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ON THE BOUNDEDNESS OF CERTAIN BILINEAR OSCILLATORY INTEGRAL OPERATORS

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Abstract. We prove the global $L^2 \times L^2 \to L^1$ boundedness of bilinear oscillatory integral operators with amplitudes satisfying a Hörmander-type condition and phases satisfying appropriate growth as well as the strong non-degeneracy conditions. This is an extension of the corresponding result of R. Coifman and Y. Meyer for bilinear pseudodifferential operators, to the case of oscillatory integral operators.

1. Introduction

A *bilinear oscillatory integral operator* T_σ^φ is an operator which is defined to act on Schwartz functions f and g by the formula

$$T_\sigma^\varphi(f,g)(x) = \iint \sigma(x,\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{i\varphi(x,\xi,\eta)}\,đ\xi đ\eta, \tag{1}$$

where đ is Lebesgue measure normalised by the factor $(2\pi)^{-n}$ and

$$\widehat{f}(\xi) = \int f(x)e^{-ix.\xi}\,\mathrm{d}x$$

is the Fourier transform of f. The function $\sigma\colon \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ is called the *amplitude* or the *symbol* of T_σ^φ and $\varphi\colon \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is the *phase function* or *phase*.

Objects closely related to oscillatory integral operators are Fourier integral operators. Indeed *bilinear Fourier integral operators* take the same form (1), but the difference between them and oscillatory integral operators lies in the assumptions made on the phase functions. Phases of Fourier integral operators are assumed to be homogeneous of degree one in the frequency (ξ and η) variables (and consequently, even when assumed to be otherwise smooth, have a singularity at $\xi, \eta = 0$), but for oscillatory integral operators there is no such homogeneity assumption on the phases.

An investigation of the boundedness of bilinear Fourier integral operators with phase functions of the form $\varphi(x,\xi,\eta) = \varphi_1(x,\xi)+\varphi_2(x,\eta)$ was made by L. Grafakos and M. Peloso [11], in which case it is convenient to write

$$T_\sigma^{\varphi_1,\varphi_2} := T_\sigma^\varphi.$$

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Of particular interest to us is Proposition 2.4 in [11]. There the authors assume that $\sigma(x,\xi,\eta)$ satisfies the estimate

$$\begin{aligned}&|\partial_\xi^\alpha\partial_\eta^\beta\partial_x^\gamma\sigma(x,\xi,\eta)|\leqslant C_{\alpha,\beta,\gamma}(1+|\xi|+|\eta|)^{m-|\alpha|-|\beta|}\\&\text{for } m\in\mathbb{R} \text{ and each triple of multi-indices } \alpha,\beta,\gamma,\end{aligned}\tag{2}$$

and is compactly supported in the x variable. They also assume the phase functions $\varphi_j(x,\xi)\in\mathcal{C}^\infty(\mathbb{R}^n\times(\mathbb{R}^n\setminus\{0\}))$ are homogeneous of degree one in the frequency variable ξ and verify the *non-degeneracy condition*

$$\left|\det\,\partial^2_{x,\xi}\varphi_j(x,\xi)\right|\neq 0\tag{3}$$

for $j=1,2$. They then deduce that the bilinear operator (1) is bounded from $L^{q_1}\times L^{q_2}\to L^r$ with $\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{r}$ and $1<q_1,\, q_2<2$, provided that the order

$$m\leqslant -(n-1)\left(\left(\frac{1}{q_1}-\frac{1}{2}\right)+\left(\frac{1}{q_2}-\frac{1}{2}\right)\right).\tag{4}$$

This result for Fourier integral operators was extended to the full Banach range of exponents $1\leqslant q_1,\, q_2\leqslant\infty$, and the assumption of compact support in x was eliminated by S. Rodríguez-López and W. Staubach in [17]. This result, contained in Theorem 5.12 of [17], concentrates on amplitudes which were only measurable in x, replacing (2) with

$$\left\|\partial_\xi^\alpha\partial_\eta^\beta\sigma(\cdot,\xi,\eta)\right\|_{L^\infty(\mathbb{R}^n)}\leqslant C_{\alpha,\beta}(1+|\xi|+|\eta|)^{m-|\alpha|-|\beta|}.\tag{5}$$

As one would expect for a global result, the non-degeneracy condition (3) was replaced with the *strong non-degeneracy condition*

$$\left|\det\,\partial^2_{x,\xi}\varphi_j(x,\xi)\right|\geqslant c_j>0\tag{6}$$

for $j=1,2$. If one, in addition, assumes that φ_1 and φ_2 are positively homogeneous (of degree 1) real-valued functions in $C^\infty(\mathbb{R}^n\times\mathbb{R}^n\setminus\{0\})$ satisfying (6) and the growth condition $\sup_{(x,\,\xi)\in\mathbb{R}^n\times\mathbb{R}^n\setminus\{0\}}|\xi|^{-1+|\alpha|}|\partial_\xi^\alpha\partial_x^\beta\varphi(x,\xi)|\leqslant C_{\alpha,\beta}$, for any pair of multi-indices α and β with $|\alpha|+|\beta|\geqslant 2$, then the corresponding bilinear operator $T_\sigma^{\varphi_1,\varphi_2}$ is bounded from $L^{q_1}\times L^{q_2}\to L^r$ provided that $\frac{1}{r}=\frac{1}{q_1}+\frac{1}{q_2}$, $1\leqslant q_1,\, q_2\leqslant\infty$ and

$$m<-(n-1)\left(\left|\frac{1}{q_1}-\frac{1}{2}\right|+\left|\frac{1}{q_2}-\frac{1}{2}\right|\right).\tag{7}$$

The change of the inequality in (4) to the strict inequality in (7) is typical when considering amplitudes only measurable in x satisfying (5) rather than smooth amplitudes satisfying (2).

Of direct relevance to this paper is another result in [17] which proved a corresponding boundedness result for oscillatory integral operators. This result, again contained in Theorem 5.12, states that in the case that σ verifies estimate (2) with $m<0$, and if the phases $\varphi_j\in\mathcal{C}^\infty(\mathbb{R}^n\times\mathbb{R}^n)$ (are not assumed to be homogeneous) satisfy (6) and the condition

$$|\partial_x^\alpha\partial_\xi^\beta\varphi_j(x,\xi)|\leqslant C_{\alpha,\beta}\text{ for all }\alpha\text{ and }\beta\text{ with }2\leqslant|\alpha|+|\beta|\tag{8}$$

for both $j=1,2$, then $T_\sigma^{\varphi_1,\varphi_2}$ is bounded from $L^2\times L^2\to L^1$. These phase functions were introduced by K. Asada and D. Fujiwara in [1].

However, given that the amplitude is assumed to be smooth and satisfy (2), one would expect such a result to hold up to and including the end-point $m=0$. The main goal of this paper is to confirm that this is the case. Most of the methods we employ are different to those used in [11] and [17], essentially due to the fact we must take advantage of more subtle cancellation properties at the end-point $m = 0$.

We will consider phases where (8) is replaced with the slightly weaker condition

$$(9) \qquad |\partial_x^\alpha \partial_\xi^\beta \varphi_j(x,\xi)| \leqslant C_{\alpha,\beta} \text{ for all } \alpha \text{ and } \beta \text{ with } 1 \leqslant |\alpha| \text{ and } 1 \leqslant |\beta|,$$

for $j = 1, 2$. In [19], M. Ruzhansky and M. Sugimoto considered a class of phases which, in particular, included those satisfying (9).

Let us now state the main result of this paper.

Theorem 1.1. *Suppose that σ satisfies estimate* (2) *with $m = 0$ and that φ_1 and φ_2 are real-valued functions in $\mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ that verify the strong non-degeneracy condition* (6) *and* (9)*. Then there exists a constant C such that*

$$\|T_\sigma^{\varphi_1,\varphi_2}(f,g)\|_{L^1(\mathbb{R}^n)} \leqslant C\|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}$$

for all Schwartz functions f and g.

The theorem is proved, as is often the case, by decomposing the operator into pieces supported in various frequency regimes and applying different methods in the different regimes. The first is when either ξ or η is small. In this case we can write the bilinear operator as an iteration of linear operators, as in [15], and then apply techniques for linear operators. The second is when both ξ and η are large. Due to the symmetry of the operator and the fact we are interested in a bound in terms of the L^2-norm of both f and g, we can further reduce this case to when $|\eta| \leqslant 2|\xi|$. Here we use a decomposition introduced in [7] by R. Coifman and Y. Meyer for the proof of the corresponding result for pseudodifferential operators. However, where they go on to use Carleson measure techniques, we must combine a quadratic $T(1)$-Theorem of M. Christ and J.-L. Journé [6] with commutator-type estimates.

It is a natural question to ask about the general case of $L^p \times L^q \to L^r$ boundedness of bilinear oscillatory integral operators, for example when $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. This general problem introduces a myriad of new difficulties that need to be handled by using more general and heavier machinery from harmonic analysis. A complete solution to this problem for the case of bilinear Fourier integral operators is the subject of our paper [16].

We will end this introduction by motivating the assumptions we have placed on the phase functions. In Section 2 we lay out the basic notions and tools we will require in the proof of Theorem 1.1. In Section 3 we provide the details of this proof, stating the commutator results we make use of in Theorem 3.2. Finally, in Section 4 we prove Theorem 3.2 using an asymptotic expansion, formulated as Proposition 4.1.

To keep the notation as simple as possible, constants which can be easily estimated by given parameters are all denoted by C, even though the precise values will vary from line to line. We also use the notation $A \lesssim B$ if there exists a constant C such that $A \leqslant CB$. For clarity, we sometimes indicate the parameters on which a constant depends as subscripts.

Examples and motivation. All phase functions of the form $x \cdot \xi + \Psi(\xi)$ with an arbitrary smooth Ψ are easily seen to satisfy (6) and (9). This includes the phase of Schrödinger-type $\varphi_j(x,\xi) = x \cdot \xi + |\xi|^2$ and that (when $n = 1$) arising from the Korteweg-de Vries equation $\varphi_j(x,\xi) = x\xi + \xi^3$, although the Korteweg-de Vries phase does not satisfy the more restrictive condition (8). However, the $L^2 \times L^2 \to L^1$ boundedness of operators with these types of phases can be easily reduced to the case of the bilinear pseudodifferential result of Coifman and Meyer, by viewing the operator as a composition of a multiplier with amplitude $e^{i\Psi(\xi)}$ composed with a bilinear pseudodifferential operator.

As a next step, one could consider operators with phases $\varphi_j(x,\xi) = x\cdot\xi + \lambda_j(x,\xi)$ with $\lambda_j(x,\xi) \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. We will illustrate here that such phases cannot easily be understood with previously available techniques without slightly artificial additional assumptions. Following the freezing of the variables argument of Coifman-Meyer (see [7], page 157), one introduces the function

$$F(x,y) = \chi(x)\chi(y) \iint e^{i\lambda_1(y,\xi)+i\lambda_2(y,\eta)}\, \sigma(x,\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{ix\cdot(\xi+\eta)}\, đ\xi đ\eta,$$

where $\chi(x)$ is a smooth cut-off function. Using Sobolev's embedding theorem we have that $|F(x,x)| \lesssim \sum_{|\alpha|\leqslant n+1}\int |\partial_y^\alpha F(x,y)|\,dy$ and the desired $L^2 \times L^2 \to L^1$ boundedness of the oscillatory integral operator with phases $x\cdot\xi + \lambda_j(x,\xi)$ would follow by estimating the L^1-norm of $F(x,x)$ with an upper bound in terms of the L^2-norms of f and g. Given the smoothness and the compact support in y, this in turn amounts to obtaining the aforementioned upper bound for $\|\partial_y^\alpha F(x,y)\|_{L^1_x}$. Now using Leibniz's rule and the chain rule, matters reduce to the study of L^1_x boundedness of operators of the form

$$\chi(x)\partial_y^\alpha\chi(y) \iint \sigma(x,\xi,\eta)\mathscr{F}(e^{i\lambda_1(y,D)}f)(\xi)\, \mathscr{F}(e^{i\lambda_2(y,D)}g)(\eta)e^{ix\cdot(\xi+\eta)}\, đ\xi đ\eta \tag{10}$$

and of the form

$$\begin{aligned}\chi(x)\partial_y^{\alpha_1}\chi(y) \iint &[\partial_y^{\alpha_2}(\lambda_1(y,\xi)+\lambda_2(y,\eta))]^k\, \sigma(x,\xi,\eta)\\ &\times\mathscr{F}(e^{i\lambda_1(y,D)}f)(\xi)\, \mathscr{F}(e^{i\lambda_2(y,D)}g)(\eta)e^{ix\cdot(\xi+\eta)}\, đ\xi đ\eta,\end{aligned} \tag{11}$$

where $\mathscr{F}$ denotes the Fourier transform. The operator in (10) is a bilinear pseudodifferential operator with symbol $\chi(x)\partial_y^\alpha\chi(y)\sigma(x,\xi,\eta)$ whose $L^2\times L^2 \to L^1$ boundedness (when e.g. σ verifies (2)) is a consequence of the classical Coifman-Meyer theorem (see [7]) and the unitarity of $e^{i\lambda_j(y,D)}$ in L^2. The operator in (11) is a bilinear pseudodifferential operator with the symbol

$$a_y(x,\xi,\eta) := \chi(x)\partial_y^\alpha\chi(y)[\partial_y^\beta(\lambda_1(y,\xi)+\lambda_2(y,\eta))]^k\, \sigma(x,\xi,\eta),$$

where y is considered as a parameter. However if one starts with an amplitude $\sigma(x,\eta,\xi)$ verifying merely estimate (2), then for instance the uniform boundedness of $\partial_y^\beta\lambda_j(y,\xi)$ would not be enough to ensure the boundedness of the bilinear pseudodifferential operator with symbol $a_y(x,\xi,\eta)$. In order to produce the desired boundedness, one could, for example, assume the validity of the estimates

$$\sup_{x,\xi,\eta} |\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x,\xi,\eta)| \leqslant C_{\alpha,\beta,\gamma},$$

$$\sup_{x,\xi} |\partial_x^\alpha \partial_\xi^\beta \lambda_j(x,\xi)| \leqslant C_{\alpha,\beta},$$

$$\sup_x \int \Big(\int |\partial_\xi^\alpha \sigma(x,\xi,\eta)|^2 \, d\eta \Big)^{\frac{1}{2}} d\xi \leqslant C_\alpha \quad \text{and}$$

$$\sup_x \int \Big(\int |\partial_\eta^\alpha \sigma(x,\xi,\eta)|^2 \, d\xi \Big)^{\frac{1}{2}} d\eta \leqslant C_\alpha$$

for all α and β and γ (even though, strictly speaking, the estimate for λ_j above only requires $|\alpha| \geqslant 1$). Then the $L^2 \times L^2 \to L^1$ boundedness of the operator with the symbol $a_y(x,\xi,\eta)$ will be a consequence of a theorem due to A. Benyi and R. Torres [3]. Obviously, given the fact that the symbol $a_y(x,\xi,\eta)$ is localized in both x and y and that the phases λ_j are smooth, one could weaken the assumptions above to some small extent, but we will not pursue this here.

To further motivate our investigation in this paper, we would like to bring the reader's attention to a class of phases satisfying (6) and (9) given by

$$\varphi_j(x,\xi) = x\cdot\xi + (u\cdot\xi)\langle x\rangle + (v\cdot x)\langle\xi\rangle, \qquad u,v\in\mathbb{R}^n \text{ and } |u|+|v|<1,$$

where the notation $\langle\cdot\rangle$ stands for $(1+|\cdot|^2)^{1/2}$. It is worth noting that that if we set $\lambda_j(x,\xi) = (u\cdot\xi)\langle x\rangle + (v\cdot x)\langle\xi\rangle$, then it is clear that estimate $|\nabla_x\lambda_j(x,\xi)| \leqslant C$ is not valid, and so the operator with phases $\varphi_j(x,\xi)$ does not fit into the realm of the previous example. An attempt to eliminate the growth in x by replacing $\langle x\rangle$ in the example above would not always reduce the operator to one which could be treated by the bilinear pseudodifferential theory. For example, consider an operator in dimension one given by

$$T_\varepsilon(f,g)(x) = \iint_{\mathbb{R}^2} a(x,\xi,\eta) e^{i\varepsilon\xi\sin x} \widehat{f}(\xi)\widehat{g}(\eta) e^{ix(\xi+\eta)} \,đ\xi đ\eta \quad \text{for } \varepsilon\in(0,1).$$

Once again, viewing this operator as a bilinear pseudodifferential operator, we observe that it has the symbol $a(x,\xi,\eta)e^{i\varepsilon\xi\sin x}$ which at best belongs to the class $S^0_{0,1}(1,2)$ (again see Definition 2.1 below). This is not a favourable class of symbols to study, even for linear operators. Moreover setting $\lambda(x,\xi) = \varepsilon\xi\sin x$, we still will not have the boundedness of $\partial_x\lambda$, and therefore not even the boundedness result sketched above could be applied to this case. Viewing the oscillatory factor $e^{i\varepsilon\xi\sin x}$ as part of the phase, on the other hand, allows us to conclude the operator is nevertheless bounded.

Theorem 1.1 can also be seen as a step towards understanding (1) where the phase $\varphi(x,\xi,\eta)$ does not separate in the frequency variables. Indeed, suppose now that $\varphi\in\mathcal{C}^\infty(\mathbb{R}^n\times(\mathbb{R}^{2n}\setminus\{0\}))$ is positively homogeneous of degree 1 in $\mathbb{R}^{2n}\setminus\{0\}$ and satisfies a suitable non-degeneracy condition. One can use Lemma 1.2.3 in [9] to reduce the study of such phases to ones of the form

$$\varphi(x,\xi,\eta) = ix\cdot\xi + i\Psi(x)\cdot\eta + i\Theta(x,\xi,\eta),$$

where Θ satisfies (2) with $m=1$ away from the origin and $\Psi = (\Psi_1,\dots,\Psi_n) \in \mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R}^n)$ satisfies

$$\inf_x |\det D_x\Psi(x)| > 0 \quad \text{and} \quad \max_{j=1,\dots,n} \sup_x \big|\partial_x^\beta \Psi_j(x)\big| \lesssim 1, \text{ for } |\beta|\geqslant 1.$$

Although Theorem 1.1 cannot be applied to such phases, it could if Θ, which encodes how far we are from the pseudodifferential case, was slightly better behaved, that is to say, satisfied (2) with $m = 0$. Therefore Theorem 1.1 gives us an understanding of operators (1) with a non-separable phase function provided that phase is sufficiently close to the separable case.

2. Definitions and tools from linear and bilinear theory

Here we recall some definitions and tools from the linear and bilinear theory of oscillatory integral operators which will be needed in what follows. The basic definition of amplitudes and symbols goes back to Hörmander [13].

Definition 2.1. Let $m \in \mathbb{R}$, $0 \leqslant \delta \leqslant 1$, $0 \leqslant \rho \leqslant 1$ and $d \in \{1, 2\}$. A function $\sigma \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^{dn})$ belongs to the class $S^m_{\rho,\delta}(n, d)$ if for all multi-indices α and β, there exist constants $C_{\alpha,\beta}$ such that

$$|\partial^\alpha_\Xi \partial^\beta_x \sigma(x, \Xi)| \leqslant C_{\alpha,\beta} \langle \Xi \rangle^{m-\rho|\alpha|+\delta|\beta|}, \qquad \text{for all } (x, \Xi) \in \mathbb{R}^n \times \mathbb{R}^{dn}.$$

We would like to remark that the notation $S^m_{\rho,\delta}(n, 1)$ is to emphasise that the operators associated to these amplitudes are linear. Equally, amplitudes associated to bilinear operators are denoted $S^m_{\rho,\delta}(n, 2)$ (see [15]). In particular, any function satisfying (2) belongs to the class $S^m_{1,0}(n, 2)$.

As mentioned above, we also need to assume that the phase function satisfies the so-called *strong non-degeneracy* condition, which is known to be a necessary condition for the global boundedness of linear oscillatory and Fourier integral operators.

Definition 2.2 (The strong non-degeneracy condition)**.** A real-valued phase $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ satisfies the strong non-degeneracy condition, if there exists a positive constant c such that

$$\left|\det\, \partial^2_{x,\xi} \varphi(x, \xi)\right| \geqslant c, \qquad \text{for all } (x,\, \xi) \in \mathbb{R}^n \times \mathbb{R}^n.$$

Given an amplitude $\sigma(x, \xi)$ and a phase function $\varphi(x, \xi)$ a *linear oscillatory integral operators* is one given by

$$T^\varphi_\sigma(f)(x) = \int \sigma(x, \xi) \widehat{f}(\xi) e^{i\varphi(x,\xi)} \,đ\xi, \tag{12}$$

for a Schwartz function f. When the phase function $\varphi(x, \xi)$ is assumed to be positively homogeneous of degree 1 in the frequency variable ξ, then the operator (12) is the well-known Fourier integral operators of Hörmander which appear naturally in the study of hyperbolic partial differential equations.

In the context of our current investigation, the following result regarding the global boundedness of linear oscillatory integral operators will be useful to us. It is a reformulation of Theorem 2.1 in [19].

Theorem 2.3. *If $\sigma \in S^0_{0,0}(n, 1)$, and φ satisfies* (9) (*with $\varphi = \varphi_j$*) *and the strong non-degeneracy condition, then the operator T^φ_σ is bounded on $L^2(\mathbb{R}^n)$ and its norm is bounded by a constant depending only on n, c in Definition* 2.2*, a finite number of $C_{\alpha,\beta}$ appearing in* (9) *and Definition* 2.1*.*

Actually this theorem is proved in [19] under weaker assumptions on the phase function. However, we will make use of the stronger assumptions stated here to prove commutator estimates, so for simplicity we state the result as we will use it, rather than in the full generality of [19].

In [17] the boundedness of linear oscillatory integral operators with amplitudes in the class $L^pS^m_\rho(n,1)$ is proved. This amplitude class was first introduced by N. Michalowski, D. Rule and W. Staubach in [15]. Here we recall the definition of the class $L^pS^m_\rho(n,1)$.

Definition 2.4. Let $1 \leqslant p \leqslant \infty$, $m \in \mathbb{R}$ and $0 \leqslant \rho \leqslant 1$ be parameters. A symbol $\sigma\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ belongs to the class $L^pS^m_\rho(n,1)$ if for each multi-index α there exists a constant C_α such that

$$\sup_{\xi\in\mathbb{R}^n} \langle \xi\rangle^{-m+\rho|\alpha|} \|\partial_\xi^\alpha \sigma(\cdot,\xi)\|_{L^p(\mathbb{R}^n)} \leqslant C_\alpha.$$

Here we also define the associated seminorms

$$|a|_{p,m,s} = \sum_{|\alpha|\leqslant s} \sup_{\xi\in\mathbb{R}^n} \langle \xi\rangle^{\rho|\alpha|-m} \left\|\partial_\xi^\alpha a(\cdot,\xi)\right\|_{L^p(\mathbb{R}^n)}. \tag{13}$$

The motivation for introducing the class $L^pS^m_\rho(n,1)$ in [15] is that it proves useful in the study of bilinear operators. Indeed, the global regularity of bilinear Fourier integral operators with amplitudes that are neither compactly supported nor smooth in the first variable is studied in [17] in part by proving the boundedness of linear operators with amplitudes in the aforementioned classes.

3. $L^2 \times L^2 \to L^1$ boundedness of bilinear oscillatory integral operators

We are now ready to prove our main result, Theorem 1.1. We introduce a smooth function $\mu\colon \mathbb{R}^n \to \mathbb{R}$ such that $\mu(\xi) = 0$ for $|\xi| \leqslant 5$ and $\mu(\xi) = 1$ for $|\xi| \geqslant 6$. Observe that

$$\begin{aligned} T_\sigma^{\varphi_1,\varphi_2}(f,g) = {} & T_\sigma^{\varphi_1,\varphi_2}(\mu(D)f,\mu(D)g) + T_\sigma^{\varphi_1,\varphi_2}(\mu(D)f,(1-\mu)(D)g) \\ & + T_\sigma^{\varphi_1,\varphi_2}((1-\mu)(D)f,g), \end{aligned} \tag{14}$$

where $\mu(D)$ and $(1-\mu)(D)$ denote the Fourier multiplier operators given by $(\mu(D)f)\widehat{\ }(\xi) = \mu(\xi)\widehat{f}(\xi)$ and $((1-\mu)(D)f)\widehat{\ }(\xi) = (1-\mu)(\xi)\widehat{f}(\xi)$ respectively.

To estimate the L^1-norm of the last two terms in (14) we can make use of linear boundedness results by viewing the bilinear operator as an iteration of linear operators. We can write $T_\sigma^{\varphi_1,\varphi_2}((1-\mu)(D)f,g)$ as

$$\begin{aligned} & T_\sigma^{\varphi_1,\varphi_2}((1-\mu)(D)f,g)(x) \\ & \quad = \int \left(\int \sigma(x,\xi,\eta)\widehat{g}(\eta)e^{i\varphi_2(x,\eta)}\,\text{đ}\eta\right)(1-\mu)(\xi)\widehat{f}(\xi)e^{i\varphi_1(x,\xi)}\,\text{đ}\xi \\ & \quad = \int \mathfrak{a}_g(x,\xi)\widehat{f}(\xi)e^{i\varphi_1(x,\xi)}\,\text{đ}\xi = T_{\mathfrak{a}_g}^{\varphi_1}(f)(x), \end{aligned}$$

where

$$\mathfrak{a}_g(x,\xi) = (1-\mu)(\xi)\int \sigma(x,\xi,\eta)\widehat{g}(\eta)e^{i\varphi_2(x,\eta)}\,\text{đ}\eta.$$

Using the fact that $\sigma \in S^0_{1,0}(n,2) \subset S^0_{0,0}(n,2)$, we get

$$\sup_{x,\xi,\eta\in\mathbb{R}^n} |\partial_\xi^\alpha \partial_\eta^\beta \partial_x^\gamma \sigma(x,\xi,\eta)| \lesssim 1.$$

Therefore, bearing in mind the support properties of $\mathfrak{a}_g$, applying Theorem 2.3 we find that

$$\sup_{\xi\in\mathbb{R}^n} \langle\xi\rangle^{-m+|\alpha|} \|\partial_\xi^\alpha \mathfrak{a}_g(\cdot,\xi)\|_{L^2(\mathbb{R}^n)} \lesssim \|g\|_{L^2(\mathbb{R}^n)}.$$

Thus $\mathfrak{a}_g \in L^2S^m_1(n,1)$ for any $m<0$, and by the very definition of $\mathfrak{a}_g$ its ξ-support is compact. Now the fact that $T^{\varphi_1}_{\mathfrak{a}_g}$ is a bounded operator from $L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ with norm of size $\|g\|_{L^2(\mathbb{R}^n)}$ is a consequence of the following lemma.

Lemma 3.1. *Assume that φ satisfies the strong non-degeneracy condition and* (9). *Suppose that $a \in L^2S^m_1$ for some $m<0$ is such that $\operatorname{supp}_\xi a(x,\xi)$ is compact. Then the oscillatory integral T_a defined as in* (12) *is bounded from L^2 to L^1.*

Proof. Consider a closed cube Q of side-length L such that $\operatorname{supp}_\xi a(x,\xi) \subset \operatorname{Int}(Q)$. We extend $a(x,\cdot)|_Q$ periodically with period L to $\widetilde{a}(x,\xi) \in \mathcal{C}^\infty(\mathbb{R}^n_\xi)$. Let $\eta \in \mathcal{C}^\infty_0$ with $\operatorname{supp}\eta \subset Q$ and $\eta = 1$ on ξ-support of $a(x,\xi)$, so we have $a(x,\xi) = \widetilde{a}(x,\xi)\eta(\xi)$. Now if we expand $\widetilde{a}(x,\xi)$ in a Fourier series, setting $f_k(x) = f(x - \frac{2\pi k}{L})$ for any $k \in \mathbb{Z}^n$, we can write

$$T_a f(x) = \sum_{k\in\mathbb{Z}^n} a_k(x) T_\eta(f_k)(x), \tag{15}$$

where

$$a_k(x) = \frac{1}{L^n}\int_{\mathbb{R}^n} a(x,\xi) e^{-\frac{2\pi i}{L}\langle k,\xi\rangle}\,\mathrm{d}\xi,$$

and $T_\eta(f_k)(x) = (2\pi)^{-n}\int \eta(\xi) e^{i\varphi(x,\xi)}\widehat{f_k}(\xi)\,\mathrm{d}\xi$. It follows at once from Theorem 2.3 that T_η is bounded on L^2. For $k \neq 0$ take $l = 1,\dots,n$ such that $|k_l| \neq 0$, where k_l is the l-th component of k. Integration by parts yields

$$a_k(x) = \frac{c_{n,N}}{|k_l|^N}\int_{\mathbb{R}^n} \partial^N_{\xi_l} a(x,\xi) e^{-\frac{2\pi i}{L}\langle k,\xi\rangle}\,\mathrm{d}\xi.$$

Observe also that, by the hypothesis on the amplitude and definition of the semi-norms (13),

$$\int_{\mathbb{R}^n} \left\|\partial^N_{\xi_l} a(\cdot,\xi)\right\|_{L^2}\,\mathrm{d}\xi \leqslant c_{n,N}\,|a|_{2,m,N}\,,$$

for $N \geqslant n+1$. Furthermore also using the assumption that the ξ-support of $a(x,\xi)$ is compact in the integral defining $a_k(x)$, we have $\|a_k\|_{L^2} \leqslant \frac{C}{L^n}$ as well. Combining this and the estimate above, we obtain

$$\|a_k\|_{L^2} \leqslant c_{n,N}\,|a|_{2,m,N}\,(1+|k|)^{-N}. \tag{16}$$

The Minkowski and Cauchy-Schwarz inequalities yield

$$\|T_a f\|_{L^1} \leqslant \sum_{k\in\mathbb{Z}^n} \|a_k T_\eta(f_k)\|_{L^1} \leqslant \sum_{k\in\mathbb{Z}^n} \|a_k\|_{L^2}\,\|T_\eta(f_k)\|_{L^2}\,. \tag{17}$$

On the other hand, since T_η is bounded on L^2 and translations are isometries on L^2, we have that $\|T_\eta(f_k)\|_{L^2} \leqslant c_{\eta,\varphi}\,\|f\|_{L^2}$. Therefore (16) yields

$$\|T_a f\|_{L^1} \lesssim |a|_{2,m,N} \sum_{k\in\mathbb{Z}^n} (1+|k|)^{-N}\,\|f\|_{L^2} \approx \|f\|_{L^2}\,,$$

which completes the proof. ☐

Thus Lemma 3.1 yields that

$$\|T^{\varphi_1,\varphi_2}_\sigma((1-\mu)(D)f, g)\|_{L^1(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}.$$

Similarly, interchanging the roles of f and g in the previous argument, we can see that

$$\|T^{\varphi_1,\varphi_2}_\sigma(\mu(D)f, (1-\mu)(D)g)\|_{L^1(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}.$$

We now turn our attention to $T_\sigma^{\varphi_1,\varphi_2}(\mu(D)f,\mu(D)g)$. We observe that without loss of generality we may assume that

$$\varphi_1(x,0)=\varphi_2(x,0)=0. \tag{18}$$

This is because

$$T_\sigma^{\varphi_1,\varphi_2}(f,g)(x)=e^{i\varphi_1(x,0)+i\varphi_2(x,0)}T_\sigma^{\widetilde{\varphi}_1,\widetilde{\varphi}_2}(f,g)(x),$$

where $\widetilde{\varphi}_j(x,\xi)=\varphi_j(x,\xi)-\varphi_j(x,0)$ for j=1,2. Since multiplication by $e^{i\varphi_1(x,0)+i\varphi_2(x,0)}$ is an isometry, and $\widetilde{\varphi}_j$ will inherit directly from φ_j exactly the same properties that φ_j satisfies (namely the strong non-degeneracy condition and (9)), the question of boundedness is reduced to the case of an operator with phases which satisfy (18).

Let us introduce two smooth cut-off functions $\chi,\nu\colon\mathbb{R}^{2n}\to\mathbb{R}$ such that $\chi(\xi,\eta)=1$ for $|(\xi,\eta)|\leqslant 1$ and $\chi(\xi,\eta)=0$ for $|(\xi,\eta)|\geqslant 2$, and $\nu(\xi,\eta)=0$ for $2|\xi|\leqslant|\eta|$ and $\nu(\xi,\eta)=1$ for $2|\eta|\leqslant|\xi|$.

Defining

$$\begin{aligned}\sigma_0(x,\xi,\eta)&=\chi(\xi,\eta)\sigma(x,\xi,\eta),\\ \sigma_1(x,\xi,\eta)&=(1-\chi(\xi,\eta))\nu(\xi,\eta)\sigma(x,\xi,\eta)\quad\text{and}\\ \sigma_2(x,\xi,\eta)&=(1-\chi(\xi,\eta))(1-\nu(\xi,\eta))\sigma(x,\xi,\eta),\end{aligned}$$

we have that $\sigma_0,\sigma_1,\sigma_2$ belong to the class $S^0_{1,0}(n,2)$ and we can decompose

$$T_\sigma^{\varphi_1,\varphi_2}(\mu(D)f,\mu(D)g)=T_{\sigma_1}^{\varphi_1,\varphi_2}(\mu(D)f,\mu(D)g)+T_{\sigma_2}^{\varphi_1,\varphi_2}(\mu(D)f,\mu(D)g). \tag{19}$$

We observe that it suffices to control the L^1-norm of merely one of these terms, say $T_{\sigma_1}^{\varphi_1,\varphi_2}(\mu(D)f,\mu(D)g)$, because once again the other can be controlled in the same way by interchanging the roles of f and g.

Following the analysis of [5] and [7] we introduce an even real-valued smooth function ψ whose Fourier transform is supported on the annulus $\{\xi\,|\,1/2\leqslant|\xi|\leqslant 2\}$ such that

$$\int_0^\infty|\widehat{\psi}(t\xi)|^2\frac{\mathrm{d}t}{t}=1,\qquad\text{for }\xi\neq 0.$$

Let θ be another real-valued smooth function whose Fourier transform is equal to one on the ball $\{\xi\,|\,|\xi|\leqslant 4\}$ and supported on $\{\xi\,|\,|\xi|\leqslant 5\}$. Then

$$\begin{aligned}&T_{\sigma_1}^{\varphi_1,\varphi_2}(\mu(D)f,\mu(D)g)(x)\\ &=\iiint_0^\infty\sigma_{1,t}(x,t\xi,t\eta)\widehat{\psi}(t\xi)\widehat{\theta}(t\eta)\mu(\xi)\widehat{f}(\xi)\mu(\eta)\widehat{g}(\eta)e^{i\varphi_1(x,\xi)+i\varphi_2(x,\eta)}\frac{\mathrm{d}t}{t}\,đ\xi đ\eta,\end{aligned} \tag{20}$$

for $\sigma_{1,t}(x,\xi,\eta):=\sigma_1(x,\xi/t,\eta/t)\widehat{\psi}(\xi)\widehat{\theta}(\eta)$. Using the Fourier inversion formula,

$$\sigma_{1,t}(x,\xi,\eta)=\iint e^{i\xi\cdot u+i\eta\cdot v}m(t,x,u,v)\frac{\mathrm{d}u\mathrm{d}v}{(1+|u|^2+|v|^2)^N}, \tag{21}$$

where

$$m(t,x,u,v):=\iint e^{-i\xi\cdot u-i\eta\cdot v}\big[(1-\Delta_\xi-\Delta_\eta)^N\sigma_{1,t}(x,\xi,\eta)\big]đ\xi đ\eta$$

for any large fixed $N\in\mathbb{N}$.

Since the (ξ,η)-support of $\sigma_{1,t}$ is contained in a compact set independent of t and x, and all (x,ξ,η)-derivatives are bounded independently of t, we see that

$\partial_x^\alpha m(t,x,u,v)$ is bounded for each multi-index α. Combining this with (20) and (21) we arrive at the representation

$$\begin{aligned} &T^{\varphi_1,\varphi_2}_{\sigma_1}(\mu(D)f,\mu(D)g)(x) \\ &= \iiint_0^\infty T^{\varphi_1}_\mu(P^v_t(f))(x)T^{\varphi_2}_\mu(Q^u_t(g))(x)\frac{m(t,x,u,v)}{(1+|u|^2+|v|^2)^N}\frac{\mathrm{d}t}{t}\mathrm{d}u\mathrm{d}v \end{aligned} \tag{22}$$

for any large fixed $N\in\mathbb{N}$. Here

$$T^{\varphi_j}_\mu(f)(x)=\int \mu(\xi)\widehat{f}(\xi)e^{i\varphi_j(x,\xi)}\text{đ}\xi \quad \text{for } j=1,2,$$

$P^v_t(f)=\theta^v_t*f$ and $Q^u_t(g)=\psi^u_t*g$, with $\theta^v_t(x)=t^{-n}\theta(\frac{x}{t}+v)$ and $\psi^u_t(x)=t^{-n}\psi(\frac{x}{t}+u)$.

We observe that

$$T^{\varphi_1}_\mu(P^v_t(f))(x)=\int \mu(\xi)\widehat{\theta}(t\xi)e^{it\xi\cdot v}\widehat{f}(\xi)e^{i\varphi_1(x,\xi)}\text{đ}\xi.$$

Since $\mu(\xi)=0$ for $|\xi|\leqslant 5$ and $\widehat{\theta}(t\xi)=0$ for $|t\xi|\geqslant 5$, then $\mu(\xi)\widehat{\theta}(t\xi)=0$ for $t>1$ and consequently $T^{\varphi_1}_\mu(P^v_t(f))(x)=0$ for $t>1$. Using this fact, together with (22) and duality, to bound $\|T^{\varphi_1,\varphi_2}_{\sigma_1}(f,g)\|_{L^1(\mathbb{R}^n)}$ it suffices to control

$$\left|\iint_0^1 T^{\varphi_1}_\mu(P^v_t(f))(x)T^{\varphi_2}_\mu(Q^u_t(g))(x)b(x)m(t,x,u,v)\frac{\mathrm{d}t\mathrm{d}x}{t}\right|$$

with at most polynomial growth in u and v for arbitrary $b\in L^\infty(\mathbb{R}^n)$. Introducing the radial function ψ_0 whose Fourier transform is compactly supported on an annulus and equal to one on the support of the Fourier transform of ψ, we define $\psi_{0,t}(x)=t^{-n}\psi_0(x/t)$ and $Q_{0,t}(g)=\psi_{0,t}*g$. Let $M^{u,v}_t$ denote the L^2-bounded operator which is multiplication by $b(x)m(t,x,u,v)$, that is, $M^{u,v}_t(f)(x)=b(x)m(t,x,u,v)f(x)$, and let $T^{\varphi_2,*}_\mu$ denote the adjoint operator of $T^{\varphi_2}_\mu$. Then $Q^u_t(g)=Q^u_t(Q_{0,t}(g))$ and so, using the Cauchy-Schwarz inequality,

$$\begin{aligned} &\left|\iint_0^1 T^{\varphi_1}_\mu(P^v_t(f))(x)T^{\varphi_2}_\mu(Q^u_t(g))(x)b(x)m(t,x,u,v)\frac{\mathrm{d}t\mathrm{d}x}{t}\right| \\ &=\left|\iint_0^1 (Q^u_tT^{\varphi_2,*}_\mu M^{u,v}_tT^{\varphi_1}_\mu P^v_t)(f)(x)Q_{0,t}(g)(x)\frac{\mathrm{d}t\mathrm{d}x}{t}\right| \\ &\leqslant\left(\iint_0^1 |(Q^u_tT^{\varphi_2,*}_\mu M^{u,v}_tT^{\varphi_1}_\mu P^v_t)(f)(x)|^2\frac{\mathrm{d}t\mathrm{d}x}{t}\right)^{1/2}\left(\iint_0^\infty |Q_{0,t}(g)(x)|^2\frac{\mathrm{d}t\mathrm{d}x}{t}\right)^{1/2} \\ &\lesssim\left(\iint_0^1 |(Q^u_tT^{\varphi_2,*}_\mu M^{u,v}_tT^{\varphi_1}_\mu P^v_t)(f)(x)|^2\frac{\mathrm{d}t\mathrm{d}x}{t}\right)^{1/2}\|g\|_{L^2(\mathbb{R}^n)}. \end{aligned}$$

The last inequality follows by repeated application of Plancherel's Theorem and the fact that ψ_0 is supported on an annulus. Indeed,

$$\begin{aligned} \left(\iint_0^\infty |Q_{0,t}(g)(x)|^2\frac{\mathrm{d}t\mathrm{d}x}{t}\right)^{1/2} &=\left(\int_0^\infty\!\!\int |\widehat{\psi}_0(t\xi)\widehat{g}(\xi)|^2\frac{\text{đ}\xi\mathrm{d}t}{t}\right)^{1/2} \\ &=\left(\iint_0^\infty |\widehat{\psi}_0(t\xi)|^2\frac{\mathrm{d}t}{t}|\widehat{g}(\xi)|^2\text{đ}\xi\right)^{1/2} \\ &\lesssim\left(\int |\widehat{g}(\xi)|^2\text{đ}\xi\right)^{1/2}=\left(\int |g(x)|^2\mathrm{d}x\right)^{1/2}. \end{aligned} \tag{23}$$

Therefore, the proof of Theorem 1.1 will be complete if we can prove the quadratic estimate

$$\left(\iint_0^1 |(Q_t^u T_\mu^{\varphi_2,*} M_t^{u,v} T_\mu^{\varphi_1} P_t^v)(f)(x)|^2 \frac{\mathrm{d}t\mathrm{d}x}{t}\right)^{1/2} \lesssim \|b\|_{L^\infty(\mathbb{R}^n)} \|f\|_{L^2(\mathbb{R}^n)}. \tag{24}$$

To obtain (24), we wish to apply a quadratic $T(1)$-theorem. This requires two hypotheses: kernel estimates and a cancellation condition (the $T(1)$ condition). Unfortunately it is not clear how to demonstrate either of these hypotheses for the operator $Q_t^u T_\mu^{\varphi_2,*} M_t^{u,v} T_\mu^{\varphi_1} P_t^v$ which appears in (24). However, let us suppose for a moment that we could commute operators at will. Define the operators M_b and M_m by $M_b(f)(x) = b(x)f(x)$ and $M_m(f)(x) = m(t,x,u,v)f(x)$, respectively. Because $M_t^{u,v} = M_m M_b$, commuting operators would lead us to consider $T_\mu^{\varphi_2,*} M_m Q_t^u M_b P_t^v T_\mu^{\varphi_1}$. This composition is much more amenable to the method mentioned above. This is because $T_\mu^{\varphi_2,*} M_m$ is bounded on $L^2(\mathbb{R}^n)$ uniformly in t (by virtue of Theorem 2.3) and appears on the left of the composition, and so can be disregarded when trying to prove a quadratic estimate. Furthermore $T_\mu^{\varphi_1}$ is t-independent, L^2-bounded (again by virtue of Theorem 2.3) and appears on the right of the composition, so can also be disregarded. This leaves us with the task of proving a quadratic estimate for $Q_t^u M_b P_t^v$. This operator does satisfy kernel estimates and the $T(1)$ condition in this context is simply the fact that $Q_t^u M_b P_t^v(1) = Q_t^u(b)$ gives rise to a Carleson measure, which is well-known. For the remainder of the paper, we will fill in the details of this heuristic argument.

The following theorem gives us three equalities. The first makes precise the extent to which we may commute $T_\mu^{\varphi_1}$ and P_t^v. Although we cannot commute Q_t^u and $T_\mu^{\varphi_2,*}$ in a manner which is acceptable to us, the second equality says there exist operators U_t^u and R_t such that $Q_t^u T_\mu^{\varphi_2,*} = U_t^{u,*} R_t$ modulo an acceptable error, where R_t has the same properties as Q_t^u and U_t^u is an operator bounded uniformly in t. The third equality shows us that the commutator of R_t and M_m is sufficiently well-behaved.

Theorem 3.2. *For $0 < t \leqslant 1$, $u, v \in \mathbb{R}^n$, there exist operators $W_{1,t}^v$, $W_{2,t}^u$, $W_{3,t}^{u,v}$, U_t^u, V_t^v and a radial smooth function ρ supported in an annulus centred at the origin such that*

(i) $[T_\mu^{\varphi_1}, P_t^v] := T_\mu^{\varphi_1} P_t^v - P_t^v T_\mu^{\varphi_1} = V_t^v + W_{1,t}^v$,
(ii) $T_\mu^{\varphi_2} Q_t^u = W_{2,t}^u + R_t U_t^u$, *and*
(iii) $[R_t, M_m] := R_t M_m - M_m R_t = W_{3,t}^{u,v}$,

where R_t is the multiplier operator defined by

$$R_t(f) = \int \rho(t\xi)\widehat{f}(\xi) e^{ix\cdot\xi} \mathrm{đ}\xi.$$

Moreover, there exists $\varepsilon > 0$ such that for any $0 < t \leqslant 1$,

$$\|W_{1,t}^v(f)\|_{L^2(\mathbb{R}^n)} + \|W_{2,t}^u(f)\|_{L^2(\mathbb{R}^n)} + \|W_{3,t}^{u,v}(f)\|_{L^2(\mathbb{R}^n)} \lesssim t^\varepsilon \|f\|_{L^2(\mathbb{R}^n)},$$

$$\|U_t^u(f)\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{R}^n)} \quad \text{and}$$

$$\iint_0^1 |V_t^v(f)(x)|^2 \frac{\mathrm{d}t}{t} \mathrm{d}x \lesssim \|f\|^2_{L^2(\mathbb{R}^n)}. \tag{25}$$

The implicit constants here depend on ε, but not on t and only polynomially on u and v.

We will postpone the proof of this theorem until Section 4. Using Theorem 3.2 (i), we can compute

$$
\begin{aligned}
&\iint_0^1 |(Q_t^u T_\mu^{\varphi_2,*} M_t^{u,v} T_\mu^{\varphi_1} P_t^v)(f)(x)|^2 \frac{\mathrm{d}t\mathrm{d}x}{t} \\
&\lesssim \iint_0^1 |(Q_t^u T_\mu^{\varphi_2,*} M_t^{u,v} P_t^v T_\mu^{\varphi_1})(f)(x)|^2 \frac{\mathrm{d}t\mathrm{d}x}{t} \\
&\quad + \iint_0^1 |(Q_t^u T_\mu^{\varphi_2,*} M_t^{u,v} V_t^v)(f)(x)|^2 \frac{\mathrm{d}t\mathrm{d}x}{t} \\
&\quad + \iint_0^1 |(Q_t^u T_\mu^{\varphi_2,*} M_t^{u,v} W_{1,t}^v)(f)(x)|^2 \frac{\mathrm{d}t\mathrm{d}x}{t} \\
&\lesssim \iint_0^1 |(Q_t^u T_\mu^{\varphi_2,*} M_t^{u,v} P_t^v T_\mu^{\varphi_1})(f)(x)|^2 \frac{\mathrm{d}t\mathrm{d}x}{t} + \|b\|^2_{L^\infty(\mathbb{R}^n)} \iint_0^1 |V_t^v(f)(x)|^2 \frac{\mathrm{d}t\mathrm{d}x}{t} \\
&\quad + \|b\|^2_{L^\infty(\mathbb{R}^n)} \iint_0^1 |W_{1,t}^v(f)(x)|^2 \frac{\mathrm{d}t\mathrm{d}x}{t} \\
&\lesssim \iint_0^1 |(Q_t^u T_\mu^{\varphi_2,*} M_t^{u,v} P_t^v T_\mu^{\varphi_1})(f)(x)|^2 \frac{\mathrm{d}t\mathrm{d}x}{t} + \|b\|^2_{L^\infty(\mathbb{R}^n)} \|f\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned}
$$

Using Theorem 3.2 (ii) we have

$$
\begin{aligned}
&\iint_0^1 |(Q_t^u T_\mu^{\varphi_2,*} M_t^{u,v} P_t^v T_\mu^{\varphi_1})(f)(x)|^2 \frac{\mathrm{d}t\mathrm{d}x}{t} \\
&\lesssim \iint_0^1 |(U_t^{u,*} R_t M_t^{u,v} P_t^v T_\mu^{\varphi_1})(f)(x)|^2 \frac{\mathrm{d}t\mathrm{d}x}{t} + \iint_0^1 |(W_{2,t}^{u,*} M_t^{u,v} P_t^v T_\mu^{\varphi_1})(f)(x)|^2 \frac{\mathrm{d}t\mathrm{d}x}{t} \\
&\lesssim \iint_0^1 |(R_t M_t^{u,v} P_t^v T_\mu^{\varphi_1})(f)(x)|^2 \frac{\mathrm{d}t\mathrm{d}x}{t} + \sup_{t>0} \int |(M_t^{u,v} P_t^v T_\mu^{\varphi_1})(f)(x)|^2 \mathrm{d}x \\
&\lesssim \iint_0^1 |(R_t M_t^{u,v} P_t^v T_\mu^{\varphi_1})(f)(x)|^2 \frac{\mathrm{d}t\mathrm{d}x}{t} + \|b\|^2_{L^\infty(\mathbb{R}^n)} \|f\|_{L^2(\mathbb{R}^n)}.
\end{aligned}
$$

Finally, using Theorem 3.2 (iii) we have

$$
\begin{aligned}
&\iint_0^1 |(R_t M_t^{u,v} P_t^v T_\mu^{\varphi_1})(f)(x)|^2 \frac{\mathrm{d}t\mathrm{d}x}{t} = \iint_0^1 |(R_t M_m M_b P_t^v T_\mu^{\varphi_1})(f)(x)|^2 \frac{\mathrm{d}t\mathrm{d}x}{t} \\
&\lesssim \iint_0^1 |(M_m R_t M_b P_t^v T_\mu^{\varphi_1})(f)(x)|^2 \frac{\mathrm{d}t\mathrm{d}x}{t} + \iint_0^1 |(W_{3,t}^{u,v} M_b P_t^v T_\mu^{\varphi_1})(f)(x)|^2 \frac{\mathrm{d}t\mathrm{d}x}{t} \\
&\lesssim \iint_0^1 |(M_m R_t M_b P_t^v T_\mu^{\varphi_1})(f)(x)|^2 \frac{\mathrm{d}t\mathrm{d}x}{t} + \sup_{t>0} \int |(M_b P_t^v T_\mu^{\varphi_1})(f)(x)|^2 \mathrm{d}x \\
&\lesssim \iint_0^1 |(R_t M_b P_t^v T_\mu^{\varphi_1})(f)(x)|^2 \frac{\mathrm{d}t\mathrm{d}x}{t} + \|b\|^2_{L^\infty(\mathbb{R}^n)} \|f\|_{L^2(\mathbb{R}^n)}.
\end{aligned}
$$

Given that $T_\mu^{\varphi_1}$ is an L^2-bounded operator and independent of t, to prove (24) we only need to prove

$$
\iint_0^\infty |(R_t M_b P_t^v)(f)(x)|^2 \frac{\mathrm{d}t\mathrm{d}x}{t} \lesssim \|b\|^2_{L^\infty(\mathbb{R}^n)} \|f\|^2_{L^2(\mathbb{R}^n)}. \tag{26}
$$

Now we recall Theorem 1 from [6], which is the quadratic $T(1)$-theorem that will be useful to us.* It concerns a family of operators $\{T_t\}_{t>0}$ which are defined as integration against a kernel:

$$T_t(f)(x) = \int K_t(x,y)f(y)\mathrm{d}y. \tag{27}$$

Estimates of significance for the kernel K_t are

$$|K_t(x,y)| \leqslant \frac{C}{t^n}\frac{1}{(1+|x-y|/t)^{n+1}} \quad \text{and} \tag{28}$$

$$|K_t(x,y+h) - K_t(x,y)| \leqslant \frac{C}{t^n}\frac{|h/t|}{(1+|x-y|/t)^{n+1}} \tag{29}$$

for $|h| \leqslant t/2$.

Theorem 3.3. *Suppose the kernel of T_t given by* (27) *satisfies estimates* (28) *and* (29) *for some constant $C < \infty$ together with the Carleson measure condition*

$$\sup_Q \frac{1}{|Q|}\int_0^{\ell(Q)}\int_Q |T_t(1)(x)|^2 \frac{\mathrm{d}t\mathrm{d}x}{t} \leqslant C. \tag{30}$$

Then the quadratic estimate

$$\iint_0^\infty |T_t(f)(x)|^2 \frac{\mathrm{d}t\mathrm{d}x}{t} \lesssim \|f\|^2_{L^2(\mathbb{R}^n)}$$

holds.

Now, we have that

$$|x|^N|\partial^\alpha\theta(x+v)| \lesssim |x+v|^N|\partial^\alpha\theta(x+v)| + |v|^N|\partial^\alpha\theta(x+v)| \lesssim 1+|v|^N$$

for any $N \geqslant 0$ and multi-index α, since θ is a Schwartz function. Therefore

$$|\partial^\alpha\theta_t^v(x)| \lesssim \frac{\langle v\rangle^N}{t^{n+|\alpha|}\langle x/t\rangle^N}$$

and so $\theta_t^v(x-y)$ satisfies (28) with a constant C that only depends polynomially on v. By the mean-value theorem, $\theta_t^v(x-y)$ also satisfies (29). It is even easier to see that the kernel of R_t, which we can write in the form $m_t(x-y) := \rho(t\cdot)\check{\ }(x-y)$, also satisfies (28). From these estimates it follows that the kernel of $(R_tM_bP_t^v)$,

$$K_t(x,y) = \int m_t(x-z)b(z)\theta_t^v(z-y)\mathrm{d}z, \tag{31}$$

also satisfies estimates (28) and (29). This may be seen by splitting the integral in (31) into integration over two half-planes H_x and H_y containing x and y, respectively, with the common boundary being the hyperplane perpendicular to the line segment passing through x and y and containing the mid-point $(x+y)/2$. Clearly

*This result is implicit in the work of R. Coifman and Y. Meyer [8]. Now there are also generalisations in the form of local $T(b)$-theorems; see, for example, S. Hofmann [14].

then $|y-z| \geqslant |x-y|/2$ when $z \in H_x$ and $|x-z| \geqslant |x-y|/2$ when $z \in H_y$, so we may conclude

$$\begin{aligned}
&\left|\int m_t(x-z)b(z)\theta_t^v(z-y)\mathrm{d}z\right| \\
&= \left|\int_{H_x} m_t(x-z)b(z)\theta_t^v(z-y)\mathrm{d}z + \int_{H_y} m_t(x-z)b(z)\theta_t^v(z-y)\mathrm{d}z\right| \\
&\lesssim \int_{H_x} \langle (x-z)/t\rangle^{-(n+1)}\langle (z-y)/t\rangle^{-(n+1)} t^{-2n}\mathrm{d}z \\
&\quad + \int_{H_y} \langle (x-z)/t\rangle^{-(n+1)}\langle (z-y)/t\rangle^{-(n+1)} t^{-2n}\mathrm{d}z \\
&\lesssim \langle (x-y)/t\rangle^{-(n+1)} \int_{H_x} \langle (x-z)/t\rangle^{-(n+1)} t^{-2n}\mathrm{d}z \\
&\quad + \langle (x-y)/t\rangle^{-(n+1)} \int_{H_y} \langle (z-y)/t\rangle^{-(n+1)} t^{-2n}\mathrm{d}z \\
&\lesssim t^{-n}\langle (x-y)/t\rangle^{-(n+1)}.
\end{aligned}$$

This proves (28), and (29) follows similarly (see [12] and also page A-36 in [10]).

The measure

$$|(R_t M_b P_t^v)(1)(x)|^2 \frac{\mathrm{d}t\mathrm{d}x}{t} = |R_t(b)(x)|^2 \frac{\mathrm{d}t\mathrm{d}x}{t}$$

is a Carleson measure, since $b \in L^\infty(\mathbb{R}^n) \subset \mathrm{BMO}(\mathbb{R}^n)$ and m_t has mean-value zero (see Theorem 3 on page 159 of [21]), whence we have estimate (30). Therefore, we may apply Theorem 3.3 and obtain (26) as required. Thus we can now move on to the only remaining task in the proof of Theorem 1.1, which is to prove Theorem 3.2.

4. Proof of Theorem 3.2

Both Q_t^u and P_t^v are convolutions with smooth functions and hence multipliers. To deal with both such operators together we define

$$R_t(f) = \int \rho(t\xi)\widehat{f}(\xi)e^{ix\cdot\xi}đ\xi,$$

where we have in mind that ρ will eventually be assumed to have properties similar to those of $\widehat{\psi}$ or $\widehat{\theta}$. To be precise, it is sufficient to assume $(x,\xi) \mapsto \rho(\xi)$ belongs to $S^0_{1,0}(n,1)$. Define

$$T^\varphi_{a_t}(f)(x) = \int a_t(\xi)\widehat{f}(\xi)e^{i\varphi(x,\xi)}d\xi, \tag{32}$$

for a smooth amplitude $(x,\xi) \mapsto a_t(\xi)$ which belongs to $S^0_{0,0}(n,1)$ uniformly in $t \in (0,1)$ and φ satisfying the strong non-degeneracy condition, (9) and the equation

$\varphi(x,0)=0$. Now we can compute

$$\begin{aligned}(R_tT^{\varphi}_{a_t})(f)(x) &= \int \rho(t\eta)\left(\int\left(\int a_t(\xi)\widehat{f}(\xi)e^{i\varphi(y,\xi)}đ\xi\right)e^{-iy\cdot\eta}\mathrm{d}y\right)e^{ix\cdot\eta}đ\eta\\ &= \iiint \rho(t\eta)a_t(\xi)e^{i(x-y)\cdot\eta+i\varphi(y,\xi)-i\varphi(x,\xi)}\widehat{f}(\xi)e^{i\varphi(x,\xi)}đ\eta đ\xi\mathrm{d}y\\ &:= \int \sigma_t(x,\xi)\widehat{f}(\xi)e^{i\varphi(x,\xi)}đ\xi,\end{aligned}$$

where

$$\sigma_t(x,\xi) := a_t(\xi)\iint \rho(t\eta)e^{i(x-y)\cdot\eta+i\varphi(y,\xi)-i\varphi(x,\xi)}đ\eta\mathrm{d}y. \tag{33}$$

Therefore $R_tT^{\varphi}_{a_t}$ can be represented as an oscillatory integral operator with phase function φ and amplitude σ_t. Now we would like to understand the behaviour of σ_t. To this end we start with the following proposition.

Proposition 4.1. *Assume that $\varphi\in\mathcal{C}^\infty(\mathbb{R}^n\times\mathbb{R}^n)$ is strongly non-degenerate and satisfies* (9) *and the equation $\varphi(x,0)=0$. Then for each $\varepsilon\in(\frac12,1)$ there exist $M=M(\varepsilon)$ and $\mu=\mu(\varepsilon)$, both greater than zero, such that* (33) *can be written as*

$$\sigma_t(x,\xi)=\rho(t\nabla_x\varphi(x,\xi))a_t(\xi)+\sum_{0<|\alpha|<M}\frac{t^{|\alpha|}}{\alpha!}\,\sigma_\alpha(t,x,\xi)+t^{\mu}r(t,x,\xi), \tag{34}$$

for $t\in[0,1]$, where for all multi-indices β and γ one has

$$\begin{aligned}|\partial^\gamma_\xi\partial^\beta_x\sigma_\alpha(t,x,\xi)| &\leqslant C_{\alpha,\beta,\gamma}\,t^{-\varepsilon|\alpha|} \quad \textit{and}\\ |\partial^\alpha_\xi\partial^\beta_x r(t,x,\xi)| &\leqslant C_{\alpha,\beta}.\end{aligned}$$

Proof. We shall begin by showing that the assumptions on the phase function φ imply that for all $x,\xi\in\mathbb{R}^n$ and $|\alpha|,|\beta|\geqslant 1$ one has

$$C_1|\xi|\leqslant|\nabla_x\varphi(x,\xi)|\leqslant C_2|\xi|, \tag{35}$$

$$|\partial^\alpha_x\varphi(x,\xi)|\leqslant C_\alpha\langle\xi\rangle \quad \text{and} \tag{36}$$

$$|\partial^\alpha_\xi\partial^\beta_x\varphi(x,\xi)|\leqslant C_{\alpha,\beta}. \tag{37}$$

Once this is done we shall then prove (34) using only (35), (36) and (37).

Condition (37) is exactly (9), so it is immediate. The mean value theorem, (37) and the fact $\varphi(x,0)=0$ imply that

$$|\nabla_x\varphi(x,\xi)|=|\nabla_x\varphi(x,\xi)-\nabla_x\varphi(x,0)|\lesssim|\xi|.$$

On the other hand, Schwartz's global inverse function theorem can be used just as in the proof of Proposition 1.2.4 (1) in [9] to prove that a consequence of the strong non-degeneracy condition and the fact that $\varphi(x,0)=0$ is

$$|\xi|\lesssim|\nabla_x\varphi(x,\xi)-\nabla_x\varphi(x,0)|=|\nabla_x\varphi(x,\xi)|.$$

Thus (35) follows. Finally,

$$|\partial^\alpha_x\varphi(x,\xi)|=|\partial^\alpha_x\varphi(x,\xi)-\partial^\alpha_x\varphi(x,0)|,$$

so the mean value theorem and (37) imply (36) for $|\alpha|\geqslant 1$.

Now, it remains to prove (34) using (35), (36) and (37). We first note that it is sufficient to prove the proposition in the special case $a_t(\xi)\equiv 1$ in (33). The general case may be obtained by multiplying (34) (with $a_t(\xi)$ replaced by the constant 1) by $a_t(\xi)$ and observing that all the claimed properties hold for the product since $a_t \in S^0_{0,0}(n,1)$.

The proof follows the presentation in [20] and [18]. Let $\chi(x-y) \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that $0 \leqslant \chi \leqslant 1$, $\chi(x-y) = 1$ for $|x-y| < \frac{\varepsilon}{2}$ and $\chi(x-y) = 0$ for $|x-y| > \varepsilon$. We now decompose $\sigma_t(x,\xi)$ into two parts $\mathbf{I}_1(t,x,\xi)$ and $\mathbf{I}_2(t,x,\xi)$, where

$$\mathbf{I}_1(t,x,\xi) = \iint \rho(t\eta)\,(1-\chi(x-y))\, e^{i(x-y)\cdot\eta + i\varphi(y,\xi) - i\varphi(x,\xi)}\, đ\eta\, \mathrm{d}y$$

and

$$\mathbf{I}_2(t,x,\xi) = \iint \rho(t\eta)\,\chi(x-y)\, e^{i(x-y)\cdot\eta + i\varphi(y,\xi) - i\varphi(x,\xi)}\, đ\eta\, \mathrm{d}y.$$

We begin by analysing $\mathbf{I}_1(t,x,\xi)$. To this end we introduce the differential operators

$${}^tL_\eta = -i\sum_{j=1}^n \frac{x_j - y_j}{|x-y|^2}\partial_{\eta_j} \quad \text{and} \quad {}^tL_y = \frac{1}{\langle \nabla_y\varphi(y,\xi)\rangle^2 - i\Delta_y\varphi(y,\xi)}(1-\Delta_y).$$

Because of (35), one has $|\langle \nabla_y\varphi(y,\xi)\rangle^2 - i\Delta_y\varphi(y,\xi)| \geqslant \langle \nabla_y\varphi(y,\xi)\rangle^2 \geqslant C_1\langle\xi\rangle^2$. Now integration by parts yields

$$\mathbf{I}_1(t,x,\xi) = \iint L_y^{N_2}\{e^{-iy\cdot\eta}L_\eta^{N_1}[(1-\chi(x-y))\rho(t\eta)]\}\, e^{ix\cdot\eta + i\varphi(y,\xi) - i\varphi(x,\xi)}\, đ\eta\, \mathrm{d}y.$$

Now since $t \leqslant 1$, provided $M < N_1$, we have

$$\begin{aligned}|\partial^{N_1}_{\eta_j}\rho(t\eta)| &\leqslant C_{N_1}\, t^{N_1}\langle t\eta\rangle^{-N_1} = C_{N_1}\, t^{N_1}\langle t\eta\rangle^{-M}\langle t\eta\rangle^{-(N_1-M)}\\ &\leqslant C_{N_1}\, t^{N_1}(t^2+|t\eta|^2)^{-M/2}\langle t\eta\rangle^{-(N_1-M)} \leqslant C_{N_1}\, t^{N_1-M}\langle\eta\rangle^{-M}.\end{aligned}$$

Therefore, choosing $N_2 < M < N_1$ large enough we have

$$\begin{aligned}|\mathbf{I}_1(t,x,\xi)| &\leqslant t^{N_1-M}\langle\xi\rangle^{-2N_2}\iint_{|x-y|>\varepsilon}\langle\eta\rangle^{2N_2}|x-y|^{-N_1}\langle\eta\rangle^{-M}\, đ\eta\, \mathrm{d}y\\ &\lesssim t^{N_1-M}\langle\xi\rangle^{-2N_2}.\end{aligned}$$

Estimating derivatives of $\mathbf{I}_1(t,x,\xi)$ with respect to x and ξ may introduce factors estimated by powers of $\langle\xi\rangle$, $\langle\eta\rangle$ and $|x-y|$, which can all be handled by choosing N_1 and N_2 appropriately. Therefore for all N and some $\nu > 0$

$$|\partial^\alpha_\xi\partial^\beta_x\mathbf{I}_1(t,x,\xi)| \leqslant C_{\alpha,\beta}\, t^\nu\,\langle\xi\rangle^{-N},$$

and so $t^{-\nu}\mathbf{I}_1(t,x,\xi)$ forms part of the error term $r(t,x,\xi)$ in (34).

We now proceed to the analysis of $\mathbf{I}_2(t,x,\xi)$. First we make the change of variables $\eta = \nabla_x\varphi(x,\xi) + \zeta$ in the integral defining $\mathbf{I}_2(t,x,\xi)$ and then expand $\rho(t\eta)$ in a Taylor series to obtain

$$\rho(t\nabla_x\varphi(x,\xi) + t\zeta) = \sum_{0\leqslant|\alpha|<M} t^{|\alpha|}\frac{\zeta^\alpha}{\alpha!}(\partial^\alpha_\xi\rho)(t\nabla_x\varphi(x,\xi)) + t^M\sum_{|\alpha|=M} C_\alpha\zeta^\alpha r_\alpha(t,x,\xi,\zeta),$$

where

$$r_\alpha(t,x,\xi,\zeta) = \int_0^1 (1-s)^{M-1}(\partial^\alpha_\xi\rho)(t\nabla_x\varphi(x,\xi) + st\zeta)\, ds.$$

If we set

$$\Phi(x,y,\xi) = \varphi(y,\xi) - \varphi(x,\xi) + (x-y)\cdot\nabla_x\varphi(x,\xi),$$

we obtain

$$\mathbf{I}_2(t,x,\xi) = \sum_{|\alpha|<M} \frac{t^{|\alpha|}}{\alpha!}\,\sigma_\alpha(t,x,\xi) + t^M \sum_{|\alpha|=M} C_\alpha\, R_\alpha(t,x,\xi),$$

where

$$\begin{aligned}\sigma_\alpha(t,x,\xi) &= \iint e^{i(x-y)\cdot\zeta + i\Phi(x,y,\xi)}\zeta^\alpha\,\chi(x-y)\,(\partial_\xi^\alpha \rho)(t\nabla_x\varphi(x,\xi))\,dy\,đ\zeta \\ &= (\partial_\xi^\alpha\rho)(t\nabla_x\varphi(x,\xi))\partial_y^\alpha\left[e^{i\Phi(x,y,\xi)}\chi(x-y)\right]\Big|_{y=x}\end{aligned}$$

and

$$R_\alpha(t,x,\xi) = \iint e^{i(x-y)\cdot\zeta}e^{i\Phi(x,y,\xi)}\zeta^\alpha\chi(x-y)\,r_\alpha(t,x,\xi,\zeta)\,dy\,đ\zeta.$$

We now claim that

$$\left|\partial_y^\gamma e^{i\Phi(x,y,\xi)}\Big|_{y=x}\right| \lesssim \langle\xi\rangle^{|\gamma|/2}. \tag{38}$$

We first observe that when $\gamma = 0$, (38) is obvious. To obtain (38) for $\gamma \neq 0$ we recall Faa-Di Bruno's formula:

$$\partial_y^\gamma e^{i\Phi(x,y,\xi)} = \sum_{\gamma_1+\cdots+\gamma_k=\gamma} C_\gamma(\partial_y^{\gamma_1}\Phi(x,y,\xi))\cdots(\partial_y^{\gamma_k}\Phi(x,y,\xi))\,e^{i\Phi(x,y,\xi)},$$

with the sum ranges of γ_j such that $|\gamma_j| \geqslant 1$ for $j = 1,2,\dots,k$ and $\gamma_1+\cdots+\gamma_k = \gamma$ for some $k \in \mathbb{N}$. Since $\Phi(x,x,\xi) = 0$ and $\partial_y\Phi(x,y,\xi)|_{y=x} = 0$, setting $y = x$ in the expansion above leaves only terms in which $|\gamma_j| \geqslant 2$ for all $j = 1,2,\dots,k$. But $\sum_{j=1}^k |\gamma_j| \leqslant |\gamma|$, so we actually have $2k \leqslant |\gamma|$, that is, $k \leqslant \frac{|\gamma|}{2}$. Estimate (36) on the phase tells us that $|\partial_y^{\gamma_j}\Phi(x,y,\xi)| \lesssim \langle\xi\rangle$, so

$$\left|\partial_y^\gamma e^{i\Phi(x,y,\xi)}\Big|_{y=x}\right| \lesssim \langle\xi\rangle\cdots\langle\xi\rangle \lesssim \langle\xi\rangle^k \lesssim \langle\xi\rangle^{|\gamma|/2},$$

which is (38).

If we use the fact that $t \leqslant 1$ and estimate (35) on the phase function φ, we obtain

$$|\sigma_\alpha(t,x,\xi)| \lesssim \langle t\nabla_x\varphi(x,\xi)\rangle^{-|\alpha|}\langle\xi\rangle^{\frac{|\alpha|}{2}} \lesssim t^{-\frac{|\alpha|}{2}}\langle t\nabla_x\varphi(x,\xi)\rangle^{-|\alpha|}\langle t\xi\rangle^{\frac{|\alpha|}{2}} \lesssim t^{-\frac{|\alpha|}{2}}$$

when $|\alpha| > 0$ and, clearly, $\sigma_\alpha(t,x,\xi) = \rho(t\nabla_x\varphi(x,\xi))$ when $\alpha = 0$.

The derivatives of σ_α with respect to x or ξ do not change the estimates when applied to ρ by the assumptions of the lemma. When applied to $\partial_y^\alpha e^{i\Phi(x,y,\xi)}|_{y=x}$ they do not change estimates since $|\partial_\xi^\beta\partial_x^\alpha\varphi(x,\xi)| \leqslant C_{\alpha\beta}$. Therefore for all multi-indices β, $\gamma \in \mathbb{Z}_+$,

$$|\partial_\xi^\beta\partial_x^\gamma\sigma_\alpha(t,x,\xi)| \leqslant C_{\beta,\gamma}t^{-\frac{|\alpha|}{2}},$$

as required.

To estimate the remainder, R_α, we take $g \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ such that $g(x) = 1$ for $|x| < r/2$ and $g(x) = 0$ for $|x| > r$, for some small $r > 0$ to be chosen later. We then decompose

$$\begin{aligned} R_\alpha(t,x,\xi) &= R^{I}_\alpha(t,x,\xi) + R^{II}_\alpha(t,x,\xi) \\ &= \iint e^{i(x-y)\cdot\zeta} g\left(\frac{\zeta}{\langle\xi\rangle}\right) D^\alpha_y \left[e^{i\Phi(x,y,\xi)}\,\chi(x-y)\,r_\alpha(t,x,\xi,\zeta)\right] dy\,đ\zeta \\ &\quad + \iint e^{i(x-y)\cdot\zeta} \left(1-g\left(\frac{\zeta}{\langle\xi\rangle}\right)\right) D^\alpha_y \left[e^{i\Phi(x,y,\xi)}\,\chi(x-y)\,r_\alpha(t,x,\xi,\zeta)\right] dy\,đ\zeta. \end{aligned}$$

As a preamble to estimating $R^I_\alpha(t,x,\xi)$, we note that the inequality

$$\langle\xi\rangle \leqslant 1+|\xi| \leqslant \sqrt{2}\langle\xi\rangle$$

and estimate (35) yield

$$\begin{aligned} \langle t\nabla_x\varphi(x,\xi)+ts\zeta\rangle &\leqslant (C_2\sqrt{2}+r)\langle t\xi\rangle \quad \text{and} \\ \sqrt{2}\langle t\nabla_x\varphi(x,\xi)+ts\zeta\rangle &\geqslant 1+|t\nabla_x\varphi|-|t\zeta| \\ &\geqslant 1+C_1|t\xi|-tr\langle\xi\rangle \\ &\geqslant (1-r)+(C_1-r)|t\xi| \geqslant (\min\{1,C_1\}-r)\langle t\xi\rangle. \end{aligned}$$

Therefore, if we choose $r<\min\{1,C_1\}$, then for any $s\in(0,1)$, $\langle t\nabla_x\varphi(x,\xi)+ts\zeta\rangle$ and $\langle t\xi\rangle$ are equivalent.

This yields that for $|\zeta|\leqslant r\langle\xi\rangle$, $r_\alpha(t,x,\xi,\zeta)$ and all of its derivatives are dominated by $\langle t\xi\rangle^{-|\alpha|}$. Furthermore, for $t\leqslant 1$, it follows from the properties of r_α that

$$\begin{aligned} \left|\partial^\beta_\zeta\left(g\left(\frac{\zeta}{\langle\xi\rangle}\right) r_\alpha(t,x,\xi,\zeta)\right)\right| &\leqslant C_{\alpha,\beta}\sum_{\gamma\leqslant\beta}\left|\partial^\gamma_\zeta g\left(\frac{\zeta}{\langle\xi\rangle}\right)\partial^{\beta-\gamma}_\zeta r_\alpha(t,x,\xi,\zeta)\right| \\ &\leqslant C_{\alpha,\beta}\sum_{\gamma\leqslant\beta} t^{|\beta|-|\gamma|}\langle\xi\rangle^{-|\gamma|}\langle t\xi\rangle^{-|\alpha|-|\beta|+|\gamma|} \\ &\leqslant C_{\alpha,\beta}t^{-\varepsilon|\alpha|}\,\langle\xi\rangle^{-|\beta|-\varepsilon|\alpha|}, \end{aligned} \tag{39}$$

for all $\varepsilon\in(\frac{1}{2},1)$.

At this point we also need estimates for $\partial^\alpha_y e^{i\Phi(x,y,\xi)}$ off the diagonal, that is, when $x\neq y$. This derivative has at most $|\alpha|$ powers of terms $\nabla_y\varphi(y,\xi)-\nabla_x\varphi(x,\xi)$, possibly also multiplied by at most $|\alpha|$ higher order derivatives $\partial^\beta_y\varphi(y,\xi)$, which can be estimated by $(|y-x|\langle\xi\rangle)^{|\alpha|}$ using (36). The term containing no difference $\nabla_y\varphi(y,\xi)-\nabla_x\varphi(x,\xi)$ is the product of at most $|\alpha|/2$ terms of the type $\partial^\beta_y\varphi(y,\xi)$, which can be estimated by $\langle\xi\rangle^{|\alpha|/2}$ in view of (36). These observations yield

$$|\partial^\alpha_y e^{i\Phi(x,y,\xi)}| \leqslant C_\alpha(1+|x-y|\langle\xi\rangle)^{|\alpha|}\langle\xi\rangle^{|\alpha|/2},$$

and therefore we also have

$$\left|\partial^\alpha_y\left[e^{i\Phi(x,y,\xi)}\chi(x-y)\right]\right| \lesssim (1+|x-y|\langle\xi\rangle)^{|\alpha|}\langle\xi\rangle^{\frac{|\alpha|}{2}}. \tag{40}$$

Now to estimate $R^I_\alpha(t,x,\xi)$, let

$$L_\zeta = \frac{(1-\langle\xi\rangle^2\Delta_\zeta)}{1+\langle\xi\rangle^2|x-y|^2}, \quad \text{so} \quad L^N_\zeta e^{i(x-y)\cdot\zeta} = e^{i(x-y)\cdot\zeta}.$$

Integration by parts with L_ζ yields

$$\begin{aligned}
&R^I_\alpha(t,x,\xi)\\
&=\iint \frac{e^{i(x-y)\cdot\zeta}\,\partial_y^\alpha\left[\chi(x-y)\,e^{i\Phi(x,y,\xi)}\right]}{(1+\langle\xi\rangle^2|x-y|^2)^N}(1-\langle\xi\rangle^2\Delta_\zeta)^N\left\{g\left(\frac{\zeta}{\langle\xi\rangle}\right)r_\alpha(t,x,\xi,\zeta)\right\}dy\,đ\zeta\\
&=\iint \frac{e^{i(x-y)\cdot\zeta}\,\partial_y^\alpha\left[\chi(x-y)\,e^{i\Phi(x,y,\xi)}\right]}{(1+\langle\xi\rangle^2|x-y|^2)^N}\\
&\qquad\times\sum_{|\beta|\leqslant 2N} c_\beta\langle\xi\rangle^{|\beta|}\left\{\partial_\zeta^\beta\left(g\left(\frac{\zeta}{\langle\xi\rangle}\right)r_\alpha(t,x,\xi,\zeta)\right)\right\}dy\,đ\zeta.
\end{aligned}$$

Using estimates (39), (40) and that the size of the support of $g(\zeta/\langle\xi\rangle)$ in ζ is bounded by $(r\langle\xi\rangle)^n$ yield

$$\begin{aligned}
|R^I_\alpha(t,x,\xi)| &\leqslant C\,t^{-\varepsilon|\alpha|}\sum_{|\beta|\leqslant 2N}\langle\xi\rangle^{n+|\beta|}\langle\xi\rangle^{-\varepsilon|\alpha|-|\beta|}\langle\xi\rangle^{\frac{|\alpha|}{2}}\\
&\quad\times\int_{|x-y|<\varepsilon}\frac{(1+|x-y|\langle\xi\rangle)^{|\alpha|}}{(1+\langle\xi\rangle^2|x-y|^2)^N}\mathrm{d}y\\
&\leqslant C\,t^{-\varepsilon|\alpha|}\langle\xi\rangle^{2n+(\frac12-\varepsilon)|\alpha|}
\end{aligned}$$

if we choose $2N>n$, and the constant C is independent of t (because of (39)). The derivatives of $R^I_\alpha(t,x,\xi)$ with respect to x and ξ give an extra power of ζ under the integral. This amounts to taking more y-derivatives, yielding a higher power of $\langle\xi\rangle$. However, for a given number of derivatives of the remainder $R^I_\alpha(t,x,\xi)$, we are free to choose $M=|\alpha|$ as large as we like and therefore the higher power of $\langle\xi\rangle$ will not cause a problem. Thus for all multi-indices β, $\gamma\in\mathbb{Z}_+$, all $\varepsilon\in(\frac12,1)$ and all $|\alpha|>\frac{4n}{2\varepsilon-1}$, we have

$$|\partial_\xi^\beta\partial_x^\gamma R^I_\alpha(t,x,\xi)|\leqslant C_{\beta,\gamma}t^{-\varepsilon|\alpha|},$$

where the constant $C_{\beta,\gamma}$ does not depend on t.

Finally, to estimate $R^{II}_\alpha(t,x,\xi)$ one defines

$$\Psi(x,y,\xi,\zeta)=(x-y)\cdot\zeta+\Phi(x,y,\xi)=(x-y)\cdot(\nabla_x\varphi(x,\xi)+\zeta)+\varphi(y,\xi)-\varphi(x,\xi).$$

It follows from (35) and (36) that if we choose $\varepsilon<r/8C_0$, then since $|x-y|<\varepsilon$ on the support of χ, one has (using that we are in the region $|\zeta|\geqslant\frac r2\langle\xi\rangle$)

$$\begin{aligned}
&|\nabla_y\Psi|=|-\zeta+\nabla_y\varphi-\nabla_x\varphi|\leqslant 2C_2(|\zeta|+\langle\xi\rangle),\quad\text{and}\\
&|\nabla_y\Psi|\geqslant|\zeta|-|\nabla_y\varphi-\nabla_x\varphi|\geqslant\frac12|\zeta|+\left(\frac r4-C_0|x-y|\right)\langle\xi\rangle\geqslant C(|\zeta|+\langle\xi\rangle).
\end{aligned}$$

Now, using (36), for any β we have the estimate

$$|\partial_y^\beta(e^{-i\Phi(x,y,\xi)}\partial_y^\gamma e^{i\Phi(x,y,\xi)})(x,y,\xi)|\lesssim\langle\xi\rangle^{|\gamma|}. \tag{41}$$

For $M=|\alpha|>0$ we also observe that

$$|r_\alpha(t,x,\xi,\zeta)|\leqslant C_\alpha. \tag{42}$$

For the differential operator defined to be ${}^tL_y = i|\nabla_y\Psi|^{-2}\sum_{j=1}^n(\partial_{y_j}\Psi)\partial_{y_j}$, induction shows that L_y^N has the form

$$L_y^N = \frac{1}{|\nabla_y\Psi|^{4N}}\sum_{|\beta|\leqslant N} P_{\beta,N}\partial_y^\beta, \quad \text{where} \quad P_{\beta,N} = \sum_{|\mu|=2N} c_{\beta\mu\delta_j}(\nabla_y\Psi)^\mu\partial_y^{\delta_1}\Psi\cdots\partial_y^{\delta_N}\Psi,$$

$|\mu| = 2N$, $|\delta_j| \geqslant 1$ and $\sum_{j=M}^N|\delta_j| + |\beta| = 2N$. It follows from (36) that $|P_{\beta,N}| \leqslant C(|\zeta| + \langle\xi\rangle)^{3N}$. Now Leibniz's rule yields

$$\begin{aligned}
R_\alpha^{I\!I}(t,x,\xi) &= \iint e^{i(x-y)\cdot\zeta}\left(1 - g\left(\frac{\zeta}{\langle\xi\rangle}\right)\right) r_\alpha(x,\xi,\zeta)D_y^\alpha\left[e^{i\Phi(x,y,\xi)}\chi(x-y)\right]\mathrm{d}y\,\text{đ}\zeta\\
&= \iint e^{i\Psi(x,y,\xi,\zeta)}\left(1 - g\left(\frac{\zeta}{\langle\xi\rangle}\right)\right) r_\alpha(t,x,\xi,\zeta)\\
&\quad\times\sum_{\gamma_1+\gamma_2=\alpha}(e^{-i\Phi(x,y,\xi)}D_y^{\gamma_1}e^{i\Phi(x,y,\xi)})\,D_y^{\gamma_2}\chi(x-y)\mathrm{d}y\,\text{đ}\zeta\\
&= \iint e^{i\Psi(x,y,\xi,\zeta)}|\nabla_y\Psi|^{-4N}\sum_{|\beta|\leqslant N}P_{\beta,N}(x,y,\xi,\zeta)\left(1 - g\left(\frac{\zeta}{\langle\xi\rangle}\right)\right) r_\alpha(t,x,\xi,\zeta)\\
&\quad\times\sum_{\gamma_1+\gamma_2=\alpha}\partial_y^\beta[(e^{-i\Phi(x,y,\xi)}D_y^{\gamma_1}e^{i\Phi(x,y,\xi)})\,D_y^{\gamma_2}\chi(x-y)]\mathrm{d}y\,\text{đ}\zeta.
\end{aligned}$$

It follows now from (41) and (42) that

$$\begin{aligned}
|R_\alpha^{I\!I}(t,x,\xi)| &\leqslant C\int_{|\zeta|\geqslant\frac{r}{2}\langle\xi\rangle}\int_{|x-y|<\varepsilon}(|\zeta| + \langle\xi\rangle)^{-N}\langle\xi\rangle^{|\alpha|}\,\mathrm{d}y\,\text{đ}\zeta\\
&\leqslant C\langle\xi\rangle^{|\alpha|}\int_{|\zeta|\geqslant\frac{r}{2}\langle\xi\rangle}|\zeta|^{-N}\text{đ}\zeta\\
&\leqslant C\langle\xi\rangle^{|\alpha|+n-N},
\end{aligned}$$

which yields the desired estimate when $N > |\alpha| + n$. For the derivatives of $R_\alpha^{I\!I}(t,x,\xi)$, we can get, in a similar way to the case for R_α^I, an extra power of ζ, which can be taken care of by choosing N large and using the fact that $|x-y| < \varepsilon$. Therefore for all multi-indices β, $\gamma \in \mathbb{Z}_+$,

$$|\partial_\xi^\beta\partial_x^\gamma R_\alpha^{I\!I}(t,x,\xi)| \leqslant C_{\beta,\gamma}, \tag{43}$$

where the constant $C_{\beta,\gamma}$ does not depend on t. The proof of Proposition 4.1 is now complete. □

To prove (i) of Theorem 3.2, we apply Proposition 4.1 with $\varphi = \varphi_1$, $a_t = \mu$ and $R_t = P_t^v$. This shows us that $P_t^vT_\mu^{\varphi_1}$ is an oscillatory integral operator with phase φ_1 and amplitude of the form

$$\mu(\xi)\widehat{\theta}(t\nabla_x\varphi(x,\xi))e^{it\nabla_x\varphi(x,\xi)\cdot v} + \sum_{0<|\alpha|<M}\frac{t^{|\alpha|}}{\alpha!}\sigma_\alpha(t,x,\xi) + t^\mu r(t,x,\xi).$$

We define $W_{1,t}^v$ to be equal to the oscillatory integral operator with phase φ_1 and amplitude

$$-\left(\sum_{0<|\alpha|<M}\frac{t^{|\alpha|}}{\alpha!}\sigma_\alpha(t,x,\xi) + t^\mu r(t,x,\xi)\right) \tag{44}$$

and V_t^v to be equal to the oscillatory integral operator with phase φ_1 and amplitude

$$\mu(\xi)\left(\widehat{\theta}(t\xi)e^{it\xi\cdot v}-\widehat{\theta}(t\nabla_x\varphi(x,\xi))e^{it\nabla_x\varphi(x,\xi)\cdot v}\right). \tag{45}$$

Clearly then $[T_\mu^{\varphi_1},P_t^v]=V_t^v+W_{1,t}^v$.

By Proposition 4.1 the amplitude (44) is in $S^0_{0,0}(n,1)$ with seminorms which have a dependence on t of the form t^ε for some small $\varepsilon>0$. Therefore, using Theorem 2.3, $\|W_{1,t}^v(f)\|_{L^2(\mathbb{R}^n)}\lesssim t^\varepsilon\|f\|_{L^2(\mathbb{R}^n)}$. To complete the proof of (i) we also need to prove (25), but as this is quite long, we will first dispose of (ii) and (iii).

To prove (ii) of Theorem 3.2, we apply Proposition 4.1 with $a_t(\xi)=\widehat{\psi}(t\xi)e^{it\xi\cdot u}\mu(\xi)$ and ρ chosen to be radial, supported on an annulus and such that $\rho(t\nabla_x\varphi(x,\xi))=1$ for all $\xi\in\operatorname{supp}(a_t)$ and all $x\in\mathbb{R}^n$. This is possible again since φ_2 satisfies the hypotheses of Proposition 4.1 on the support of μ, as discussed previously. If we set U_t^u equal to the oscillatory integral operator with phase φ_2 and amplitude $\widehat{\psi}(t\xi)e^{it\xi\cdot u}\mu(\xi)$, then the amplitude of $R_tU_t^u$ is of the form

$$\begin{aligned}&\rho(t\nabla_x\varphi(x,\xi))\widehat{\psi}(t\xi)e^{it\xi\cdot u}\mu(\xi)+\sum_{0<|\alpha|<M}\frac{t^{|\alpha|}}{\alpha!}\,\sigma_\alpha(t,x,\xi)+t^\mu r(t,x,\xi)\\&=\widehat{\psi}(t\xi)e^{it\xi\cdot u}\mu(\xi)+\sum_{0<|\alpha|<M}\frac{t^{|\alpha|}}{\alpha!}\,\sigma_\alpha(t,x,\xi)+t^\mu r(t,x,\xi).\end{aligned}$$

Therefore $R_tU_t^u=T_\mu^{\varphi_2}Q_t^u-W_{2,t}^u$ and $\|W_{2,t}^u(f)\|_{L^2(\mathbb{R}^n)}\lesssim t^\varepsilon\|f\|_{L^2(\mathbb{R}^n)}$ as before. Since a_t is smooth and compactly supported, $\|U_t^u(f)\|_{L^2(\mathbb{R}^n)}\lesssim\|f\|_{L^2(\mathbb{R}^n)}$ with an implicit constant that grows at most polynomially in u (we can see this by applying, for example, Theorem 2.3).

To prove (iii) of Theorem 3.2, first observe that by the mean-value theorem we have that

$$|m(t,y,u,v)-m(t,x,u,v)|\lesssim|y-x|$$

with an implicit constant that is independent of t, u and v. This is because $\partial_x^\alpha m(t,x,u,v)$ is bounded. Using this, we compute

$$\begin{aligned}|W_{3,t}^{u,v}(f)(x)|&=|[R_t,M_m](f)(x)|\\&\lesssim\left|\int t^{-n}\check{\rho}\left(\frac{x-y}{t}\right)\big(m(t,y,u,v)-m(t,x,u,v)\big)f(y)\mathrm{d}y\right|\\&\lesssim t\int\left|t^{-n}\check{\rho}\left(\frac{x-y}{t}\right)\right|\frac{|x-y|}{t}|f(y)|\mathrm{d}y,\end{aligned}$$

where

$$\check{\rho}(x)=\int\rho(\xi)e^{ix.\xi}\,\text{đ}\xi.$$

The estimate $\|W_{3,t}^{u,v}(f)\|_{L^2(\mathbb{R}^n)}\lesssim t\|f\|_{L^2(\mathbb{R}^n)}$ now follows from Young's inequality since the L^1-norm of $x\mapsto t^{-n}\check{\rho}(x/t)x/t$ is independent of t.

Now to complete the proof of Theorem 3.2 it only remains to prove (25). Remembering that the amplitude of V_t^v is given by (45), let us define

$$\sigma_t^v(x,\xi)=\widehat{\theta}(t\xi)e^{itv\cdot\xi}-\widehat{\theta}(t\nabla_x\varphi_1(x,\xi))e^{itv\cdot\nabla_x\varphi_1(x,\xi)},$$

where recall that $\widehat{\theta}\in\mathcal{C}_0^\infty(\mathbb{R}^n)$ is such that $\operatorname{supp}\widehat{\theta}\subset\{|\xi|\leqslant 5\}$ and that $\widehat{\theta}$ is constant on $\{|\xi|\leqslant 4\}$. We want to study the validity of the quadratic estimate (25) for the oscillatory integral operator

$$V_t^v(f)(x) = \int \mu(\xi)\sigma_t^v(x,\xi)e^{i\varphi_1(x,\xi)}\widehat{f}(\xi)đ\xi.$$

Observe that

$$\begin{aligned}\sigma_t^v(x,\xi) &= \left(\widehat{\theta}(t\xi) - \widehat{\theta}(t\nabla_x\varphi_1(x,\xi))\right)e^{itv\cdot\xi} + \widehat{\theta}(t\nabla_x\varphi_1(x,\xi))\left(e^{itv\cdot\xi} - e^{itv\cdot\nabla_x\varphi_1(x,\xi)}\right)\\ &= \sigma_{I,t}(x,\xi)e^{itv\cdot\xi} + \widehat{\theta}(t\nabla_x\varphi_1(x,\xi))\sigma_{II,tv}(x,\xi),\end{aligned}$$

where

$$\begin{aligned}\sigma_{I,t}(x,\xi) &= \widehat{\theta}(t\xi) - \widehat{\theta}(t\nabla_x\varphi_1(x,\xi)) \quad \text{and}\\ \sigma_{II,tv}(x,\xi) &= e^{itv\cdot\xi} - e^{itv\cdot\nabla_x\varphi_1(x,\xi)}.\end{aligned}$$

Thus

$$V_t^v f = V_{t,I}^v f + V_{t,II}^v f,$$

where $V_{t,I}^v$ and $V_{t,II}^v$ are the oscillatory integral operators with amplitudes $\mu(\xi)\sigma_{I,t}(x,\xi)e^{itv\xi}$ and $\mu(\xi)\widehat{\theta}(t\nabla_x\varphi_1(x,\xi))\sigma_{II,tv}(x,\xi)$ respectively. Thus, it suffices to prove the desired quadratic estimate for $V_{t,I}^v$ and $V_{t,II}^v$ separately. This will finally be achieved in Propositions 4.3 and 4.6 below, but to carry out this task, we shall first prove some technical lemmas.

Lemma 4.2. *The amplitude $\mu(\xi)\sigma_{I,t}(x,\xi)$ belongs to the class $S^0_{0,0}(n,1)$, where the seminorms are uniformly bounded in t for $0 < t \leqslant 1$. Moreover there exists $\widehat{\psi} \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, supported in an annulus for which*

$$\mu(\xi)\sigma_{I,t}(x,\xi) = \mu(\xi)\sigma_{I,t}(x,\xi)\widehat{\psi}(t\xi). \tag{46}$$

Proof. The first claim is a direct consequence of the chain rule, the conditions on the phase function φ_1 and the estimate

$$\sup_{0<t\leqslant 1}\left|\partial_\xi^\alpha\partial_x^\beta\widehat{\theta}(t\nabla_x\varphi_1(x,\xi))\right| \lesssim 1, \tag{47}$$

which is valid for all multi-indices α, β.

We claim that the function $\widehat{\theta}(t\nabla_x\varphi_1(x,\xi))$ is supported in $\mathbb{R}^n \times \{|\xi| \leqslant 5/tC_1\}$ and it is constant on $\mathbb{R}^n \times \{|\xi| \leqslant 4/C_2 t\}$, where C_1 and C_2 are the constants appearing in estimate (35). Recall that $\operatorname{supp}\widehat{\theta} \subset \{|\xi| \leqslant 5\}$ and that $\widehat{\theta}$ is constant on $\{|\xi| \leqslant 4\}$. Now, if $|\xi| > \frac{5}{tC_1}$, then $|\nabla_x\varphi_1(x,\xi)| \geqslant C_1|\xi| > 5/t$, which yields that for any $x \in \mathbb{R}^n$, $\widehat{\theta}(t\nabla_x\varphi_1(x,\xi)) = 0$. On the other hand, if $|\xi| < \frac{4}{tC_2}$, then $|\nabla_x\varphi_1(x,\xi)| \leqslant C_2|\xi| < 4/t$, which yields that for any $x \in \mathbb{R}^n$, $\widehat{\theta}(t\nabla_x\varphi_1(x,\xi))$ is constant and $\sigma_{I,t}(x,\xi) = 0$.

The last assertion follows by taking $\widehat{\psi} \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ such that it is equal to one on the set $\{\frac{4}{C_2} \leqslant |\xi| \leqslant \frac{5}{C_1}\}$ and it is equal to 0 on $\{|\xi| \leqslant r\}$ with $r < 4/C_2$. ☐

Proposition 4.3. *For any $f \in L^2$,*

$$\sup_{v\in\mathbb{R}^n}\left(\int_0^1 \left\|V_{t,I}^v f\right\|_{L^2}^2 \frac{\mathrm{d}t}{t}\right)^{\frac{1}{2}} \lesssim \|f\|_{L^2}.$$

Proof. By the previous lemma, $V_{t,I}f(x)=T_{\sigma_t}\left(\psi_t^v * f\right)(x)$, where $\widehat{\psi_t^v}(\xi)=\widehat{\psi}(t\xi)e^{itv\xi}$ and T_{σ_t} is an oscillatory integral with amplitude $\sigma_t(x,\xi) = \mu(\xi)\sigma_{I,t}(x,\xi)$.

Now Lemma 4.2 and Theorem 2.3 yield

$$\sup_{0<t<1} \|T_{\sigma_t}\|_{L^2\to L^2} = \mathfrak{c} < +\infty,$$

which, by using the properties of ψ, implies that

$$\begin{aligned}\int_0^1 \int_{\mathbb{R}^n} \left|T_{\sigma_t}\left(\psi_t^v * f\right)\right|^2 \,\mathrm{d}x\frac{\mathrm{d}t}{t} &\leqslant \mathfrak{c}\int_0^1 \int_{\mathbb{R}^n} \left|\psi_t^v * f(x)\right|^2 \,\mathrm{d}x\frac{\mathrm{d}t}{t} \\ &\leqslant \mathfrak{c}\int_0^\infty \int_{\mathbb{R}^n} \left|\psi_t^v * f(x)\right|^2 \,\mathrm{d}x\frac{\mathrm{d}t}{t} \lesssim \mathfrak{c}\,\|f\|_{L^2}^2 .\end{aligned}$$

□

Lemma 4.4. *The amplitude* $\widehat{\theta}(t\nabla_x\varphi_1(x,\xi))\sigma_{II,tv}(x,\xi)\mu(\xi)$ *belongs to the class* $S^0_{0,0}(n,1)$ *uniformly in* t *for* $0<t\leqslant 1$, *and*

$$\left|\partial_x^\alpha\partial_\xi^\beta\left(\mu(\xi)\widehat{\theta}(t\nabla_x\varphi_1(x,\xi))\sigma_{II,tv}(x,\xi)\right)\right| \lesssim P_{\alpha,\beta}(t\,|v|\,|\xi|). \tag{48}$$

where $P_{\alpha,\beta}(r) = \sum_{j=1}^{\max(|\beta|+|\alpha|,1)} r^j$.

Proof. We claim that, for any multi-indices α,β,

$$\left|\partial_\xi^\alpha\partial_x^\beta\sigma_{II,tv}(x,\xi)\right| \lesssim P_{\alpha,\beta}(t\,|v|\,|\xi|), \quad \text{for } |\xi|\geqslant 1.$$

For $\alpha=\beta=0$, we have by (35)

$$\left|e^{itv\cdot\xi} - e^{itv\cdot\nabla_x\varphi_1(x,\xi)}\right| \leqslant t\,|v|\,|\xi-\nabla_x\varphi_1(x,\xi)| \lesssim t\,|v|\,|\xi|\,.$$

Once again the chain rule and (35), (36), (37) yield that, for $|\alpha|+|\beta|\geqslant 1$,

$$\left|\partial_\xi^\alpha\partial_x^\beta\left(e^{itv\cdot\nabla_x\varphi_1(x,\xi)}\right)\right| \lesssim P_{\alpha,\beta}(t\,|v|) \leqslant P_{\alpha,\beta}(t\,|v|\,|\xi|), \quad \text{for } |\xi|\geqslant 1.$$

A similar estimate holds for $e^{iv\cdot\xi}$. From these estimates and (47) the lemma follows.

□

Lemma 4.5. *Let* $t\leqslant 1$ *and let* $0<s<\infty$. *Let* $\psi\in\mathcal{C}^\infty(\mathbb{R}^n)$ *such that* $\widehat{\psi}$ *is supported in* $\{5/C_1\leqslant|\xi|\leqslant 20/C_1\}$ *and such that* $\int_0^\infty|\widehat{\psi}(s\xi)|^2\frac{\mathrm{d}s}{s}=1$ *for* $\xi\neq 0$. *Consider*

$$a_{s,t}(x,\xi) = \mu(\xi)\widehat{\psi}(s\xi)\widehat{\theta}(t\nabla_x\varphi_1(x,\xi))\sigma_{II,tv}(x,\xi).$$

Then

$$\left|\partial_x^\alpha\partial_\xi^\beta a_{s,t}(x,\xi)\right| \lesssim P_{\alpha,\beta}(|v|)\min\left(\frac{s}{t},\frac{t}{s}\right), \tag{49}$$

with $P_{\alpha,\beta}$ *as above.*

Proof. Observe that $a_{s,t}$ is supported in

$$D_{s,t} = \left\{\frac{5}{sC_1}\leqslant|\xi|\leqslant\frac{20}{sC_1}\right\}\cap\left\{|\xi|\leqslant\frac{5}{tC_1}\right\}\cap\{|\xi|\geqslant 1\}\,.$$

Then if $t\geqslant s$, $\widehat{\psi}(s\xi)\widehat{\theta}(t\nabla_x\varphi_1(x,\xi))=0$, so (49) trivially holds. For $s>t$ and $\xi\in D_{s,t}$, (48) yields

$$\left|\partial_x^\alpha\partial_\xi^\beta\left(\mu(\xi)\widehat{\theta}(t\nabla_x\varphi_1(x,\xi))\sigma_{II,tv}(x,\xi)\right)\right| \lesssim P_{\alpha,\beta}(t\,|v|\,|\xi|)\leqslant P_{\alpha,\beta}\left(\frac{t\,|v|}{s}\right)\leqslant\frac{t}{s}P_{\alpha,\beta}(|v|).$$

Since for $|\xi| \geqslant 1$ and for any multi-index α, $|\partial_\xi^\alpha \widehat{\psi}(s\xi)| \lesssim 1$ uniformly in s, the Leibniz rule yields (49). □

Proposition 4.6. *Let $\{V^v_{t,II}\}_{0<t\leqslant 1, v\in\mathbb{R}^n}$ be the family of operators defined by*

$$V^v_{t,II}f(x) = \int \mu(\xi)\widehat{\theta}(t\nabla_x\varphi_1(x,\xi))\sigma_{II,tv}(x,\xi)\widehat{f}(\xi)e^{i\varphi_1(x,\xi)}\,đ\xi.$$

There is a polynomial $\mathcal{P}$ such that for any $v \in \mathbb{R}^n$ and any $f \in L^2$,

$$\left(\int_0^1 \left\|V^v_{t,II}f\right\|^2_{L^2}\frac{\mathrm{d}t}{t}\right)^{\frac{1}{2}} \lesssim \mathcal{P}(|v|)\,\|f\|_{L^2}.$$

Proof. The previous lemma and Theorem 2.3 yield that there exist two polynomials $\mathcal{P}_1$ and $\mathcal{P}_2$ such that, for any $v \in \mathbb{R}^n$,

$$\sup_{0<t\leqslant 1}\left\|V^v_{t,II}\right\|_{L^2\to L^2} \lesssim \mathcal{P}_1(|v|)$$

and, for $0 < t \leqslant 1$, $0 < s < \infty$,

$$\left\|V^v_{t,II}Q_s\right\|_{L^2\to L^2} \lesssim \mathcal{P}_2(|v|)\min\left(\frac{s}{t},\frac{t}{s}\right),$$

where Q_s denotes the convolution operator with kernel $\psi_s(x) = s^{-n}\psi(x/s)$, where ψ satisfies the conditions in Lemma 4.5. Hence, defining $V^v_{t,II} = 0$ for $t > 1$, we can apply Corollary 8.6.4 in [10] to conclude that

$$\left(\int_0^1 \left\|V^v_{t,II}f\right\|^2_{L^2}\frac{\mathrm{d}t}{t}\right)^{\frac{1}{2}} \lesssim (\mathcal{P}_1(|v|)+\mathcal{P}_2(|v|))\,\|f\|_{L^2}.$$

□

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References

[1] Kenji Asada and Daisuke Fujiwara, *On some oscillatory integral transformations in $L^2(\mathbf{R}^n)$*, Japan. J. Math. (N.S.) **4** (1978), no. 2, 299–361. MR528863 (80d:47076)

[2] Árpád Bényi, Diego Maldonado, Virginia Naibo, and Rodolfo H. Torres, *On the Hörmander classes of bilinear pseudodifferential operators*, Integral Equations Operator Theory **67** (2010), no. 3, 341–364, DOI 10.1007/s00020-010-1782-y. MR2660466 (2011g:47111)

[3] Árpád Bényi and Rodolfo H. Torres, *Almost orthogonality and a class of bounded bilinear pseudodifferential operators*, Math. Res. Lett. **11** (2004), no. 1, 1–11, DOI 10.4310/MRL.2004.v11.n1.a1. MR2046194 (2005a:35293)

[4] Frédéric Bernicot and Pierre Germain, *Bilinear oscillatory integrals and boundedness for new bilinear multipliers*, Adv. Math. **225** (2010), no. 4, 1739–1785, DOI 10.1016/j.aim.2010.03.032. MR2680189 (2011k:42025)

[5] Alberto P. Calderón, *Commutators, singular integrals on Lipschitz curves and applications*, Proceedings of the International Congress of Mathematicians (Helsinki, 1978), Acad. Sci. Fennica, Helsinki, 1980, pp. 85–96. MR562599 (82f:42016)

[6] Michael Christ and Jean-Lin Journé, *Polynomial growth estimates for multilinear singular integral operators*, Acta Math. **159** (1987), no. 1-2, 51–80, DOI 10.1007/BF02392554. MR906525 (89a:42024)

[7] Ronald R. Coifman and Yves Meyer, *Au delà des opérateurs pseudo-différentiels* (French), Astérisque, vol. 57, Société Mathématique de France, Paris, 1978. With an English summary. MR518170 (81b:47061)

[8] Ronald R. Coifman and Yves Meyer, *Nonlinear harmonic analysis, operator theory and P.D.E*, Beijing lectures in harmonic analysis (Beijing, 1984), Ann. of Math. Stud., vol. 112, Princeton Univ. Press, Princeton, NJ, 1986, pp. 3–45. MR864370 (88e:42030)

[9] David Dos Santos Ferreira and Wolfgang Staubach, *Global and local regularity of Fourier integral operators on weighted and unweighted spaces*, Mem. Amer. Math. Soc. **229** (2014), no. 1074, xiv+65. MR3185074

[10] Loukas Grafakos, *Classical and modern Fourier analysis*, Pearson Education, Inc., Upper Saddle River, NJ, 2004. MR2449250

[11] Loukas Grafakos and Marco M. Peloso, *Bilinear Fourier integral operators*, J. Pseudo-Differ. Oper. Appl. **1** (2010), no. 2, 161–182, DOI 10.1007/s11868-010-0011-4. MR2679898 (2011m:42015)

[12] Loukas Grafakos and Rodolfo H. Torres, *Discrete decompositions for bilinear operators and almost diagonal conditions*, Trans. Amer. Math. Soc. **354** (2002), no. 3, 1153–1176 (electronic), DOI 10.1090/S0002-9947-01-02912-9. MR1867376 (2002i:42024)

[13] Lars Hörmander, *Pseudo-differential operators and hypoelliptic equations*, Singular integrals (Proc. Sympos. Pure Math., Vol. X, Chicago, Ill., 1966), Amer. Math. Soc., Providence, R.I., 1967, pp. 138–183. MR0383152 (52 #4033)

[14] Steve Hofmann, *A local Tb theorem for square functions*, Perspectives in partial differential equations, harmonic analysis and applications, Proc. Sympos. Pure Math., vol. 79, Amer. Math. Soc., Providence, RI, 2008, pp. 175–185. MR2500492 (2010b:42025)

[15] Nicholas Michalowski, David Rule, and Wolfgang Staubach, *Multilinear pseudodifferential operators beyond Calderón-Zygmund theory*, J. Math. Anal. Appl. **414** (2014), no. 1, 149–165, DOI 10.1016/j.jmaa.2013.12.062. MR3165300

[16] Salvador Rodríguez-López, David Rule, and Wolfgang Staubach, *A Seeger-Sogge-Stein theorem for bilinear Fourier integral operators*, Adv. Math. **264** (2014), 1–54, DOI 10.1016/j.aim.2014.07.009. MR3250279

[17] Salvador Rodríguez-López and Wolfgang Staubach, *Estimates for rough Fourier integral and pseudodifferential operators and applications to the boundedness of multilinear operators*, J. Funct. Anal. **264** (2013), no. 10, 2356–2385, DOI 10.1016/j.jfa.2013.02.018. MR3035059

[18] Michael Ruzhansky and Mitsuru Sugimoto, *Weighted Sobolev L^2 estimates for a class of Fourier integral operators*, Math. Nachr. **284** (2011), no. 13, 1715–1738, DOI 10.1002/mana.200910080. MR2832678 (2012h:45012)

[19] Michael Ruzhansky and Mitsuru Sugimoto, *Global L^2-boundedness theorems for a class of Fourier integral operators*, Comm. Partial Differential Equations **31** (2006), no. 4-6, 547–569, DOI 10.1080/03605300500455958. MR2233032 (2007e:35310)

[20] Mikhail A. Shubin, *Pseudodifferential operators and spectral theory*, 2nd ed., Springer-Verlag, Berlin, 2001. Translated from the 1978 Russian original by Stig I. Andersson. MR1852334 (2002d:47073)

[21] Elias M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy; Monographs in Harmonic Analysis, III. MR1232192 (95c:42002)

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