COUNTING MINIMAL SURFACES
IN QUASI-FUCHSIAN THREE-MANIFOLDS

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Abstract. It is well known that every quasi-Fuchsian manifold admits at least one closed incompressible minimal surface, and at most finitely many stable ones. In this paper, we construct a quasi-Fuchsian manifold which contains at least $2^N$ such minimal surfaces. As a consequence, there exists some simple closed Jordan curve on $S^2_\infty$ such that there are at least $2^N$ disk-type complete minimal surfaces in $H^3$ sharing this Jordan curve as the asymptotic boundary.

1. Introduction

Let $M^3$ be a hyperbolic three-manifold, then we can write $M^3 = \mathbb{H}^3/\Gamma$, where $\mathbb{H}^3$ is hyperbolic three-space, and $\Gamma$ is a discrete subgroup of $\text{PSL}_2(\mathbb{C})$. The geometry of $M^3$ is largely determined by the behavior of the action of the discrete group $\Gamma$ on $S^2_\infty = \partial \mathbb{H}^3$. When the limit set of $\Gamma$ is a round circle $S^1 \subset S^2_\infty$, then the corresponding three-manifold $M^3$ is called Fuchsian. In this case, $M^3$ contains a (unique) totally geodesic surface $\Sigma$ and $M^3$ is a warped product space $M^3 = \Sigma \times \mathbb{R}$. The moduli space of Fuchsian manifolds (denoted by $\mathcal{F}_g(\Sigma)$) is isometric to Teichmüller space, the space of hyperbolic metrics on the surface $\Sigma$ under the equivalence induced by the group of diffeomorphisms isotopic to the identity. This space $\mathcal{F}_g(\Sigma)$ is a manifold of dimension $6g - 6$, where $g \geq 2$ is the genus of $\Sigma$. In the case when the limit set of $\Gamma$ lies in a Jordan curve, $M^3 = \mathbb{H}^3/\Gamma$ is called quasi-Fuchsian. This is an important class of hyperbolic three-manifolds and $M^3$ is diffeomorphic to $\Sigma \times \mathbb{R}$, where $\Sigma$ is a closed surface. The celebrated Simultaneous Uniformization Theorem of Bers ([Ber72]) implies that each quasi-Fuchsian manifold is determined by a pair of Riemann surfaces in Teichmüller space. The space of quasi-Fuchsian manifolds, called the quasi-Fuchsian space, is diffeomorphic to the product of Teichmüller spaces, hence of dimension $12g - 12$.

All surfaces in this paper, when referred to as being contained in a quasi-Fuchsian manifold, are smooth, closed, incompressible and of genus greater than one.

A closed surface embedded in a hyperbolic three-manifold is called incompressible if it induces a fundamental group injection. Incompressible surfaces play fundamental roles in the study of hyperbolic three-manifolds ([Has05, Rub07]), and among these surfaces, minimal ones often provide important geometric information about the ambient three-manifolds (see for instance [Rub05]). Analogs of these surfaces and hyperbolic three-manifolds are also central objects to study in anti-de-Sitter geometry (see [KS07] and many others).
In this paper, we investigate counting problems on minimal surfaces in quasi-Fuchsian manifolds. We quickly describe our goal of this work: Any counting problem starts with a (non-)existence theorem. It is a fundamental fact, proved by Schoen-Yau ([SY79]) and Sacks-Uhlenbeck ([SU82]), that any quasi-Fuchsian manifold \( M^3 \) contains an area-minimizing immersed incompressible surface and this surface is shown to be embedded by Freedman-Hass-Scott ([FHS83]). Uhlenbeck ([Uhl83]) further considered a subclass of quasi-Fuchsian manifolds which admit a minimal surface of principal curvatures lying in the interval \((-1, 1)\), and such quasi-Fuchsian manifolds are later ([KS07]) called almost Fuchsian. She showed, among other results, that any almost Fuchsian manifold contains exactly one minimal surface. Therefore one can parametrize the space of almost Fuchsian manifolds by studying these minimal surfaces (see also [Tau04]), and obtain important information on almost Fuchsian manifolds such as volume of the convex core and Hausdorff dimension of the limit set ([HW13]), as well as relate these problems to natural metrics on Teichmüller space ([GHW10]). It was expected (see for instance [Uhl83] page 7) that any (fixed) quasi-Fuchsian manifold contains at most finitely many (incompressible) minimal surfaces. This was indeed the case for stable ones as shown by Anderson ([And83]) by the method of geometric measure theory. One of the authors here ([Wan12b]) showed that there exist quasi-Fuchsian manifolds which admit more than one minimal surface. The purpose of this paper is to construct quasi-Fuchsian manifolds that admit arbitrarily many minimal surfaces, or more precisely, to prove the following main result:

**Theorem 1.1.** For any given positive integer \( N \), there exists a quasi-Fuchsian manifold \( M^3 \) that contains at least \( 2^N \) distinct (closed and incompressible) minimal surfaces, and each of them is embedded in \( M^3 \).

We remark here that for different integers \( N_1 \) and \( N_2 \), the minimal surfaces in the quasi-Fuchsian manifolds from our construction might have different genera. In fact, as \( N \) gets large, the genera of the minimal surfaces are expected to get very large as well. There is also a different type of counting problem for closed minimal surfaces in hyperbolic three-manifolds studied in [HL12].

When we lift this quasi-Fuchsian manifold to \( \mathbb{H}^3 \), a direct consequence of Theorem [1.1] is that we obtain a Jordan curve (not smooth since it is the limit set of a quasi-Fuchsian group) which bounds many disk-type complete minimal surfaces, namely,

**Theorem 1.2.** For any given positive integer \( N \), there exists a Jordan curve \( \Lambda \) on \( S^2_\infty \), such that there exist at least \( 2^N \) distinct complete minimal surfaces in \( \mathbb{H}^3 \), sharing the boundary \( \Lambda \) at the infinity.

This is related to a type of “asymptotic Plateau problem” studied by Morrey ([Mor48]), Almgren-Simon ([AS79]) and Anderson ([And83]) via geometric measure theory, and more recently Coskunuzer ([Cos09]) from a more topological approach. Our result is restricted to \( n = 3 \), which is very different from higher dimensions. Moreover, many results on this asymptotic Plateau problem concern the regularity of the prescribed simple closed curve at infinity (see for example [HL87],[GS00] and many others). The Jordan curve in our construction is the limit set of some quasi-Fuchsian group, therefore, except in the Fuchsian case, this curve is well known to contain no rectifiable arcs (see for instance [Leh64],[Ber72]).
Outline of the construction. Our construction can be outlined as follows: we separate the unit ball $B^3$ into $N + 1$ chambers by $N$ parallel Euclidean circles and these circles are slightly fattened to form bands (see Figure 3); bands are then connected by narrow bridges to form one piecewise smooth Jordan curve $\Lambda$ (see Figure 4) on the sphere at infinity $S^2_{\infty}$. There will be two disjoint circles next to every bridge to form a minimal catenoid; we then cover $\Gamma$ by an even number of small circles on $S^2_{\infty}$, and they induce, via inversions (or reflections), a quasi-Fuchsian group $\Gamma_A$ whose limit set is within a small neighborhood of the Jordan curve $\Lambda$, and hence this gives rise to a quasi-Fuchsian manifold (notably this construction of quasi-Fuchsian groups was known to Poincaré and Fricke-Klein); at last, there are more than $2^N$ distinct ways to arrange pairs of circles next to the bridges to bound minimal catenoids (see Figure 6), and for each arrangement, we obtain a compact region of mean convex boundary inside the convex core of the corresponding quasi-Fuchsian manifold, which allows us to apply the results of Schoen-Yau and Sacks-Unlenbeck to trap a minimal surface in this region.

There is a construction (unpublished) due to Hass-Thurston of quasi-Fuchsian manifolds which admit many minimal surfaces (see [GW07]). Their construction works for all genera greater than one, while our construction is quite different, and of quantitative nature. Moreover, one can see in our construction that each minimal surface is obtained by removing some solid minimal catenoids in the regions within the convex core of the quasi-Fuchsian manifold. We ([HW13]) have used a simplified version of this type of construction to obtain a quasi-Fuchsian manifold which does not admit a foliation of closed surfaces of constant mean curvature.

Plan of the paper. We organize this paper as follows: Subsections 2.1, 2.2 and 2.3 consist of preliminaries where we recall quasi-Fuchsian groups, quasi-Fuchsian manifolds, minimal surfaces in quasi-Fuchsian manifolds and minimal surfaces of catenoid type in $\mathbb{H}^3$. We prove the main Theorem [1.1] in Section 3 via a construction outlined above. There are several subsections in this main section: in Subsection 3.1, we obtain a condition when two disjoint circles on $S^2_{\infty}$ bound a minimal catenoid; in Subsection 3.2, we follow the classical construction of quasi-Fuchsian groups by using inversions of circles which bound open disks covering some piecewise smooth Jordan curve; in Subsection 3.3, we find a minimal surface in the corresponding quasi-Fuchsian manifold in the region of the convex core with some solid catenoid removed; Subsection 3.4 contains a topological lemma which will be used later to show the minimal surfaces obtained from the construction are distinct; in Subsection 3.5, we assemble these results to finalize the proof.

2. Preliminaries

In this section, we collect some important facts and describe some key properties on quasi-Fuchsian groups, quasi-Fuchsian manifolds and their minimal surfaces.

2.1. Quasi-Fuchsian groups and quasi-Fuchsian manifolds. In this paper, we will work in the ball model ($B^3$) of the hyperbolic three-space $\mathbb{H}^3$, i.e.,

$$B^3 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 < 1\},$$

equipped with metric

$$ds^2 = \frac{4(dx^2 + dy^2 + dz^2)}{(1 - r^2)^2},$$
where \( r = \sqrt{x^2 + y^2 + z^2} \). The hyperbolic space \( \mathbb{H}^3 \) has a natural compactification: \( \overline{\mathbb{H}}^3 = \mathbb{H}^3 \cup S^2_{\infty} \), where \( S^2_{\infty} \cong \mathbb{C} \cup \{ \infty \} \) is the Riemann sphere. The orientation-preserving isometry group of the three-ball \( \mathbb{B}^3 \) is denoted by \( \text{M"ob}(\mathbb{B}^3) \), which consists of Möbius transformations that preserve the unit ball (see [MT98, Theorem 1.7]). It’s well known that \( \text{M"ob}(\mathbb{B}^3) \cong \text{Isom}^+(\mathbb{H}^3) \), which is isomorphic to \( \text{PSL}_2(\mathbb{C}) \).

Suppose that \( X \) is a subset of \( \mathbb{B}^3 \). We define the asymptotic boundary of \( X \) by

\[
\partial_\infty X = \overline{X} \cap S^2_{\infty},
\]

where \( \overline{X} \) is the closure of \( X \) in \( \overline{\mathbb{H}}^3 \).

Using the above notation, if \( P \) is a geodesic plane in \( \mathbb{B}^3 \), then \( P \) is perpendicular to \( S^2_{\infty} \) and its asymptotic boundary \( C \mathrel{\overset{\text{def}}{=}} \partial_\infty P \) is an Euclidean circle on \( S^2_{\infty} \). We also say that \( P \) is asymptotic to \( C \).

A (torsion-free) discrete subgroup \( \Gamma \) of \( \text{Isom}^+(\mathbb{H}^3) \) is called a Kleinian group, and

\[
M^3 \equiv M^3_+ = \mathbb{H}^3/\Gamma
\]

is a complete hyperbolic three-manifold with the fundamental group \( \pi_1(M^3) \cong \Gamma \).

For any Kleinian group \( \Gamma \), \( \forall p \in \mathbb{H}^3 \), its orbit set

\[
\Gamma(p) = \{ g(p) \mid g \in \Gamma \}
\]

has accumulation points on the Riemann sphere \( S^2_{\infty} \), which are called the limit points of \( \Gamma \). The limit set of \( \Gamma \), denoted by \( \Lambda_\Gamma \), is the closure of the limit points on \( S^2_{\infty} \), i.e., \( \Lambda_\Gamma = \overline{\Gamma(p)} \cap S^2_{\infty} \). It is known that \( \Lambda_\Gamma \) is independent of the choice of the reference point \( p \), and it is a closed \( \Gamma \)-invariant subset of \( S^2_{\infty} \). The open set \( \Omega_\Gamma = S^2_{\infty} \setminus \Lambda_\Gamma \) is called the domain of discontinuity. The Kleinian group \( \Gamma \) acts properly discontinuously on \( \Omega_\Gamma \), and the quotient \( \Omega_\Gamma / \Gamma \) is a finite union of Riemann surfaces of finite type if \( \Gamma \) is finitely generated (see [Mar74]).

For any Kleinian group \( \Gamma \), the convex hull of the limit set \( \Lambda_\Gamma \), denoted by \( \text{Hull}(\Lambda_\Gamma) \), is the smallest convex subset in \( \mathbb{H}^3 \) whose closure in \( \mathbb{H}^3 \cup S^2_{\infty} \) contains \( \Lambda_\Gamma \). The quotient space \( \mathcal{C}_\Gamma = \text{Hull}(\Lambda_\Gamma) / \Gamma \) is called the convex core of \( M^3 \).

A Kleinian group \( \Gamma \) is quasi-Fuchsian if \( S^2_{\infty} \setminus \Lambda_\Gamma \) has exactly two components, denoted by \( \Omega_\pm \), such that each component is invariant under \( \Gamma \). If \( \Gamma \) is a quasi-Fuchsian group, then its limit set \( \Lambda_\Gamma \) is a Jordan curve, and the quotient space \( M = \mathbb{H}^3 / \Gamma \), which is called a quasi-Fuchsian manifold, is diffeomorphic to \( \Sigma \times \mathbb{R} \), where \( \Sigma = \Omega_+ / \Gamma \) or \( \Omega_- / \Gamma \) is a finitely punctured compact surface (see [Mar74] Lemma 3.2 and 3.3).

We always assume that the quasi-Fuchsian group \( \Gamma \) is torsion-free. Therefore \( \Omega_+ / \Gamma \) and \( \Omega_- / \Gamma \) are closed surfaces.

### 2.2. Minimal surfaces in quasi-Fuchsian manifolds

Suppose that \( \Sigma \) is a surface (compact or complete) and that \( M \) is a 3-dimensional Riemannian manifold (compact or complete). An immersion \( f : \Sigma \to M \) is called a minimal surface if its mean curvature is identically equal to zero.

If \( \Sigma \) is a closed minimal surface, then it is of least area if its area is less than that of any other surfaces in the same homotopy class; it is area minimizing if its area is no larger than that of any surface in the same homology class. If \( \partial \Sigma \neq \emptyset \), we require that other surfaces should share the same boundary as \( \Sigma \). If \( \Sigma \) is a (non-compact) complete minimal surface, then it is least area or area minimizing if any compact subdomain \( K \) of \( \Sigma \) is least area or area minimizing.
Recall that a map $f : \Sigma \to M$ of a closed surface $\Sigma$ into a three-manifold $M$ is called \textit{incompressible} if $f_* : \pi_1(\Sigma) \to \pi_1(M)$ is injective.

The convex core of the quasi-Fuchsian group $\Gamma$ is a compact hyperbolic three-manifold, whose boundary is convex with respect to the inward normal vector. Schoen-Yau ([SY79]) and Sacks-Uhlenbeck ([SU82]) proved that such a hyperbolic three-manifold always contains an (immersed) incompressible least area minimal surface. By the work of Freedman, Hass and Scott ([FHS83, Theorem 5.1]), this minimal surface is actually embedded. Here we should remark that the ambient hyperbolic three-manifold appeared in the main results in [FY79][SU82][FHS83] and the works of Meeks-Yau ([MY82a,MY82b]), these results are stated to be compact without boundary, i.e., closed. But from the discussions in [FHS83] p. 635 and the works of Meeks-Yau ([MY82a,MY82b]), these results can be extended to the compact hyperbolic three-manifold whose boundaries are \textit{convex} or \textit{mean convex}.

\subsection*{2.3. Minimal catenoids in $\mathbb{H}^3$}

Minimal surfaces of catenoid type in $\mathbb{H}^3$ whose asymptotic boundary consists of two circles on $S^2_\infty$ will play an important role in the construction: they serve as barriers among minimal surfaces. In this subsection, we describe the existence of these minimal catenoids. The properties of minimal catenoids can be found in many articles, for example, [Mor81,Gom87,BSE10,Seo11,Wan12a].

Let $G$ be the subgroup of $\text{M"ob}(\mathbb{B}^3)$ that leaves a geodesic $\gamma \subset \mathbb{B}^3$ pointwisely fixed; then we call $G$ the \textit{spherical group} of $\mathbb{B}^3$ and $\gamma$ the \textit{rotation axis} of $G$. A (connected) surface $\Pi$ in $\mathbb{B}^3$ that is invariant under $G$ is called a \textit{spherical surface} or a \textit{surface of revolution}. If $\Pi$ is minimal, then it is called a \textit{spherical minimal catenoid} or a \textit{minimal catenoid}.

For two circles $C_1$ and $C_2$ in $\mathbb{H}^3$, if there is a geodesic $\gamma$ such that each of the circles $C_1$ and $C_2$ is invariant under the group of rotations that fixes $\gamma$ pointwisely, then $C_1$ and $C_2$ are said to be \textit{coaxial}, and $\gamma$ is called the \textit{rotation axis} of $C_1$ and $C_2$. If $C_1$ and $C_2$ are two disjoint circles on $S^2_\infty$, then they are always coaxial. In this case, we want to know whether there exists a spherical minimal surface that is asymptotic to $C_1 \cup C_2$. This existence depends on two kinds of distances between two disjoint circles on $S^2_\infty$, which we now describe.

Equipping the Riemann sphere $S^2_\infty$ with the spherical metric $\rho$, then every geodesic on $(S^2_\infty, \rho)$ is just a great circle. The orientation preserving isometry group of $S^2_\infty$, denoted by $\text{Isom}^+(S^2_\infty)$, is $\text{SO}(3)$ which is a subgroup of $\text{M"ob}(\mathbb{B}^3)$. Suppose $C_1$ and $C_2$ are disjoint (round) circles in $S^2_\infty$ which bound disjoint subdisks $\Delta_1$ and $\Delta_2$ of $S^2_\infty$ with injective radii $\leq \pi/2$. For $i = 1, 2$, let $P_i$ be the geodesic plane in $\mathbb{B}^3$ such that $\partial_\infty P_i = C_i$. Then we may define two distances between $C_1$ and $C_2$ by

\begin{align*}
    d_L(C_1, C_2) &= \text{dist}(P_1, P_2), \\
    \rho(C_1, C_2) &= \min \{ \rho(p, q) \mid p \in C_1 \text{ and } q \in C_2 \}
\end{align*}

where $\text{dist}(\cdot, \cdot)$ is the hyperbolic distance in $\mathbb{B}^3$ (or $\mathbb{H}^3$). The distance $d_L$ is invariant under $\text{M"ob}(\mathbb{B}^3)$, whereas the spherical distance $\rho$ is only invariant under $\text{SO}(3)$.

Suppose that $G$ is the spherical group of $\mathbb{B}^3$ with rotation axis $\gamma_0$, a geodesic along on the $y$-axis; then $\mathbb{B}^3/G \cong \mathbb{B}^2_+$, where

\begin{equation}
    \mathbb{B}^2_+ = \{(x, y, z) \in \mathbb{B}^3 \mid z = 0, \ y \geq 0\}.
\end{equation}

If $\Pi$ is a spherical minimal catenoid in $\mathbb{B}^3$ with respect to the axis $\gamma_0$, then the curve $\sigma = \Pi \cap \mathbb{B}^2_+$ is called the \textit{generating curve} of $\Pi$. Gomes ([Gom87]) proved
that $\sigma$ is symmetric about the $y$-axis up to isometries. Moreover, he proved the following theorem:

**Theorem 2.1** ([Gom87 Proposition 3.2]). There exists a finite constant $d_0 > 0$ such that for two disjoint circles $C_1, C_2 \subset S^2_\infty$, if $d_L(C_1, C_2) \leq d_0$, then there exist a minimal surface $\Pi$ which is a surface of revolution asymptotic to $C_1 \cup C_2$.

In this paper, we require the existence of least area minimal catenoids in order to apply the results in [MYS20]. This is fulfilled by:

**Theorem 2.2** ([Wan12a Theorem 1.2]). There exists a finite constant $d_1 > 0$ such that for two disjoint circles $C_1, C_2 \subset S^2_\infty$, if $d_L(C_1, C_2) \leq d_1$, then there exist a least area minimal surface $\Pi$ which is a surface of revolution asymptotic to $C_1 \cup C_2$.

3. Main construction

This section is devoted to constructing a quasi-Fuchsian manifold which admits at least $2^N$ minimal surfaces, for any prescribed positive integer $N$. Let us outline the organization of this section. In Subsection 3.1, in order to utilize Theorems 2.1 2.2 to find a minimal catenoid asymptotic to two disjoint circles on $S^2_\infty$, we show how to compute the distance $d_L$ (see the equation (2.1)) between two circles (Lemma 3.1); in Subsection 3.2, we use inversions with respect to disjoint circles on $S^2_\infty$ to construct a quasi-Fuchsian group whose limit set is contained within a small neighborhood of a prescribed (piecewise smooth) Jordan curve on $S^2_\infty$ (Theorem 3.3); in Subsection 3.3, we use the notion of fundamental polyhedron (Definition 3.8) for a quasi-Fuchsian group to find a compact hyperbolic three-manifold with mean convex boundary, which allows us to apply fundamental results of Schoen-Yau, Saks-Uhlenbeck to find a closed minimal surface (Theorem 3.9); finally in Subsection 3.4, we combine the results in previous subsections to prove our main theorem.

3.1. Distances between circles on $S^2_\infty$. We have introduced two distances (see equations (2.1) and (2.2)) between two disjoint circles on $S^2_\infty$. The purpose of this subsection is to find the relationship between these two distances, and use it to find conditions under which two circles determine a minimal catenoid.

We may calculate the above two distances as follows. As in Subsection 2.3, let $C_1$ and $C_2$ be two disjoint circles on $S^2_\infty$, and $P_i$ be the geodesic planes in $\mathbb{B}^3$ and $\Delta_i$ be the disks on $S^2_\infty$ with injective radius $\leq \pi/2$ such that $\partial_\infty P_i = C_i$ and $\partial \Delta_i = C_i$ for $i = 1, 2$ and such that $\Delta_1 \cap \Delta_2 = \emptyset$. Now let $C$ be the great circle that passes the centers of $\Delta_1$ and $\Delta_2$, and we also mark $p_1$ and $p_2$ as the intersection points of $C_1$ and $C$, while $q_1$ and $q_2$ are the intersection points of $C_2$ and $C$. See Figure 1.

Let $P$ be the geodesic plane asymptotic to $C$ and $\gamma_i = P_i \cap P$. Then, from the definitions, we have $d_L(C_1, C_2) = \text{dist}(\gamma_1, \gamma_2)$, and the spherical distance $\rho(C_1, C_2)$ is equal to the shorter length of two geodesic segments on $C$ between $\Delta_1$ and $\Delta_2$.

One of our key lemmas is the following:

**Lemma 3.1.** Suppose that $C_1$ and $C_2$ are two disjoint circles on $S^2_\infty$ such that $\Delta_1 \cap \Delta_2 = \emptyset$, where $\Delta_i \subset S^2_\infty$ is a (spherical) disk bounded by $C_i$ with injective radius $\leq \pi/2$. Then

$$d_L(C_1, C_2) \to 0, \quad \rho(C_1, C_2) \to 0.$$  

In particular, if $C_1$ and $C_2$ are tangent at one point on $S^2_\infty$, then $d_L(C_1, C_2) = 0$. 

We note, in this lemma, $\rho(C_1, C_2) \to 0$ means that the spherical distance between two closed sets $C_1$ and $C_2$ approaches zero while the injective radii of two disks $\Delta_1$ and $\Delta_2$ stay unchanged.

**Proof.** Without lost of generality, we may assume that the center of the disk $\Delta_1$ is the north pole, i.e., $(0, 0, 1)$, and the center of $\Delta_2$ lies on the great circle $C = \{(x, 0, z) \in \mathbb{R}^3 \mid x^2 + z^2 = 1\}$. In fact, using at most two rotations along some axes, we can obtain the required picture. Since these two rotations are elements of $SO(3)$, the two distances (2.1) and (2.2) are preserved.

As above, we suppose that $C_1 \cap C = \{p_1, p_2\}$ and $C_2 \cap C = \{q_1, q_2\}$. We order them on the great circle $C$ as $p_1, p_2, q_1$ and $q_2$. See Figure 2.

Then it is easy to see that $\rho(C_1, C_2) = \min\{\rho(p_2, q_1), \rho(p_1, q_2)\}$. We may assume that $\rho(C_1, C_2) = \rho(p_2, q_1)$.

For $i = 1, 2$, let $\gamma_i = P_i \cap P$, where $P = \{(x, 0, z) \in \mathbb{R}^3 \mid x^2 + z^2 \leq 1\}$, and $P_i$ is the geodesic plane asymptotic to $C_i$. Then $\partial_{\infty} \gamma_1 = \{p_1, p_2\}$ and $\partial_{\infty} \gamma_2 = \{q_1, q_2\}$. Since $\text{dist}(P_1, P_2) = \text{dist}(\gamma_1, \gamma_2)$, we have $d_L(C_1, C_2) = \text{dist}(\gamma_1, \gamma_2)$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Spherical distance between two circles}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Stereographic projection of two circles}
\end{figure}
There exists a Möbius transformation $\Phi \in \text{Möb}(\mathbb{R}^2)$ that satisfies the following properties (see [MT98 pp.19-20]):

- $\Phi|\mathbb{B}^3 : \mathbb{B}^3 \to \mathbb{H}^3$ is an isometry, and
- $\Phi|S^2_\infty \setminus \{e_3\} : S^2_\infty \setminus \{e_3\} \to \mathbb{C}$ is the stereographic projection,

where $\mathbb{R}^3 = \mathbb{R}^3 \cup \{\infty\}$ and $e_3 = (0,0,1)$.

For $i = 1, 2$, let $\gamma_i = \Phi(\gamma_i)$, and let $u_i = \Phi(p_i)$ and $v_i = \Phi(q_i)$, respectively. Then $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are two geodesics in the $xy$-plane. Since $\Phi$ preserves the hyperbolic metrics, we have

$$\text{dist}(\tilde{\gamma}_1, \tilde{\gamma}_2) = d_L(C_1, C_2).$$

Suppose the points $u_1$, $u_2$, $v_1$ and $v_2$ are contained on the $x$-axis in the following order: $u_1, v_2, v_1, u_2$. Since $\partial_\infty \tilde{\gamma}_1 = \{u_1, u_2\}$ and $\partial_\infty \tilde{\gamma}_2 = \{v_1, v_2\}$, applying [Bea93 Eq. (7.23.1)], we have

$$\tanh^2[\text{dist}(\tilde{\gamma}_1, \tilde{\gamma}_2)] = \frac{1}{[u_1, v_2, v_1, u_2]} = \frac{(u_1 - v_2)(v_1 - u_2)}{(u_1 - v_1)(v_2 - u_2)}.
$$

Then we have the following consequences:

$$\rho(C_1, C_2) \to 0 \implies q_1 \to p_2 \implies v_1 \to u_2 \implies \text{dist}(\tilde{\gamma}_1, \tilde{\gamma}_2) \to 0,$$

and then $d_L(C_1, C_2) \to 0$ because of \[(3.1)\].

This lemma enables us to conclude that if we can arrange two disjoint circles on $S^2_\infty$ sufficiently close with respect to the (spherical) distance $\rho$, then we can apply Theorem 2.1 (and Theorem 2.2) to find a (least area) minimal catenoid asymptotic to these two circles.

### 3.2. Constructing quasi-Fuchsian groups via inversions.

In this subsection, we use inversions of circles on $S^2_\infty$ to generate a quasi-Fuchsian group and we obtain a quasi-Fuchsian manifold in the quotient.

Suppose that $C_1$ and $C_2$ are two circles on $S^2_\infty$. Let $f_i$ be the inversions with respect to to the circle $C_i$; then each inversion is of order 2, i.e., $f_i^2 = \text{id}$, for $i = 1, 2$. Consider the Möbius transformation $g = f_1 \circ f_2$. We have:

**Lemma 3.2.** For this element $g$, there are three possibilities:

1. If $C_1 \cap C_2 = 0$, then $g$ is a loxodromic element, whose fixed points are contained in the disks $\Delta_1$ and $\Delta_2$, respectively.
2. If $C_1$ and $C_2$ are tangent at some point $z_0 \in \mathbb{R}^2$, then $g$ is a parabolic element with exactly one fixed point $z_0$.
3. If $C_1$ and $C_2$ are orthogonal, and $C_1 \cap C_2 = \{z_1, z_2\}$, then $g$ is an elliptic element of order 2 with fixed points $\{z_1, z_2\}$.

**Proof.** We will only need to prove the third case, for which we work in the upper half space model. Suppose $C_1$ is a circle with radius $r_1$ whose center is the origin, and $C_2$ is a circle of radius $r_2$ whose center is at the point $(a,0)$, where

$$a = (r_1^2 + r_2^2)^{1/2}.$$

For any $x \in \mathbb{R}^2$, let $x^* = x/|x|^2$, where $|x|$ is the Euclidean norm. Then the inversions with respect to $C_1$ and $C_2$ have the following forms:

$$f_1(x) = r_1^2 x^* \quad \text{and} \quad f_2(x) = a + r_2^2 (x - a)^*.$$
We need to show that these two inversions commute, namely, \( f_1 \circ f_2 = f_2 \circ f_1 \). It is clear that
\[
(3.2) \quad f_2 \circ f_1 (x) = a + r_2^2 (r_1^2 x^* - a)^*.
\]
Direct computation shows
\[
(3.3) \quad |f_2(x)|^2 = \frac{|r_1^2 x^* - a|^2 |x|^2}{|x - a|^2}
\]
and
\[
(3.4) \quad |r_1^2 x^* - a|^2 - r_2^2 = \frac{r_1^2 (|x - a|^2 - r_2^2)}{|x|^2}.
\]
Therefore, we have
\[
f_1 \circ f_2 (x) = \frac{r_1^2 |x - a|^2}{|r_1^2 x^* - a|^2 |x|^2} (a + r_2^2 (x - a)^*)
\]
\[
= \frac{1}{|r_1^2 x^* - a|^2} \left( \frac{r_1^2 |x - a|^2 a + r_2^2 r_2^2 (x - a)}{|x|^2} \right)
\]
\[
= \frac{1}{|r_1^2 x^* - a|^2} \left( \frac{r_1^2 (|x - a|^2 - r_2^2 a + r_2^2 r_2^2 x^*)}{|x|^2} \right)
\]
\[
= \frac{1}{|r_1^2 x^* - a|^2} ( (|r_1^2 x^* - a|^2 - r_2^2 a) + r_2^2 r_2^2 x^*)
\]
\[
= a + r_2^2 (r_1^2 x^* - a)^*.
\]
This agrees with \((3.2)\). Since that \( f_i^2 = \text{id} \) for \( i = 1, 2 \), we have
\[
g^2 = f_1 \circ f_2 \circ f_1 \circ f_2 = f_1 \circ f_1 \circ f_2 \circ f_2 = \text{id}.
\]
Recall that any inversion with respect to some circle fixes that circle pointwisely; then it is not hard to see that \( \{\zeta_1, \zeta_2\} \) are the points that are fixed by both \( f_1 \) and \( f_2 \). This completes the proof of the lemma. \( \square \)

For any piecewisely smooth simple closed curve \( \Lambda \) on \( S^2_\infty \), we can cover it by finitely many small disks. The above lemma allows us to construct a quasi-Fuchsian group \( \Gamma \), whose limit set \( \Lambda \) is within some small distance of the curve \( \Lambda \).

We now proceed with this construction as follow: let \( \mathcal{N}_\delta (\Lambda) \subset S^2_\infty \) be a \( \delta \)-neighborhood of \( \Lambda \), where \( \delta \) is a small positive number. Then we can cover the curve \( \Lambda \) by a family of finitely many open disks \( \{\Delta_i\}_{i=1}^{2L} \) on \( S^2_\infty \), where \( L \geq 3 \), which satisfy the following conditions:

- For each disk \( \Delta_i \), its center is on \( \Lambda \) and its injective radius is \( \leq \delta \).
- Let \( C_i = \partial \Delta_i \), each circle \( C_i \) intersects \( C_{i+1} \) at an angle of \( \pi/2 \) and \( C_i \cap C_j = \emptyset \) for \( |i - j| \geq 2 \), here \( C_{2L+1} \overset{\text{def}}{=} C_1 \).
- Suppose that \( C_i \cap C_{i+1} = \{z_i, z_i'\}, i = 1, \ldots, 2L \).

It is easy to see that we have \( \Lambda \subset \bigcup_{i=1}^{2L} \Delta_i \subset \mathcal{N}_\delta (\Lambda) \).

Let \( f_i \) be the inversion with respect to the circle \( C_i \), and let \( g_i = f_i \circ f_{i+1} \), \( i = 1, 2, \ldots, 2L \), where \( C_{2L+1} = C_1 \). Let \( \Gamma' \) be the group generated by the inversions \( f_1, f_2, \ldots, f_{2L} \). Let \( \Gamma_0 \) be the group whose elements are the product of an even number of inversions \( f_i \). Clearly \( \Gamma_0 \) is a subgroup of \( \Gamma' \) of index 2. The main result in this subsection is the following theorem.
**Theorem 3.3.** Let $\Gamma'$ and $\Gamma_0$ be the above groups, and let $\Lambda_{\Gamma'}$ be the limit set of $\Gamma'$. Then we have

1. $\Lambda_{\Gamma'}$ is a Jordan curve that passes through all of the points $\{z_i, z'_i\}_{i=1}^{2L}$. Moreover $\Gamma_0$ is a quasi-Fuchsian group and $\Lambda_{\Gamma_0} = \Lambda_{\Gamma'} \subset \mathcal{N}_0(\Lambda)$.

2. The group $\Gamma_0$ has the following presentation:

$$\Gamma_0 = \langle g_1, \ldots, g_{2L} \mid g_1^2 = 1, \ldots, g_{2L}^2 = 1, g_1 \cdots g_{2L} = 1 \rangle.$$  

3. The quotient space $\mathcal{O} = \Omega_+ / \Gamma_0$ is a genus zero orbifold that has $2L$ cone points with cone angle $\pi$, where $\Omega_+ = S^2_{\infty} \setminus \Lambda_{\Gamma_0}$.

**Remark 3.4.** This construction of quasi-Fuchsian groups was known at the end of the nineteenth century by Poincaré and Fricke-Klein (see [Poi83, Poi85] and [FK65 pp. 399-445]; see also [Ber72 p. 263]).

**Proof.** (1). Similar to the discussion in [Mag74, Chapter IV] or [MSW02, Chapter 6], one can show that $\Lambda_{\Gamma'}$ is a Jordan curve contained in $\bigcup \Delta_i \subset \mathcal{N}_0(\Lambda)$. By the construction, $\Gamma_0$ is a subgroup of $\Gamma'$ with index 2, so $\Gamma_0$ is a normal subgroup of $\Gamma'$. Then it is well known (see for instance [MT98 Lemma 2.22]) that they have the same limit set, namely, $\Lambda_{\Gamma_0} = \Lambda_{\Gamma'}$.

(2). By Lemma 3.2 we have $g_1^2 = 1, \ldots, g_{2L}^2 = 1$, and $g_1 \cdots g_{2L} = f_1^2 = 1$. In order to show that $\Gamma_0$ is generated by $g_1, \ldots, g_{2L}$, we only need to consider the special case, i.e., suppose that $h = f_i f_j$, here $|i - j| = k \geq 2$. Then we may write $h$ in the form

$$h = \begin{cases} g_i g_{i+1} \cdots g_{j-1} & \text{if } i < j, \\ g_{i-1} \cdots g_{i-k+1} g_j & \text{if } i > j, \end{cases}$$

which implies that any element of $\Gamma_0$ can be written by the product of finitely many $g_i, i = 1, \ldots, 2L$.

(3). The domain $S^2_{\infty} \setminus \bigcup_{i=1}^{2L} \Delta_i$ has two components, denoted by $E$ and $E'$, where $E$ is a $2L$-polygon with vertices $z_1, z_2, \ldots, z_{2L}$ and $E'$ is also a $2L$-polygon with vertices $z'_1, z'_2, \ldots, z'_{2L}$. Both $E$ and $E'$ are orbifolds that contain $2L$ corner points with corner angle $\pi/2$. Each of them is a fundamental domain of $\Gamma'$ in $S^2_{\infty}$. Recall that $\Gamma_0$ is a subgroup of $\Gamma'$ with index 2 and let $\mathcal{O}$ be the double cover of $E$ or $E'$. Then $\Gamma_0$ is the fundamental group of $\mathcal{O}$ (see [Sco83 pp. 423-424]) and $\Omega_+ / \Gamma_0 = \mathcal{O}$ is a 2-sphere that has $2L$ cone points with cone angle $\pi$.

It is well known that $\mathcal{O}$ can be doubly covered by a closed surface $\Sigma$ of genus $g = L - 1 \geq 2$. Therefore $\Gamma_0$ contains a torsion-free subgroup $\Gamma$ of index 2. Therefore again this subgroup is normal and we have $\Lambda_{\Gamma} = \Lambda_{\Gamma_0}$, and $\mathbb{H}^3 / \Gamma \cong \Sigma \times \mathbb{R}$ is quasi-Fuchsian. In particular, we have showed:

**Corollary 3.5.** Suppose that $\Lambda$ is a simple closed (piecewise smooth) curve on $S^2_{\infty}$. Then for any positive number $\delta$, there exists a torsion-free quasi-Fuchsian group $\Gamma$ such that its limit set $\Lambda_{\Gamma}$ is contained in a $\delta$-neighborhood of $\Lambda$. Furthermore, suppose that $\Sigma$ is a finite type surface such that $\mathbb{H}^3 / \Gamma \cong \Sigma \times \mathbb{R}$; then genus($\Sigma$) = $g(\Lambda, \delta)$. Here $g(\Lambda, \delta)$ is a positive integer depending only on $\Lambda$ and $\delta$.

### 3.3. Fundamental polyhedron.

The goal of this subsection is to obtain a compact hyperbolic three-manifold $Y$ whose boundary is mean convex with respect to the inward normal vector. This is achieved by arranging disjoint circles on $S^2_{\infty}$ in a
We may assume that we can use minimal catenoids (asymptotic to two disjoint circles) as barrier surfaces. Let us start with two definitions.

**Definition 3.6.** Suppose that \( \Lambda_1, \ldots, \Lambda_n \) are a family of disjoint Jordan curves on \( S^2_\infty \), where \( n \geq 3 \). We say that \( \Lambda_1, \ldots, \Lambda_n \) are in good position if each Jordan curve bounds a topological disk on \( S^2_\infty \) that does not contain any other Jordan curves in the family.

First we need a lemma on separation:

**Lemma 3.7.** Let \( C_1, C_2 \) and \( C \) be three disjoint circles on \( S^2_\infty \) that are in good position. Suppose that \( \Pi \) is the minimal catenoid with \( \partial_\infty \Pi = C_1 \cup C_2 \), and \( P \) is the geodesic plane with \( \partial_\infty P = C \); then \( \Pi \cap P = \emptyset \).

**Proof.** We may assume that \( P \) is on the \( xy \)-plane, i.e., \( P = \{(x, y, 0) \mid x^2 + y^2 < 1\} \). Otherwise, we may find a Möbius transformation \( g \in \text{Möb}( \mathbb{B}^3 ) \) such that \( g(P) \) is on the \( xy \)-plane (see [MT98 Proposition 1.3]). In addition, we assume that \( C_1 \) and \( C_2 \) are above the \( xy \)-plane. We now need to prove that \( \Pi \) is above \( P \).

If \( \Pi \cap P = \emptyset \), then we are done. Otherwise, for each \( t \in (-1, 0) \), let \( P(t) \) be the geodesic plane that is perpendicular to the \( z \)-axis at the point \((0, 0, t)\). Obviously \( P = P(0) \). It is easy to see that the family of the geodesic planes \( \{P(t)\}_{-1 < t \leq 0} \) foliates the lower half ball, i.e., \( \{(x, y, z) \in \mathbb{B}^3 \mid -1 < z \leq 0\} \). If \( \Pi \cap P \neq \emptyset \), then we always can find a number \( t_0 \in (-1, 0) \) such that \( P(t_0) \) is tangent to \( \Pi \) at some point in \( \mathbb{B}^3 \). This but is impossible by the maximum principle, therefore \( P \) and \( \Pi \) must be disjoint.

Suppose that \( \Gamma \) is a quasi-Fuchsian group such that \( \mathbb{B}^3/\Gamma \cong \Sigma \times \mathbb{R} \), where \( \Sigma \) is a closed surface of genus \( \geq 2 \). The limit set of \( \Gamma \) is denoted by \( \Lambda_\Gamma \), and \( \Lambda_\Gamma \) separates \( S^2_\infty \) into two components, namely, \( S^2_\infty \setminus \Lambda_\Gamma = \Omega_+ \cup \Omega_- \).

**Definition 3.8** (see section 2.1.2 in [MT98]). A closed convex set \( \mathcal{P} \) in the hyperbolic space \( \mathbb{B}^3 \) bounded by a finite collection of hyperbolic planes is called a fundamental polyhedron for a quasi-Fuchsian group \( \Gamma \), if it satisfies the following conditions:

1. \( \bigcup_{g \in \Gamma} g(\mathcal{P}) = \mathbb{B}^3 \).
2. \( g(\text{Int } \mathcal{P}) \cap \text{Int } \mathcal{P} = \emptyset \) for any non-trivial element \( g \in \Gamma \); here \( \text{Int } \mathcal{P} \) is the interior of \( \mathcal{P} \) in \( \mathbb{B}^3 \).
3. For each side \( S \) of \( \mathcal{P} \), there is another side \( S' \) and an element \( g \in \Gamma \) such that \( g(S) = S' \).
4. For any compact subset \( K \subset \mathbb{B}^3 \), \( \{g \in \Gamma \mid g(\mathcal{P}) \cap K \neq \emptyset\} \) is a finite set.

The relatively closed sets \( Q_\pm = \overline{\mathcal{P}} \cap \Omega_\pm \) are fundamental domains of \( \Gamma \) in \( \Omega_\pm \), respectively. If \( \Gamma \) is a quasi-Fuchsian group, then we have \( \mathcal{P} \cong Q_+ \times \mathbb{R} \). Now we proceed with the following theorem which finds a closed minimal surface in some submanifold of \( M^3 = \mathbb{H}^3/\Gamma \).

**Theorem 3.9.** Let \( \Lambda_\Gamma \) be the limit set of some torsion-free quasi-Fuchsian group \( \Gamma \) such that \( \mathbb{B}^3/\Gamma \) is homotopic to a closed surface of genus at least two. If \( \mathcal{P} \subset \mathbb{B}^3 \) is the fundamental polyhedra of the group \( \Gamma \) and \( Q_\pm = \overline{\mathcal{P}} \cap S^2_\infty \subset \Omega_\pm \) are the fundamental domains of \( \Gamma \) in \( \Omega_\pm \), respectively, where \( \Omega_\pm = S^2_\infty \setminus \Lambda_\Gamma \). Suppose that
Π₁, . . . , Πₙ are finite disjoint least area minimal catenoids in \( \mathbb{B}^3 \) that satisfy the following conditions:

1. The asymptotic boundary of each minimal catenoid must either both be contained in \( Q_+ \) or both be contained in \( Q_- \).
2. If \( n = 1 \), then \( \partial_\infty \Pi_1 \) and \( \Lambda_\Gamma \) are in good position; if \( n \geq 2 \), then the asymptotic boundaries \( \partial_\infty \Pi_1, \ldots, \partial_\infty \Pi_n \) are in good position.
3. \( \Lambda_\Gamma \) is null-homotopic in the component of \( \mathbb{B}^3 \cup \Pi_l \) which contains \( \Lambda_\Gamma \).

Then there exists a \( \Gamma \)-invariant minimal disk with asymptotic boundary \( \Lambda_\Gamma \), which is disjoint from \( g(\Pi_l) \), for all \( g \in \Gamma \) and all \( l \in \{1, \ldots, n\} \). Furthermore, the disk is embedded in \( \mathbb{B}^3 \).

**Proof.** Our strategy is to remove some solid catenoids from \( \mathbb{B}^3 \) to obtain a submanifold of the quasi-Fuchsian manifold \( M^3 = \mathbb{B}^3 / \Gamma \).

We let \( T_l \) be the solid catenoid bounded by \( \Pi_l \) for each \( l = 1, \ldots, n \). By the assumptions and Lemma 3.7, we know that these solid catenoids lie inside the fundamental polyhedron of the quasi-Fuchsian group \( \Gamma \), namely, \( T_l \subset \mathcal{P} \) for each \( l = 1, \ldots, n \). Therefore, in what follows we abuse our notion to use \( T_l \) to denote the corresponding solid catenoid in \( M^3 \) as well.

From assumption of the theorem, we may assume that \( \partial_\infty \Pi_l \subset Q_+ \) for \( l = 1, \ldots, m \), and \( \partial_\infty \Pi_l \subset Q_- \) for \( l = m + 1, \ldots, n \), where \( m \leq n \) are positive integers.

Since \( T_l \subset \mathcal{P} \), we also have \( g(T_l) \subset g(\mathcal{P}) \) for all \( g \in \Gamma \) and \( l = 1, \ldots, n \). Let

\[
X_\infty = \left\{ \mathbb{B}^3 \setminus \bigcup_{g \in \Gamma} \bigcup_{l=1}^n g(T_l) \right\} \setminus S^2_\infty,
\]

then \( X_\infty \) is a \( \Gamma \)-invariant subset of \( \mathbb{H}^3 \). Let \( M_\infty = X_\infty / \Gamma \). Since \( T_l \subset \mathcal{P} \) for all \( l = 1, \ldots, n \), it is clear that \( M_\infty \) is a submanifold of the quasi-Fuchsian manifold \( M^3 = \mathbb{H}^3 / \Gamma \). By the construction, \( M_\infty \) is a homogeneously regular three-manifold in the sense of Morrey (Mor48), whose boundary is mean convex with respect to the inward normal vector.

Now let \( Y = \mathcal{C}_\Gamma \cap M_\infty \), where \( \mathcal{C}_\Gamma \) is the convex core of the quasi-Fuchsian manifold \( M^3 \). Then \( Y \) is a compact hyperbolic three-manifold whose boundary is mean convex with respect to the inward normal vector (for the precise meaning of positive mean curvature with respect to inward normal vector; see §7 of [FHSS83]).

By Lemma 3.11 which we will prove later, this hyperbolic three-manifold \( Y \) contains a closed surface that is incompressible. As discussed in Subsection 2.2, we can now apply the results of Schoen-Yau, Sacks-Uhlenbeck, Freedman-Hass-Scott, to conclude \( Y \) also contains an embedded least area surface, denoted by \( \Sigma \), which is incompressible. Lifting this minimal surface \( \Sigma \) to \( \mathbb{H}^3 \), we obtain an embedded \( \Gamma \)-invariant minimal disk \( \tilde{\Sigma} \) in \( X_\infty \) with asymptotic boundary \( \Lambda_\Gamma \).

**Remark 3.10.** In general, the universal cover, \( \tilde{\Sigma} \), of an area minimizing surface \( \Sigma \) in \( M^3 = \mathbb{H}^3 / \Gamma \) is not necessarily a least area minimal disk in \( \mathbb{H}^3 \) whose asymptotic boundary is \( \Lambda_\Gamma \).

To complete the proof for Theorem 3.9 we prove the following lemma:

**Lemma 3.11.** There exists a closed incompressible surface \( F \) contained in \( Y \).
Proof. The convex core of a quasi-Fuchsian manifold has two boundary components, namely we can write \( \partial \mathcal{C}_\Gamma = F_+ \cup F_- \), where \( F_\pm \) is the pleated surface that faces the surface \( \Omega_{\pm}/\Gamma \), respectively.

Let \( W_\pm \) be the submanifold of \( M^3 \) that is bounded by \( F_\pm \) and the conformal infinity \( \Omega_{\pm}/\Gamma \), respectively. Obviously \( \mathcal{C}_\Gamma = \partial W_+ \cap W_- \). Then \( \partial W_\pm = F_\pm \) is convex with respect to the inward normal vector. For all \( l = 1, \ldots, n \), since the catenoid \( \Pi_l = \partial T_l \) is a least area minimal surface, by [MYS2a, Theorem 1], \( \Pi_l \subset W_+ \) if \( 1 \leq l \leq m \), and \( \Pi_l \subset W_- \) if \( m+1 \leq l \leq n \). Therefore \( T_l \subset W_+ \) if \( 1 \leq l \leq m \), and \( T_l \subset W_- \) if \( m+1 \leq l \leq n \).

Since \( \Lambda_\Gamma \) is null-homotopic in the component of \( \mathbb{B}^3 \setminus \bigcup \Pi_l \) which contains \( \Lambda_\Gamma \), there exists a disk \( D \) that is asymptotic to \( \Lambda_\Gamma \) and that is disjoint from \( \{ T_1, \ldots, T_n \} \). In particular, \( D \cap \mathcal{P} \) is disjoint from \( \{ T_1, \ldots, T_n \} \). Then we can perturb \( D \cap \mathcal{P} \) to get a disk \( \Delta \subset \mathcal{P} \) so that \( \Delta \) is still disjoint from \( \{ T_1, \ldots, T_n \} \) and

\[
D = \bigcup_{g \in \Gamma} g(\Delta)
\]

is a \( \Gamma \)-invariant disk asymptotic to \( \Lambda_\Gamma \), and it is disjoint from \( g(T_l) \) for all \( g \in \Gamma \) and all \( l \in \{ 1, \ldots, n \} \).

Let \( F' = D/\Gamma \); then clearly \( F' \) is a closed incompressible surface. If \( F' \) is contained in \( Y \), then we are done. Otherwise, by the above discussion, \( F_+ \) is disjoint from \( T_l \) for all \( m+1 \leq l \leq n \) and \( F_- \) is disjoint from \( T_l \) for all \( 1 \leq l \leq m \), so \( F' \) is isotopic to an incompressible surface \( F \subset Y \) via an isotopy \( \varphi_t : M_\infty \to M_\infty \) such that \( \varphi_0 = \text{id}_{M_\infty} \) and each \( \varphi_t \) is a homeomorphism, where \( 0 \leq t \leq 1 \).

3.4. A topological lemma. We will prove a topological lemma in this subsection (see Lemma 3.12). This will be very useful when we show the distinctiveness of minimal surfaces from the construction.

Let us recall some notation. Let \( \Lambda \) be a Jordan curve on the sphere at infinity \( S_\infty^2 \), and \( \Omega_\pm = S_\infty^2 \setminus \Lambda \). Suppose there are two disjoint solid minimal catenoids \( T_\pm \) with boundary circles \( \partial_\infty T_\pm = \Delta_\pm' \cup \Delta_\pm'' \subset \Omega_\pm \). Let \( \delta_\pm \subset \mathbb{B}^3 \cup S_\infty^2 \) be two (unknotted) simple closed curves constructed as follows: take two points \( p_1' \in \Delta_+'' \) and \( p_2'' \in \Delta_\pm'' \). Let \( \alpha_+ \subset T_+ \) be a curve with asymptotic endpoints \( p_1' \) and \( p_2'' \), and let \( \beta_+ \subset \Omega_+ \) be a simple curve with endpoints \( p_1' \) and \( p_2'' \). Let \( \delta_+ = \alpha_+ \cup \beta_+ \). Similarly we can define the loop \( \delta_- \).

We say that solid minimal catenoids \( T_+ \) and \( T_- \) are linked (or unlinked) in \( \mathbb{B}^3 \cup S_\infty^2 \) if \( \delta_+ \) and \( \delta_- \) are linked (or unlinked) loops. See Figure for the linked case.

Lemma 3.12. We have the following statements:

1. If solid minimal catenoids \( T_+ \) and \( T_- \) are linked in \( \mathbb{B}^3 \cup S_\infty^2 \), then the Jordan curve \( \Lambda \) is essential in the space \( \mathbb{B}^3 \setminus (T_+ \cup T_-) \).
2. If \( T_+ \) and \( T_- \) are unlinked in \( \mathbb{B}^3 \cup S_\infty^2 \), then \( \Lambda \) is null-homotopic in the space \( \mathbb{B}^3 \setminus (T_+ \cup T_-) \).

Remark 3.13. If the boundary circles \( \partial_\infty T_+ \) and \( \partial_\infty T_- \) are contained in the same component of \( S_\infty^2 \setminus \Lambda \), then \( \Lambda \) is always null-homotopic in \( \mathbb{B}^3 \setminus (T_+ \cup T_-) \).

Proof of Lemma 3.12. We will only prove statement (1) as the other follows easily. We consider the equivalent figure of Figure (see Exercise I.6 in [Ro90, Chapter 3]): Let \( X \) denote the solid cylinder in Figure and let \( V \) be the quotient space
of \(X\) obtained by gluing the top and the bottom disks of \(X\). We also identify \(\Delta'_+\) with \(\Delta'_-\) and \(\Delta''_+\) with \(\Delta''_-\) in the quotient space \(V\); then \(J = T_+ \cup T_-\) is a solid torus contained in \(V\).

If the Jordan curve \(\Lambda\) is null-homotopic in \(X \setminus (T_+ \cup T_-)\), then it is also null-homotopic in \(V \setminus J\). But this is impossible according to Proposition G.3 in [Rol90, Chapter 3]. Therefore \(\Lambda\) is not contractible in \(\mathbb{B}^3 \setminus (T_+ \cup T_-)\).

3.5. **Proof of the main theorem.** In this subsection, we use results of the previous subsections to prove our main Theorem 1.1, namely, for a prescribed integer \(N\), we construct a quasi-Fuchsian manifold \(M^3 = \mathbb{H}^3 / \Gamma\) such that it contains at least \(2^N\) many closed incompressible minimal surfaces.

All figures in this subsection correspond to the case \(N = 3\) for simplicity.

**Proof of Theorem 1.1** The construction consists of four steps which we now describe.
Step 1. Consider the unit ball $\mathbb{B}^3$ in the $xyz$-space. We divide the interval $[-1,1]$ on the $z$-axis into $N+1$ subintervals of equal length. Let $\varepsilon > 0$ be a sufficiently small number and $H_i^\pm$ be the (Euclidean) horizontal planes

$$z = 1 - \frac{2i}{N+1} \pm \varepsilon, \quad 1 \leq i \leq N.$$ 

Let $C_i^\pm = H_i^\pm \cap S^2_\infty$ be a pair of parallel circles on $S^2_\infty$, $1 \leq i \leq N$.

By Lemma 3.1 and Theorem 2.1 we may assume that $\varepsilon$ is sufficiently small so that we can find a pair of circles above $C_i^+$ and below $C_i^-$, respectively, that bound a spherical minimal catenoid for each $1 \leq i \leq N$ (see Figure 5). By Theorem 2.2 we may assume each minimal catenoid is of least area.

![Figure 5. 2N parallel circles (N = 3)](image)

Step 2. We connect each pair of parallel circles $C_i^\pm$ by a narrow bridge $B_i$, $(1 \leq i \leq N)$, and connect each pair of parallel circles $C_i^-$ and $C_{i+1}^+$ by a narrow bridge $B_j'$ $(1 \leq j \leq N - 1)$; then we obtain a piecewise smooth Jordan curve $\Lambda \subset S^2_\infty$ (see Figure 6).

![Figure 6. 2N - 1 narrow bridges (N = 3)](image)
By Lemma 3.1 and Theorem 2.2, we may assume that the \(2N-1\) bridges are again sufficiently narrow so that we can find a pair of circles around each bridge that bound a spherical least area minimal catenoid.

By the construction, there are a total of \(6N-2\) disjoint circles, and they are in good position. We also obtain \(3N-1\) solid minimal catenoids in \(\mathbb{B}^3\) for each pair of disjoint circles next to the same bridge.

**Step 3.** Now we construct the quasi-Fuchsian group \(\Gamma\) by the method in Subsection 3.2. In particular, we cover the Jordan curve \(\Lambda\) in Step 2 by \(2L\) small disks \(\{\Delta_1, \Delta_2, \ldots, \Delta_{2L}\}\) on \(S^2_\infty\). We can choose these disks sufficiently small such that \(\bigcup\Delta_i\) is disjoint from the \(6N-2\) circles in Step 2 (see Figure 7).

![Figure 7. 3N - 1 catenoids (N = 3)](image)

Now we are in position to apply Theorem 3.3 to use the inversions with respect to the circles \(\{\partial \Delta_1, \partial \Delta_2, \ldots, \partial \Delta_{2L}\}\) to construct a torsion-free quasi-Fuchsian group \(\Gamma\) whose limit set \(\Lambda_\Gamma\) is contained in \(\bigcup_{i=1}^{2L} \Delta_i\).

It is easy to see that \(\bigcup_{i=1}^{2L} \Delta_i\) separates the sphere \(S^2_\infty\) into two components, denoted by \(E_+\). By the construction, it is clear that for each pair of the \(6N-2\) circles, either both are contained in \(E_+\) or both are in \(E_-\). Therefore, for each pair of these circles, either both are in \(Q_+\) or both are in \(Q_-\), where \(E_+ \subset Q_+ \subset \Omega_+\) are the asymptotic boundary of a fundamental polyhedron \(\mathcal{P}\) of \(\Gamma\).

**Step 4.** Now we can choose the \(3N-1\) pairs of circles in different ways to form \(2N-1\) minimal catenoids (see Figure 8 for the case when \(N = 3\)). Let us fix some notation: each Jordan curve in Figure 8 represents the limit set \(\Lambda_\Gamma\) of the quasi-Fuchsian group \(\Gamma\). Let the circles next to bridges \(B'_j\) \((1 \leq j \leq N-1)\) be fixed. Therefore the \(N-1\) minimal catenoids bounded by these circles are fixed as well. Now we are left with \(2N\) minimal catenoids: \(N\) bounded by each pair of two disjoint circles next to the bridge \(B_i\) \((1 \leq i \leq N)\), and the other \(N\), each bounded by a circle above the circle \(C^+_i\) and a circle below the circle \(C^-_i\) \((1 \leq i \leq N)\), as in Figure 5. Namely, for each bridge \(B_i\), \(1 \leq i \leq N\), there are two choices of minimal catenoids. Hence we have a choice of \(2^N\) many different ways to obtain \(2N-1\) minimal catenoids, which can be seen from the following: let \(\Pi'_i, \Pi^B_i, \text{ and } \Pi^C_i\) be the minimal catenoids asymptotic to the circles near the bridge \(B'_i\), the bridge \(B_i\) and the circle \(C_i\), respectively. For each function \(\varphi : \{1, \cdots, N\} \rightarrow \{B, C\}\),
Figure 8. $2^N$ minimal surfaces ($N = 3$)
let $\Pi = \bigcup_{i=1}^{N} \Pi_i^{(i)}$. We see that for each $i$, $\Pi_i$ is a collection of $N$ minimal catenoids. Such a collection, together with $\bigcup_{i=1}^{N-1} \Pi'_i$, is then a collection of $2N - 1$ minimal catenoids. Since there are $2^N$ choices of $i$, there are $2^N$ such collections. For example, in Figure 8, where $N = 3$, there are 8 total choices, each case leading to 5 minimal catenoids.

Now for each case, it is easy to verify that $\Lambda_{\Gamma}$ bounds a disk, which consists of strips and/or disks connected by narrow bridges, and are disjoint from $2N - 1$ solid catenoids. Therefore it is null-homotopic in the subspace of $B^3$ with $2N - 1$ solid catenoids removed. For example, when $N = 3$, we consider the top right figure in Figure 8. It is easy to see that $\Lambda_{\Gamma}$ bounds a disk consisting of two strips and two disks connected by 3 narrow bridges; therefore, $\Lambda_{\Gamma}$ is null-homotopic in the subspace of $B^3$ with 5 solid catenoids removed (see Figure 9).

![Figure 9](image-url)

**Figure 9.** $\Lambda_{\Gamma}$ is null-homotopic in the subspace of $B^3$ with $2N - 1$ solid catenoids removed ($N = 3$)

Now we apply Theorem 3.9. For each case, there is a $\Gamma$-invariant minimal disk with asymptotic boundary $\Lambda_{\Gamma}$. Therefore we obtain $2^N$ $\Gamma$-invariant minimal disks in $B^3$, denoted by $D_1, \ldots, D_{2^N}$, which all share the asymptotic boundary $\Lambda_{\Gamma}$.

We claim that these minimal disks $D_1, \ldots, D_{2^N}$ are distinct. To illustrate this, we once again consider the case when $N = 3$. We count the minimal surfaces in Figure 8 from left to right and from top to bottom. Recall that each minimal disk $D_i$ ($i = 1, \ldots, 8$) is isotopic to the disk in the $i$-th subspace of $B^3$ with 5 catenoids removed in Figure 8. We can now show, for example, that $D_2$ (see Figure 9) is distinct from the other seven minimal disks. Indeed, the disk $D_2$ is distinct from $D_1$, since $D_1$ intersects the bottom catenoid in Figure 9 by Lemma 3.12; similarly, by Lemma 3.12, each $D_i$, $i = 3, \ldots, 8$, intersects either the top catenoid, the middle catenoid or both in Figure 9. Therefore $D_1, \ldots, D_8$ are distinct minimal disks. The general case is similar, therefore these minimal disks $D_1, \ldots, D_{2^N}$ from our construction are distinct.

Now we set $\Sigma_i = D_i/\Gamma$, $i = 1, \ldots, 2^N$; then $\Sigma_1, \ldots, \Sigma_{2^N}$ are distinct embedded incompressible minimal surfaces in the quasi-Fuchsian manifold $M^3 = B^3/\Gamma$. This completes the proof of our main theorem. $\square$
In Step 4, the minimal disks $D_1, \ldots, D_{2N}$ share the asymptotic boundary $\Gamma$, which is the Jordan curve justifying Theorem 1.2.

**Remark 3.14 (On the genus of the surface).** By our construction, we use $2L$ small disks of injective radii $< \delta$ to cover the Jordan curve $\Lambda$. Then by Theorem 3.3, we can construct a torsion-free quasi-Fuchsian group $\Gamma$ so that $\mathbb{H}^3/\Gamma \cong F \times \mathbb{R}$, where $F$ is a closed surface with $\text{genus}(F) = L - 1$. One sees that $\text{length}(\Lambda)/\delta$ is large, and therefore so is the genus of $F$, and $L \to \infty$ as $N \to \infty$.

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