attention, see for instance [1,2,15,16].

We have two simple byproducts regarding homothetically shrinking solitons. We show the non-existence of homothetically shrinking solitons in dimensions four and lower, and a gap theorem. Let $A(x)$ be a homothetically shrinking soliton centered at $(0, 1)$. Then we have the identity
\begin{equation}
\int_{\mathbb{R}^n} |x|^2 |F|^2 G(x) dx = 2(n - 4) \int_{\mathbb{R}^n} |F|^2 G(x) dx.
\end{equation}
It immediately implies the following

**Proposition 1.5.** When $n = 2, 3$, or $4$, there exists no homothetically shrinking soliton such that $|F|$ is uniformly bounded and not identically zero.

Råde [19] proved that the Yang-Mills flow, over a compact Riemannian manifold of dimension $n = 2$ or $3$, exists for all time and converges to a Yang-Mills connection. However if the base manifold has dimension five or above, Naito [18] showed that the Yang-Mills flow can develop a singularity in finite time, see also [11]. It is unclear yet whether the Yang-Mills flow over a four-dimensional manifold develops a singularity in finite time. For partial results in this dimension, see [9,13,20] and the references therein. Together with Weinkove’s blowup analysis for Type-I singularities of the Yang-Mills flow, Proposition 1.5 shows that the Yang-Mills flow cannot develop a singularity of Type-I. This was actually a known fact, see for instance [12].

Gap theorems for Yang-Mills connections over spheres was considered in [8]. Gap theorems for various kinds of self-similar solutions have also been obtained, see for instance [5,14,23]. By (1.5), we have the following gap result for homothetically shrinking solitons.

**Theorem 1.6.** Let $A(x)$ be a homothetically shrinking soliton centered at $(0, 1)$. If $|F|^2 < \frac{n}{2(n-1)}$, then $(E, A)$ is flat.
analysis for Type-I singularities of the Yang-Mills flow. In Section 3, we consider the $F$-functional and its first variation. Section 4 is devoted to the calculation of the second variation of the $F$-functional, i.e. Theorem 1.4. In Section 5, we study the $F$-stability of homothetically shrinking solitons and prove Theorem 1.6. In Section 6, we introduce the $\lambda$-entropy and prove Proposition 1.2. In the last section, we prove that entropy-stability implies $F$-stability, i.e. Theorem 1.3.

We would like to point out that although we assume, for simplicity, that the homothetically shrinking solitons have uniform bounds on $|\nabla^k A|$, our statements except Theorem 1.3 are still straightforwardly valid if $|\nabla^k A|$ has polynomial growth.

Many results in this paper have also been obtained by Kelleher and Streets [17].

2. Preliminaries

In this section we briefly introduce the Yang-Mills flow and its singularity. We shall introduce the blowup analysis for Type-I singularities, which was carried out by Weinkove [22]. It leads to the main object in this paper, i.e. the homothetically shrinking soliton.

Let $(M, g)$ be a closed $n$-dimensional Riemannian manifold. Let $G$ be a compact Lie group and $P(M, G)$ a principal bundle over $M$ with the structure group $G$. We fix a $G$-vector bundle $E_M = P(M, G) \times \rho \mathbb{R}^r$, associated to $P(M, G)$ with a $G$-valued 1-form. Using Latin letters for the manifold indices, one may write a connection $A$ in the form of $A = A^i dx^i$, where $A^i \in \mathfrak{so}(r)$. Using Greek letters for the bundle indices, one may also write $A = A_{\alpha i} dx^i$. The curvature of the connection $A$ is locally a $\mathfrak{g}$-valued 2-form $F = \frac{1}{2} F_{ij} dx^i \wedge dx^j$, and $F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$.

The Yang-Mills functional, defined on the space of connections, is given by

$$YM(A) = \frac{1}{2} \int_M |F|^2 d\mu_g,$$

where

$$|F|^2 = \frac{1}{2} g^{ik} g^{jl} (F_{ij}, F_{kl}) = \frac{1}{2} g^{ik} g^{jl} F^\alpha_{ij\beta} F^\alpha_{kl\beta},$$

and

$$F^\alpha_{ij\beta} = \partial_i A^\alpha_{j\beta} - \partial_j A^\alpha_{i\beta} + A^\alpha_{\gamma i} A^\gamma_{j\beta} - A^\alpha_{\gamma j} A^\gamma_{i\beta}.$$

Let $\nabla$ denote the covariant differentiation on $\Gamma(E_M)$ associated to the connection $A$, and also the covariant differentiation on $\mathfrak{g}$-valued $p$-forms induced by $A$ and the Levi-Civita connection of $(M, g)$. Curvature $F$ satisfies the Bianchi identity $d \nabla F = 0$, where $d \nabla$ denotes the covariant exterior differentiation. Let $(d \nabla)^*$ denote the formal adjoint of $d \nabla$. A connection $A$ is a critical point of the Yang-Mills functional, called a Yang-Mills connection, if and only if it is a solution of the Yang-Mills equation $(d \nabla)^* F = 0$. The Yang-Mills equation can also be written as

$$\nabla^p F^\alpha_{pj\beta} = 0.$$

In normal coordinates of $(M, g)$, we have

$$\nabla^p F^\alpha_{pj\beta} = \partial_p F^\alpha_{pj\beta} + A^\alpha_{\gamma p} F^\gamma_{pj\beta} - F^\alpha_{pj\gamma} A^\gamma_{p\beta}.$$
As the $L^2$-gradient flow of the Yang-Mills functional, the Yang-Mills flow is defined by

$$
\frac{dA}{dt} = -(d\nabla)^* F.
$$

Assume $A(x, t)$ is a smooth solution to the Yang-Mills flow for $0 \leq t < T$, and as $t \to T$ the curvature blows up, i.e. $\limsup_{t \to T} \max_{x \in M} |F(x, t)| = \infty$. If there exists a positive constant $C$ such that

$$
|F(x, t)| \leq \frac{C}{T - t},
$$

one says that the Yang-Mills flow develops a Type-I singularity, or a rapidly forming singularity. Otherwise one says that the Yang-Mills flow develops a Type-II singularity. If (2.3) is satisfied and $x_0$ is a point such that $\limsup_{t \to T} |F(x_0, t)| = \infty$, we call $(x_0, T)$ a Type-I singularity.

Let $A(x, t)$ be a smooth solution to the Yang-Mills flow and $(x_0, T)$ a Type-I singularity. We now follow [22] introducing the blowup procedure around $(x_0, T)$. Let $B_r(x_0)$ be a small geodesic ball centered at $x_0$ and of radius $r$ over which $E_M$ is trivial. For simplicity we identify a sequence of positive numbers tending to zero. For each $i$, one gets a Yang-Mills flow $A^{\lambda_i}(y, s)$ by setting

$$
A^{\lambda_i}(y, s) = \lambda_i A_p(\lambda_i y, T + \lambda_i^2 s)dy^p, \quad y \in B_r/\lambda_i(0), s \in [-\lambda_i^{-2}T, 0).
$$

(An alternative way of obtaining a sequence of blowups of $A(x, t)$ is to rescale the metric around the singular point $x_0$.) Let $x = \lambda_i y$ and $t = T + \lambda_i^2 s$. By the assumption (2.3), the curvature of $A^{\lambda_i}$ satisfies

$$
|F^{\lambda_i}(y, s)| = \lambda_i^2 |F(x, t)| = |s|^{-1}(T - t)|F(x, t)| \leq C|s|^{-1}.
$$

Let $h = h_0^\alpha_i$ be a gauge transformation which acts on connections by

$$
h^* \nabla = h^{-1} \circ \nabla \circ h,
$$

or equivalently,

$$
h^* A = h^{-1}dh + h^{-1}Ah.
$$

Note that gauge transformations preserve Yang-Mills flows. Hence $h^* A^{\lambda_i}(y, s)$ defines a solution to the Yang-Mills flow. Weinkove [22] proved the following

**Theorem 2.1.** Let $(x_0, T)$ be a Type-I singularity of the Yang-Mills flow $A(x, t)$ over $M$. Then there exists a sequence of blowups $A^{\lambda_i}(y, s)$ defined by (2.4) and a sequence of gauge transformations $h_i$ such that $h_i^* A^{\lambda_i}(y, s)$ converges smoothly on any compact set to a flow $\bar{A}(y, s)$. Here $\bar{A}(y, s)$, defined on a trivial $G$-vector bundle over $\mathbb{R}^n \times (-\infty, 0)$, is a solution to the Yang-Mills flow, which has non-zero curvature and satisfies

$$
\nabla^p \bar{F}_{pj} - \frac{1}{2|s|^2} y^p \bar{F}_{pj} = 0.
$$

In Theorem 2.1, $h_i$ are chosen as suitable Coulomb gauge transformations so that for any $s < 0$ and $k \geq 1$, $|\nabla^k h_i^* A^{\lambda_i}|$ is uniformly bounded. The bounds do not depend on $i$. Hence for any $s < 0$ and $k \geq 1$, $|\nabla^k \bar{A}|$ is uniformly bounded.
A solution \( A(y,s) \) to the Yang-Mills flow, defined on a trivial bundle over \( \mathbb{R}^n \times (-\infty,0) \), is called a homothetically shrinking soliton if it satisfies

\[
A_{i\beta}^\alpha(y,s) = \frac{1}{\sqrt{|s|}}A_{i\beta}^\alpha\left(\frac{y}{\sqrt{|s|}},-1\right)
\]

for any \( y \in \mathbb{R}^n \) and \( s < 0 \); for more details see [22]. The limiting Yang-Mills flow \( \widetilde{A}(y,s) \) is actually a homothetically shrinking soliton. In fact via an exponential gauge for \( \widetilde{A}(y,s) \), in which \( y^p \widetilde{A}_{p\beta}^\alpha = 0 \), \( (2.5) \) and \( (2.6) \) are equivalent for the Yang-Mills flow \( \widetilde{A}(y,s) \).

One of the main ingredients of Theorem 2.1 is the monotonicity formula for the Yang-Mills flow; see [7][12][18]. In the simplest case that \( A(x,t) \) is a solution to the Yang-Mills flow over \( \mathbb{R}^n \), one can define

\[
\Phi_{x_0,t_0}(A(x,t)) = (t_0 - t)^2 \int_{\mathbb{R}^n} |F(x,t)|^2 G_{x_0,t_0}(x,t)dx.
\]

Here \( t_0 > 0, t \in [0, \min\{T,t_0\}] \), and \( G_{x_0,t_0}(x,t) = [4\pi(t_0-t)]^{-\frac{n}{2}} \exp(-\frac{|x-x_0|^2}{4(t_0-t)}) \) is the backward heat kernel. The monotonicity formula of the Yang-Mills flow reads

\[
d\frac{d}{dt}\Phi_{x_0,t_0}(A(x,t)) = -2(t_0-t)^2 \int_{\mathbb{R}^n} |\nabla^p F_{pj} - \frac{1}{2(t_0-t)}(x-x_0)^p F_{pj}|^2 G_{x_0,t_0}(x,t)dx.
\]

The monotonicity \( \Phi_{x_0,t_0} \) is non-increasing in \( t \), and is preserved if and only if

\[
|\nabla^p F_{pj} - \frac{1}{2(t_0-t)}(x-x_0)^p F_{pj}| = 0.
\]

For the limiting Yang-Mills flow \( \widetilde{A}(y,s) \) obtained in Theorem 2.1 and any \((x_0,t_0) \in \mathbb{R}^n \times (0, +\infty)\), one can translate it into

\[
A(x,t) = A_p(x,t)dx^p = \widetilde{A}_p(x-x_0, t-t_0)dx^p;
\]

then \( A(x,t) \) is a solution to the Yang-Mills flow and \( (2.9) \) is satisfied. On the other hand if a connection \( A(x) \) on a trivial \( G \)-vector bundle over \( \mathbb{R}^n \) satisfies

\[
|\nabla^p F_{pj} - \frac{1}{2t_0}(x-x_0)^p F_{pj}| = 0,
\]

then, in the exponential gauge for \( A(x) \), i.e. a gauge such that \((x-x_0)^p A_p(x) = 0\), the flow of connections given by

\[
A_p(x,t) := \sqrt{t_0 - t} A_p(x_0) + \sqrt{\frac{t_0}{t_0 - t}}(x-x_0)
\]

is a solution to the Yang-Mills flow which satisfies \( (2.9) \). All these amount to saying that limiting flows \( \widetilde{A}(y,s) \), homothetically shrinking solitons \( A(x) \) and homothetically shrinking Yang-Mills flows are the same thing.

From now on we assume that \( E \) is a trivial \( G \)-vector bundle over \( \mathbb{R}^n \).

**Definition 2.1.** A connection \( A(x) \) on \( E \) is called a homothetically shrinking soliton centered at \((x_0,t_0)\) if it satisfies

\[
|\nabla^p F_{pj} - \frac{1}{2t_0}(x-x_0)^p F_{pj}| = 0.
\]
Let $A(x)$ be a homothetically shrinking soliton centered at $(x_0, t_0)$ and $A(x, t)$ the Yang-Mills flow initiating from $A(x)$. In an exponential gauge such that $(x - x_0)^p A_p(x, t) = 0$, we have for any $\lambda > 0$ and any $t < t_0$ that $A_j(x, t) = \lambda A_j(\lambda(x-x_0) + x_0, \lambda^2(t-t_0) + t_0)$.

3. $\mathcal{F}$-FUNCTIONAL AND ITS FIRST VARIATION

In this section we define the $\mathcal{F}$-functional of connections on the trivial $G$-vector bundle $E$ over $\mathbb{R}^n$. Homothetically shrinking solitons are critical points of the $\mathcal{F}$-functional. We shall prove necessary integral identities for homothetically shrinking solitons. As a corollary of one of these identities, we give a proof of the fact that the Yang-Mills flow in dimension four cannot develop a Type-I singularity.

For convenience, we set two $\mathfrak{g}$-valued 1-forms $J$ and $X$, respectively, by

$$J := \nabla^p F_{pj} dx^j, \quad X := i_{x-x_0} F = (x-x_0)^p F_{pj} dx^j.$$ 

According to (2.11), $A(x)$ is a homothetically shrinking soliton centered at $(x_0, t_0)$ if and only if

$$J = \frac{1}{2t_0} X.$$ 

We also set

$$S_{x_0, t_0} = \{ A(x) : A \text{ is a homothetically shrinking soliton centered at } (x_0, t_0) \}.$$ 

Note that for any $k \geq 1$, any time-slice $\tilde{A}(\cdot, s)$ in Theorem 2.1 satisfies $\sup |\nabla^k \tilde{A}(\cdot, s)| < \infty$.

**Definition 3.1.** For any $x_0 \in \mathbb{R}^n$, $t_0 > 0$, the $\mathcal{F}$-functional with respect to $(x_0, t_0)$ is defined by

$$\mathcal{F}_{x_0,t_0}(A) = t_0^2 \int_{\mathbb{R}^n} |F|^2(4\pi t_0)^{-\frac{n}{2}} e^{-\frac{|x-x_0|^2}{4t_0}} dx.$$ 

We now compute the first variation of the $\mathcal{F}$-functional. Consider a differentiable 1-parameter family $(x_s, t_s, A_s)$, where $A_0 = A$. Denote

$$i_s = \frac{d}{ds} t_s, \quad x_s = \frac{d}{ds} x_s, \quad \theta_s = \frac{d}{ds} A_s,$$

and

$$G_s(x) = (4\pi t_s)^{-\frac{n}{2}} e^{-\frac{|x-x_s|^2}{4t_s}}.$$ 

**Proposition 3.1.** Assume $|\nabla^k A_s| < \infty$ for any $k \geq 1$ and $\int_{\mathbb{R}^n} (|\theta_s|^2 + |\nabla \theta_s|^2) G_s dx < \infty$. The first variation of the $\mathcal{F}$-functional is given by

$$\frac{d}{ds} \mathcal{F}_{x_s,t_s}(A_s) = \int_{\mathbb{R}^n} i_s \left( \frac{4-n}{2} t_s + \frac{1}{4} |x-x_s|^2 |F_s|^2 G_s(x) dx ight)$$

$$+ \int_{\mathbb{R}^n} \frac{1}{2} t_s(x_s(x-x_s) |F_s|^2 G_s(x) dx$$

$$- \frac{2t_s^2}{\pi} \theta_s - \frac{X_s}{2t_s} G_s(x) dx.$$ 

(3.3)
Proof. Note that
\[
\frac{\partial}{\partial s} G_s(x) = \left( -\frac{n}{2} \frac{i_s}{t_s} + \frac{i_s |x-x_s|^2}{4t_s^2} + \frac{\langle \dot{x}_s, x-x_s \rangle}{2t_s} \right) G_s(x),
\]
and
\[
\frac{\partial}{\partial s} |F_s|^2 = F_{ij\beta}^\alpha (\nabla_i \theta_{j\beta}^\alpha - \nabla_j \theta_{i\beta}^\alpha),
\]
so we have
\[
\frac{d}{ds} F_{x_s, t_s}(A_s) = \int_{\mathbb{R}^n} 2t_s i_s |F_s|^2 G_s(x) dx + \int_{\mathbb{R}^n} t_s^2 F_{ij\beta}^\alpha (\nabla_i \theta_{j\beta}^\alpha - \nabla_j \theta_{i\beta}^\alpha) G_s(x) dx
\]
\[
+ \int_{\mathbb{R}^n} t_s^2 |F_s|^2 \left( -\frac{n}{2} \frac{i_s}{t_s} + \frac{i_s |x-x_s|^2}{4t_s^2} + \frac{\langle \dot{x}_s, x-x_s \rangle}{2t_s} \right) G_s(x) dx
\]
\[
= \int_{\mathbb{R}^n} 2t_s i_s |F_s|^2 G_s(x) dx + \int_{\mathbb{R}^n} 2t_s^2 F_{ij\beta}^\alpha \nabla_i \theta_{j\beta}^\alpha G_s(x) dx
\]
\[
+ \int_{\mathbb{R}^n} t_s^2 |F_s|^2 \left( -\frac{n}{2} \frac{i_s}{t_s} + \frac{i_s |x-x_s|^2}{4t_s^2} + \frac{\langle \dot{x}_s, x-x_s \rangle}{2t_s} \right) G_s(x) dx.
\]
Let \( \eta(x) \) be a cutoff function on \( \mathbb{R}^n \). By integration by parts, we have
\[
\int_{\mathbb{R}^n} 2t_s^2 F_{ij\beta}^\alpha \nabla_i \theta_{j\beta}^\alpha G_s(x) \eta(x) dx
\]
\[
= \int_{\mathbb{R}^n} -2t_s^2 \theta_{j\beta}^\alpha \nabla_i F_{ij\beta}^\alpha G_s \eta + F_{ij\beta}^\alpha \partial_i (G_s) \eta + F_{ij\beta}^\alpha \partial_i \theta_i \eta \eta \eta dx
\]
\[
(3.4) \quad = \int_{\mathbb{R}^n} -2t_s^2 \theta_{j\beta}^\alpha \nabla_i F_{ij\beta}^\alpha \eta - \frac{(x-x_s)^i}{2t_s} F_{ij\beta}^\alpha \eta \eta + F_{ij\beta}^\alpha \partial_i \theta_i \eta \eta G_s dx.
\]
Let \( \eta_l(x) = 1 \) for \( |x| \leq l \), and cut off to zero linearly on \( B_{l+1} \setminus B_l \). Taking \( \eta = \eta_l \) in (3.4) and applying Lebesgue’s dominated convergence theorem, we get
\[
(3.5) \quad \int_{\mathbb{R}^n} 2t_s^2 F_{ij\beta}^\alpha \nabla_i \theta_{j\beta}^\alpha G_s(x) dx = \int_{\mathbb{R}^n} \theta_{j\beta}^\alpha [-2t_s^2 \nabla_i F_{ij\beta}^\alpha + t_s (x-x_s)^i F_{ij\beta}^\alpha] G_s dx.
\]
Hence we get
\[
\frac{d}{ds} F_{x_s, t_s}(A_s) = \int_{\mathbb{R}^n} 2t_s i_s |F_s|^2 G_s(x) dx
\]
\[
+ \int_{\mathbb{R}^n} \theta_{j\beta}^\alpha [-2t_s^2 \nabla_i F_{ij\beta}^\alpha + t_s (x-x_s)^i F_{ij\beta}^\alpha] G_s(x) dx
\]
\[
+ \int_{\mathbb{R}^n} t_s^2 |F_s|^2 \left( -\frac{n}{2} \frac{i_s}{t_s} + \frac{i_s |x-x_s|^2}{4t_s^2} + \frac{\langle \dot{x}_s, x-x_s \rangle}{2t_s} \right) G_s(x) dx
\]
\[
= \int_{\mathbb{R}^n} i_s \left( \frac{4-n}{2} t_s + \frac{1}{4} |x-x_s|^2 \right) |F_s|^2 G_s(x) dx
\]
\[
+ \int_{\mathbb{R}^n} \frac{1}{2} t_s \langle \dot{x}_s, x-x_s \rangle |F_s|^2 G_s(x) dx
\]
\[
- \int_{\mathbb{R}^n} 2t_s^2 \langle \theta_s, J_{\eta} \rangle \frac{X_s}{2t_s} G_s(x) dx.
\]
From Proposition 3.1, we have the following

**Corollary 3.1.** A connection \( A(x) \) is a critical point of \( \mathcal{F}_{x_0,t_0} \) if and only if \( A(x) \) is a homothetically shrinking soliton centered at \((x_0,t_0)\).

We shall check that \((A(x),x_0,t_0)\) is a critical point of the \( F \)-functional \((\tilde{A},x,t) \mapsto \mathcal{F}_{x,t}(\tilde{A}) \) if and only if \( A(x) \) is a homothetically shrinking soliton centered at \((x_0,t_0)\).

To check this we need some identities for homothetically shrinking solitons. We also need such identities in the calculation of the second variation of the \( F \)-functional in the next section. Denote

\[
G(x) = (4\pi t_0)^{-\frac{n}{2}} e^{-\frac{|x-x_0|^2}{4t_0}}.
\]

**Lemma 3.2.** Let \( A(x) \) be a homothetically shrinking soliton centered at \((x_0,t_0)\) such that \( \sup |F(x)| \leq \infty \). Let \( \varphi = \varphi^p \partial_p \) be a vector field on \( \mathbb{R}^n \) such that \( |\varphi| \) is a polynomial in \(|x-x_0|\), and \( V \) a vector in \( \mathbb{R}^n \). Then we have

\[
\int_{\mathbb{R}^n} \varphi^p(x-x_0)^p |F|^2 G(x) dx = \int_{\mathbb{R}^n} \left[ 2t_0 \partial_p (\varphi^p) |F|^2 - 4t_0 \partial_i \varphi^p F_{\alpha \beta}^{p \cdot \beta} F_{ij}^{\alpha} \right] G(x) dx.
\]

In particular,

(a) \( \int_{\mathbb{R}^n} |x-x_0|^2 |F|^2 G(x) dx = \int_{\mathbb{R}^n} 2(n-4)t_0 |F|^2 G(x) dx \);

(b) \( \int_{\mathbb{R}^n} (x-x_0)^k |F|^2 G(x) dx = 0 \);

(c) \( \int_{\mathbb{R}^n} |x-x_0|^4 |F|^2 G(x) dx = \int_{\mathbb{R}^n} [4(n-2)(n-4)t_0^2 |F|^2 - 32t_0^3 |J|^2] G dx \);

(d) \( \int_{\mathbb{R}^n} |x-x_0|^2 \langle V, x-x_0 \rangle |F|^2 G(x) dx = 0 \);

(e) \( \int_{\mathbb{R}^n} (x-x_0, V)^2 |F|^2 G dx = \int_{\mathbb{R}^n} (2t_0 |V|^2 |F|^2 - 4t_0 \langle V^i F_{ij}, V^p F_{pj} \rangle) G dx \).

**Proof.** Let \( \eta(x) \) be a cutoff function on \( \mathbb{R}^n \). By integration by parts, we get

\[
\int_{\mathbb{R}^n} \varphi^p(x-x_0)^p |F|^2 G(x) \eta(x) dx
\]

\[
= \int_{\mathbb{R}^n} -2t_0 \varphi^p |F|^2 \partial_p G(x) \eta(x) dx
\]

\[
= \int_{\mathbb{R}^n} 2t_0 [\partial_p (\varphi^p) |F|^2 \eta + \varphi^p \partial_p (|F|^2) \eta + \varphi^p |F|^2 \partial_p \eta] G(x) dx.
\]

By integration by parts we have

\[
\int_{\mathbb{R}^n} 4t_0 \varphi^p F_{\alpha \beta}^{p \cdot \beta} F_{ij}^{\alpha} \eta G dx = \int_{\mathbb{R}^n} 4t_0 \varphi^p \left[ \nabla_i (F_{\alpha \beta}^{p \cdot \beta} F_{ij}^{\alpha}) - \nabla_i F_{\alpha \beta}^{p \cdot \beta} F_{ij}^{\alpha} \right] \eta G dx
\]

\[
= \int_{\mathbb{R}^n} -4t_0 F_{\alpha \beta}^{p \cdot \beta} F_{ij}^{\alpha} \left[ \partial_i \varphi^p - \frac{(x-x_0)^i}{2t_0} \varphi^p \right] G \eta dx
\]

\[
- \int_{\mathbb{R}^n} 2t_0 \varphi^p \left[ \nabla_i F_{\alpha \beta}^{p \cdot \beta} F_{ij}^{\alpha} + \nabla_j F_{\alpha \beta}^{p \cdot \beta} F_{ij}^{\alpha} \right] G \eta dx
\]

\[
- \int_{\mathbb{R}^n} 4t_0 \varphi^p F_{\alpha \beta}^{p \cdot \beta} F_{ij}^{\alpha} \partial_i \eta G dx.
\]
It then follows from the Bianchi identity that
\[
\int_{\mathbb{R}^n} 4t_0 \varphi^p F^{\alpha}_{\beta j} J_{j}^{\alpha} G dx
\]
\[
= \int_{\mathbb{R}^n} -4t_0 F^{\alpha}_{\beta j} \varphi^{p} [\partial_i \varphi^p - \frac{(x - x_0)^i}{2t_0} \varphi^p] G dx
\]
\[- \int_{\mathbb{R}^n} 2t_0 \varphi^p \nabla_p F^{\alpha}_{ij} F^{\alpha}_{ij} G dx - \int_{\mathbb{R}^n} 4t_0 \varphi^p F^{\alpha}_{\beta j} F^{\alpha}_{ij} G \partial_i \eta dx
\]
\[= \int_{\mathbb{R}^n} -4t_0 F^{\alpha}_{\beta j} \varphi^{p} [\partial_i \varphi^p - \frac{(x - x_0)^i}{2t_0} \varphi^p] G dx
\]
\[= - \int_{\mathbb{R}^n} 2t_0 \varphi^p \partial_p (|F|^2) G \eta dx - \int_{\mathbb{R}^n} 4t_0 \varphi^p F^{\alpha}_{\beta j} F^{\alpha}_{ij} G \partial_i \eta dx,
\]
i.e.
\[
\int_{\mathbb{R}^n} 2t_0 \varphi^p \partial_p (|F|^2) G \eta dx = - \int_{\mathbb{R}^n} 4t_0 \varphi^p F^{\alpha}_{\beta j} J_{j}^{\alpha} G \eta dx
\]
\[- \int_{\mathbb{R}^n} 4t_0 F^{\alpha}_{\beta j} \varphi^{p} [\partial_i \varphi^p - \frac{(x - x_0)^i}{2t_0} \varphi^p] G \eta dx
\]
\[- \int_{\mathbb{R}^n} 4t_0 \varphi^p F^{\alpha}_{\beta j} F^{\alpha}_{ij} G \partial_i \eta dx.
\]
Thus we have
\[
\int_{\mathbb{R}^n} \varphi^p (x - x_0)^p |F|^2 G(x) \eta(x) dx
\]
\[
= \int_{\mathbb{R}^n} 2t_0 [\partial_p (\varphi^p)] |F|^2 \eta + \varphi^p \partial_p (|F|^2) \eta + \varphi^p |F|^2 \partial_p \eta) G(x) dx
\]
\[
= \int_{\mathbb{R}^n} 2t_0 \partial_p (\varphi^p) |F|^2 G \eta dx - \int_{\mathbb{R}^n} 4t_0 \varphi^p F^{\alpha}_{\beta j} J_{j}^{\alpha} G \eta dx
\]
\[- \int_{\mathbb{R}^n} 4t_0 F^{\alpha}_{\beta j} \varphi^{p} [\partial_i \varphi^p - \frac{(x - x_0)^i}{2t_0} \varphi^p] G \eta dx
\]
\[- \int_{\mathbb{R}^n} 4t_0 \varphi^p F^{\alpha}_{\beta j} F^{\alpha}_{ij} G \partial_i \eta dx + \int_{\mathbb{R}^n} 2t_0 \varphi^p |F|^2 G \partial_p \eta dx
\]
\[= \int_{\mathbb{R}^n} [2t_0 \partial_p (\varphi^p)] |F|^2 - 4t_0 \partial_i \varphi^p F^{\alpha}_{\beta j} F^{\alpha}_{ij} G \eta dx
\]
\[- \int_{\mathbb{R}^n} 4t_0 \varphi^p F^{\alpha}_{\beta j} (J_{j}^{\alpha} - \frac{1}{2t_0} X^{\alpha}_{j} \beta) G \eta dx
\]
\[- \int_{\mathbb{R}^n} 4t_0 \varphi^p F^{\alpha}_{\beta j} F^{\alpha}_{ij} G \partial_i \eta dx + \int_{\mathbb{R}^n} 2t_0 \varphi^p |F|^2 G \partial_p \eta dx.
\]
Therefore for a homothetically shrinking soliton centered at \((x_0, t_0),\)
\[
\int_{\mathbb{R}^n} \varphi^p (x - x_0)^p |F|^2 G \eta dx = \int_{\mathbb{R}^n} [2t_0 \partial_p (\varphi^p)] |F|^2 - 4t_0 \partial_i \varphi^p F^{\alpha}_{\beta j} F^{\alpha}_{ij} G \eta dx
\]
\[- \int_{\mathbb{R}^n} 4t_0 \varphi^p F^{\alpha}_{\beta j} F^{\alpha}_{ij} G \partial_i \eta dx + \int_{\mathbb{R}^n} 2t_0 \varphi^p |F|^2 G \partial_p \eta dx.
\]
(3.6)
Applying to (3.6) with $\eta(x) = \eta_l(x)$, where $\eta_l(x) = 1$ for $|x| \leq l$ and is cut off to zero linearly on $B_{l+1} \setminus B_l$, we get

\begin{equation}
\int_{\mathbb{R}^n} \varphi^p(x - x_0)^p |F|^2 G dx = \int_{\mathbb{R}^n} [2t_0 \partial_p (\varphi^p)|F|^2 - 4t_0 \partial_i \varphi^p F^\alpha_{ij} F^\alpha_{ij}] G dx.
\end{equation}

Taking $\varphi^p = (x - x_0)^p$, by (3.7) we get

\[\int_{\mathbb{R}^n} |x - x_0|^2 |F|^2 G(x) dx = \int_{\mathbb{R}^n} 2(n - 4)t_0|x - x_0|^2 G(x) dx.\]

Taking $\varphi^p = \delta^p_k$, by (3.7) we get for any $k = 1, \ldots, n$,

\[\int_{\mathbb{R}^n} (x - x_0)^k |F|^2 G(x) dx = 0.\]

Taking $\varphi^p = |x - x_0|^2(x - x_0)^p$, by (3.7) and (a) we get

\[\int_{\mathbb{R}^n} |x - x_0|^4 |F|^2 G(x) dx = \int_{\mathbb{R}^n} [2t_0(n + 2)|x - x_0|^2 |F|^2 - 8t_0|x - x_0|^2 |F|^2 - 8t_0|X|^2] G dx = \int_{\mathbb{R}^n} [4(n - 2)(n - 4)t_0^2|F|^2 - 32t_0^2|J|^2] G dx.
\]

Taking $\varphi^p = |x - x_0|^2 V_p$, by (3.7) and (b) we get

\[\int_{\mathbb{R}^n} |x - x_0|^2 \langle V, x - x_0 \rangle |F|^2 G(x) dx = \int_{\mathbb{R}^n} -16t_0^2 \langle J_j, V^p F_{pj} \rangle G dx.
\]

On the other hand taking $\varphi^p = \langle V, x - x_0 \rangle (x - x_0)^p$, by (3.7) and (b) we get

\[\int_{\mathbb{R}^n} |x - x_0|^2 \langle V, x - x_0 \rangle |F|^2 G(x) dx = \int_{\mathbb{R}^n} -8t_0^2 \langle J_j, V^i F_{ij} \rangle G dx.
\]

Thus we have

\[\int_{\mathbb{R}^n} |x - x_0|^2 \langle V, x - x_0 \rangle |F|^2 G(x) dx = \int_{\mathbb{R}^n} (J_j, V^p F_{pj}) G dx = 0.
\]

Taking $\varphi^p = \langle V, x - x_0 \rangle V_p$, by (3.7) we get

\[\int_{\mathbb{R}^n} \langle x - x_0, V \rangle^2 |F|^2 G dx = \int_{\mathbb{R}^n} (2t_0|V|^2|F|^2 - 4t_0 \langle V^i F_{ij}, V^p F_{pj} \rangle) G dx.
\]

By the first variation formula (3.3), and (a) and (b) of Lemma 3.2 we get the following

**Corollary 3.2.** $(A(x), x_0, t_0)$ is a critical point of the $F$-functional if and only if $A(x)$ is a homothetically shrinking soliton centered at $(x_0, t_0)$. 

**Corollary 3.3.** When $n = 2, 3, \text{ or } 4$, there exists no homothetically shrinking soliton such that $|F|$ is uniformly bounded and not identically zero. In particular in dimension four, the Yang-Mills flow on $E_M$ cannot develop a singularity of Type-I.
Proof. The first part follows from Lemma 3.2 (a). By Weinkove’s result 22 (see also Section 2) at a Type-I singularity of a Yang-Mills flow one can obtain a homothetically shrinking soliton on a trivial G-vector bundle over $\mathbb{R}^n$ whose curvature is uniformly bounded and non-zero. Therefore in dimension four if a Type-I singularity occurs, it would contradict with the non-existence of such a homothetically shrinking soliton.

4. SECOND VARIATION OF $\mathcal{F}$-FUNCTIONAL

We now compute the second variation of the $\mathcal{F}$-functional at a homothetically shrinking soliton $A(x)$. Let $d^\nabla$ denote the covariant exterior differentiation on $\mathfrak{g}$-valued forms and $(d^\nabla)^*$ denote the formal adjoint of $d^\nabla$. For a $\mathfrak{g}$-valued 1-form $\theta$, let

\[
R(\theta_j) = R(\theta)(\partial_j) := [F_{ij}, \theta_i],
\]

and

\[
L\theta := -t_0[(d^\nabla)^*d^\nabla \theta + \mathcal{R}(\theta) + \iota_{\frac{1}{2t_0}(x-x_0)}d^\nabla \theta].
\]

We also introduce the space

\[
W_G^{2,2} := \{ \theta : \int_{\mathbb{R}^n} (|\theta|^2 + |\nabla \theta|^2 + |L\theta|^2)G(x)dx < \infty \}.
\]

Denote

\[
i_s|_{s=0} = q, \quad \dot{x}_s|_{s=0} = V, \quad \theta = \frac{d}{ds}|_{s=0}A_s,
\]

\[
\mathcal{F}''_{x_0,t_0}(q,V,\theta) = \frac{d^2}{ds^2}|_{s=0}\mathcal{F}_{x_s,t_s}(A_s).
\]

Proposition 4.1. Let $A(x)$ be a homothetically shrinking soliton in $\mathcal{S}_{x_0,t_0}$; see (3.1). Then for any $\theta \in W_G^{2,2}$, we have

\[
\frac{1}{2t_0}\mathcal{F}''_{x_0,t_0}(q,V,\theta) = \int_{\mathbb{R}^n} (-L\theta - 2qJ - \iota_VF,\theta)Gdx - \int_{\mathbb{R}^n} (q^2|J|^2 + \frac{1}{2}|\iota_VF|^2)Gdx.
\]

Proof. Recall that

\[
\frac{d}{ds}\mathcal{F}_{x_s,t_s}(A_s) = \int_{\mathbb{R}^n} i_s \left( \frac{4-n}{2} t_s + \frac{1}{4}|x-x_s|^2 |F_s|^2 G_s(x)dx \right.
\]

\[+ \int_{\mathbb{R}^n} \left( \frac{1}{2} t_s \langle \dot{x}_s, x-x_s \rangle |\dot{F}_s|^2 G_s(x)dx \right.
\]

\[- \int_{\mathbb{R}^n} 2t_s^2 \langle J_s - \frac{X_0}{2t_s}, \theta_s \rangle G_s(x)dx.
\]

By the assumption that $A(x) \in \mathcal{S}_{x_0,t_0}$ and Lemma 3.2 (a), (b), we have

\[
\mathcal{F}''_{x_0,t_0}(q,V,\theta) = \int_{\mathbb{R}^n} \left[ q \left( \frac{4-n}{2} q - \frac{1}{2}\langle x-x_0, V \rangle \right) + \frac{1}{2} t_0 \langle V, -V \rangle \right] |F|^2 Gdx
\]

\[+ \int_{\mathbb{R}^n} \left[ q \left( \frac{4-n}{2} t_0 + \frac{1}{4}|x-x_0|^2 \right) + \frac{1}{2} t_0 \langle V, x-x_0 \rangle \right] \frac{\partial |F|^2}{\partial s}|_{s=0} Gdx
\]

\[+ \int_{\mathbb{R}^n} \left[ q \left( \frac{4-n}{2} t_0 + \frac{1}{4}|x-x_0|^2 \right) + \frac{1}{2} t_0 \langle V, x-x_0 \rangle \right] \frac{\partial G_s}{\partial s}|_{s=0} Gdx
\]

\[- \int_{\mathbb{R}^n} 2t_0^2 \left( \frac{\partial}{\partial s}|_{s=0} \langle J_s - \frac{X_0}{2t_s}, \theta \rangle \right) Gdx.
\]
Note that

\[
\frac{\partial |\mathcal{F}_s|^2}{\partial s} \bigg|_{s=0} = F_{ij}\theta^\alpha_j - \nabla_j \theta^\alpha_j = 2F_{ij}\nabla_j \theta^\alpha_j,
\]

\[
\frac{\partial G_s}{\partial s} \bigg|_{s=0} = \left(-\frac{n}{2} q + \frac{q|x - x_0|^2}{4t_0^2} + \frac{\langle V, x - x_0 \rangle}{2t_0} \right)G(x),
\]

\[
\frac{\partial}{\partial s} \bigg|_{s=0} f_{\alpha \beta} = \nabla_p \nabla \theta^\alpha - \nabla_p \partial_j \theta^\alpha + \theta^\alpha \partial^\gamma p_j \theta^\gamma - F_{\alpha \gamma p_j \theta^\gamma},
\]

\[
\frac{\partial}{\partial s} \bigg|_{s=0} \left(-\frac{1}{2t_s} X^\alpha_{j\beta} \right) = \frac{q}{2t_0^2} X^\alpha_{j\beta} + \frac{1}{2t_0} V_k F_{k\alpha j \beta} - \frac{1}{2t_0} (x - x_0)^k (\nabla_k \theta^\alpha_{j\beta} - \nabla_j \theta^\alpha_{k\beta}).
\]

Thus we get

\[
\mathcal{F}_{x_0, t_0}^\alpha (q, V, \theta) = \int_{\mathbb{R}^n} [q\left(\frac{4}{2} - \frac{1}{2} \langle x - x_0, V \rangle \right) - \frac{1}{2} t_0 |V|^2] |\mathcal{F}|^2 Gdx
\]

\[
+ \int_{\mathbb{R}^n} \left[q\left(\frac{4}{2} - \frac{1}{2} \langle x - x_0, V \rangle \right) - \frac{1}{2} t_0 |V|^2 \right] 2F_{ij\beta} \nabla_i \theta^\alpha_j Gdx
\]

\[
+ \int_{\mathbb{R}^n} \left[q\left(\frac{4}{2} - \frac{1}{2} \langle x - x_0, V \rangle \right) - \frac{1}{2} t_0 |V|^2 \right] |\mathcal{F}|^2
\]

\[
\times \left(-\frac{n}{2} q + \frac{q|x - x_0|^2}{4t_0^2} + \frac{\langle V, x - x_0 \rangle}{2t_0} \right) Gdx
\]

\[
- \int_{\mathbb{R}^n} 2t_0^2 \nabla_p (\nabla_i \theta^\alpha_j - \nabla_j \theta^\alpha_i) + \theta^\alpha p_j F_{\gamma p_j \theta^\gamma} - F_{\alpha \gamma p_j \theta^\gamma} \theta^\alpha_j Gdx
\]

\[
- \int_{\mathbb{R}^n} 2t_0^2 \left(\frac{q}{2t_0^2} X^\alpha_{j\beta} + \frac{1}{2t_0} V_k F_{k\alpha j \beta} - \frac{1}{2t_0} (x - x_0)^k (\nabla_k \theta^\alpha_{j\beta} - \nabla_j \theta^\alpha_{k\beta}) \right) \theta^\alpha_j Gdx.
\]

By integration by parts, we have

\[
\int_{\mathbb{R}^n} \left[q\left(\frac{4}{2} - \frac{1}{2} \langle x - x_0, V \rangle \right) - \frac{1}{2} t_0 |V|^2 \right] 2F_{ij\beta} \nabla_i \theta^\alpha_j Gdx
\]

\[
= \int_{\mathbb{R}^n} -2q\left(\frac{4}{2} - \frac{1}{2} \langle x - x_0, V \rangle \right) - \frac{1}{2} t_0 \langle V, x - x_0 \rangle |J - \frac{1}{2t_0} X, \theta \rangle Gdx
\]

\[
- \int_{\mathbb{R}^n} 2\frac{1}{2} q(x - x_0)^i - \frac{1}{2} t_0 V^i F_{ij\beta} \theta^\alpha_j Gdx
\]

\[
= \int_{\mathbb{R}^n} \left[-q(x - x_0)^i - t_0 V^i \right] F_{ij\beta} \theta^\alpha_j Gdx.
\]
Then by using Lemma 3.2, we have

\[ \mathcal{F}_{x_0,t_0}''(q,V,\theta) = \int_{\mathbb{R}^n} \left[ \frac{4-n}{2} q^2 - \frac{1}{2} t_0 |V|^2 \right] |F|^2 Gdx \]

\[ + \int_{\mathbb{R}^n} \left[ -q(x-x_0)^i - t_0 V^i \right] F_{ij\beta}^\alpha \theta_{j\beta}^\alpha Gdx \]

\[ + \int_{\mathbb{R}^n} \left[ \frac{n-4}{2} q^2 |F|^2 - 2 t_0 q^2 |J|^2 \right] Gdx \]

\[ + \int_{\mathbb{R}^n} \frac{1}{4} (2 t_0 |V|^2 |F|^2 - 4 t_0 \langle V^i F_{ij}, V^p F_{pj} \rangle) Gdx \]

\[ - \int_{\mathbb{R}^n} 2 t_0^2 \left[ \nabla_p (\nabla_p \theta_{j\beta}^\alpha - \nabla_j \theta_{p\beta}^\alpha) + \theta_{p\gamma}^\alpha F_{p\gamma j\beta}^\alpha - F_{p\gamma j\beta}^\alpha \theta_{p\beta}^\alpha \right] Gdx \]

\[ - \int_{\mathbb{R}^n} [q(x-x_0)^i + t_0 V^i] F_{ij\beta}^\alpha \theta_{j\beta}^\alpha Gdx \]

\[ + \int_{\mathbb{R}^n} t_0 (x-x_0)^k (\nabla_k \theta_{j\beta}^\alpha - \nabla_j \theta_{k\beta}^\alpha) \theta_{j\beta}^\alpha Gdx. \]

Thus,

\[ \mathcal{F}_{x_0,t_0}''(q,V,\theta) = \int_{\mathbb{R}^n} \left[ -2q(x-x_0)^i - 2t_0 V^i \right] F_{ij\beta}^\alpha \theta_{j\beta}^\alpha Gdx \]

\[ - \int_{\mathbb{R}^n} 2 t_0 q^2 |J|^2 Gdx - \int_{\mathbb{R}^n} t_0 \langle V^i F_{ij}, V^p F_{pj} \rangle Gdx \]

\[ - \int_{\mathbb{R}^n} 2 t_0^2 \left[ \nabla_p (\nabla_p \theta_{j\beta}^\alpha - \nabla_j \theta_{p\beta}^\alpha) + \theta_{p\gamma}^\alpha F_{p\gamma j\beta}^\alpha - F_{p\gamma j\beta}^\alpha \theta_{p\beta}^\alpha \right] Gdx \]

\[ + \int_{\mathbb{R}^n} t_0 (x-x_0)^k (\nabla_k \theta_{j\beta}^\alpha - \nabla_j \theta_{k\beta}^\alpha) \theta_{j\beta}^\alpha Gdx. \]

Note that

\[ (d^V)^* d^V \theta_j = - \nabla_p (\nabla_p \theta_j - \nabla_j \theta_p), \]

\[ R(\theta_j) = [F_{pj}, \theta_p] = F_{pj} \theta_p - \theta_p F_{pj}, \]

\[ i_{x-x_0} d^V \theta_j = (x-x_0)^k (\nabla_k \theta_j - \nabla_j \theta_k), \]

so we have

\[ \mathcal{F}_{x_0,t_0}''(q,V,\theta) = \int_{\mathbb{R}^n} 2 t_0^2 \langle (d^V)^* d^V \theta_j + R(\theta_j) + i \frac{1}{2 t_0} (x-x_0)^k d^V \theta_j, \theta_j \rangle Gdx \]

\[ - \int_{\mathbb{R}^n} 2 t_0 (2q J_j + V^i F_{ij}, \theta_j) Gdx \]

\[ -2 t_0 \int_{\mathbb{R}^n} (q^2 |J|^2 + \frac{1}{2} |i_{V} F|^2) Gdx. \]

Let

\[ L = - t_0 \langle (d^V)^* d^V + R + i \frac{1}{2 t_0} (x-x_0)^k d^V \rangle; \]

then we have

\[ \frac{1}{2 t_0} \mathcal{F}_{x_0,t_0}''(q,V,\theta) = \int_{\mathbb{R}^n} \langle -L \theta - 2q J - i_{V} F, \theta \rangle Gdx - \int_{\mathbb{R}^n} (q^2 |J|^2 + \frac{1}{2} |i_{V} F|^2) Gdx. \]
5. F-stability and its characterization

In this section we define the F-stability for homothetically shrinking solitons in $S_{x_0,t_0}$. The operator $L$ admits eigenfields $J$ and $iV F$ of eigenvalues $-1$ and $-\frac{1}{2}$, respectively. F-stability is equivalent to the semi-positiveness of $-L$ modulo the vector space spanned by $J$ and $iV F$. Let $C_0^\infty(\Omega^1 \otimes g)$, or simply $C_0^\infty$, denote the space of $g$-valued 1-forms with compact support on $\mathbb{R}^n$. The space $C_0^\infty$ is dense in $W^{2,2}_G$.

**Definition 5.1.** A homothetically shrinking soliton $A \in S_{x_0,t_0}$ is called F-stable if for any $\theta$ in $C_0^\infty$, or equivalently in $W^{2,2}_G$, there exist a real number $q$ and a vector $V$ such that

$$F''_{x_0,t_0}(q,V,\theta) \geq 0.$$ 

Given a homothetically shrinking soliton $A \in S_{x_0,t_0}$ with an exponential gauge, the rescaling

$$\tilde{A}_i(x) = \sqrt{\alpha}_0 A_i(\sqrt{\alpha}_0 x + x_0)$$

is a homothetically shrinking soliton in $S_{0,1}$. Without loss of generality, in the remainder of this section we let $x_0 = 0$ and $t_0 = 1$. Then

$$G(x) = (4\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4}}$$

and

$$L \theta = -[(d\nabla)^* d\nabla \theta + R(\theta) + i_\theta d\nabla \theta].$$

The operator $L$ is self-adjoint in the following sense: for any $\theta, \eta \in W^{2,2}_G$,

$$\int_{\mathbb{R}^n} \langle L \theta, \eta \rangle G dx = -\int_{\mathbb{R}^n} \langle [d\nabla \theta,d\nabla \eta] + \langle R(\theta),\eta \rangle \rangle G dx = \int_{\mathbb{R}^n} \langle \theta, L \eta \rangle G dx.$$ 

A $g$-valued 1-form $\theta \in W^{2,2}_G$ is called an eigenfield of $L$ and of eigenvalue $\lambda$ if $L \theta = -\lambda \theta$. We denote the eigenfield space of eigenvalue $\lambda$ by $E_\lambda$.

**Proposition 5.1.** Let $A$ be a homothetically shrinking soliton in $S_{0,1}$. Then

$$L J = J,$$

and

$$L(iV F) = \frac{1}{2} iV F, \quad \forall V \in \mathbb{R}^n.$$ 

**Proof.** Note that

$$J_j = \nabla_p F_{pj} = \frac{1}{2} x^p F_{pj},$$

$$L = -(d\nabla)^* d\nabla - R - i_\theta d\nabla,$$

and

$$L J_j = \nabla_p \nabla_p J_j - \nabla_p \nabla_j J_p - [F_{pj}, J_p] - \frac{1}{2} (d\nabla J)(x^p \partial_p, \partial_j)$$

$$= \nabla_p J_j - \nabla_p J_j - [F_{pj}, J_p] - \frac{1}{2} x^p (\nabla_p J_j - \nabla_j J_p).$$

We have

$$\nabla_p J_j = \nabla_p \left( \frac{1}{2} x^q F_{qj} \right) = \frac{1}{2} F_{pj} + \frac{1}{2} x^q \nabla_p F_{qj}.$$ 

Then

$$\nabla_p \nabla_j J_p = \nabla_p \left( -\frac{1}{2} F_{pj} + \frac{1}{2} x^q \nabla_j F_{pq} \right) = -\frac{1}{2} J_j - \frac{1}{2} x^q \nabla_p \nabla_j F_{pq},$$
and by using the Bianchi identity and the Ricci formula, we get
\[
\nabla_p \nabla_p J_j = \nabla_p F_{p j} + \frac{1}{2} x^q \nabla_p \nabla_p F_{q j}
\]
\[
= \nabla_p F_{p j} + \frac{1}{2} x^q \nabla_p (-\nabla_q F_{j p} - \nabla_j F_{p q})
\]
\[
= \nabla_p F_{p j} - \frac{1}{2} x^q (\nabla_q \nabla_p F_{j p} + F_{p q} F_{j p} - F_{j p} F_{p q}) - \frac{1}{2} x^q \nabla_p \nabla_j F_{p q}
\]
\[
= J_j + \frac{1}{2} x^q \nabla_q J_j + [J_p, F_{j p}] - \frac{1}{2} x^q \nabla_p \nabla_j F_{p q}.
\]
Hence
\[
L J_j = \frac{3}{2} J_j + \frac{1}{2} x^p \nabla_j J_p.
\]
The identity (5.2) then follows from
\[
\frac{1}{2} x^p \nabla_j J_p = \frac{1}{2} \nabla_j (x^p J_p) - \frac{1}{2} J_j = \frac{1}{2} \nabla_j (\frac{1}{2} x^p x^q F_{p q}) - \frac{1}{2} J_j = -\frac{1}{2} J_j.
\]

We now prove (5.3). By using the Bianchi identity and the Ricci formula, we get
\[
\nabla_p \nabla_p (V^q F_{q j}) = V^q \nabla_p (-\nabla_q F_{j p} - \nabla_j F_{p q})
\]
\[
= -V^q (\nabla_q \nabla_p F_{j p} + F_{p q} F_{j p} - F_{j p} F_{p q}) - V^q \nabla_p \nabla_j F_{p q}
\]
\[
= V^q \nabla_q (\frac{1}{2} x^p F_{p j}) + [V^q F_{q p}, F_{j p}] + \nabla_p \nabla_j (V^q F_{q p});
\]
hence
\[
L (V^q F_{q j}) = \nabla_p \nabla_p (V^q F_{q j}) - \nabla_p \nabla_j (V^q F_{q p}) - [F_{p j}, V^q F_{q p}]
\]
\[
= \frac{1}{2} x^p [\nabla_p (V^q F_{q j}) - \nabla_j (V^q F_{q p})]
\]
\[
= V^q \nabla_q (\frac{1}{2} x^p F_{p j}) - \frac{1}{2} x^p [\nabla_p (V^q F_{q j}) - \nabla_j (V^q F_{q p})]
\]
\[
= \frac{1}{2} V^q F_{q j} + \frac{1}{2} x^p V^q (\nabla_q F_{p j} + \nabla_p F_{j q} + \nabla_j F_{p q})
\]
\[
= \frac{1}{2} V^q F_{q j}.
\]

\[\square\]

**Corollary 5.1.** Let \( A \) be a homothetically shrinking soliton in \( S_{0,1} \). If \(|F|^2 < \frac{n}{2(n-1)}\), then \((E, A)\) is flat.

**Proof.** Note that \( J_j = \nabla^p F_{p j} = \frac{1}{2} x^p F_{p j} \). By integration by parts, we have
\[
\int_{\mathbb{R}^n} (d^\nabla)^* d^\nabla J + \frac{i}{2} d^\nabla J, J)Gdx = \int_{\mathbb{R}^n} |d^\nabla J|^2 Gdx.
\]
On the other hand by (5.2), we have
\[
\int_{\mathbb{R}^n} (d^\nabla)^* d^\nabla J + \frac{i}{2} d^\nabla J, J)Gdx = \int_{\mathbb{R}^n} (LJ - J, J)Gdx
\]
\[
= -\int_{\mathbb{R}^n} |J|^2 Gdx - \int_{\mathbb{R}^n} (F_{i j}, J_i, J_j)Gdx.
\]
For any $B, C \in \text{so}(r)$, we have $|\langle [B, C], J \rangle| \leq |B||C|$; see Lemma 2.30 in [3]. Hence

\[
|\langle [F_{ij}, J_i], J_j \rangle| \leq |F_{ij}, J_i||J_j| = 2 \sum_{i<j} |F_{ij}, J_i||J_j| \leq 2 \sqrt{\sum_{i<j} |F_{ij}, J_i|^2} \sqrt{\frac{1}{2}(|J_i|^4 - \sum_k |J_k|^4)} \leq 2 |F| \sqrt{\frac{1}{n} (1 - \frac{1}{n}) |J|^4} = 2 \sqrt{\frac{2(n-1)}{n} |F||J|^2}
\]

and

\[
\int_{\mathbb{R}^n} |\nabla J|^2 G dx \leq \int_{\mathbb{R}^n} \left( \frac{2(n-1)}{n} |F| - 1 \right) |J|^2 G dx.
\]

If $|F|^2 < \frac{n}{2(n-1)}$, one then gets $J = 0$. Note that if $A \in \mathcal{S}_{0,1}$ has $J = 0$, then for any $t_0 > 0$ we have $J = \frac{1}{2t_0} X$. Hence Lemma [5.2] (a) holds for any $t_0 > 0$ and $F$ vanishes.

**Theorem 5.2.** Let $A$ be a homothetically shrinking soliton in $\mathcal{S}_{0,1}$. Then it is $F$-stable if and only if the following properties are satisfied:

1. $E_{-1} = \{ cJ , \ c \in \mathbb{R} \}$;
2. $E_{-\frac{1}{2}} = \{ iVF , \ V \in \mathbb{R}^n \}$;
3. $E_\lambda = \{ 0 \}$, for any $\lambda < 0$ and $\lambda \neq -1, -\frac{1}{2}$.

**Proof.** Let $\theta$ be a $g$-value $1$-form in $W^{2,2}_G$ of the form

\[
\theta = aJ + iWF + \tilde{\theta}, \quad a \in \mathbb{R}, W \in \mathbb{R}^n,
\]

and satisfying

\[
\int_{\mathbb{R}^n} \langle \tilde{\theta}, J \rangle G dx = \int_{\mathbb{R}^n} \langle \tilde{\theta}, iVF \rangle G dx = 0, \quad \forall V \in \mathbb{R}^n.
\]

Then it follows from Proposition 4.1, Proposition 5.1 and (5.1) that

\[
\frac{1}{2} F_{0,1}''(q, V, \theta) = \int_{\mathbb{R}^n} \langle -L\theta - 2qJ - iVF, \theta \rangle G dx - \int_{\mathbb{R}^n} (q^2 |J|^2 + \frac{1}{2} |iVF|^2) G dx
\]

\[
= \int_{\mathbb{R}^n} \langle -aJ - \frac{1}{2} iWF - L\tilde{\theta}, aJ + iWF + \tilde{\theta} \rangle G dx + \int_{\mathbb{R}^n} \langle -2qJ - iVF, aJ + iWF + \tilde{\theta} \rangle G dx
\]

\[
- \int_{\mathbb{R}^n} (q^2 |J|^2 + \frac{1}{2} |iVF|^2) G dx
\]

\[
= -(a + q)^2 \int_{\mathbb{R}^n} |J|^2 G dx - \frac{1}{2} \int_{\mathbb{R}^n} |iVF|^2 G dx
\]

\[
+ \int_{\mathbb{R}^n} \langle -L\tilde{\theta}, \tilde{\theta} \rangle G dx.
\]

Letting $q = -a, V = -W$, one has the equivalence. \qed
6. Entropy and Entropy-stability

We now introduce $\lambda$-entropy of connections on the trivial $G$-vector bundle $E$ over $\mathbb{R}^n$. We shall show that along the Yang-Mills flow, the entropy is non-increasing. We also prove that the entropy of a homothetically shrinking soliton $A(x) \in S_{x_0, t_0}$ is achieved exactly by $F_{x_0, t_0}(A)$, provided that $i_V F \neq 0$ for any non-zero vector $V \in \mathbb{R}^n$.

**Definition 6.1.** Let $A(x)$ be a connection on $E$. We define the entropy by

\[
\lambda(A) = \sup_{x_0 \in \mathbb{R}^n, t_0 > 0} F_{x_0, t_0}(A).
\]

We first consider the invariance property of the entropy.

**Proposition 6.1.** The entropy $\lambda$ is invariant under translations and rescalings.

*Proof.* Let $A(x)$ be a connection on $E$. A translation of $A(x)$ is a new connection, denoted by $\tilde{A}(x)$, of the form

\[
\tilde{A}_i(x) = A_i(x + x_1),
\]

where $x_1$ is a point in $\mathbb{R}^n$. For any $x_0 \in \mathbb{R}^n$ and $t_0 > 0$, we have

\[
F_{x_0 - x_1, t_0}(\tilde{A}) = F_{x_0, t_0}(A).
\]

Hence

\[
\lambda(\tilde{A}) = \lambda(A).
\]

A rescaling of $A(x)$ is a new connection, denoted by $A^c(x)$, of the form

\[
A^c_i(x) = c^{-1} A_i(c^{-1} x),
\]

where $c$ is a positive number. Then, by setting $y = c^{-1} x$, we have

\[
F_{cx_0, c^2 t_0}(A^c) = (c^2 t_0)^2 \int_{\mathbb{R}^n} |F^c(x)|^2 (4\pi c^2 t_0)^{-\frac{n}{2}} e^{-\frac{|x - cx_0|^2}{4c^2 t_0}} dx
\]

\[
= (c^2 t_0)^2 \int_{\mathbb{R}^n} c^{-4}|F(c^{-1} x)|^2 (4\pi c^2 t_0)^{-\frac{n}{2}} e^{-\frac{|c^{-1} x - x_0|^2}{4c^2 t_0}} dx
\]

\[
= (c^2 t_0)^2 \int_{\mathbb{R}^n} c^{-4}|F(y)|^2 (4\pi c^2 t_0)^{-\frac{n}{2}} e^{-\frac{|y - cx_0|^2}{4c^2 t_0}} c^n dy
\]

\[
= t_0^2 \int_{\mathbb{R}^n} |F(y)|^2 (4\pi t_0)^{-\frac{n}{2}} e^{-\frac{|y - x_0|^2}{4t_0}} dy
\]

\[
= F_{x_0, t_0}(A).
\]

Hence

\[
\lambda(A^c) = \lambda(A).
\]

\[\square\]

In the case that $A(x)$ is a homothetically shrinking soliton, Proposition 6.1 explains why in Theorem 5.2 $J$ and $i_V F$ do not violate the F-stability.

**Proposition 6.2.** Let $A(x, t)$ be a solution to the Yang-Mills flow on $E$. Then the entropy $\lambda(A(x, t))$ is non-increasing in $t$. 
Proof. Let \( t_1 < t_2 < T \). Here \( T \) denotes the first singular time of the Yang-Mills flow. By \((6.1)\), for any given \( \epsilon > 0 \) there exists \((x_0, t_0)\) such that

\[ \lambda(A(x, t_2)) - \epsilon \leq F_{x_0, t_0}(A(x, t_2)). \]

Note that for any \( c > 0 \) and \( 0 \leq t < T \), we have

\[ F_{x_0, c}(A(x, t_2)) = \Phi_{x_0, c+t}(A(x, t)). \]

By \((6.3)\), the monotonicity formula \((2.8)\), and the definition of entropy, we have

\[ F_{x_0, t_0}(A(x, t_2)) = \Phi_{x_0, t_0+t_2}(A(x, t_1)) \leq \lambda(A(x, t_1)). \]

Together with \((6.2)\), we see that

\[ \lambda(A(x, t_2)) \leq \lambda(A(x, t_1)). \]

\[ \square \]

**Definition 6.2.** A homothetically shrinking soliton \( A(x) \) is called entropy-stable if it is a local minimum of the entropy, among all perturbations \( \tilde{A}(x) \), such that \( ||\tilde{A} - A||_{C^1} \) is sufficiently small.

In general the entropy \( \lambda(A) \) is not attained by any \( F_{x_0, t_0}(A) \). However if \( A \in S_{x_0, t_0} \) and \( i_V F \neq 0 \) for any \( V \in \mathbb{R}^n \), we will show that \( \lambda(A) \) is attained exactly by \( F_{x_0, t_0}(A) \). We first examine the geometric meaning of \( i_V F = 0 \) in the case that \( A(x) \) is a homothetically shrinking soliton.

**Proposition 6.3.** If \( A(x) \) is a homothetically shrinking soliton satisfying \( i_V F = 0 \) for some non-zero vector \( V \), then \( A(x) \) is defined on a hyperplane perpendicular to \( V \).

*Proof.* Without loss of generality we assume \( A(x) \) is centered at \((0, 1)\) and let \( A(x, t) \) be the homothetically shrinking Yang-Mills flow with \( A(x, 0) = A(x) \). In the exponential gauge, i.e. a gauge such that \( x^i A_j(x) = 0 \), we have for any \( t < 1 \) and \( \lambda > 0 \) that

\[ A_j(x, t) = \lambda A_j(\lambda x, \lambda^2(t - 1) + 1) = \frac{1}{\sqrt{1-t}} A_j\left(\frac{x}{\sqrt{1-t}}, 0\right) = \frac{1}{\sqrt{1-t}} A_j\left(\frac{x}{\sqrt{1-t}}\right) \]

and

\[ F_{ij}(x, t) = \frac{1}{1-t} F_{ij}(\frac{x}{\sqrt{1-t}}). \]

Moreover the exponential gauge is uniform for all \( t < 1 \), i.e. \( x^j A_j(x, t) = 0 \).

By assumption we have \( i_V F(x) = 0 \). For simplicity let \( V = \frac{\partial}{\partial x^j} \). Then by \((6.5)\) we have

\[ F_{jil}(x, t) = 0, \forall j. \]

Note that \( A(x, t) \) is a homothetically shrinking Yang-Mills flow; hence

\[ J_i(x, t) = \frac{1}{2(1-t)} x^j F_{jil}(x, t) = 0. \]

Then

\[ \frac{\partial}{\partial t} A_l(x, t) = J_l(x, t) = 0. \]

In particular,

\[ A_l(x, t') = A_l(x, t), \forall t, t' < 1. \]
Then by (6.4), we have for any \( \lambda > 0 \) and \( t < 1 \) that
\[
A_t(x, t) = \lambda A_t(\lambda x, \lambda^2 (t - 1) + 1) = \lambda A_t(\lambda x, t).
\]
Letting \( \lambda \to 0 \), we see that
\[
(6.6) \quad \lambda A_t(\lambda x, t) = 0.
\]
Note that
\[
0 = F_{ij}(x, t) = \partial_t A_j - \partial_j A_t + A_t A_j - A_j A_t = \partial_t A_j(x, t),
\]
so for any \( j \), we have
\[
\partial_t A_j(x, t) = 0
\]
and
\[
(6.7) \quad A_j(x + cV, t) = A_j(x, t), \quad \forall c \in \mathbb{R}.
\]
For example if \( V = \frac{\partial}{\partial x^n} \), then by (6.6) and (6.7) we have
\[
A(x^1, \ldots, x^{n-1}, x^n) = A_1(x^1, \ldots, x^{n-1}, 0, t)dx^1 + \cdots + A_{n-1}(x^1, \ldots, x^{n-1}, 0, t)dx^{n-1}.
\]
In particular for \( V = \frac{\partial}{\partial x^n} \) and in the exponential gauge, we have
\[
(6.8) \quad A(x^1, \ldots, x^{n-1}, x^n) = A_1(x^1, \ldots, x^{n-1}, 0)dx^1 + \cdots + A_{n-1}(x^1, \ldots, x^{n-1}, 0)dx^{n-1}.
\]
This means that \( A(x) \) is defined on a hyperplane perpendicular to \( V \), i.e. \( A(x) \) descends to a trivial \( G \)-vector bundle over a hyperplane \( V^\perp \).

The following proposition is analogous to a corresponding result for self-shrinkers of the mean curvature flow; see [8]. We follow closely the arguments given in [8].

**Proposition 6.4.** Let \( A(x) \) be a homothetically shrinking soliton centered at \((0, 1)\) such that \( i_VF \neq 0 \) for any non-zero \( V \). Then the function \((x_0, t_0) \mapsto F_{x_0, t_0}(A)\) attains its strict maximum at \((0, 1)\). In fact for any given \( \epsilon > 0 \), there exists a constant \( \delta > 0 \) such that
\[
(6.9) \quad \sup\{F_{x_0, t_0}(A) : |x_0| + |\log t_0| \geq \epsilon \} < \lambda(A) - \delta.
\]
In particular, the entropy of \( A \) is achieved by \( F_{x_0, t_0}(A) \).

**Proof.** We first show that \((0, 1)\) is a local maximum of the function \((x_0, t_0) \mapsto F_{x_0, t_0}(A)\). That is, to show
\[
F'_{0,1}(q, V, 0) = 0, \quad \forall q, V,
\]
and
\[
F''_{0,1}(q, V, 0) = 0, \quad \forall (q, V) \neq (0, 0).
\]
In fact by the first variation formula (3.3) and Lemma 3.2 (a), (b), we have
\[
\frac{d}{ds}|_{s=0} F_{x_s, t_s}(A) = F'_{0,1}(q, V, 0) = 0.
\]
Let \( x_s = sV, t_s = 1 + sq \). Note that \( J \neq 0 \). Otherwise \( F \) would be vanishing, as shown in the proof of Corollary 5.1 which violates the assumption that \( i_VF \neq 0 \) for any non-zero \( V \). Then by the second variation formula (4.4), we have for any \((q, V) \neq (0, 0)\) that
\[
\frac{1}{2} F''_{0,1}(q, V, 0) = -\int_{\mathbb{R}^n} (q^2 |J|^2 + \frac{1}{2} |i_VF|^2) Gdx < 0.
\]
For any fixed \((y, T)\), where \(y \in \mathbb{R}^n\) and \(T > 0\), we set
\[
x_s = sy, \quad t_s = 1 + (T - 1)s^2.
\]
Note that \((x_s, t_s), s \in [0, 1]\), is a path from \((0, 1)\) to \((y, T)\). Let
\[
g(s) = F_{x_s, t_s}(A).
\]
The remainder of the proof is to show that \(g'(s) \leq 0\) for \(s \in [0, 1]\).

By the first variation formula \((3.3)\), we have
\[
g'(s) = \int_{\mathbb{R}^n} \tilde{t}(s) \left\{ \frac{4 - n}{2} t_s + \frac{1}{4} |x - x_s|^2 \right\} |F|^2 G_s(x) dx
+ \int_{\mathbb{R}^n} \frac{1}{2} t_s (\dot{x}_s - x_s) |F|^2 G_s(x) dx.
\]
In the same way as in the proof of Lemma \(3.2\) for vector fields \(\varphi\) on \(\mathbb{R}^n\) we have
\[
\int_{\mathbb{R}^n} \varphi^p (x - x_s)^p |F|^2 G_s(x) dx
= \int_{\mathbb{R}^n} [2t_s \partial_p (\varphi^p) |F|^2 - 4t_s \partial_{ij}[\varphi^p F^\alpha_{pj}] F^\alpha_{ij}] G_s dx
- \int_{\mathbb{R}^n} 4t_s \varphi^p F^\alpha_{pj} (J^\alpha_{j \beta} - \frac{1}{2t_s} X^\alpha_{j \beta}) G_s dx,
\]
where
\[
X^\alpha_{j \beta} = (x - x_s)^p F^\alpha_{pj}.
\]
Taking \(\varphi = \frac{\partial}{\partial x^i}\) and noting that \(J_j = \frac{x^p}{2} F^\alpha_{pj}\), we get
\[
\int_{\mathbb{R}^n} (x - x_s)^p |F|^2 G_s(x) dx = -\int_{\mathbb{R}^n} 4t_s F^\alpha_{pj} (J^\alpha_{j \beta} - \frac{1}{2t_s} X^\alpha_{j \beta}) G_s dx
= -\int_{\mathbb{R}^n} 4t_s F^\alpha_{pj} \left( \frac{1}{2} x^i - \frac{1}{2t_s} (x - x_s)^i \right) F^\alpha_{ij \beta} G_s dx.
\]
Taking \(\varphi(x) = x - x_s\), we get
\[
\int_{\mathbb{R}^n} |x - x_s|^2 |F|^2 G_s dx
= \int_{\mathbb{R}^n} [2(n - 4)t_s |F|^2] G_s dx - \int_{\mathbb{R}^n} 4t_s X^\alpha_{j \beta} (J^\alpha_{j \beta} - \frac{1}{2t_s} X^\alpha_{j \beta}) G_s dx
= \int_{\mathbb{R}^n} [2(n - 4)t_s |F|^2 + 2|X|^2] G_s dx - \int_{\mathbb{R}^n} 2t_s X^\alpha_{j \beta} x^i F^\alpha_{ij \beta} G_s dx.
\]
Hence we have
\[
g'(s) = -\int_{\mathbb{R}^n} \frac{n - 4}{2} t_s \tilde{t}_s |F|^2 G_s(x) dx
+ \frac{1}{4} t_s \left[ \int_{\mathbb{R}^n} [2(n - 4)t_s |F|^2 + 2|X|^2] G_s dx - \int_{\mathbb{R}^n} 2t_s X^\alpha_{j \beta} x^i F^\alpha_{ij \beta} G_s dx \right]
- t_s y^p \int_{\mathbb{R}^n} t_s F^\alpha_{pj \beta} (x^i - \frac{1}{t_s} (x - x_s)^i) F^\alpha_{ij \beta} G_s dx
= \frac{1}{2} t_s \left[ \int_{\mathbb{R}^n} |X|^2 G_s dx - \int_{\mathbb{R}^n} t_s X^\alpha_{j \beta} x^i F^\alpha_{ij \beta} G_s dx \right]
- t_s y^p \int_{\mathbb{R}^n} t_s F^\alpha_{pj \beta} (x^i - \frac{1}{t_s} (x - x_s)^i) F^\alpha_{ij \beta} G_s dx.
Set \( z = x - x_s = x - sy \). We have \( x = z + sy \) and \( X_j = z^i F_{ij} \). Then we get

\[
g'(s) = \frac{1}{2} \dot{t}_s \int_{\mathbb{R}^n} (1 - t_s) |X|^2 G_s dx - \int_{\mathbb{R}^n} t_s X_j \alpha s y_i F_{ij} \alpha G_s dx \\
- t_s y^p \int_{\mathbb{R}^n} t_s F^\alpha_{pj} (z^i + sy^i - \frac{1}{t_s} z^i) F_{ij} \alpha G_s dx \\
= \frac{1}{2} \dot{t}_s \int_{\mathbb{R}^n} (1 - t_s) |X|^2 G_s dx - \int_{\mathbb{R}^n} t_s X_j \alpha s y_i F_{ij} \alpha G_s dx \\
- t_s y^p \int_{\mathbb{R}^n} (t_s - 1) F^\alpha_{pj} X_j \alpha G_s dx - t_s \int_{\mathbb{R}^n} s y^p F^\alpha_{pj} y^i F_{ij} \alpha G_s dx \\
= \frac{1}{2} \dot{t}_s (1 - t_s) \int_{\mathbb{R}^n} |X|^2 G_s dx - (\frac{1}{2} st_s t_s + t_s (t_s - 1)) \int_{\mathbb{R}^n} \langle X_j, y^i F_{ij} \rangle G_s dx \\
- st_s \int_{\mathbb{R}^n} |y^i F_{ij}|^2 G_s dx.
\]

For \( t_s = 1 + (T - 1)s^2 \), we have

\[
g'(s) = -s\left[(T-1)^2s^2 \int_{\mathbb{R}^n} |X|^2 G_s dx + 2(T-1)s \int_{\mathbb{R}^n} \langle X_j, y^i F_{ij} \rangle G_s dx \right] \\
+ t_s s \int_{\mathbb{R}^n} |y^i F_{ij}|^2 G_s dx \\
= -s \int_{\mathbb{R}^n} [(T-1)s X_j + t_s y^i F_{ij}]^2 G_s dx \\
\leq 0.
\]

\[
\square
\]

7. Entropy-stability and F-stability

In this section we shall show that the entropy-stability of a homothetically shrinking soliton such that \( i_V F \neq 0 \) for any non-zero \( V \) implies F-stability.

**Theorem 7.1.** Let \( A(x) \) be a homothetically shrinking soliton in \( S_{0,1} \) such that \( i_V F \neq 0 \) for any non-zero \( V \). If \( A(x) \) is entropy-stable, then it is F-stable.

**Proof.** We argue by contradiction. Assume that \( A(x) \) is F-unstable. By the definition of F-stability there exists a 1-parameter family of connections \( A_s(x), s \in [-\epsilon, \epsilon] \), with \( \theta_s(x) := \frac{d}{ds} A_s(x) \in C_0^\infty \), such that for any deformation \( (x_s, t_s) \) of \( (x_0 = 0, t_0 = 1) \), we have

\[
(7.1) \quad \frac{d^2}{ds^2}|_{s=0} F_{x_s,t_s}(A_s) < 0.
\]

We start from this to show that \( A \) is entropy-unstable. Let

\[
H : \mathbb{R}^n \times \mathbb{R}^+ \times [-\epsilon, \epsilon], \quad H(y, T, s) = F_{y,T}(A_s).
\]
In fact we will show that there exists $\epsilon_0 > 0$ such that for $s$ with $0 < |s| \leq \epsilon_0$, 
\begin{equation}
\sup_{y,T} \ H(y, T, s) < H(0, 1, 0).
\end{equation}
Hence for $s$ with $0 < |s| \leq \epsilon_0$, $\lambda(A_s) < \lambda(A)$, which contradicts our assumption.

Step 1. We prove that there exists $\epsilon_1 > 0$ such that for any $s$ with $0 < |s| \leq \epsilon_1$, 
\begin{equation}
\sup_{y,T} \{H(y, T, s) : |y| \leq \epsilon_1, |\log T| \leq \epsilon_1\} < H(0, 1, 0).
\end{equation}
By the assumption that $A(x) \in S_{0, 1}$ and Corollary 3.2 we have \[\nabla H(0, 1, 0) = 0.\]
For any $y \in \mathbb{R}^n, a \in \mathbb{R}$ and $b \in \mathbb{R}$, $(sy, 1 + as, bs)$ is a curve through $(0, 1, 0)$. In the case of $b \neq 0$, by (7.1) we have
\[\frac{d^2 H}{ds^2}|_{s=0}(sy, 1 + as, bs) = \frac{d^2}{ds^2}|_{s=0}F_{syy, 1+as}(A_{bs}) = b^2 \frac{d^2}{ds^2}|_{s=0}F_{syy, 1+a_s}(A_s) < 0.\]
For $b = 0$ and $(a, y) \neq (0, 0)$, we have
\[\frac{d^2 H}{ds^2}|_{s=0}(sy, 1 + as, 0) = \frac{d^2}{ds^2}|_{s=0}F_{syy, 1+as}(A) = -2 \int_{\mathbb{R}^n} (a^2 |J|^2 + \frac{1}{2}|i_y F|^2) G dx < 0,\]
where we used the assumption that $i_y F \neq 0$ for $y \neq 0$ and its implication that $J \neq 0$. Hence the Hessian of $H$ at $(0, 1, 0)$ is negative definite and $H$ has a local strict maximum at $(0, 1, 0)$. Thus there exists $\epsilon_1 \in (0, \epsilon]$ such that if $0 < |y| + |\log T| + |s| \leq 3\epsilon_1$, then $H(y, T, s) < H(0, 1, 0)$. In particular for any $s$ with $0 < |s| \leq \epsilon_1$, we have 
\[\sup_{y,T} \{H(y, T, s) : |y| \leq \epsilon_1, |\log T| \leq \epsilon_1\} < H(0, 1, 0).\]

Step 2. We prove that there exists $R_0 > 0$ such that 
\begin{equation}
\sup_{T,s} H(y, T, s) < H(0, 1, 0), \quad \text{for} \quad |y| \geq R_0.
\end{equation}
Denote the support of $\theta_s$ by $\Omega_s$ and $\Omega = \bigcup_{s \in [-\epsilon, \epsilon]} \Omega_s$. Then on $\mathbb{R}^n \setminus \Omega$, $F_s = F$. Hence
\begin{align*}
H(y, T, s) &= T^2 \int_{\Omega} |F_s|^2(4\pi T)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4T}} dx + T^2 \int_{\mathbb{R}^n \setminus \Omega} |F|^2(4\pi T)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4T}} dx \\
&\leq T^2 \int_{\Omega} |F_s|^2(4\pi T)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4T}} dx + H(y, T, 0).
\end{align*}
Note that for $|y| \geq 1$, there exists $\delta > 0$ such that $H(y, T, 0) \leq H(0, 1, 0) - \delta$; see Proposition 6.4. Let $M = \sup\{|F_s(x)|^2 : s \in [-\epsilon, \epsilon], x \in \mathbb{R}^n\}$, $D = \sup_{x \in \Omega}|x|$ and $|\Omega| = \int_\Omega dx$. Then for $|y| \geq D + R$ with $R \geq 1$, we have

$$H(y, T, s) \leq M|\Omega|T^2(4\pi T)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4T}} + H(0, 1, 0) - \delta.$$  

Let $f(r) = r^n e^{-\frac{r^2}{4T}}$, $r > 0$, which is uniformly bounded. Note that $n \geq 5$. Then as $R \to \infty$, $T^4 e^{-\frac{R^2}{4T}} = f\left(\frac{T}{R^2}\right)R^4 \to 0$, uniformly in $T > 0$. Hence we can choose sufficiently large $R$ such that for $|y| \geq D + R := R_0$, we have $H(y, T, s) \leq H(0, 1, 0) - \frac{\delta}{2}$.

**Step 3.** We prove that there exists $T_0 > 0$ such that

$$\text{(7.5)} \quad \sup_{y,s} H(y, T, s) < H(0, 1, 0), \quad \text{for } |\log T| \geq T_0.$$  

We first consider the case that $T$ is large. Note that for any $T > 0$,

$$H(y, T, s) = T^2 \int_\Omega |F_s|^2(4\pi T)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4T}} dx + T^2 \int_{\mathbb{R}^n \setminus \Omega} |F|^2(4\pi T)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4T}} dx$$

$$\leq T^2 \int_\Omega |F_s|^2(4\pi T)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4T}} dx + H(y, T, 0)$$

$$\leq M|\Omega|T^2(4\pi T)^{-\frac{n}{2}} + H(y, T, 0).$$

By Proposition 6.4 there exists $\delta > 0$ such that $H(y, T, 0) \leq H(0, 1, 0) - \delta$ when $T \geq 2$. Hence there exists $T_1 \geq 2$ such that

$$H(y, T, s) \leq H(0, 1, 0) - \frac{\delta}{2}, \quad \text{for } T \geq T_1.$$  

Note that $M = \sup\{|F_s(x)|^2 : s \in [-\epsilon, \epsilon], x \in \mathbb{R}^n\}$. Hence for any $T > 0$, we have

$$H(y, T, s) = H_{y,T}(A_s) = T^2 \int_{\mathbb{R}^n} |F_s|^2(4\pi T)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4T}} dx \leq MT^2.$$  

Thus there exists $T_2 > 0$ such that

$$\text{(7.7)} \quad \sup_{y,s} H(y, T, s) < H(0, 1, 0), \quad \text{for } T \leq T_2.$$  

Combing (7.6) and (7.7), we get (7.5).

**Step 4.** Set

$$U = \{(y, T) : |y| \leq R_0, |\log T| \leq T_0\} \setminus \{(y, T) : |y| < \epsilon_1, |\log T| < \epsilon_1\}.$$  

We now prove that there exists $\epsilon_0 \leq \epsilon_1$ such that for any $s$ with $|s| \leq \epsilon_0$,

$$\sup_U \{H(y, T, s) : (y, T) \in U\} < H(0, 1, 0).$$

Note that $U$ is a compact set which does not contain $(0,1)$. By Proposition 6.4 there exists $\delta > 0$ such that

$$\sup_U H(y, T, 0) \leq H(0, 1, 0) - \delta.$$
By the first variation formula (3.3) of the $F$-functional, we have
\[
\frac{d}{ds} H(y, T, s) = -2T^2 \int_{\mathbb{R}^n} \langle J(A_s) - \frac{1}{2T} i_{x-y} F(A_s), \theta_s \rangle G_{y,T}(x) dx.
\]
Since $\theta_s$ is compactly supported, $\partial_s H$ is continuous in all three variables $y, T, s$. Therefore there exists $0 < \epsilon_0 \leq \epsilon_1$ such that if $|s| \leq \epsilon_0$, then
\[
\sup U H(y, T, s) \leq H(0, 1, 0) - \frac{\delta}{2}.
\]
This proves (7.8). Combining (7.3), (7.4), (7.5) and (7.8), we get (7.2) and complete the proof. □

REFERENCES


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