TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 367, Number 11, November 2015, Pages 7797–7836
S 0002-9947(2015)06270-9
Article electronically published on February 18, 2015

A JUMP-TYPE SDE APPROACH TO REAL-VALUED SELF-SIMILAR MARKOV PROCESSES

LEIF DÖRING

ABSTRACT. In his 1972 paper, John Lamperti characterized all positive self-similar Markov processes as time-changes of exponentials of Lévy processes. In the past decade the problem of representing all non-negative self-similar Markov processes that do not necessarily have zero as a trap has been solved gradually via connections to ladder height processes and excursion theory.

Motivated by a recent article of Chaumont, Panti, and Rivero, we represent via jump-type SDEs the symmetric real-valued self-similar Markov processes that only decrease the absolute value by jumps and leave zero continuously.

Our construction of these self-similar processes involves a pseudo excursion construction and singular stochastic calculus arguments ensuring that solutions to the SDEs spend zero time at zero to avoid problems caused by a "bang-bang" drift.

1. INTRODUCTION AND MAIN RESULTS

1.1. The representation problem for self-similar Markov processes. Dating back to Lamperti's seminal article [27], the study of self-similar Markov processes (originally called semi-stable processes by Lamperti) with values in a subset E of $\mathbb{R}$ has attracted a lot of attention. In what follows, we will only discuss self-similar Markov processes in $E=\mathbb{R}$ and $E=[0,\infty)$ and denote by $\mathbb{D}$ the space of càdlàg functions $\omega:\mathbb{R}_+\to E$ (right continuous with left limits) endowed with the Borel sigma-field $\mathcal{D}$ generated by Skorokhod's topology. A strong Markov family $(P^z)_{z\in E}$ on $(\mathbb{D},\mathcal{D})$ is called self-similar of index $a>0$ if the coordinate process $Z_t(\omega):=\omega(t), t\geq 0$, fulfills the following scaling property:

$$\text{the law of } (c^{-a}Z_{ct})_{t\geq 0} \text{ under } P^z \text{ is } P^{c^{-a}z} \tag{1.1}$$

for all $c>0$ and $z\in E$. In this article we will mostly assume $a=1$, the change of index can be performed taking the power a (resp. $1/a$). We will say that Z (or alternatively the strong Markov family $(P^z)_{z\in E}$) is a

- positive self-similar Markov process if $E=[0,\infty)$ and Z is trapped at zero,
- non-negative self-similar Markov process if $E=[0,\infty)$,
- $\mathbb{R}\backslash\{0\}=:\mathbb{R}_*$-valued self-similar Markov process if $E=\mathbb{R}$ and Z is trapped at zero,
- real-valued self-similar Markov process if $E=\mathbb{R}$.

According to this definition, a positive self-similar Markov process is not really a positive process but the above definition seems to be the most rigorous to separate the appearing cases. Let us denote by $T_0=\inf\{t\geq 0: Z_t=0\}$ the first hitting time

Received by the editors October 27, 2012 and, in revised form, August 9, 2013.
2010 *Mathematics Subject Classification.* Primary 60G18; Secondary 60G55.
The author was supported by the Fondation Science Matématiques de Paris.

of zero and by $(Z_t^\dagger)_{t\geq 0} := (Z_{t\wedge T_0})_{t\geq 0}$ the process obtained from Z by absorption at 0. It is easy to see that $Z^\dagger$ is a positive self-similar Markov process so that in our notation absorbing at zero turns a non-negative self-similar Markov process into a positive self-similar Markov process and analogously a real-valued into a $\mathbb{R}_*$-valued self-similar Markov process.

We are interested in the following problem:

Problem. Represent all positive, non-negative, $\mathbb{R}_*$- and real-valued self-similar Markov processes.

The first three instances of the problem have been resolved and only the last remains open.

(I) Lamperti's representation of positive self-similar Markov processes. The fundamental result in the representation theory of self-similar Markov processes is Lamperti's representation obtained in [27]. Lamperti showed that there is a bijection between positive self-similar Markov processes and Lévy processes, possibly killed at an independent exponential time ζ. For a Lévy process ξ, Lamperti's representation for a positive self-similar Markov process of index a takes the form

$$Z_t = z\exp\big(\xi_{\tau(tz^{-a})}\big), \qquad 0\leq t< T_0, \tag{1.2}$$

where the random time-change is given by the generalized inverse of the exponential functional of ξ, that is

$$\tau(t) := \inf\left\{ s\geq 0 : \int_0^s \exp\left(a\xi_r\right)dr > t\right\}.$$

It is important to add that T_0 is finite almost surely for all initial conditions $z>0$ precisely if ξ drifts to $-\infty$ and in this case $\tau(T_0 z^{-a})=\infty$ (see Section 2 of [27]). Consequently, if we suppose that ξ is set to $-\infty$ at the killing time ζ, then equation (1.2) is equally valid for $t\geq 0$ with $Z_t=0$ for any $t\geq T_0$. A consequence of Lamperti's representation is that the Feller property holds on $(0,\infty)$ for any non-negative self-similar strong Markov process.

Example 1.1. The only positive self-similar Markov processes (with self-similarity index $a=1$) with continuous sample paths are solutions to the stochastic differential equations (SDEs)

$$Z_t = c\,dt + \sigma\sqrt{Z_t}\,dB_t, \quad t\leq T_0, \tag{1.3}$$

for $c\in\mathbb{R}, \sigma\geq 0$. Their Lamperti transformed Lévy processes are $\xi_t = \big(c-\frac{\sigma^2}{2}\big)t + \sigma B_t$.

Let us mention one recent application of Lamperti's representation in the study of stable Lévy processes due to Kuznetsov and Pardo [26]. From (1.2) one can express the distribution of the running maximum $S_1 := \sup_{t\leq 1} X_t$ of an α-stable process X in terms of the first hitting time $\hat{T}_0$ of zero of the self-similar process $\hat{X} := 1+X$. It is a direct consequence of Lamperti's representation that $\hat{T}_0$ is equal to the exponential functional $\int_0^\infty \exp(\alpha\hat{\xi}_s)\,ds$, where $\hat{\xi}$ is the Lévy process corresponding to $\hat{X}$ through Lamperti's representation. In this particular case it was possible to identify $\hat{\xi}$ as a so-called hypergeometric Lévy process for which Mellin transform techniques can be employed to give expressions for the density of the exponential functional and consequently for S_1.

(II) Representation of non-negative self-similar Markov processes. We mentioned above that absorbing a non-negative self-similar Markov process Z at zero yields a unique positive self-similar Markov process $Z^\dagger$. As a consequence, the representation problem for non-negative self-similar Markov processes is equivalent to finding all self-similar extensions of positive self-similar Markov processes. More precisely, the task is to find all self-similar Markov families $(\bar{P}^z)_{z\geq 0}$ under which (i) the absorbed coordinate process $\omega^\dagger(t) = \omega(t \wedge T_0)$ is distributed according to ω under $(P^z)_{z\geq 0}$ and (ii) 0 is not a trap. The question has been resolved in recent years; first, if the corresponding Lévy process ξ drifts to $-\infty$ (i.e. $T_0 < \infty$ a.s.), and later if ξ fluctuates or drifts to $+\infty$ (i.e. $T_0 = \infty$ a.s.).

It has first been proved by Rivero [32], [33] and Fitzsimmons [19] that positive self-similar Markov processes of index 1 that hit zero in finite time can be extended uniquely to a non-negative self-similar Markov process that leaves zero continuously if and only if the Cramér type condition

$$\text{there is a } 0 < \theta < 1 \text{ such that } \Psi(\theta) = 0 \tag{1.4}$$

holds. Here, and in what follows, whenever well-defined,

$$\Psi(\theta) = \log E\big(e^{\theta \xi_1}; \zeta > 1\big), \quad \theta \geq 0,$$

denotes the Laplace exponent of the Lévy process ξ (killed at ζ) that occurs in Lamperti's representation. The proofs of Rivero and Fitzsimmons are based on Blumenthal's general theory of Markov extensions developed in [11]. In fact, both authors construct self-similar excursion measures $\mathbf{n}$ that are compatible with the given positive self-similar Markov process. The term leaving zero continuously is then used for the property

$$\mathbf{n}(X_0 > 0) = 0. \tag{1.5}$$

For a compact explanation of the needed excursion theory we refer to the introduction of Fitzsimmons [19].

For positive self-similar Markov processes that do not hit zero in finite time, the representation problem is to give conditions when (and how) the family $(P^z)_{z>0}$ can be extended continuously to $z = 0$ so that the extended process remains self-similar and leaves zero. This challenging question was answered subsequently in Bertoin and Caballero [6], Bertoin and Yor [8], Caballero and Chaumont [12], and Chaumont et al. [14]: The law $(P^z)_{z>0}$ of a positive self-similar Markov process extends continuously to the initial condition $z = 0$ if and only if

$$\text{the overshoot process } (\xi_{\bar{T}_x} - x)_{x\geq 0} \text{ converges weakly as } x \to \infty, \tag{1.6}$$

where $\bar{T}_x = \inf\{t \geq 0 : \xi_t \geq x\}$ is the first exceedence time of x. If the overshoot process does not converge, then the laws P^z converge, as $z \to 0$, to the degenerate law concentrated at $+\infty$.

The necessity of condition (1.6) is relatively easy to prove and the main difficulty lies in the construction of the limiting law P^0 if condition (1.6) is valid. For the simplest construction of the non-degenerate limiting law P^0 via Lévy processes started from $-\infty$, we refer to Bertoin and Savov [7]. A jump-type SDE approach which motivated the present article was developed in [17].

(III) Representation of $\mathbb{R}_*$-valued self-similar Markov processes. In contrast to self-similar Markov processes with non-negative sample paths, less is known about the representation of self-similar Markov processes with real-valued sample paths. The analogue to Lamperti's representation, called the Lamperti-Kiu representation, has recently been proved in Chaumont et al. [13] completing earlier work of Kiu [24], [25] and Chybiryakov [15]. To the best knowledge of the author the representation of real-valued self-similar Markov processes that leave zero remains open.

The main idea of the Lamperti-Kiu representation is as follows: due to the assumed càdlàg property of sample paths, the times H_k of the k-th change of sign

$$H_0 = 0, \quad H_k = \inf\{t > H_{k-1} : Z_t Z_{t-} < 0\}, \quad k \geq 1, \tag{1.7}$$

can only accumulate at T_0. In the random intervals $[H_k, H_{k+1})$ the real-valued self-similar Markov process reduces to a strictly positive or strictly negative self-similar Markov process to which Lamperti's transformation can be applied and leads to two (possibly different) sequences $\xi^{+,k}$ and $\xi^{-,k}$ of Lévy processes. Using the strong Markov property of Z, independence of the sequence $\xi^{\pm,k}$ follows so that the Lamperti-Kiu representation is obtained by glueing a sequence of Lamperti representations. A crucial additional ingredient are jumps ΔZ_{H_k} that determine the random initial condition for the positive/negative self-similar Markov processes on $[H_k, H_{k+1})$. Again by the strong Markov property it was shown that those jumps are independent and the rate of their occurrence is determined by a random time-change as in Lamperti's representation (1.2). Loosely speaking, the time-change accelerates all jumps with a rate $1/|Z_{s-}|$ and, consequently, Z changes sign infinitely often before hitting zero. The jumps ΔZ_{H_k} add many difficulties to the representation and prevent a straightforward adaptation of arguments developed for positive self-similar Markov processes.

Our main results are for symmetric real-valued self-similar Markov processes, that is the law of $-Z$ under P^z is P^{-z}. In the symmetric case the Lamperti-Kiu transformed Lévy processes satisfy $\xi^{+,k} \stackrel{\mathcal{L}}{=} \xi^{-,k}$.

A formal description of the Lamperti-Kiu representation is rather unpleasant since the change of sign is coded in the underlying Lévy process via an additional complex direction. We follow a different approach based on jump-type SDEs that have a more compact form.

Notation. Solutions to SDEs are always considered on a stochastic basis $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, P)$ that is rich enough to carry all appearing Brownian motions and Poisson point processes and satisfies the usual conditions. All SDEs are driven by a $(\mathcal{G}_t)$-standard Wiener process B and an independent $(\mathcal{G}_t)$-Poisson random measure $\mathcal{N}$. We will use weak solutions, i.e. $(\mathcal{G}_t)_{t\geq 0}$ adapted stochastic processes $(Z_t)_{t\geq 0}$ with almost surely càdlàg sample paths that satisfy an SDE in integrated form almost surely. If additionally Z is adapted to the augmented filtration generated by B and $\mathcal{N}$, then Z is said to be a strong solution. Pathwise uniqueness holds if any two weak solutions with same initial condition defined on the same probability space with the same standard Wiener process and Poisson random measure are indistinguishable. In several SDEs the sign-function

$$\operatorname{sign}(x) := \mathbf{1}_{\{x>0\}} - \mathbf{1}_{\{x\leq 0\}}$$

is used. We say that a stochastic process Z does not spend time at zero if almost surely

$$\int_0^\infty \mathbf{1}_{\{Z_s=0\}}\,ds = 0.$$

Here is a reformulation of the main result of [13] via jump-type SDEs which we only state for symmetric $\mathbb{R}_*$-valued self-similar Markov processes with the additional assumption

$$\textbf{(A)} \quad P^z(|Z_s| \le |Z_{s-}|,\ \forall s \ge 0) = 1, \quad z \in \mathbb{R}_*.$$

Assumption **(A)** excludes the possibility that jumps of Z increase the absolute value or, equivalently, that the Lévy processes $\xi^{+,k} \overset{\mathcal{L}}{=} \xi^{-,k}$ of the Lamperti-Kiu representation have positive jumps. A general non-symmetric version without Assumption **(A)** is given below in Proposition 2.1.

Proposition 1.2. ***(I)*** *There is a bijection between symmetric $\mathbb{R}_*$-valued self-similar Markov processes satisfying Assumption* ***(A)*** *and the set of quintuples (a, σ^2, Π, q, V) consisting of*

- *a triplet (a, σ^2, Π) of a spectrally negative Lévy process killed at rate $q \ge 0$ with Laplace exponent Ψ,*
- *a finite measure $V(du)$ on $[-1, 0)$.*

(II) *For a quintuple (a, σ^2, Π, q, V) a symmetric real-valued self-similar Markov process satisfying Assumption* ***(A)*** *issued from $z \in \mathbb{R}_*$ can be constructed as a strong solution to*

$$\begin{aligned} Z_t = z + \left(\Psi(1) + \int_{-1}^0 (u-1)V(du)\right) \int_0^t \operatorname{sign}(Z_s)\,ds + \sigma \int_0^t \sqrt{|Z_s|}\,dB_s \\ + \int_0^t \int_0^{\frac{1}{|Z_{s-}|}} \int_{-1}^1 Z_{s-}(u-1)(\mathcal{N} - \mathcal{N}')(ds, dr, du) \end{aligned} \tag{1.8}$$

for $t \le T_0$. Here, B is a standard Brownian motion and $\mathcal{N}$ is an independent Poisson point process on $(0,\infty) \times (0,\infty) \times [-1,1]$ with intensity measure $\mathcal{N}'(ds, dr, du) = ds \otimes dr \otimes \bar{\Pi}(du)$ according to the piecewise definition

- *$\bar{\Pi}_{|(0,1]}(du)$ is the image measure under $\mathbb{R}_- \ni u \mapsto e^u$ of Π,*
- *$\bar{\Pi}(\{0\}) = q$,*
- *$\bar{\Pi}_{|[-1,0)}(du) = V(du)$.*

Let us briefly explain the ingredients of the jump-type SDE (1.8). Comparing with Example 1.1 the so-called "bang-bang" drift and the Brownian part might not be surprising since in the intervals $[H_k, H_{k+1})$, the restrictions of Z (resp. $-Z$) have to be positive self-similar Markov processes. The jumps of the Poissonian integral are such that

$$Z_{s-} \mapsto Z_{s-} + Z_{s-}(u-1) = Z_{s-}u, \tag{1.9}$$

and u is chosen according to the measure $\bar{\Pi}$ which looks a bit complicated. We chose this formulation since it allows us to explain the three occurring jump possibilities for self-similar Markov processes with only one stochastic integral:

- If $u > 0$, then Z does not change sign and consequently these are the jumps corresponding to a piecewise Lamperti transformation in $[H_k, H_{k+1})$. If the Lévy measure Π is infinite, i.e. $\Pi((-\infty, 0)) = \infty$, then also $\bar{\Pi}$ is infinite with a pole only at $+1$ so that small jumps (i.e. $\Delta Z_s \approx 0$) accumulate.
- If $u = 0$, then Z jumps to zero which is equivalent to a jump to $-\infty$ (killing) for the Lévy process in Lamperti's representation (1.2).
- If $u < 0$, then Z changes sign and the jump-times are precisely the H_k from (1.7). The finiteness of the intensity measure $V(du)$ is equivalent to the non-accumulation of H_k away from T_0.

The dr-integral is included to dynamically accelerate the jump rate by $1/|Z_{s-}|$. Hence, on the zero set of solutions all jumps come with infinite rate and the jumps are not changing sign even with "double-infinite" rate if Π is infinite. Such explosions of the jump rate are the main difficulty of the SDE (1.8) when studied for all $t \geq 0$ or issued from $z = 0$.

Definition 1.3. **(a)** For a symmetric $\mathbb{R}_*$-valued self-similar Markov process the quintuple (a, σ^2, Π, q, V) appearing in Proposition 1.2 (or appearing equivalently in the Lamperti-Kiu representation of [13]) is called the corresponding Lamperti-Kiu quintuple.

(b) A quintuple (a, σ^2, Π, q, V) is called the Lamperti-Kiu quintuple of a symmetric real-valued self-similar Markov process Z if it is the Lamperti-Kiu quintuple for the $\mathbb{R}_*$-valued self-similar Markov process $Z^\dagger$ obtained from Z by absorption at zero.

1.2. **Main result.** In what follows we study real-valued self-similar Markov processes issued from zero continuously. Technical difficulties force us to assume symmetry so that some properties can be reduced to the radial part which is a positive self-similar Markov process. A definition of leaving zero continuously can be given via the classical case of positive processes (see (1.5) above) without introducing excursion theory for the real-valued case.

Definition 1.4. A symmetric real-valued self-similar Markov process Z is said to leave zero continuously if the positive self-similar Markov process $|Z|$ leaves zero continuously.

The striking feature of the SDE (1.8) compared to the time-change Lamperti-Kiu representation is that the form of possible extensions after hitting zero can be guessed immediately. If possible, they should be solutions to the same SDE for all $t \geq 0$. Here is the main result of this article:

Theorem 1.5. ***(I)*** *There is a bijection between symmetric real-valued self-similar Markov processes that leave zero continuously and satisfy Assumption **(A)** and the set of quintuples (a, σ^2, Π, q, V) consisting of*

- *a triplet (a, σ^2, Π) of a spectrally negative Lévy process killed at rate $q \geq 0$ with Laplace exponent Ψ,*
- *a finite measure $V(du)$ on $[-1, 0)$*

that satisfy

$$\Psi(1) + \int_{-1}^{0} (|u| - 1) V(du) > 0. \tag{1.10}$$

(II) *Given a quintuple* (a, σ^2, Π, q, V) *satisfying* (1.10)*, a corresponding symmetric real-valued self-similar Markov process that leaves zero continuously and satisfies Assumption* ***(A)*** *can be constructed as a weak solution to the SDE* (1.8) *for* $t \geq 0$.

The necessity of condition (1.10) can be found easily by a reduction to condition (1.4) for positive self-similar Markov processes. The difficult part of the proof is a construction of a real-valued self-similar Markov process that leaves zero continuously with Lamperti-Kiu quintuple (a, σ^2, Π, q, V) whenever condition (1.10) is valid. Our reformulation of the Lamperti-Kiu representation given in Proposition 1.2 turns out to be useful since it gives flexibility for the construction via approximation procedures and semi-martingale calculus. The approximation is rather non-standard (and might remind the reader of constructions of skew Brownian motion) since the non-continuity of the "bang-bang" drift causes problems in weak convergence arguments. Limiting points of the approximating sequences might become trivial (trapped at zero) and it is precisely condition (1.10) that ensures this is not the case.

Remark 1.6. It is surprising that condition (1.10) is sufficient for the existence of solutions to the SDE (1.8) that leave zero. Since Ψ does not depend on V, the quintuple (a, σ^2, Π, q, V) can be chosen such that

$$\left(\Psi(1) + \int_{-1}^{0} (u-1)V(du)\right) < 0 < \left(\Psi(1) + \int_{-1}^{0} (|u|-1)V(du)\right). \tag{1.11}$$

Then the SDE (1.8) has martingale terms that vanish at zero and a drift that points towards the origin. In such a situation it is impossible to find non-negative solutions to SDEs since positive martingales are absorbed at zero. Real-valued solutions, however, can exist. Precisely the jumps crossing the origin cause this effect; if $V = 0$ solutions can leave 0 if and only if the drift points away from the origin.

It is important to note that the SDE (1.8) behaves very differently at zero and away from zero. Away from zero the coefficients are locally Lipschitz continuous so that pathwise uniqueness holds and strong solutions exist. Only when solutions hit zero the drift and the jumps are problematic. Consequently, the main task of the proofs is to give a construction and uniqueness statement for solutions issued from zero.

1.3. **Connection to other self-similar SDEs.**

Theorem 1.5 extends the results of [17] obtained for positive self-similar Markov processes with an assumption similar to Assumption **(A)**. The techniques utilized here need to be different from those of [17] since the drift coefficient is not even Hölder continuous so that standard arguments for SDEs do not apply. In particular, due to (1.11) solutions to (1.8) need to be constructed even if the drift points towards zero, which forces us to leave straightforward arguments and to combine more specific stochastic calculus arguments with general martingale problem techniques. The main result of [17] was stronger in the sense that pathwise uniqueness could be proved for their SDEs and solutions are automatically strong. Consequently, the constructed non-negative self-similar Markov processes are deterministic functionals of a Brownian motion and a Poisson point process so that we can speak of a strong representation.

Remark 1.7. Possible uniqueness statements for the SDE (1.8) issued from $z=0$ need to be in a restricted sense as one can see in the simplest special case:

$$dZ_t = \mathrm{sign}(Z_t)\,dt + 2\sqrt{|Z_t|}\,dB_t, \quad Z_0 = 0. \tag{1.12}$$

Given a Brownian motion W, then three weak solutions to the SDE (1.12) can be defined explicitly:

$$Z_t^{(1)} = \mathrm{sign}(W_t)W_t^2, \quad Z_t^{(2)} = W_t^2, \quad Z_t^{(3)} = -W_t^2.$$

Of course, already the one-dimensional marginal distributions differ for the $Z^{(i)}$. Nonetheless, restricted to symmetric solutions (this rules out $Z^{(2)}$ and $Z^{(3)}$) one can easily deduce the uniqueness for the one-dimensional marginals as can be seen as follows: Taking expectations, the symmetry assumption and Fubini's theorem give

$$\begin{aligned} 0 = E[Z_t] &= E\Big[\int_0^t \mathrm{sign}(Z_s)\,ds\Big] \\ &= \int_0^t \big(P(Z_s>0) - P(Z_s<0)\big)\,ds - E\Big[\int_0^t \mathbf{1}_{\{Z_s=0\}}\,ds\Big] \\ &= E\Big[\int_0^t \mathbf{1}_{\{Z_s=0\}}\,ds\Big] \end{aligned}$$

so that solutions do not spend time at zero. But then the semi-martingale local time of Z at zero vanishes almost surely:

$$0 \le L_t = \lim_{\varepsilon\to 0}\frac{4}{\varepsilon}\int_0^t \mathbf{1}_{\{Z_s\le\varepsilon\}}|Z_s|\,ds \le \lim_{\varepsilon\to 0} 4\int_0^t \mathbf{1}_{\{Z_s\le\varepsilon\}}\,ds = 4\int_0^t \mathbf{1}_{\{Z_s=0\}}\,ds = 0.$$

Hence, Tanaka's formula applied to (1.12) shows that the absolute value of any symmetric solution satisfies

$$X_t = t + 2\int_0^t \sqrt{X_s}\,d\bar{B}_s \tag{1.13}$$

with the Brownian motion $\bar{B}_t = \int_0^t \mathrm{sign}(Z_s)\,dB_s$. Now strong (we only need weak) uniqueness for (1.13) implies weak uniqueness for the absolute value of solutions to (1.12), hence, uniqueness for one-dimensional marginals of symmetric solutions.

The simple example (1.12) shows that one can only expect uniqueness statements for (1.8) among symmetric solutions.

There might be more sophisticated arguments to prove better uniqueness statements among symmetric solutions, such as the arguments developed in Bass et al. [4] for the self-similar SDE

$$dZ_t = |Z_t|^\beta\,dB_t, \quad t \ge 0, \tag{1.14}$$

for $\beta < 1/2$. They work under the restriction to solutions that do not spend time at zero, a property which is also crucial in all our arguments. For (1.14) this restriction rules out the possibility of solutions in the spirit of $Z^{(2)}$ and $Z^{(3)}$ (positive martingales are trapped at zero) of the previous example since those were necessarily trapped at zero.

Note that the index of self-similarity of (1.14) is $a = \frac{1}{2-2\beta} < 1$ and the Hölder continuity of the coefficient becomes worse when the self-similarity index decreases. For the representation problem of real-valued self-similar Markov processes the assumption $a = 1$ could be imposed without loss of generality but it would be interesting to see whether pathwise uniqueness among symmetric solutions holds for the generalized version of the SDE (1.8) that should describe all symmetric real-valued self-similar Markov processes of index a:

$$\begin{aligned} Z_t = z + \sigma \int_0^t |Z_s|^{1-\frac{1}{2a}}\, dB_s \\ + \left(\Psi(a) + \int_{-\infty}^0 (u-1)V(du)\right) \int_0^t \operatorname{sign}(Z_s) Z_s^{1-\frac{1}{a}}\, ds \\ + \int_0^t \int_0^{\frac{1}{|Z_{s-}|^a}} \int_{\mathbb{R}} Z_{s-}(u-1)(\mathcal{N} - \mathcal{N}')(ds, dr, du), \end{aligned}$$

with the same definitions as in Proposition 1.2. This generalization of (1.8) can be derived from the Lamperti-Kiu representation as we do in Section 2.1 for the special case $a = 1$. For the drift and diffusive coefficients the self-similarity index $a = 1$ separates between a regime of Hölder continuity ($a > 1$) and a regime with singular drift ($a < 1$). Moreover, for all $a > 0$ we find lack of monotonicity in the Poissonian integral and it seems that this term forces the biggest trouble.

Organization of the article. In Section 2.1 we prove the jump-type SDE reformulation of the Lamperti-Kiu representation for real-valued self-similar Markov processes. The proofs are given for the more general setup without symmetry and without Assumption **(A)**. The construction of solutions to (1.8) that leave zero is presented in Section 2.2. Self-similarity and the strong Markov property are deduced from moment equations which imply uniqueness of one-dimensional marginals for solutions to the SDE (1.8). Finally, the link to the representation problem of self-similar Markov processes is given in Section 2.3.

2. Proofs

2.1. **Lamperti-Kiu representation via jump-type SDEs.** We now state and prove a jump-type SDE formulation of the Lamperti-Kiu representation in the general case.

Proposition 2.1. *(I) There is a bijection between $\mathbb{R}_*$-valued self-similar Markov processes and two quintuples*

$$(a_+, \sigma_+^2, \Pi_+, q_+, V_+) \quad \text{and} \quad (a_-, \sigma_-^2, \Pi_-, q_-, V_-)$$

consisting of

- *two triplets $(a_\pm, \sigma_\pm^2, \Pi_\pm)$ of Lévy processes killed at rates $q_\pm \geq 0$ with Laplace exponents $\Psi_\pm$,*
- *two finite measures $V_\pm(du)$ on $(-\infty, 0)$.*

(II) *Given two quintuples $(a_\pm,\sigma^2_\pm,\Pi_\pm,q_\pm,V_\pm)$, a real-valued self-similar Markov process issued from $z\in\mathbb{R}_*$ can be constructed as a strong solution to*

$$\begin{aligned}
Z_t = z &+ \Bigg[\bigg(a_+ + \frac{\sigma_+^2}{2} + \int_{|u|\le 1}\big(e^u-1-u\big)\,\Pi_+(du)\bigg)\int_0^t \mathbf{1}_{\{Z_s>0\}}\,ds\\
&+\sigma_+\int_0^t\sqrt{|Z_s|}\mathbf{1}_{\{Z_s>0\}}\,dB_+(s)\\
&+\int_0^t\int_0^{\frac{1}{|Z_{s-}|}}\int_{\mathbb{R}\setminus(e^{-1},e)}\mathbf{1}_{\{Z_{s-}>0\}}Z_{s-}\big(u-1\big)\,\mathcal{N}_+(ds,dr,du)\\
&+\int_0^t\int_0^{\frac{1}{|Z_{s-}|}}\int_{e^{-1}}^{e}\mathbf{1}_{\{Z_{s-}>0\}}Z_{s-}\big(u-1\big)\,(\mathcal{N}_+-\mathcal{N}_+{}')(ds,dr,du)\Bigg]\\
&+\Bigg[\bigg(a_- + \frac{\sigma_-^2}{2} + \int_{|u|\le 1}\big(e^u-1-u\big)\,\Pi_-(du)\bigg)\int_0^t \mathbf{1}_{\{Z_s<0\}}\,ds\\
&+\sigma_-\int_0^t\sqrt{|Z_s|}\mathbf{1}_{\{Z_s<0\}}\,dB_-(s)\\
&+\int_0^t\int_0^{\frac{1}{|Z_{s-}|}}\int_{\mathbb{R}\setminus(e^{-1},e)}\mathbf{1}_{\{Z_{s-}<0\}}Z_{s-}\big(u-1\big)\,\mathcal{N}_-(ds,dr,du)\\
&+\int_0^t\int_0^{\frac{1}{|Z_{s-}|}}\int_{e^{-1}}^{e}\mathbf{1}_{\{Z_{s-}<0\}}Z_{s-}\big(u-1\big)\,(\mathcal{N}_--\mathcal{N}_-{}')(ds,dr,du)\Bigg],\quad t\le T_0.
\end{aligned}$$

Here, $B_\pm$ are standard Brownian motions, $\mathcal{N}_\pm$ are Poisson point processes on $(0,\infty)\times(0,\infty)\times(-\infty,\infty)$ with intensity measure $\mathcal{N}'_\pm(ds,dr,du)=ds\otimes dr\otimes\bar\Pi_\pm(du)$ according to the piecewise definition

- $\bar\Pi^\pm_{|(0,\infty)}(du)$ *are the image measures under $\mathbb{R}\ni u\mapsto e^u$ of $\Pi_\pm$,*
- $\bar\Pi^\pm(\{0\})=q_\pm$,
- $\bar\Pi^\pm_{|(-\infty,0)}(du)=V_\pm(du)$,

and the processes $B_+,B_-,\mathcal{N}_+,\mathcal{N}_-$ are independent.

Before proving the proposition, let us quickly consider part (II) for two special cases.

Example 2.2. With the choice $z>0$ and

$$\begin{aligned}(a_+,\sigma_+^2,\Pi_+,q_+,V_+)&=(a,\sigma^2,\Pi,q,0),\\(a_-,\sigma_-^2,\Pi_-,q_-,V_-)&=(0,0,0,0,0)\end{aligned}\tag{2.1}$$

zero is not crossed and, dropping the subscripts, the SDE simplifies to

$$\begin{aligned}
Z_t = z &+ \bigg(a+\frac{\sigma^2}{2}+\int_{|u|\le 1}\big(e^u-1-u\big)\,\Pi(du)\bigg)t+\sigma\int_0^t\sqrt{Z_s}\,dB(s)\\
&+\int_0^t\int_0^{\frac{1}{Z_{s-}}}\int_{\mathbb{R}_+\setminus(e^{-1},e)}Z_{s-}\big(u-1\big)\,\mathcal{N}(ds,dr,du)\\
&+\int_0^t\int_0^{\frac{1}{Z_{s-}}}\int_{e^{-1}}^{e}Z_{s-}\big(u-1\big)\,(\mathcal{N}-\mathcal{N}')(ds,dr,du),\quad t\le T_0.
\end{aligned}$$

Under the additional assumption $\int_{\mathbb{R}_+\setminus(e^{-1},e)}(u-1)\,\bar{\Pi}(du)=\int_{|u|>1}(e^u-1)\,\Pi(du)<\infty$ one can use the Lévy-Khintchin representation to simplify by adding and subtracting the finite compensator integral to get

$$
\begin{aligned}
Z_t &= z+\Psi(1)\,t+\sigma\int_0^t \sqrt{Z_s}\,dB_s \\
&\quad+\int_0^t\int_0^{\frac{1}{Z_{s-}}}\int_0^\infty Z_{s-}\big(u-1\big)\,(\mathcal{N}-\mathcal{N}')(ds,dr,du),\quad t\le T_0.
\end{aligned}
\tag{2.2}
$$

The SDE (2.2) was derived in [17] as a reformulation of Lamperti's transformation for positive self-similar Markov processes; strong existence and pathwise uniqueness for $t\ge 0$ was proved in [17] and also by Li and Pu [28].

The next example shows that Proposition 1.2 is a special case of Proposition 2.1.

Example 2.3. Let us assume the symmetry

$$(a_+,\sigma_+^2,\Pi_+,q_+,V_+)=(a_-,\sigma_-^2,\Pi_-,q_-,V_-)$$

and that jumps are directed only towards the origin, i.e. $\bar{\Pi}_\pm$ vanishes on the complement of $[-1,1]$. The assumption on $\bar{\Pi}_\pm$ implies that the non-compensated integrals can be compensated; adding and subtracting the compensator integrals as in Example 2.2 yields the needed drift. Finally, the noises $B_\pm,\mathcal{N}_\pm$ can be replaced in this special case by $B,\mathcal{N}$ due to the symmetry assumption and the independence of increments. The resulting equation is (1.8).

Proof of Proposition 2.1. To save notation, let us assume throughout the proof that $z>0$; for $z<0$ the arguments follow the same lines interchanging odd and even.

We start with a reminder of the main result of [13]: the Lamperti-Kiu representation formulated as time-changed exponential of a complex-valued Lévy process. Let $\xi^\pm$ be real-valued Lévy processes with triplets $(a_\pm,\sigma_\pm^2,\Pi_\pm)$ killed at rates $q_\pm$ (formalized here as jump to $-\infty$ without causing technical complication since the process will be absorbed at its first occurrence) as in the formulation of the proposition, and let $\zeta^\pm$ be exponential random variables with parameters $p_\pm=V_\pm((-\infty,0))$. If we denote (without confusion) by $V_\pm$ respectively negative random variables with probability distribution $V_\pm(du)/V_\pm(\mathbb{R})$, then $U^\pm:=\log(|V_\pm|)$ are real-valued random variables (for the trivial case $V_\pm(\mathbb{R})=0$ we define $U^\pm=0$ and note that $\zeta^\pm$ becomes trivial). Further, suppose that $\xi^\pm,\zeta^\pm,U^\pm$ are independent. We consider the sequence $\big((\xi^k,\zeta^k,U^k),k\ge 0\big)$ given by

$$
(\xi^k,\zeta^k,U^k)=\begin{cases}(\xi^{+,k},\zeta^{+,k},U^{+,k}): & k \text{ even (including } k=0),\\ (\xi^{-,k},\zeta^{-,k},U^{-,k}): & k \text{ odd,}\end{cases}
\tag{2.3}
$$

where $(\xi^{\pm,k},\zeta^{\pm,k},U^{\pm,k})\overset{\mathcal{L}}{=}(\xi^\pm,\zeta^\pm,U^\pm)$ are independent. Let $(H_k,k\ge 0)$ be the sequence defined by

$$H_0=0,\quad H_k=\sum_{i=0}^{k-1}\zeta^i,\quad k\ge 1,$$

and $(N_t,t\ge 0)$ the alternating renewal type process

$$N_t=\max\big\{k\ge 0: H_k\le t\big\}.$$

The Lamperti-Kiu representation can now be stated as

$$Z_t = z \exp\left(\mathcal{E}_{\tau(t|z|^{-1})}\right), \quad t \le T_0, \tag{2.4}$$

where (a verbal description of $\mathcal{E}$ was given below (1.7))

$$\mathcal{E}_t = \xi^{N_t}_{t-H_{N_t}} + \sum_{k=1}^{N_t} \xi^{k-1}_{H_k - H_{k-1}} + \sum_{k=1}^{N_t} \left(U^{k-1} + i\pi\right) \tag{2.5}$$

and

$$\tau(t) := \inf\left\{ s \ge 0 : \int_0^s \left|\exp\left(\mathcal{E}_r\right)\right| dr > t \right\}.$$

Theorem 6 of [13] states that (2.4) defines an $\mathbb{R}_*$-valued self-similar Markov process issued from z and conversely every $\mathbb{R}_*$-valued self-similar Markov process can be represented via (2.4) with two quintuples as in the statement of part (I) of the proposition. Recall that throughout this article we suppose the index of self-similarity is 1.

Note that, if $V^{\pm} = 0$, then $N_t = 0$ and (2.4) simplifies to Lamperti's representation (1.2).

The rest of the proof is concerned with part (II), the reformulation of (2.4) via jump-type SDEs. The main idea is to write the appearing Lévy processes $\xi^{\pm,k}$ in Lévy-Itō form, apply Itō's formula, and include the time-change via a transformation of the driving noises.

Since the sequence of Lévy processes is independent and runs on disjoint time-intervals, the same driving noises can be used for all $k \ge 0$. Let us now fix the notation needed for the Lévy-Itō representations of the $\xi^{\pm,k}$: The occurring Brownian motions are denoted by $W^1_{\pm}$ and the Poisson point processes with intensities $ds \otimes \Pi_{\pm}(du)$ by $\mathcal{N}^1_{\pm}$ so that

$$\begin{aligned}
\xi^k_{t-H_k} = \xi^{+,k}_{t-H_k} &= a_+ \,(t - H_k) + \sigma_+ \, W^1_+(t - H_k) \\
&\quad + \int_0^{t-H_k} \int_{|u|\le 1} u \,(\mathcal{N}^1_+ - \mathcal{N}^{1\prime}_+)(ds, du) \\
&\quad + \int_0^{t-H_k} \int_{|u|>1} u \, \mathcal{N}^1_+(ds, du), \quad t \in (H_k, H_{k+1}),
\end{aligned} \tag{2.6}$$

if k is even (including $k = 0$) and

$$\begin{aligned}
\xi^k_{t-H_k} = \xi^{-,k}_{t-H_k} &= a_- \,(t - H_k) + \sigma_- \, W^1_-(t - H_k) \\
&\quad + \int_0^{t-H_k} \int_{|u|\le 1} u \,(\mathcal{N}^1_- - \mathcal{N}^{1\prime}_-)(ds, du) \\
&\quad + \int_0^{t-H_k} \int_{|u|>1} u \, \mathcal{N}^1_-(ds, du), \quad t \in (H_k, H_{k+1}),
\end{aligned} \tag{2.7}$$

if k is odd. Next, since the jumps of $\mathcal{E}$ in the imaginary direction and the jumps $U^{\pm,k}$ come at the same times (those times were denoted by H_k), they can be added according to the same Poisson point processes $\mathcal{M}^1_{\pm}$ on $(0, \infty) \times \mathbb{R}$ with intensity measures $\mathcal{M}^{1\prime}_{\pm}(ds, du) = p_{\pm} ds \otimes U^{\pm}(du)$, where $U^{\pm}(du)$ denotes the probability law of $U^{\pm}$:

$$\Delta \mathcal{E}_{H_k} = \Delta\Bigg(\int_0^{H_k} \int_{\mathbb{R}} u \, \mathcal{M}_+^1(ds, du) + i \int_0^{H_k} \int_{\mathbb{R}} \pi \, \mathcal{M}_+^1(ds, du) \Bigg) \tag{2.8}$$

if $k \geq 1$ is odd and

$$\Delta \mathcal{E}_{H_k} = \Delta\Bigg(\int_0^{H_k} \int_{\mathbb{R}} u \, \mathcal{M}_-^1(ds, du) + i \int_0^{H_k} \int_{\mathbb{R}} \pi \, \mathcal{M}_-^1(ds, du) \Bigg) \tag{2.9}$$

if $k \geq 1$ is even. Here, and in the sequel, we shall use the abbreviation $\Delta \mathcal{E}_t = \mathcal{E}_t - \mathcal{E}_{t-}$ for the jump of $\mathcal{E}$ at time t. In order to decide if k is odd or even, we include indicator functions based on the observation

$$\begin{aligned} Im(\mathcal{E}_{H_{k-}})/\pi \text{ is odd} \quad &\Longleftrightarrow \quad k \geq 1 \text{ is even}, \\ Im(\mathcal{E}_{H_{k-}})/\pi \text{ is even} \quad &\Longleftrightarrow \quad k \geq 1 \text{ is odd}, \end{aligned}$$

so that it follows inductively on the H_k that (2.5) combined with (2.6), (2.7), (2.8), (2.9) implies the Lévy-Itō type representation

$$\begin{aligned} \mathcal{E}_t = \Bigg[& a_+ \int_0^t \mathbf{1}_{\{Im(\mathcal{E}_s)/\pi \text{ even}\}} \, ds + \sigma_+ \int_0^t \mathbf{1}_{\{Im(\mathcal{E}_s)/\pi \text{ even}\}} \, dW_+^1(s) \\ & + \int_0^t \int_{|u| \leq 1} \mathbf{1}_{\{Im(\mathcal{E}_{s-})/\pi \text{ even}\}} u \, (\mathcal{N}_+^1 - \mathcal{N}_+^{1\prime})(ds, du) \\ & + \int_0^t \int_{|u| > 1} \mathbf{1}_{\{Im(\mathcal{E}_{s-})/\pi \text{ even}\}} u \, \mathcal{N}_+^1(ds, du) \\ & + \int_0^t \int_{\mathbb{R}} \mathbf{1}_{\{Im(\mathcal{E}_{s-})/\pi \text{ even}\}} u \, \mathcal{M}_+^1(ds, du) \\ & + i \int_0^t \int_{\mathbb{R}} \mathbf{1}_{\{Im(\mathcal{E}_{s-})/\pi \text{ even}\}} \pi \, \mathcal{M}_+^1(ds, du) \Bigg] \\ & + \Bigg[a_- \int_0^t \mathbf{1}_{\{Im(\mathcal{E}_s)/\pi \text{ odd}\}} \, ds + \sigma_- \int_0^t \mathbf{1}_{\{Im(\mathcal{E}_s)/\pi \text{ odd}\}} \, dW_-^1(s) \\ & + \int_0^t \int_{|u| \leq 1} \mathbf{1}_{\{Im(\mathcal{E}_{s-})/\pi \text{ odd}\}} u \, (\mathcal{N}_-^1 - \mathcal{N}_-^{1\prime})(ds, du) \\ & + \int_0^t \int_{|u| > 1} \mathbf{1}_{\{Im(\mathcal{E}_{s-})/\pi \text{ odd}\}} u \, \mathcal{N}_-^1(ds, du) \\ & + \int_0^t \int_{\mathbb{R}} \mathbf{1}_{\{Im(\mathcal{E}_{s-})/\pi \text{ odd}\}} u \, \mathcal{M}_-^1(ds, du) \\ & + i \int_0^t \int_{\mathbb{R}} \mathbf{1}_{\{Im(\mathcal{E}_{s-})/\pi \text{ odd}\}} \pi \, \mathcal{M}_-^1(ds, du) \Bigg]. \end{aligned}$$

If now we set $\eta_t = z\exp(\mathcal{E}_t)$, apply Itō's lemma, and use the identities

$$\begin{aligned}\mathbf{1}_{\{Im(\mathcal{E}_s)/\pi \text{ even}\}} &= \mathbf{1}_{\{\eta_s>0\}},\\ \mathbf{1}_{\{Im(\mathcal{E}_s)/\pi \text{ odd}\}} &= \mathbf{1}_{\{\eta_s<0\}},\\ ze^{\xi_{s-}+(u+i\pi)} - ze^{\xi_{s-}} &= \eta_{s-}(-e^u-1),\end{aligned}$$

then we obtain

$$\begin{aligned}\eta_t = z + \Bigg[& a_+\int_0^t \eta_s \mathbf{1}_{\{\eta_s>0\}}\,ds + \frac{\sigma_+^2}{2}\int_0^t \mathbf{1}_{\{\eta_s>0\}}\eta_s\,ds + \sigma_+\int_0^t \eta_s\mathbf{1}_{\{\eta_s>0\}}\,dW_+^1(s)\\
&+\int_0^t\int_{|u|\le 1}\mathbf{1}_{\{\eta_{s-}>0\}}\eta_{s-}\big(e^u-1\big)\,(\mathcal{N}_+^1-\mathcal{N}_+^{1\prime})(ds,du)\\
&+\int_0^t\int_{|u|>1}\mathbf{1}_{\{\eta_{s-}>0\}}\eta_{s-}\big(e^u-1\big)\,\mathcal{N}_+^1(ds,du)\\
&+\int_0^t\int_{|u|\le 1}\mathbf{1}_{\{\eta_s>0\}}\eta_{s-}\big(e^u-1-u\big)\,ds\,\Pi_+(du)\\
&+\int_0^t\int_{\mathbb{R}}\eta_{s-}\mathbf{1}_{\{\eta_{s-}>0\}}\big(-e^u-1\big)\,\mathcal{M}_+^1(ds,du)\Bigg]\\
+\Bigg[& a_-\int_0^t \eta_s \mathbf{1}_{\{\eta_s<0\}}\,ds + \frac{\sigma_-^2}{2}\int_0^t \mathbf{1}_{\{\eta_s<0\}}\eta_s\,ds + \sigma_-\int_0^t \eta_s\mathbf{1}_{\{\eta_s<0\}}\,dW_-^1(s)\\
&+\int_0^t\int_{|u|\le 1}\mathbf{1}_{\{\eta_{s-}<0\}}\eta_{s-}\big(e^u-1\big)\,(\mathcal{N}_-^1-\mathcal{N}_-^{1\prime})(ds,du)\\
&+\int_0^t\int_{|u|>1}\mathbf{1}_{\{\eta_{s-}<0\}}\eta_{s-}\big(e^u-1\big)\,\mathcal{N}_-^1(ds,du)\\
&+\int_0^t\int_{|u|\le 1}\mathbf{1}_{\{\eta_{s-}<0\}}\eta_{s-}\big(e^u-1-u\big)\,ds\,\Pi_-(du)\\
&+\int_0^t\int_{\mathbb{R}}\mathbf{1}_{\{\eta_{s-}<0\}}\eta_{s-}\big(-e^u-1\big)\,\mathcal{M}_-^1(ds,du)\Bigg].\end{aligned}$$

To incorporate the time-change τ we follow closely the arguments of Proposition 3.13 of [17] for the special case (2.1). Since the arguments are almost identical, we refer for the verification of intermediate steps to the careful treatment in [17].

Let us first denote by $(t_n,\Delta_n)_{n\in\mathbb{N}}$ an arbitrary labeling of the pairs associated to jump times and jump sizes of $\big(\mathcal{E}_{\tau(tz^{-1})}\big)_{t\in[0,T_0)}$ and more precisely by $(t_n,\Delta_n^{\pm})_{n\in\mathbb{N}}$ the subset of jumps due to $\mathcal{N}_{\pm}^1$ and by $(t_n,\bar{\Delta}_n^{\pm})_{n\in\mathbb{N}}$ the subset of jumps due to $\mathcal{M}_{\pm}^1$. We can assume that we are given additionally independent Wiener processes $(\bar{W}_{\pm}(t))_{t\ge 0}$, Poisson random measures $\mathcal{P}_{\pm}^1$ on $(0,\infty)\times(0,\infty)\times\mathbb{R}$ with intensity measure $ds\otimes dr\otimes\bar{\Pi}(du)$, and Poisson point processes $\mathcal{P}_{\pm}^2$ on $(0,\infty)\times(0,\infty)\times\mathbb{R}$ with intensity measure $p_{\pm}ds\otimes dr\otimes U^{\pm}(du)$ that generate a filtration $(\mathcal{H}_t)_{t\ge 0}$. Additionally, we choose an independent sequence of random variables $(R_n)_{n\in\mathbb{N}}$ uniformly distributed on $(0,1)$ such that R_n is $\mathcal{H}_{t_n}$-measurable and independent of $\mathcal{H}_{t_n-}$ and define

$$\begin{aligned}
W^2_\pm(t) &:= \int_0^t \operatorname{sign}(Z_s)\sqrt{|Z_s|}\,dW^1_\pm(\tau(sz^{-1})) + \int_0^t \mathbf{1}_{\{Z_s=0\}}\,d\bar{W}_\pm(s),\\
\mathcal{N}^2_\pm(A_1\times A_2\times A_3) &:= \sum_{n=1}^\infty \mathbf{1}_{\{A_1\times A_2\times A_3\}}((t_n, R_n/|Z_{t_n-}|, \Delta^\pm_n))\\
&\quad + \int_{A_1}\int_{A_2}\int_{A_3} (\mathbf{1}_{\{r|Z_{s-}|>1\}} + \mathbf{1}_{\{Z_{s-}=0\}})\,\mathcal{P}^1_\pm(ds,dr,du),\\
\mathcal{M}^2_\pm(A_1\times A_2\times A_3) &:= \sum_{n=1}^\infty \mathbf{1}_{\{A_1\times A_2\times A_3\}}((t_n, R_n/|Z_{t_n-}|, \bar{\Delta}^\pm_n))\\
&\quad + \int_{A_1}\int_{A_2}\int_{A_3} (\mathbf{1}_{\{r|Z_{s-}|>1\}} + \mathbf{1}_{\{Z_{s-}=0\}})\,\mathcal{P}^2_\pm(ds,dr,du),
\end{aligned}$$

for all $A_1, A_2 \in \mathcal{B}((0,\infty))$ and $A_3 \in \mathcal{B}(\mathbb{R})$. It now follows from Lévy's characterization that the $W^2_\pm$ are $\mathcal{H}_t$-Brownian motions:

$$\begin{aligned}
\langle W^2_\pm(\cdot)\rangle_t &= \int_0^t |Z_s|\mathbf{1}_{\{Z_s\neq 0\}}\,d\tau(sz^{-1}) + \int_0^t \mathbf{1}_{\{Z_s=0\}}\,ds\\
&= \int_0^t \mathbf{1}_{\{Z_s\neq 0\}}\,ds + \int_0^t \mathbf{1}_{\{Z_s=0\}}\,ds = t.
\end{aligned}$$

Next, we have to show that the $\mathcal{N}^2_\pm$ are $\mathcal{H}_t$-Poisson point processes with intensity measures $dt\otimes dr\otimes \Pi_\pm(du)$. Applying Theorems II.1.8 and II.4.8 of [22], we need to verify

$$\begin{aligned}
&E\Big[\int_0^\infty\int_0^\infty\int_{\mathbb{R}} H(s,r,u)\,\mathcal{N}^2_\pm(ds,dr,du)\Big]\\
&\qquad = E\Big[\int_0^\infty\int_0^\infty\int_{\mathbb{R}} H(s,r,u)\,ds\,dr\,\Pi_\pm(du)\Big]
\end{aligned}\tag{2.10}$$

for every non-negative predictable function H on $\Omega\times(0,\infty)\times(0,\infty)\times\mathbb{R}$. By the definition of $\mathcal{N}^2_\pm$ we can write

$$\begin{aligned}
&E\Big[\int_0^\infty\int_0^\infty\int_{\mathbb{R}} H(s,r,u)\,\mathcal{N}^2_\pm(ds,dr,du)\Big]\\
&\quad = E\Big[\sum_{n=1}^\infty H(t_n, R_n/|Z_{t_n-}|, \Delta^\pm_n)\Big]\\
&\qquad + E\Big[\int_0^\infty\int_0^\infty\int_{\mathbb{R}} H(s,r,u)(\mathbf{1}_{\{r|Z_{s-}|>1\}} + \mathbf{1}_{\{Z_{s-}=0\}})\,\mathcal{P}^1_\pm(ds,dr,du)\Big].
\end{aligned}\tag{2.11}$$

To express the first summand we apply Theorem II.1.8 of Jacod and Shiryaev [22] to the non-negative predictable function

$$\widetilde{H}(s,r,u) := H(s, r/|Z_{s-}|, u), \quad s>0, r>0, u\in\mathbb{R},$$

and the Poisson random measure on $(0,\infty)\times(0,\infty)\times\mathbb{R}$ defined by

$$\widetilde{\mathcal{P}}^1(A_1\times A_2\times A_3) := \sum_{n=1}^\infty \mathbf{1}_{A_1\times A_2\times A_3}((t_n, R_n, \Delta_n))$$

for $A_1, A_2 \in \mathcal{B}((0,\infty))$, $A_3 \in \mathcal{B}(\mathbb{R})$, to obtain

$$E\Big[\sum_{n=1}^{\infty} H(t_n, R_n/|Z_{t_n-}|, \Delta_n)\Big] = E\Big[\int_0^\infty\int_0^\infty\int_{\mathbb{R}} \widetilde{H}(s,r,u)\,\widetilde{\mathcal{P}}^1(ds,dr,du)\Big]$$
$$= E\Big[\int_0^{T_0}\int_0^1\int_{\mathbb{R}} H(s,r/|Z_{s-}|,u)\,d\tau(sz^{-1})\,dr\,\Pi(du)\Big],$$

where the second equality holds since, by construction, the compensator measure of $\widetilde{\mathcal{P}}^1$ is

$$\mathbf{1}_{(0,T_0)}(s)\mathbf{1}_{(0,1)}(r)d\tau(sz^{-1})\,dr\,\Pi(du).$$

Utilizing a change of variable in the second coordinate of H, we can further simplify the right-hand side to

$$E\Big[\int_0^{T_0}\int_0^1\int_{\mathbb{R}} \frac{1}{|Z_{s-}|} H(s,r/|Z_{s-}|,u)\,ds\,dr\,\Pi(du)\Big]$$
$$= E\Big[\int_0^{T_0}\int_0^{1/|Z_{s-}|}\int_{\mathbb{R}} H(s,r,u)\,ds\,dr\,\Pi(du)\Big].$$

Similarly, applying Theorem II.1.8 of Jacod and Shiryaev [22] to $\mathcal{P}^1_\pm$, the second summand of the right-hand side of (2.11) equals

$$E\Big[\int_0^\infty\int_0^\infty\int_{\mathbb{R}} H(s,r,u)(\mathbf{1}_{\{r|Z_{s-}|>1\}} + \mathbf{1}_{\{Z_{s-}=0\}})\,\mathcal{P}^1_\pm(ds,dr,du)\Big]$$
$$= E\Big[\int_0^{T_0}\int_{1/|Z_s|}^\infty\int_{\mathbb{R}} H(s,r,u)\,ds\,dr\,\Pi(du)\Big]$$
$$+ E\Big[\int_{T_0}^\infty\int_0^\infty\int_{\mathbb{R}} H(s,r,u)\,ds\,dr\,\Pi(du)\Big].$$

Adding the right-hand sides of the above two equalities, by (2.11), we have (2.10).

Similarly, one can show that the $\mathcal{M}^2_\pm$ are $\mathcal{H}_t$-Poisson point processes with intensity measures $p_\pm dt \otimes dr \otimes U^\pm(du)$.

Plugging-in the new Brownian motion we obtain

$$\sigma_\pm \int_0^{\tau(tz^{-1})} \mathbf{1}_{\{\eta_s>0\}}\eta_s\,dW^1_\pm(s) = \sigma_\pm \int_0^t \mathbf{1}_{\{Z_s>0\}} Z_s\,dW^1_\pm(\tau(sz^{-1}))$$
$$= \sigma_\pm \int_0^t \mathbf{1}_{\{Z_s>0\}}\sqrt{|Z_s|}\,\mathrm{sign}(Z_s)\sqrt{|Z_s|}\,dW^1_\pm(\tau(sz^{-1}))$$
$$= \sigma_\pm \int_0^t \mathbf{1}_{\{Z_s>0\}}\sqrt{|Z_s|}\,dW^2_\pm(s), \qquad t \le T_0,$$

and analogously for the negative part. Comparing one-by-one the jumps of the Poisson point processes we also find, by construction of the new point measures,

$$\int_0^{\tau(tz^{-1})}\int_{|u|>1} \mathbf{1}_{\{\eta_{s-}>0\}}\eta_{s-}\big(e^u-1\big)\,\mathcal{N}^1_+(ds,du)$$
$$= \int_0^t\int_0^{\frac{1}{|Z_{s-}|}}\int_{|u|>1} \mathbf{1}_{\{Z_{s-}>0\}} Z_{s-}\big(e^u-1\big)\,\mathcal{N}^2_+(ds,dr,du), \quad t \le T_0,$$

and

$$\int_0^{\tau(tz^{-1})}\int_{|u|\le 1}\mathbf{1}_{\{\eta_{s-}>0\}}\eta_{s-}\big(e^u-1\big)\,(\mathcal{N}^1_+-\mathcal{N}^{1\prime}_+)(ds,du)$$
$$=\int_0^t\int_0^{\frac{1}{|Z_{s-}|}}\int_{|u|\le 1}\mathbf{1}_{\{Z_{s-}>0\}}Z_{s-}\big(e^u-1\big)\,(\mathcal{N}^2_+-\mathcal{N}^{2\prime}_+)(ds,dr,du),\quad t\le T_0,$$

and

$$\int_0^{\tau(tz^{-1})}\int_{\mathbb{R}}\mathbf{1}_{\{\eta_{s-}>0\}}\eta_{s-}\big(-e^u-1\big)\,\mathcal{M}^1_+(ds,du)$$
$$=\int_0^t\int_0^{\frac{1}{|Z_{s-}|}}\int_{\mathbb{R}}\mathbf{1}_{\{Z_{s-}>0\}}Z_{s-}\big(-e^u-1\big)\,\mathcal{M}^2_+(ds,dr,du),\quad t\le T_0,$$

and analogously for the negative parts. Finally, ordinary change of time yields

$$\frac{\sigma_\pm^2}{2}\int_0^{\tau(tz^{-1})}\eta_s\mathbf{1}_{\{\eta_s>0\}}\,ds=\frac{\sigma_\pm^2}{2}\int_0^t\mathbf{1}_{\{Z_s>0\}}\,ds,\quad t\le T_0.$$

Plugging-into the integral equation derived for η, we find that Z satisfies

$$\begin{aligned}Z_t=z+\Bigg[&\bigg(a_++\frac{\sigma_+^2}{2}+\int_{|u|\le 1}\big(e^u-1-u\big)\,\Pi_+(du)\bigg)\int_0^t\mathbf{1}_{\{Z_s>0\}}\,ds\\
&+\sigma_+\int_0^t\sqrt{|Z_s|}\mathbf{1}_{\{Z_s>0\}}\,dW^2_+(s)\\
&+\int_0^t\int_0^{\frac{1}{|Z_{s-}|}}\int_{|u|>1}\mathbf{1}_{\{Z_{s-}>0\}}Z_{s-}\big(e^u-1\big)\,\mathcal{N}^2_+(ds,dr,du)\\
&+\int_0^t\int_0^{\frac{1}{|Z_{s-}|}}\int_{|u|\le 1}\mathbf{1}_{\{Z_{s-}>0\}}Z_{s-}\big(e^u-1\big)\,(\mathcal{N}^2_+-\mathcal{N}^{2\prime}_+)(ds,dr,du)\\
&+\int_0^t\int_0^{\frac{1}{|Z_{s-}|}}\int_{\mathbb{R}}\mathbf{1}_{\{Z_{s-}>0\}}Z_{s-}\big(-e^v-1\big)\,\mathcal{M}^2_+(ds,dr,dv)\Bigg]\\
&+\Bigg[\bigg(a_-+\frac{\sigma_-^2}{2}+\int_{|u|\le 1}\big(e^u-1-u\big)\,\Pi_-(du)\bigg)\int_0^t\mathbf{1}_{\{Z_s<0\}}\,ds\\
&+\sigma_-\int_0^t\sqrt{|Z_s|}\mathbf{1}_{\{Z_s<0\}}\,dW^2_-(s)\\
&+\int_0^t\int_0^{\frac{1}{|Z_{s-}|}}\int_{|u|>1}\mathbf{1}_{\{Z_{s-}<0\}}Z_{s-}\big(e^u-1\big)\,\mathcal{N}^2_-(ds,dr,du)\\
&+\int_0^t\int_0^{\frac{1}{|Z_{s-}|}}\int_{|u|\le 1}\mathbf{1}_{\{Z_{s-}<0\}}Z_{s-}\big(e^u-1\big)\,(\mathcal{N}^2_--\mathcal{N}^{2\prime}_-)(ds,dr,du)\\
&+\int_0^t\int_0^{\frac{1}{|Z_{s-}|}}\int_{\mathbb{R}}\mathbf{1}_{\{Z_{s-}<0\}}Z_{s-}\big(-e^v-1\big)\,\mathcal{M}^2_-(ds,dr,dv)\Bigg],\quad t\le T_0.\end{aligned}$$

The final step is only for notational convenience: We change the coordinates for the jumps of $\mathcal{N}^2_\pm$, $\mathcal{M}^2_\pm$ in order to combine the integrals to integrals driven by $\mathcal{N}_\pm$ as in the statement of the proposition. □

2.2. **Construction of real-valued self-similar processes.** The aim of this section is to construct real-valued self-similar Markov processes that leave zero continuously with Lamperti-Kiu quintuple (a, σ^2, Π, q, V) whenever condition (1.10) is valid. We construct a symmetric approximating sequence for the martingale problem corresponding to the SDE (1.8) and use moment equations of Bertoin and Yor [9] to show that limit points are Markovian and self-similar.

Recall that for a generator $\mathcal{A}$ defined on a suitably chosen subset $\mathcal{D}(\mathcal{A})$ of the bounded and measurable functions $B(\mathbb{R})$ mapping $\mathbb{R}$ into $\mathbb{R}$ a stochastic process Z is said to be a solution to the martingale problem $(\mathcal{A}, \nu)$ corresponding to $\mathcal{A}$ with initial distribution ν if for all $f \in \mathcal{D}(\mathcal{A})$

$$M_t^f = f(Z_t) - \int_0^t \mathcal{A} f(Z_s)\, ds, \quad t \geq 0,$$

is a martingale and Z_0 is distributed according to ν. The next proposition is standard; it is included for completeness and since the used estimates will appear several times in the sequel.

Proposition 2.4. *A stochastic process Z is a weak solution to the SDE* (1.8) *issued from $z \in \mathbb{R}$ if and only if it satisfies the martingale problem $(\mathcal{A}, \delta_z)$ corresponding to the generator*

(2.12)
$$\begin{aligned}(\mathcal{A}f)(z) := &\Big(\Psi(1) + \int_{-1}^0 (u-1) V(du)\Big) \operatorname{sign}(z) f'(z) + \frac{\sigma^2}{2} |z| f''(z)\\ &+ \int_0^\infty \int_{-1}^1 \mathbf{1}_{\{r|z|\leq 1\}} \Big(f\big(uz\big) - f(z) - f'(z) z(u-1)\Big)\, dr\, \bar{\Pi}(du), \quad z \in \mathbb{R},\end{aligned}$$

acting on the infinitely differentiable functions with compact support $C_c^\infty(\mathbb{R})$.

Proof. Let us first suppose Z is a weak solution to the SDE (1.8). Applying Itō's formula with $f \in C_c^\infty(\mathbb{R})$ yields that

$$\begin{aligned}M_t^f &= f(Z_t) - f(z) - \Big(\Psi(1) + \int_{-1}^0 (u-1) V(du)\Big) \int_0^t f'(Z_s) \operatorname{sign}(Z_s)\, ds\\ &\quad - \frac{\sigma^2}{2} \int_0^t f''(Z_s) |Z_s|\, ds\\ &\quad - \int_0^t \int_0^\infty \int_{-1}^1 \big(f(Z_s + \mathbf{1}_{\{r|Z_s|\leq 1\}} Z_s(u-1)) - f(Z_s)\\ &\qquad\qquad - f'(Z_s) \mathbf{1}_{\{r|Z_s|\leq 1\}} Z_s(u-1)\big)\, ds\, dr\, \bar{\Pi}(du)\\ &= f(Z_t) - f(z) - \int_0^t \mathcal{A} f(Z_s)\, ds\end{aligned}$$

is a local martingale, where

$$\begin{aligned}M_t^f &= \sigma \int_0^t f'(Z_s) \sqrt{|Z_s|}\, dB_s\\ &\quad + 1\int_0^t \int_0^\infty \int_{-1}^1 \big(f(Z_{s-} + \mathbf{1}_{\{r|Z_{s-}|\leq 1\}} Z_{s-}(u-1)) - f(Z_{s-})\big)(\mathcal{N} - \mathcal{N}')(ds, dr, du).\end{aligned}$$

Next, we show that M^f is a true martingale: By Theorem 51 of [30] it suffices to verify $E[\sup_{t\leq T} |M_t^f|] < \infty$ for all $T > 0$. Applying the Burkholder-Davis-Gundy inequality (for the non-continuous martingale see [16], p. 287) and the simple estimate $E\big[\sup_{t\le T}|M_t^f|\big]\le 1+E\big[\sup_{t\le T}|M_t^f|^2\big]$, we obtain

$$\begin{aligned}E\big[\sup_{t\le T}|M_t^f|\big] &\le 1+2\sigma^2 E\left[\sup_{t\le T}\left|\int_0^t f'(Z_s)\sqrt{|Z_s|}\,dB_s\right|^2\right]\\&\quad+2E\left[\sup_{t\le T}\left|\int_0^t\int_0^{\frac{1}{|Z_{s-}|}}\int_{-1}^1\Big(f(Z_{s-}+Z_{s-}(u-1))-f(Z_{s-})\Big)\right.\right.\\&\qquad\qquad\left.\left.\times(\mathcal{N}-\mathcal{N}')(ds,dr,du)\right|^2\right]\\&\le 1+CE\left[\int_0^T f'(Z_s)^2|Z_s|\,ds\right]\\&\quad+CE\left[\int_0^T\int_0^{\frac{1}{|Z_{s-}|}}\int_{-1}^1\Big(f(Z_s+Z_s(u-1))-f(Z_s)\Big)^2 ds\,dr\,\bar\Pi(du)\right].\end{aligned}$$

By Taylor's formula and the boundedness of f' we find the upper bound

(2.13)
$$E\big[\sup_{t\le T}|M_t^f|\big]\le 1+\big(\sup_z f'(z)\big)^2\left(C+C\int_{-1}^1(u-1)^2\,\bar\Pi(du)\right)E\left[\int_0^T|Z_s|\,ds\right].$$

Note that the definition of $\bar\Pi$ implies

$$\begin{aligned}\int_{-1}^1(u-1)^2\,\bar\Pi(du)&=\int_{-1}^0(u-1)^2\,V(du)+\int_{-\infty}^0\big(e^u-1\big)^2\,\Pi(du)\\&\le 2V\big([-1,0)\big)+\Pi\big((-\infty,-1]\big)+C\int_{-1}^0 u^2\,\Pi(du)\end{aligned}$$

which is finite since $V(du)$ is a finite measure and Π is a Lévy measure. Hence, it suffices to find a locally integrable upper bound for $E[|Z_t|]$. First note (this is seen by a double application of Itō's rule similarly to the argument used for Corollary 2.9 using $|Z_t|=\sqrt{Z_t^2}$) that $|Z|$ satisfies almost surely the integral equation

$$\begin{aligned}|Z_t|&=z+\left(\Psi(1)+\int_{-1}^0(|u|-1)V(du)\right)\int_0^t\mathbf{1}_{\{|Z_s|>0\}}\,ds+\sigma\int_0^t\sqrt{|Z_s|}\,d\bar B_s\\&\quad+\int_0^t\int_0^{\frac{1}{|Z_{s-}|}}\int_{-1}^1|Z_{s-}|(|u|-1)(\mathcal{N}-\mathcal{N}')(ds,dr,du),\quad t\ge 0,\end{aligned}$$

for the Brownian motion $\bar B_t=\int_0^t\operatorname{sign}(Z_s)\,dB_s$. Introducing the stopping times $T_N=\inf\{t\ge 0:|Z_t|>N\}$, we obtain

$$E\big[|Z_{t\wedge T_N}|\big]\le z+\left(\Psi(1)+\int_{-1}^0(|u|-1)V(du)\right)t$$

so that by Fatou's lemma $E[|Z_t|]$ grows at most linearly. It is easy to see that the T_N tend to infinity almost surely; we can refer for instance to Proposition 2.3 of Fu and Li [20]. Consequently, the right-hand side of (2.13) is finite and M_t^f is a martingale. The first part of the proof is complete.

Conversely, suppose the law of the process Z is a solution to the martingale problem $(\mathcal{A}, \delta_z)$. By a standard stopping time argument to allow for the test-function $f(z) = z$, we have

$$Z_t = z + \left(\Psi(1) + \int_{-1}^0 (u-1)V(du)\right) \int_0^t \operatorname{sign}(Z_s)\, ds + M_t, \quad t \geq 0,$$

almost surely, for a square-integrable martingale M that we have to identify. Let $\mathcal{C}(ds, dz)$ be the optional random measure on $[0, \infty) \times \mathbb{R}$ defined by the jumps of Z:

$$\mathcal{C}(ds, dz) = \sum_{s>0} \mathbf{1}_{\{\Delta Z_s \neq 0\}} \delta_{(s, \Delta Z_s)}(ds, dz),$$

where $\Delta Z_s = Z_s - Z_{s-}$ is the jump of Z at time s. If $\mathcal{C}'$ denotes the predictable compensator of $\mathcal{C}$, then page 376 of [16] shows that

$$Z_t = z + \left(\Psi(1) + \int_{-1}^0 (u-1)V(du)\right) \int_0^t \operatorname{sign}(Z_s)\, ds + M_t^c + M_t^d \tag{2.14}$$

for a continuous martingale M^c and

$$M_t^d = \int_0^t \int_{\mathbb{R}} z\, (\mathcal{C} - \mathcal{C}')(ds, dz).$$

We now have to identify the martingales M^c and M^d. Applying Itō's formula to the semi-martingale representation (2.14) of Z yields

$$\begin{aligned} f(Z_t) = f(z) &+ \left(\Psi(1) + \int_{-1}^0 (u-1)V(du)\right) \int_0^t f'(Z_s) \operatorname{sign}(Z_s)\, ds \\ &+ \frac{1}{2} \int_0^t f''(Z_s)\, d[M_s^c, M_s^c] \\ &+ \int_0^t \big(f(Z_s + z) - f(Z_s) - f'(Z_s) z\big)\, \mathcal{C}'(ds, dz) + \text{local martingale} \end{aligned}$$

for all $f \in C_c^\infty(\mathbb{R})$. We can assume without loss of generality that the local martingale is a martingale since otherwise the rest of the proof can be carried out via localization. Comparing with the martingale problem $(\mathcal{A}, \delta_z)$ from (2.12) and using the uniqueness of the canonical decomposition for a semi-martingale, we see that $d[M_s^c, M_s^c] = \sigma^2 |Z_s| ds$ and

$$\int_0^t \int_{\mathbb{R}} F(s, z)\, \mathcal{C}'(ds, dz) = \int_0^t \int_0^\infty \int_{-1}^1 F(s, \mathbf{1}_{\{r|Z_s| \leq 1\}} Z_s(u-1))\, ds\, dr\, \bar{\Pi}(du)$$

for any non-negative Borel function F on $[0, \infty) \times \mathbb{R}$. Then we can find a Brownian motion B and an independent Poisson point process $\mathcal{N}$ on $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t \geq 0}, P)$ by applying martingale representation theorems to (2.14) (see for instance [21], page 84 and page 93). □

The construction of a solution to the martingale problem $(\mathcal{A}, \delta_z)$ is achieved with a series of lemmas. To give a rough idea how to construct solutions let us reconsider the simplest special case

$$dZ_t = \operatorname{sign}(Z_t)\, dt + 2\sqrt{|Z_t|}\, dB_t, \quad Z_0 = 0, \tag{2.15}$$

and its positive analogue

$$dZ_t = dt + 2\sqrt{Z_t}\, dB_t, \quad Z_0 = 0, \tag{2.16}$$

obtained for the absolute value. If W is a Brownian motion, then we already noted that $Z^{(1)}_t = \operatorname{sign}(W_t)W_t^2$ is a weak solution to the SDE (2.15) and furthermore $Z^{(2)}_t = W_t^2$ is also a weak solution to the SDE (2.16). Of course, $Z^{(1)}$ can be constructed from $Z^{(2)}$: given $Z^{(2)}$, $Z^{(1)}$ is obtained by reflecting every excursion at the origin with probability 1/2.

A simple construction via reflected excursions for the SDE (1.8) does not work directly if there are jumps that change sign, i.e. $V \neq 0$. In what follows we give a stochastic calculus construction that mimics the reflection idea but is robust enough to encounter jumps that change signs.

The notation of Theorem 1.5 and Proposition 2.4 will be used in the sequel without explicit repetitions.

We start with the construction of a symmetric approximating sequence for (1.8).

Lemma 2.5. *Suppose $m \in \mathbb{N}$ and that $\mathcal{M}^m$ is a Poisson point process on $(0,\infty)\times\{-\frac{1}{m},\frac{1}{m}\}$ independent of B and $\mathcal{N}$ with intensity measure $\mathcal{M}^{m\prime}(ds,dv) = ds \otimes \Sigma(dv)$, where $\Sigma(\{\frac{1}{m}\}) = \Sigma(\{-\frac{1}{m}\}) = \frac{m}{2}$. If we define*

$$\operatorname{sign}_{(0)}(x) = \mathbf{1}_{\{x>0\}} - \mathbf{1}_{\{x<0\}},$$

then, for $z\in\mathbb{R}$, there are unique strong solutions Z^m to the SDE

$$\begin{aligned} Z_t &= z + \left(\Psi(1) + \int_{-1}^0 (u-1)V(du)\right)\int_0^t \operatorname{sign}_{(0)}(Z_s)\,ds + \sigma\int_0^t \sqrt{|Z_s|}\,dB_s \\ &\quad + \left(\Psi(1) + \int_{-1}^0 (|u|-1)V(du)\right)\int_0^t\int_{\{\pm\frac{1}{m}\}} \mathbf{1}_{\{Z_{s-}=0\}} v\,\mathcal{M}^m(ds,dv) \\ &\quad + \int_0^t \int_0^{\frac{1}{|Z_{s-}|}\wedge m}\int_{[-1,1-\frac{1}{m}]} Z_{s-}(u-1)(\mathcal{N}-\mathcal{N}')(ds,dr,du), \quad t\geq 0. \end{aligned} \tag{2.17}$$

Proof. Suppressing the jumps according to the point process $\mathcal{N}$ and integrating out dr in the remaining compensator integral yields the SDE

$$\begin{aligned} Z_t &= z + \left(\Psi(1) + \int_{-1}^0 (u-1)V(du)\right)\int_0^t \operatorname{sign}_{(0)}(Z_s)\,ds + \sigma\int_0^t \sqrt{|Z_s|}\,dB_s \\ &\quad + \left(\Psi(1) + \int_{-1}^0 (|u|-1)V(du)\right)\int_0^t\int_{\{\pm\frac{1}{m}\}} \mathbf{1}_{\{Z_{s-}=0\}} v\,\mathcal{M}^m(ds,dv) \\ &\quad - \int_{[-1,1-\frac{1}{m}]} (u-1)\bar{\Pi}(du)\int_0^t \operatorname{sign}(Z_s)\big(1\wedge m|Z_s|\big)\,ds. \end{aligned} \tag{2.18}$$

A solution can be constructed by induction on the number of hits of zero: Since the coefficients are Lipschitz continuous away from zero, by standard (continuous) SDE theory there is a unique strong solution Z before the first hitting time δ_1 of zero. Let us denote by τ_1 the time of the first atom of $\mathcal{M}^m$ that is larger than δ_1. We set $Z_t = 0$ for all $t\in[\delta_1,\tau_1)$ and

$$Z_{\delta_1+\tau_1} = \pm\frac{\Psi(1) + \int_{-1}^0(|u|-1)V(du)}{m}$$

with equal probability depending on the atom of $\mathcal{M}^m$. Between $\delta_1+\tau_1$ and δ_2 the Lipschitz continuity can be exploited again to construct Z as a unique strong solution. Inductively this defines a solution process $(Z_t)_{t\geq 0}$ to (2.18). The truncation by m implies that jumps driven by $\mathcal{N}$ come by finite rate, hence, a strong solution of (2.17) can be constructed from (2.18) piecewise via the interlacing method. This is a standard argument, so we omit the details and refer for instance to the proof of Proposition 3.5 of [17]. □

The choice of the SDE (2.17) is motivated by the reflected excursion idea for a construction of solutions: whenever solutions are away from zero, they follow the original SDE (1.8), the "pseudo excursions" taking values in $\mathbb{R}_*$. At zero the pseudo excursions stop and after an exponential time a new pseudo excursion is started at a small initial state chosen symmetrically by $\mathcal{M}^m$. The symmetric restarting is needed to construct a symmetric process; non-symmetric restarting might be used to construct skew-self-similar Markov processes.

It is crucial to redefine the sign-function to be zero at zero since otherwise the constructed process is not a solution to (2.17). As m increases, the times between pseudo excursions and the new initial states tend to zero so that possible limiting processes leave zero continuously.

The construction shows that, for all z and m,

$$\int_0^\infty \mathbf{1}_{\{Z_s^m=0\}} ds = \infty, \quad a.s., \tag{2.19}$$

if the pseudo excursions hit zero in finite time. Hence, a priori it is possible that any limiting process Z of Z^m is trapped at zero. To guarantee that Z is not such a trivial solution (and to resolve technical difficulties), under condition (1.10) we are going to deduce

$$\int_0^\infty \mathbf{1}_{\{Z_s=0\}} ds = 0, \quad a.s, \tag{2.20}$$

which is a bit surprising because of (2.19). The constant in front of the stochastic integral with respect to $\mathcal{M}^m$ turns out to be crucial; it guarantees that, after taking absolute values, all drift terms become constant in the limit $m \to \infty$ from which we can deduce (2.20).

Lemma 2.6. *Suppose Z^m is as in Lemma* 2.5*; then*

(2.21)

$$\begin{aligned} |Z_t^m| = |z| &+ \left(\Psi(1) + \int_{-1}^0 (u-1)V(du)\right) t \\ &+ \left(\int_{-1}^0 (|u|-u)V(du)\right) \int_0^t \left(\mathbf{1}_{\{Z_s^m=0\}} + \left(1 \wedge m|Z_s^m|\right)\right) ds \\ &+ \sigma \int_0^t \operatorname{sign}(Z_s^m)\sqrt{|Z_s^m|}\, dB_s \\ &+ \left(\Psi(1) + \int_{-1}^0 (|u|-1)V(du)\right) \int_0^t \int_{\{\pm\frac{1}{m}\}} \mathbf{1}_{\{Z_{s-}^m=0\}} |v| \left(\mathcal{M}^m - \mathcal{M}^{m\prime}\right)(ds, dv) \\ &+ \int_0^t \int_0^{\frac{1}{|Z_{s-}^m|}\wedge m} \int_{[-1,1-\frac{1}{m}]} |Z_{s-}^m|(|u|-1)(\mathcal{N}-\mathcal{N}')(ds, dr, du), \quad t \geq 0, \end{aligned}$$

almost surely.

Proof. Let us denote by $\tau_1 < \tau_2 < \ldots$ the jump times of the Poissonian integral driven by $\mathcal{M}^m$ which are precisely the times when Z^m leaves zero. Further,

$\delta_1 < \delta_2 < ...$ denote the successive first hitting times of zero which do not accumulate since paths are càdlàg and solutions only leave zero with a jump of size $\pm\frac{\Psi(1)+\int_{-1}^{0}(|u|-1)V(du)}{m}$. If we define $ZERO := [\delta_1, \tau_1) \cup [\delta_2, \tau_2) \cup ...$, then

$$Z_s^m = 0 \quad \forall s \in ZERO \qquad \text{and} \qquad Z_s^m \neq 0 \quad \forall s \notin ZERO.$$

Consequently, $|Z_s^m| = 0$ for $s \in ZERO$ so that it suffices to apply Tanaka's formula to Z^m on $ZERO^c$. Let us first show that the semi-martingale local time at zero vanishes. The truncation by m implies that jumps are summable so that Corollary 3 on page 178 of [30] yields

$$\begin{aligned}
L_t^0 &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^t \mathbf{1}_{\{|Z_s^m|\le\varepsilon\}}\, d[Z_s^m, Z_s^m]^c \\
&= \lim_{\varepsilon\to 0}\frac{\sigma^2}{\varepsilon}\left[\sum_{j=1}^{i-1}\int_{\tau_j}^{\delta_{j+1}} \mathbf{1}_{\{|Z_s^m|\le\varepsilon\}}|Z_s^m|\, ds + \int_{\tau_i}^{t} \mathbf{1}_{\{|Z_s^m|\le\varepsilon\}}|Z_s^m|\, ds\right] \\
&\le \lim_{\varepsilon\to 0}\sigma^2\left[\sum_{j=1}^{i-1}\int_{\tau_j}^{\delta_{j+1}} \mathbf{1}_{\{|Z_s^m|\le\varepsilon\}}\, ds + \int_{\tau_i}^{t} \mathbf{1}_{\{|Z_s^m|\le\varepsilon\}}\, ds\right], \quad t \in [\tau_i, \delta_{i+1}).
\end{aligned}$$

Using dominated convergence, the right-hand side converges to zero since Z^m does not spend time at zero on $ZERO^c$. Next, Tanaka's formula can be applied without additional local time term to deduce the semi-martingale decomposition

$$\begin{aligned}
|Z_t^m| = |z| &+ \left(\Psi(1) + \int_{-1}^{0}(u-1)V(du)\right)\int_0^t \operatorname{sign}(Z_s^m)\operatorname{sign}_{(0)}(Z_s^m)\, ds \\
&+ \sigma\int_0^t \operatorname{sign}(Z_s^m)\sqrt{|Z_s^m|}\, dB_s \\
&+ \int_0^t\int_{\{\pm\frac{1}{m}\}}\left(\left|Z_{s-}^m + \left(\Psi(1) + \int_{-1}^{0}(|u|-1)V(du)\right)\right.\right. \\
&\qquad\qquad \left.\left.\times\, \mathbf{1}_{\{Z_{s-}^m=0\}}v\right| - \left|Z_{s-}^m\right|\right)\mathcal{M}^m(ds, dv) \\
&+ \int_0^t\int_0^{\frac{1}{|Z_{s-}^m|}\wedge m}\int_{[-1,1-\frac{1}{m}]}|Z_{s-}^m|(|u|-1)(\mathcal{N}-\mathcal{N}')(ds, dr, du) \\
&+ \int_0^t \left(1\wedge m|Z_s^m|\right) ds\int_{[-1,1-\frac{1}{m}]}(|u|-u)\bar{\Pi}(du).
\end{aligned}$$

Adding and subtracting the compensator integral for $\mathcal{M}^m$, we obtain as a drift

$$\begin{aligned}
&\Big(\Psi(1)+\int_{-1}^0 (u-1)V(du)\Big)\int_0^t \mathbf{1}_{\{Z_s^m\neq 0\}}\,ds\\
&\quad+\int_0^t\int_{\{\pm\frac{1}{m}\}} \mathbf{1}_{\{Z_{s-}^m=0\}}\Big(\Psi(1)+\int_{-1}^0(|u|-1)V(du)\Big)v\,\mathcal{M}^{m\prime}(ds,dv)\\
&\quad+\int_0^t \big(1\wedge m|Z_s^m|\big)\,ds\int_{[-1,1-\frac{1}{m}]}(|u|-u)\bar\Pi(du)\\
&=\Big(\Psi(1)+\int_{-1}^0 (u-1)V(du)\Big)\int_0^t \mathbf{1}_{\{Z_s^m\neq 0\}}\,ds\\
&\quad+\Big(\Big(\Psi(1)+\int_{-1}^0 (u-1)V(du)\Big)\\
&\quad+\Big(\int_{-1}^0(|u|-u)V(du)\Big)\Big)\int_0^t\int_{\{\pm\frac{1}{m}\}}\mathbf{1}_{\{Z_s^m=0\}}|v|\,\Sigma(dv)\,ds\\
&\quad+\int_0^t \big(1\wedge m|Z_s^m|\big)\,ds\int_{[-1,1-\frac{1}{m}]}(|u|-u)\bar\Pi(du)\\
&=\Big(\Psi(1)+\int_{-1}^0 (u-1)V(du)\Big)t+\Big(\int_{-1}^0(|u|-u)V(du)\Big)\int_0^t\mathbf{1}_{\{Z_s^m=0\}}\,ds\\
&\quad+\int_0^t \big(1\wedge m|Z_s^m|\big)\,ds\int_{[-1,1-\frac{1}{m}]}(|u|-u)\bar\Pi(du).
\end{aligned}$$

Using the definition of $\bar\Pi$ we can simplify the final integral to

$$\int_{[-1,1-\frac{1}{m}]}(|u|-u)\bar\Pi(du)=\int_{-1}^0(|u|-u)\bar\Pi(du)=\int_{-1}^0(|u|-u)V(du)$$

from which the claim follows. □

Next, we show that there are limits of the sequence Z^m:

Lemma 2.7. *For any $z\in\mathbb{R}$ the sequence $(Z^m)_{m\in\mathbb{N}}$ constructed in Lemma* 2.5 *is tight in the Skorokhod topology on $\mathbb{D}$.*

Proof. For the proof we apply Aldous's tightness criterion (see Aldous [1]). According to Aldous, to prove that $\{Z^m : m\in\mathbb{N}\}$ is tight in $\mathbb{D}$ it is enough to show that

(i) for every fixed $t\geq 0$, the set of random variables $\{Z_t^m : m\in\mathbb{N}\}$ is tight,
(ii) for every sequence of stopping times $(\tau_m)_{m\in\mathbb{N}}$ (with respect to the filtration $(\mathcal{G}_t)_{t\geq 0}$) bounded above by $T>0$ and for every sequence of positive real numbers $(\delta_m)_{m\in\mathbb{N}}$ converging to 0, $Z^m_{\tau_m+\delta_m}-Z^m_{\tau_m}\to 0$ in probability as $m\to\infty$.

To prove (i), by Markov's inequality it is enough to check that, for every fixed $t\geq 0$,

$$\sup_{m\in\mathbb{N}} E\big[(Z_t^m)^2\big]<\infty. \tag{2.22}$$

Using that

$$(a+b+c+d+e)^2 \le 5(a^2+b^2+c^2+d^2+e^2)$$

for $a,b,c,d,e \in \mathbb{R}$, we obtain that $E\big[(Z_t^m)^2\big]$ can be bounded by

$$\begin{aligned}
&5z^2 + 5\Big(\Psi(1) + \int_{-1}^{0}(u-1)V(du)\Big)^2 E\Big[\int_0^t \operatorname{sign}_{(0)}(Z_s^m)\,ds\Big]^2 \\
&+ 5\sigma^2 E\Big[\int_0^t \sqrt{|Z_s^m|}\,dB_s\Big]^2 \\
&+ 5\Big(\Psi(1)+\int_{-1}^{0}(|u|-1)V(du)\Big)^2 E\Big[\int_0^t\int_{\{-\frac{1}{m},\frac{1}{m}\}} \mathbf{1}_{\{Z_{s-}^m=0\}} v\,(\mathcal{M}^m-\mathcal{M}^{m\prime})(ds,dv)\Big]^2 \\
&+ 5E\Big[\int_0^t\int_0^{\frac{1}{|Z_{s-}^m|}\wedge m}\int_{[-1,1-\frac{1}{m}]} Z_{s-}^m(u-1)(\mathcal{N}-\mathcal{N}')(ds,dr,du)\Big]^2
\end{aligned}$$

which, via Itō's isometry (for the Poissonian integral see for instance page 62 in Ikeda and Watanabe [21]), can be bounded from above by

$$\begin{aligned}
&5z^2 + 5\Big(\Psi(1) + \int_{-1}^{0}(u-1)V(du)\Big)^2 t^2 + 5\sigma^2 E\Big[\int_0^t |Z_s^m|\,ds\Big] \\
&\quad + 5\,t\,\Big(\Psi(1) + \int_{-1}^{0}(|u|-1)V(du)\Big)^2 \int_{\{-\frac{1}{m},\frac{1}{m}\}} v^2\,\Sigma(dv) \\
&\quad + 5E\Big[\int_0^t\int_0^{\frac{1}{|Z_s^m|}}\int_{[-1,1-\frac{1}{m}]} (Z_s^m)^2(u-1)^2\,ds\,dr\,\bar{\Pi}(du)\Big] \\
&\le 5z^2 + 5\Big(\Psi(1) + \int_{-1}^{0}(u-1)V(du)\Big)^2 t^2 + 5\Big(\Psi(1) + \int_{-1}^{0}(|u|-1)V(du)\Big)^2 \frac{t}{m} \\
&\quad + \Big(5\sigma^2 + 5\int_{-1}^{1}(u-1)^2\,\bar{\Pi}(du)\Big)\int_0^t E[|Z_s^m|]\,ds
\end{aligned}$$

so that the estimate $E[|Z_s^m|] \le 1 + E[(Z_s^m)^2]$ combined with Gronwall's inequality yields the claim (the finiteness of $E[(Z_s^m)^2]$ for fixed $s \ge 0$ is checked easily as in the proof of Proposition 2.4).

Now we turn to (ii) and prove the stronger statement that $Z^m_{\tau_m+\delta_m} - Z^m_{\tau_m}$ converges to 0 in L^2 as $m \to \infty$. Namely, by the SDE (2.17) and the splitting of summands as before,

$$
\begin{aligned}
&E\big[\big|Z^m_{\tau_m+\delta_m}-Z^m_{\tau_m}\big|^2\big]\\
&\le 4\bigg(\Psi(1)+\int_{-1}^0 (u-1)V(du)\bigg)^2 E\bigg[\int_{\tau_m}^{\tau_m+\delta_m}\operatorname{sign}_{(0)}(Z^m_s)\,ds\bigg]^2\\
&\quad+4\sigma^2 E\bigg[\int_{\tau_m}^{\tau_m+\delta_m}\sqrt{|Z^m_s|}\,dB_s\bigg]^2\\
&\quad+4\bigg(\Psi(1)+\int_{-1}^0 (|u|-1)V(du)\bigg)^2\\
&\quad\times E\bigg[\int_{\tau_m}^{\tau_m+\delta_m}\int_{\{-\frac{1}{m},\frac{1}{m}\}}\mathbf{1}_{\{Z^m_{s-}=0\}}v\,(\mathcal{M}^m-\mathcal{M}^{m\prime})(ds,dv)\bigg]^2\\
&\quad+4E\bigg[\int_{\tau_m}^{\tau_m+\delta_m}\int_0^{\frac{1}{|Z^m_{s-}|}\wedge m}\int_{[-1,1-\frac{1}{m}]}Z^m_{s-}(u-1)(\mathcal{N}-\mathcal{N}')(ds,dr,du)\bigg]^2.
\end{aligned}
$$

The first summand can be estimated by $C\delta_m^2$ and, hence, can be neglected. By Proposition 3.2.10 in Karatzas and Shreve [23] we obtain

$$
E\bigg[\int_{\tau_m}^{\tau_m+\delta_m}\sqrt{|Z^m_s|}\,dB_s\bigg]^2=E\bigg[\int_{\tau_m}^{\tau_m+\delta_m}|Z^m_s|\,ds\bigg],
$$

and, by Theorem II.1.33 in Jacod and Shiryaev [22], the optimal stopping theorem, using the same argument as in the proof of (3.2.22) in Karatzas and Shreve [23], yields

$$
\begin{aligned}
&E\bigg[\int_{\tau_m}^{\tau_m+\delta_m}\int_0^{\frac{1}{|Z^m_{s-}|}\wedge m}\int_{[-1,1-\frac{1}{m}]}Z^m_{s-}(u-1)(\mathcal{N}-\mathcal{N}')(ds,dr,du)\bigg]^2\\
&\quad\le E\bigg[\int_{\tau_m}^{\tau_m+\delta_m}\int_0^{\frac{1}{|Z^m_s|}}\int_{[-1,1-\frac{1}{m}]}(Z^m_s)^2(u-1)^2\,ds\,dr\,\bar\Pi(du)\bigg]\\
&\quad\le E\bigg[\int_{\tau_m}^{\tau_m+\delta_m}|Z^m_s|\,ds\bigg]\int_{-1}^1(u-1)^2\bar\Pi(du).
\end{aligned}
$$

For the integral with respect to $\mathcal{M}$ a similar argument gives the upper bound $C_3\delta_m$. In total this shows that (we can suppose that $\delta_m\le 1$)

$$
\begin{aligned}
E\big[|Z^m_{\tau_m+\delta_m}-Z^m_{\tau_m}|^2\big]&\le C_1\delta_m^2+C_2\delta_m+C_3E\bigg[\int_{\tau_m}^{\tau_m+\delta_m}|Z^m_s|\,ds\bigg]\\
&\le C_1\delta_m^2+C_2\delta_m+C_3\delta_m E\big[\sup_{s\le T+1}|Z^m_s|\big].
\end{aligned}
$$

Hence, the proof is complete if we can show that $E\big[\sup_{s\le T+1}|Z^m_s|\big]$ is bounded in m. Proceeding similarly to (i), replacing Itō's isometry with the Burkholder-Davis-Gundy inequality, we obtain

$$
\begin{aligned}
&E\big[\sup_{s\le T+1}|Z^m_s|\big]\\
&\le 1+E\big[\sup_{s\le T+1}(Z^m_s)^2\big]\\
&\le 1+5z^2+5(T+1)^2\Big(\Psi(1)+\int_{-1}^0(u-1)V(du)\Big)^2+C_1E\Big[\int_0^{T+1}|Z^m_s|\,ds\Big]\\
&\quad +C_2\Big(\Psi(1)+\int_{-1}^0(|u|-1)V(du)\Big)^2E\Big[\int_0^{T+1}\int_{\{-\frac1m,\frac1m\}}\mathbf{1}_{\{Z^m_{s-}=0\}}v^2\,\mathcal{M}^{m\prime}(ds,dv)\Big]\\
&\quad +C_3E\Big[\int_0^{T+1}\int_0^{\frac{1}{|Z^m_s|}\wedge m}\int_{[-1,1-\frac1m]}(Z^m_s)^2(u-1)^2\mathcal{N}'(ds,dr,du)\Big]\\
&\le 1+5z^2+5(T+1)^2\Big(\Psi(1)+\int_{-1}^0(u-1)V(du)\Big)^2+C_1\int_0^{T+1}E\big[|Z^m_s|\big]\,ds\\
&\quad +C_2\Big(\Psi(1)+\int_{-1}^0(|u|-1)V(du)\Big)^2(T+1)\frac1m\\
&\quad +C_3\int_0^{T+1}E\big[|Z^m_s|\wedge m(Z^m_s)^2\big]\,ds\int_{[-1,1]}(u-1)^2\,\bar\Pi(du)\\
&\le C_4(T+1)^2+C_5\int_0^{T+1}E\big[|Z^m_s|\big]\,ds.
\end{aligned}
$$

The right-hand side is finite due to the exponential second moment bound obtained from the Gronwall inequality in (i). □

We next prepare for the convergence proof of Z^m along subsequences. It is crucial to deduce, a priori, that all limiting points do not spend time at zero in order to control the discontinuity of the sign-function at zero. Since we cannot deduce this property for the limiting points of Z^m directly, we show it for $|Z^m|$.

Lemma 2.8. *Suppose Z denotes a limiting point of the tight sequence $(Z^m)_{m\in\mathbb{N}}$ constructed in Lemma 2.5. Then $|Z|$ is a weak solution to the SDE*

$$
\begin{aligned}
X_t&=z+\Big(\Psi(1)+\int_{-1}^0(|u|-1)V(du)\Big)t+\sigma\int_0^t\sqrt{X_s}\,dB_s\\
&\quad+\int_0^t\int_0^{\frac{1}{X_{s-}}}\int_{-1}^1X_{s-}(|u|-1)(\mathcal{N}-\mathcal{N}')(ds,dr,du),\quad t\ge 0.
\end{aligned}
\tag{2.23}
$$

Proof. Let us suppose that along the subsequence m_k we have weak convergence of Z^{m_k} to Z and, due to the continuous mapping theorem, also weak convergence of $|Z^{m_k}|$ to $|Z|$. We first derive a martingale problem for $|Z^{m_k}|, m_k\in\mathbb{N}$, from which we then derive the claimed statement.

Step A): Proceeding exactly as in the first part of the proof of Proposition 2.4, one derives from Lemma 2.6 that $|Z^m|$ solves the martingale problem $(|\mathcal{A}^{m_k}|, \delta_{|z|})$ with

$$
\begin{aligned}
&(|\mathcal{A}^{m_k}|f)(x)\\
:= &\bigg(\Big(\Psi(1)+\int_{-1}^0 (u-1)V(du)\Big)\\
&\quad+\Big(\int_{-1}^0 (|u|-u)V(du)\Big)\big(\mathbf{1}_{\{x=0\}}+(1\wedge m_k x)\big)\bigg)f'(x)\\
&+\frac{\sigma^2}{2}xf''(x)+\int_{\{\pm\frac{1}{m_k}\}}\mathbf{1}_{\{x=0\}}\bigg(f\bigg(\Big(\Psi(1)+\int_{-1}^0(|u|-1)V(du)\Big)|u|\bigg)\\
&\qquad-f(0)-f'(0)\Big(\Psi(1)+\int_{-1}^0(|u|-1)V(du)\Big)|u|\bigg)\Sigma(dv)\\
&+\int_0^{m_k}\int_{[-1,1-\frac{1}{m_k}]}\mathbf{1}_{\{rx\le 1\}}\Big(f(|u|x)-f(x)-f'(x)x(|u|-1)\Big)\,dr\,\bar\Pi(du),\quad x\ge 0,
\end{aligned}
$$

for $f\in C_c^\infty[0,\infty)$.

Step B): We show that the limit point $|Z|$ solves the martingale problem $(|\mathcal{A}|, \delta_{|z|})$, with

$$
\begin{aligned}
(|\mathcal{A}|f)(x) := &\Big(\Psi(1)+\int_{-1}^0(|u|-1)V(du)\Big)f'(x)+\frac{\sigma^2}{2}xf''(x)\\
&+\int_0^\infty\int_{-1}^1\mathbf{1}_{\{rx\le 1\}}\Big(f(|u|x)-f(x)-f'(x)x(|u|-1)\Big)\,dr\,\bar\Pi(du),\quad x\ge 0,
\end{aligned}
$$

for $f\in C_c^\infty[0,\infty)$. By Skorokhod's representation theorem we may assume that Z^{m_k} converges to Z (resp. $|Z^{m_k}|$ to $|Z|$) almost surely in $\mathbb{D}$ (possibly on a different probability space). By Proposition 3.5.2 of [18], the almost sure convergence yields that $P(\bar\Omega)=1$, where

$$
\bar\Omega := \Big\{\omega\in\Omega : \lim_{k\to\infty}|Z_t^{m_k}(\omega)| = |Z_t(\omega)| \text{ for } t\ge 0 \text{ at which } (Z_u(\omega))_{u\ge 0} \text{ is continuous}\Big\}.
$$

In what follows we let $\omega\in\bar\Omega$ be fixed and show that

$$
\lim_{k\to\infty}\int_0^t (|\mathcal{A}^{m_k}|f)(|Z_s^{m_k}|)(\omega)\,ds = \int_0^t (|\mathcal{A}|f)(|Z_s|)(\omega)\,ds,\qquad t\ge 0. \tag{2.24}
$$

In Step B1) we verify the pointwise convergence

$$
(|\mathcal{A}^{m_k}|f)(|Z_s^{m_k}|)(\omega) \overset{m_k\to\infty}{\longrightarrow} (|\mathcal{A}|f)(|Z_s|)(\omega)
$$

for $s\le t$ fixed and in Step B2) we verify the convergence of (2.24) via dominated convergence.

Let us introduce the notation

$$
D(\omega) := \big\{t\ge 0 : (Z_u(\omega))_{u\ge 0} \text{ is continuous at } t\big\},
$$

so that $\lim_{k\to\infty}|Z_t^{m_k}(\omega)| = |Z_t(\omega)|$ for all $t\in D(\omega)$ and furthermore $[0,\infty)\setminus D(\omega)$ is at most countable since Z has càdlàg paths.

Step B1a): The pointwise convergence for the drift part is trivial.

Step B1b): The pointwise convergence for the diffusive part is trivial.

Step B1c): For the integral with respect to Σ we apply Taylor's formula to find

$$\begin{aligned}&\int_{\{\pm\frac{1}{m_k}\}}\mathbf{1}_{\{Z_s^{m_k}=0\}}\bigg(f\bigg(\bigg(\Psi(1)+\int_{-1}^{0}(|u|-1)V(du)\bigg)|v|\bigg)\\&\qquad-f(0)-f'(0)\bigg(\Psi(1)+\int_{-1}^{0}(|u|-1)V(du)\bigg)|v|\bigg)\Sigma(dv)\\&\le\frac{1}{2}\bigg(\Psi(1)+\int_{-1}^{0}(|u|-1)V(du)\bigg)^2\sup_z f''(z)\int_{\{-\frac{1}{m_k},\frac{1}{m_k}\}}|v|^2\,\Sigma(dv)\\&=\frac{1}{2}\bigg(\Psi(1)+\int_{-1}^{0}(|u|-1)V(du)\bigg)^2\frac{\sup_z f''(z)}{m_k},\end{aligned}$$

so that pointwise convergence to zero for $m_k\to\infty$ is verified.

Step B1d): For the integral with respect to $\bar{\Pi}$ we use that, for all $s\in D(\omega)$, the integrand

$$\begin{aligned}(r,u)\mapsto\ &\mathbf{1}_{\{u\in[-1,1-\frac{1}{m_k}]\}}\mathbf{1}_{\{r|Z_s^{m_k}(\omega)|\le 1\}}\\&\times\big[f\big(|u||Z_s^{m_k}(\omega)|\big)-f(|Z_s^{m_k}(\omega)|)-f'(|Z_s^{m_k}(\omega)|)|Z_s^{m_k}(\omega)|(|u|-1)\big]\end{aligned}$$

converges, as $k\to\infty$, pointwise to

$$\begin{aligned}(r,u)\mapsto\mathbf{1}_{\{|u|\le 1\}}\mathbf{1}_{\{r|Z_s(\omega)|\le 1\}}\big[&f(|u||Z_s(\omega)|)-f(|Z_s(\omega)|)\\&-f'(|Z_s(\omega)|)|Z_s(\omega)|(|u|-1)\big].\end{aligned}$$

Step B2): Since

$$\lim_{k\to\infty}(|\mathcal{A}^{m_k}|f)(|Z_s^{m_k}(\omega)|)=(|\mathcal{A}|f)(|Z_s(\omega)|),\quad\omega\in\bar{\Omega},$$

is verified for any $s\le t$, it remains to justify the change of limit and integration in (2.24). For the first two summands this is clear, for the third we use dominated convergence with the estimate given in Step B1c). We only need to deal with the integral with respect to $\bar{\Pi}$: By Taylor expansion of second order, we can derive the upper bound

$$\begin{aligned}&\sup_{k\in\mathbb{N}}\mathbf{1}_{\{u\in[-1,1-\frac{1}{m_k}]\}}\mathbf{1}_{\{r|Z_s^{m_k}(\omega)|\le 1\}}\\&\qquad\Big|f(|u||Z_s^{m_k}(\omega)|)-f(|Z_s^{m_k}(\omega)|)\mathbf{1}_{\{|u|\ge\varepsilon_k\}}-f'(Z_s^{m_k}(\omega))|Z_s^{m_k}(\omega)|(|u|-1)\Big|\\&\le\frac{1}{2}\mathbf{1}_{\{|u|\le 1\}}\sup_z|f''(z)|\sup_{k\in\mathbb{N}}\mathbf{1}_{\{r|Z_s^{m_k}(\omega)|\le 1\}}|Z_s^{m_k}(\omega)|^2(|u|-1)^2\\&\le\frac{1}{2}\mathbf{1}_{\{|u|\le 1\}}\sup_z|f''(z)|\sup_{k\in\mathbb{N}}\left(Z_s^{m_k}(\omega)\wedge\frac{1}{r}\right)^2(|u|-1)^2\\&\le\frac{1}{2}\mathbf{1}_{\{|u|\le 1\}}\sup_z|f''(z)|\sup_{k\in\mathbb{N}}\left(\sup_{s\le t}Z_s^{m_k}(\omega)\wedge\frac{1}{r}\right)^2(|u|-1)^2.\end{aligned}$$

Note that for the final line we used that $x \mapsto \sup_{s\le t} x_s$ is a continuous functional on the Skorokhod space so that the convergence of Z^{m_k} implies $\sup_{k\in\mathbb{N}} \sup_{s\le t} Z^{m_k}_s(\omega) =: C_t(\omega) < \infty$. Thus, the integral with respect to $\bar\Pi$ in $(|\mathcal{A}^{m_k}|f)(|Z^{m_k}_s(\omega)|)$ is bounded from above by

$$\begin{aligned}
&\frac{1}{2}\sup_z |f''(z)| \int_0^\infty \left(C_t(\omega)\wedge\frac{1}{r}\right)^2 dr \int_{-1}^1 (|u|-1)^2\,\bar\Pi(du)\\
&\le C\sup_z |f''(z)| \left(C_t(\omega)^2 + \int_1^\infty r^{-2}dr\right)\int_{-1}^1 (|u|-1)^2\,\bar\Pi(du),
\end{aligned}$$

which is finite and independent of s. Using the dominated convergence theorem we have convergence for the third summand of $\mathcal{A}^{m_k}f$.

Step C): To conclude the proof let us write

$$\begin{aligned}
M^{m_k}_t &= f(|Z^{m_k}_t|) - \int_0^t |\mathcal{A}^{m_k}|f(|Z^{m_k}_s|)\,ds, \quad t\ge 0,\\
M_t &= f(|Z_t|) - \int_0^t |\mathcal{A}|f(|Z_s|)\,ds, \quad t\ge 0,
\end{aligned}$$

for which we know that the M^{m_k} are martingales with respect to the filtrations generated by Z^{m_k}. The martingale property of M with respect to its own filtration follows by Jacod and Shiryaev [22, Corollary IX.1.19]. To check the conditions of this result we have to show that M^{m_k} converges weakly in $\mathbb{D}$ as $k\to\infty$ to M and that there is some $b\ge 0$ such that $|\Delta M^{m_k}_t| \le b$ for all $t>0$, $m\in\mathbb{N}$, almost surely. Using that $|Z^{m_k}|$ converges weakly in $\mathbb{D}$ to $|Z|$ as $k\to\infty$ and that f is continuous and bounded, we have $f(|Z^{m_k}|)$ converges weakly in $\mathbb{D}$ to $f(|Z|)$ as $k\to\infty$. Since the integral in the definition of M is continuous, by Jacod and Shiryaev [22, Proposition VI.1.23], we obtain that M^{m_k} converges weakly in $\mathbb{D}$ as $k\to\infty$ to M. Further, almost surely for all $t\ge 0$,

$$|\Delta M^{m_k}_t| = |f(|Z^{m_k}_t|) - f(|Z^{m_k}_{t-}|)| \le 2\sup_z |f(z)| < \infty.$$

□

Corollary 2.9. *Suppose Z denotes a limiting point of the tight sequence $(Z^m)_{m\in\mathbb{N}}$ constructed in Lemma* 2.5*. Then almost surely Z does not spend time at zero.*

Proof. Utilizing Lemma 2.8 it is enough to show that any non-negative weak solution to the SDE (2.23) does not spend time at zero. Without further assumptions on the jump measure $\bar\Pi$ the jumps of a solution X to the SDE (2.23) are not summable, thus, we cannot directly resort to a simple local time argument based on the occupation time formula (compare for instance Section IV.6 of [30]). Instead, we use an Itō formula argument that was used in a more specific situation in [5].

The argument is based on the trivial fact $\sqrt{X_t^2} = X_t$ and a double use of Itō's formula, once applied to a smooth function and once to a singular function. The singular use gives an additional term from which the claim follows. Here is the simple direction applying Itō's formula to the $C^2([0,\infty))$-function $f(x)=x^2$:

$$\begin{aligned}X_t^2 &= z^2+\Bigg(2\bigg(\Psi(1)+\int_{-1}^0(|u|-1)V(du)\bigg)\\&\quad+\sigma^2+\int_{-1}^1(u^2-2|u|+1)\,\bar\Pi(du)\Bigg)\int_0^t X_s\,ds\\&\quad+2\sigma\int_0^t X_s^{3/2}\,dB_s+\int_0^t\int_0^{\frac{1}{X_{s-}}}\int_{-1}^1 X_{s-}^2(u^2-1)\,(\mathcal{N}-\mathcal{N}')(ds,dr,du),\quad t\geq 0.\end{aligned}$$

Next, we proceed with the reverse direction: Suppose we could apply Itō's formula with $f(x)=\sqrt{x}$ and the convention $\frac{0}{0}=0$ to the semi-martingale decomposition derived for X_t^2. Then

$$\begin{aligned}X_t &= z+\bigg(\Psi(1)+\int_{-1}^0(|u|-1)V(du)\bigg)\int_0^t\mathbf{1}_{\{X_s>0\}}\,ds+\sigma\int_0^t\sqrt{X_s}\,dB_s\\&\quad+\int_0^t\int_0^{\frac{1}{X_{s-}}}\int_{-1}^1 X_{s-}(|u|-1)(\mathcal{N}-\mathcal{N}')(ds,dr,du),\quad t\geq 0,\end{aligned}\tag{2.25}$$

so that comparing the drifts of (2.25) and (2.23) implies the claim by taking differences. To verify equation (2.25) rigorously, we approximate $f(x)=\sqrt{x}$ on $[0,\infty)$ by the $C^2([0,\infty))$-functions $f^\varepsilon(x)=\sqrt{x+\varepsilon}$. Applying Itō's formula to the semi-martingale decomposition derived for X_t^2 gives

$$\begin{aligned}&\sqrt{X_t^2+\varepsilon}\\&=\sqrt{z^2+\varepsilon}+\Bigg(2\bigg(\Psi(1)+\int_{-1}^0(|u|-1)V(du)\bigg)\\&\quad+\sigma^2+\int_{-1}^1(u^2-2|u|+1)\,\bar\Pi(du)\Bigg)\frac{1}{2}\int_0^t(X_s^2+\varepsilon)^{-\frac12}X_s\,ds\\&\quad+\sigma\int_0^t(X_s^2+\varepsilon)^{-\frac12}X_s^{\frac32}\,dB_s-\frac{\sigma^2}{2}\int_0^t(X_s^2+\varepsilon)^{-\frac32}X_s^3\,ds\\&\quad+\int_0^t\int_0^{\frac{1}{X_{s-}}}\int_{-1}^1\Big(\sqrt{X_{s-}^2u^2+\varepsilon}-\sqrt{X_{s-}^2+\varepsilon}\Big)\,(\mathcal{N}-\mathcal{N}')(ds,dr,du)\\&\quad+\int_0^t\int_0^{\frac{1}{X_s}}\int_{-1}^1\Big(\sqrt{X_s^2u^2+\varepsilon}-\sqrt{X_s^2+\varepsilon}-\frac12(X_s^2+\varepsilon)^{-\frac12}X_s^2(u^2-1)\Big)\,ds\,dr\,\bar\Pi(du)\\&=:\sqrt{z^2+\varepsilon}+I_t^{1,\varepsilon}+I_t^{2,\varepsilon}+I_t^{3,\varepsilon}+I_t^{4,\varepsilon}+I_t^{5,\varepsilon}.\end{aligned}$$

Since the left-hand side converges to X_t almost surely as $\varepsilon\to 0$, it suffices to find a subsequence ε_k along which the summands $I_t^{1,\varepsilon_k},\dots,I_t^{5,\varepsilon_k}$ converge almost surely to the summands of (2.25).

For the drift we directly obtain the almost sure convergence

$$\begin{aligned}I_t^{1,\varepsilon}\overset{\varepsilon\to0}{\longrightarrow}\Bigg(\bigg(\Psi(1)+\int_{-1}^0(|u|-1)V(du)\bigg)\\+\frac{\sigma^2}{2}+\frac12\int_{-1}^1(u^2-2|u|+1)\,\bar\Pi(du)\Bigg)\int_0^t\mathbf{1}_{\{X_s>0\}}\,ds\end{aligned}$$

by dominated convergence. To show convergence of $I^{2,\varepsilon}$ we first use Itō's isometry to obtain

$$E\Big[\Big(I_t^{2,\varepsilon}-\sigma\int_0^t\sqrt{X_s}\,dB_s\Big)^2\Big]=\sigma^2E\Big[\int_0^t\big((X_s^2+\varepsilon)^{-\frac12}X_s^{\frac32}-X_s^{\frac12}\big)^2\,ds\Big]. \tag{2.26}$$

If we define $g_\varepsilon(x)=(x^2+\varepsilon)^{-\frac12}x^{\frac32}-x^{\frac12}$, then

$$\frac{\partial}{\partial\varepsilon}g_\varepsilon(x)=-\frac12(x^2+\varepsilon)^{-\frac32}x^{\frac32}\leq 0,\quad x\geq 0.$$

Since $g_\varepsilon(x)$ converges pointwise to zero as ε tends to zero, the right-hand side of (2.26) converges to zero by monotone convergence so that $I_t^{2,\varepsilon}$ converges to $\sigma\int_0^t\sqrt{X_s}\,dB_s$ in L^2. The almost sure convergence

$$I_t^{3,\varepsilon}\overset{\varepsilon\to0}{\longrightarrow}-\frac{\sigma^2}{2}\int_0^t\mathbf{1}_{\{X_s>0\}}\,ds$$

is proved as before and cancels in the limit the second summand of $I^{1,\varepsilon}$. To show convergence of $I_t^{4,\varepsilon}$ we use the Itō isometry for Poissonian integrals to find

$$\begin{aligned}&E\Big[\Big(I_t^{4,\varepsilon}-\int_0^t\int_0^{\frac{1}{X_{s-}}}\int_{-1}^1X_{s-}(|u|-1)(\mathcal{N}-\mathcal{N}')(ds,dr,du)\Big)^2\Big]\\&=E\Big[\int_0^t\int_{-1}^1\int_0^{\frac{1}{X_s}}\Big(\sqrt{X_s^2u^2+\varepsilon}-\sqrt{X_s^2+\varepsilon}-X_s(|u|-1)\Big)^2\,ds\,dr\,\bar\Pi(du)\Big].\end{aligned}$$

With $h_\varepsilon(x)=\sqrt{x^2u^2+\varepsilon}-\sqrt{x^2+\varepsilon}-x(|u|-1)$ we claim that

$$\frac{\partial}{\partial\varepsilon}h_\varepsilon(x)=\frac12\big((x^2u^2+\varepsilon)^{-\frac12}-(x^2+\varepsilon)^{-\frac12}\big)\geq0\quad\text{for all } x\geq0,|u|\leq1.$$

Hence, the L^2-convergence of $I_t^{4,\varepsilon}$ to $\int_0^t\int_0^{\frac{1}{X_{s-}}}\int_{-1}^1X_{s-}(|u|-1)(\mathcal{N}-\mathcal{N}')(ds,dr,du)$ follows again from monotone convergence.

Finally, if we rewrite $I_t^{5,\varepsilon}$ as

$$\begin{aligned}I_t^{5,\varepsilon}=&\int_0^t\int_{-1}^1\frac{1}{X_s}\Big(\sqrt{X_s^2u^2+\varepsilon}-\sqrt{X_s^2+\varepsilon}\Big)\,ds\,\bar\Pi(du)\\&-\int_0^t\int_{-1}^1\frac12(X_s^2+\varepsilon)^{-\frac12}X_s(u^2-1)\,ds\,\bar\Pi(du),\end{aligned}$$

then the first summand converges by monotone convergence shown as above and the second summand by dominated convergence:

$$I_t^{5,\varepsilon}\overset{\varepsilon\to0}{\longrightarrow}\int_{-1}^1\Big(|u|-1-\frac12(u^2-1)\Big)\bar\Pi(du)\int_0^t\mathbf{1}_{\{X_s>0\}}\,ds,\quad a.s.,$$

so that the third summand of $I^{1,\varepsilon}$ is cancelled in the limit.

Choosing a common subsequence ε_k such that all terms converge almost surely completes the proof. □

The previous lemma is the key to completing the construction of solutions to (1.8). First, it is clear that the limiting processes are non-trivial (i.e. not absorbed at zero) and, secondly, the problems in the weak convergence argument caused by the discontinuity of the sign-functions disappear.

Lemma 2.10. *Suppose Z denotes a limiting point of the tight sequence $(Z^m)_{m\in\mathbb{N}}$ constructed in Lemma* 2.5. *Then Z is a weak solution to the SDE* (1.8).

Proof. Taking into account Proposition 2.4 it suffices to show that any weak limiting point Z of the tight sequence $(Z^m)_{m\in\mathbb{N}}$ constructed in Lemma 2.5 satisfies the martingale problem $(\mathcal{A},\delta_z)$. The proof is along the same lines as the proof of Lemma 2.8 changing the state-space from $[0,\infty)$ to $\mathbb{R}$ and the generators to

$$
\begin{aligned}
&(\mathcal{A}^m f)(z)\\
&:= \Big(\Psi(1)+\int_{-1}^0 (u-1)V(du)\Big)\,\mathrm{sign}_{(0)}(z)f'(z)+\frac{\sigma^2}{2}|z|f''(z)\\
&\quad+\int_{\{\pm\frac{1}{m}\}}\mathbf{1}_{\{z=0\}}\Big[f\Big(\Big(\Psi(1)+\int_{-1}^0(|u|-1)V(du)\Big)v\Big)\\
&\qquad -f(0)-f'(0)\Big(\Psi(1)+\int_{-1}^0(|u|-1)V(du)\Big)v\Big]\Sigma(dv)\\
&\quad+\int_0^m\int_{[-1,1-\frac{1}{m}]}\mathbf{1}_{\{r|z|\le 1\}}\Big(f(uz)-f(z)-f'(z)z(u-1)\Big)\,dr\,\bar{\Pi}(du)
\end{aligned}
$$

for $f\in C_c^\infty((-\infty,\infty))$, and

$$
\begin{aligned}
&(\mathcal{A}f)(z)\\
&:= \Big(\Psi(1)+\int_{-1}^0 (u-1)V(du)\Big)\,\mathrm{sign}(z)f'(z)+\frac{\sigma^2}{2}|z|f''(z)\\
&\quad+\int_0^\infty\int_{-1}^1\mathbf{1}_{\{r|z|\le 1\}}\Big(f(uz)-f(z)-f'(z)z(u-1)\Big)\,dr\,\bar{\Pi}(du)
\end{aligned}
$$

for $f\in C_c^\infty((-\infty,\infty))$. Comparing with the proof of Lemma 2.8, the ony difference occurs in Step B1a) because the pointwise convergence for the drift coefficients fails since (i) the sign-function is defined differently for the approximating martingale problem and the limit martingale problem and (ii) both sign-functions are discontinuous. Both problems are avoidable via Corollary 2.9 applied for the first and fourth equality

$$
\begin{aligned}
\lim_{m\to\infty}\int_0^t \mathrm{sign}_{(0)}\big(Z_s^m(\omega)\big)\,ds &= \lim_{m\to\infty}\int_0^t \mathrm{sign}_{(0)}\big(Z_s^m(\omega)\big)\mathbf{1}_{\{Z_s(\omega)\neq 0\}}\,ds\\
&=\int_0^t \mathrm{sign}_{(0)}\big(Z_s(\omega)\big)\mathbf{1}_{\{Z_s(\omega)\neq 0\}}\,ds\\
&=\int_0^t \mathrm{sign}\big(Z_s(\omega)\big)\mathbf{1}_{\{Z_s(\omega)\neq 0\}}\,ds\\
&=\int_0^t \mathrm{sign}\big(Z_s(\omega)\big)\,ds.
\end{aligned}
$$

For the pointwise convergence of the integrands we used the continuity of the sign-function away from zero and dominated convergence to interchange limits and integration. The proof is now complete. ☐

Now that we have constructed processes Z^z started from $z \in \mathbb{R}$ that are weak solutions to (1.8) and symmetric by construction, we need to show that the family $(Z^z)_{z\in\mathbb{R}}$ is

(i) Markovian,
(ii) self-similar.

Both statements are derived from a weak uniqueness statement for (1.8) for which we had to impose Assumption **(A)**.

For initial condition zero, we derive moment equations for (1.8) from which, due to Assumption **(A)**, the well-posedness of the moment problem for one-dimensional marginals of symmetric solutions can be deduced. For initial conditions different from zero pathwise uniqueness before hitting zero holds. Combining the two uniqueness statements, uniqueness for one-dimensional marginals of solutions issued from the same initial condition follows. The Markov property is then a consequence of martingale problem theory and the self-similarity can be deduced from the self-similar structure of the coefficients in (1.8).

Proposition 2.11. *Denote by Z a limiting point of the tight sequence $(Z^m)_{m\in\mathbb{N}}$ with initial conditions $Z_0^m = z$ constructed in Lemma 2.5. Then Z is Markovian.*

Proof. Since we showed that Z is a weak solution to the SDE (1.8) we start with a weak uniqueness statement for any solution X to the SDE (1.8) that satisfies the symmetry property

$$P(X_{T_0+t} \in A) = P(X_{T_0+t} \in -A), \quad t \geq 0, A \in \mathcal{B}(\mathbb{R}), \tag{2.27}$$

and do not spend time at zero. Both properties hold for Z: the first by construction as a weak limit of the symmetric Z^m defined by (2.17), the latter by Corollary 2.9.

Step 1a): Let us first deduce the almost sure dichotomy $T_0 < \infty$ or $T_0 = \infty$ for the first hitting time of X at zero. A singular application of Itō's formula as in the proof of Corollary 2.9 using $|X| = \sqrt{X^2}$ yields the semi-martingale decomposition

(2.28)
$$\begin{aligned}|X_t| &= |z| + \left(\Psi(1) + \int_{-1}^0 (|u|-1)V(du)\right)t \\ &\quad + \sigma\int_0^t \sqrt{|X_s|}\,d\bar{B}_s + \int_0^t\int_0^{\frac{1}{|X_{s-}|}}\int_{-1}^1 |X_{s-}|(|u|-1)(\mathcal{N}-\mathcal{N}')(ds,dr,du)\end{aligned}$$

for $t \geq 0$ and with the Brownian motion $\bar{B}_t = \int_0^t \operatorname{sign}(X_s)dB_s$. We find that (2.28) coincides with (2.2) so that $|X|$ is a positive self-similar Markov process. Since the first hitting times at zero of X and $|X|$ coincide, the claimed dichotomy follows from Lamperti's dichotomy (Section 3 of [27]) for positive self-similar Markov processes.

Step 1b): Pathwise uniqueness holds for (1.8) up to first hitting zero: Suppose that X^1, X^2 are two solutions driven by the same noises B and $\mathcal{N}$ and set $T_{\frac{1}{n}} = \inf\{t \geq 0 : |X_t^1| \leq \frac{1}{n} \text{ or } |X_t^2| \leq \frac{1}{n}\}$ for $\frac{1}{n} < |X_0^i|$. Then,

$$P(X_t^1 = X_t^2 \text{ for all } t < T_{\frac{1}{n}}) = 1$$

since all integrands are locally Lipschitz continuous away from zero. Letting n tend to infinity and using the right-continuity of solutions we find that

$$P(X_t^1 = X_t^2 \text{ for all } t < T_0) = 1$$

and in particular that

$$T_0 = \inf\{t \geq 0 : X_t^1 = 0\} = \inf\{t \geq 0 : X_t^2 = 0\}.$$

In what follows we assume $T_0 < \infty$ almost surely; otherwise, we can directly proceed with Step 2) since pathwise uniqueness implies uniqueness in law.

Step 1c): The pathwise uniqueness implies that solutions to (1.8) are strong up to T_0 and, consequently, T_0 is a stopping time for B and $\mathcal{N}$. We denote by $\tilde{B}$ and $\tilde{N}$ the noises shifted by T_0. Due to the strong Markov property of the Brownian motion and the Poisson point process, $\tilde{B}$ is a Brownian motion and $\tilde{\mathcal{N}}$ a Poisson point process with same intensity as $\mathcal{N}$. Furthermore, we define the shifted process $(\tilde{X}_t = X_{T_0+t})_{t\geq 0}$ that satisfies

$$\begin{aligned}\tilde{X}_t = &\left(\Psi(1) + \int_{-1}^{0} (u-1)V(du)\right) \int_0^t \operatorname{sign}(\tilde{X}_s)\, ds + \sigma \int_0^t \sqrt{|\tilde{X}_s|}\, d\tilde{B}_s \\ &+ \int_0^t \int_0^{\frac{1}{|\tilde{X}_{s-}|}} \int_{-1}^{1} \tilde{X}_{s-}(u-1)(\tilde{\mathcal{N}} - \tilde{\mathcal{N}}')(ds, dr, du), \quad t \geq 0.\end{aligned} \tag{2.29}$$

In other words, $\tilde{X}$ is a weak solution to the SDE (1.8) with respect to the noises $\tilde{B}$ and $\tilde{\mathcal{N}}$ issued from zero. Furthermore, the symmetry condition (2.27) implies the symmetry condition

$$P\big(\tilde{X}_t \in A\big) = P\big(\tilde{X}_t \in -A\big), \quad t \geq 0, A \in \mathcal{B}(\mathbb{R}), \tag{2.30}$$

and clearly $\tilde{X}$ does not spend time at zero since X does not.

Step 1d): Next, we need all solutions to the SDE (2.29) that do not spend time at zero and satisfy the symmetry condition (2.30) to have the same one-dimensional marginals. The symmetry assumption implies that the one-dimensional laws $\tilde{X}_t$ are uniquely determined by $|\tilde{X}_t|$. Since $|\tilde{X}_t|$ satisfies the SDE (2.28) with $z = 0$ and the noises $\tilde{B}$ and $\tilde{\mathcal{N}}$ it suffices to show that the moment problem for (2.28) with $z = 0$ is well-posed; this follows from the main result of [3] since the jumps of

$$t \mapsto \int_0^t \int_0^{\frac{1}{|\tilde{X}_{s-}|}} \int_{-1}^{1} |\tilde{X}_{s-}|(|u|-1)(\tilde{\mathcal{N}} - \tilde{\mathcal{N}}')(ds, dr, du)$$

are negative (this is the reason for our Assumption **(A)**). It was shown in [3] that the k-th moments of $\tilde{X}$ equal $C_k t^k$ for some constants C_k that decrease sufficiently fast so that the moment problem is well-posed.[1] Furthermore, the C_k only depend on (a, σ^2, Π, q, V) but not on the solution, thus, well-posedness of the moment problem implies the uniqueness of one-dimensional marginals.

Step 1e): Let us now suppose X^1 and X^2 are two weak solutions to (1.8) that do not spend time at zero and both satisfy the symmetry condition (2.27). We split according to

$$P(X_t^i \in A) = P(X_t^i \in A\,,\, t \leq T_0^i) + P(X_t^i \in A\,,\, t > T_0^i)$$

[1]The moment formulas go back to Bertoin and Yor [9] for positive self-similar Markov processes; at this stage of the proof the self-similarity is not proved so we use that the moment formulas can be derived directly from (2.29) via Itō's formula. Even more, the self-similarity for solutions to (1.8) is proved below via the moment formulas for self-similar Markov processes.

and show that

$$P(X^1_t \in A\,,\, t \le T^1_0) = P(X^2_t \in A\,,\, t \le T^2_0), \tag{2.31}$$

$$P(X^1_t \in A\,,\, t > T^1_0) = P(X^2_t \in A\,,\, t > T^2_0). \tag{2.32}$$

Equality (2.31) follows from the pathwise uniqueness before hitting zero so that we only need to verify (2.32). Using the defining equation for the X^i and the definition of $\tilde{X}^i$ above, one can rewrite

$$P(X^i_t \in A\,,\, t > T^i_0) = P(\tilde{X}^i_{t-T^i_0} \in A\,,\, t > T^i_0).$$

Integrating out $P(T^i_0 \in ds)$ (note that $P(T^1_0 \in ds) = P(T^2_0 \in ds)$ has the same law as shown in Step 1b)) we obtain (2.32) from Step 1d) since $\tilde{X}^i_0 = 0$.

Step 2): The uniqueness of one-dimensional marginals for symmetric weak solutions to (1.8) now implies the Markov property for the weak solution Z by martingale problem arguments such as in the proof of Theorem 4.4.2 of [18]. The required measurability $z \mapsto P^z$ is a consequence of the construction: the measurability (even continuity) in the initial condition holds for the Z^m by construction and since the pointwise limit of measurable functions remains measurable the measurability for the limit follows. □

Proposition 2.12. *Denote by Z^z a limiting point of the tight sequence $(Z^m)_{m\in\mathbb{N}}$ with initial conditions $z \in \mathbb{R}$ constructed in Lemma* 2.5. *Then the family $(Z^z)_{z\in\mathbb{R}}$ is a real-valued self-similar Markov family with Lamperti-Kiu quintuple (a, σ^2, Π, q, V).*

Proof. For $c > 0$ fixed we define $\bar{Z}^z_t := \frac{1}{c} Z^z_{ct}$, $t \ge 0$. Since Z is a weak solution to (1.8), $\bar{Z}^z$ satisfies

$$\begin{aligned}\bar{Z}^z_t = \frac{z}{c} &+ \left(\Psi(1) + \int_{-1}^0 (u-1)V(du)\right)\frac{1}{c}\int_0^{ct} \operatorname{sign}(Z^z_s)\,ds + \frac{\sigma}{c}\int_0^{ct}\sqrt{|Z^z_s|}\,dB_s\\ &+ \frac{1}{c}\int_0^{ct}\int_0^{\frac{1}{|Z^z_{s-}|}}\int_{-1}^1 Z^z_{s-}(u-1)(\mathcal{N}-\mathcal{N}')(ds,dr,du), \qquad t \ge 0,\end{aligned}$$

almost surely. We have the almost sure identities

$$\frac{\sigma}{c}\int_0^{ct}\sqrt{|Z^z_s|}\,dB_s = \sigma\int_0^t \sqrt{c^{-1}|Z^z_{cs}|}d\big(c^{-1/2}B_{cs}\big), \quad t \ge 0,$$

and

$$\begin{aligned}&\int_0^{ct}\int_0^{\frac{1}{|Z^z_{s-}|}}\int_{-1}^1 Z^z_{s-}(u-1)(\mathcal{N}-\mathcal{N}')(ds,dr,du)\\ &= \int_0^t\int_0^{\frac{1}{\frac{1}{c}|Z^z_{(cs)-}|}}\int_{-1}^1 Z^z_{(cs)-}(u-1)(\mathcal{N}-\mathcal{N}')(cds,c^{-1}dr,du), \qquad t \ge 0,\end{aligned}$$

forcing us to define the Wiener process $\bar{B}_t := \frac{1}{\sqrt{c}}B_{ct}$, $t \ge 0$, and the independent Poisson random measures $\bar{\mathcal{N}}$ on $(0,\infty)\times(0,\infty)\times[-1,1]$ by $\bar{\mathcal{N}}(ds,dr,du) := \mathcal{N}(cds, c^{-1}dr, du)$. It follows directly from the definition of a Poisson random measure that $\bar{\mathcal{N}}$ is a Poisson random measure with the same intensity measure as $\mathcal{N}$.

With these definitions, the above calculation leads to

$$\bar{Z}_t^z = \frac{z}{c} + \left(\Psi(1) + \int_{-1}^0 (u-1)V(du)\right) \int_0^t \operatorname{sign}(\bar{Z}_s^z)\,ds + \sigma \int_0^t \sqrt{|\bar{Z}_s^z|}d\bar{B}_s$$
$$+ \int_0^t \int_0^{\frac{1}{|\bar{Z}_{s-}^z|}} \int_{-1}^1 \bar{Z}_{s-}^z (u-1)(\bar{\mathcal{N}} - \bar{\mathcal{N}}')(ds, dr, du), \quad t \geq 0.$$

Hence, $Z^{z/c}$ and $\bar{Z}^z$ both satisfy the SDE (1.8) with initial condition z/c and, by Corollary 2.9, both do not spend time at zero. Furthermore both satisfy the symmetry condition (2.27) by construction of Z. But then the equality of one-dimensional marginals holds due to Step 1e) of the proof of Proposition 2.11. Finally, the Markov property proved in Proposition 2.11 implies the identification of the finite-dimensional marginals and the self-similarity is proved.

The statement about the Lamperti-Kiu quintuple is a direct consequence of the construction of Z^z via the SDE (1.8) and the definition of Lamperti-Kiu quintuples. □

2.3. **Proof of Theorem 1.5.** Let us start with a simple reformulation of the necessary and sufficient conditions for the extendability of a positive self-similar Markov process in a special situation. Recall that ξ always denotes the Lévy process with Laplace exponent Ψ for a positive self-similar Markov process.

Lemma 2.13. *Suppose $(P^z)_{z\geq 0}$ is a positive self-similar Markov process that only jumps towards the origin. Then there is a unique self-similar extension $(P^z)_{z\geq 0}$ of $(P^z)_{z>0}$ under which the canonical process leaves zero continuously precisely if $\Psi(1) > 0$.*

Proof. First note that the Lévy process ξ from Lamperti's transformation (1.2) is spectrally negative. Hence, $\Psi(\lambda) = \log E[e^{\lambda \xi_1}] < \infty$ for all $\lambda \geq 0$, for instance by the Lévy-Khintchin formula. Note furthermore that, if well-defined, the Laplace exponent $\lambda \mapsto \Psi(\lambda)$ is a convex function on $\mathbb{R}_{\geq 0}$.

First, suppose ξ drifts to $-\infty$, so that $\Psi'(0+) < 0$ and the existence of the claimed extension is equivalent to (1.4). The claim is a direct consequence of the convexity.

Next, suppose ξ does not drift to $-\infty$. Then the existence of the claimed extension is equivalent to (1.6) which is trivially fulfilled since all overshoots are zero because all jumps are negative. At the same time the convexity of Ψ and $\Psi'(0+) \geq 0$ imply $\Psi(\lambda) > 0$ for any $\lambda > 0$. Hence, the claimed equivalence is trivial in this latter case. □

To find a necessary condition for $\mathbb{R}_*$-valued self-similar Markov processes to have an extension that leaves zero continuously we want to apply the previous lemma for a suitable positive self-similar Markov process. Since we are only interested in symmetric self-similar processes the good choice is the absolute value.

Lemma 2.14. *Suppose $(P^z)_{z\in\mathbb{R}}$ is a symmetric $\mathbb{R}_*$-valued self-similar Markov process with Lamperti-Kiu quintuple (a, σ^2, Π, q, V) that satisfies Assumption **(A)** and define $(|P|^z)_{z\geq 0}$ as the law of $|Z|$ under $(P^z)_{z\in\mathbb{R}}$.*

(a) *$(|P|^z)_{z\geq 0}$ is a positive self-similar Markov process.*

(b) *The Lamperti-transformed Lévy process* $\xi^{|P|}$ *of* $(|P|^z)_{z\geq 0}$ *satisfies*

$$\Psi^{|P|}(1) = \Psi(1) + \int_{-1}^{0} (|u| - 1)V(du),$$

where Ψ *is the Laplace exponent of the Lévy process* ξ *with triplet* (a, σ^2, Π) *killed at rate* q.

Proof. **(a)** The Markov property for $(|P|^z)_{z\geq 0}$ is inherited from $(P^z)_{z\in\mathbb{R}}$ due to the symmetry assumption. The self-similarity carries over trivially.

(b) To determine $\Psi^{|P|}(1)$ we use Proposition 2.1 twice. First recall from Proposition 1.2 that P^z can be expressed by

$$\begin{aligned} Z_t = z &+ \left(\Psi(1) + \int_{-1}^{0} (u-1)V(du)\right) \int_0^t \operatorname{sign}(Z_s)\, ds + \sigma \int_0^t \sqrt{|Z_s|}\, dB_s \\ &+ \int_0^t \int_0^{\frac{1}{|Z_{s-}|}} \int_{-1}^{1} Z_{s-}(u-1)(\mathcal{N} - \mathcal{N}')(ds, dr, du), \quad t \leq T_0. \end{aligned}$$

Taking absolute values $|Z| = \sqrt{Z^2}$ by twice applying Itō's rule similarly to the argument utilized in Corollary 2.9 we find that $|P|^z$ can be expressed by

$$\begin{aligned} |Z_t| = &\left(\Psi(1) + \int_{-1}^{0} (|u|-1)V(du)\right) t + \sigma \int_0^t \sqrt{|Z_s|}\, d\bar{B}_s \\ &+ \int_0^t \int_0^{\frac{1}{|Z_{s-}|}} \int_{-1}^{1} |Z_{s-}|(|u|-1)(\mathcal{N} - \mathcal{N}')(ds, dr, du), \quad t \leq T_0, \end{aligned}$$

with the Brownian motion $\bar{B}_t := \int_0^t \operatorname{sign}(Z_s)\, dB_s$. Equivalently, we can write

$$\begin{aligned} |Z_t| = &\left(\Psi(1) + \int_{-1}^{0} (|u|-1)V(du)\right) t + \sigma \int_0^t \sqrt{|Z_s|}\, d\bar{B}_s \\ &+ \int_0^t \int_0^{\frac{1}{|Z_{s-}|}} \int_{0}^{1} |Z_{s-}|(u-1)(\bar{\mathcal{N}} - \bar{\mathcal{N}}')(ds, dr, du), \quad t \leq T_0, \end{aligned} \tag{2.33}$$

where $\bar{\mathcal{N}}$ has intensity $ds \otimes dr \otimes \big(\bar{\Pi}(du) + V(-du)\big)$ on $(0,\infty) \times (0,\infty) \times (0,1)$. Comparing with (2.2) we can read off the Lévy triplet for $\xi^{|P|}$ and in particular the Laplace exponent evaluated at 1. ☐

We can now finish the proof of our main result.

Proof of Theorem 1.5. Recall from Proposition 1.2 that the $\mathbb{R}_*$-valued self-similar symmetric Markov families obtained from real-valued self-similar Markov families by absorption at zero are completely characterized by Lamperti-Kiu quintuples (a, σ^2, Π, q, V).

To see that condition (1.10) is necessary we apply Lemmas 2.13 and 2.14: Suppose $(P^z)_{z\in\mathbb{R}}$ is a real-valued self-similar Markov process that leaves zero continuously. Then the Markov family $(|P^\dagger|^z)_{z\geq 0}$ obtained by absorption at zero is a positive self-similar Markov family with a self-similar extension that leaves zero continuously. The Laplace exponent of the Lamperti transformed Lévy process satisfies

$$\Psi^{|P^\dagger|}(1) = \Psi(1) + \int_{-1}^{0} (|u| - 1)V(du)$$

which, as we showed in Lemma 2.13 has to be strictly positive.

Conversely, if condition (1.10) is satisfied for a given quintuple (a, σ^2, Π, q, V), then by Proposition 2.12, we constructed in Section 2.2 a real-valued self-similar Markov process with Lamperti-Kiu triplet (a, σ^2, Π, q, V). Furthermore, the solutions Z^z leave zero continuously since the integrand of the Poissonian integral is zero at zero. □

Acknowledgement

The author would like to thank Jean Bertoin and Zenghu Li for interesting discussions on the subject and Matyas Barczy for careful reading of earlier manuscripts. Furthermore, the author thanks Zenghu Li for the very kind hospitality at Beijing Normal University, where part of this work was carried out. The careful reading by an anonymous referee was very helpful in enhancing the presentation.

References

[1] David Aldous, *Stopping times and tightness*, Ann. Probability **6** (1978), no. 2, 335–340. MR0474446 (57 #14086)

[2] David Applebaum, *Lévy processes and stochastic calculus*, 2nd ed., Cambridge Studies in Advanced Mathematics, vol. 116, Cambridge University Press, Cambridge, 2009. MR2512800 (2010m:60002)

[3] Mátyás Barczy and Leif Döring, *On entire moments of self-similar Markov processes*, Stoch. Anal. Appl. **31** (2013), no. 2, 191–198, DOI 10.1080/07362994.2013.741397. MR3021485

[4] Richard F. Bass, Krzysztof Burdzy, and Zhen-Qing Chen, *Pathwise uniqueness for a degenerate stochastic differential equation*, Ann. Probab. **35** (2007), no. 6, 2385–2418, DOI 10.1214/009117907000000033. MR2353392 (2009c:60144)

[5] J. Berestycki, L. Döring, L. Mytnik and L. Zambotti, *On existence and uniqueness for self-similar SDEs driven by stable processes*, arXiv:1111.4388 (2012)

[6] Jean Bertoin and Maria-Emilia Caballero, *Entrance from 0+ for increasing semi-stable Markov processes*, Bernoulli **8** (2002), no. 2, 195–205. MR1895890 (2003c:60071)

[7] Jean Bertoin and Mladen Savov, *Some applications of duality for Lévy processes in a half-line*, Bull. Lond. Math. Soc. **43** (2011), no. 1, 97–110, DOI 10.1112/blms/bdq084. MR2765554 (2011k:60168)

[8] Jean Bertoin and Marc Yor, *The entrance laws of self-similar Markov processes and exponential functionals of Lévy processes*, Potential Anal. **17** (2002), no. 4, 389–400, DOI 10.1023/A:1016377720516. MR1918243 (2003i:60082)

[9] Jean Bertoin and Marc Yor, *On the entire moments of self-similar Markov processes and exponential functionals of Lévy processes* (English, with English and French summaries), Ann. Fac. Sci. Toulouse Math. (6) **11** (2002), no. 1, 33–45. MR1986381 (2004d:60119)

[10] Jean Bertoin and Marc Yor, *Exponential functionals of Lévy processes*, Probab. Surv. **2** (2005), 191–212, DOI 10.1214/154957805100000122. MR2178044 (2007b:60116)

[11] R. M. Blumenthal, *On construction of Markov processes*, Z. Wahrsch. Verw. Gebiete **63** (1983), no. 4, 433–444, DOI 10.1007/BF00533718. MR705615 (84j:60081)

[12] M. E. Caballero and L. Chaumont, *Weak convergence of positive self-similar Markov processes and overshoots of Lévy processes*, Ann. Probab. **34** (2006), no. 3, 1012–1034, DOI 10.1214/009117905000000611. MR2243877 (2008c:60035)

[13] Loïc Chaumont, Henry Pantí, and Víctor Rivero, *The Lamperti representation of real-valued self-similar Markov processes*, Bernoulli **19** (2013), no. 5B, 2494–2523, DOI 10.3150/12-BEJ460. MR3160562

[14] Loïc Chaumont, Andreas Kyprianou, Juan Carlos Pardo, and Víctor Rivero, *Fluctuation theory and exit systems for positive self-similar Markov processes*, Ann. Probab. **40** (2012), no. 1, 245–279, DOI 10.1214/10-AOP612. MR2917773

[15] Oleksandr Chybiryakov, *The Lamperti correspondence extended to Lévy processes and semi-stable Markov processes in locally compact groups*, Stochastic Process. Appl. **116** (2006), no. 5, 857–872, DOI 10.1016/j.spa.2005.11.009. MR2218339 (2006m:60058)

[16] Claude Dellacherie and Paul-André Meyer, *Probabilités et potentiel. Chapitres V à VIII*, Théorie des martingales. [Martingale theory], Revised edition, Actualités Scientifiques et Industrielles [Current Scientific and Industrial Topics], vol. 1385, Hermann, Paris, 1980 (French). MR566768 (82b:60001)

[17] Leif Döring and Mátyás Barczy, *A jump type SDE approach to positive self-similar Markov processes*, Electron. J. Probab. **17** (2012), no. 94, 39. MR2994842

[18] Stewart N. Ethier and Thomas G. Kurtz, *Markov processes*, Characterization and convergence, Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, John Wiley & Sons Inc., New York, 1986. MR838085 (88a:60130)

[19] P. J. Fitzsimmons, *On the existence of recurrent extensions of self-similar Markov processes*, Electron. Comm. Probab. **11** (2006), 230–241, DOI 10.1214/ECP.v11-1222. MR2266714 (2008h:60143)

[20] Zongfei Fu and Zenghu Li, *Stochastic equations of non-negative processes with jumps*, Stochastic Process. Appl. **120** (2010), no. 3, 306–330, DOI 10.1016/j.spa.2009.11.005. MR2584896 (2011d:60178)

[21] Nobuyuki Ikeda and Shinzo Watanabe, *Stochastic differential equations and diffusion processes*, North-Holland Mathematical Library, vol. 24, North-Holland Publishing Co., Amsterdam, 1981. MR637061 (84b:60080)

[22] Jean Jacod and Albert N. Shiryaev, *Limit theorems for stochastic processes*, 2nd ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 288, Springer-Verlag, Berlin, 2003. MR1943877 (2003j:60001)

[23] Ioannis Karatzas and Steven E. Shreve, *Brownian motion and stochastic calculus*, 2nd ed., Graduate Texts in Mathematics, vol. 113, Springer-Verlag, New York, 1991. MR1121940 (92h:60127)

[24] Sun Wah Kiu, *Two dimensional semi-stable Markov processes*, Ann. Probability **3** (1975), no. 3, 440–448. MR0388563 (52 #9399)

[25] Sun Wah Kiu, *Semistable Markov processes in* $\mathbf{R}^n$, Stochastic Process. Appl. **10** (1980), no. 2, 183–191, DOI 10.1016/0304-4149(80)90020-4. MR587423 (82g:60098)

[26] A. Kuznetsov and J. C. Pardo, *Fluctuations of stable processes and exponential functionals of hypergeometric Lévy processes*, Acta Appl. Math. **123** (2013), 113–139, DOI 10.1007/s10440-012-9718-y. MR3010227

[27] John Lamperti, *Semi-stable Markov processes. I*, Z. Wahrsch. Verw. Gebiete **22** (1972), 205–225. MR0307358 (46 #6478)

[28] Zenghu Li and Fei Pu, *Strong solutions of jump-type stochastic equations*, Electron. Commun. Probab. **17** (2012), no. 33, 13, DOI 10.1214/ECP.v17-1915. MR2965746

[29] Pierre Patie, *Exponential functional of a new family of Lévy processes and self-similar continuous state branching processes with immigration* (English, with English and French summaries), Bull. Sci. Math. **133** (2009), no. 4, 355–382, DOI 10.1016/j.bulsci.2008.10.001. MR2532690 (2010j:60118)

[30] Philip E. Protter, *Stochastic integration and differential equations,* Stochastic Modelling and Applied Probability, 2nd ed., Applications of Mathematics (New York), vol. 21, Springer-Verlag, Berlin, 2004. MR2020294 (2005k:60008)

[31] Daniel Revuz and Marc Yor, *Continuous martingales and Brownian motion*, 3rd ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 293, Springer-Verlag, Berlin, 1999. MR1725357 (2000h:60050)

[32] Víctor Rivero, *Recurrent extensions of self-similar Markov processes and Cramér's condition*, Bernoulli **11** (2005), no. 3, 471–509, DOI 10.3150/bj/1120591185. MR2146891 (2006d:60118)

[33] Víctor Rivero, *Recurrent extensions of self-similar Markov processes and Cramér's condition. II*, Bernoulli **13** (2007), no. 4, 1053–1070, DOI 10.3150/07-BEJ6082. MR2364226 (2008k:60173)

[34] Ken-iti Sato, *Lévy processes and infinitely divisible distributions*, Cambridge Studies in Advanced Mathematics, vol. 68, Cambridge University Press, Cambridge, 1999. Translated from the 1990 Japanese original; Revised by the author. MR1739520 (2003b:60064)

[35] Toshio Yamada and Shinzo Watanabe, *On the uniqueness of solutions of stochastic differential equations.*, J. Math. Kyoto Univ. **11** (1971), 155–167. MR0278420 (43 #4150)

Laboratoire de Probabilités et Modèles Aléatoires, Université Paris VI, 4, Place Jussieu, 75005 Paris, France

E-mail address: `leif.doering@googlemail.com`