A C*-ALGEBRA APPROACH TO COMPLEX SYMMETRIC OPERATORS

KUNYU GUO, YOUQING JI, AND SEN ZHU

Abstract. In this paper, certain connections between complex symmetric operators and anti-automorphisms of singly generated C*-algebras are established. This provides a C*-algebra approach to the norm closure problem for complex symmetric operators. For \( T \in \mathcal{B}(\mathcal{H}) \) satisfying \( C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\} \), we give several characterizations for \( T \) to be a norm limit of complex symmetric operators. As applications, we give concrete characterizations for weighted shifts with nonzero weights to be norm limits of complex symmetric operators. In particular, we prove a conjecture of Garcia and Poore. On the other hand, it is proved that an essentially normal operator is a norm limit of complex symmetric operators if and only if it is complex symmetric. We obtain a canonical decomposition for essentially normal operators which are complex symmetric.

1. Introduction

Throughout this paper, we let \( \mathbb{C} \), \( \mathbb{R} \), \( \mathbb{Z} \) and \( \mathbb{N} \) denote the set of complex numbers, the set of real numbers, the set of integers and the set of positive integers respectively. We always denote by \( \mathcal{H} \) a complex separable infinite dimensional Hilbert space endowed with the inner product \( \langle \cdot, \cdot \rangle \), and by \( \mathcal{B}(\mathcal{H}) \) the algebra of all bounded linear operators on \( \mathcal{H} \). We let \( \mathcal{K}(\mathcal{H}) \) denote the ideal of compact operators on \( \mathcal{H} \), and let \( \pi \) denote the canonical quotient map of \( \mathcal{B}(\mathcal{H}) \) onto \( \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H}) \). For \( A \in \mathcal{B}(\mathcal{H}) \), we let \( C^*(A) \) denote the C*-algebra generated by \( A \) and the identity operator on \( \mathcal{H} \). We let \( \sigma(A) \) and \( \sigma_e(A) \) denote the spectrum and the essential spectrum of \( A \) respectively.

1.1. Complex symmetric operators and their norm closure problem. In this subsection, let us give a brief introduction to complex symmetric operators and their norm closure problem.

Definition 1.1. A map \( C \) on \( \mathcal{H} \) is called an anti-unitary operator if \( C \) is conjugate-linear, invertible and \( \langle Cx, Cy \rangle = \langle y, x \rangle \) for all \( x, y \in \mathcal{H} \). If, in addition, \( C^{-1} = C \), then \( C \) is called a conjugation on \( \mathcal{H} \).

Definition 1.2. An operator \( T \in \mathcal{B}(\mathcal{H}) \) is said to be complex symmetric if there is a conjugation \( C \) on \( \mathcal{H} \) such that \( CTC = T^* \).

The study of complex symmetric operators was initiated by Garcia and Putinar [18,19] and has recently received much attention. Many significant results concerning the internal structure of complex symmetric operators have been obtained...
Complex symmetric operators have many motivations in function theory, matrix analysis and other areas; in particular, complex symmetric operators are closely related to the study of truncated Toeplitz operators, which was initiated in Sarason’s seminal paper [35] and has led to rapid progress in related areas [3,5,6,20,21,36,37]. The reader is referred to [18,20] for more about the history of this topic and its connections to other subjects.

Following Garcia and Poore [16], we denote by \( \text{CSO} \) the set of all complex symmetric operators on \( \mathcal{H} \). People have recently paid much attention to the structure of the set \( \text{CSO} \). Among other things, people consider the closures of \( \text{CSO} \) in several important topologies, including the weak operator topology (\( \text{wot} \)), the strong operator topology (\( \text{sot} \)) and the norm topology. Garcia and Poore [16] recently proved that \( \text{CSO} \) is dense in \( B(\mathcal{H}) \) with respect to both \( \text{wot} \) and \( \text{sot} \). A fortiori, things become very complicated.

In the following, we let \( \overline{\text{CSO}} \) denote the norm closure of \( \text{CSO} \). Although \( \text{CSO} \) encompasses many important special operators, the set \( \overline{\text{CSO}} \) is indeed nowhere dense in \( B(\mathcal{H}) \). In fact, one can easily verify that each operator in \( \text{CSO} \) is biquasitriangular. Recall that an operator \( A \) is said to be biquasitriangular if there exists no \( \lambda \in \mathbb{C} \) such that \( A - \lambda \) is semi-Fredholm and \( \text{ind}(A - \lambda) \neq 0 \) (see [28, Section 6.3]). Then, using an approximation result [28, Theorem 6.17], one can see that \( \overline{\text{CSO}} \) is nowhere dense in \( B(\mathcal{H}) \).

In [23], Garcia and Wogen posed the norm closure problem for complex symmetric operators, which asked whether or not the set \( \text{CSO} \) is norm closed. Zhu, Li and Ji [48] gave a negative answer to the norm closure problem by proving that the Kakutani shift is not complex symmetric but belongs to \( \overline{\text{CSO}} \). Almost immediately, using the unilateral shift and its adjoint, Garcia and Poore [17] constructed a completely different counterexample.

Generalizing the Kakutani shift, Garcia and Poore [16] constructed some special weighted shifts in \( \overline{\text{CSO}} \setminus \text{CSO} \) which they called approximately Kakutani weighted shifts. A unilateral weighted shift \( T \in B(\mathcal{H}) \) with nonzero weights \( \{\alpha_k\}_{k=1}^\infty \) is said to be approximately Kakutani if for each \( n \geq 1 \) and \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) such that

\[
0 < |\alpha_N| < \varepsilon
\]

and

\[
1 \leq k \leq n \implies -\varepsilon < |\alpha_k| - |\alpha_{N-k}| < \varepsilon.
\]

Garcia and Poore raised the following conjecture.

**Conjecture 1.3** ([16, Conjecture 1]). Every irreducible unilateral weighted shift in \( \overline{\text{CSO}} \) is approximately Kakutani.

In general, if \( T \in \overline{\text{CSO}} \), then it follows that \( T \) is a “small perturbation” of operators in \( \text{CSO} \); however, we find that in many cases \( T \) is in fact a “small compact perturbation” of operators in \( \text{CSO} \). To be precise, we first give a definition.

Given a subset \( \mathcal{E} \) of \( B(\mathcal{H}) \), we denote by \( \overline{\mathcal{E}} \) the set of all operators \( A \in B(\mathcal{H}) \) satisfying: for any \( \varepsilon > 0 \), there exists \( K \in \mathcal{K}(\mathcal{H}) \) with \( \|K\| < \varepsilon \) such that \( A + K \in \mathcal{E} \). We call \( \overline{\mathcal{E}} \) the compact closure of \( \mathcal{E} \). It is clear that \( \mathcal{E} \subset \overline{\mathcal{E}} \subset \overline{\mathcal{E}}^{\text{c}} \) and \( \overline{\mathcal{E}}^{\text{c}} \subset [\mathcal{E} + \mathcal{K}(\mathcal{H})] \).

If we let \( W \) denote the Kakutani shift, then a minor modification of the proof of [48] Theorem 0.5] shows that \( W \in \overline{\text{CSO}}^{\text{c}} \). In a recent paper, Garcia and Poore proved that a compact operator belongs to \( \overline{\text{CSO}} \) if and only if it is complex symmetric [16 Theorem 4]. These two results motivate the following question.
Question 1.4. Does $\overline{CSO}$ coincide with $\overline{CSO^c}$?

For some special classes of operators, including completely reducible operators, essentially normal operators, hyponormal operators and many weighted shifts, this paper gives a positive answer to Question 1.4. All these results mentioned above suggest that the structure of the set $\overline{CSO}$ may admit some special form, and it needs and deserves much more study.

The aim of this paper is to provide a $C^*$-algebra approach to the norm closure problem for complex symmetric operators, which exhibits an interplay between complex symmetric operators and operator algebras. In fact, certain connections between complex symmetric operators and anti-automorphisms of singly generated $C^*$-algebras are established. Several new notions are introduced to characterize $\overline{CSO}$. Our main results apply to several special classes of operators, including completely reducible operators, irreducible operators, weighted shifts and essentially normal operators. In particular, we give a positive answer to Garcia and Poore’s conjecture. These results generalize and update some recent results on complex symmetric operators [16, 22, 47, 48].

1.2. Anti-automorphisms and g-normal operators. The proofs of our main results depend heavily on connections between complex symmetric operators and anti-automorphisms of singly generated $C^*$-algebras. Now let us show some $C^*$-algebra information contained in the notion of complex symmetry.

Let $T \in B(H)$ and $C$ be a conjugation on $H$ satisfying $CTC = T^*$. If $p(x, y)$ is a polynomial in two free variables, then it is easy to verify that $\tilde{p}(T, T^*) = Cp(T^*, T)C$, where $\tilde{p}(x, y)$ is obtained from $p(x, y)$ by conjugating each coefficient. Since $C$ is isometric, it follows that

$$\|p(T^*, T)\| = \|\tilde{p}(T, T^*)\|.$$  

This was first observed by Garcia, Lutz and Timotin [15, Question 1]. They asked whether the converse holds; that is, if $T \in B(H)$ satisfies (1.1) for every polynomial $p(\cdot, \cdot)$ in two free variables, does it follow that $T$ is complex symmetric?

Definition 1.5. For convenience, we say that an operator $T$ is g-normal if it satisfies

$$\|p(T^*, T)\| = \|\tilde{p}(T, T^*)\|$$

for any polynomial $p(\cdot, \cdot)$ in two free variables. So, by the above discussion, each complex symmetric operator is g-normal. In particular, each normal operator is g-normal.

It is easy to see that each norm limit of g-normal operators is still g-normal. So each operator in $\overline{CSO}$ is g-normal. By [48, Theorem 0.5], the Kakutani shift $W$ satisfies $W \in \overline{CSO \setminus CSO}$. This shows that $CSO \subseteq \overline{CSO} \subseteq \{\text{g-normal operators on } H\}$, which gives a negative answer to the question of Garcia, Lutz and Timotin.

In view of the above discussion, the following is perhaps an appropriate revision of the question of Garcia, Lutz and Timotin.

Question 1.6. If $T \in B(H)$ is g-normal, does it follow that $T \in \overline{CSO}$?
As we shall see later, although the answer to Question 1.6 is in general negative, g-normality of operators is closely related to complex symmetry. In fact, we shall prove that if \( T \in \mathcal{B}(\mathcal{H}) \) and \( C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\} \), then \( T \in \mathcal{CSO} \) if and only if \( T \) is g-normal (Theorem 2.1). In particular, it follows that if \( A \in \mathcal{B}(\mathcal{H}) \), then \( A^{(\infty)} \) is a norm limit of complex symmetric operators if and only if \( A \) is g-normal.

This paper depends heavily on the observation that the g-normality of an operator \( T \) implies the existence of anti-automorphisms on \( C^*(T) \). Recall that an anti-automorphism \( \rho \) of a \( C^\ast \)-algebra is said to be involutory if \( \rho \) is an isometric and densely defined. Hence \( \rho \) can be extended to a map on \( C^*(T) \), which is also denoted by \( \rho \). One can check that \( \rho \) is a conjugate-linear map, \( \rho(X^*) = \rho(X)^* \) and \( \rho(XY) = \rho(X)\rho(Y) \) for \( X, Y \in C^*(T) \). So, if we define \( \varphi(X) = \rho(X)^* \) for \( X \in C^*(T) \), then \( \varphi \) is an anti-automorphism of \( C^*(T) \) and \( \varphi(T) = T \).

The following lemma shows that there must exist involutory anti-automorphisms on the \( C^\ast \)-algebra generated by a g-normal operator. Hence each \( C^\ast \)-algebra generated by an operator in \( \mathcal{CSO} \) possesses a real structure. This makes it possible for us to use \( C^\ast \)-algebra methods to study complex symmetric operators and their norm closure problem.

**Lemma 1.7.** An operator \( T \in \mathcal{B}(\mathcal{H}) \) is g-normal if and only if there exists an anti-automorphism \( \varphi \) of \( C^*(T) \) such that \( \varphi(T) = T \).

**Proof.** “\( \Rightarrow \)” Assume that \( T \) is g-normal. Then the map
\[
\begin{align*}
\rho : C^*(T) &\rightarrow C^*(T), \\
p(T^*, T) &\mapsto \overline{p}(T, T^*)
\end{align*}
\]
is isometric and densely defined. Hence \( \rho \) can be extended to a map on \( C^*(T) \), which is also denoted by \( \rho \). One can check that \( \rho \) is a conjugate-linear map, \( \rho(X^*) = \rho(X)^* \) and \( \rho(XY) = \rho(X)\rho(Y) \) for \( X, Y \in C^*(T) \). So, if we define \( \varphi(X) = \rho(X)^* \) for \( X \in C^*(T) \), then \( \varphi \) is an anti-automorphism of \( C^*(T) \) and \( \varphi(T) = T \).

“\( \Leftarrow \)” Let \( \varphi \) be an anti-automorphism of \( C^*(T) \) satisfying \( \varphi(T) = T \). Then \( \varphi(T^*) = T^* \) and, given a polynomial \( p(\cdot, \cdot) \) in two free variables, one can see
\[
\varphi(p(T^*, T)) = \overline{p}(T, T^*)^*.
\]
Since each anti-automorphism of \( C^*(T) \) is isometric, it follows that \( \|p(T^*, T)\| = \|\overline{p}(T, T^*)\| \). So \( T \) is g-normal. \( \square \)

One of our main results characterizes when an essentially normal operator is g-normal; in fact, a decomposition theorem is given (Theorem 5.1). Recall that an operator \( A \in \mathcal{B}(\mathcal{H}) \) is said to be essentially normal if \( A^*A - AA^* \in \mathcal{K}(\mathcal{H}) \). For an irreducible operator \( T \), we shall prove in Section 3 (Theorem 3.23) that \( T \) is g-normal if and only if \( T \) is an AUET operator (the notion will be introduced in the next subsection).
1.3. UET operators and AUET operators. Besides $g$-normal operators, two other important classes of operators, namely UET operators and AUET operators, are closely related to our results. To give the definitions, we need to define transposes of Hilbert space operators.

**Definition 1.8.** Let $T \in \mathcal{B}(\mathcal{H})$. An operator $A \in \mathcal{B}(\mathcal{H})$ is called a transpose of $T$ if $A = CT^*C$ for some conjugation $C$ on $\mathcal{H}$.

The notion “transpose” for operators is in fact a generalization of that for matrices. Let $T \in \mathcal{B}(\mathcal{H})$. Assume that $C$ is a conjugation on $\mathcal{H}$. Then there exists an orthonormal basis (ONB, for short) $\{e_n\}$ of $\mathcal{H}$ such that $Ce_n = e_n$ for all $n$ (see [15, Lemma 1]). Thus $T$ has a matrix representation $[a_{i,j}]$ with respect to $\{e_n\}$, where $a_{i,j} = \langle Te_j, e_i \rangle$. Set $A = CT^*C$. Note that

$$\langle Ae_i, e_j \rangle = \langle CT^*Ce_i, e_j \rangle = \langle CT^*e_i, e_j \rangle = \langle Ce_j, T^*e_i \rangle = \langle e_j, T^*e_i \rangle = \langle Te_j, e_i \rangle.$$ 

Thus the matrix representation of $A$ with respect to $\{e_n\}$ is just the transpose of the matrix $[a_{i,j}]$. So, given an operator $T$, a transpose of $T$ is obtained from $T$ by transposing the matrix representation of $T$ with respect to some ONB.

By the above discussion, an operator may have more than one transpose. In fact, any two transposes of an operator are unitarily equivalent. Assume that $A, B, T \in \mathcal{B}(\mathcal{H})$ and $A, B$ are two transposes of $T$. Then there are two conjugations $C$ and $D$ on $\mathcal{H}$ such that $A = CT^*C$ and $B = DT^*D$. Set $U = CD$. Then it is easy to see that $U \in \mathcal{B}(\mathcal{H})$ is unitary and $AU = (CT^*C)(CD) = CT^*D = (CD)(DT^*D) = UB$; that is, $A, B$ are unitarily equivalent.

We often write $T^t$ to denote a transpose of $T$. In general, there is no ambiguity especially when we write $T \cong T^t$ or $T \cong_a T^t$. Here and in what follows, the notation $\cong$ denotes unitary equivalence, and $\cong_a$ denotes approximate unitary equivalence. As usual, given two representations $\rho_1$ and $\rho_2$ of a $C^*$-algebra, we also write $\rho_1 \cong \rho_2$ ($\rho_1 \cong_a \rho_2$) to denote that $\rho_1$ and $\rho_2$ are unitarily equivalent (approximately unitarily equivalent, respectively).

**Definition 1.9.** An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be UET if $T \cong T^t$, and $T$ is said to be AUET if $T \cong_a T^t$.

By definitions, each complex symmetric operator is UET. But the converse does not hold (see Example 3.16).

The notion of UET operators has its motivations in linear algebra. In his problem book [27, Proposition 159], Halmos asked when a matrix is unitarily equivalent to its transpose (UET). There are matrices that are not UET (see [22]). Recently, Garcia and Tener [22, Theorem 1.2] gave a canonical decomposition for UET matrices. As an application, they gave a canonical decomposition for complex symmetric operators on finite dimensional Hilbert spaces.

In Section 5, we give a characterization for an essentially normal operator to be UET (Proposition 5.7); in particular, one of our main results gives a canonical decomposition for essentially normal operators which are UET (Theorem 6.1). Also we give a canonical decomposition for essentially normal operators which are complex symmetric (Theorem 2.8). The notion of AUET operators is useful for us to characterize $\text{CSO}$. In fact, when $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$, we shall prove that $T \in \overline{\text{CSO}}$ if and only if $T$ is AUET (Theorem 2.1).

**Remark 1.10.** So far we have introduced some new classes of operators including UET operators, AUET operators and $g$-normal operators. All these operators are
closely related to complex symmetric operators. In fact, in many special cases, they can be used to characterize norm limits of complex symmetric operators. As we shall see later (Corollary 3.11 and Lemma 3.12), the inclusion relations among them can be summarized as follows:

\[ \text{CSO} \subset \left[ \text{CSO} \cup \{ \text{UET operators} \} \right] \subset \{ \text{AUET operators} \} \subset \{ \text{g-normal operators} \}. \]

Moreover, each inclusion relation above is proper (see Examples 3.13, 3.16 and 3.17).

2. Main results

In this section, we shall list the main results of this paper.

The first main result of this paper focuses on those operators \( T \in B(\mathcal{H}) \) satisfying \( C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}. \)

**Theorem 2.1.** Let \( T \in B(\mathcal{H}) \) and assume that \( C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\} \). Then the following are equivalent:

- (i) \( T \in \text{CSO} \);
- (ii) \( T \in \text{CSO}^c \);
- (iii) \( \exists A \in \text{CSO} \) such that \( A \sim_a T \);
- (iv) \( T \sim_a T^t \);
- (v) \( T \) is g-normal.

**Remark 2.2.**

(i) Theorem 2.1 applies to many special classes of operators such as completely reducible operators, many weighted shifts and irreducible operators. Recall that an operator is said to be completely reducible if its lattice of reducing subspaces has no nonzero minimal elements ([13]).

(ii) Using the unilateral shift, Garcia and Poore [17] constructed an operator \( T \in \text{CSO} \setminus \text{CSO}^c \). One can check that their operator \( T \) is g-normal and \( C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\} \). Then it follows from Theorem 2.1 that \( T \in \text{CSO}^c \).

(iii) In Theorem 2.1 when \( C^*(T) \cap \mathcal{K}(\mathcal{H}) \neq \{0\} \), it is possible that neither (iv) nor (v) is equivalent to (i) (see Examples 3.16 and 3.17). But we do not know whether (ii) or (iii) is equivalent to (i) for general \( T \in B(\mathcal{H}) \).

It is clear that an operator \( T \) is g-normal if and only if \( T^{(\infty)} \) is g-normal. So the following corollary is immediate from Theorem 2.1.

**Corollary 2.3.** If \( T \in B(\mathcal{H}) \), then \( T^{(\infty)} \) is a norm limit of complex symmetric operators if and only if \( T \) is g-normal.

As an application of Theorem 2.1, we shall characterize when a weighted shift with nonzero weights belongs to \( \text{CSO} \). Recall that a (forward) weighted shift \( T \) on \( \mathcal{H} \) with weight sequence \( \{w_n\} \) is the operator defined by \( T e_n = w_n e_{n+1} \) for all \( n \), where \( \{e_n\} \) is an ONB of \( \mathcal{H} \). If the index \( n \) runs over positive integers, then \( T \) is called a unilateral weighted shift, while if \( n \) runs over integers, then \( T \) is called a bilateral weighted shift. According to a result of Shields [38, Corollary 1], each weighted shift is unitarily equivalent to a weighted shift with nonnegative weights. So we need only deal with weighted shifts with positive weights.

Let \( T \) be a bilateral weighted shift with positive weights \( \{w_i\}_{i \in \mathbb{Z}} \). For each \( n \geq 1 \), the \( n \)-spectrum of \( T \) (denoted by \( \Sigma_n(T) \)) is defined to be the closure (in the usual topology on \( \mathbb{R}^n \)) of the set

\[ \{(w_{i+1}, w_{i+2}, \cdots, w_{i+n}) : i \in \mathbb{Z}\}. \]
This notion was first introduced to estimate the distance between unitary orbits of invertible bilateral weighted shifts \[31\].

Given a subset \( G \) of \( \mathbb{R}^n \), we denote
\[
G^t = \{ (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{R}^n : (\alpha_n, \alpha_{n-1}, \cdots, \alpha_1) \in G \}.
\]

For a weighted shift \( T \) with positive weights, although the equality \( C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\} \) generally does not hold, we still have the following three theorems which completely characterize weighted shifts with positive weights in \( \overline{CSO} \). In particular, Theorem 2.4 answers Conjecture [1.3] in the positive.

**Theorem 2.4.** Let \( T \in B(\mathcal{H}) \) be a unilateral weighted shift with positive weights. Then the following are equivalent:

(i) \( T \in \mathcal{CSO} \);
(ii) \( T \in \overline{CSO} \);
(iii) \( \exists A \in \mathcal{CSO} \) such that \( A \cong_a T \);
(iv) \( T \cong_a T^* \);
(v) \( T \) is g-normal;
(vi) \( T \) is approximately Kakutani.

**Theorem 2.5.** Let \( T \in B(\mathcal{H}) \) be a bilateral weighted shift with positive weights \( \{w_i\}_{i \in \mathbb{Z}} \). If \( T \) is reducible or \( C^*(T) \cap \mathcal{K}(\mathcal{H}) \neq \{0\} \), then the following are equivalent:

(i) \( T \in \mathcal{CSO} \);
(ii) \( T \in \overline{CSO} \);
(iii) \( T \cong T^* \);
(iv) \( T \) is g-normal;
(v) \( \exists k \in \mathbb{Z} \) such that \( w_i = w_{k-i} \) for all \( i \in \mathbb{Z} \).

**Theorem 2.6.** Let \( T \in B(\mathcal{H}) \) be a bilateral weighted shift with positive weights. Then the following are equivalent:

(i) \( T \in \mathcal{CSO} \);
(ii) \( T \in \overline{CSO} \);
(iii) \( \exists A \in \mathcal{CSO} \) such that \( A \cong_a T \);
(iv) \( T \cong_a T^* \);
(v) \( T \) is g-normal;
(vi) \( \Sigma_n(T)^t = \Sigma_n(T) \) for all \( n \geq 1 \).

**Example 2.7.** Let \( G \) be the set of all rational numbers in \((0,1]\). Since \( G \) is denumerable, one can construct a bilateral weighted shift \( T \) with positive weights such that \( \Sigma_n(T) = [0,1]^n \) for all \( n \geq 1 \). Thus \( \Sigma_n(T)^t = \Sigma_n(T) \) for all \( n \geq 1 \). By Theorem 2.6, it follows that \( T \in \overline{CSO} \).

In general, g-normality, UET property and complex symmetry are quite different. To see the difference, we characterize in Sections 5 and 6 when an essentially normal operator is g-normal or UET. The following theorem gives a canonical decomposition for essentially normal operators in \( \overline{CSO} \).

**Theorem 2.8.** Let \( T \in B(\mathcal{H}) \) be essentially normal. Then the following are equivalent:

(i) \( T \in \mathcal{CSO} \);
(ii) \( T \in \overline{CSO} \);
(iii) $T$ is unitarily equivalent to a direct sum of (some of the summands may be absent): normal operators, irreducible complex symmetric operators and operators with form of $A \oplus A^t$, where $A$ is irreducible and not complex symmetric.

A fundamental question about complex symmetric operators is how to develop a model theory [14]. A natural thought is to decompose complex symmetric operators into “simple blocks” and then represent them in concrete terms. Some known results suggest that truncated Toeplitz operators may play the role of “simple blocks” [20].

Let $T \in \mathcal{B}(\mathcal{H})$ be complex symmetric and $M$ be a nontrivial reducing subspace of $T$. It is known that each normal operator is complex symmetric [18]. If $T$ is normal, then $T|_M$ must be complex symmetric; if $T$ is not normal, it is possible that $T|_M$ is not complex symmetric (see [22]). This motivates the following definition:

**Definition 2.9.** Let $T \in \mathcal{B}(\mathcal{H})$ be complex symmetric. $T$ is said to be **completely complex symmetric** if $T$ is reducible and $T|_M$ is complex symmetric for any nontrivial reducing subspace $M$ of $T$; $T$ is called a **minimal complex symmetric operator** if there exists no nontrivial reducing subspace $M$ of $T$ such that $T|_M$ is complex symmetric.

Thus each normal operator on Hilbert spaces of dimension greater than 1 is completely complex symmetric. Note that each operator on a Hilbert space of dimension 1 is normal, irreducible and hence a minimal complex symmetric operator. Thus each normal operator is either completely complex symmetric or a minimal complex symmetric operator. On the other hand, if $A$ is irreducible and not complex symmetric, we shall prove later that $A \oplus A^t$ is a minimal complex symmetric operator (Proposition 7.6). So Theorem 2.8 shows that if an essentially normal operator $T$ is complex symmetric, then $T$ can be written as a direct sum of completely complex symmetric operators and minimal complex symmetric operators. In Section 4 we shall show some completely complex symmetric operators which are nonnormal (Proposition 4.14).

**Remark 2.10.** The equivalence of (i) and (ii) in Theorem 2.8 strengthens [16, Theorem 4], which deals with compact operators. The equivalence of (ii) and (iii) in Theorem 2.8 is a generalization of [22, Proposition 3.6], which deals with complex symmetric operators on finite dimensional Hilbert spaces.

The rest of this paper is organized as follows. In Section 3, we shall give some concrete examples and the proof of Theorem 2.1 in addition, we shall also discuss anti-automorphisms of singly generated $C^*$-algebras. Section 4 is devoted to characterizing which weighted shifts belong to $CSO$; the proofs of Theorems 2.4, 2.5 and 2.6 are provided in this section. In Sections 5 and 6 we shall characterize respectively which essentially normal operators are g-normal and which essentially normal operators are UET. Using results in Sections 5 and 6 we shall prove Theorem 2.8 in the last section.

### 3. Proof of Theorem 2.1

Given $A, B \in \mathcal{B}(\mathcal{H})$, we denote $[A, B] = AB - BA$.

Let $T \in \mathcal{B}(\mathcal{H})$. Denote $M = \bigcap_{m,n \geq 1} \ker[T^{*m}, T^n]$. Then $M$ and $M^\perp$ both reduce $T$. In fact, $T|_M$ is normal and $T|_{M^\perp}$ is abnormal ([28, page 116]).
such that $A|_{\mathcal{N}}$ is normal. We call $T|_{\mathcal{M}}$ the normal part of $T$ and $T|_{\mathcal{M}^\perp}$ the abnormal part of $T$, denoted by $T_{\text{nor}}$ and $T_{\text{abnor}}$ respectively.

**Lemma 3.1** ([18], page 1295). Each normal operator is complex symmetric.

**Lemma 3.2.** An operator $T \in \mathcal{B}(\mathcal{H})$ is complex symmetric if and only if $T_{\text{abnor}}$ is complex symmetric.

**Proof.** By Lemma 3.1 the sufficiency is clear. We need only prove the necessity.

Denote $M = \bigcap_{m,n \geq 1} \ker[T^*m, T^n]$. Assume that $C$ is a conjugation on $\mathcal{H}$ and $CT^*C = T$. Thus, for any $m, n \geq 1$, we have $C[T^*m, T^n]C = -[T^*n, T^m]$. Hence we deduce that $C(\ker[T^*n, T^m]) = \ker[T^*m, T^n]$. Since $m, n \geq 1$ are arbitrary, we obtain $C(M) = M$. Noting that $C$ is a conjugation, we deduce that $C(M^\perp) = M^\perp$ and $D = C|_{\mathcal{M}^\perp}$ is a conjugation. It follows from $CT^*C = T$ that $DT_{\text{abnor}}D = T_{\text{abnor}}$. This completes the proof.

**Proposition 3.3.** Let $T \in \mathcal{B}(\mathcal{H})$ be hyponormal. Then the following are equivalent:

(i) $T \in \overline{CSO}$;
(ii) $T$ is g-normal;
(iii) $T$ is normal;
(iv) $T \in CSO$.

**Proof.** Note that each normal operator is complex symmetric. By definition, the implications “(iii)⇒(iv)⇒(i)⇒(ii)" are obvious.

“(ii)⇒(iii)”. Since $T$ is g-normal, by Lemma 1.7 the map $\varphi$ defined as

$$\varphi : C^*(T) \rightarrow C^*(T),$$

$$p(T^*, T) \mapsto \tilde{p}(T, T^*)^*$$

is an anti-automorphism of $C^*(T)$ and $\varphi(T) = T$. Thus the map $\varphi$ preserves the *-operation and preserves the spectra of operators. Hence an operator $X \in C^*(T)$ is positive if and only if $\varphi(X)$ is positive. Set $A = [T^*, T]$. Since $T$ is hyponormal, it follows that $A \geq 0$; furthermore, $-A = \varphi(A) \geq 0$. So $A = 0$ and $T$ is normal.

**Lemma 3.4.** Let $T \in \mathcal{B}(\mathcal{H})$ and assume that $T = A^{(m)} \oplus B^{(n)}$, where $1 \leq m, n \leq \infty$. Then $T$ is g-normal if and only if $A \oplus B$ is g-normal.

**Proof.** Set $R = A \oplus B$. Let $p(\cdot, \cdot)$ be a polynomial in two free variables. It is easy to check that

$$\|p(T^*, T)\| = \max\{|\|p(A^*, A)\|, \|p(B^*, B)\|\} = \|p(R^*, R)\|$$

and

$$\|\tilde{p}(T, T^*)\| = \max\{|\|\tilde{p}(A, A^*)\|, \|\tilde{p}(B, B^*)\|\} = \|\tilde{p}(R, R^*)\|.$$
Then it is easy to see that Proposition 3.10. Let \( \rho \) and \( \mathcal{H} \) be nondegenerate representations of \( \mathcal{A} \) on separable Hilbert spaces. Then the following are equivalent:

\[
\rho \in \mathcal{B}(\mathcal{H}) \quad \text{if and only if there exists an anti-unitary operator } D \text{ on } \mathcal{H} \text{ such that } DT = T^* D.
\]

Proof. “\( \Rightarrow \)”. If \( T \) is UET, then there exist unitary \( U \in \mathcal{B}(\mathcal{H}) \) and a conjugation \( C \) on \( \mathcal{H} \) such that \( U^* TU = CT^* C \). Set \( D = CU^* \). Then \( D \) is an anti-unitary operator on \( \mathcal{H} \) and \( DT = T^* D \).

“\( \Leftarrow \)”. Arbitrarily choose a conjugation \( C \) on \( \mathcal{H} \). If \( D \) is an anti-unitary operator on \( \mathcal{H} \) such that \( DT = T^* D \), then \( (CD)T = (CT^* C)(CD) \). Set \( U = CD \). Then \( U \in \mathcal{B}(\mathcal{H}) \) is unitary. It follows that \( T \cong CT^* C \) \( \square \).

Lemma 3.9 (\cite{10}, Theorem II.5.8). Let \( \mathcal{A} \) be a separable \( C^* \)-algebra, and let \( \rho_1 \) and \( \rho_2 \) be nondegenerate representations of \( \mathcal{A} \) on separable Hilbert spaces. Then the following are equivalent:

(i) \( \rho_1 \cong a \rho_2 \),

(ii) \( \text{rank } \rho_1(X) = \text{rank } \rho_2(X) \) for all \( X \in \mathcal{A} \).

Proposition 3.10. Let \( T \in \mathcal{B}(\mathcal{H}) \). Then the following are equivalent:

(i) \( T \) is AUET.

(ii) There exists a sequence \( \{D_n\} \) of anti-unitary operators on \( \mathcal{H} \) such that \( \lim_n \|D_n T - T^* D_n\| = 0 \).

(iii) There exists an anti-automorphism \( \varphi \) of \( C^*(T) \) such that \( \varphi(T) = T \) and \( \text{rank } \varphi(X) = \text{rank } X \) for all \( X \in C^*(T) \).

Proof. “\( \text{(i)} \Rightarrow \text{(ii)} \)”. Since \( T \) is AUET, we can find a sequence \( \{U_n\} \) of unitary operators and a conjugation \( C \) on \( \mathcal{H} \) such that \( U_n^* TU_n \rightarrow CT^* C \). Hence \( D_n := CU_n^* \) is an anti-unitary operator on \( \mathcal{H} \) for \( n \geq 1 \) and \( D_n = U_n C \). One can check that \( \lim_n \|D_n T - T^* D_n\| = 0 \).
“(ii)⇒(iii)”. By the hypothesis, we have \( \lim_n D_n T^* D_n^{-1} = T \) and \( \lim_n D_n T D_n^{-1} = T^* \). Then, given a polynomial \( p(\cdot, \cdot) \) in two free variables, it can be verified that

\[
\bar{p}(T, T^*) = \lim_n D_n p(T^*, T) D_n^{-1}.
\]

Since each \( D_n \) is isometric, \( T \) is g-normal. By Lemma 1.7, the map \( \varphi \) defined by

\[
\varphi : C^*(T) \to C^*(T), \quad p(T^*, T) \mapsto \bar{p}(T, T^*)
\]

is an anti-automorphism of \( C^*(T) \). Moreover, \( \varphi \) is involutory and \( \varphi(T) = T \). In view of (3.1), we deduce that \( \varphi(X) = \lim_n D_n X^* D_n^{-1} \) for \( X \in C^*(T) \). By the lower semi-continuity of the rank in approximation (see [28, Proposition 1.12]), it follows that \( \text{rank } X = \liminf_n \text{rank } D_n X^* D_n^{-1} \geq \text{rank } \varphi(X) \).

Moreover, we have \( \text{rank } \varphi(X) \geq \text{rank } \varphi^2(X) \). Since \( \varphi \) is involutory, it follows that \( \text{rank } \varphi(X) = \text{rank } X \) for all \( X \in C^*(T) \).

“(iii)⇒(i)”. Let \( C \) be a conjugation on \( \mathcal{H} \). For each \( X \in C^*(T) \), define \( \rho(X) = C \varphi(X)^* C \). It is easily seen that \( \rho \) is a faithful representation of \( C^*(T) \) on \( \mathcal{H} \) and

\[
\text{rank } \rho(X) = \text{rank } \varphi(X) = \text{rank } X = \text{rank } \text{Id}(X)
\]

for all \( X \in C^*(T) \), where \( \text{Id}(\cdot) \) is the identity representation on \( \mathcal{H} \). Noting that \( \text{Id}(\cdot) \) and \( \rho \) are both nondegenerate, it follows from Lemma 3.9 that \( \rho \cong \text{Id} \). Furthermore we obtain \( T \cong CT^*C \), that is, \( T \) is AUET. □

By Lemma 1.7, the following corollary is immediate from Proposition 3.10.

**Corollary 3.11.** If \( T \in \mathcal{B}(\mathcal{H}) \) is AUET, then \( T \) is g-normal.

**Lemma 3.12.** If \( T \in \overline{\text{CSO}} \), then \( T \) is AUET.

**Proof.** Since \( T \in \overline{\text{CSO}} \), there exists a sequence \( \{T_n\} \) of complex symmetric operators such that \( T_n \to T \). For each \( n \), since \( T_n \in \text{CSO} \), we can choose a conjugation \( C_n \) on \( \mathcal{H} \) such that \( C_n T_n C_n = T_n^* \). One can verify that \( C_n T C_n \to T^* \). In view of Proposition 3.10, this implies that \( T \) is AUET. □

**Example 3.13.** The Kakutani shift is a unilateral weighted shift with weight sequence \( \{w_n\}_{n=1}^\infty \), where

\[
w_n = \frac{1}{\gcd\{n, 2n\}}, \quad n \geq 1.
\]

Here \( \gcd\{i, j\} \) denotes the greatest common divisor of \( i \) and \( j \).

Denote by \( W \) the Kakutani shift. By [18, Theorem 0.5], \( W \) is a norm limit of complex symmetric operators and hence it is AUET. However, note that \( \dim \ker W = 0 < 1 = \dim \ker W^* = \dim \ker DW^* D^{-1} \) for any anti-unitary operator \( D \). Thus, by Lemma 3.8, \( W \) is not UET. This example shows that

\[
\overline{\text{CSO}} \not\subseteq \{ \text{UET operators} \} \subset \{ \text{AUET operators} \}.
\]
Theorem 3.14. Let $S \in \mathcal{B}(\mathcal{H})$ be the unilateral shift defined by $Se_i = e_{i+1}$ for $i \geq 1$, where $\{e_i\}_{i=1}^{\infty}$ is an ONB of $\mathcal{H}$. Assume that $T \in \mathcal{B}(\mathcal{H})$ and define

$$R_T = \begin{bmatrix} S^* & T \\ 0 & S \end{bmatrix} \mathcal{H}$$

Then

(i) $R_T$ is complex symmetric if and only if $\langle Te_i, e_j \rangle = \langle Te_j, e_i \rangle$ for all $i, j \geq 1$;

(ii) $R_T$ is UET if and only if there exists $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $
\lambda\langle Te_i, e_j \rangle = \langle Te_j, e_i \rangle$ for all $i, j \geq 1$.

Proof. (i) $\iff$. For each $x \in \mathcal{H}$ with $x = \sum_{i=1}^{\infty} \alpha_i e_i$, define $Cx = \sum_{i=1}^{\infty} \overline{\alpha_i} e_i$. Then $C$ is a conjugation on $\mathcal{H}$. One can verify that $CS = SC$ and $CS^* = S^*C$. Since $\langle Te_i, e_j \rangle = \langle Te_j, e_i \rangle$ for all $i, j$, we also have $CTC = T^*$. Define

$$D = \begin{bmatrix} 0 & C \\ C^* & 0 \end{bmatrix} \mathcal{H}$$

Then $D$ is a conjugation on $\mathcal{H} \oplus \mathcal{H}$ and one can check that $DR_TD = R_T^*$. $\iff$. Assume that $C$ is a conjugation on $\mathcal{H} \oplus \mathcal{H}$ and $CR_TC = R_T^*$. For convenience, we write

$$R_T = \begin{bmatrix} S^* & T \\ 0 & S \end{bmatrix} \mathcal{H}_1$$

where $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$. Note that $CR_T^nC = (R_T^n)^n$ for all $n \geq 1$. So $C(\ker R_T^n) = \ker (R_T^n)$ for all $n \geq 1$. It follows that $C(\bigvee_n \ker R_T^n) = \bigvee_n \ker (R_T^n)$, that is, $C(\mathcal{H}_1) = \mathcal{H}_2$. Since $C^{-1} = C$, we have $C(\mathcal{H}_2) = \mathcal{H}_1$. Hence $C$ admits the following matrix representation:

$$C = \begin{bmatrix} 0 & E \\ D & 0 \end{bmatrix}$$

Also one can see that $D$ is an anti-unitary operator on $\mathcal{H}$ and $E = D^{-1}$. Since $CR_TC = R_T^*$, a straightforward computation shows that $DS = SD$, $DS^* = S^*D$ and $DTD = T^*$. For $n \geq 1$, $DS^n = S^nD$ implies that $D(S^n) = (S^n)^nD$. Since $D$ is invertible and $\ker (S^n) = \bigvee \{e_i : 1 \leq i \leq n\}$, we deduce that $D(\ker (S^n)) = \ker (S^n)$. Since $D$ is isometric, there exists a sequence $\{\lambda_i\}$ of complex numbers with $|\lambda_i| = 1$ such that $De_i = \lambda_i e_i$ for all $i$.

Now fix an $i \geq 1$. Hence

$$\lambda_{i+1} e_{i+1} = De_{i+1} = DSe_i = SDe_i = \lambda_i e_i = \lambda_i e_{i+1}.$$ 

So we have $\lambda_i = \lambda_{i+1}$. Thus the sequence $\{\lambda_i\}$ is constant.

On the other hand, for given $i, j \geq 1$, one can verify that

$$\langle T^* e_i, e_j \rangle = \langle DTDe_i, e_j \rangle = \langle D^{-1} e_j, TDe_i \rangle = \langle T^* D^{-1} e_j, De_i \rangle = \langle T^* e_j, e_i \rangle,$$

that is, $\langle Te_i, e_j \rangle = \langle Te_j, e_i \rangle$. This completes the proof.

(ii) For each $x \in \mathcal{H}$ with $x = \sum_{i=1}^{\infty} \alpha_i e_i$, define $Cx = \sum_{i=1}^{\infty} \overline{\alpha_i} e_i$. This defines a conjugation on $\mathcal{H}$. It is easy to verify that $CS = SC$ and $CS^* = S^*C$. Define

$$D = \begin{bmatrix} 0 & C \\ C^* & 0 \end{bmatrix} \mathcal{H}$$

Thus $D$ is a conjugation on $\mathcal{H} \oplus \mathcal{H}$. 
“⇐”. For $i, j \geq 1$, we have

$$\langle CT^*Ce_i, e_j \rangle = \langle Ce_j, T^*Ce_i \rangle = \langle e_j, T^*e_i \rangle = \langle Te_j, e_i \rangle = \lambda \langle e_i, e_j \rangle.$$  

It follows that $CT^*C = \lambda T$.

Define

$$U = \begin{bmatrix} \lambda I & 0 \\ 0 & I \end{bmatrix} \mathcal{H}$$

where $I$ is the identity operator on $\mathcal{H}$. So $U \in B(\mathcal{H}(2))$ is unitary and

$$DR_TD = \begin{bmatrix} CSC & 0 \\ CTC & CS^*C \end{bmatrix} = \begin{bmatrix} S & 0 \\ \lambda T^* & S^* \end{bmatrix}^* = (UR_TU^*)^* = UR_T^*U^*;$$

that is, $DR_TD = UR_T^*U^*$. Hence $R_T$ is UET.

“⇒”. Since $D$ is a conjugation on $\mathcal{H} \oplus \mathcal{H}$, the operator

$$A := DR_T^*D = \begin{bmatrix} CS^*C & CT^*C \\ 0 & CSC \end{bmatrix} = \begin{bmatrix} S^* & CT^*C \\ 0 & S \end{bmatrix}$$

is a transpose of $R_T$. For convenience, we write

$$A = \begin{bmatrix} S^* & CT^*C \\ 0 & S \end{bmatrix} \mathcal{H}_1 \oplus \mathcal{H}_2 \text{ and } R_T = \begin{bmatrix} S^* & T \\ 0 & S \end{bmatrix} \mathcal{H}_1 \oplus \mathcal{H}_2,$$

where $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$. Since $R_T$ is UET, we can choose a unitary operator $U$ on $\mathcal{H}_1 \oplus \mathcal{H}_2$ such that $UR_T = AU$. Thus for each $n \geq 1$ we have $UR_T^n = A^nU$ and $U(\ker R_T^n) = \ker A^n$. Furthermore we have $U(\bigvee_n \ker R_T^n) = \bigvee_n \ker A^n$. Since $\bigvee_n \ker R_T^n = \bigvee_n \ker A^n = \mathcal{H}_1$, it follows that $U(\mathcal{H}_1) = \mathcal{H}_1$.

On the other hand, since $R_T^*U^* = U^*A^*$, using a similar argument as above, we can prove that $U^*(\bigvee_n \ker(A^*n)) = \bigvee_n \ker(R_T^*)^n$. Noting that $\bigvee_n \ker(R_T^*)^n = \bigvee_n \ker(A^n)^* = \mathcal{H}_2$, we obtain $U^*(\mathcal{H}_2) = \mathcal{H}_2$, that is, $U(\mathcal{H}_2) = \mathcal{H}_2$. Then $U$ admits the matrix representation

$$U = \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} \mathcal{H}_1 \oplus \mathcal{H}_2.$$

Since $UR_T = AU$, it follows that $U_1S^* = S^*U_1$, $U_2S = SU_2$ and $U_1T = CT^*CU_2$. Noting that $S$ is irreducible and each $U_i$ is unitary, there exist $\lambda_1, \lambda_2 \in \mathbb{C}$ with $|\lambda_1| = |\lambda_2| = 1$ such that $U_1 = \lambda_1 I$ and $U_2 = \lambda_2 I$. Thus we conclude that $CT^*C = \lambda T$, where $\lambda = \lambda_1/\lambda_2$. One can check that $\lambda \langle e_i, e_j \rangle = \langle Te_j, e_i \rangle$ for all $i, j \geq 1$. This completes the proof. \qed

Remark 3.15. (i) In Theorem 3.14 when $S$ is replaced by any unilateral weighted shift with positive weights, the results remain true.

(ii) In Theorem 3.14 if $T \neq 0$, then $R_T$ is irreducible. This shows that there exist irreducible operators which are UET but not complex symmetric.

Example 3.16. Let $A$ be the unilateral weighted shift defined as $Ae_i = \frac{1}{i}e_{i+1}$ for $i \geq 1$, where $\{e_i\}_{i=1}^\infty$ is an ONB of $\mathcal{H}$. Define a finite-rank operator $F$ on $\mathcal{H}$ as

$$Fe_1 = -e_2, Fe_2 = e_1 \text{ and } Fe_i = 0 \text{ for } i \geq 3.$$  

Then the operator $K$ on $\mathcal{H} \oplus \mathcal{H}$ given by

$$K = \begin{bmatrix} A^* & F \\ 0 & A \end{bmatrix} \mathcal{H} \oplus \mathcal{H}$$
is compact. By Theorem 3.14 and Remark 3.15 (i), \( K \) is UET but not complex symmetric. Then, by [16, Theorem 4], \( K \) is not a norm limit of complex symmetric operators. This combined with Lemma 3.12 shows that

\[
CSO \subseteq \{ \text{UET operators} \} \not\subseteq CO \subseteq \{ \text{AUET operators} \}.
\]

Now, using \( K \), we shall construct an AUET operator which is neither UET nor a norm limit of complex symmetric operators.

Let \( W \in \mathcal{B}(\mathcal{H}) \) be the Kakutani shift defined in Example 3.13. Define

\[
T = K \oplus (2I + W),
\]

where \( I \) is the identity operator on \( \mathcal{H} \). Thus \( K' \oplus (2I + W') \) is a transpose of \( T \).

We claim that \( T \) is not UET. In fact, if not, then \( K \oplus (2I + W) \cong K' \oplus (2I + W') \). Note that \( \sigma(2I + W') = \sigma(2I + W), \sigma(K) = \sigma(K') \) and \( \sigma(2I + W) \cap \sigma(K) = \emptyset \). By Rosenblum’s Theorem [28, Corollary 3.20], it follows that \( K \cong K' \) and \( 2I + W \cong 2I + W' \); in particular, \( W \cong W' \). By Example 3.13, \( W \) is AUET but not UET, a contradiction.

In view of Theorem 3.14 and Remark 3.15 (i), \( K \) is UET and hence AUET. Thus \( T \) is AUET.

Now it remains to prove that \( T \) is not a norm limit of complex symmetric operators. For a proof by contradiction, we assume that \( T \) is a norm limit of complex symmetric operators. Then there exists a sequence \( \{ C_n \} \) of conjugations on \( \mathcal{H}^{(2)} \) such that \( C_nT^*C_n - T \to 0 \) as \( n \) tends to \( \infty \). For each \( n \), we assume that

\[
C_n = \begin{bmatrix} C_{1,1}^n & C_{1,2}^n & C_{1,3}^n \\ C_{2,1}^n & C_{2,2}^n & C_{2,3}^n \\ C_{3,1}^n & C_{3,2}^n & C_{3,3}^n \end{bmatrix} \mathcal{H}
\]

Since \( C_nT^*C_n - T \to 0 \), we have

\[
K^* \left[ \begin{bmatrix} C_{1,1}^n \\ C_{2,1}^n \end{bmatrix} \right] - \left[ \begin{bmatrix} C_{1,1}^n \\ C_{2,1}^n \end{bmatrix} \right] (2I + W) \to 0.
\]

Arbitrarily choose a conjugation \( E \) on \( \mathcal{H}^{(2)} \). Then

\[
(EK^*E) \left( E \left[ \begin{bmatrix} C_{1,1}^n \\ C_{2,1}^n \end{bmatrix} \right] \right) - \left( E \left[ \begin{bmatrix} C_{1,1}^n \\ C_{2,1}^n \end{bmatrix} \right] \right) (2I + W) \to 0.
\]

Since \( \sigma(EK^*E) = \sigma(K) \) and \( \sigma(K) \cap \sigma(2I + W) = \emptyset \), using Rosenblum’s Theorem again, one can see that

\[
\left\| E \left[ \begin{bmatrix} C_{1,1}^n \\ C_{2,1}^n \end{bmatrix} \right] \right\| \to 0.
\]

So \( \| C_{1,1}^n \| + \| C_{2,1}^n \| \to 0 \). Similarly one can prove that \( \| C_{3,1}^n \| + \| C_{3,2}^n \| \to 0 \). Thus

\[
\| C_{1,1}^n \| + \| C_{2,1}^n \| + \| C_{3,1}^n \| + \| C_{3,2}^n \| \to 0.
\]

For each \( n \geq 1 \), denote

\[
D_n = \begin{bmatrix} C_{1,1}^n & C_{1,2}^n \\ C_{2,1}^n & C_{2,2}^n \end{bmatrix} \mathcal{H}
\]

Then one can deduce that \( D_nK^*D_n \to K \) and \( \{ D_n^2 \} \) converges to the identity operator on \( \mathcal{H}^{(2)} \). So \( D_n \) is conjugate-linear and invertible provided that \( n \) is large enough.

Since \( K \) is compact and \( \overline{\text{ran}K} + \text{ran}K^* = \mathcal{H}^{(2)} \), using a similar argument as in the proof of [16, Theorem 4], one can prove that there exists a subsequence
{n_j} of \mathbb{N} such that \{D_{n_j}\} converges to a conjugation \(D\) on \(\mathcal{H}^{(2)}\). Noting that \(D_{n_j}K^*D_{n_j} - K \to 0\), we obtain \(DK^*D = K\), contradicting the fact that \(K\) is not complex symmetric. Thus we have proved that \(T\) is an AUET operator; however, \(T\) is neither UET nor a norm limit of complex symmetric operators. This shows that

\[\{\text{UET operators}\} \subset \{\text{g-normal operators}\}.\]

**Example 3.17.** Let \(S\) be the unilateral shift defined by \(Se_i = e_{i+1}\) for \(i \geq 1\), where \(\{e_i\}_{i=1}^{\infty}\) is an ONB of \(\mathcal{H}\). Define \(T = S^{(2)} \oplus S^*\). By Theorem 3.14 (i), \(S \oplus S^*\) is complex symmetric and hence g-normal. By Corollary 3.5, it follows that \(T\) is g-normal. Note that each AUET operator is biquasitriangular and \(T\) is not biquasitriangular. We deduce that \(T\) is not AUET. This example combined with Corollary 3.11 implies that

\[\{\text{AUET operators}\} \subset \{\text{g-normal operators}\}.\]

**Proposition 3.18.** Let \(A\) be a \(C^*\)-subalgebra of \(\mathcal{B}(\mathcal{H})\) satisfying \(\mathcal{K}(\mathcal{H}) \subset A\). If \(\varphi\) is an anti-automorphism of \(A\), then there exists an anti-unitary operator \(D\) on \(\mathcal{H}\) such that

\[\varphi(X) = DX^*D^{-1}, \forall X \in A.\]

**Proof.** Arbitrarily choose a conjugation \(C\) on \(\mathcal{H}\) and define \(\rho(X) = C\varphi(X)^*C\) for \(X \in C^*(T)\). Then it is easy to see that \(\rho\) is a faithful representation of \(A\) on \(\mathcal{H}\). Since \(\mathcal{K}(\mathcal{H}) \subset A\), we have

\[\mathcal{K}(\mathcal{H}) = C\mathcal{K}(\mathcal{H})C \subset CAC = \rho(A).\]

It follows that \(\rho\) is irreducible. Then, by [11 Corollary 5.41], there exists a unitary \(U \in \mathcal{B}(\mathcal{H})\) such that \(\rho(X) = U^*XU\) for \(X \in A\). Then \(\varphi(X) = C\rho(X)^*C = CU^*X^*UC\). Set \(D = CU^*\). Then \(D\) is an anti-unitary operator on \(\mathcal{H}\) and \(\varphi(X) = DX^*D^{-1}\) for \(X \in A\). \(\square\)

**Corollary 3.19.** Let \(T \in \mathcal{B}(\mathcal{H})\) and assume that \(\mathcal{K}(\mathcal{H}) \subset C^*(T)\). Then \(T\) is g-normal if and only if \(T\) is UET.

**Proof.** The sufficiency follows from Corollary 3.11. It suffices to prove the necessity.

\[\implies\text{“}\implies\text{“}\quad .\quad \text{Since} \; T \; \text{is g-normal, by Lemma 1.7, there is an anti-automorphism} \; \varphi \; \text{of} \; C^*(T) \; \text{such that} \; \varphi(T) = T. \; \text{Since} \; \mathcal{K}(\mathcal{H}) \subset C^*(T), \; \text{by Proposition 3.18 there exists an anti-unitary operator} \; D \; \text{on} \; \mathcal{H} \; \text{such that} \]

\[\varphi(X) = DX^*D^{-1}, \forall X \in C^*(T).\]

In particular, \(T = \varphi(T) = DT^*D^{-1}\). By Lemma 3.8, it follows that \(T\) is UET. \(\square\)

**Proposition 3.20.** Let \(T \in \mathcal{B}(\mathcal{H})\) and assume that \(C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}\). If \(\rho\) is an anti-automorphism of \(C^*(T)\), then there exists a sequence \(\{D_n\}\) of anti-unitary operators on \(\mathcal{H}\) such that

\[\rho(X) = \lim_n D_nX^*D_n^{-1}, \forall X \in C^*(T).\]

**Proof.** Arbitrarily choose a conjugation \(C\) on \(\mathcal{H}\) and define \(\varphi(X) = C\rho(X)^*C\) for \(X \in C^*(T)\). Then \(\varphi\) is a faithful, nondegenerate representation of \(C^*(T)\) on \(\mathcal{H}\). Noting that \(C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}\), we have

\[\text{rank} \; \varphi(X) = \text{rank} \; X = \text{rank} \; \text{Id}(X).\]
for all \( X \in C^*(T) \), where \( \text{Id}(\cdot) \) is the identity representation of \( C^*(T) \). By Lemma 3.9, we have \( \varphi \cong_a \text{Id} \). Hence there exists a sequence \( \{U_n\} \) of unitary operators on \( \mathcal{H} \) such that \( \varphi(X) = \lim_n U_n^*XU_n \) for \( X \in C^*(T) \). Thus we have
\[
\rho(X) = C\varphi(X)^*C = \lim_n (CU_n^*)X^*(U_nC)
\]
for \( X \in C^*(T) \).

For each \( n \), set \( D_n = CU_n^* \). Then \( \{D_n\} \) satisfies all requirements. \( \square \)

**Corollary 3.21.** Let \( T \in \mathcal{B}(\mathcal{H}) \) be irreducible. If \( \rho \) is an anti-automorphism of \( C^*(T) \), then there exists a sequence \( \{D_n\} \) of anti-unitary operators on \( \mathcal{H} \) such that
\[
\rho(X) = \lim_n D_nX^*D_n^{-1}, \forall X \in C^*(T).
\]

**Proof.** Since \( T \) is irreducible, we have either \( \mathcal{K}(\mathcal{H}) \subset C^*(T) \) or \( C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\} \). In view of Propositions 3.18 and 3.20, one can see the conclusion. \( \square \)

**Corollary 3.22.** If \( T \in \mathcal{B}(\mathcal{H}) \) and \( C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\} \), then \( T \) is g-normal if and only if \( T \cong_a T^t \).

**Proof.** By Corollary 3.11, we need only prove the necessity.

Since \( T \) is g-normal, by Lemma 1.7, there is an anti-automorphism \( \varphi \) of \( C^*(T) \) such that \( \varphi(T) = T \). By Proposition 3.20, there exists a sequence \( \{D_n\} \) of anti-unitary operators on \( \mathcal{H} \) such that
\[
\varphi(X) = \lim_n D_nX^*D_n^{-1}, \forall X \in C^*(T).
\]
In particular, we have \( T = \varphi(T) = \lim_n D_nT^*D_n^{-1} \). In view of Proposition 3.10, it follows that \( T \cong_a T^t \). This completes the proof. \( \square \)

**Theorem 3.23.** If \( T \in \mathcal{B}(\mathcal{H}) \) is irreducible, then \( T \) is g-normal if and only if \( T \cong_a T^t \).

**Proof.** The sufficiency follows from Corollary 3.11. It remains to prove the necessity. Since \( T \) is irreducible, we have either \( C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\} \) or \( \mathcal{K}(\mathcal{H}) \subset C^*(T) \). By Corollaries 3.19 and 3.22, it follows in either case that \( T \cong_a T^t \). \( \square \)

**Remark 3.24.** By Example 3.17, there exists a reducible operator which is g-normal but not biquasitriangular. It is easy to check that each AUET operator is biquasitriangular. So Theorem 3.23 does not hold when \( T \) is reducible.

Now we can give the proof of Theorem 2.1

**Proof of Theorem 2.1** By the discussion in Subsection 1.2, “(i) \( \Rightarrow \) (v)” is clear.

The equivalence of (iv) and (v) follows from Corollary 3.22.

“(iii) \( \Rightarrow \) (ii)” By Proposition 4.21, \( A \cong_a T \) implies that for any \( \varepsilon > 0 \) there exists \( K \in \mathcal{K}(\mathcal{H}) \) with \( \|K\| < \varepsilon \) such that \( T + K \cong A \). Hence \( T + K \in \text{CSO} \). This implies that \( T \in \overline{\text{CSO}}^\circ \).

The implication “(ii) \( \Rightarrow \) (i)” is trivial. Now it remains to prove “(iv) \( \Rightarrow \) (iii)”.

“(iv) \( \Rightarrow \) (iii)” For \( X \in C^*(T) \), define \( \rho_1(X) = X \) and \( \rho_2(X) = X^\infty \). Then \( \rho_1 \) and \( \rho_2 \) are two nondegenerate faithful representations of \( C^*(T) \). Since \( C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\} \), we deduce that \( \text{rank } \rho_1(X) = \text{rank } \rho_2(X) \) for all \( X \in C^*(T) \). Then, by Lemma 3.9, \( \rho_1 \cong_a \rho_2 \). Hence \( T = \rho_1(T) \cong_a \rho_2(T) = T^\infty \). Noting that \( T \cong_a T^t \), we have
\[
T \cong_a T^\infty \cong (T \oplus T)^\infty \cong_a (T \oplus T^t)^\infty.
\]
By Lemma \[3.6\] \((T \oplus T^*)(\infty)\) is complex symmetric. Arbitrarily choose an operator \(A\) on \(H\) satisfying \(A \cong (T \oplus T^*)(\infty)\). Hence \(A \in CSO\) and \(T \cong_a A\). This completes the proof. \[\square\]

4. Proofs of Theorems 2.4, 2.5 and 2.6

Let \(\{A_i : 1 \leq i \leq n\}\) be a commuting family of normal operators on \(H\). Denote by \(C^*(A_1, A_2, \cdots, A_n)\) the \(C^*\)-algebra generated by \(A_1, A_2, \cdots, A_n\) and the identity \(I\). The joint spectrum of the \(n\)-tuple \((A_1, A_2, \cdots, A_n)\) is defined as the set of \(n\)-tuples of scalars \((\lambda_1, \lambda_2, \cdots, \lambda_n)\) such that the ideal of \(C^*(A_1, A_2, \cdots, A_n)\) generated by \(A_1 - \lambda_1, A_2 - \lambda_2, A_n - \lambda_n\) is different from \(C^*(A_1, A_2, \cdots, A_n)\) (see [29, Definition 3.1.13]). We let \(\sigma(A_1, A_2, \cdots, A_n)\) denote the joint spectrum of \((A_1, A_2, \cdots, A_n)\).

**Proposition 4.1.** Let \(W \in B(H)\) be a unilateral weighted shift with positive weights \(\{d_i\}_{i=0}^\infty\). If \(W \cong_a W^*\), then there exists a subsequence \(\{n_i\}_{i=1}^\infty\) of \(\mathbb{N}\) such that \(d_{n_i+n_i} \to 0\), and, for each \(k \geq 0\), we have \(d_{n_i-k} \to d_k\).

**Proof.** Since \(W \cong_a W^*\), we can choose a sequence \(\{U_n\}_{n=1}^\infty\) of unitary operators on \(H\) such that \(U_n^\ast WU_n \to W^*\). As consequences, we have for each \(k \geq 1\) that \(U_n^\ast W^k U_n \to |(W^k)^*|\) and \(U_n^\ast (W^k)^* U_n \to |W^k|\).

Denote \(A_0 = |W^*|\) and \(A_k = |W^k|\) for \(k \geq 1\). Denote \(B_0 = |W|\) and \(B_k = |(W^k)^*|\) for \(k \geq 1\). So, for each \(k \geq 0\), one can see that

\[
U_n^\ast A_k U_n \to B_k.
\]

Without loss of generality, we may assume that \(W e_i = d_i e_{i+1}\) for all \(i \geq 0\), where \(\{e_i\}_{i=0}^\infty\) is an ONB of \(H\). Note that all \(A_k\)'s and \(B_k\)'s are diagonal operators with respect to \(\{e_i\}\). For each \(k \geq 0\), we assume that

\[
A_k = \text{diag}\{a_0^{(k)}, a_1^{(k)}, a_2^{(k)}, \cdots\} \quad \text{and} \quad B_k = \text{diag}\{b_0^{(k)}, b_1^{(k)}, b_2^{(k)}, \cdots\}
\]

with respect to \(\{e_i\}\).

Let \(k \geq 1\) be fixed. Note that \((A_0, A_1, \cdots, A_k)\) and \((B_0, B_1, \cdots, B_k)\) are two \((k+1)\)-tuples of commuting diagonal operators. In view of (4.1), there is an isomorphism \(\rho\) from \(C^*(A_0, A_1, \cdots, A_k)\) onto \(C^*(B_0, B_1, \cdots, B_k)\) such that \(\rho(A_i) = B_i\) for \(0 \leq i \leq k\). Thus we have

\[
\sigma(A_0, A_1, \cdots, A_k) = \sigma(B_0, B_1, \cdots, B_k).
\]

For \(i \geq 0\), define \(\omega_i(X) = x_i\) if \(X \in C^*(A_0, A_1, \cdots, A_k)\) and

\[
X = \text{diag}\{x_0, x_1, x_2, \cdots\}
\]

with respect to \(\{e_i\}\). Then each \(\omega_i\) is a multiplicative linear functional on the \(C^*\)-algebra \(C^*(A_0, A_1, \cdots, A_k)\). Moreover, \(\{\omega_i : i \geq 0\}\) is dense in the maximal ideal space of \(C^*(A_0, A_1, \cdots, A_k)\). Then, by [29, Theorem 3.1.14], it follows from (4.2) that

\[
\sigma(A_0, A_1, \cdots, A_k) = \overline{\{(a_i^{(0)}, a_i^{(1)}, \cdots, a_i^{(k)}) : i \geq 0\}},
\]

where the closure is taken in the usual topology on \(\mathbb{R}^n\). Similarly we have

\[
\sigma(B_0, B_1, \cdots, B_k) = \overline{\{(b_i^{(0)}, b_i^{(1)}, \cdots, b_i^{(k)}) : i \geq 0\}}.
\]
We choose the desired subsequence \( \{ n_k \} \) of \( \mathbb{N} \) as follows.

**Step 1.** The choice of \( n_1 \).

Note that \( B_1 = A_0 = \text{diag}\{0, d_0, d_1, d_2, \cdots\} \) and \( B_0 = A_1 = \text{diag}\{d_0, d_1, d_2, \cdots\} \) with respect to \( \{ e_i \} \). In view of (4.4) and (4.5), \((0, d_0) \in \sigma(A_0, A_1) \) and \( \sigma(B_0, B_1) \) is the closure of \( \{(d_0, 0)\} \cup \{(d_{i+1}, d_i) : i \geq 0\} \). By (4.3), there exists \( i \) such that 
\[
|d_{i+1} + |d_i - d_0| < \frac{1}{2}.
\]
Denote \( n_1 = i \).

**Step 2.** The choice of \( n_2 \).

Note that \( A_2 = \text{diag}\{d_0, d_1, d_2, d_3, \cdots\} \) and \( B_2 = \text{diag}\{0, 0, d_0, d_1, d_2, \cdots\} \) with respect to \( \{ e_i \} \). In view of (4.4) and (4.5), \((0, d_0, d_0d_1) \in \sigma(A_0, A_1, A_2) \) and \( \sigma(B_0, B_1, B_2) \) is the closure of 
\[
\{(d_0, 0, 0), (d_1, d_0, 0)\} \cup \{(d_{i+1}, d_i, d_{i-1}d_i) : i \geq 1\}.
\]
By (4.3), there exists \( i > n_1 \) such that
\[
\max\{|d_{i+1} - d_0|, |d_i - d_0d_1|\} < \frac{d_0}{24(1 + \|W\|)(1 + d_0)};
\]
furthermore, we have
\[
d_{i+1} + |d_i - d_0| + |d_{i-1} - d_1| \\
\leq \frac{1}{24} + \frac{1}{24} + \frac{|d_0d_1 - d_0d_{i-1}|}{d_0} \\
\leq \frac{1}{12} + \frac{|d_0d_1 - d_{i-1}d_i| + |d_{i-1}d_i - d_0d_{i-1}|}{d_0} \\
\leq \frac{1}{12} + \frac{|d_{i-1}d_i - d_0d_{i-1}|}{d_0} \\
\leq \frac{1}{6} + \frac{\|W\| \cdot |d_i - d_0|}{d_0} < \frac{1}{6} + \frac{1}{12} = \frac{1}{22}.
\]
Denote \( n_2 = i \).

**Step 3.** The choice of \( n_3 \).

Note that \( A_3 = \text{diag}\{d_0, d_1, d_2, d_3, d_4, \cdots\} \) and 
\( B_3 = \text{diag}\{0, 0, 0, d_0d_1d_2, d_1d_2d_3, d_2d_3d_4, \cdots\} \)
with respect to \( \{ e_i \} \). In view of (4.4) and (4.5), \((0, 0, d_0, d_0d_1, d_0d_1d_2) \in \sigma(A_0, \cdots, A_3) \) and \( \sigma(B_0, \cdots, B_3) \) is the closure of 
\[
\{(d_0, 0, 0, 0), (d_1, d_0, 0, 0), (d_2, d_1, d_0d_1, 0)\} \cup \{(d_{i+1}, d_i, d_{i-1}d_i, d_{i-2}d_{i-1}d_i) : i \geq 2\}.
\]
By (4.3), there exists \( i > n_2 \) such that
\[
\max\{|d_{i+1} - d_0|, |d_i - d_0d_1|, |d_{i-1}d_i - d_0d_{i-1}d_i|, |d_{i-2}d_{i-1}d_i - d_0d_1d_2|\}
\]
is small enough to guarantee 
\[
d_{i+1} + |d_i - d_0| + |d_{i-1} - d_1| + |d_{i-2} - d_2| < \frac{1}{22}.
\]
Denote \( n_3 = i \).

Using a similar argument as above, one can choose a subsequence \( \{ n_k \}_{k=1}^{\infty} \) of \( \mathbb{N} \) such that
\[
d_{n_k+1} + |d_{n_k} - d_0| + |d_{n_k-1} - d_1| + \cdots + |d_{n_k-k+1} - d_k-1| < \frac{1}{2^k}
\]
for each \( k \geq 1 \). Thus \( \{ n_k \} \) is the desired subsequence of \( \mathbb{N} \).
Remark 4.2. Let $W$ be a unilateral weighted shift with positive weights. By Proposition 4.4, if $W \cong_a W^*$, then $W$ is approximately Kakutani.

Proof of Theorem 4.4 By definitions, “(iii) $\Rightarrow$ (ii) $\Rightarrow$ (i)” are obvious.

Without loss of generality, we may directly assume that $Te_i = w_ie_{i+1}$ for all $i \geq 1$, where $\{e_i\}_{i \in \mathbb{N}}$ is an ONB of $\mathcal{H}$ and $w_i > 0$ for all $i$. Thus we can define a conjugation $C$ on $\mathcal{H}$ satisfying $Ce_i = e_i$ for all $i$. Noting that $CT^*C = T^*$, we deduce that $T^*$ is a transpose of $T$. Then the implication “(i) $\Rightarrow$ (iv)” follows from Lemma 3.12. Since $T$ is irreducible, “(iv) $\iff$ (v)” follows from Theorem 3.23.

“(iv) $\iff$ (iii)” Since $T \cong_a T^*$, we claim that $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$. In fact, if not, then $\mathcal{K}(\mathcal{H}) \subset C^*(T)$. Note that $T \cong_a T^*$ induces an automorphism $\rho$ of $C^*(T)$ satisfying $\rho(T) = T^*$. Since $\mathcal{K}(\mathcal{H}) \subset C^*(T)$, it follows from [11, Corollary 5.41] that $\rho$ is unitarily implemented. Thus $T \cong T^*$, a contradiction. So $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$. $T \cong_a T^*$ implies that $T$ is AUET. By Theorem 2.1, there exists $A \in \text{CSO}$ such that $A \cong_a T$.

The implications “(vi) $\Rightarrow$ (i)” and “(iv) $\Rightarrow$ (vi)” follow from [16, Theorem 10] and Proposition 4.4, respectively. This completes the proof.

Corollary 4.3. If $T \in \mathcal{B}(\mathcal{H})$ is an irreducible unilateral weighted shift and $T \in \overline{\text{CSO}}$, then $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$.

In the rest of this section, we deal with bilateral weighted shifts with positive weights.

Proposition 4.4. Let $T \in \mathcal{B}(\mathcal{H})$ be a bilateral weighted shift with positive weights. If $T$ is reducible, then $T$ is invertible and completely reducible; moreover, $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$.

Proof. Assume that $\{w_i\}_{i \in \mathbb{Z}}$ is the weight sequence of $T$. Since $T$ is reducible, by [26, Problem 159], $\{w_i\}$ is periodic. Thus $\inf w_i > 0$ and $T$ is invertible. By [13, Lemma 2.5], if $T$ is completely reducible, then $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$. So it suffices to prove that $T$ is completely reducible.

Now we may assume that $\{w_i\}$ is of period $n$. When $n = 1$, $\{w_i\}$ is constant; in this case, $T$ is normal without eigenvalues and hence completely reducible. In the rest of this proof we deal with the case that $n > 1$.

Without loss of generality, we assume that $Te_i = w_ie_{i+1}$ for $i \in \mathbb{Z}$, where $\{e_i\}_{i \in \mathbb{Z}}$ is an ONB of $\mathcal{H}$. Let $U$ be the bilateral shift on $\mathcal{H}$ defined by $Ue_i = e_{i+1}$ for $i \in \mathbb{Z}$. Set

$$A = \begin{bmatrix}
    0 & w_1U & w_2U & \cdots & w_nU \\
    w_1I & 0 & \cdots & \cdots & \cdots \\
    w_2I & w_1I & 0 & \cdots & \cdots \\
    \vdots & \ddots & \ddots & \ddots & \ddots \\
    w_{n-1}I & \cdots & w_2I & w_1I & 0
\end{bmatrix}
\begin{bmatrix}
    \mathcal{H} \\
    \mathcal{H} \\
    \vdots \\
    \mathcal{H} \\
\end{bmatrix}
\begin{bmatrix}
    w_1U \\
    0 \\
    \cdots \\
    0 \\
\end{bmatrix}
\begin{bmatrix}
    \mathcal{H} \\
    \mathcal{H} \\
    \vdots \\
    \mathcal{H}
\end{bmatrix}
\begin{bmatrix}
    H \\
    H \\
    \vdots \\
    H
\end{bmatrix},
$$

where $I$ is the identity operator on $\mathcal{H}$ and all omitted entries are zero. Then $A \in \mathcal{B}(\mathcal{H}^{(n)})$ is invertible. Since $w_i = w_{i+n}$ for all $i \in \mathbb{Z}$, it is easy to see that $A \cong T$. So we need only prove that $A$ is completely reducible.
Let $P \in \mathcal{B}(\mathcal{H}^{(n)})$ be a nonzero projection which commutes with $A$. Assume that $P$ admits the following matrix representation:

$$
P = \begin{bmatrix}
P_{1,1} & P_{1,2} & \cdots & P_{1,n} \\
P_{2,1} & P_{2,2} & \cdots & P_{2,n} \\
\vdots & \vdots & \ddots & \vdots \\
P_{n,1} & P_{n,2} & \cdots & P_{n,n}
\end{bmatrix} \mathcal{H}
$$

Since $PA = AP$, a straightforward matrical calculation shows that there exist positive numbers $\{\lambda_{i,j}\}_{1 \leq i, j \leq n}$ satisfying

$$(4.6) \quad \frac{P_{l,l+k}}{\lambda_{l,l+k}} = \frac{P_{n-k,n}}{\lambda_{n-k,n}} = \frac{P_{n-k+s,s}}{\lambda_{n-k+s,s}}$$

whenever $1 \leq k \leq n - 1$, $1 \leq l \leq n - k$ and $1 \leq s \leq k$; in particular, we have $P_{1,1} = P_{2,2} = \cdots = P_{n,n}$.

On the other hand, since $P$ is self-adjoint, $PA = AP$ implies that $P|A| = |A|P$. Noting that

$$(4.7) \quad |A| = \begin{bmatrix} w_1 I & & \\
& w_2 I & \\
& & \ddots & \\
& & & w_n I \end{bmatrix} \mathcal{H}$$

it follows from $P|A| = |A|P$ that

$$1 \leq i, j \leq n, w_i \neq w_j \implies "P_{i,j} = 0".$$
Proof. Since $TV = VT^*$, one can see that $|T|^V = V|T^*|$. 

(i) It is easy to check that $|T|e_j = w_i e_j$ and $|T^*|e_j = w_{j-1} e_j$ for all $j \in \mathbb{Z}$. Then, given $k \in \mathbb{Z}$, we have

$$w_{k-1} V e_k = V(w_{k-1} e_k) = V|T^*|e_k = |T|^V e_k;$$

that is, $V e_k \in \ker(|T| - w_{k-1}) = \sqrt{\{e_j : w_j = w_{k-1}\}}$.

(ii) For any $s, t \in \mathbb{Z}$, we have

$$w_{s-1} \langle V e_{s-1}, e_{t+1} \rangle = \langle V(w_{s-1} e_{s-1}), e_{t+1} \rangle = \langle VT^* e_s, e_{t+1} \rangle = \langle TV e_s, e_{t+1} \rangle = \langle V e_s, T^* e_{t+1} \rangle = \langle V e_s, w_i e_t \rangle = w_t \langle V e_s, e_t \rangle.$$ 

Thus $\langle V e_{s-1}, e_{t+1} \rangle = 0$ if and only if $\langle V e_s, e_t \rangle \neq 0$. Using an obvious inductive argument, one can see the conclusion. \hfill $\Box$

Lemma 4.6 ([17, Theorem 4.4]). Let $\{e_i\}_{i \in \mathbb{Z}}$ be an ONB of $\mathcal{H}$ and $T \in \mathcal{B}(\mathcal{H})$ with $T e_i = w_i e_{i+1}$ for $i \in \mathbb{Z}$. If $w_i \neq 0$ for all $i \in \mathbb{Z}$, then $T$ is complex symmetric if and only if there exists $k \in \mathbb{Z}$ such that $|w_{k-j}| = |w_j|$ for all $j \in \mathbb{Z}$.

Theorem 4.7. Let $T \in \mathcal{B}(\mathcal{H})$ be a bilateral weighted shift with positive weights $\{w_i\}_{i \in \mathbb{Z}}$. If $C^*(T) \cap \ker(\mathcal{H}) \neq \{0\}$, then the following are equivalent:

(i) $T \in \text{CSO}$;

(ii) $T \in \text{CSO}$;

(iii) $T \cong T^*$;

(iv) $T \cong_a T^*$;

(v) $T$ is g-normal;

(vi) $\exists k \in \mathbb{Z}$ such that $w_i = w_{k-i}$ for all $i \in \mathbb{Z}$.

Proof. Without loss of generality, we may directly assume that $T e_i = w_i e_{i+1}$ for all $i$, where $\{e_i\}_{i \in \mathbb{Z}}$ is an ONB of $\mathcal{H}$. Thus we can define a conjugation $C$ on $\mathcal{H}$ satisfying $C e_i = e_i$ for all $i$. Noting that $CT^* C = T^*$, we deduce that $T^*$ is a transpose of $T$. Then, by Corollary 3.11 and Lemma 3.12, the implications “(i)$\implies$(iv)$\implies$(v)” are obvious.

By Proposition 4.4, it follows from $C^*(T) \cap \mathcal{K}(\mathcal{H}) \neq \{0\}$ that $T$ is irreducible and $\mathcal{K}(\mathcal{H}) \subset C^*(T)$. So the equivalence between (iii) and (v) is given by Corollary 3.19.

The equivalence between (ii) and (vi) is given by Lemma 4.6. The implication “(ii)$\implies$(i)” is trivial. Now it remains to prove “(iii)$\implies$(vi)”.

“(iii)$\implies$(vi)” Since $T \cong T^*$, we can choose a unitary operator $V$ on $\mathcal{H}$ such that $TV = VT^*$. Thus there exists $k \in \mathbb{Z}$ such that $\langle V e_k, e_k \rangle \neq 0$. Then, by Lemma 4.5 (ii), $\langle V e_{j+1}, e_{k-j} \rangle \neq 0$ for all $j \in \mathbb{Z}$. By Lemma 4.5 (i), it follows that $w_j = w_{k-j}$ for all $j \in \mathbb{Z}$. This completes the proof. \hfill $\Box$

Lemma 4.8 ([31, Prop. 2.2.14 and Thm. 3.1.7]). Let $A, B \in \mathcal{B}(\mathcal{H})$ be two bilateral weighted shifts with positive weights. If $A, B$ are both invertible, then $A \cong_a B$ if and only if $\Sigma n(A) = \Sigma n(B)$ for all $n \geq 1$.

Lemma 4.9 ([32, Lemma 2.4.1]). Let $A \in \mathcal{B}(\mathcal{H})$ have the polar decomposition $A = U|A|$. If $\rho$ is any representation of $C^*(A)$ on $\mathcal{H}_\rho$ for which $\ker \rho(A) = \ker \rho(A)^* = \{0\}$, then $\rho$ has an extension to a representation of $C^*(A, U)$ on the same Hilbert space $\mathcal{H}_\rho$. 
Theorem 4.10. Let $A, B \in \mathcal{B}(\mathcal{H})$ be two bilateral weighted shifts with positive weights. Then $A \cong_{a} B$ if and only if $\Sigma_{n}(A) = \Sigma_{n}(B)$ for all $n \geq 1$.

Proof. Without loss of generality, we assume that $\{e_{i}\}_{i \in \mathbb{Z}}$ is an ONB of $\mathcal{H}$ and $Ae_{i} = w_{i}e_{i+1}, Be_{i} = v_{i}e_{i+1},$ where $w_{i}, v_{i} > 0$ for all $i \in \mathbb{Z}$.

$\implies$. Arbitrarily choose an $\varepsilon > 0$. Set 

$$a_{i} = \begin{cases} w_{i}, & \text{if } w_{i} \geq \varepsilon, \\ \varepsilon, & \text{if } w_{i} < \varepsilon, \end{cases} \quad b_{i} = \begin{cases} v_{i}, & \text{if } v_{i} \geq \varepsilon, \\ \varepsilon, & \text{if } v_{i} < \varepsilon. \end{cases}$$

Define $A_{\varepsilon}e_{i} = a_{i}e_{i+1}$ and $B_{\varepsilon}e_{i} = b_{i}e_{i+1}$ for $i \in \mathbb{Z}$. Then $A_{\varepsilon}, B_{\varepsilon}$ are two invertible bilateral weighted shifts and $\max\{\|A_{\varepsilon} - A\|, \|B_{\varepsilon} - B\|\} \leq \varepsilon$.

Since $\Sigma_{n}(A) = \Sigma_{n}(B)$ for all $n \geq 1$, it is not difficult to see that $\Sigma_{n}(A_{\varepsilon}) = \Sigma_{n}(B_{\varepsilon})$ for all $n \geq 1$. So, by Lemma 4.8, there exists a unitary $U \in \mathcal{B}(\mathcal{H})$ such that $\|U^{*}A_{\varepsilon}U - B_{\varepsilon}\| < \varepsilon$. Hence

$$\|U^{*}A_{\varepsilon}U - B\| \leq \|U^{*}(A - A_{\varepsilon})U\| + \|U^{*}A_{\varepsilon}U - B_{\varepsilon}\| + \|B_{\varepsilon} - B\| < 3\varepsilon.$$ 

Since $\varepsilon$ was arbitrarily chosen, we deduce that $A \cong_{a} B$.

$\impliedby$. If $A$ is reducible, then, by Proposition 4.4, $A$ is invertible. Since $A \cong_{a} B$, it follows that $B$ is also invertible. In view of Lemma 4.8 we obtain $\Sigma_{n}(A) = \Sigma_{n}(B)$ for all $n \geq 1$. When $B$ is reducible, the proof is similar. So, in the following, we may assume that $A$ and $B$ are both irreducible.

Denote by $U$ the bilateral shift on $\mathcal{H}$ defined by $Ue_{i} = e_{i+1}$ for all $i \in \mathbb{Z}$. Thus $A = U\|A\|$ and $B = U\|B\|$ are respectively the polar decomposition of $A$ and the polar decomposition of $B$.

Since $A \cong_{a} B$, there exists an isomorphism $\rho : C^{*}(A) \rightarrow C^{*}(B)$ so that $\rho(A) = B$. Thus $\rho$ is a faithful representation of $C^{*}(A)$ on $\mathcal{H}$. Note that $\ker B = \{0\} = \ker B^{*}$. Then, by Lemma 4.9, $\rho$ has an extension to a representation $\rho'$ of $C^{*}(A, U)$ on $\mathcal{H}$. Noting that 

$$B = \rho'(A) = \rho'(U)\rho'(\|A\|) = \rho'(U)\rho(\|A\|) = \rho'(U)\|B\|,$$

one can deduce that $\rho'(U) = U$. Thus, for any polynomial $p(\cdot, \cdot, \cdot)$ in three free variables, we have $\|p(B, B^{*}, U)\| \leq \|p(A, A^{*}, U)\|$. By the symmetry, we can also prove that $\|p(A, A^{*}, U)\| \leq \|p(B, B^{*}, U)\|$. Hence $\rho' : C^{*}(A, U) \rightarrow C^{*}(B, U)$ is an isomorphism.

Define $\tilde{A} = U(\|A\| + I)$ and $\tilde{B} = U(\|B\| + I)$. It is easy to see that $C^{*}(A, U) = C^{*}(\tilde{A}), C^{*}(B, U) = C^{*}(\tilde{B})$ and $\rho'(\tilde{A}) = \tilde{B}$. Since $A$ and $B$ are both irreducible, it follows from [26, Problem 159] that $\tilde{A}$ and $\tilde{B}$ are both irreducible. Thus we have either $\mathcal{K}(\mathcal{H}) \subset C^{*}(\tilde{A})$ or $C^{*}(\tilde{A}) \cap \mathcal{K}(\mathcal{H}) = \{0\}$.

Case 1. $\mathcal{K}(\mathcal{H}) \subset C^{*}(\tilde{A})$. In this case, since $\rho'$ is a faithful representation of $C^{*}(\tilde{A})$, by [11, Corollary 5.41], it must be unitarily implemented; that is, there exists a unitary $U \in \mathcal{B}(\mathcal{H})$ such that $\rho'(X) = U^{*}UX$ for all $X \in C^{*}(\tilde{A})$. In particular, $\tilde{B} = U^{*}\tilde{A}U$. Note that $\tilde{A}, \tilde{B}$ are both invertible bilateral weighted shifts with positive weights. Thus, by Lemma 4.8 we have $\Sigma_{n}(\tilde{A}) = \Sigma_{n}(\tilde{B})$ for all $n \geq 1$. So we conclude that $\Sigma_{n}(A) = \Sigma_{n}(B)$ for all $n \geq 1$. 


Case 2. $C^*(\tilde{A}) \cap K(H) = \{0\}$. In this case, since $\tilde{B}$ is irreducible, we claim that $C^*(\tilde{B}) \cap K(H) = \{0\}$. In fact, if not, then $K(H) \subset C^*(\tilde{B})$ and, using a similar argument as in Case 1, one can prove that $\tilde{A} \cong \tilde{B}$. Hence $K(H) \subset C^*(\tilde{A})$, a contradiction. By the claim, we have rank $X = \operatorname{rank} \rho'(X)$ for all $X \in C^*(\tilde{A})$. By Lemma 3.3 it follows that $\rho' \cong_a \text{Id}$, where $\text{Id}(\cdot)$ is the identity representation of $C^*(\tilde{A})$. So we obtain $\tilde{A} \cong_a \tilde{B}$. By Lemma 3.8 it implies that $\Sigma_n(\tilde{A}) = \Sigma_n(\tilde{B})$ for all $n \geq 1$; that is, $\Sigma_n(A) = \Sigma_n(B)$ for all $n \geq 1$. This completes the proof. \qed

**Corollary 4.11.** Let $T \in B(H)$ be a bilateral weighted shift with positive weights $\{w_i\}_{i \in \mathbb{Z}}$. Then $T \cong_a T^*$ if and only if $\Sigma_n(T)^t = \Sigma_n(T)$ for all $n \geq 1$.

**Proof.** Note that $T^*$ is unitarily equivalent to a bilateral weighted shift $A$ with weights $\{v_i\}_{i \in \mathbb{Z}}$, where $v_i = w_{-i}$ for all $i \in \mathbb{Z}$. It is easy to see that $\Sigma_n(A) = \Sigma_n(T)^t$ for all $n \geq 1$. Then, by Theorem 4.10 we have

$$T \cong_a T^* \iff T \cong_a A \iff \Sigma_n(T) = \Sigma_n(T)^t, \forall n \geq 1.$$ 

\qed

**Theorem 4.12.** Let $T \in B(H)$ be a bilateral weighted shift with positive weights $\{w_i\}_{i \in \mathbb{Z}}$. If $T$ is irreducible, then the following are equivalent:

(i) $T \in \overline{C^*}$;

(ii) $T \in C^*$;

(iii) $T$ is $g$-normal;

(iv) $T \cong_a T^*$;

(v) $T \cong T^*$;

(vi) $\exists k \in \mathbb{Z}$ such that $w_i = w_{k-i}$ for all $i \in \mathbb{Z}$.

**Proof.** We first note that $T^*$ is also a transpose of $T$.

The implication “(vi)\implies(ii)” follows from Lemma 4.6. By definition, the implications “(ii)\implies(v)\implies(iv)” are obvious.

Since $T$ is irreducible, by [26] Problem 159, $\{w_i\}$ is periodic. By Proposition 4.4 $T$ is invertible and $C^*(T) \cap K(H) = \{0\}$. Thus, by Theorem 2.7 (i), (iii) and (iv) are equivalent. Now it remains to prove “(iv)\implies(vi)”.

“(iv)\implies(vi)”. Note that $T^*$ is unitarily equivalent to a bilateral weighted shift $A$ with weights $\{v_i\}_{i \in \mathbb{Z}}$, where $v_i = w_{-i}$ for all $i \in \mathbb{Z}$. Then we have $T \cong_a A$. It follows from Theorem 3.10 that $\Sigma_i(T) = \Sigma_i(A)$ for all $i \geq 1$.

We may assume that $\{w_i\}$ is of period $n$. Since $\Sigma_n(T) = \Sigma_n(A) = \Sigma_n(T)^t$, there exists $i \in \mathbb{Z}$ such that

$$(w_1, w_2, \ldots, w_n) = (w_{i+n}, w_{i+n-1}, \ldots, w_{i+1}).$$

**Case 1.** $n$ divides $i$. Noting that $\{w_i\}_{i \in \mathbb{Z}}$ is of period $n$, it follows that

$$(w_1, w_2, \ldots, w_n) = (w_n, w_{n-1}, \ldots, w_1).$$

So $w_i = w_{n+1-i}$ for all $i \in \mathbb{Z}$.

**Case 2.** $n$ does not divide $i$. Thus there exists $1 \leq m < n$ such that

$$(w_1, w_2, \ldots, w_n) = (w_m, w_{m-1}, \ldots, w_1, w_n, w_{n-1}, \ldots, w_{m+1}).$$

Noting that $\{w_i\}_{i \in \mathbb{Z}}$ is of period $n$, we deduce that $w_i = w_{m+1-i}$ for all $i \in \mathbb{Z}$. This completes the proof. \qed
Remark 4.13. Summarizing the results of Theorems 4.7 and 4.12 one can obtain Theorem 2.5.

Proposition 4.14. Let $T \in \mathcal{B}(\mathcal{H})$ be a reducible bilateral weighted shift with positive weights $\{w_i\}_{i \in \mathbb{Z}}$. If $T \in \text{CSO}$, then $T$ is completely complex symmetric.

Proof. Since $T$ is reducible, by [26, Problem 159], $\{w_i\}$ is periodic and we may assume that $\{w_i\}$ is of period $n$. If $n = 1$, then $T$ is normal without eigenvalues and hence completely complex symmetric. It suffices to give the proof in the case that $n > 1$.

Let $U$ be the bilateral shift on $\mathcal{H}$ defined by $Ue_i = e_{i+1}$ for $i \in \mathbb{Z}$. Set

$$A = \begin{bmatrix}
0 & w_1I & \cdots & w_nU \\
I & 0 & \cdots & 0 \\
0 & I & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
w_nI & 0 & \cdots & I
\end{bmatrix},$$

where $I$ is the identity operator on $\mathcal{H}$ and all omitted entries are zero. Then $A \in \mathcal{B}(\mathcal{H}(n))$. Since $w_i = w_{i+n}$ for all $i \in \mathbb{Z}$, it is easy to see that $A \cong T$. So we need only prove that $A$ is completely complex symmetric.

Let $P$ be the projection of $\mathcal{H}(n)$ onto $M$. Then $PA = AP$. By the proof of Proposition 4.14 $P$ can be written as $P = P_0(n)$, where $P_0$ is a projection on $\mathcal{H}$ commuting with $U$.

Since $U$ is unitary and hence complex symmetric, there exists a conjugation $D$ on $\mathcal{H}$ such that $DUD = U^*$. Then, for each polynomial $p(\cdot, \cdot)$ in two free variables, we have $Dp(U^*, U)D = p(U^*, U)^*$. Note that there exists a sequence $\{p_n\}$ of polynomials in two free variables such that $\{p_n(U^*, U)\}$ converges to $P_0$ in the weak operator topology (see [9, page 282, Thm. 7.8]). It follows that $DP_0D = P_0$.

Since $T$ is complex symmetric, by the proof of "(iv) $\Rightarrow$ (vi)" in Theorem 4.12 we have either (a)

$$(w_1, w_2, \cdots, w_n) = (w_k, w_{k-1}, \cdots, w_1, w_n, w_{n-1}, \cdots, w_{k+1})$$

for some $1 \leq k < n$, or (b) $(w_1, w_2, \cdots, w_n) = (w_n, w_{n-1}, \cdots, w_1)$. Thus the rest of the proof is divided into two cases.

In case (a), we set

$$C = \begin{bmatrix}
UD & \cdots & UD \\
UD & \ddots & UD \\
UD & \cdots & UD
\end{bmatrix}$$

where

$$\begin{aligned}
\mathcal{H}_1 \\
\vdots \\
\mathcal{H}_k \\
\mathcal{H}_{k+1} \\
\vdots \\
\mathcal{H}_n
\end{aligned}$$
where $\mathcal{H}_1 = \cdots = \mathcal{H}_n = \mathcal{H}$ and all omitted entries are zero. In case (b), set

$$ C = \begin{bmatrix} UD & \mathcal{H}_1 \\ \vdots & \vdots \\ D & \mathcal{H}_n \end{bmatrix} $$

In either case, one can verify that $C$ is a conjugation on $\mathcal{H}^{(n)}$ and $CAC = A^*$. Since $DP_0 = P_0D$ and $P_0U = UP_0$, a direct calculation shows that $CP = PC$, that is, $M$ reduces $C$. Set $\bar{C} = C|_M$. Then $\bar{C}$ is a conjugation and it follows from $CAC = A^*$ that $\bar{C}(A|M)\bar{C} = (A|M)^*$. So $A|M$ is complex symmetric. This completes the proof.

**Remark 4.15.** Let $T$ be a bilateral weighted shift with positive weights. Proposition 4.14 shows that if $T$ is complex symmetric, then $T$ is either completely complex symmetric or a minimal complex symmetric operator.

**Proof of Theorem 2.6.** If $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$, then, by Theorem 2.1, (i)-(v) are equivalent. By Corollary 4.11, $T \cong a T^*$ if and only if $\Sigma_n(T) = \Sigma_n(T)^t$ for all $n \geq 1$. So, in this case, (i)-(vi) are all equivalent.

If $C^*(T) \cap \mathcal{K}(\mathcal{H}) \neq \{0\}$, then, by Proposition 4.4, $T$ is irreducible and $\mathcal{K}(\mathcal{H}) \subseteq C^*(T)$. In view of Theorem 4.7, one can see the following implications:

$$(i) \iff (iv) \iff (v) \iff (iii) \implies (ii) \implies (i);$$

that is, (i)-(v) are all equivalent. On the other hand, it follows from Corollary 4.11 that (iv) and (vi) are equivalent. This completes the proof. 

### 5. On essentially normal operators which are G-normal

The main theorem of this section is the following theorem which characterizes when an essentially normal operator is g-normal. In fact, we obtain a canonical decomposition for such operators.

**Theorem 5.1.** Let $T \in \mathcal{B}(\mathcal{H})$ be essentially normal. Then $T$ is g-normal if and only if it is unitarily equivalent to a direct sum of the following three kinds of g-normal operators (some of the summands may be absent):

(i) normal operators;

(ii) irreducible UET operators;

(iii) operators with form of $A^{(m)} \oplus (A^t)^{(n)}$, where $A$ is irreducible, not UET and $m, n \in \mathbb{N}$.

**Corollary 5.2.** If $T \in \mathcal{B}(\mathcal{H})$ is essentially normal, then $T$ is g-normal if and only if $T_{abnor}$ is g-normal.

**Proof.** Since each normal operator is g-normal, the sufficiency is evident.

“$\Longrightarrow$”. Note that if $R$ is an irreducible operator, then $R$ is either normal or abnormal. If $T$ is essentially normal and g-normal, then, by Theorem 5.1 $T_{abnor}$ is unitarily equivalent to a direct sum of irreducible UET operators and operators with a form of $A^{(m)} \oplus (A^t)^{(n)}$, where $A$ is irreducible but not UET. In view of Theorem 5.1 $T_{abnor}$ is still g-normal.
Corollary 5.3. If $T \in \mathcal{B}(\mathcal{H})$ is essentially normal, then $T$ is g-normal if and only if $T^{(\infty)}$ is complex symmetric.

Proof. The sufficiency follows from Corollary 3.5. “$\Longrightarrow$”. If $R \in \mathcal{B}(\mathcal{H})$ is UET, then it is easy to see that

$$R^{(\infty)} \cong (R \oplus R)^{(\infty)} \cong (R \oplus R^t)^{(\infty)}.$$ 

If $Q = A^{(m)} \oplus (A^t)^{(n)}$, then $Q^{(\infty)} = A^{(\infty)} \oplus (A^t)^{(\infty)} = (A \oplus A^t)^{(\infty)}$.

By the above discussion and Theorem 5.1 if $T$ is g-normal, then $T^{(\infty)}$ is unitarily equivalent to a direct sum of normal operators and operators with a form of $B \oplus B^t$. By Lemmas 3.1 and 3.6 $T^{(\infty)}$ is complex symmetric. $\square$

To give the proof of Theorem 6.1 we need to make some preparations. Let $\{A_i\}_{i \in \Lambda}$ be a family of $C^*$-algebras. Following Arveson [1], we let $\sum_{i \in \Lambda} A_i$ denote the direct sum of $\{A_i\}_{i \in \Lambda}$.

Lemma 5.4 ([10], Theorem I.10.8). Let $A$ be a $C^*$-subalgebra of $\mathcal{K}(\mathcal{H})$. Then there are Hilbert spaces $\mathcal{H}_0, \mathcal{H}_i$ for $i \in \Lambda$ and nonnegative integers $n_i$ so that

$$\mathcal{H} \cong \mathcal{H}_0 \oplus \left( \bigoplus_{i \in \Lambda} \mathcal{H}_i^{(n_i)} \right) \quad \text{and} \quad A \cong 0 \oplus \sum_{i \in \Lambda} \mathcal{K}(\mathcal{H}_i)^{(n_i)}.$$ 

Corollary 5.5. Let $T \in \mathcal{B}(\mathcal{H})$. If $T$ is essentially normal, then $T \cong N \oplus (\bigoplus_{i \in \Lambda} T_i^{(n_i)})$, where $N$ is normal, each $T_i \in \mathcal{B}(\mathcal{H}_i)$ is irreducible with $\mathcal{K}(\mathcal{H}_i) \subset C^*(T_i)$ and $T_i \not\cong T_j$ whenever $i \neq j$; moreover, $C^*(T) \cap \mathcal{K}(\mathcal{H}) \cong 0 \oplus \sum_{i \in \Lambda} \mathcal{K}(\mathcal{H}_i)^{(n_i)}$.

Proof. Denote $A = C^*(T) \cap \mathcal{K}(\mathcal{H})$. Then $A$ is a $C^*$-subalgebra of $\mathcal{K}(\mathcal{H})$. By Lemma 5.4 there are Hilbert spaces $\mathcal{H}_0, \mathcal{H}_i$ for $i \in \Lambda$ and nonnegative integers $n_i$ so that

$$\mathcal{H} \cong \mathcal{H}_0 \oplus \left( \bigoplus_{i \in \Lambda} \mathcal{H}_i^{(n_i)} \right) \quad \text{and} \quad A \cong 0 \oplus \sum_{i \in \Lambda} \mathcal{K}(\mathcal{H}_i)^{(n_i)}.$$ 

Then there exists an operator $A$ acting on $\tilde{\mathcal{H}} := \mathcal{H}_0 \oplus \left( \bigoplus_{i \in \Lambda} \mathcal{H}_i^{(n_i)} \right)$ so that $T \cong A$ and

$$C^*(A) \cap \mathcal{K}(\tilde{\mathcal{H}}) = 0 \oplus \sum_{i \in \Lambda} \mathcal{K}(\mathcal{H}_i)^{(n_i)}.$$ 

(5.1)

Since $C^*(A) \cap \mathcal{K}(\tilde{\mathcal{H}})$ is an ideal of $C^*(A)$, one can see that $A = T_0 \oplus \left( \bigoplus_{i \in \Lambda} T_i^{(n_i)} \right)$ with respect to the decomposition $\tilde{\mathcal{H}} = \mathcal{H}_0 \oplus \left( \bigoplus_{i \in \Lambda} \mathcal{H}_i^{(n_i)} \right)$, where $T_0 \in \mathcal{B}(\mathcal{H}_0)$ and $T_i \in \mathcal{B}(\mathcal{H}_i)$ for $i \in \Lambda$; moreover, $\mathcal{K}(\mathcal{H}_i) \subset C^*(T_i)$ for each $i \in \Lambda$. So each $T_i$ is irreducible. On the other hand, it is evident that $T_i \not\cong T_j$ whenever $i, j \in \Lambda$ and $i \neq j$.

Since $A \cong T$ is essentially normal, $A^*A - AA^*$ is compact. By (5.1), it follows that $T_0^*T_0 - T_0T_0^* = 0$, that is, $T_0$ is normal. Denote $N = T_0$. This completes the proof. $\square$

Lemma 5.6 ([28], Proposition 4.27). Let $S, T \in \mathcal{B}(\mathcal{H})$ and assume that $T$ is essentially normal. Then $T \cong_a S$ if and only if $S_{abnor} \cong T_{abnor}$, $\sigma_e(S) = \sigma_e(T)$ and $\dim \ker(\lambda - S) = \dim \ker(\lambda - T)$ for all $\lambda \in [\sigma(S) \cup \sigma(T)] \setminus \sigma_e(T)$. 

Proposition 5.7. Let $T \in \mathcal{B}(\mathcal{H})$ be essentially normal. Then the following are equivalent:

(i) $T$ is UET;
(ii) $T$ is AUET;
(iii) there exists an anti-automorphism $\varphi$ of $C^*(T)$ such that $\varphi(T) = T$ and $\text{rank } \varphi(X) = \text{rank } X$ for all $X \in C^*(T)$.

Proof. The implication “(i) $\implies$ (ii)” is clear, and the equivalence of (ii) and (iii) follows from Proposition 3.10

“(ii) $\implies$ (i)” Let $T = N \oplus A$, where $N \in \mathcal{B}(\mathcal{H}_1)$ and $A \in \mathcal{B}(\mathcal{H}_2)$ are the normal part and the abnorm part of $T$ respectively. Arbitrarily choose a conjugation $C_1$ on $\mathcal{H}_1$ and a conjugation $C_2$ on $\mathcal{H}_2$. Then $S := (C_1 N^* C_1) \oplus (C_2 A^* C_2)$ is an transpose of $T$. Since $T$ is AUET, we have $T \cong_a S$.

Note that $S_{nor} = C_1 N^* C_1$ and $S_{abnor} = C_2 A^* C_2$. Since $N$ is normal, it follows that $N \cong C_1 N^* C_1$, that is, $T_{nor} \cong S_{nor}$. Since $T$ is essentially normal and $T \cong_a S$, it follows from Lemma 5.6 that $T_{abnor} \cong S_{abnor}$. Thus $T \cong S$ and $T$ is UET. □

Proof of Theorem 5.1. By Corollary 3.5, Lemma 3.6, and Corollary 3.11, the sufficiency is obvious. We need only prove the necessity.

By Corollary 5.3, we may directly assume that $T = N \oplus \bigoplus_{i \in \Lambda} T_i^{(n_i)}$, where $N \in \mathcal{B}(\mathcal{H}_0)$ is normal, each $T_i \in \mathcal{B}(\mathcal{H}_i)$ is irreducible with $\mathcal{K}(\mathcal{H}_i) \subset C^*(T_i)$ and $T_i \not\cong T_j$ whenever $i \neq j$. Moreover, $C^*(T) \cap \mathcal{K}(\mathcal{H}) = 0 \oplus \sum_{i \in \Lambda} \mathcal{K}(\mathcal{H}_i)^{(n_i)}$. Since $T$ is essentially normal, it is obvious that $1 \leq n_i < \infty$ for all $i$.

Noting that $T$ is g-normal, it follows from Corollary 3.5 that $S := N \oplus \bigoplus_{i \in \Lambda} T_i$ is also g-normal. Denote $\tilde{\mathcal{H}} = \mathcal{H}_0 \oplus \bigoplus_{i \in \Lambda} \mathcal{H}_i$. Thus $S \in \mathcal{B}(\tilde{\mathcal{H}})$ and $C^*(S) \cap \mathcal{K}(\tilde{\mathcal{H}}) = 0 \oplus \sum_{i \in \Lambda} \mathcal{K}(\mathcal{H}_i)$. Moreover, by Lemma 1.7, the map $\rho$ defined by

$$\rho : C^*(S) \longrightarrow C^*(S), \quad p(S^*, S) \longrightarrow \tilde{\rho}(S, S^*)$$

is an anti-automorphism of $C^*(S)$. Note that $\rho^{-1} = \rho$.

Claim 1. rank $X = \text{rank } \rho(X)$ for all $X \in C^*(S)$.

Let $p(\cdot, \cdot)$ be a polynomial in two free variables. Noting that $S$ is essentially normal, it can be verified that $\rho(S^*, S) = \tilde{\rho}(S, S^*) + K$ for some $K \in \mathcal{K}(\tilde{\mathcal{H}})$. So $\pi(\rho(S^*, S)) = \pi(\rho(S, S^*))$, where $\pi$ is the canonical quotient map of $\mathcal{B}(\tilde{\mathcal{H}})$ onto $\mathcal{B}(\mathcal{H})/\mathcal{K}(\tilde{\mathcal{H}})$. Furthermore, one can deduce that $\pi(X) = \pi(\rho(X))$ for all $X \in C^*(S)$. It follows that an operator $X \in C^*(S)$ is compact if and only if $\rho(X)$ is compact.

We first note that if $P$ is a rank-one projection in $C^*(S)$, then $\rho(P)$ is a minimal projection of $C^*(S)$ and $\rho(P)$ is compact. Since $C^*(S) \cap \mathcal{K}(\mathcal{H}) = 0 \oplus \sum_{i \in \Lambda} \mathcal{K}(\mathcal{H}_i)$, one can see that $\rho(P)$ is of rank one. It shows that $\rho$ maps each rank-one projection in $C^*(S)$ to another rank-one projection. On the other hand, if $Y$ is a finite-rank positive operator in $C^*(S)$, then $Y$ can be written as $Y = \sum_{i=1}^m \lambda_i P_i$, where $\lambda_i > 0$, $\{P_j\}_{j=1}^n$ are pairwise orthogonal projections in $C^*(S)$ and rank $P_i = 1$ for all $i$. Noting that $\rho$ is an anti-automorphism, we have

$$\text{rank } \rho(Y) = \text{rank } \sum_{i=1}^m \lambda_i \rho(P_i) = \sum_{i=1}^m \text{rank } \rho(P_i) = \sum_{i=1}^m \text{rank } P_i = \text{rank } Y.$$  

Now fix an operator $X \in C^*(S)$. We shall prove that rank $\rho(X) = \text{rank } X$. For a proof by contradiction, we assume that rank $\rho(X) \neq \text{rank } X$. Noting that $\rho^{-1} = \rho$,
without loss of generality, we may directly assume that rank $X < \text{rank } \rho(X)$. Thus rank $X < \infty$. Denote $Z = \rho(X)$. Since $\rho$ is an anti-automorphism of $C^*(S)$, we have $\rho(X^*) = Z^*$ and $\rho(X^*X) = ZZ^*$. Using functional calculus, we obtain $\rho(|X|) = |Z^*|$. Since $|X|$ is positive and rank $|X| = \text{rank } X < \infty$, by the discussion in the last paragraph, we have

$$\text{rank } X = \text{rank } |X| = \text{rank } |Z^*| = \text{rank } Z,$$

a contradiction. This proves Claim 1.

**Claim 2.** There exists an anti-unitary operator $D$ on $\tilde{H}$ such that $\rho(X) = DX^*D^{-1}$ for $X \in C^*(S)$. In particular, $S = DS^*D^{-1}$ and $S^* = DSD^{-1}$.

Since $\rho$ is an anti-automorphism of $C^*(S)$ and $\rho(S) = S$, in view of Proposition 5.7 Claim 1 implies that $S$ is UET. By Lemma 3.8 there is an anti-unitary operator $D$ on $\tilde{H}$ such that $DS = S^*D$. Thus, given a polynomial $p(\cdot, \cdot)$ in two free variables, it is easy to see that

$$\rho(S, S^*) = Dp(S^*, S)D^{-1}.$$

It follows that $\rho(X) = DX^*D^{-1}$ for $X \in C^*(S)$. This proves Claim 2.

For convenience we denote $\mathcal{A} = C^*(S)$. Note that $DAD^{-1} = \mathcal{A}$ and $D^{-1}AD = \mathcal{A}$.

**Claim 3.** For each $i \in \Lambda$, there exists a unique $\tau_i \in \Lambda$ such that $D(\mathcal{H}_i) = \mathcal{H}_{\tau_i}$ and $D(\mathcal{H}_{\tau_i}) = \mathcal{H}_i$.

Now fix an $i \in \Lambda$. Arbitrarily choose a unit vector $e_i \in \mathcal{H}_i$ and set $P_i = e_i \otimes e_i$. Then $P_i \in C^*(S)$. Denote $Q_i = \rho(P_i)$ and $f_i = De_i$. For $x \in \mathcal{H}$, we have

$$Q_ix = DP_iD^{-1}x = D(\langle D^{-1}x, e_i \rangle e_i) = \langle e_i, D^{-1}x \rangle f_i = \langle x, De_i \rangle f_i = (f_i \otimes f_i)(x).$$

So we obtain $Q_i = f_i \otimes f_i$. Note that $\mathcal{A} \cap \mathcal{K}(\tilde{H}) = 0 \oplus \sum_{j \in \Lambda} \mathcal{K}(\mathcal{H}_j)$. Since $Q_i \in \mathcal{A}$ is of rank one, there exists $\tau_i \in \Lambda$ such that $Q_i \in \mathcal{K}(\mathcal{H}_{\tau_i})$. So $f_i \in \mathcal{H}_{\tau_i}$. Thus

$$\mathcal{H}_{\tau_i} = [\mathcal{A}f_i] = [\mathcal{A}(De_i)] = D(\langle D^{-1}AD \rangle e_i) = D[\mathcal{A}e_i] = D(\mathcal{H}_i).$$

Noting that $\rho = \rho^{-1}$, we have $DQ_iD^{-1} = P_i$ and $Df_i = \alpha_i e_i$ for some $\alpha_i \in \mathbb{C}$ with $|\alpha_i| = 1$. So

$$\mathcal{H}_i = [\mathcal{A}e_i] = [\mathcal{A}(Df_i)] = D([\langle D^{-1}AD \rangle f_i]) = D[\mathcal{A}f_i] = D(\mathcal{H}_{\tau_i}).$$

This proves Claim 3.

**Claim 4.** The map $\tau : i \mapsto \tau_i$ is bijective on $\Lambda$ and $\tau^{-1} = \tau$.

Let $i, j \in \Lambda$ with $i \neq j$. If $\tau_i = \tau_j$, then $\mathcal{H}_i = D(\mathcal{H}_{\tau_i}) = D(\mathcal{H}_{\tau_j}) = \mathcal{H}_j$, a contradiction. Given a $j \in \Lambda$, since $\mathcal{H}_j = D(\mathcal{H}_{\tau_j}) = \mathcal{H}_{\tau(\tau_j)}$, we have $j = \tau(\tau_j) = \tau^2(j)$. This means that $\tau$ is bijective and $\tau^{-1} = \tau$.

By Claim 4, $\tau$ induces a partition $\Lambda = \bigcup_{r \in \Gamma} \Lambda_r$, where each $\Lambda_r$ can be written as $\Lambda_r = \{j, \tau_j\}$ for some $j \in \Lambda$. So $S = N \oplus \bigoplus_{r \in \Gamma} \bigoplus_{j \in \Lambda_r} T_{\tau_j}^{(n_j)}$. Thus

$$T = N \oplus \left( \bigoplus_{r \in \Gamma} \bigoplus_{j \in \Lambda_r} T_{\tau_j}^{(n_j)} \right).$$

**Claim 5.** $T_j \cong T_{\tau_j}$ for each $j \in \Lambda$. 
Let \( j \in \Lambda \) be fixed. Denote \( D_j = D|_{\mathcal{H}_j} \). Then \( D_j : \mathcal{H}_j \to \mathcal{H}_{\tau_j} \) is an anti-unitary operator. Since \( DS = S^*D \), we have \( (D|_{\mathcal{H}_i})(S|_{\mathcal{H}_i}) = (S^*|_{\mathcal{H}_{\tau_j}})(D|_{\mathcal{H}_i}) \), that is, \( D_iT_j = T_{\tau_j}^*D_j \). Arbitrarily choose a conjugation \( E \) on \( \mathcal{H}_{\tau_j} \). Thus \( (ED_j)T_j = (ET_{\tau_j}^*E)(ED_j) \). Noting that \( ED_j \) is unitary, we obtain \( T_j \cong T_{\tau_j}^* \).

**Claim 6.** If \( i \in \Lambda \) and \( i \neq \tau_i \), then \( T_i \) is not UET.

In fact, if not, then \( T_i^t \cong T_i \). By Claim 5, \( T_i \cong T_{\tau_i}^t \). So we have \( T_i \cong T_{\tau_i} \), contradicting the hypothesis that \( T_i \not\cong T_s \) whenever \( l \neq s \).

Now we can conclude the proof. Let \( r \in \Gamma \) be fixed.

If \( \text{card } \Lambda_r = 1 \) and \( k \in \Lambda_r \), then \( \tau_k \). By Claim 5, \( T_{k} \cong T_{\tau_k}^t = T_{k}^t \). Hence \( T_{k} \) is an irreducible UET operator and \( \bigoplus_{k \in \Lambda_r} T_{i_k}^{(n_i)} = T_{k}^{(n_k)} \).

If \( \text{card } \Lambda_r = 2 \) and \( k \in \Lambda_r \), then \( k \neq \tau_k \). By Claim 6, \( T_k \) is not UET. So \( T_k \oplus T_{\tau_k} \cong T_k \oplus T_{k}^t \). Hence \( \bigoplus_{j \in \Lambda_r} T_{j}^{(n_j)} \) is unitarily equivalent to an operator with a form of \( A^{(m)} \oplus (A^t)^{(n)} \), where \( A \) is irreducible, not UET and \( m, n \geq 1 \).

In view of (5.2), we complete the proof. \( \square \)

6. ON ESSENTIALLY NORMAL OPERATORS WHICH ARE UET

The main result of this section is the following theorem which gives a canonical decomposition for essentially normal operators which are UET.

**Theorem 6.1.** Let \( T \in \mathcal{B}(\mathcal{H}) \) be essentially normal. Then the following are equivalent:

(i) \( T \cong T_t \);

(ii) \( T \cong_a T^t \);

(iii) \( T \) is unitarily equivalent to a direct sum of (some of the summands may be absent): normal operators, irreducible UET operators and operators with the form of \( A \oplus A^t \), where \( A \) is irreducible and not UET.

To give the proof of Theorem 6.1, we need to make some preparations. The proof of the following result is immediate, so we omit it.

**Lemma 6.2.** Let \( T \in \mathcal{B}(\mathcal{H}) \). Then \( T \) is UET if and only if \( T_{abnor} \) is UET.

**Proposition 6.3.** Let \( S \) be the unilateral shift on \( \mathcal{H} \) defined by

\[
Se_i = e_{i+1}, \forall i \geq 1,
\]

where \( \{e_i\}_{i=1}^{\infty} \) is an ONB of \( \mathcal{H} \). Set \( T = S^{(m)} \oplus (S^*)^{(n)} \), where \( 1 \leq m, n \leq \infty \). Then the following are equivalent:

(i) \( T \) is complex symmetric;

(ii) \( T \cong T^* \);

(iii) \( T \cong_a T^* \);

(iv) \( T \) is \( g \)-normal and rank \( p(T^*, T) = \text{rank } \overline{\nu}(T, T^*) \) for any polynomial \( p(\cdot, \cdot) \);

(v) \( m = n \).

**Proof.** We first note that \( T^* \) is a transpose of \( T \). By definition, the implications “(i) \( \implies \) (ii) \( \implies \) (iii)” are obvious.

“(iii) \( \implies \) (iv)”. Since \( T \) is AUET, by Proposition 3.10, there exists an anti-automorphism \( \varphi \) of \( C^*(T) \) such that \( \varphi(T) = T \) and rank \( \varphi(X) = \text{rank } X \) for all \( X \in C^*(T) \). For each polynomial \( p(\cdot, \cdot) \) in two free variables, one can see that \( \varphi(p(T^*, T)) = \overline{\nu}(T, T^*)^* \). Thus the conclusion follows.
we may also assume that \( \tilde{\rho}(T, T^*) = m \). By the hypothesis, we have \( m = n \).

“\((v)\) follows from (i)” Note that \( S^* \) is a transpose of \( S \). Then, by Lemma 3.6, \( S \oplus S^* \) is complex symmetric. Since \( m = n \), one can see that \( T = (S \oplus S^*)^{(n)} \) and \( T \) is complex symmetric.

\[ \square \]

**Question 6.4**. Let \( T \in \mathcal{B}(\mathcal{H}) \). If \( T \) is g-normal and rank \( p(T^*, T) = \mathrm{rank} \tilde{p}(T, T^*) \) for any polynomial \( p(\cdot, \cdot) \) in two free variables, then does it follow that \( T \) is AUET?

**Proof of Theorem 6.1**. By Proposition 5.7, the implication “\( (i) \Leftrightarrow (ii) \)” is obvious. The implication “\( (iii) \Leftrightarrow (i) \)” follows from Lemma 3.6.

“\( (ii) \Rightarrow (iii) \)” Assume that \( T \) is UET. In view of Lemma 6.2, it follows that \( T_{abnor} \) is UET. Thus we may directly assume that \( T \) is abnormal. By Corollary 5.5, we may also assume that \( T = \bigoplus_{i \in \Lambda} T_i^{(ni)} \), where each \( T_i \in \mathcal{B}(\mathcal{H}_i) \) is irreducible with \( \mathcal{K}(\mathcal{H}_i) \subset C^*(T_i) \) and \( T_i \neq T_j \) whenever \( i \neq j \). Moreover, \( C^*(T) \cap \mathcal{K}(\mathcal{H}) = \sum_{i \in \Lambda} \mathcal{K}(\mathcal{H}_i)^{(ni)} \).

Since \( T \) is g-normal, by Lemma 1.7, the map \( \varphi \) defined by

\[
\varphi : C^*(T) \rightarrow C^*(T), \quad p(T^*, T) \mapsto \tilde{p}(T, T^*)^*
\]

is an anti-automorphism of \( C^*(T) \). Since \( T \cong T^t \), there is an anti-unitary operator \( D \) on \( \mathcal{H} \) such that \( DT = T^*D \). It follows that \( \varphi(X) = DX^*D^{-1} \) for \( X \in C^*(T) \). So \( \text{rank } \varphi(X) = \text{rank } X \) for all \( X \in C^*(T) \).

Noting that \( T \) is g-normal, it follows from Corollary 3.5 that \( S := \bigoplus_{i \in \Lambda} T_i \) is also g-normal. Thus the map \( \rho \) defined by

\[
\rho : C^*(S) \rightarrow C^*(S), \quad p(S^*, S) \mapsto \tilde{p}(S, S^*)^*
\]

is an anti-automorphism of \( C^*(S) \). Denote \( \widetilde{\mathcal{H}} = \bigoplus_{i \in \Lambda} \mathcal{H}_i \). Then \( S \in \mathcal{B}(\widetilde{\mathcal{H}}) \) and \( C^*(S) \cap \mathcal{K}(\widetilde{\mathcal{H}}) = \sum_{i \in \Lambda} \mathcal{K}(\mathcal{H}_i) \).

Let \( X, Y \in C^*(S) \). Assume that \( X = \bigoplus_{i \in \Lambda} X_i \) and \( Y = \bigoplus_{i \in \Lambda} Y_i \) with respect to the decomposition \( \mathcal{H} = \bigoplus_{i \in \Lambda} \mathcal{H}_i \). By definition, if \( \rho(X) = Y \), then \( X, Y \in C^*(T) \) and \( \varphi(X) = Y \), where \( X = \bigoplus_{i \in \Lambda} X_i^{(ni)} \) and \( Y = \bigoplus_{i \in \Lambda} Y_i^{(ni)} \).

In the following, we are going to establish some facts about \( C^*(S) \) and \( \rho \). Since the proof follows the same lines as the proof of Theorem 5.1, we omit it.

(a) \( S \) is UET and there is an anti-unitary operator \( E \) on \( \mathcal{H} \) such that \( \rho(X) = EX^*E^{-1} \) for \( X \in C^*(S) \).

(b) For each \( i \in \Lambda \), choose a unit vector \( e_i \in \mathcal{H}_i \) and denote \( P_i = e_i \otimes e_i \). Then \( P_i \subset C^*(S) \). Denote \( Q_i = \rho(P_i) \) and \( f_i = Ee_i \) for \( i \in \Lambda \). One can verify that \( Q_i = f_i \otimes f_i \).

(c) For each \( i \in \Lambda \), there exists a unique \( \tau_i \in \Lambda \) such that \( \mathcal{H}_i = E(\mathcal{H}_{\tau_i}) \) and \( \mathcal{H}_{\tau_i} = E(\mathcal{H}_i) \). Thus \( f_i \in \mathcal{H}_{\tau_i} \) and \( Q_i \in \mathcal{K}(\mathcal{H}_{\tau_i}) \).

(d) The map \( \tau : i \mapsto \tau_i \) is bijective on \( \Lambda \) and \( \tau^{-1} = \tau \).

(e) \( T_i \cong T_{\tau_i}^t \) for each \( i \in \Lambda \).

(f) If \( i \in \Lambda \) and \( i \neq \tau_i \), then \( T_i \) is not UET.

For each \( i \in \Lambda \), note that \( P_i, Q_i \in C^*(S) \) and \( \rho(P_i) = Q_i \). Then, by definition, we have \( P_i^{(ni)}, Q_i^{(ni)} \subset C^*(T) \) and \( \varphi(P_i^{(ni)}) = Q_i^{(ni)} \). Since \( \text{rank } \varphi(X) = \text{rank } X \) for all \( X \in C^*(T) \), one can deduce that \( n_i = n_{\tau_i} \).
By statement (d), \( \tau \) induces a partition \( \Lambda = \bigcup_{r \in \Gamma} \Lambda_r \), where each \( \Lambda_r \) can be written as \( \Lambda_r = \{ j, \tau_j \} \) for some \( j \in \Lambda \). So \( S \) is the direct sum of \( \bigoplus_{i \in \Lambda_r} T_i, r \in \Gamma \). Thus

\[
T = \bigoplus_{r \in \Gamma} \bigoplus_{i \in \Lambda_r} T_i^{(n_i)}.
\]

Now we can conclude the proof. Let \( r \in \Gamma \) be fixed.

If \( \text{card} \Lambda_r = 1 \) and \( k \in \Lambda_r \), then \( k = \tau_k \). By statement (e), \( T_k \equiv T_k^\tau = T_k^\rho \).

Hence \( T_k \) is an irreducible UET operator and \( \bigoplus_{i \in \Lambda_r} T_i^{(n_i)} = T_k^{(n_k)} \).

If \( \text{card} \Lambda_r = 2 \) and \( k \in \Lambda_r \), then \( k \neq \tau_k \). Hence \( \bigoplus_{i \in \Lambda_r} T_i^{(n_i)} = T_k^{(n_k)} \oplus T_\tau^{(n_\tau_k)} \). Since we have proved that \( n_k = n_\tau_k \), it follows from statement (e) that \( \bigoplus_{i \in \Lambda_r} T_i^{(n_i)} \equiv (T_k \oplus T_\tau)^{(n_k)} \). By statement (f), \( \bigoplus_{i \in \Lambda_r} T_i^{(n_i)} \) is unitarily equivalent to a direct sum of operators with a form of \( A \oplus A' \), where \( A \) is irreducible and not UET.

In view of (6.1), we complete the proof. \( \square \)

7. PROOF OF THEOREM 2.8

This section is mainly devoted to the proof of Theorem 2.8.

**Lemma 7.1** ([16], Theorem 4). Let \( T \in \mathcal{K}(\mathcal{H}) \) and \( \{ C_n \}_{n=1}^\infty \) be a sequence of conjugations on \( \mathcal{H} \) satisfying \( C_n T^* C_n - T \to 0 \) as \( n \to \infty \). If \( P \) is the projection of \( \mathcal{H} \) onto \( \text{ran} T + \text{ran} T^* \), then there exists a subsequence \( \{ n_j \}_{j=1}^\infty \) of \( \mathbb{N} \) such that \( \{ PC_{n_j} | \text{ran} P \}_{j=1}^\infty \) converges to a conjugation on \( \text{ran} P \).

Using a similar argument as in the proof of Lemma 7.1 one can prove the following result.

**Corollary 7.2.** Let \( T \in \mathcal{K}(\mathcal{H}) \) and \( \{ C_n \}_{n=1}^\infty \) be a sequence of conjugations on \( \mathcal{H} \) satisfying \( C_n T^* C_n + T \to 0 \) as \( n \to \infty \). If \( P \) is the projection of \( \mathcal{H} \) onto \( \text{ran} T + \text{ran} T^* \), then there exists a subsequence \( \{ n_j \}_{j=1}^\infty \) of \( \mathbb{N} \) such that \( \{ PC_{n_j} | \text{ran} P \}_{j=1}^\infty \) converges to a conjugation on \( \text{ran} P \).

**Theorem 7.3.** If \( T \in \mathcal{B}(\mathcal{H}) \) is essentially normal, then \( T \in \text{CSO} \) if and only if \( T \in \text{CSO} \).

**Proof.** We need only prove the necessity. Since \( T \in \text{CSO} \), it follows from the proof of Lemma 6.12 that there exists a sequence \( \{ C_n \} \) of conjugations on \( \mathcal{H} \) such that \( C_n T^* C_n \to T \) as \( n \) tends to \( \infty \).

Now let \( m, l \geq 1 \) be fixed. Let \( P_{m,l} \) be the projection of \( \mathcal{H} \) onto the subspace spanned by \( \text{ran}[T^{*m}, T^l] \) and \( \text{ran}[T^{*m}, T^l]^* \). It is easy to verify that

\[
C_n[T^{*m}, T^l]C_n + [T^{*m}, T^l]^* \to 0 \quad \text{as} \quad n \to \infty.
\]

Since \( T \) is essentially normal, it follows that \( [T^{*m}, T^l] \in \mathcal{K}(\mathcal{H}) \). Then, by Corollary 7.2, \( \{ C_n \} \) has a subsequence \( \{ C_{n_j} \} \) such that \( \{ P_{m,l} C_{n_j} | \text{ran} P_{m,l} \} \) converges to a conjugation on \( \text{ran} P_{m,l} \). Hence, for any \( x \in \text{ran} P_{m,l} \), \( \{ C_{n_j} x \}_{j \geq 1} \) converges to a vector in \( \text{ran} P_{m,l} \).

Since \( \{ \text{ran} P_{m,l} : m, l \geq 1 \} \) is at most denumerable, applying the diagonal process we can find a subsequence \( \{ n_j \} \) of \( \mathbb{N} \) satisfying: for any \( m, l \geq 1 \),

\[
x \in \text{ran} P_{m,l} \implies \{ C_{n_j} x \} \text{ converges to a vector in } \text{ran} P_{m,l}.
\]
Denote by $M_0$ the subset of $\mathcal{H}$ consisting of all finite linear combinations of vectors in $\bigcup_{m,l \geq 1} \text{ran}P_{m,l}$. Then for each $x \in M_0$ the sequence $\{C_{n_j}x\}$ converges to a vector in $M_0$.

Denote $M = \overline{M_0}$. Then $M$ is in fact the subspace of $\mathcal{H}$ spanned by all $\text{ran}P_{m,l}$ $(m, l \geq 1)$; moreover, $M, M^\perp$ both reduce $T$, $A := T|_M$ is abnormal and $N := T|_{M^\perp}$ is normal. In order to complete the proof, we need only prove that $(\text{ii})$ is normal. We give the rest of the proof by proving the following three claims.

Claim 1. For each $x \in M$, $\{C_{n_j}x\}$ converges to a vector in $M$.

Note that $\{C_{n_j}x\}$ converges to a vector in $M_0$ for each $x \in M_0$. Define $E_x = \lim_j C_{n_j}x$ for $x \in M_0$. Thus $E$ is a conjugate-linear map on $M_0$. Noting that $\|Ex\| = \|x\|$ for each $x \in M_0$ and $M_0$ is a dense subset of $M$, $E$ can be extended to an isometric map on $M$, denoted by $C_M$. So $C_Mx = \lim_j C_{n_j}x \in M$ for each $x \in M$.

Claim 2. $C_M$ is a conjugation on $M$.

It is obvious that $C_M$ is conjugate-linear and isometric. By the polarization identity, it implies $\langle C_Mx, C_My \rangle = \langle y, x \rangle$ for all $x, y \in M$. So it suffices to prove that $C_M^2x = x$ for all $x \in M$. Now fix an $x \in M$. Since $x$ and $C_Mx$ both belong to $M$, given $\varepsilon > 0$, there exists $j_0$ such that $\|C_{n_{j_0}}x - C_Mx\| + \|C_{n_{j_0}}C_Mx - C_M^2x\| < \varepsilon$. Then

$$\|C_M^2x - x\| \leq \|C_M^2x - C_{n_{j_0}}C_Mx\| + \|C_{n_{j_0}}C_Mx - C_M^2x\|$$

$$= \|C_M^2x - C_{n_{j_0}}C_Mx\| + \|C_Mx - C_{n_{j_0}}x\| < \varepsilon.$$

Since $\varepsilon$ is arbitrary, we deduce that $C_M^2x = x$. Thus $C_M$ is a conjugation on $M$.


Fix an $x \in M$. Since $\lim_j C_{n_j}x = C_Mx \in M$, we have

$$A^*x = T^*x = \lim_j C_{n_j}TC_{n_j}x = \lim_j C_{n_j}TC_Mx$$

$$= \lim_j C_{n_j}AC_Mx = C_MAC_Mx.$$ 

It follows that $A^* = C_MA_CM$, that is, $A$ is complex symmetric.

Proposition 7.4. Let $T = A^{(n)}$, where $n \in \mathbb{N}$ and $A \in B(\mathcal{H})$ is irreducible. If $M$ is a nonzero reducing subspace of $T$, then the following are equivalent:

(i) $T|_M \cong A$;
(ii) $T|_M$ is irreducible;
(iii) there exists a nonzero $(\alpha_1, \cdots, \alpha_n) \in \mathbb{C}^n$ such that $M = \{\bigoplus_{i=1}^n \alpha_i \xi : \xi \in \mathcal{H}\}$.

Proof. The implication “(i)$\implies$(ii)” is obvious.

“(ii)$\implies$(iii)”. Denote by $P$ the projection of $\mathcal{H}^{(n)}$ onto $M$. Then $PT = TP$ and $PT^* = T^*P$. We may assume that $P$ admits the following matrix representation:

$$P = 
\begin{bmatrix}
P_{1,1} & P_{1,2} & \cdots & P_{1,n} \\
P_{2,1} & P_{2,2} & \cdots & P_{2,n} \\
\vdots & \vdots & \ddots & \vdots \\
P_{n,1} & P_{n,2} & \cdots & P_{n,n}
\end{bmatrix}
\mathcal{H}$$

where $\mathcal{H}$ is a nonzero reducing subspace of $T$. Then for each $x \in M$ the sequence $\{C_{n_j}x\}$ converges to a vector in $M_0$. 
It follows that \( AP_{i,j} = P_{i,j}A \) and \( AP_{i,j}^* = P_{i,j}^*A \) for all \( i, j \). Let \( i, j \) be fixed. Thus
\[
AP_{i,j}P_{i,j}^* = P_{i,j}P_{i,j}^*A \quad \text{and} \quad AP_{i,j}^*P_{i,j} = P_{i,j}^*P_{i,j}A.
\]
Noting that \( P_{i,j}P_{i,j}^* \) is positive and \( A \) is irreducible, \( \sigma(P_{i,j}P_{i,j}^*) \) is a singleton set. Similarly \( \sigma(P_{i,j}^*P_{i,j}) \) is also a singleton set. So there exists unitary \( U \in \mathcal{B}(\mathcal{H}) \) and \( \lambda \in \mathbb{C} \) such that \( P_{i,j} = \lambda U \). If \( \lambda \neq 0 \), then we obtain \( AU = UA \); by the irreducibility of \( A \), we obtain \( U = e^{i\theta}I \) for some \( \theta \in \mathbb{R} \), where \( I \) is the identity operator on \( \mathcal{H} \). Thus we conclude that there exists \( \lambda_{i,j} \in \mathbb{C} \) such that \( P_{i,j} = \lambda_{i,j}I \). Set \( R = [\lambda_{i,j}]_{1 \leq i, j \leq n} \). Then \( R \) is a nonnegative-definite matrix, \( R^2 = R \) and \( P = R \otimes I \).

We claim that \( \text{rank } R = 1 \). In fact, if not, then we can choose an \( n \times n \) nonnegative-definite matrix \( R_1 \) such that \( R_1^2 = R_1 \), \( \text{rank } R_1 = 1 \), and \( R_1 \leq R \). Then \( \text{ran} R_1 \otimes I \) is a nonzero reducing subspace of \( T \) and \( \text{ran} R_1 \otimes I \subseteq \text{ran} P \), contradicting the fact that \( T|_{\text{ran} P} \) is irreducible. So we have \( \text{rank } R = 1 \). Then there exists \( 1 \leq j_0 \leq n \) such that each column of \( R \) is a scalar multiple of the \( j_0 \)-th column. For \( 1 \leq i \leq n \), set \( \alpha_i = \lambda_{i,j_0} \). Then one can see that
\[
M = \text{ran} P = \{ \bigoplus_{i=1}^n \alpha_i x : x \in \mathcal{H} \}.
\]

“(iii)\( \Rightarrow \)(i)”. Set \( \delta = \sqrt{\sum_{i=1}^n |\alpha_i|^2} \). For \( \xi \in \mathcal{H} \), define \( U \xi = \bigoplus_{i=1}^n \frac{\alpha_i}{\delta} \xi \). Then one can see that \( U : \mathcal{H} \rightarrow M \) is a unitary operator. For each \( \xi \in \mathcal{H} \), we have
\[
UA\xi = \bigoplus_{i=1}^n \frac{\alpha_i}{\delta} A\xi = (T|_M)\bigoplus_{i=1}^n \frac{\alpha_i}{\delta} \xi = (T|_M)U\xi.
\]
This implies that \( T|_M \cong A \). \( \square \)

**Corollary 7.5.** Let \( M = A^{(n)} \), where \( n \in \mathbb{N} \) and \( A \in \mathcal{B}(\mathcal{H}) \) is irreducible. If \( M \) is a nonzero reducing subspace of \( T \), then there exists \( 1 \leq m \leq n \) such that \( T|_M \cong A^{(m)} \).

**Proof.** Let \( P \) be the projection of \( \mathcal{H}^{(n)} \) onto \( M \). It can be seen from the proof of “(ii)\( \Rightarrow \)(iii)” in Proposition 7.4 that \( P = R \otimes I \), where \( I \) is the identity operator on \( \mathcal{H} \) and \( R \) is an \( n \times n \) nonnegative-definite matrix satisfying \( R^2 = R \).

Denote \( m = \text{rank } R \). So \( 1 \leq m \leq n \). Then there exist nonnegative-definite matrices \( R_1, \cdots, R_m \) with rank \( R_i = 1 \) and \( R_i^2 = R_i \) for all \( 1 \leq i \leq m \) such that \( R = \sum_{i=1}^m R_i \) and \( R_iR_j = 0 \) whenever \( i \neq j \). For each \( 1 \leq i \leq m \), set \( P_i = R_i \otimes I \). Then \( P_iP_j = 0 \) whenever \( i \neq j \), \( P_iT = TP_i \) and \( \text{ran} P_i \) reduces \( T \) for each \( i \). Hence \( M = \bigoplus_{i=1}^m \text{ran} P_i \). Let \( i \) be fixed. Since \( P_i = R_i \otimes I \), there exists a nonzero \( (\alpha_1, \cdots, \alpha_n) \in \mathbb{C}^n \) such that
\[
\text{ran} P_i = \{ \bigoplus_{j=1}^n \alpha_j \xi : \xi \in \mathcal{H} \}.
\]

By Proposition 7.4, \( T|_{\text{ran} P_i} \cong A \). Thus we conclude that \( T|_M \cong A^{(m)} \). \( \square \)

**Proposition 7.6.** Let \( T = A \oplus A^t \), where \( A \in \mathcal{B}(\mathcal{H}) \) is irreducible. If \( A \) is not complex symmetric, then \( T \) is a minimal complex symmetric operator.

**Proof.** It is obvious that \( T \) is complex symmetric. Assume that \( M \) is a nontrivial reducing subspace of \( T \). Denote by \( P \) the projection of \( \mathcal{H}^{(2)} \) onto \( M \). We shall prove that \( T|_M \) is not complex symmetric.
Case 1. $A$ is UET. In this case, we have $T \cong A^{(2)}$. We may directly assume that $T = A^{(2)}$. Since $A$ is irreducible and $P$ is a projection commuting with $T$, using a similar argument as in the proof of “(ii)$\Rightarrow$(iii)” in Proposition 7.4, one can prove that $P = R \otimes I$, where $I$ is the identity operator on $\mathcal{H}$ and $R$ is a $2 \times 2$ nonnegative-definite matrix satisfying $R^2 = R$.

Since $\text{ran} P \neq \{0\}$ and $\text{ran} P \neq \mathcal{H}^{(2)}$, we have rank $R = 1$. Then there exists a nonzero $(\alpha_1, \alpha_2) \in \mathbb{C}^2$ such that

$$M = \{ \bigoplus_{i=1}^{2} \alpha_i \xi : \xi \in \mathcal{H} \}.$$ 

By Proposition 7.4 $T|_{M} \cong A$. Since $A$ is not complex symmetric, this completes the proof in Case 1.

Case 2. $A$ is not UET. In this case, we have $A \not\cong A^t$. For convenience we write

$$T = \begin{bmatrix} A & 0 \\ 0 & A^t \end{bmatrix}_{\mathcal{H}_1 \mathcal{H}_2},$$

where $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$. Then, by [12, Theorem 2.1], $T$ has only four reducing subspaces. Then $\{0\}, \mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_1 \oplus \mathcal{H}_2$ are all reducing subspaces of $T$. Since $M$ is nontrivial, we have either $M = \mathcal{H}_1$ or $M = \mathcal{H}_2$. Hence $T|_{M} = A$ or $T|_{M} = A^t$. Since $A$ is not complex symmetric, it follows that $T|_{M}$ is not complex symmetric. □

In view of [47, Theorems 3.1 and 4.1], the following result immediately follows from Propositions 4.14 and 7.6.

**Corollary 7.7.** Let $T \in B(\mathcal{H})$ be a weighted shift (unilateral or bilateral) and assume that $T \in \text{CSO}$. Then $T$ is either completely complex symmetric or a direct sum of the following two kinds of minimal complex symmetric operators: irreducible complex symmetric operators and operators with a form of $A \oplus A^t$, where $A$ is irreducible and not complex symmetric.

**Proposition 7.8.** Let $T = A^{(n)}$, where $n \in \mathbb{N}$ and $A \in B(\mathcal{H})$ is irreducible. Then $T$ is complex symmetric if and only if exactly one of the following holds:

(i) $A$ is complex symmetric;

(ii) $n$ is even, and $A$ is UET and not complex symmetric.

**Proof.** “$\Leftarrow$”. If $A$ is complex symmetric, then the conclusion is evident. If $A$ is UET and $n$ is even, then

$$T = A^{(n)} = (A \oplus A)^{(n/2)} \cong (A \oplus A^t)^{(n/2)}.$$ 

By Lemma 3.6 $T$ is complex symmetric.

“$\Rightarrow$”. Now we assume that $T$ is complex symmetric and $A$ is not complex symmetric. We shall prove that $A$ is UET and $n$ is even. By the hypothesis, there is a conjugation $C$ on $\mathcal{H}^{(n)}$ such that $CTC = T^*$. For convenience, we write

$$T = \begin{bmatrix} A & \cdot & \cdot & \cdot \\ \cdot & A & \cdot & \cdot \\ \cdot & \cdot & \ddots & \cdot \\ \cdot & \cdot & \cdot & A \end{bmatrix}_{\mathcal{H}_1 \mathcal{H}_2 \cdots \mathcal{H}_n},$$

where $\mathcal{H}_1 = \cdots = \mathcal{H}_n = \mathcal{H}$. 

Denote $M = C(\mathcal{H}_1)$. Since $C^{-1} = C$, one can see that $\mathcal{H}_1 = C(M)$. So we have

$$T(M) = CT^*C(M) = CT^*(\mathcal{H}_1) \subset C(\mathcal{H}_1) = M$$

and

$$T^*(M) = CTC(M) = CT(\mathcal{H}_1) \subset C(\mathcal{H}_1) = M.$$ 

It follows that $M$ is a nonzero reducing subspace of $T$. Denote $D = C|_M$. Then $D : M \to \mathcal{H}_1$ is an anti-unitary operator. Since $CT = T^*C$, we have $(C|_M)(T|_M) = (T^*|_{\mathcal{H}_1})(C|_M)$, that is, $D(T|_M) = A^*D$. Arbitrarily choose a conjugation $E$ on $\mathcal{H}_1$. Thus $T|_M = D^{-1}A^*D = (D^{-1}E)(EA^*E)(ED)$. Noting that $ED$ is unitary and $(ED)^{-1} = D^{-1}E$, we obtain $T|_M \cong EA^*E$. So $T|_M$ is irreducible and, by Proposition 7.4, we have $T|_M \cong A$; furthermore, we have $A \cong EA^*E$ (i.e., $A$ is UET). Also there exists a nonzero $(\alpha_1, \cdots, \alpha_n) \in \mathbb{C}^n$ such that

$$M = \{ \bigoplus_{i=1}^{n} \alpha_ix : x \in \mathcal{H} \}.$$ 

Note that $C(\mathcal{V}(M, \mathcal{H}_1)) = \mathcal{V}(M, \mathcal{H}_1)$ and $C(\{M, \mathcal{H}_1\}^\perp) = \{M, \mathcal{H}_1\}^\perp$. Thus $\mathcal{V}(M, \mathcal{H}_1)$ is a common reducing subspace of $T$ and $C$. So $T|_{\mathcal{V}(M, \mathcal{H}_1)}$ and $T|_{\mathcal{H}_1}^\perp$ are both complex symmetric. Also it can be seen that $\mathcal{V}(M, \mathcal{H}_1) = \mathcal{H}_1 \oplus M_1$, where $M_1 = \{ \bigoplus_{i=2}^{n} \alpha_ix : x \in \mathcal{H}_i \} \subset \bigoplus_{i=2}^{n} \mathcal{H}_i$.

We claim that $(\alpha_2, \cdots, \alpha_n) \neq 0$. Otherwise, we have $M_1 = \{ 0 \}$ and $\mathcal{H}_1 = M$. Thus

$$T|_{\mathcal{V}(M, \mathcal{H}_1)} = T|_{\mathcal{H}_1} = A$$

is complex symmetric, contradicting the hypothesis that $A$ is not complex symmetric. Thus $M_1 \neq \{ 0 \}$. It is easy to see that $M_1$ is a nonzero reducing subspace of $T$ and, by Proposition 7.4, we have $T|_{M_1} \cong A$.

By Corollary 7.8, there exists some positive integer $m$ less than $n$ such that $T|_{\mathcal{H}_1}^\perp \cong A^{(m)}$. So $T \cong A^{(2)} \oplus (T|_{\mathcal{H}_1}^\perp) = A^{(2+m)}$. Since $T = A^{(n)}$ and $A$ is irreducible, one can deduce that $m = n - 2$. This shows that $T|_{\mathcal{H}_1}^\perp \cong A^{(n-2)}$. Note that $T|_{\mathcal{H}_1}^\perp$ is complex symmetric. If $n$ is odd, then, using an inductive argument, it can eventually be proved that $A$ is complex symmetric, contradicting the hypothesis. So $n$ is even. Since we have proved that $A$ is UET, this completes the proof.

**Remark 7.9.** Let $T = A^{(n)}$, where $n \in \mathbb{N}$ and $A \in B(\mathcal{H})$ is irreducible. Assume that $T$ is complex symmetric. Then $T$ is a direct sum of minimal complex symmetric operators. In fact, if $A$ is complex symmetric, the conclusion is obvious; if $A$ is not complex symmetric, then, by Proposition 7.8, $A$ is UET and $n$ is even. So

$$T = A^{(n)} = (A \oplus A)^{(n/2)} \cong (A \oplus A^t)^{(n/2)}.$$ 

By Proposition 7.6, $A \oplus A^t$ is a minimal complex symmetric operator. So, in this case, $T$ is also a direct sum of minimal complex symmetric operators.

**Proof of Theorem 2.8.** By Theorem 7.8, (i) and (ii) are equivalent. “(iii)⇒(ii)” follows from Lemmas 3.3 and 3.6. It suffices to prove “(ii)⇒(iii)”.

By Lemma 3.2, we may directly assume that $T$ is abnormal. Then, by Corollary 5.5, we may also assume that $T = \bigoplus_{i \in \Lambda} T_i^{(n_i)}$, where each $T_i \in B(\mathcal{H}_i)$ is irreducible with $K(\mathcal{H}_i) \subset C^*(T_i)$ and $T_i \not\cong T_j$ whenever $i \neq j$. Moreover, $C^*(T) \cap K(\mathcal{H}) = \sum_{i \in \Lambda} K(\mathcal{H}_i)^{(n_i)}$. 
Since $T$ is g-normal, by Lemma 1.7, the map defined by
\[
\varphi : C^*(T) \rightarrow C^*(T),
\]
\[
p(T*, T) \mapsto \overline{p(T, T*)}
\]
is an anti-automorphism of $C^*(T)$. Since $T$ is complex symmetric, there is a conjugation $D$ on $\mathcal{H}$ such that $DT = T^* D$. It follows that $\varphi(X) = DX^* D$ for $X \in C^*(T)$. So rank $X = \text{rank } \varphi(X)$ for all $X \in C^*(T)$.

Noting that $T$ is g-normal, it follows from Corollary 3.5 that $S := \bigoplus_{i \in \Lambda} T_i$ is also g-normal. Thus the rank of $\rho$ defined by
\[
\rho : C^*(S) \rightarrow C^*(S),
\]
\[
p(S^*, S) \mapsto \overline{p(S, S*)}
\]
is an anti-automorphism of $C^*(S)$. Denote $\tilde{\mathcal{H}} = \bigoplus_{i \in \Lambda} \mathcal{H}_i$. Then $S \in \mathcal{B}(\tilde{\mathcal{H}})$ and $C^*(S) \cap K(\tilde{\mathcal{H}}) = \sum_{i \in \Lambda} K(\mathcal{H}_i)$.

Let $X, Y \in C^*(S)$. Assume that $X = \bigoplus_{i \in \Lambda} X_i$ and $Y = \bigoplus_{i \in \Lambda} Y_i$ with respect to the decomposition $\mathcal{H} = \bigoplus_{i \in \Lambda} \mathcal{H}_i$. By definition, if $\rho(X) = Y$, then $X, Y \in C^*(T)$ and $\varphi(X) = Y$, where $X = \bigoplus_{i \in \Lambda} X_i^{(n_i)}$ and $Y = \bigoplus_{i \in \Lambda} Y_i^{(n_i)}$.

Denote $A = C^*(S)$. Just as we have proved in the proof of Theorem 5.1, the following statements hold:

(a) $S$ is UET and hence there is an anti-unitary operator $E$ on $\tilde{\mathcal{H}}$ such that $\rho(X) = EX^* E^{-1}$ for $X \in A$. Then $EA E^{-1} = A$ and $E^{-1} AE = A$.
(b) For each $i \in \Lambda$, there exists a unique $\tau_i \in A$ such that $\mathcal{H}_i = E(\mathcal{H}_{\tau_i})$ and $\mathcal{H}_{\tau_i} = E(\mathcal{H}_i)$.
(c) The map $\tau : i \mapsto \tau_i$ is bijective on $\Lambda$ and $\tau^{-1} = \tau$.
(d) $T_i \cong T_{\tau_i}^t$ for each $i \in \Lambda$.
(e) If $i \in \Lambda$ and $i \neq \tau_i$, then $T_i$ is not UET.

By the above statements, we have the following claim.

Claim 1. If $i \in \Lambda$ and $x \in \mathcal{H}_i$ with $\|x\| = 1$, then $\rho(x \otimes x) \in K(\mathcal{H}_{\tau_i})$.

Fix an $i \in \Lambda$ and a unit vector $x \in \mathcal{H}_i$. Denote $y = Ex$. In view of (b), we obtain $y \in \mathcal{H}_{\tau_i}$. Note that $\rho(X) = EX^* E^{-1}$ for all $X \in A$. Thus, for each $z \in \tilde{\mathcal{H}}$, we have
\[
\rho(x \otimes x)(z) = E(x \otimes x)E^{-1}(z) = E((E^{-1}z, x)x) = (x, E^{-1}z)Ex = (z, Ex)y = (y \otimes y)(z).
\]
It follows that $\rho(x \otimes x) = y \otimes y \in K(\mathcal{H}_{\tau_i})$. This proves Claim 1.

Claim 2. $n_i = n_{\tau_i}$ and $D(\mathcal{H}_i^{(n_i)}) = \mathcal{H}_{\tau_i}^{(n_{\tau_i})}$ for all $i \in \Lambda$.

Let $i \in \Lambda$ be fixed. Arbitrarily choose a rank-one projection $P \in K(\mathcal{H}_i)$. Then, by Claim 1, $Q := \rho(P)$ is a rank-one projection in $K(\mathcal{H}_{\tau_i})$. Noting that $P^{(n_i)}, Q^{(n_{\tau_i})} \in C^*(T)$, we have
\[
DP^{(n_i)}D = \varphi(P^{(n_i)}) = Q^{(n_{\tau_i})} \in K(\mathcal{H}_{\tau_i}^{(n_{\tau_i})}).
\]
Thus rank $P^{(n_i)} = \text{rank } Q^{(n_{\tau_i})}$. So $n_i = n_{\tau_i}$ and $D(\text{ran}P^{(n_i)}) \subset \text{ran}Q^{(n_{\tau_i})} \subset \mathcal{H}_{\tau_i}^{(n_{\tau_i})}$. Since $P$ was arbitrarily chosen in $K(\mathcal{H}_i)$, we deduce that $D(\mathcal{H}_i^{(n_i)}) \subset \mathcal{H}_{\tau_i}^{(n_{\tau_i})}$.
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A \(C^\ast\)-algebra approach to CSOS


School of Mathematical Sciences, Fudan University, Shanghai 200433, People’s Republic of China

E-mail address: kgyuo@fudan.edu.cn

Department of Mathematics, Jilin University, Changchun 130012, People’s Republic of China

E-mail address: jiyq@jlu.edu.cn

Department of Mathematics, Jilin University, Changchun 130012, People’s Republic of China

E-mail address: senzhu@163.com