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NODAL SOLUTIONS FOR $(p, 2)$-EQUATIONS

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ABSTRACT. In this paper, we study a nonlinear elliptic equation driven by the sum of a p-Laplacian and a Laplacian ($(p, 2)$-equation), with a Carathéodory $(p-1)$-(sub-)linear reaction. Using variational methods combined with Morse theory, we prove two multiplicity theorems providing precise sign information for all the solutions (constant sign and nodal solutions). In the process, we prove two auxiliary results of independent interest.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the Dirichlet $(p, 2)$-equation

$$-\triangle_p u(z) - \mu\triangle u(z) = f(z, u(z)) \text{ in } \Omega, \ u|_{\partial\Omega} = 0, \tag{1.1}$$

where $\mu \geq 0$. In (1.1) we assume $p \in (2, \infty)$ when $\mu > 0$ and $p \in (1, \infty)$ when $\mu = 0$. Also, for $r \in (1, \infty)$, by $\triangle_r$ we denote the r-Laplace differential operator defined by

$$\triangle_r u = div\left(\|Du\|_{\mathbb{R}^N}^{r-2} Du\right), \text{ for all } u \in W_0^{1,r}(\Omega).$$

Note that the differential operator in (1.1) is not homogeneous when $\mu > 0$. The reaction $f : \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function (i.e., for all $x \in \mathbb{R}$, $z \to f(z, x)$ is measurable and for a.a. $z \in \Omega$, $x \to f(z, x)$ is continuous) and exhibits subcritical growth in the x-variable. Our hypotheses on $f(z, x)$ make the energy functional of problem (1.1) coercive. Our aim in this paper is to prove two multiplicity theorems for problem (1.1). In the first theorem, we establish the existence of three nontrivial solutions and in the second (with $f(t, x) = f(x)$, $\mu > 0$), we produce four nontrivial smooth solutions. In both theorems, we provide precise sign information for all the solutions produced.

Multiplicity results for coercive problems driven by the p-Laplacian (i.e., $\mu = 0$) were proved by Liu [21], Liu-Liu [22] and Papageorgiou-Papageorgiou [24].

Our approach is variational, based on the critical point theory, combined with suitable truncation techniques and the use of Morse theory (critical groups). In particular, the second multiplicity theorem relies heavily on the works of Cingolani-Vannella [10], [11] which deal with the critical groups of the energy functional of a $(p, 2)$-equation. Finally, we should mention that an existence result for $(p, 2)$ equations can be found in the work of Cingolani-Degiovanni [9]. We mention that

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$(p,2)$-equations are important in quantum physics (see Benci-D'Avenia-Fortunato-Pisani [4]) and in the study of reaction diffusion systems. Consider the parabolic equation

$$u_t = div\ \left[S(u)\,Du\right] + f(z,u) \tag{1.2}$$

where $S(u) = \|Du\|_{\mathbb{R}^N}^{r-2} + 1$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$. This equation arises in many physical applications, such as plasma physics and chemical reaction design; see Cherfils-Ilyasov [8]. In such applications u describes a concentration, the first term on the right hand side of (1.2) (the divergence term) corresponds to diffusion with a diffusion coefficient $S(u)$, while the second term (the function f) relates to source and loss properties. The stationary version of (1.2) leads to a $(p,2)$-equation. Typically, in physical applications, the term $f(z,x)$ has a polynomial form in x.

In the next section, for the convenience of the reader, we survey the main mathematical tools that we will use in this paper and prove some auxiliary results of independent interest.

2. Mathematical background and auxiliary results

Let $(X, \|.\|)$ be a Banach space and X^* its topological dual. By $\langle \cdot,\cdot \rangle$ we denote the duality brackets for the pair (X^*, X).

Given $\varphi \in C^1(X)$, we say that φ satisfies the *Cerami condition* (the C-condition, for short) if the following is true:

> "every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $\{\varphi(x_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and
>
> $$(1 + \|x_n\|)\,\varphi'(x_n) \to 0 \text{ in } X^* \text{ as } n \to \infty$$
>
> admits a strongly convergent subsequence."

This condition is a generalization of the more common Palais-Smale condition. Nevertheless, it suffices to prove a deformation lemma and derive from it the minimax theorems of critical point theory (see, for example, Papageorgiou-Kyritsi [25]). In particular, we can state the following slight extension of the well known "mountain pass theorem".

Theorem 1. *If $(X, \|.\|)$ is a Banach space, $\varphi \in C^1(X)$ satisfies the C-condition, $x_0,\ x_1 \in X$ and $\rho > 0$ are such that $\|x_1 - x_0\| > \rho > 0$,*

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf\{\varphi(x) : \|x - x_0\| = \rho\} =: \eta_\rho,$$

and

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \varphi(\gamma(t)),\ \textit{where } \Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = x_0,\ \gamma(1) = x_1\},$$

then $c \geq \eta_\rho$ and c is a critical value of φ; i.e., there exists $x^* \in X$ such that $\varphi'(x^*) = 0$ and $\varphi(x^*) = c$.

The following notion from the theory of nonlinear operators of monotone type will help us verify the C-condition. Here and in the sequel, $\xrightarrow{w}$ designates weak convergence.

Definition 1. A map $A : X \to X^*$ is said to be of type $(S)_+$ if for every sequence $\{u_n\}_{n\geq 1} \subseteq X$ such that $u_n \xrightarrow{w} u$ *in* X and

$$\limsup_{n\to\infty} \langle A(u_n), u_n - u \rangle \leq 0,$$

one has

$$u_n \to u \text{ in } X \text{ as } n \to \infty.$$

In the analysis of problem (1.1), we will use the Sobolev space $W_0^{1,p}(\Omega)$ and the Banach space

$$C_0^1(\overline{\Omega}) = \left\{u \in C^1(\overline{\Omega}) : u\,|_{\partial\Omega} = 0\right\}.$$

The latter is an ordered Banach space with positive cone

$$C_+ = \left\{u \in C_0^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\right\}.$$

This cone has a nonempty interior, given by

$$int\ C_+ = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega, \frac{\partial u}{\partial n}(z) < 0 \text{ for all } z \in \partial\Omega\right\},$$

where by $n(.)$ we denote the outward unit normal on $\partial\Omega$.

Let $\|.\|_p$ denote the norm of $L^p(\Omega)$ or $L^p(\Omega, \mathbb{R}^N)$. Throughout this work, for every $u \in W_0^{1,p}(\Omega)$, we set

$$\|u\| = \|Du\|_p$$

(by virtue of the Poincaré inequality) and for every $x \in \mathbb{R}$, we set

$$x^{\pm} = \max\{\pm x, 0\}.$$

The notation $\|.\|$ will also be used to denote the $\mathbb{R}^N$-norm, but this will not create any confusion, since it will always be clear from the context which norm we use.

If $u \in W_0^{1,p}(\Omega)$, then $u^{\pm} \in W_0^{1,p}(\Omega)$ and

$$|u| = u^+ + u^-,\ u = u^+ - u^-.$$

Also, by $|.|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$. Finally, for any measurable function $h : \Omega \times \mathbb{R} \to \mathbb{R}$, we denote by N_h the corresponding Nemytskii map, i.e.,

$$N_h(u)(.) = h(., u(.)) \text{ for all } u \in W_0^{1,p}(\Omega).$$

Let $r \in (1, \infty)$ and consider the map $A_r : W_0^{1,r}(\Omega) \to W_0^{-1,r'}(\Omega)$ $\left(\frac{1}{r} + \frac{1}{r'} = 1\right)$ defined by

$$\langle A_r(u), y\rangle = \int_\Omega \|Du\|^{r-2}(Du, Dy)_{\mathbb{R}^N}\,dz \text{ for all } u,\ y \in W_0^{1,r}(\Omega). \tag{2.1}$$

If $r = 2$, then we set $A_2 =: A$. The next proposition can be found in Papageorgiou-Kyritsi [25] (p. 314):

Proposition 1. *If* $r \in (1, \infty)$ *and* $A_r : W_0^{1,r}(\Omega) \to W_0^{-1,r'}(\Omega)$ *is defined by* (2.1), *then* A_r *is continuous, monotone (hence maximal monotone) and of type* $(S)_+$.

Remark. Note that $A \in L\left(H_0^1(\Omega), H^{-1}(\Omega)\right)$ (recall that $A := A_2$).

Next we recall some basic facts about the spectrum of the negative Dirichlet p-Laplacian which is denoted by $-\triangle_p^D$. We consider the following weighted eigenvalue problem with weight $m \in L^\infty(\Omega)_+$, $m \neq 0$:

$$\begin{cases} -\triangle_p u(z) = \widehat{\lambda} m(z)\,|u(z)|^{p-2}\,u(z) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{2.2}$$

Every $\widehat{\lambda} \in \mathbb{R}$ for which problem (2.2) has a nontrivial solution u is said to be an eigenvalue of $-\triangle_p^D$ and u a corresponding eigenfunction. The smallest eigenvalue is denoted by $\widehat{\lambda}_1(p, m)$ and it is positive, isolated (i.e., we can find $\varepsilon > 0$ such that the interval $\left[\widehat{\lambda}_1(p,m), \widehat{\lambda}_1(p,m)+\varepsilon\right)$ contains no other eigenvalue), and simple (i.e., the corresponding eigenspace is one-dimensional). Moreover, $\widehat{\lambda}_1(p,m)$ admits the following variational characterization:

$$\widehat{\lambda}_1(p,m) = \inf\left\{ \frac{\|Du\|_p^p}{\int_\Omega m\,|u|^p\,dz} : u \in W_0^{1,p}(\Omega),\ u \neq 0 \right\}. \tag{2.3}$$

In (2.3) the infimum is attained on the corresponding one dimensional eigenspace.

By $\widehat{u}_1$ we denote the L^p-normalized (i.e., $\|\widehat{u}_1\|_p = 1$) eigenfunction associated with $\widehat{\lambda}_1(p,m) > 0$. From (2.3) it is clear that $\widehat{u}_1$ does not change sign, and so we may assume that $\widehat{u}_1 \geq 0$. Nonlinear regularity theory (see, for example, Papageorgiou-Kyritsi [25] (pp. 311-312)) implies that

$$\widehat{u}_1 \in C_+.$$

In fact the nonlinear strong maximum principle of Vazquez [27] implies that

$$\widehat{u}_1 \in int\ C_+.$$

If $u \in W_0^{1,p}(\Omega)$ is an eigenfunction corresponding to an eigenvalue $\widehat{\lambda} \neq \widehat{\lambda}_1(p,m)$, then $u \in C_0^1(\overline{\Omega})$ and u must change sign.

Since the p-Laplacian is a $(p-1)$-homogeneous operator, the *Ljusternik-Schnirelmann* theory provides an increasing sequence $\left\{\widehat{\lambda}_n(p,m)\right\}_{n\geq 1}$ of eigenvalues of $-\triangle_p^D$ such that $\widehat{\lambda}_n(p,m) \to \infty$. These eigenvalues are known as the "LS-eigenvalues" of $-\triangle_p^D$.

If $p=2$ (linear eigenvalue problem), then these are all the eigenvalues and they are denoted by $\{\lambda_n(m), n\geq 1\}$ and by $\{\lambda_n, n \geq 1\}$ if $m \equiv 1$.

If $p \neq 2$ (nonlinear eigenvalue problem), we do not know if this is the case. Let $\sigma(p,m)$ denote the set of all eigenvalues of (2.2). It is easily seen that $\sigma(p,m)$ is closed. So, since $\widehat{\lambda}_1(p,m) > 0$ is isolated, we can define

$$\widehat{\lambda}_2^*(p,m) := \inf\left\{\widehat{\lambda} : \widehat{\lambda} \in \sigma(p,m), \widehat{\lambda} > \widehat{\lambda}_1(p,m)\right\} > \widehat{\lambda}_1(p,m).$$

Evidently, $\widehat{\lambda}_2^*(p,m)$ is the second eigenvalue of $-\triangle_p^D$. We have

$$\widehat{\lambda}_2^*(p,m) = \widehat{\lambda}_2(p,m);$$

i.e., for $-\triangle_p^D$ the second eigenvalue and the second LS-eigenvalue coincide.

The function $m \to \widehat{\lambda}_n(p,m)$ from $L^\infty(\Omega)_+ \setminus \{0\}$ into $(0,+\infty)$ is continuous and exhibits the following monotonicity properties:

- If $m(z) \leq m'(z)$ a.e. in Ω and $m \neq m'$, then $\widehat{\lambda}_1(p,m') < \widehat{\lambda}_1(p,m)$.
- If $m(z) < m'(z)$ a.e. in Ω, then $\widehat{\lambda}_2(p,m') < \widehat{\lambda}_2(p,m)$.

If $p=2$, then the unique continuation property of the eigenspaces $E(\lambda_n(m))$ implies that:

- If $m(z) \leq m'(z)$ a.e. on Ω and $m \neq m'$, then $\lambda_n(m') < \lambda_n(m)$ for all $n \geq 1$.

If $m \equiv 1$, then we write

$$\widehat{\lambda}_n(p) := \widehat{\lambda}_n(p,1) \text{ and } \lambda_n := \lambda_n(1)\ \ \forall n \geq 1.$$

The *Ljusternik-Schnirelmann* theory gives us a variational characterization of $\widehat{\lambda}_2(p)$. For our purposes this characterization is not convenient. Instead, we will use an alternative one due to Cuesta-de Figueiredo-Gossez [12]). So, let

$$\partial B_1^{L^p} = \left\{ u \in L^p(\Omega) : \|u\|_p = 1 \right\}, \ M = W_0^{1,p}(\Omega) \cap \partial B_1^{L^p}$$

and

$$\Gamma_0 = \{\gamma \in C([-1,1], M) : \gamma(-1) = -\widehat{u}_1, \gamma(1) = \widehat{u}_1\}.$$

Then we have the following:

Proposition 2.

$$\widehat{\lambda}_2(p) = \inf_{\gamma\in\Gamma_0} \max_{u\in\gamma([-1,1])} \|Du\|^p.$$

The next simple lemma can be found in Papageorgiou-Kyritsi [25] (p. 356).

Lemma 1. *If* $\theta \in L^\infty(\Omega)_+$, $\theta(z) \le \widehat{\lambda}_1(p)$ *a.e. in* Ω *and* $\theta \ne \widehat{\lambda}_1(p)$, *then there exists* $\xi_0 > 0$ *such that*

$$\|Du\|_p^p - \int_\Omega \theta |u|^p \, dz \ge \xi_0 \|u\|^p \ \textit{for all}\ u \in W_0^{1,p}(\Omega).$$

Next we prove an auxiliary result which relates Hölder and Sobolev local minimizers for a large class of C^1-functionals on $W_0^{1,p}(\Omega)$. We can prove this result for a more general differential operator than the $(p,2)$-operator. So, we introduce the following hypotheses:

$\mathbf{H}(a)$: $a(y) = a_0(\|y\|)y$ for all $y \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$, $a \in C^1(\mathbb{R}^N \setminus \{0\}, \mathbb{R}^N) \cap C(\mathbb{R}^N, \mathbb{R}^N)$, $a(0) = 0$ and

(i) there exists $C_0 > 0$ and $\eta \ge 0$ such that

$$C_0 (\eta + \|y\|)^{p-2} \|\xi\|^2 \le (\nabla a(y)\xi, \xi)_{\mathbb{R}^N}$$

for all $y \in \mathbb{R}^N \setminus \{0\}$, all $\xi \in \mathbb{R}^N$;

(ii) $\|\nabla a(y)\| \le C_1 (\eta + \|y\|)^{p-2}$ for all $y \in \mathbb{R}^N \setminus \{0\}$, with $C_1 > 0$ and $\eta \ge 0$ as in (i);

(iii) if $G_0(t) = \int_0^t a_0(s) s ds$, then the function $t \to G_0\left(t^{\frac{1}{2}}\right)$ is convex.

Remarks. These hypotheses imply that $a(.)$ is strictly monotone. To see this, let $y, y' \in \mathbb{R}^N$. Then we have

$$a(y) - a(y') = \int_0^1 \nabla a(y' + t(y - y'))(y - y') \, dt,$$

hence

$$\begin{aligned}(a(y) - a(y'), y - y')_{\mathbb{R}^N} &= \int_0^1 (\nabla a(y' + t(y - y'))(y - y'), y - y')_{\mathbb{R}^N} \, dt \\ &\ge C_0 \left[\int_0^1 (\eta + \|ty + (1-t)y'\|)^{p-2} dt\right] \|y - y'\|^2;\end{aligned}$$

therefore $a(.)$ is strictly monotone.

Also, we have

$$a(y) = \int_0^1 \frac{d}{dt} a(ty)\,dt = \int_0^1 (\nabla a(ty), y)_{\mathbb{R}^N}$$
$$\leq C_1 (\eta + \|y\|)^{p-2} \|y\| \text{ (see } \mathbf{H}(a)(ii))$$
$$\leq \widehat{C}_1 (1 + \|y\|)^{p-1} \text{ for some } \widehat{C}_1 > 0, \text{ all } y \in \mathbb{R}^N. \tag{2.4}$$

In addition, we have

$$(a(y), y)_{\mathbb{R}^N} = \int_0^1 (\nabla a(ty) y, y)_{\mathbb{R}^N} \geq C_0 \int_0^1 (\eta + \|y\|)^{p-2} \|y\|\,dt$$
$$\geq \frac{C_0}{p-1} \|y\|^{p-1} \text{ for all } y \in \mathbb{R}^N. \tag{2.5}$$

Finally, if $G(y) = G_0(\|y\|)$, then $G'(y) = a(y)$ for all $y \in \mathbb{R}^N$, $G(0) = 0$ and from (2.4) and (2.5) it follows that

$$\frac{C_0}{(p-1)p} \|y\|^p \leq G(y) \leq \widehat{C}_0 (1 + \|y\|)^p \text{ for some } \widehat{C}_0 > 0, \text{ all } y \in \mathbb{R}^N. \tag{2.6}$$

Examples. The following maps satisfy hypotheses $\mathbf{H}(a)$:

$$a(y) = \|y\|^{p-2} y, \ 1 < p < \infty \text{ (the } p - \text{Laplace differential operator)},$$
$$a(y) = \|y\|^{p-2} y + \mu y, \ \mu \geq 0, \ 2 < p \text{ (the } (p,2) - \text{differential operator)},$$
$$a(y) = \|y\|^{p-2} y \left(1 + \frac{1}{1 + \|y\|^p}\right), \ 1 < p < \infty.$$

Let $f_0 : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function with subcritical growth in $x \in \mathbb{R}$, i.e.,

$$|f_0(z, x)| \leq \alpha(z) + C|x|^{r-1} \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}$$

with $\alpha \in L^\infty(\Omega)_+$, $C > 0$, $1 < r < p^*$, where p^* is the critical Sobolev exponent, i.e.,

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p \geq N. \end{cases}$$

We set $F_0(z, x) = \int_0^x f_0(z, s)\,ds$ and consider the C^1-functional $\psi_0 : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_0(u) = \int_\Omega G(Du)\,dz - \int_\Omega F_0(z, u)\,dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

The next result was first proved by Brézis-Nirenberg [6] when $G(y) = \frac{1}{2}\|y\|^2$ and was extended by Garcia Azorero-Manfredi-Peral Alonso [16] to the case $G(y) = \frac{1}{p}\|y\|^p$, $1 < p < \infty$. Our proof here is simpler than in both the aforementioned works.

Proposition 3. *If hypotheses* $\mathbf{H}(a)(i)$, (ii) *hold,* ψ_0 *is as above and* $u_0 \in W_0^{1,p}(\Omega)$ *is a local* $C_0^1(\overline{\Omega})$*-minimizer of* ψ_0 *(i.e., there exists* $\rho_0 > 0$ *such that* $\psi_0(u_0) \leq \psi_0(u_0 + h)$ *for all* $h \in C_0^1(\overline{\Omega})$ *with* $\|h\|_{C_0^1(\overline{\Omega})} \leq \rho_0$*), then* $u_0 \in C_0^{1,\beta}(\overline{\Omega})$ *for some* $\beta \in (0,1)$ *and* u_0 *is a local* $W_0^{1,p}(\Omega)$*-minimizer of* ψ_0 *(i.e., there exists* $\rho_1 > 0$ *such that* $\psi_0(u_0) \leq \psi_0(u_0 + h)$ *for all* $h \in W_0^{1,p}(\Omega)$ *with* $\|h\| \leq \rho_1$*).*

Proof. Let $h \in C_0^1\left(\overline{\Omega}\right)$ and let $t > 0$ be small. Then we have $\psi_0\left(u_0\right) \le \psi_0\left(u_0 + th\right)$ and so

$$0 \le \left\langle \psi_0'\left(u_0\right), h\right\rangle. \tag{2.7}$$

Since $h \in C_0^1\left(\overline{\Omega}\right)$ is arbitrary and $C_0^1\left(\overline{\Omega}\right)$ is dense in $W_0^{1,p}\left(\Omega\right)$, we obtain

$$\psi_0'\left(u_0\right) = 0,$$

hence

$$V_a\left(u_0\right) = N_{f_0}\left(u_0\right),$$

where

$$\left\langle V_a\left(u\right), y\right\rangle = \int_\Omega \left(a\left(Du\right), Dy\right)_{\mathbb{R}^N} dz \text{ for all } u, y \in W_0^{1,p}\left(\Omega\right);$$

therefore

$$-div\ a\left(Du_0\left(z\right)\right) = f_0\left(z, u_0\left(z\right)\right) \text{ a.e. in } \Omega, u_0\left|_{\partial\Omega}\right.= 0. \tag{2.8}$$

Invoking Theorem 7.1 (p. 286) of Ladyzhenskaya-Uraltseva [19], we have $u_0 \in L^\infty\left(\Omega\right)$, and then we can apply Theorem 1 of Lieberman [20] and conclude that $u_0 \in C_0^{1,\beta}\left(\overline{\Omega}\right)$ with $\beta \in (0,1)$.

Next we show that u_0 is a local $W_0^{1,p}\left(\Omega\right)$-minimizer of ψ_0. We proceed indirectly. So, suppose that u_0 is not a local $W_0^{1,p}\left(\Omega\right)$-minimizer of ψ_0. For $\varepsilon > 0$, let

$$\overline{B}_\varepsilon^r = \left\{u \in W_0^{1,p}\left(\Omega\right) : \left\|u\right\|_r \le \varepsilon\right\}$$

and consider the minimization problem

$$\inf\left\{\psi_0\left(u_0 + h\right) : h \in \overline{B}_\varepsilon^r\right\} = m_0^\varepsilon > -\infty. \tag{2.9}$$

Since by hypothesis u_0 is not a local $W_0^{1,p}\left(\Omega\right)$-minimizer of ψ_0, we have

$$m_0^\varepsilon < \psi_0\left(u_0\right). \tag{2.10}$$

Let $\left\{h_n\right\}_{n\ge1} \subset \overline{B}_\varepsilon^r$ be a minimizing sequence for problem (2.9). From (2.6) and the growth condition on $f_0\left(z,.\right)$, it follows that $\left\{h_n\right\}_{n\ge1} \subset W_0^{1,p}\left(\Omega\right)$ is bounded. So, by passing to a suitable subsequence if necessary, we may assume that

$$h_n \overset{w}{\to} h_\varepsilon \text{ in } W_0^{1,p}\left(\Omega\right) \text{ and } h_n \to h_\varepsilon \text{ in } L^r\left(\Omega\right) \text{ as } n \to \infty, \tag{2.11}$$

hence $h_\varepsilon \in \overline{B}_\varepsilon^r$. Exploiting the compact embedding of $W_0^{1,p}\left(\Omega\right)$ into $L^r\left(\Omega\right)$, we can check that ψ_0 is sequentially weakly lower semicontinuous. Hence

$$\psi_0\left(u_0 + h_\varepsilon\right) \le \lim_{n\to\infty} \psi_0\left(u_0 + h_n\right) = m_0^\varepsilon,$$

hence $\psi_0\left(u_0 + h_\varepsilon\right) = m_0^\varepsilon$ (see (2.9)), and so $h_\varepsilon \ne 0$ (see (2.10)).

So, the minimization problem (2.9) has a solution $h_\varepsilon \in \overline{B}_\varepsilon^r \setminus \{0\}$. Invoking the Lagrange multiplier rule (see Ioffe-Tichomirov [18], p. 74), we can find $\lambda_\varepsilon \le 0$ such that

$$\psi_0'\left(u_0 + h_\varepsilon\right) = \lambda_\varepsilon \left|h_\varepsilon\right|^{r-2} h_\varepsilon,$$

hence

$$V_a\left(u_0 + h_\varepsilon\right) = N_{f_0}\left(u_0 + h_\varepsilon\right) + \lambda_\varepsilon \left|h_\varepsilon\right|^{r-2} h_\varepsilon;$$

therefore

$$-div\ a\left(D\left(u_0+h_\varepsilon\right)(z)\right)=f_0\left(z,\left(u_0+h_\varepsilon\right)(z)\right)+\lambda_\varepsilon\left|h_\varepsilon(z)\right|^{r-2}h_\varepsilon(z)\ \text{ a.e. in } \Omega. \tag{2.12}$$

From (2.8) and (2.12) it follows that if

$$\beta_\varepsilon(z,y):=a(y)-a\left(Du_0(z)\right),$$

then

$$-div\ \beta_\varepsilon\left(z,D\left(u_0+h_\varepsilon\right)(z)\right)=f_0\left(z,\left(u_0+h_\varepsilon\right)(z)\right)-f_0\left(z,u_0(z)\right)+\lambda_\varepsilon\left|h_\varepsilon(z)\right|^{r-2}h_\varepsilon(z)\ \text{ a.e. in } \Omega. \tag{2.13}$$

Since $\{h_\varepsilon\}_{\varepsilon\in(0,1]}$ is bounded in $L^r(\Omega)$, using the growth hypothesis on $f_0(z,.)$, the fact that $u_0\in C_0^1\left(\overline{\Omega}\right)$ and recalling that $\lambda_\varepsilon\leq 0$ for all $\varepsilon\in(0,1]$, from Theorem 7.1 of Ladyzhenskaya-Uraltseva [19], (p. 286) we have

$$h_\varepsilon\in L^\infty(\Omega)\ \text{ and }\ \|h_\varepsilon\|_\infty\leq M_1\ \text{for some } M_1>0,\ \text{all } \varepsilon\in(0,1]. \tag{2.14}$$

Then we can apply Theorem 1 of Lieberman [20] and conclude that $h_\varepsilon\in C_0^1\left(\overline{\Omega}\right)$ for all $\varepsilon\in(0,1]$.

Let $\mu\geq 1$ and consider the function $|h_\varepsilon|^\mu h_\varepsilon$. Then

$$D\left(|h_\varepsilon|^\mu h_\varepsilon\right)=(\mu+1)|h_\varepsilon|^\mu Dh_\varepsilon,$$

hence $|h_\varepsilon|^\mu h_\varepsilon\in W_0^{1,p}(\Omega)$. Using this as a test function in (2.13) and taking into account the monotonicity of a, we obtain

$$\begin{aligned}0&\leq(\mu+1)\int_\Omega\left(\beta_\varepsilon\left(z,D\left(u_0+h_\varepsilon\right)(z)\right),Dh_\varepsilon\right)_{\mathbb{R}^N}|h_\varepsilon|^\mu dz\\&=\int_\Omega\left(f_0\left(z,u_0+h_\varepsilon\right)-f_0\left(z,u_0\right)\right)|h_\varepsilon|^\mu h_\varepsilon dz+\lambda_\varepsilon\int_\Omega|h_\varepsilon|^{\mu+r}dz\\&\leq M_2\int_\Omega|h_\varepsilon|^{\mu+1}dz+\lambda_\varepsilon\|h_\varepsilon\|_{\mu+r}^{\mu+r}\ \text{ for some } M_2>0 \text{ (see (2.14))}\\&\leq M_2|\Omega|_N^{\frac{r-1}{\mu+r}}\|h_\varepsilon\|_{\mu+r}^{\mu-1}+\lambda_\varepsilon\|h_\varepsilon\|_{\mu+r}^{\mu+r}\ ,\end{aligned}$$

where the last inequality is obtained via Hölder's inequality with exponents $\frac{\mu+r}{\mu+1}$ and $\frac{\mu+r}{r-1}$. Then

$$-\lambda_\varepsilon\|h_\varepsilon\|_{\mu+1}^{\mu+r}\leq M_2|\Omega|_N^{\frac{r-1}{\mu+r}}\|h_\varepsilon\|_{\mu+r}^{\mu+1},$$

hence

$$-\lambda_\varepsilon\|h_\varepsilon\|_{\mu+r}^{r-1}\leq M_2|\Omega|_N^{\frac{r-1}{\mu+r}}\ \text{ for all } \mu\geq 1,\ \text{all } \varepsilon\in(0,1].$$

Letting $\mu\to\infty$, we obtain

$$-\lambda_\varepsilon\|h_\varepsilon\|_\infty^{r-1}\leq M_2 \text{ for all } \varepsilon\in(0,1]. \tag{2.15}$$

We set

$$\widehat{f}_\varepsilon(z,x)=f_0(z,x)+\lambda_\varepsilon\left|h_\varepsilon(z)\right|^{r-2}h_\varepsilon(z).$$

Then (2.12) becomes

$$-div\ a\left(D\left(u_0+h_\varepsilon\right)(z)\right)=\widehat{f}_\varepsilon\left(z,\left(u_0+h_\varepsilon\right)(z)\right)\ \text{ a.e. in } \Omega. \tag{2.16}$$

From (2.14) and (2.15), we have

$$\begin{aligned}\left|\widehat{f}_{\varepsilon}\left(z,\left(u_0+h_{\varepsilon}\right)(z)\right)\right| &\leq \left|f_0\left(z,\left(u_0+h_{\varepsilon}\right)(z)\right)\right|-\lambda_{\varepsilon}\left\|h_{\varepsilon}\right\|_{\infty}^{r-1}\\ &\leq M_3+M_2 \text{ for some } M_3>0, \text{ all } \varepsilon\in(0,1].\end{aligned}$$

So, we can apply Theorem 1 of Lieberman [20] and infer that there is $\gamma\in(0,1)$ such that

$$u_0+h_{\varepsilon}\in C_0^{1,\gamma}\left(\overline{\Omega}\right) \text{ and } \left\|u_0+h_{\varepsilon}\right\|_{C_0^{1,\gamma}\left(\overline{\Omega}\right)}\leq M_4 \text{ for some } M_4>0, \text{ all } \varepsilon\in(0,1]. \tag{2.17}$$

Since $C_0^{1,\gamma}\left(\overline{\Omega}\right)$ is embedded compactly in $C_0^{1}\left(\overline{\Omega}\right)$, from (2.17) it follows that

$$u_0+h_{\varepsilon}\to u_0 \text{ in } C_0^{1}\left(\overline{\Omega}\right) \text{ as } \varepsilon\to 0^{+} \text{ (recall } h_{\varepsilon}\to 0 \text{ in } L^{r}(\Omega)).$$

Since, by hypothesis, u_0 is a local $C_0^{1}\left(\overline{\Omega}\right)$-minimizer of ψ_0, we have

$$\psi_0\left(u_0\right)\leq\psi_0\left(u_0+h_{\varepsilon}\right) \text{ for all small } \varepsilon\in(0,1]. \tag{2.18}$$

On the other hand, since h_{ε} is a solution of (2.9) and because of (2.10), we have

$$\psi_0\left(u_0+h_{\varepsilon}\right)<\psi_0\left(u_0\right) \text{ for all } \varepsilon\in(0,1]. \tag{2.19}$$

Comparing (2.18) and (2.19), we reach a contradiction. This proves the proposition. ☐

Next we consider the auxiliary Dirichlet problem

$$-div\ a\left(Du(z)\right)=\widehat{f}_0\left(z,u(z)\right) \text{ in } \Omega,\ u\left.\right|_{\partial\Omega}=0. \tag{2.20}$$

The hypotheses on the reaction $\widehat{f}_0(z,x)$ are the following:

$\mathbf{H}\left(\widehat{f}_0\right)$: $\widehat{f}_0:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that $\widehat{f}_0(z,0)=0$ for a.a. $z\in\Omega$ and

(*i*) $\left|\widehat{f}_0(z,x)\right|\leq a(z)+c|x|^{r-1}$ for a.a. $z\in\Omega$, all $x\in\mathbb{R}$ with $a\in L^{\infty}(\Omega)_{+}$, $c>0$, $1<r<p^{*}$;

(*ii*) for a.a. $z\in\Omega$ we have

$$x\to\frac{\widehat{f}_0(z,x)}{x} \text{ is strictly decreasing on } (0,+\infty),$$

$$x\to\frac{\widehat{f}_0(z,x)}{x} \text{ is strictly increasing on } (-\infty,0)$$

and

$$\widehat{f}_0(z,x)\,x\geq-\overline{c}_0|x|^{p} \text{ for a.a. } z\in\Omega, \text{ all } x\in\mathbb{R}, \text{ with } \overline{c}_0>0.$$

Proposition 4. *If hypotheses* $\mathbf{H}(a)$ *and* $\mathbf{H}\left(\widehat{f}_0\right)$ *hold, then problem* (2.20) *has at most one positive solution* $u^{*}\in int\ C_{+}$ *and at most one negative solution* $v^{*}\in -int\ C_{+}$.

Proof. We prove the uniqueness of the positive solution (if it exists). The proof of the uniqueness of the negative solution is similar.

So, let $\sigma_{+}:L^{1}(\Omega)\to\overline{\mathbb{R}}=\mathbb{R}\cup\{\infty\}$ be the integral functional defined by

$$\sigma_{+}(u)=\begin{cases}\int_{\Omega}G\left(Du^{\frac{1}{2}}\right)dz & \text{if } u\geq 0, u^{\frac{1}{2}}\in W_0^{1,p}(\Omega),\\ +\infty & \text{otherwise.}\end{cases} \tag{2.21}$$

Then σ_+ is convex (see $\mathbf{H}(a)(iii)$ and Benguria-Brézis-Lieb [5], Lemma 4) and lower semicontinuous (by Fatou's lemma).

Let $u \in W_0^{1,p}(\Omega)$ be a nontrivial positive solution of (2.20). Then the nonlinear regularity theory (see [18] and [19]) implies that $u \in C_+$. Moreover, hypothesis $\mathbf{H}\left(\widehat{f}_0\right)(ii)$ implies that

$$div\ a(Du(z)) \leq \overline{c}_0 u(z)^{p-1} \text{ a.e. in } \Omega,$$

hence

$$u \in int\ C_+ \tag{2.22}$$

(see Pucci-Serrin [26], p. 120). Note that $u^2 \geq 0$ and $\left(u^2\right)^{\frac{1}{2}} \in W_0^{1,p}(\Omega)$. So, u^2 is in the effective domain of the functional σ_+. Let $h \in C_0^1\left(\overline{\Omega}\right)$ and $t > 0$ small. Then $u^2 + th \in int\ C_+$ (see (2.22)), and so the Gateaux derivative of σ_+ at u^2 in the direction h exists. Using the chain rule and integration by parts, we obtain

$$\sigma_+'\left(u^2\right)(h) = -\frac{1}{2}\int_\Omega \frac{div\ a(Du)}{u} h dz. \tag{2.23}$$

If $w \in W_0^{1,p}(\Omega)$ is another nontrivial positive solution of (2.20), then $w \in int\ C_+$ and from the convexity of the functional σ_+ and (2.23), we have

$$\int_\Omega \left(\frac{-div\ a(Du)}{u} + \frac{div\ a(Dw)}{w}\right)(u-w)\,dz \geq 0,$$

hence

$$\int_\Omega \left(\frac{\widehat{f}_0(z,u(z))}{u} - \frac{\widehat{f}_0(z,w(z))}{w}\right)(u-w)\,dz \geq 0;$$

therefore $u = w$ (see hypothesis $\mathbf{H}\left(\widehat{f}_0\right)(ii)$). □

A slight modification of the proof of Proposition 2.6 of Arcoya-Ruiz [2], in order to take into account the presence of the linear term $-\mu\triangle u$, leads to the following strong comparison principle.

The following notation will be used: for any functions h, $g \in L^\infty(\Omega)$, we say that $h \prec g$ if, for any $K \subset \Omega$ compact, we can find $\varepsilon > 0$ such that

$$h(z) + \varepsilon \leq g(z) \text{ for a.a. } z \in K.$$

Evidently, if h, $g \in C(\Omega)$ and $h(z) < g(z)$ for all $z \in \Omega$, then $h \prec g$.

Proposition 5. *If $\xi \geq 0$, h, $g \in L^\infty(\Omega)$, $h \prec g$ and u, $v \in C_0^1\left(\overline{\Omega}\right)$ are solutions of*

$$-\triangle_p u(z) - \mu\triangle u(z) + \xi|u(z)|^{p-2}u(z) = h(z) \text{ in } \Omega,$$
$$-\triangle_p v(z) - \mu\triangle v(z) + \xi|v(z)|^{p-2}v(z) = g(z) \text{ in } \Omega$$

with $v \in int\ C_+$, then

$$v - u \in int\ C_+.$$

Next let us recall a few basic definitions and facts from Morse theory and from the works of Cingolani-Vannella [10], [11], which we will need in the sequel.

So, let (Y_1, Y_2) be a topological pair with $Y_2 \subset Y_1 \subset X$. For every integer $k \geq 0$, we denote by $H_k(Y_1, Y_2)$ the k^{th}-relative singular homology group with integer coefficients for the topological pair (Y_1, Y_2).

Recall that $H_k(Y_1, Y_2) = 0$ for all $k < 0$. For $\varphi \in C^1(X)$ and $c \in \mathbb{R}$, we introduce the following sets:

$$\varphi^c = \{x \in X : \varphi(x) \leq c\},\ K_\varphi = \{x \in X : \varphi'(x) = 0\}$$

and

$$K_\varphi^c = \{x \in K_\varphi : \varphi(x) = c\}.$$

The *critical groups of* φ at an isolated critical point $x \in X$ with $\varphi(x) = c$ (i.e., $x \in K_\varphi^c$) are defined by

$$C_k(\varphi, x) = H_k(\varphi^c \cap U, (\varphi^c \cap U) \setminus \{x\}) \text{ for all integers } k \geq 0,$$

where U is a neighborhood of x such that $K_\varphi \cap \varphi^c \cap U = \{x\}$. The excision property of singular homology theory implies that the above definition of critical groups is independent of the particular choice of the neighborhood U.

Suppose that $\varphi \in C^1(X)$ satisfies the C-condition and $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. The *critical groups of* φ *at infinity* are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \text{ for all integers } k \geq 0.$$

The second deformation lemma (which is valid since φ satisfies the C-condition; see Papageorgiou-Kyritsi [25] p. 349) implies that this definition is independent of the choice of the level $c < \inf \varphi(K_\varphi)$.

Suppose K_φ is finite. We set

$$M(t, x) = \sum_{k \geq 0} rank\ C_k(\varphi, x)\, t^k \text{ for all } t \in \mathbb{R},\ \text{all } x \in K_\varphi$$

and

$$P(t, \infty) = \sum_{k \geq 0} rank\ C_k(\varphi, \infty)\, t^k \text{ for all } t \in \mathbb{R}.$$

Then the Morse relation reads

$$\sum_{x \in K_\varphi} M(t, x) = P(t, \infty) + (1 + t)\, Q(t) \tag{2.24}$$

where

$$Q(t) := \sum_{k \geq 0} \beta_k t^k$$

is a formal series in $t \in \mathbb{R}$ with nonnegative integer coefficients.

Suppose $X = H$ is a Hilbert space, $x \in H$, U is a neighborhood of x and $\varphi \in C^2(U)$. If $x \in K_\varphi$, then the "Morse index" of x is defined to be the supremum of the dimensions of the vector subspaces of H on which $\varphi''(x)$ is negative definite. Also, we say that $x \in K_\varphi$ is nondegenerate if $\varphi''(x)$ is invertible. The critical groups of φ at a nondegenerate critical point $x \in H$ with Morse index m are given by

$$C_k(\varphi, x) = \delta_{k,m}\mathbb{Z} \text{ for all integers } k \geq 0, \tag{2.25}$$

where $\delta_{k,m}$ is the usual Kronecker symbol

$$\delta_{k,m} = \begin{cases} 1 & \text{if} \quad k = m, \\ 0 & \text{if} \quad k \neq m. \end{cases}$$

As we already mentioned in the introduction, in the second multiplicity theorem we will assume that $f(z,x) = f(x)$, $\mu > 0$ and will use tools and results from the works of Cingolani-Vannella [10], [11].

So, let $f \in C^1(\mathbb{R})$ with

$$|f'(x)| \leq C\left(1 + |x|^{r-1}\right) \text{ for all } x \in \mathbb{R}$$

with $C > 0$ and $p \leq r \leq p^*$. We set $F(x) = \int_0^x f(s)\,ds$ and consider the C^2-functional $\varphi : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi(u) = \frac{1}{p}\|Du\|_p^p + \frac{\mu}{2}\|Du\|_2^2 - \int_\Omega F(u(z))\,dz \ \forall u \in W_0^{1,p}(\Omega).$$

From [10], [11] we know that

$$\begin{aligned}\langle \varphi''(u)y, v\rangle = &\int_\Omega \left(\mu + \|Du\|^{p-2}\right)(Dy, Dv)_{\mathbb{R}^N}\,dz \\ &+ (p-2)\int_\Omega \|Du\|^{p-4}(Du, Dy)_{\mathbb{R}^N}(Du, Dv)_{\mathbb{R}^N}\,dz \\ &- \int_\Omega f'(u)yv dz \text{ for all } u, y, v \in W_0^{1,p}(\Omega).\end{aligned} \tag{2.26}$$

As before, by $\langle .,. \rangle$ we denote the duality brackets for the pair $(W^{-1,p'}(\Omega), W_0^{1,p}(\Omega))$, where $W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*$, $\frac{1}{p} + \frac{1}{p'} = 1$.

Suppose $u_0 \in K_\varphi$. Then the nonlinear regularity theory (see [19], [20]) implies that $u_0 \in C_0^1\left(\overline{\Omega}\right)$. Then

$$b(.) = \|Du_0(.)\|^{\frac{p-4}{2}} Du_0(.) \in L^\infty(\Omega).$$

Let H_b be the closure of $C_0^\infty(\Omega)$ under the inner product

$$(y,v)_b = \int_\Omega \left[\left(\mu + b^2\right)(Dy, Dv)_{\mathbb{R}^N} + (p-2)(b, Dy)_{\mathbb{R}^N}(b, Dv)_{\mathbb{R}^N}\right]dz.$$

Let $\|.\|_b$ be the norm induced by the inner product $(.,.)_b$. Then the norm $\|.\|_b$ is equivalent to the usual norm $\|.\|_{H_0^1(\Omega)}$ and so H_b is isomorphic to $H_0^1(\Omega)$. Hence $W_0^{1,p}(\Omega)$ is embedded into H_b continuously.

Let $L_b \in \mathcal{L}(H_b, H_b^*)$ be defined by

$$\langle L_b(u), v\rangle = (u,v)_b - \int_\Omega f'(u_0)uv\ dz \text{ for all } u, v \in H_b.$$

Then L_b is a Fredholm operator of index zero and it is the extension of $\varphi''(u_0)$ on H_b. We consider the orthogonal direct sum decomposition

$$H_b = H^- \oplus H^0 \oplus H^+,$$

where H^-, H^0, H^+ are respectively the negative, null and positive spaces according to the spectral decomposition of L_b in $L^2(\Omega)$. Note that $\dim H^-$, $\dim H^0 < \infty$. Since $u_0 \in C_0^1\left(\overline{\Omega}\right)$, from standard regularity theory, we have

$$H^- \oplus H^0 \subseteq W_0^{1,p}(\Omega) \cap L^\infty(\Omega).$$

If we set $W := H^+ \cap W_0^{1,p}(\Omega)$, $V := H^- \oplus H^0$, then we have the direct sum decomposition

$$W_0^{1,p}(\Omega) = V \oplus W$$

and

$$\langle \varphi''(u_0)y, y\rangle \geq \beta\|y\|_b^2 \text{ for some } \beta > 0, \text{ all } y \in W$$

(see Cingolani-Vannella [10], p. 279).

3. Three solutions theorem

In this section we prove a three solutions theorem for problem (1.1), providing precise sign information for all the solutions. For this multiplicity theorem, we will impose the following conditions on the reaction $f(z,x)$:

$\mathbf{H}^1(f)$: $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that $f(z,0)=0$ for a.a. $z\in\Omega$ and

(i) $|f(z,x)|\le a(z)+c|x|^{r-1}$ for a.a. $z\in\Omega$, all $x\in\mathbb{R}$ with $a\in L^\infty(\Omega)_+$, $c>0$, $p<r<p^*$;

(ii) if $F(z,x)=\int_0^x f(z,s)\,ds$, then there exists $\theta\in L^\infty(\Omega)_+$, $\theta(z)\le\widehat{\lambda}_1(p)$ a.e. in Ω, $\theta\neq\widehat{\lambda}_1(p)$ and

$$\limsup_{x\to\pm\infty}\frac{pF(z,x)}{|x|^p}\le\theta(z)\ \text{ uniformly for a.a. } z\in\Omega;$$

(iii) if $\mu>0$ and $2<p$, then there exist $h\in L^\infty(\Omega)$ and an integer $m\ge 2$ such that

$$\lim_{x\to 0}\frac{f(z,x)}{x}=\mu h(z)\ \text{ uniformly for a.a. } z\in\Omega,$$
$$h(z)\in[\lambda_m,\lambda_{m+1}]\ \text{ a.e. in } \Omega, h\neq\lambda_m, h\neq\lambda_{m+1}$$

and

$$f(z,x)x\ge C_2x^2-C_3|x|^p\ \text{ for a.a. } z\in\Omega,\ \text{all } x\in\mathbb{R}, \text{with } C_2,C_3>0;$$

if $\mu=0$ and $1<p<\infty$, then there exist β_0, $\widehat{\beta}_0$ such that

$$\widehat{\lambda}_2(p)<\beta_0\le\liminf_{x\to 0}\frac{f(z,x)}{|x|^{p-2}x}\le\limsup_{x\to 0}\frac{f(z,x)}{|x|^{p-2}x}\le\widehat{\beta}_0\ \text{ uniformly for a.a. } z\in\Omega;$$

(iv) for every $\rho>0$, there exists $\xi_\rho>0$ such that

$$f(z,x)x+\xi_\rho|x|^p\ge 0 \text{ for a.a. } z\in\Omega,\ \text{all } x\in\mathbb{R} \text{ with } |x|\le\rho.$$

Remarks. Hypothesis $\mathbf{H}^1(f)(ii)$ is a nonuniform nonresonance condition at $\pm\infty$ with respect to the principal eigenvalue $\widehat{\lambda}_1(p)$. For $\mu>0$ and $2<p$, hypothesis $\mathbf{H}^1(f)(iii)$ implies the existence of a concave (sublinear) term in the reaction. If $\mu=0$ and $1<p<\infty$, then hypothesis $\mathbf{H}^1(f)(iii)$ implies that for a.a. $z\in\Omega$, $f(z,.)$ is $(p-1)$-linear near zero. Hypothesis $\mathbf{H}^1(f)(iv)$ is weaker than assuming a sign condition on $f(z,.)$. Note that $\mathbf{H}^1(f)(iv)$ follows from $\mathbf{H}^1(f)(iii)$ in case $\mu>0$, $p>2$.

Examples. The following functions satisfy hypotheses $\mathbf{H}^1(f)$ (for the sake of simplicity we drop the z-dependence):

$$f_1(x)=\theta|x|^{p-2}x+\eta x \text{ if } \mu=1,\ 2<p,\ \text{with } \theta<\widehat{\lambda}_1(p),\ \eta\in(\lambda_m,\lambda_{m+1}), m\ge 2;$$

$$f_2(x)=\begin{cases}\beta|x|^{p-2}x-c|x|^{r-2}x & \text{if } |x|\le 1\\ \theta|x|^{p-2}x & \text{if } |x|>1\end{cases}\ \text{if } \mu=0,\ 1<p<\infty,$$
$$\text{with } \theta<\widehat{\lambda}_1(p)<\widehat{\lambda}_2(p)<\beta,\ p<r, c=\beta-\theta;$$

$$f_3(x)=\begin{cases}\eta x+x^2\ln|x| & \text{if } |x|\le 1\\ \theta|x|x+cx & \text{if } |x|>1\end{cases}\ \text{if } \mu=1,\ p=3,$$
$$\text{with } \eta\in(\lambda_m,\lambda_{m+1}),\ m\ge 2,\ \theta<\widehat{\lambda}_1(3),\ c=\eta-\theta.$$

First we produce two nontrivial constant sign smooth solutions.

Proposition 6. *If hypotheses* $\mathbf{H}^1(f)$ *hold, then problem* (1.1) *has at least two constant sign smooth solutions:* $u_0 \in int\ C_+$, $v_0 \in -int\ C_+$.

Proof. Let $f_\pm(z,x) = f(z,\pm x^\pm)$ and set $F_\pm(z,x) = \int_0^x f_\pm(z,s)\,ds$. We consider the C^1-functionals $\varphi_\pm : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_\pm(u) = \frac{1}{p}\|Du\|_p^p + \frac{\mu}{2}\|Du\|_2^2 - \int_\Omega F_\pm(z,u(z))\,dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

First we show the existence of a positive solution. By hypotheses $\mathbf{H}^1(f)(i),(ii)$, given $\varepsilon > 0$, we can find $C_\varepsilon > 0$ such that

$$F_+(z,x) \le \frac{1}{p}(\theta(z)+\varepsilon)|x|^p + C_\varepsilon \text{ for a.a. } z \in \Omega, \text{all } x \in \mathbb{R}. \tag{3.1}$$

Then for all $u \in W_0^{1,p}(\Omega)$ we have

$$\begin{aligned}\varphi_+(u) &= \frac{1}{p}\|Du\|_p^p + \frac{\mu}{2}\|Du\|_2^2 - \int_\Omega F_+(z,u(z))\,dz \\ &\ge \frac{1}{p}\left[\|Du\|_p^p - \int_\Omega \theta|u|^p\,dz\right] - \frac{\varepsilon}{\widehat{\lambda}_1(p)}\|u\|^p - C_\varepsilon|\Omega|_N \\ &\qquad\qquad \text{(see (3.1) and (2.3))} \\ &\ge \frac{1}{p}\left[\xi_0 - \frac{\varepsilon}{\widehat{\lambda}_1(p)}\right]\|u\|^p - C_\varepsilon|\Omega|_N \text{ (see Lemma 1).}\end{aligned} \tag{3.2}$$

Choosing $\varepsilon \in \left(0, \xi_0\widehat{\lambda}_1(p)\right)$, from (3.2) we infer that φ_+ is coercive. Also, exploiting the compact embedding of $W_0^{1,p}(\Omega)$ into $L^p(\Omega)$, we show that φ_+ is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\varphi_+(u_0) = \inf\left\{\varphi_+(u) : u \in W_0^{1,p}(\Omega)\right\} =: m_+. \tag{3.3}$$

First we assume that $\mu > 0$, $2 < p < \infty$. Then hypothesis $\mathbf{H}^1(f)(iii)$ implies that given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that

$$f_+(z,x) \ge (\mu h(z) - \varepsilon)x \text{ for a.a. } z \in \Omega, \text{ all } 0 \le x \le \delta,$$

hence

$$F_+(z,x) \ge \frac{1}{2}(\mu h(z) - \varepsilon)x^2 \text{ for a.a. } z \in \Omega, \text{ all } 0 \le x \le \delta. \tag{3.4}$$

Let $\widetilde{u}_1 \in int\ C_+$ be the L^2-normalized (i.e., $\|\widetilde{u}_1\|_2 = 1$) principal eigenfunction of $-\triangle^D$ and choose $t \in (0,1)$ small such that

$$t\widetilde{u}_1(z) \in [0,\delta] \text{ for all } z \in \overline{\Omega}.$$

Then

$$\begin{aligned}\varphi_+(t\widetilde{u}_1) &= \frac{t^p}{p}\|D\widetilde{u}_1\|_p^p + \frac{t^2\mu}{2}\|D\widetilde{u}_1\|_2^2 - \int_\Omega F_+(z,t\widetilde{u}_1(z))\,dz \\ &\le \frac{t^p}{p}\|D\widetilde{u}_1\|_p^p + \frac{t^2\mu}{2}\int_\Omega (\lambda_1 - h(z))\widetilde{u}_1^2(z)\,dz + \frac{\varepsilon t^2}{2}\end{aligned}$$

(see (3.4)). The hypothesis on h (see $\mathbf{H}^1(f)(iii)$) and the fact that $\widetilde{u}_1 \in int\ C_+$, $m \geq 2$ imply that

$$\gamma_0 := \int_\Omega (h(z) - \lambda_1)\, \widetilde{u}_1^2(z)\, dz > 0.$$

Then

$$\varphi_+(t\widetilde{u}_1) \leq \frac{t^p}{p} \|D\widetilde{u}_1\|_p^p + \frac{t^2}{2}(\varepsilon - \mu\gamma_0). \tag{3.5}$$

Choosing $\varepsilon \in (0, \mu\gamma_0)$ and making $t \in (0,1)$ even smaller if necessary, from (3.5) we infer that

$$\varphi_+(t\widetilde{u}_1) < 0 \text{ (recall that } p > 2),$$

hence

$$\varphi_+(u_0) = m_+ < 0 = \varphi_+(0) \ \text{ (see (3.3))};$$

therefore

$$u_0 \neq 0.$$

Next assume that $\mu = 0$, $1 < p < \infty$. Then by virtue of hypothesis $\mathbf{H}^1(f)(iii)$ we can find $\beta_1 \in \left(\widehat{\lambda}_2(p), \beta_0\right)$ and $\delta > 0$ such that

$$f_+(z,x) \geq \beta_1 x^{p-1} \text{ for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \delta,$$

hence

$$F_+(z,x) \geq \frac{\beta_1}{p} x^p \text{ for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \delta. \tag{3.6}$$

Let $\widehat{u}_1 \in int\ C_+$ be the L^p-normalized (i.e., $\|\widehat{u}_1\|_p = 1$) principal eigenfunction of $-\triangle_p^D$. Choose $t \in (0,1)$ small such that

$$t\widehat{u}_1(z) \in [0, \delta] \text{ for all } z \in \overline{\Omega}.$$

Then

$$\begin{aligned}
\varphi_+(t\widehat{u}_1) &= \frac{t^p}{p} \|D\widehat{u}_1\|_p^p - \int_\Omega F_+(z, t\widehat{u}_1(z))\, dz \\
&\leq \frac{t^p}{p} \widehat{\lambda}_1(p) - \frac{\beta_1}{p} t^p \ \text{ (see (3.6) and recall that } \|\widehat{u}_1\|_p = 1) \\
&= \frac{t^p}{p} \left[\widehat{\lambda}_1(p) - \beta_1\right] < 0,
\end{aligned}$$

hence

$$\varphi_+(u_0) = m_+ < 0 = \varphi_+(0) \ \text{ (see (3.3))};$$

therefore

$$u_0 \neq 0.$$

Thus, in both cases, we have $u_0 \neq 0$. From (3.3), we derive

$$\varphi_+'(u_0) = 0,$$

hence

$$A_p(u_0) + \mu A(u_0) = N_f(u_0). \tag{3.7}$$

On (3.7) we act with $-u_0^- \in W_0^{1,p}(\Omega)$ and obtain $u_0 \geq 0$, $u_0 \neq 0$. Then from (3.7) we have

$$-\triangle_p u_0(z) - \mu \triangle u_0(z) = f(z, u_0(z)) \text{ a.e. in } \Omega, u_0|_{\partial\Omega} = 0. \tag{3.8}$$

From the nonlinear regularity theory (see [19], [20]) it follows that $u_0 \in C_+ \setminus \{0\}$. Let $\rho = \|u_0\|_\infty$ and let $\xi_\rho > 0$ be as postulated by hypothesis $\mathbf{H}^1(f)(iv)$. Then

$$\triangle_p u_0(z) + \mu \triangle u_0(z) \le \xi_\rho u_0^{p-1}(z) \text{ a.e. in } \Omega$$

(see (3.8) and $\mathbf{H}^1(f)(iv)$), hence

$$u_0 \in int\ C_+$$

(see Pucci-Serrin [26], p. 120). Similarly, working with φ_- we obtain a second nontrivial constant sign smooth solution,

$$v_0 \in -int\ C_+.$$

□

Next we show that problem (1.1) has extremal constant sign smooth solutions; i.e., there exist a smallest nontrivial positive solution and a biggest nontrivial negative solution. To this end, we first consider the auxiliary Dirichlet problem

$$-\triangle_p u(z) - \mu \triangle u(z) = C_2 u_2(z) - C_3 |u(z)|^{p-2} u(z) \text{ in } \Omega,\ u|_{\partial\Omega} = 0, \tag{3.9}$$

where $\mu > 0$, $p > 2$, and C_2, C_3 are as in $\mathbf{H}^1(f)(iii)$.

Proposition 7. *Problem* (3.9) *has a unique nontrivial positive solution* $u^* \in int\ C_+$, *and consequently* $-u^* \in -int\ C_+$ *is the unique nontrivial negative solution of* (3.9).

Proof. Let $\psi_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\psi_+(u) = \frac{1}{p}\|Du\|_p^p + \frac{\mu}{2}\|Du\|_2^2 - \frac{C_2}{2}\|u^+\|_2^2 + \frac{C_3}{p}\|u^+\|_p^p \text{ for all } u \in W_0^{1,p}(\Omega).$$

Since $p > 2$, it is clear that ψ_+ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $u^* \in W_0^{1,p}(\Omega)$ such that

$$\psi_+(u^*) = \inf\left\{\psi_+(u) : u \in W_0^{1,p}(\Omega)\right\} =: m_+^*. \tag{3.10}$$

As in the proof of Proposition 6, exploiting the fact that $p > 2$, we have

$$\psi_+(u^*) = m^* < 0 = \psi_+(0) \text{ (see (3.10));}$$

therefore

$$u^* \neq 0.$$

From (3.10) we have

$$\psi_+'(u^*) = 0,$$

hence

$$A_p(u^*) + \mu A(u^*) = C_2(u^*)^+ - C_3\left[(u^*)^+\right]^{p-1}. \tag{3.11}$$

On (3.11) we act with $-(u^*)^- \in W_0^{1,p}(\Omega)$ and obtain $u^* \ge 0$, $u^* \neq 0$. Then (3.11) becomes

$$A_p(u^*) + \mu A(u^*) = C_2(u^*) - C_3(u^*)^{p-1},$$

hence

$$-\triangle_p u^*(z) - \mu \triangle u^*(z) = C_2 u^*(z) - C_3(u^*(z))^{p-1} \text{ a.e. in } \Omega,\ u^*|_{\partial\Omega} = 0. \tag{3.12}$$

Nonlinear regularity theory (see [19], [20]) implies that $u^* \in C_+ \setminus \{0\}$. From (3.12) we infer

$$\triangle_p u^*(z) + \mu \triangle u^*(z) \le C_3(u^*(z))^{p-1} \text{ a.e. in } \Omega,$$

hence (see Pucci-Serrin [26], p. 120)

$$u^* \in int\ C_+.$$

The uniqueness of u^* follows from Proposition 4.

The fact that (3.9) is odd implies that $-u^* \in -int\ C_+$ is the unique nontrivial negative solution of (3.9) . □

Now let $\mathcal{S}_+$ (respectively, $\mathcal{S}_-$) be the set of nontrivial positive (respectively, negative) solutions of (1.1) . From Proposition 6 we know that both sets are nonempty.

Lemma 2. *If hypotheses* $\mathbf{H}^1(f)$ *hold and* $\widetilde{u} \in \mathcal{S}_+$ *(respectively,* $\widetilde{v} \in \mathcal{S}_-$*), then* $\widetilde{u} \geq u^*$ *(respectively,* $\widetilde{v} \leq -u^*$*).*

Proof. Let $\widetilde{u} \in \mathcal{S}_+$. We introduce the following Carathéodory function:

$$h_+(z,x) = \begin{cases} 0 & \text{if } x<0, \\ C_2x - C_3x^{p-1} & \text{if } 0 \leq x \leq \widetilde{u}(z), \\ C_2\widetilde{u}(z) - C_3(\widetilde{u}(z))^{p-1} & \text{if } \widetilde{u}(z) < x. \end{cases} \tag{3.13}$$

Let $H_+(z,x) = \int_0^x h_+(z,s)\,ds$ and consider the C^1-functional $\xi_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\xi_+(u) = \frac{1}{p}\|Du\|_p^p + \frac{\mu}{2}\|Du\|_2^2 - \int_\Omega H_+(z,u(z))\,dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

It is clear that ξ_+ is coercive. Also it is sequentially weakly lower semicontinuous. Therefore, we can find $\widehat{y} \in W_0^{1,p}(\Omega)$ such that

$$\xi_+(\widehat{y}) = \inf\left\{\xi_+(y) : y \in W_0^{1,p}(\Omega)\right\} =: \widehat{m}_+. \tag{3.14}$$

As before (see the proof of Proposition 6), since $p > 2$, we have

$$\xi_+(\widehat{y}) = \widehat{m}_+ < 0 = \xi_+(0) \text{ (see (3.14))};$$

therefore

$$\widehat{y} \neq 0.$$

From (3.14) we have

$$\xi_+'(\widehat{y}) = 0,$$

hence

$$A_p(\widehat{y}) + \mu A(\widehat{y}) = N_{h_+}(\widehat{y}). \tag{3.15}$$

On (3.15) we act with $-(\widehat{y})^- \in W_0^{1,p}(\Omega)$ and obtain $\widehat{y} \geq 0$, $\widehat{y} \neq 0$. Also on (3.15) we act with $(\widehat{y}-\widetilde{u})^+ \in W_0^{1,p}(\Omega)$ and derive

$$\begin{aligned} &\left\langle A_p(\widehat{y}), (\widehat{y}-\widetilde{u})^+\right\rangle + \mu\left\langle A(\widehat{y}), (\widehat{y}-\widetilde{u})^+\right\rangle \\ &= \int_\Omega h_+(z,\widehat{y})(\widehat{y}-\widetilde{u})^+\,dz \\ &= \int_\Omega \left(C_2\widetilde{u} - C_3\widetilde{u}^{p-1}\right)(\widehat{y}-\widetilde{u})^+\,dz \text{ (see (3.13))} \\ &\leq \int_\Omega f(z,\widetilde{u})(\widehat{y}-\widetilde{u})^+\,dz \text{ (see } \mathbf{H}^1(f)(iii)) \\ &= \left\langle A_p(\widetilde{u}), (\widehat{y}-\widetilde{u})^+\right\rangle + \mu\left\langle A(\widetilde{u}), (\widehat{y}-\widetilde{u})^+\right\rangle, \end{aligned}$$

hence

$$\int_{\{\widehat{y}>\widetilde{u}\}}\left(\|D\widehat{y}\|^{p-2}D\widehat{y}-\|D\widetilde{u}\|^{p-2}D\widetilde{u},D\widehat{y}-D\widetilde{u}\right)_{\mathbb{R}^N}+\mu\left\|(\widehat{y}-\widetilde{u})^+\right\|^2\leq 0.$$

It follows that $|\{\widehat{y}>\widetilde{u}\}|_N=0$, i.e.,

$$\widehat{y}\leq\widetilde{u}.$$

Hence, we have proved that

$$\widehat{y}\in[0,\widetilde{u}]:=\left\{u\in W_0^{1,p}(\Omega):0\leq u(z)\leq\widetilde{u}(z)\ \text{ a.e. in }\Omega\right\}.$$

Then equation (3.15) becomes

$$A_p(\widehat{y})+\mu A(\widehat{y})=C_2\widehat{y}-C_3\widehat{y}^{p-1}\ \text{(see (3.13))},$$

hence

$$-\triangle_p\widehat{y}(z)-\mu\triangle\widehat{y}(z)=C_2\widehat{y}(z)-C_3(\widehat{y}(z))^{p-1}\ \text{ a.e. in }\Omega,\ \widehat{y}\mid_{\partial\Omega}=0,$$
$$\widehat{y}\neq 0,$$

hence $\widehat{y}=u^*$ (see Proposition 7); therefore

$$u^*\leq\widetilde{u}.$$

Similarly, if $\widetilde{v}\in\mathcal{S}_-$, then

$$\widetilde{v}\leq -u^*.$$

□

Now we are ready to produce the extremal constant sign smooth solutions for problem (1.1).

Proposition 8. *If hypotheses* $\mathbf{H}^1(f)$ *hold, then problem* (1.1) *has a smallest positive solution* $u_+\in int\ C_+$ *and a biggest negative solution* $v_-\in -int\ C_+$.

Proof. Exploiting the monotonicity of the map $u\to A_p(u)+\mu A(u)$, as in Lemma 4.3 of Filippakis-Kristaly-Papageorgiou [15], we show that $\mathcal{S}_+$ is downward directed; i.e., if $u,\ y\in\mathcal{S}_+$, then we can find $w\in\mathcal{S}_+$ such that $w\leq\min\{u,y\}$.

First we assume that $\mu>0$, $2<p<\infty$. Let $C\subset\mathcal{S}_+$ be a chain (i.e., a totally ordered subset of $\mathcal{S}_+$). Then from Dunford-Schwartz [14] (p. 336), we can find $\{u_n\}_{n\geq 1}\subseteq C$ such that

$$\inf C=\inf_{n\geq 1}u_n.$$

Since C is totally ordered, Lemma 1.1.5 (p. 15) of Heikkila-Lakshmikantham [17] implies that we can take $\{u_n\}_{n\geq 1}$ to be decreasing. Then

$$A_p(u_n)+\mu A(u_n)=N_f(u_n)\ \text{ and } u^*\leq u_n\leq u_1 \text{ for all } n\geq 1 \tag{3.16}$$

(see Lemma 2), hence $\{u_n\}_{n\geq 1}\subseteq W_0^{1,p}(\Omega)$ is bounded. So, we may assume that

$$u_n\overset{w}{\to}u \text{ in } W_0^{1,p}(\Omega)\ \text{ and } u_n\to u \text{ in } L^r(\Omega)\ \text{ as } n\to\infty. \tag{3.17}$$

On (3.16) we act with u_n-u, pass to the limit as $n\to\infty$ and use (3.17). Then

$$\lim_{n\to\infty}\left[\langle A_p(u_n),u_n-u\rangle+\mu\langle A(u_n),u_n-u\rangle\right]=0,$$

hence

$$\limsup_{n\to\infty}\left[\langle A_p(u_n),u_n-u\rangle+\mu\langle A(u),u_n-u\rangle\right]\leq 0$$

(from the monotonicity of A); therefore

$$\limsup_{n\to\infty} \langle A_p(u_n), u_n - u\rangle \le 0,$$

and by Proposition 1 we conclude that

$$u_n \to u \text{ in } W_0^{1,p}(\Omega),\ u = \inf C. \tag{3.18}$$

Passing to the limit as $n \to \infty$ in (3.16) and using (3.18), we obtain

$$A_p(u) + \mu A(u) = N_f(u) \text{ and } u^* \le u,$$

hence $u \in \mathcal{S}_+$. Since C was an arbitrary chain in $\mathcal{S}_+$, from the Kuratowski-Zorn lemma we infer that $\mathcal{S}_+$ has a minimal element $u_+ \in \mathcal{S}_+$. Recalling that $\mathcal{S}_+$ is downward directed and $\mathcal{S}_+ \subseteq int\ C_+$, we infer that $u_+ \in int\ C_+$ is the smallest nontrivial positive solution of (1.1).

Similarly, $\mathcal{S}_-$ is upward directed; i.e., if $v, y \in \mathcal{S}_-$, then we can find $w \in \mathcal{S}_-$ such that $\max\{v, y\} \le w$ (see Lemma 4.4 of Filippakis-Kristaly-Papageorgiou [15]). Then, arguing as above, via the Kuratowski-Zorn lemma, we can produce $v_- \in \mathcal{S}_- \subseteq -int\ C_+$, the smallest nontrivial negative solution of (1.1).

Next suppose $\mu = 0$, $1 < p < \infty$. The argument remains essentially the same, only now we cannot make use of Lemma 2, so we need a different approach to show that $u \neq 0$. More precisely, we have

$$A_p(u_n) = N_f(u_n) \text{ for all } n \ge 1 \tag{3.19}$$

(recall that $\mu = 0$; see (3.16)). From (3.19) as before, using the boundedness of $\{u_n\}_{n\ge1} \subseteq W_0^{1,p}(\Omega)$ and Proposition 1, we obtain

$$u_n \to u \text{ in } W_0^{1,p}(\Omega)$$

and conclude that

$$A_p(u) = N_f(u).$$

We need to show that $u \neq 0$ in order to conclude that $\inf C = u \in \mathcal{S}_+$ and apply the Kuratowski-Zorn lemma. Arguing by contradiction, suppose that $u = 0$. Then $\|u_n\| \to 0$. Let

$$y_n = \frac{u_n}{\|u_n\|},\ n \ge 1.$$

We have $\|y_n\| = 1$ for all $n \ge 1$, and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega) \text{ and } y_n \to y \text{ in } L^p(\Omega) \text{ as } n \to \infty. \tag{3.20}$$

From (3.19) we have

$$A_p(y_n) = \frac{N_f(u_n)}{\|u_n\|^{p-1}} \text{ for all } n \ge 1. \tag{3.21}$$

Note that

$$\left\{\frac{N_f(u_n)}{\|u_n\|^{p-1}}\right\}_{n\ge1} \subseteq L^{p'}(\Omega) \text{ is bounded.}$$

So, acting on (3.21) with $y_n - y$ and passing to the limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \langle A_p(y_n), y_n - y\rangle = 0,$$

and by Proposition 1 we conclude that

$$y_n \to y \text{ in } W_0^{1,p}(\Omega),\text{ and so } \|y\| = 1,\ y \ge 0. \tag{3.22}$$

Using hypothesis $\mathbf{H}^1(f)(iii)$ (the $\mu = 0$, $1 < p < \infty$ case) and reasoning as in Aizicovici-Papageorgiou-Staicu [1] (see the proof of Proposition 31), we obtain

$$\frac{N_f(u_n)}{\|u_n\|^{p-1}} \xrightarrow{w} hy^{p-1} \text{ in } L^{p'}(\Omega) \text{ with } \beta_0 \le h(z) \le \widehat{\beta}_0 \text{ a.e. in } \Omega. \tag{3.23}$$

So, if in (3.21) we pass to the limit as $n \to \infty$ and use (3.22) and (3.23), we obtain

$$A_p(y) = hy^{p-1},$$

hence

$$-\triangle_p y(z) = h(z)[y(z)]^{p-1} \text{ a.e. in } \Omega, \ y|_{\partial\Omega} = 0. \tag{3.24}$$

Since $\beta_0 > \widehat{\lambda}_2(p)$, from (3.23) it follows that $\widehat{\lambda}_2(p,h) < 1$, and so (3.24) implies that $y = 0$, which contradicts (3.22). Therefore $u \neq 0$ and this takes care of the case $\mu = 0$, $1 < p < \infty$. □

Having produced the extremal constant sign solutions of (1.1), we can use them to establish the existence of a nodal (sign changing) solution for problem (1.1).

Proposition 9. *If hypotheses* $\mathbf{H}^1(f)$ *hold, then problem* (1.1) *admits a nodal solution* $y_0 \in C_0^1(\overline{\Omega})$.

Proof. Let $u_+ \in int\ C_+$ and $v_- \in -int\ C_+$ be the two extremal constant sign solutions of (1.1) obtained in Proposition 8. We introduce the following truncation of the reaction $f(z,.)$:

$$\widehat{f}(z,x) = \begin{cases} f(z, v_-(z)) & \text{if } x < v_-(z), \\ f(z,x) & \text{if } v_-(z) \le x \le u_+(z), \\ f(z,u_+(z)) & \text{if } u_+(z) < x. \end{cases} \tag{3.25}$$

This is a Carathéodory function. Let $\widehat{F}(z,x) = \int_0^x \widehat{f}(z,s)\,ds$ and consider the C^1-functionals $\widehat{\varphi} : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\varphi}(u) = \frac{1}{p}\|Du\|_p^p + \frac{\mu}{2}\|Du\|_2^2 - \int_\Omega \widehat{F}(z,u(z))\,dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

Also, let $\widehat{f}_\pm(z,x) = \widehat{f}(z, \pm x^\pm)$ and $\widehat{F}_\pm(z,x) = \int_0^x \widehat{f}_\pm(z,s)\,ds$ and consider the C^1- functionals $\widehat{\varphi}_\pm : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\varphi}_\pm(u) = \frac{1}{p}\|Du\|_p^p + \frac{\mu}{2}\|Du\|_2^2 - \int_\Omega \widehat{F}_\pm(z,u(z))\,dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

Let $u \in K_{\widehat{\varphi}}$. Then

$$A_p(u) + \mu A(u) = N_{\widehat{f}}(u). \tag{3.26}$$

On (3.26) we act with $(u - u_+)^+ \in W_0^{1,p}(\Omega)$ and obtain

$$\begin{aligned} &\left\langle A_p(u), (u-u_+)^+ \right\rangle + \mu \left\langle A(u), (u-u_+)^+ \right\rangle \\ &= \int_\Omega \widehat{f}(z,u)(u-u_+)^+\,dz \\ &= \int_\Omega f(z,u_+)(u-u_+)^+\,dz \text{ (see (3.25))} \\ &= \left\langle A_p(u_+), (u-u_+)^+ \right\rangle + \mu \left\langle A(u), (u-u_+)^+ \right\rangle, \end{aligned}$$

hence

$$\int_{\{u>u_+\}}\left(\|Du\|^{p-2}Du-\|Du_+\|^{p-2}Du_+,Du-Du_+\right)_{\mathbb{R}^N}dz+\mu\left\|D\left(u-u_+\right)^+\right\|_2^2=0,$$

therefore

$$u\leq u_+.$$

Similarly, acting on (3.26) with $(v_- - u)^+ \in W_0^{1,p}(\Omega)$ and reasoning as above, we obtain

$$v_-\leq u.$$

Hence

$$u\in[v_-,u_+]:=\left\{u\in W_0^{1,p}(\Omega):v_-(z)\leq u(z)\leq u_+(z)\ \text{ a.e. in }\Omega\right\}$$

and so

$$K_{\widehat{\varphi}}\subseteq[v_-,u_+].$$

In a similar fashion, we show that

$$K_{\widehat{\varphi}_+}\subseteq[0,u_+]:=\left\{u\in W_0^{1,p}(\Omega):0\leq u(z)\leq u_+(z)\ \text{ a.e. in }\Omega\right\}$$

and

$$K_{\widehat{\varphi}_-}\subseteq[v_-,0]:=\left\{u\in W_0^{1,p}(\Omega):v_-(z)\leq u(z)\leq 0\text{ a.e. in }\Omega\right\}.$$

In fact, due to the extremality of u_+ and v_-, we conclude that

$$K_{\widehat{\varphi}}\subseteq[v_-,u_+],K_{\widehat{\varphi}_+}=\{0,u_+\}\ \text{ and }K_{\widehat{\varphi}_-}=\{v_-,0\}. \tag{3.27}$$

Claim 1. u_+ and v_- are local minimizers of $\widehat{\varphi}$.

From (3.25) it is clear that $\widehat{\varphi}_+$ is coercive. Also, it is sequentially weakly lower semicontinuous. Therefore, by the Weierstrass theorem, we can find $\widehat{u}_0\in W_0^{1,p}(\Omega)$ such that

$$\widehat{\varphi}_+(\widehat{u}_0)=\inf\left\{\widehat{\varphi}_+(u):u\in W_0^{1,p}(\Omega)\right\}=:\widehat{m}_+. \tag{3.28}$$

As in the proof of Proposition 6, using hypothesis $\mathbf{H}^1(f)(iii)$, we show that

$$\widehat{\varphi}_+(\widehat{u}_0)=\widehat{m}_+<0=\widehat{\varphi}_+(0)\ \text{ (see (3.28))},$$

hence

$$\widehat{u}_0\neq 0.$$

From (3.28) we have $\widehat{u}_0\in K_{\widehat{\varphi}_+}\setminus\{0\}$ and so $\widehat{u}_0=u_+$ (see (3.27)). Recall that $u_+\in int\ C_+$ and note that

$$\widehat{\varphi}\,|_{C_+}=\widehat{\varphi}_+\,|_{C_+}.$$

Therefore u_+ is a local $C_0^1(\overline{\Omega})$-minimizer of $\widehat{\varphi}$. Invoking Proposition 3, we infer that u_+ is a local $W_0^{1,p}(\Omega)$-minimizer of $\widehat{\varphi}$. Similarly for $v_-\in -int\ C_+$, using this time the functional $\widehat{\varphi}_-$. This proves Claim 1.

Without any loss of generality, we may assume that $\widehat{\varphi}(v_-)\leq\widehat{\varphi}(u_+)$. Also, using Claim 1 and reasoning as in Aizicovici-Papageorgiou-Staicu [1], we can find $\rho\in(0,1)$ small such that

$$\widehat{\varphi}(v_-)\leq\widehat{\varphi}(u_+)<\inf\{\widehat{\varphi}(u):\|u-u_+\|=\rho\}=:\widehat{\eta}_\rho. \tag{3.29}$$

It is clear from (3.25) that $\widehat{\varphi}$ is coercive, and so it satisfies the C-condition. This fact and (3.29) permit the use of Theorem 1 (the mountain pass theorem). So, we obtain $y_0 \in K_{\widehat{\varphi}}$ such that

$$\widehat{\varphi}(v_-) \leq \widehat{\varphi}(u_+) < \widehat{\eta}_\rho \leq \widehat{\varphi}(y_0),$$

hence

$$y_0 \in [v_-, u_+] \setminus \{v_-, u_+\} \text{ and } y_0 \in C_0^1\left(\overline{\Omega}\right)$$

by nonlinear regularity and (3.27).

Evidently (see (3.25)), if we show that $y_0 \neq 0$, then the extremality of u_+ and v_- implies that $y_0 \in C_0^1\left(\overline{\Omega}\right)$ is nodal.

First suppose $\mu > 0$, $2 < p < \infty$.

Claim 2. $C_k(\widehat{\varphi}, 0) = \delta_{k,d_m}\mathbb{Z}$ for all $k \geq 0$ where $d_m = \dim \oplus_{i=1}^m E(\lambda_i)$, where $E(\lambda_i)$ is the eigenspace corresponding to λ_i.

Let $\psi : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the C^2-functional defined by

$$\psi(u) = \frac{1}{p}\|Du\|_p^p + \frac{\mu}{2}\|Du\|_2^2 - \frac{\mu}{2}\int_\Omega h(z)\,u^2(z)\,dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

By virtue of hypothesis $\mathbf{H}^1(f)(iii)$ (the $\mu > 0$, $2 < p < \infty$ case), given $\varepsilon > 0$, we can find $\rho \in (0,1)$ such that if

$$\overline{B}_\rho^c = \left\{u \in C_0^1\left(\overline{\Omega}\right) : \|u\|_{C_0^1(\overline{\Omega})} \leq \rho\right\},$$

then

$$\|\widehat{\varphi} - \psi\|_{C_0^1(\overline{B}_\rho^c)} \leq \varepsilon.$$

It follows that

$$C_k\left(\widehat{\varphi}\,|_{C_0^1(\overline{\Omega})}, 0\right) = C_k\left(\psi\,|_{C_0^1(\overline{\Omega})}, 0\right) \tag{3.30}$$

(see Chang [7], p. 336). Since $C_0^1\left(\overline{\Omega}\right)$ is dense in $W_0^{1,p}(\Omega)$, from Palais [23] we have (3.31)

$$C_k\left(\widehat{\varphi}\,|_{C_0^1(\overline{\Omega})}, 0\right) = C_k(\widehat{\varphi}, 0) \text{ and } C_k\left(\psi\,|_{C_0^1(\overline{\Omega})}, 0\right) = C_k(\psi, 0) \text{ for all } k \geq 0.$$

Moreover, from Theorem 1.1 of Cingolani-Vannella [10], we have

$$C_k(\psi, 0) = \delta_{k,d_m}\mathbb{Z} \text{ for all } k \geq 0 \text{ with } d_m = \dim \bigoplus_{i=1}^m E(\lambda_i). \tag{3.32}$$

From (3.30), (3.31) and (3.32), we conclude that

$$C_k(\widehat{\varphi}, 0) = \delta_{k,d_m}\mathbb{Z} \text{ for all } k \geq 0.$$

This proves Claim 2.

Recall that $y_0 \in C_0^1\left(\overline{\Omega}\right)$ is a critical point of $\widehat{\varphi}$ of mountain pass type. Hence

$$C_1(\widehat{\varphi}, y_0) \neq 0. \tag{3.33}$$

Comparing (3.33) with Claim 2, we conclude that $y_0 \neq 0$. Therefore $y_0 \in C_0^1\left(\overline{\Omega}\right)$ is nodal.

Next we assume that $\mu = 0$, $1 < p < \infty$.

Since $y_0 \in C_0^1\left(\overline{\Omega}\right)$ is a critical point of $\widehat{\varphi}$ of mountain pass type, we have

$$\widehat{\varphi}(y_0) = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \widehat{\varphi}(\gamma(t)) \tag{3.34}$$

where

$$\Gamma=\left\{\gamma\in C\left([0,1],W_0^{1,p}(\Omega)\right):\gamma(0)=v_-,\ \gamma(1)=u_+\right\}$$

(see Theorem 1). According to (3.34), in order to establish the nontriviality of y_0 it suffices to construct a path $\gamma_*\in\Gamma$ such that

$$\widehat{\varphi}\,|_{\gamma_*}<0.$$

By virtue of $\mathbf{H}^1(f)(iii)$ we can find $\delta,\,\delta_0>0$ such that

$$\widehat{\lambda}_2(p)+\delta\leq\frac{f(z,x)}{|x|^{p-2}x}\text{ for a.a. }z\in\Omega,\text{ all }|x|\leq\delta_0,$$

hence

$$\frac{1}{p}\left(\widehat{\lambda}_2(p)+\delta\right)|x|^p\leq F(z,x)\text{ for a.a. }z\in\Omega,\text{ all }|x|\leq\delta_0. \tag{3.35}$$

Let

$$\partial B_1^{L^p}=\left\{u\in L^p(\Omega):\|u\|_p=1\right\},\ M=W_0^{1,p}(\Omega)\cap\partial B_1^{L^p}\text{ and }M_c=M\cap C_0^1\left(\overline{\Omega}\right).$$

We endow M with the relative $W_0^{1,p}(\Omega)$-topology and M_c with the relative $C_0^1\left(\overline{\Omega}\right)$-topology. Then M_c is dense in M and $C([-1,1],M_c)$ is dense in $C([-1,1],M)$. Also, let

$$\Gamma_0=\{\gamma_0\in C([-1,1],M):\gamma_0(-1)=-\widehat{u}_1,\ \gamma_0(1)=\widehat{u}_1\}$$

and

$$\Gamma_0^c=\{\gamma_0\in C([-1,1],M_c):\gamma_0(-1)=-\widehat{u}_1,\ \gamma_0(1)=\widehat{u}_1\}.$$

Evidently Γ_0^c is dense in Γ_0. From Proposition 2, we see that we can find $\widehat{\gamma}_0\in\Gamma_0^c$ such that

$$\max\left\{\|Du\|_p^p:u\in\widehat{\gamma}_0([-1,1])\right\}\leq\widehat{\lambda}_1(p)+\delta. \tag{3.36}$$

Since $\widehat{\gamma}_0\in\Gamma_0^c$ and $v_-\in-int\ C_+$, $u_+\in int\ C_+$, we can find $\tau>0$ small such that

$$v_-(z)\leq\tau u(z)\leq u_+(z)\text{ and }\tau|u(z)|\leq\delta_0\text{ for all }z\in\overline{\Omega}\text{ and all }u\in\widehat{\gamma}_0([-1,1]). \tag{3.37}$$

Then for any $u\in\widehat{\gamma}_0([-1,1])$, we have

$$\begin{aligned}\widehat{\varphi}(\tau u)&=\frac{\tau^p}{p}\|Du\|_p^p-\int_\Omega\widehat{F}(z,\tau u(z))\,dz\\&=\frac{\tau^p}{p}\|Du\|_p^p-\int_\Omega F(z,\tau u(z))\,dz\text{(see (3.25) and (3.37)).}\end{aligned}$$

If we set $\gamma_0=\tau\widehat{\gamma}_0$, then γ_0 is a continuous path in $W_0^{1,p}(\Omega)$ which joins $-\tau\widehat{u}_1$ and $\tau\widehat{u}_1$ and

$$\widehat{\varphi}\,|_{\gamma_0}<0. \tag{3.38}$$

Next we produce a continuous path in $W_0^{1,p}(\Omega)$ which joins $-\tau\widehat{u}_1$ and u_+, and along which $\widehat{\varphi}$ is strictly negative. To this end, let

$$a=m_+=\inf_{W_0^{1,p}(\Omega)}\widehat{\varphi}<0=\widehat{\varphi}(0)=b. \tag{3.39}$$

Since $\widehat{\varphi}$ is coercive, it satisfies the C-condition. So, invoking the second deformation theorem (see Papageorgiou-Kyritsi [25], p. 349) we can find a continuous map $h: [0,1]\times\left(\widehat{\varphi}_+^b\backslash K_{\widehat{\varphi}_+}^b\right)\to\widehat{\varphi}_+^b$ such that $h(t,.)\mid_{\widehat{\varphi}_+^a}=Id\mid_{\widehat{\varphi}_+^a}$ and

$$h\left(1,\widehat{\varphi}_+^b\backslash K_{\widehat{\varphi}_+}^b\right)\subseteq\widehat{\varphi}_+^a=\{u_+\}\ \text{(see (3.27) and (3.39))},\tag{3.40}$$

$$\widehat{\varphi}_+\left(h\left(t,u\right)\right)\leq\widehat{\varphi}_+\left(h\left(s,u\right)\right)\ \text{for all } 0\leq s\leq t\leq 1,\ \text{all } u\in W_0^{1,p}\left(\Omega\right).\tag{3.41}$$

From (3.38) we see that $\tau\widehat{u}_1\in\widehat{\varphi}_+^b\backslash K_{\widehat{\varphi}_+}^b$, and so we can define the continuous path

$$\gamma_+\left(t\right)=h\left(t,\tau\widehat{u}_1\right)^+\ \text{ for all } t\in[0,1].$$

We have

$$\gamma_+\left(0\right)=h\left(0,\tau\widehat{u}_1\right)^+=\left(\tau\widehat{u}_1\right)^+=\tau\widehat{u}_1.$$

So, γ_+ is a continuous path in $W_0^{1,p}\left(\Omega\right)$ which joins $\tau\widehat{u}_1$ and u_+. Moreover, since

$$\widehat{\varphi}\mid_{W_+}=\widehat{\varphi}_+\mid_{W_+}\ \left(\text{where } W_+:=\left\{u\in W_0^{1,p}\left(\Omega\right):u\left(z\right)\geq 0 \text{ a.e. in } \Omega\right\}\right)$$

we have

$$\widehat{\varphi}\left(\gamma_+\left(t\right)\right)=\widehat{\varphi}_+\left(\gamma_+\left(t\right)\right)\leq\widehat{\varphi}_+\left(\tau\widehat{u}_1\right)=\widehat{\varphi}\left(\tau\widehat{u}_1\right)<0 \text{ for all } t\in[0,1]$$

(see (3.38)), hence

$$\widehat{\varphi}\mid_{\gamma_+}<0.\tag{3.42}$$

Similarly, we produce a continuous path γ_- in $W_0^{1,p}\left(\Omega\right)$ which joins $-\tau\widehat{u}_1$ and v_- such that

$$\widehat{\varphi}\mid_{\gamma_-}<0.\tag{3.43}$$

We concatenate γ_-, γ_0, γ_+ and obtain a continuous path γ_* in $W_0^{1,p}\left(\Omega\right)$, which joins v_- and u_+ and for which we have

$$\widehat{\varphi}\mid_{\gamma_*}<0\ \text{(see (3.38), (3.42), (3.43))}.$$

This implies that $y_0\neq 0$, hence $y_0\in C_0^1\left(\overline{\Omega}\right)$ is nodal. □

So, we have proved the following multiplicity result (three solutions theorem) for problem (1.1).

Theorem 2. *If hypotheses* $\mathbf{H}^1(f)$ *hold, then problem* (1.1) *has at least three nontrivial smooth solutions:*

$$u_0\in int\ C_+, v_0\in -int\ C_+,\ \textit{and } y_0\in C_0^1\left(\overline{\Omega}\right)\ \textit{nodal.}$$

Moreover, problem (1.1) *admits extremal nontrivial constant sign solutions.*

Remark. The part of the above theorem corresponding to the case $\mu=0$, $1<p<\infty$ improves Theorem 1.2 of Liu [21], where the hypotheses on the reaction $f(z,x)$ are more restrictive and no sign information is provided for the third solution.

4. Four solutions theorem

In this section, under stronger hypotheses on the reaction $f(z,x) = f(x)$, we produce a second nodal solution for a total of four nontrivial solutions.

Now $\mu > 0$, $2 < p < \infty$, $f(z,x) = f(x)$ (i.e., f is z-independent) and the problem under consideration is the following:

$$-\triangle_p u(z) - \mu \triangle u(z) = f(u(z)) \text{ in } \Omega,\ u\,|_{\partial\Omega} = 0. \tag{4.1}$$

The hypotheses on $f(.)$ are the following:

$\mathbf{H}^2(f)$: $f \in C^1(\mathbb{R})$, $f(0) = 0$ and

(i) $|f'(x)| \le C\left(1 + |x|^{r-2}\right)$ for all $x \in \mathbb{R}$ with $C > 0$, and $p < r < p^*$;

(ii) if $F(x) = \int_0^x f(s)\,ds$, then

$$\limsup_{x\to\pm\infty} \frac{pF(x)}{|x|^p} < \widehat{\lambda}_1(p);$$

(iii) there exists an integer $m \ge 2$ such that $\frac{f'(0)}{\mu} \in (\lambda_m, \lambda_{m+1})$ and

$f(x)x \ge C_2 x^2 - C_3 |x|^p$ for all $x \in \mathbb{R}$, with $C_2, C_3 > 0$;

(iv) for every $\rho > 0$, there exists $\xi_\rho > 0$ such that $x \to f(x) + \xi_\rho |x|^{p-2} x$ is nondecreasing on $[-\rho, \rho]$.

Examples. The following function satisfies hypotheses $\mathbf{H}^2(f)$ with $\mu = 1$:

$$f(x) = \begin{cases} \eta x & \text{if} \quad |x| \le 1 \\ \theta |x|^{p-2} x + c|x|^{q-2} x & \text{if} \quad |x| > 1 \end{cases} \text{ with } \eta \in (\lambda_m, \lambda_{m+1}), m \ge 2,$$

$$\eta = c + \theta, 0 < \theta < \widehat{\lambda}_1(p),\ q \in (1,2), c = \tfrac{\theta(p-2)}{2-q} > 0.$$

We have the following multiplicity theorem.

Theorem 3. *If hypotheses* $\mathbf{H}^2(f)$ *hold, then problem* (4.1) *has at least four nontrivial smooth solutions:*

$$u_0 \in int\ C_+, v_0 \in -int\ C_+, \text{ and } y_0, \widehat{y} \in C_0^1\left(\overline{\Omega}\right) \text{ nodal.}$$

Moreover, problem (4.1) *admits extremal nontrivial constant sign solutions.*

Proof. From Theorem 2, we already have three nontrivial smooth solutions:

$$u_0 \in int\ C_+, v_0 \in -int\ C_+, \text{ and } y_0 \in C_0^1\left(\overline{\Omega}\right) \text{ nodal,}$$

as well as extremal nontrivial constant sign solutions. Without any loss of generality we may assume that $u_0 \in int\ C_+$ and $v_0 \in -int\ C_+$ are these extremal nontrivial constant sign solutions.

Let $\varphi : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the C^2-energy functional for the problem (4.1) defined by

$$\varphi(u) = \frac{1}{p}\|Du\|_p^p + \frac{\mu}{2}\|Du\|_2^2 - \int_\Omega F(u(z))\,dz \text{ for all } u \in W_0^{1,p}(\Omega),$$

with $F(x) = \int_0^x f(s)\,ds$. From Cingolani-Vannella [11] (Lemma 2.2), we know that we can find $\rho > 0$ and a C^2-function $\xi : V \cap \overline{B}_\rho(0) \to \mathbb{R}$ (recall $V = H^- \oplus H^0$ and $\overline{B}_\rho(0) = \left\{u \in W_0^{1,p}(\Omega) : \|u\| \le \rho\right\}$ such that

$$\langle \xi''(0)v, u\rangle = \langle \varphi''(y_0)v, u\rangle \text{ for all } u, v \in W_0^{1,p}(\Omega).$$

As before, by $\langle .,. \rangle$ we denote the duality brackets for the pair $(W^{-1,p'}(\Omega), W_0^{1,p}(\Omega))$, where $W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*$, $\frac{1}{p}+\frac{1}{p'}=1$. Moreover, $\xi''(0)$ is a Fredholm operator and $\ker \xi''(0) = H^0$. Recall that $y_0 \in [v_0, u_0]$ (see the proof of Proposition 9 and recall that we have assumed that u_0 and v_0 are the extremal nontrivial constant sign solutions of (4.1)).

Let $\rho = \max\{\|u_0\|_\infty, \|v_0\|_\infty\}$ and let $\xi_\rho > 0$ be as postulated by hypothesis $\mathbf{H}^2(f)(iv)$. Then

$$\begin{aligned}(4.2)\qquad & -\triangle_p y_0(z) - \mu\triangle y_0(z) + \xi_\rho |y_0(z)|^{p-2} y_0(z) \\ & = f(y_0(z)) + \xi_\rho |y_0(z)|^{p-2} y_0(z) \\ & \leq f(u_0(z)) + \xi_\rho u_0^{p-1}(z) \ \text{ (since } y_0 \leq u_0\text{, see } \mathbf{H}^2(f)(iv)) \\ & = -\triangle_p u_0(z) - \mu\triangle u_0(z) + \xi_\rho u_0(z)^{p-1} \ \text{ a.e. in } \Omega.\end{aligned}$$

If $\gamma(y) = \|y\|^{p-2} y + \mu y$, then

$$\nabla\gamma(y) = \|y\|^{p-2}\left(I + (p-2)\frac{y\otimes y}{\|y\|^2}\right) + \mu I$$

and

$$\begin{aligned}(\nabla\gamma(Dy_0(z))\tau,\tau)_{\mathbb{R}^N} &= \|Dy_0(z)\|^{p-2}\left(\|\tau\|^2 + (p-2)\frac{(Dy_0(z),\tau)^2_{\mathbb{R}^N}}{\|Dy_0(z)\|^2}\right) + \mu\|\tau\|^2 \\ &\geq \mu\|\tau\|^2 > 0 \text{ for all } \tau \in \mathbb{R}^N\setminus\{0\}.\end{aligned}$$

So, from (4.2) and the tangency principle of Pucci-Serrin ([26], p. 35), we infer that

$$y_0(z) < u_0(z) \ \text{ for all } z \in \Omega.$$

Then invoking (4.2) and Proposition 5, we conclude that $u_0 - y_0 \in int\ C_+$. In a similar fashion, we show that $y_0 - v_0 \in int\ C_+$. Therefore

$$(4.3)\qquad y_0 \in int_{C_0^1(\overline{\Omega})}[v_0, u_0].$$

Let $\widehat{\varphi} : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional introduced in the proof of Proposition 9 (truncation of $f(.)$ at $\{u_0(z), v_0(z)\}$). Then

$$\widehat{\varphi}\mid_{[v_0,u_0]} = \varphi\mid_{[v_0,u_0]}.$$

So, from (4.3) it follows that

$$C_k\left(\widehat{\varphi}\mid_{C_0^1(\overline{\Omega})}, y_0\right) = C_k\left(\varphi\mid_{C_0^1(\overline{\Omega})}, y_0\right) \text{ for all } k \geq 0,$$

hence

$$(4.4)\qquad C_k(\widehat{\varphi}, y_0) = C_k(\varphi, y_0) \text{ for all } k \geq 0 \text{ (see Palais [23]).}$$

Recall (see the proof of Proposition 9) that $y_0 \in K_{\widehat{\varphi}}$ is of mountain pass type. Hence

$$C_1(\widehat{\varphi}, y_0) \neq 0 \text{ (see Chang [7]);}$$

therefore

$$(4.5)\qquad C_1(\varphi, y_0) \neq 0 \text{ (see (4.4)).}$$

From Cingolani-Vannella ([10], p. 286), we know that

$$(4.6)\qquad C_k(\varphi, y_0) = C_k(\xi, 0) \text{ for all } k \geq 0,$$

hence

$$C_1(\xi,0)\neq 0 \text{ (see (4.5))}; \tag{4.7}$$

therefore

$$\alpha := \dim H^- \leq 1.$$

Also, let $\alpha_0 = \dim H^0$. First suppose that $\alpha_0 = 0$. Then $u = 0$ is a nondegenerate critical point of ξ with "Morse index" α and so

$$C_k(\xi,0) = \delta_{k,\alpha}\mathbb{Z} \text{ for all } k \geq 0 \text{ (see (2.25))},$$

hence $\alpha = 1$ (see (4.7)); therefore

$$C_k(\varphi,y_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \geq 0 \text{ (see (4.6))}.$$

Next suppose that $\alpha_0 > 0$. Then $u = 0$ is a degenerate critical point of φ. Invoking the Shifting Theorem (see Chang [7], p. 333), we have

$$C_k(\xi,0) = C_{k-\alpha}\left(\widehat{\xi},0\right) \text{ for all } k \geq 0, \tag{4.8}$$

where $\widehat{\xi} = \xi\mid_{H^0}$. There are two possibilities: $\alpha = 1$ or $\alpha = 0$.

If $\alpha = 1$, then from (4.7) and (4.8), we have

$$C_0\left(\widehat{\xi},0\right) \neq 0,$$

hence

$$C_k(\xi,0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \geq 0$$

(see (4.8) and Bartsch [3], Proposition 2.3); therefore

$$C_k(\varphi,y_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \geq 0 \text{ (see (4.6))}.$$

If $\alpha = 0$, then from (4.8) we have

$$C_k(\xi,0) = C_k\left(\widehat{\xi},0\right) \text{ for all } k \geq 0. \tag{4.9}$$

If $\widehat{u} \in H^0 = \ker \xi''(0)$, then from Cingolani-Vannella [11], Lemma 2.2, we have

$$\int_\Omega \left(\mu + (p-1)\,b^2(z)\right)\|D\widehat{u}\|^2\,dz = \int_\Omega f'(y_0)\,\widehat{u}^2dz \tag{4.10}$$

where $b(z) = \|Dy_0(z)\|^{\frac{p-4}{2}} Dy_0(z)$, $b \in L^\infty(\Omega)$. We consider the weighted linear eigenvalue problem

$$-div\left(\left(\mu + (p-1)\,b^2(z)\right)Du(z)\right) = \widetilde{\lambda} f'(y_0(z))\,u(z) \text{ in } \Omega,\ u\mid_{\partial\Omega}=0. \tag{4.11}$$

Then from (4.10) and de Figueiredo [13], we deduce that $\widetilde{\lambda}_1 \leq 1$ and it is simple. (Here $\widetilde{\lambda}_1$ denotes the first eigenvalue of (4.11).) So, $\alpha_0 = 1$ and it follows that

$$C_1\left(\widehat{\xi},0\right) \neq 0,$$

hence

$$C_k(\xi,0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \geq 0 \text{ (see Bartsch [3], Proposition 2.5)};$$

therefore

$$C_k(\varphi,y_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \geq 0 \text{ (see (4.6))}.$$

From all of the above, we conclude that

$$C_k(\varphi, y_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \geq 0,$$

hence

$$C_k(\widehat{\varphi}, y_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \geq 0 \text{ (see (4.4))}. \tag{4.12}$$

From the proof of Proposition 9 (see Claim 1), we know that u_0 and v_0 are local minimizers of $\widehat{\varphi}$, hence

$$C_k(\widehat{\varphi}, u_0) = C_k(\widehat{\varphi}, v_0) = \delta_{k,0}\mathbb{Z} \text{ for all } k \geq 0 \text{ (see Chang [7])}. \tag{4.13}$$

Moreover, from the proof of Proposition 9 (see Claim 2), we have

$$C_k(\widehat{\varphi}, 0) = \delta_{k,d_m}\mathbb{Z} \text{ for all } k \geq 0. \tag{4.14}$$

Finally, since $\widehat{\varphi}$ is coercive (see (3.25)), we have

$$C_k(\widehat{\varphi}, \infty) = \delta_{k,0}\mathbb{Z} \text{ for all } k \geq 0. \tag{4.15}$$

Suppose $K_{\widehat{\varphi}} = \{0, u_0, v_0, y_0\}$. Then from (4.12), (4.13), (4.14), (4.15) and the Morse relation with $t = -1$ (see (2.24)), we have

$$(-1)^{d_m} + 2(-1)^0 + (-1)^1 = (-1)^0,$$

a contradiction. So, we can find $\widehat{y} \in K_{\widehat{\varphi}}$, $\widehat{y} \notin \{0, u_0, v_0, y_0\}$. From (3.32) and the nonlinear regularity theory, we conclude that $\widehat{y} \in C_0^1(\overline{\Omega})$ is a second nodal solution of (4.1). ☐

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